ANALYSIS & PDE

Volume 9 No. 6 2016

ELINA ANDRIYANOVA AND SHIBING CHEN

BOUNDARY $C^{1,\alpha}$ REGULARITY OF POTENTIAL FUNCTIONS IN OPTIMAL TRANSPORTATION WITH QUADRATIC COST

msp

ANALYSIS AND PDE
Vol. 9, No. 6, 2016
dx.doi.org/10.2140/apde.2016.9.1483

BOUNDARY $C^{1,\alpha}$ REGULARITY OF POTENTIAL FUNCTIONS IN OPTIMAL TRANSPORTATION WITH QUADRATIC COST

ELINA ANDRIYANOVA AND SHIBING CHEN

We provide a different proof for the global $C^{1,\alpha}$ regularity of potential functions in the optimal transport problem, which was originally proved by Caffarelli. Our method applies to a more general class of domains.

1. Introduction

We study the global $C^{1,\alpha}$ regularity of potential functions in optimal transportation with quadratic cost. Let Ω and Ω^* be the source and target domains associated with densities $1/C < f < C$ and $1/C < g < C$, respectively, where C is a positive constant. The optimal transport problem with quadratic cost is about finding a map $T : \Omega \to \Omega^*$ among all measure-preserving maps minimizing the transportation cost

$$\int_\Omega |x - Tx|^2 \, dx.$$

Here the term "measure-preserving" means that $\int_{T^{-1}(B)} f = \int_B g$ for any Borel set $B \subset \Omega^*$. Brenier [1991] proved that one can find a convex function u such that

$$T(x) = Du(x) \quad \text{for a.e. } x \in \Omega.$$

Indeed, the convex function u satisfies $\int_{(\partial u)^{-1}B} f = \int_B g$ for any Borel set $B \subset \Omega$, where ∂u is the standard subgradient map of the convex function u. We call u a *Brenier solution* of the optimal transport problem if it satisfies the property above. When the target domain Ω^* is convex, Caffarelli proved that $\partial u(\Omega) = \Omega^*$ and that u is an *Alexandrov solution*, namely u satisfies $(1/C)|A \cap \Omega| \le |\partial u(A)| \le C|A \cap \Omega|$ for any Borel set $A \subset \Omega$. Moreover, if we extend u to $\mathbb{R}^n$ via

$$\tilde{u} := \sup\{L \mid L \text{ is linear, } L|_\Omega \le u, \ L(z) = u(z) \text{ for some } z \in \Omega\},$$

then $\tilde{u}$ is a globally Lipschitz convex solution of

$$C^{-1}\chi_\Omega \le \det \tilde{u}_{ij} \le C\chi_\Omega.$$

We will still use u to denote this extended function. Caffarelli [1992b] proved interior $C^{1,\alpha}$ regularity by using his techniques for studying the standard Monge–Ampère-type equation; see [Caffarelli 1990a; 1990b; 1991].

This work is supported by the Australian Research Council. The authors would like thank Professor Xu-Jia Wang for sharing his proof of Lemma 2.5.
MSC2010: 35B45, 35J60, 49Q20, 35J96.
Keywords: optimal transport, quadratic cost, boundary regularity.

Then, Caffarelli [1992a] proved the boundary $C^{1,\alpha}$ regularity result under the condition that both Ω and Ω^* are convex. Below we will briefly discuss the main ideas involved in his proof. First, Caffarelli established a fundamental property of convex functions, namely the existence of sections centred at a given point (see the statement of Lemma 2.5). Then, he proved that such sections are decaying geometrically, namely there exists a constant δ such that

$$S_{\delta h}(y) \subset \tfrac{3}{4} S_h(x) \quad \text{for any } \; y \in \tfrac{1}{2} S_h(x). \tag{1-1}$$

Here $S_h(x)$ denotes the section of u centred at x with height h. From (1-1) we obtain the quantitative strict convexity estimate

$$u(z) \geq u(x) + Du(x)\cdot(z-x) + C|z-x|^{\beta} \quad \text{for any } \; x, z \in \overline{\Omega}, \tag{1-2}$$

for some $\beta > 1$. From (1-2), it is easy to check that u^*, the standard Legendre transform of u, is $C^{1,\alpha}$ on $\overline{\Omega}^*$. Recall the well-known fact that u^* is indeed the potential function of the optimal transport problem from Ω^* to Ω. Therefore, by switching the role of u and u^* one can show the global $C^{1,\alpha}$ regularity of u.

The convexity of domains is crucial in Caffarelli's approach. Indeed, the convexity of Ω ensures that u^* is an Alexandrov solution, while the convexity of Ω^* ensures that the sections of u^*, centred at some point in $\overline{\Omega}^*$, have some doubling property. Here we provide a different proof of the global $C^{1,\alpha}$ result. Instead of deducing the $C^{1,\alpha}$ regularity of u from the strict convexity of u^*, we prove the $C^{1,\alpha}$ regularity of u directly. Moreover, our method works for a slightly more general class of domains, namely we allow the source to be a domain obtained by removing finitely many disjoint convex subsets from a convex domain.

We would like to mention that in recent years the regularity of optimal transport maps has attracted much interest and there are many important works related to it; to cite a few, see [Figalli and Loeper 2009; Liu 2009; Trudinger and Wang 2009b; 2009a; Figalli and Rifford 2009; Loeper 2011; Loeper and Villani 2010; Liu et al. 2010; Kim and McCann 2010; Figalli et al. 2010; 2011; 2012; 2013a; 2013b].

The rest of the paper is organized as follows. In Section 2 we introduce some notations and preliminaries, and state the main results. Section 3 is devoted to the proof of global C^1 regularity. In the last section we complete the proof of the main results.

2. Preliminaries and main result

The main result of this paper is the following theorem:

Theorem 2.1. *Let Ω and Ω^* be two bounded domains in $\mathbb{R}^n$, $n \geq 2$, and f and g be densities of two positive probability measures defined in Ω and Ω^*, respectively, satisfying $C^{-1} \leq f, g \leq C$ for a positive constant C. Assume that Ω^* is convex and Ω is Lipschitz.*

(i) *If, for any given $x \in \overline{\Omega}$, there exists a small ball $B_{r_x}(x)$ such that, for any convex set $\omega \subset B_{r_x}(x)$ centred in Ω, we have $\int_\omega f \leq C \int_{\omega/2} f$ for some constant C independent of ω, then the potential function u is $C^1(\overline{\Omega})$.* (Here f is defined to be 0 outside Ω.)

(ii) *If Ω is a domain obtained by removing finitely many disjoint convex subsets from a convex set, then the potential function u is $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

Remark 2.2. (a) It is easy to see that in Theorem 2.1(i) we allow Ω to be any polytope (not necessarily convex). We also note that the C^1 regularity always holds in dimension two without any condition on Ω. This is a classical result of Alexandrov; see also [Figalli and Loeper 2009].

(b) One may want to prove higher regularity when the densities are smooth; however, in view of the following simple example we see that this is impossible. Let the dimension be $n = 2$. Let $\Omega := B_2 - B_1$, with uniform probability density, and let $\Omega^* := B_{\sqrt{3}}$, with uniform probability density. Then by symmetry it is easy to compute that the optimal transport map is $T(x) = \sqrt{|x|^2 - 1}\, x/|x|$, which is only $C^{1/2}$ on $\partial B_1 \subset \partial\Omega$.

In the following we will use $S_h(x_0)$ to denote a section of u with height h, namely

$$S_h(x_0) := \{x \mid u < p \cdot (x - x_0) + h\},$$

where p is chosen so that x_0 is the centre of mass of $S_h(x_0)$. We say a point $x_0 \in \overline{\Omega}$ is localized (with respect to u) if, for any sequences $h_k \to 0$ and $x_k \to x_0$ satisfying $x_0 \in S_{h_k}(x_k)$, we have that $S_{h_k}(x_k)$ shrinks to the point $x_0 \in \overline{\Omega}$.

Now we record a fundamental property of convex sets.

Lemma 2.3 (John's lemma). *Let $U \subset \mathbb{R}^n$ be a bounded, convex domain with its centre of mass at the origin. There exists an ellipsoid E, also centred at the origin, such that*

$$E \subset U \subset n^{3/2} E.$$

The original John's lemma does not require that the ellipsoid is centered at the origin, and the constant $n^{3/2}$ can be replaced by n. We refer the reader to [Liu and Wang 2015] for a simple proof of the existence and uniqueness of such an ellipsoid.

By John's lemma we can show the following property of convex functions:

Lemma 2.4. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let L be a supporting function of u. Then any extreme point of $\{u = L\}$ is localized.*

Proof. Suppose to the contrary that there exists an extreme point x_0 of $\{u = L\}$ which is not localized. Then there exist sequences $x_k \to x_0$ and $h_k \to 0$ such that $x_0 \in S_{h_k}(x_k)$, and that $S_{h_k}(x_k)$ contains a segment of length greater than or equal to some positive constant δ. Since $S_{h_k}(x_k)$ is convex and centred at x_k, by John's lemma there exists a unit vector ξ_k such that I_k, the segment connecting $x_k - \delta/(2n^{3/2})\,\xi_k$ and $x_k + \delta/(2n^{3/2})\,\xi_k$, is contained in $S_{h_k}(x_k)$. Denote by L_k the defining function of $S_{h_k}(x_k)$, namely $S_{h_k}(x_k) = \{u \le L_k\}$. Then it is easy to see that DL_k is bounded; hence, by passing to a subsequence, $L_k \to L_\infty$ for some linear function L_∞. Also by passing to a subsequence we may assume $\xi_k \to \xi_\infty$ for some unit vector ξ_∞. Then u is linear on I_∞, which is the segment connecting $x_0 - \delta/(2n^{3/2})\,\xi_\infty$ and $x_0 + \delta/(2n^{3/2})\,\xi_\infty$. Hence $I_\infty \subset \{u = L\}$, which contradicts the assumption that x_0 is an extreme point of $\{u = L\}$. □

The following property of sections of convex functions was proved by Caffarelli [1992a]. Here we provide a different proof by using a well-known fact that if a continuous map from a ball to itself fixes the boundary then it must be surjective. We learned this method from Wang; see [Sheng et al. 2004, Section 4].

Lemma 2.5. *Let* $u : \mathbb{R}^n \to [0, \infty]$ *be a convex function. Assume that*:

(1) $u(0) = 0,\ u \geq 0$.

(2) *u is finite in a neighbourhood of* 0.

(3) *The graph of u contains no complete lines.*

Then for $h > 0$ *there exists a slope* p *such that the centre of mass of the section*

$$S_{h,p} := \{x \mid u \leq x \cdot p + h\}$$

is defined and equal to 0.

Proof. Let

$$\begin{cases} u_k(x) = u(x) & \text{in } B_k, \\ u_k = \infty & \text{in } \mathbb{R}^n - B_k. \end{cases} \tag{2-1}$$

We only need to show the existence of sections $S^k := \{x \mid u_k \leq x \cdot p_k + h\}$ centred at 0 with bounded p_k. Then $S_{h,p} = \lim_{k\to\infty} S^k$ is the desired section in the lemma.

Take a large ball B_r. For any $p \in B_r$, let z_p be the centre of mass of the section $S_p := \{x \mid u_k(x) \leq x \cdot p + h\}$. Then we obtain a mapping $M_1 : p \to z_p$ from B_r to $\mathbb{R}^n$. If $p \in \partial B_r$, it is easy to see that $p \cdot z_p > 0$ provided r is sufficiently large.

If there is no $p \in B_r$ such that $z_p = 0$, then we can define a mapping $M_2 : z_p \to t_p z_p$, where $t_p > 0$ is a constant such that $t_p z_p \in \partial B_r$. We then obtain a continuous mapping $M = M_2 \circ M_1$ from B_r to ∂B_r with the property that

$$p \cdot M(p) > 0 \quad \text{on } \partial B_r. \tag{2-2}$$

To get a contradiction, we extend the mapping M to B_{2r} as follows. For any point $p \in \partial B_{2r}$, let $p_1 = p,\ p_0 = \frac{1}{2} p \in \partial B_r$ and $p_t = (1-t)p_0 + p_1$. We extend the mapping M to B_{2r} by letting $M(p_t) = (1-t)M(p_0) + tp_1$. Then, by (2-2), $M(p) \neq 0$ on B_{2r} and M is the identity mapping on ∂B_{2r}. This is a contradiction.

Hence, for each $k > 0$, there exists a $p_k \in \mathbb{R}^n$ such that $S^k := \{x \mid u_k \leq x \cdot p_k + h\}$ is centred at 0. Moreover, $|p_k| \leq C$ for some constant independent of k. Indeed, we can argue as follows: By rotating the coordinates we may assume $p_k = (a, 0, \dots, 0)$ with $a > 0$. Let $\alpha^+ = \sup\{x_1 \mid (x_1, 0, \dots, 0) \in S^k\}$ and $\alpha^- = -\inf\{x_1 \mid (x_1, 0, \dots, 0) \in S^k\}$. Then $\alpha^+/\alpha^- \to \infty$ as $a \to \infty$. Since S_k is centred at 0, a cannot be too large. □

The following Alexandrov-type estimates were proved by Caffarelli [1996]:

Lemma 2.6. *Let u be a convex solution of*

$$\det D^2 u = d\mu$$

in the convex domain S with $u = 0$ *on* ∂S. *Assume S is normalized, namely* $B_1 \subset S \subset n^{3/2} B_1$. *Assume* $d\mu(S) \leq \theta\, d\mu\big(\frac{1}{2} S\big)$ *for some constant* θ, *where* $\frac{1}{2} S$ *is a dilation of S with respect to the origin.*

(a) $(1/C)|\inf_S u|^n \leq d\mu(S) \leq C|\inf_S u|^n$, *where C is a constant depending only on* θ.

(b) $|u(x)|^n \leq C\, d\mu(S)\, d(x, \partial S)$.

3. Global C^1 regularity

In this section, we prove Theorem 2.1(i).

Lemma 3.1. *Suppose u is a globally Lipschitz convex function. Assume that u is C^1 at all of the extreme points of a convex set $K = \{u = L\}$, where L is a linear function satisfying $u \geq L$ and $u(y) = L(y)$ for some $y \in \mathbb{R}^n$. Then u is C^1 on K.*

Proof. By subtracting L we may assume $K = \{u = 0\}$. If K is a bounded convex set, then for any $x \in K$ we have

$$x = \sum_{i=1}^{k} \lambda_i x_i,$$

where x_i, $i = 1, \ldots, k$, are extreme points of K, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since u is C^1 at x_i, $i = 1, \ldots, k$, we have $0 \leq u(z) = o(z - x_i)$, $i = 1, \ldots, k$. Now, by convexity we have

$$0 \leq u(z) = u\biggl(\sum_{i=1}^{k} \lambda_i (z - x + x_i)\biggr) \leq \sum_{i=1}^{k} \lambda_i u(z - x + x_i) = \sum_{i=1}^{k} \lambda_i o(z - x) = o(z - x).$$

Hence, u is C^1 at x.

If K is unbounded, it is well-known that $K = \operatorname{covext}[K] + \operatorname{rc}[K]$, where $\operatorname{covext}[K]$ is the convex hull of the extreme points of K, and $\operatorname{rc}[K] := \lim_{\lambda\downarrow 0} \lambda K$ is the recession cone of K. Hence we need only to show that u is C^1 at points represented by $x = x_0 + q$, where x_0 is an extreme point of K and $q \in \operatorname{rc}[K]$. For any $M \geq 0$, by using the facts that u is Lipschitz and $x_1 := x_0 + Mq \in K$ we have that $u(z - x + x_1) \leq C|z - x|$. By convexity we have

$$u(z) = u\biggl(\frac{M-1}{M}(z - x + x_0) + \frac{1}{M}(z - x + x_1)\biggr) \leq \frac{M-1}{M} o(|z - x|) + \frac{C}{M}|z - x|.$$

By letting $M \to \infty$ we have $0 \leq u(z) \leq o(|z - x|)$. Hence u is C^1 at x. □

Since u is convex, for any unit vector γ the lateral derivatives

$$\partial_\gamma^+ u(x) =: \lim_{t\searrow 0} t^{-1}(u(x + t\gamma) - u(x)) \quad \text{and} \quad \partial_\gamma^- u(x) =: \lim_{t\searrow 0} t^{-1}(u(x) - u(x - t\gamma))$$

exist. To prove that $u \in C^1(\overline{\Omega})$, it suffices to prove that

$$\partial_\gamma^+ u(x_0) = \partial_\gamma^- u(x_0) \tag{3-1}$$

at any point $x_0 \in \partial\Omega$ for any unit vector γ. By convexity, it suffices to prove this for $\xi = \xi_k$ for all $k = 1, 2, \ldots, n$, where ξ_k, $k = 1, \ldots, n$, are any n linearly independent unit vectors.

Proof of Theorem 2.1(i). By Lemmas 3.1 and 2.4 we only need to show that u is C^1 at localized points. Assume to the contrary that u is not C^1 at $x_0 \in \partial\Omega$. Let us assume that $x_0 = 0$, $u(0) = 0$, $u \geq 0$ and $\partial_1^+ u(0) > \partial_1^- u(0) = 0$. Since $\partial\Omega$ is Lipschitz, we may also assume that $-te_1 \in \Omega$ for $t \in (0, 1)$, where e_1 is the first coordinate direction.

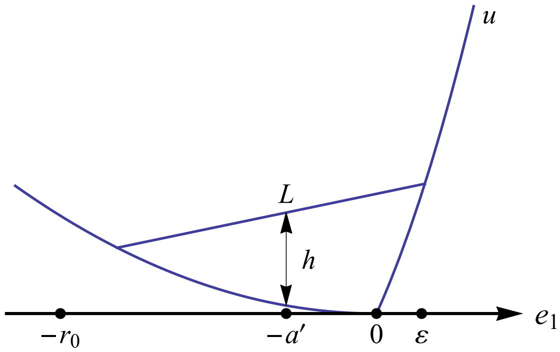

Figure 1. Centred section.

Now we consider a section $S_h(x')$, where $x' = (-a', 0, \dots, 0)$ for some small constant $0 < a' < \frac{1}{2}r_0$, where $r_0 := r_{x_0}$ is the radius in the condition of Theorem 2.1(i). Note that by John's lemma there exists an ellipsoid E with centre x' such that $E \subset S_h(x') \subset n^{3/2}E$. Since u is Lipschitz and $\partial_1^+ u(0) > 0$, we have that $C^{-1}\varepsilon \le u(\varepsilon e_1) \le C\varepsilon$ for any small positive ε, where C is a positive constant. Since $\partial_1^- u(0) = 0$, we have $u(-Ma'e_1) = o(a')$, where $M = 2n^{3/2}$. Hence, we can choose small ε and a' so that the following properties hold:

(1) $o(a') = u(-Ma'e_1) \le C^{-1}\varepsilon \ll a'$,

(2) εe_1 is on the boundary of some section $S_h(x')$, and

(3) $S_h(x') \subset B_{r_0}(0)$.

The existence of such a section $S_h(x')$ in (2) follows from the property that a centred section, say $S_h(x)$, various continuously with respect to the height h; see [Caffarelli and McCann 2010, Lemma A.8], and (3) follows from the assumption that $x_0 = 0$ is localized.

Let L be the defining linear function of $S_h(x')$; by (1) it is easy to see that L is increasing in the e_1 direction (see Figure 1); hence,

$$(L - u)(0) \ge (L - u)(x') = h. \tag{3-2}$$

Since $\int_{S_h(x')} f \le C \int_{\frac{1}{2}S_h(x')} f$, we have that

$$(L - u)(0) \le C\left(\frac{\varepsilon}{a'}\right)^{\frac{1}{n}} h, \tag{3-3}$$

contradicting (3-2), since $a' \gg \varepsilon$. Here we have followed the argument of [Caffarelli 1996]. Indeed, let A be an affine transform normalizing $S_h(x')$; then $v := (u - L)(A^{-1}x)/h$ satisfies $\det D^2 v = f(A^{-1}x)/h^n$ in $A(S_h(x'))$ and $v = 0$ on $\partial S_h(x')$. Hence, by applying Lemma 2.6 to v and translating back to u we get (3-3).

Hence, u must be C^1 at any localized point x_0. Therefore $u \in C^1(\mathbb{R}^n)$. $\square$

Remark 3.2. The proof of Theorem 2.1(i) shares some similarities with the proof of C^1 regularity for the obstacle problem in [Savin 2005] (see Proposition 2.8 in that paper).

4. Global $C^{1,\alpha}$ regularity

In this section, we prove Theorem 2.1(ii). First we point out that to prove $u \in C^{1,\alpha}(\overline{\Omega})$, it suffices to prove that there exist constants $C > 0$, $\alpha \in (0,1)$ and $r > 0$ such that, for any point $x_0 \in \overline{\Omega}$,

$$u(x) - \ell_{x_0}(x) \le C|x - x_0|^{1+\alpha} \tag{4-1}$$

for every $x \in B_r(x_0) \cap \overline{\Omega}$. From (4-1) one can prove that $u \in C^{1,\alpha}(\overline{\Omega})$, using the convexity of u. In the following we will show that a relaxed version of (4-1) is enough to show $u \in C^{1,\alpha}(\overline{\Omega})$, and it has the advantage of avoiding some annoying limiting picture.

By the assumption of Theorem 2.1(ii) we write $\Omega = U - \sum_{i=1}^k C_i$, where U is an open convex set, and C_i, $i = 1, \dots, k$, are closed disjoint convex subsets of U; see Figure 2. Given any $x \in \overline{\Omega}$, we introduce the function

$$\rho_x(t) := \sup\left\{ u(z) - u(x) - Du(x)\cdot(z-x) \;\middle|\; |z-x| = t,\ x + s\frac{z-x}{|z-x|} \in \overline{\Omega} \text{ for any } s \in [0, r_0] \right\}, \tag{4-2}$$

where r_0 is a fixed small positive constant depending on Ω, and its smallness will be clear in the proof of Lemma 4.1. Indeed, we need to take r_0 small enough that $B_{r_0}(x) \cap \partial U$ can be represented as the graph of some Lipschitz function for any $x \in \partial U$ with the Lipschitz constant independent of x, and that

$$r_0 \ll \min\{\operatorname{dist}(\partial U, \partial C_i),\ \operatorname{dist}(\partial C_j, \partial C_l) \mid i = 1, \dots, k,\ 1 \le j \ne l \le k\}.$$

Lemma 4.1. *Suppose that there exist $r > 0$ and $\delta \in (0,1)$ such that for any $x \in \overline{\Omega}$ we have*

$$\rho_x\left(\tfrac{1}{2}t\right) \le \tfrac{1}{2}(1-\delta)\rho_x(t) \tag{4-3}$$

whenever $t \le r$. Then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

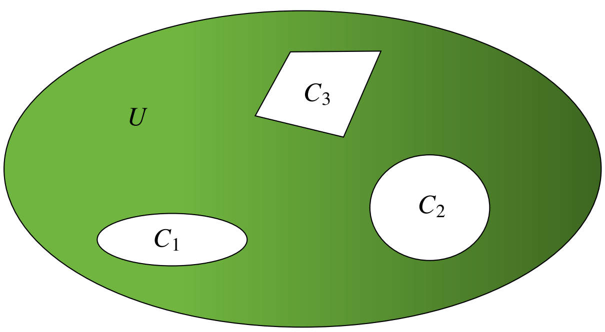

Figure 2. Domain Ω.

Proof. For $t = r/2^k$, we have

$$\rho_x(t) \le \frac{(1-\delta)^k}{2^k}\rho_x(r) \le \frac{t}{r}(1-\delta)^{\log(r/t)/\log 2}\rho_x(r) \le Ct^{1+\alpha}, \tag{4-4}$$

where C depends on r, δ and $\rho_x(r)$, and $\alpha = -\log(1-\delta)/\log 2$.

Suppose $x, y \in \overline{\Omega}$ and $|x-y| \ll r \ll r_0$. We need to consider two cases:

(a) x, y are close to ∂U.

(b) x, y are close to ∂C_i for some $1 \le i \le k$.

We will deal with case (a) first; case (b) follows from a similar argument. Without loss of generality we may assume that $B_{3r_1} \subset U$ for some small fixed r_1, that $r_0 \ll r_1$, and that $\operatorname{dist}(\partial B_{3r_1}, \partial U) \gg r_1$. Denote by $\mathscr{C}_{x,r_1}$ the convex hull of x and B_{r_1}. By convexity, $\mathscr{C}_{x,3r_1} \subset U$. Then we prove the following claim:

Claim 1. For any $z \in B_{r/2}(x) \cap \mathscr{C}_{x,2r_1}$, we have $|Du(x) - Du(z)| \le C|x-z|^\alpha$.

Proof of Claim 1. Observe that $\operatorname{dist}(z, \partial\mathscr{C}_{x,3r_1}) \ge (1/C)|x-z|$ for some large constant C. Hence, $B_{(1/C)|x-z|}(z) \subset B_r \cap \mathscr{C}_{x,3r_1}$. Now, for any $\tilde{z} \in \partial B_{(1/C)|x-z|}(z)$, by (4-4) we have that

$$u(\tilde{z}) \le u(x) + Du(x)\cdot(\tilde{z}-x) + C|\tilde{z}-x|^{1+\alpha}. \tag{4-5}$$

By convexity we also have

$$u(\tilde{z}) \ge u(z) + Du(z)\cdot(\tilde{z}-z) \tag{4-6}$$

and

$$u(z) \ge u(x) + Du(x)\cdot(z-x). \tag{4-7}$$

By (4-5), (4-6) and (4-7) we have

$$(Du(z) - Du(x))\cdot(\tilde{z}-z) \le C|\tilde{z}-x|^{1+\alpha}. \tag{4-8}$$

Note that $|\tilde{z}-z| \approx |\tilde{z}-x| \approx |z-x|$ provided $\tilde{z} \in \partial B_{(1/C)|x-z|}(z)$ and C is sufficiently large. Since (4-8) holds for any $\tilde{z} \in \partial B_{(1/C)|x-z|}(z)$, it follows that $|Du(x) - Du(z)| \le C|x-z|^\alpha$. □

Now suppose $|x-y| \ll r$. If either $y \in \mathscr{C}_{x,2r_1}$ or $x \in \mathscr{C}_{y,2r_1}$ holds, then by Claim 1 we have $|Du(x) - Du(y)| \le C|x-y|^\alpha$. Otherwise one may find a point $z \in \mathscr{C}_{x,r_1} \cap \mathscr{C}_{y,r_1}$ such that $|z-x| \approx |z-y| \approx |x-y|$. Then by applying the estimate in Claim 1 we have

$$|Du(x) - Du(y)| \le |Du(x) - Du(z)| + |Du(y) - Du(z)| \le C(|x-z|^\alpha + |y-z|^\alpha) \le C|x-y|^\alpha.$$

We can prove case (b) by a similar argument. Indeed, $\partial C_1 \cap B_r(x)$ can be represented as the graph of some Lipschitz function for any fixed $x \in \partial C_1$ provided $r \ll r_0$. Then, by the assumption that the C_i are disjoint, it is easy to find a small ball $B_{3r_1} \subset \Omega$ such that $\mathscr{C}_{z,3r_1} \subset \Omega$ for any $z \in B_r(x) \cap \overline{\Omega}$. Then, by a similar argument to the proof of case (a), we can show that $|Du(x) - Du(y)| \le C|x-y|^\alpha$ provided $|x-y| \ll r$. □

The following lemma shows that the centred sections are well-localized provided the heights are sufficiently small.

Lemma 4.2. *There exists a height $h_0 > 0$ such that, for any $x \in \overline{\Omega}$, the section $S_h(x)$ intersects at most one of ∂U, ∂C_i, $i = 1, \dots, m$, provided $h \le h_0$.*

Proof. Suppose to the contrary there exist sequences $x_k \in \overline{\Omega}$ and $h_k \to 0$, such that $S_{h_k}(x_k)$ intersects at least two of ∂U, ∂C_i, $i = 1, \dots, m$. Passing to a subsequence we may assume $x_k \to y \in \overline{\Omega}$. Since u is strictly convex in the interior of Ω, we have either $y \in \partial U$ or $y \in \partial C_i$ for some i. Denote by L_k the defining function of $S_{h_k}(x_k)$, namely $S_{h_k}(x_k) = \{u \le L_k\}$. Then, passing to a subsequence we may assume $L_k \to L$ for some affine function L, and $S_{h_k}(x_k) \to S \subset \{u \le L\}$. It follows from the properties of $S_{h_k}(x_k)$ that:

(i) S is centred at y.

(ii) S intersects at least two of ∂U, ∂C_i, $i = 1, \dots, m$.

(iii) $L(y) = \lim_{k\to\infty} L_k(x_k) = \lim_{k\to\infty} u(x_k) + h_k = u(y)$.

By (i) and (iii) we have that $S \subset \{u = L\}$. Then by (ii) we see that S passes through the interior of Ω, which contradicts the fact that u is strictly convex in the interior of Ω. □

Proof of Theorem 2.1(ii). *Step 1.* The main observation in this step is that if (4-3) is violated for small δ, then u is close to a linear function on a segment connecting x and some point $z_\delta \in \overline{\Omega}$. Hence, if (4-3) is violated for arbitrary r, δ, then one can find a sequence of points x_k such that u is more and more linear around x_k in some direction as $k \to \infty$. The "almost linearity" will be clear if we perform blow-up and an affine transform on u properly restricted to some carefully chosen section around x_k, and a line segment will appear on the graph of the limiting function. The detailed argument goes as follows.

To prove $\rho_x(t) \le Ct^{1+\alpha}$ for any $x \in \overline{\Omega}$ and any $t \le r$, by Lemma 4.1 we assume to the contrary that there exist sequences $t_k \le 1/k$, $\delta_k = 1/k$ and $x_k \in \overline{\Omega}$ such that

$$\rho_{x_k}\left(\tfrac{1}{2}t_k\right) \ge \tfrac{1}{2}(1 - 1/k)\rho_{x_k}(t_k). \tag{4-9}$$

Suppose the supremum in (4-2) $\left(\text{when } x = x_k \text{ and } t = \tfrac{1}{2}t_k\right)$ is attained at $\tfrac{1}{2}(x_k + z_k) \in \overline{\Omega}$; by the definition of ρ_x we see that $\overline{z_k x_k} \subset \overline{\Omega}$, where $\overline{z_k x_k}$ denotes the segment connecting z_k and x_k. By passing to a subsequence, we may assume $x_k \to x_\infty \in \partial\Omega$.

Choosing sections. For each k, let $S_{h_k}(x_k)$ be a section of u with centre x_k, where h_k is chosen so that $z_k \in \partial S_{h_k}(x_k)$. Similar to the proof of Theorem 2.1(i), the existence of such a section follows from the property that a centred section, say $S_h(x)$, varies continuously with respect to the height h; see [Caffarelli and McCann 2010, Lemma A.8] for a proof. It is easy to see that $h_k \to 0$.

Normalization. Let L_k be the defining function of $S_{h_k}(x_k)$. We normalize the section $S_{h_k}(x_k)$ by a linear transformation T_k, and let $S_k = T_k(S_{h_k}(x_k))$. Note that $T_k(x_k) = 0$ and $B_1 \subset S_k \subset n^{3/2}B_1$. Also we let $u_k = (u - L_k)(T_k^{-1}x)/h_k$. Then u_k solves

$$\begin{cases} \det D^2 u_k = f_k & \text{in } S_k, \\ u_k = 0 & \text{on } \partial S_k, \end{cases} \tag{4-10}$$

where $f_k = h_k^{-n}(\det T_k)^{-1} f(T_k^{-1}x)/g(Du(T_k^{-1}x))$. After a rotation of coordinates, we may assume $T_k(z_k)$ is on the x_1-axis.

Linearity estimate. Let

$$v_k(x) := u(x) - Du(x_k)\cdot(x-x_k) - u(x_k);$$

from (4-9) we have that $v_k\big(\frac{1}{2}(x_k+z_k)\big) \ge \frac{1}{2}(1-1/k)v_k(z_k)$. Let

$$\tilde{L}_k(x) := L_k(x) - Du(x_k)\cdot(x-x_k) - u(x_k).$$

Then we have that $S_{h_k}(x_k) = \{v_k \le \tilde{L}_k\}$. Since $S_{h_k}(x_k)$ is centred at x_k, $z_k \in \partial S_{h_k}(x_k)$, $v_k \ge 0$ and $\tilde{L}_k(x_k) = h_k$, by John's lemma we have that $0 \le \tilde{L}_k(z_k) \le 2n^{3/2}h_k$. Now,

$$(v_k-\tilde{L}_k)\big(\tfrac{1}{2}(x_k+z_k)\big) - \tfrac{1}{2}\Big(1-\frac{1}{k}\Big)((v_k-\tilde{L}_k)(x_k)+(v_k-\tilde{L}_k)(z_k)) \ge -\frac{1}{2k}(\tilde{L}_k(x_k)+\tilde{L}_k(z_k)) \ge -\frac{3n^{3/2}}{2k}h_k.$$

Since $v_k - \tilde{L}_k = u - L_k$, from the above estimate and the definition of u_k we have

$$u_k\big(\tfrac{1}{2}T_k z_k\big) \ge \tfrac{1}{2}\Big(1-\frac{1}{k}\Big)(u_k(0)+u_k(T_k z_k)) - \frac{3n^{3/2}}{2k}. \tag{4-11}$$

Limiting problem. Now, by convexity we may take limits $S_k \to S_\infty$ and $u_k \to u_\infty$. Let f_∞ be the weak limit of f_k. Then u_∞ satisfies $\det D^2 u_\infty = f_\infty$ in the Alexandrov sense. Let $z_\infty := \lim_{k\to\infty} T_k(z_k)$. By (4-11) we have

$$u_\infty = L \ \text{ on the segment connecting } \ 0 \ \text{ and } \ z_\infty, \tag{4-12}$$

where L is a supporting function of u_∞ at 0.

Step 2. In this step, we need to consider two situations:

(a) $x_\infty \in \partial C_i$ for some $1 \le i \le k$.

(b) $x_\infty \in \partial U$.

In each case, a contradiction is obtained at some carefully chosen extreme point (denoted by y) of $\{u_\infty = L\}$. Heuristically, we can choose a section of u_∞ (denoted by S) around y such that y is much closer to ∂S in one direction than in the opposite direction. Hence, on one hand the Alexandrov-type estimate Lemma 2.6(a) shows that h, the height of the section S, should not be too small. On the other hand, Lemma 2.6(b) shows that h is very small, which is a contradiction.

We deal with case (a) first.

Proof in case (a). Note that since $x_\infty \in \partial C_i$ for some $1 \le i \le k$ and $h_k \to 0$ as $k \to \infty$, by Lemma 4.2 we have that the support of f_k can be represented by $S_k - A_k$ when k is large, where A_k is an open convex subset of S_k. Let the convex set A_∞ be the limit of the A_k. Then $S_\infty - A_\infty$ is the support of f. Since the centre of mass of S_∞ is 0 and $0 \in S_\infty - A_\infty$, we have that the volume of $S_\infty - A_\infty$ is positive. Hence, it is easy to see that there exists a constant C such that $C^{-1}\chi_{S_\infty - A_\infty} \le f_\infty \le C\chi_{S_\infty - A_\infty}$.

Since $\overline{z_k x_k} \subset \bar{\Omega}$, we have $\overline{0 z_\infty} \cap A_\infty = \varnothing$.

Subcase 1: $\{u_\infty = L\}$ contains an interior point of $S_\infty - \bar{A}_\infty$.

Subcase 2: $\{u_\infty = L\} \cap S_\infty \subset \bar{A}_\infty$.

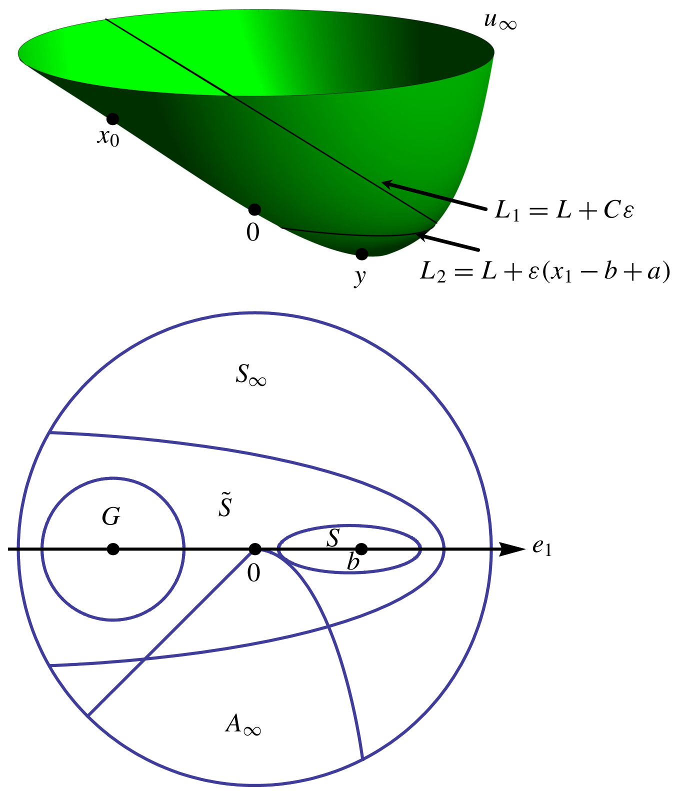

Figure 3. Two related sections.

For subcase 1, take $x_0 \in (S_\infty - \bar{A}_\infty) \cap \{u = L\}$. Take δ sufficiently small that $B_\delta(x_0) \subset S_\infty - \bar{A}_\infty$.

Choosing an extreme point. Let $y \in \{u = L\}$ be the point such that:

(1) $u_\infty(y) = \inf_{\{u=L\}} u_\infty$.

(2) y is an extreme point of the convex set $\{u_\infty = L\} \cap \{u_\infty = u(y)\}$.

It is easy to see that y is an extreme point of $\{u_\infty = L\}$.

Cutting a suitable section. By rotating the coordinates we may assume that $\{u_\infty = L\} \subset \{x_1 \leq b\}$ for some constant $b > 0$, and that $\{u_\infty = L\} \cap \{x_1 = b\} = \{y\}$. Then we consider the section $S = \{u_\infty < L + \varepsilon(x_1 - b + a)\}$ (see Figure 3), where we fix a sufficiently small and then take $\varepsilon \ll a$, so that $S \Subset S_\infty$ and $a \gg d := \max\{x_1 \mid (x_1, 0, \ldots, 0) \in S\} - b$.

Using Alexandrov estimates to obtain a contradiction. On one hand, by the Alexandrov estimate we have

$$|S|^2 > C\frac{a}{d}\varepsilon^n. \tag{4-13}$$

On the other hand, we consider another section $\tilde{S} = \{u_\infty < L + C\varepsilon\}$. Since u is Lipschitz, it is easy to see that $S \subset \tilde{S}$ provided C (independent of ε) is sufficiently large. By convexity we have $|B_\delta(x_0) \cap \tilde{S}| \geq C|\tilde{S}|$

for some constant C. We claim

$$|S|^2 \leq C\varepsilon^n, \tag{4-14}$$

where the constant C is independent of d. The claim follows from the following argument. Let $v = u_\infty - L - C\varepsilon$. Let $G := \tilde{S} \cap B_\delta(x_0)$. By John's lemma, there exists an affine transformation A with $\det A = 1$ such that

$$B_{\bar{r}} \subset A(G) \subset n^{3/2} B_{\bar{r}}$$

for some $\bar{r}$. Now $\bar{v} = v(A^{-1}x)$ satisfies $\det D^2\bar{v} = f_\infty(A^{-1}x) \geq C^{-1}$ in $A(G)$ and $|v| \leq C\varepsilon$ in $A(G)$. Then we have

$$C^{-1}|G| \leq \int_{G/2} f_\infty = \left|\partial\bar{v}\left(A\left(\tfrac{1}{2}G\right)\right)\right| \leq C\frac{\varepsilon^n}{\bar{r}^n}. \tag{4-15}$$

Equation (4-14) follows from (4-15) and the fact that $|\tilde{S}| \approx |G| \approx \bar{r}^n$. Since $d \ll a$, it is easy to see that (4-14) contradicts (4-13).

For subcase 2, we need to choose the extreme point more carefully.

Choosing an extreme point. Let $\tilde{K} \subset \mathbb{R}^n$ be a supporting plane of the convex set A_∞ at 0. If A_∞ is not C^1 at 0 we choose $\tilde{K}$ to be the one containing $\overline{z_\infty 0}$. Let y' be the point where u_∞ attains its minimum on $D := \{u = L\} \cap \tilde{K} \cap \bar{S}_\infty$. It is easy to check that D is a convex set, and the set $D \cap \{x \mid u(x) = u(y')\}$ is also convex. Let y be an extreme point of $D \cap \{x \mid u(x) = u(y')\}$. We claim that y is an extreme point of $\{u = L\}$. Indeed, suppose not; then there exist $y_1,\ y_2 \in \{u = L\} \cap S_\infty \subset \bar{A}_\infty$ such that $y = \frac{1}{2}(y_1 + y_2)$. Since $\tilde{K}$ is a supporting plane of A_∞ and $y \in \bar{A}_\infty$, we have that $y_1,\ y_2 \in D$. However, since $u(y) = \min\{u(x) \mid x \in D\}$, we have $y_1,\ y_2 \in D \cap \{x \mid u(x) = u(y')\}$, which contradicts the choice of y as an extreme point of $D \cap \{x \mid u(x) = u(y')\}$.

Cutting a suitable section. By subtracting L and translating the coordinates we may assume that $y = 0$, that $u_\infty \geq 0$, that $u_\infty(te_1) = 0$ for $t \in (0, 1)$, and that $u_\infty(te_1) > 0$ for $t < 0$. Let $0 < \varepsilon \ll a$ be small positive numbers. Let $S_h(ae_1)$ be a section of u_∞ with centre ae_1, where h is chosen so that $-\varepsilon e_1 \in \partial S_h(ae_1)$. Since y is an extreme point of $\{u = L\}$, we have that $S_h(ae_1) \Subset S_\infty$ provided h is sufficiently small. Note that $h \to 0$ as $\varepsilon \to 0$.

Using Alexandrov estimates to obtain a contradiction. Since A_∞ is convex, it is easy to see that

$$\int_{S_h(ae_1)} f_\infty \leq C \int_{\frac{1}{2}S_h(ae_1)} f_\infty$$

for some constant C. Let L_1 be the defining function of the section $S_h(ae_1)$, which is obviously decreasing in the e_1 direction. Hence $(L_1 - u_\infty)(0) \geq h$. Then by Lemma 2.6 we also have

$$(L_1 - u_\infty)(0) \leq C\left(\frac{\varepsilon}{a}\right)^{1/n} h,$$

which contradicts the previous estimate. □

Proof in case (b). The proof in case (b) follows from a similar argument to the proof of [Caffarelli 1992a, Lemma 4]; we sketch the argument here. Note that f_k is now supported in a convex domain $D_k \subset \bar{S}_k$.

Let $D_\infty := \lim_{k\to\infty} D_k$. We have $z_\infty \in D_\infty$. Let L be the supporting function of u_∞ at 0 such that $\overline{0z_\infty} \subset \{u_\infty = L\}$. Similarly to the proof of subcase 1 of case (a), let $y \in \{u_\infty = L\}$ be the point such that:

(1) $u_\infty(y) = \inf_{\{u_\infty = L\}} u_\infty$.

(2) y is an extreme point of the convex set $\{u_\infty = L\} \cap \{u_\infty = u(y)\}$.

It is easy to see that y is an extreme point of $\{u_\infty = L\}$. Observe that $y \in D_\infty$, since otherwise u_k has positive Monge–Ampère measure outside D_k for large k. Let $z = (1-\sigma)y + \sigma z_\infty$ for some small positive σ; we may also find a section satisfying $S_h(z) := \{u_\infty < L\} \Subset S_\infty$ and $y + \varepsilon(y - z_\infty)/|y - z_\infty| \in \partial S_h(z)$ for small $\varepsilon \ll \sigma$. Since $y \in D_\infty$, there exists a sequence $y_k \in D_k$ such that $y_k \to y$ as $k \to \infty$. Let

$$\tilde{z}_k := (1-\sigma)y_k + \sigma T(z_k);$$

it is easy to see that $\tilde{z}_k \to z$ as $k \to \infty$. Recall that $z_\infty := \lim_{k\to\infty} T(z_k)$ with $T(z_k) \in D_k$. Let $\tilde{S}_k := \{u_k \le L_k\}$ be a section of u_k centred at $\tilde{z}_k$ with height h. Then, passing to a subsequence, $\tilde{S}_k \to S_h(z)$ in Hausdorff distance. In particular, $\tilde{S}_k \Subset S_k$ provided k is sufficiently large. Then, by Lemma 2.6, we have that

$$Ch \le (L_k - u_k)(y_k) \le \left(\frac{\varepsilon}{\sigma}\right)^{1/n} h$$

for large k, which is a contradiction because $\varepsilon \ll \sigma$. □

Theorem 2.1(ii) follows from the above discussions. □

References

[Brenier 1991] Y. Brenier, "Polar factorization and monotone rearrangement of vector-valued functions", *Comm. Pure Appl. Math.* **44**:4 (1991), 375–417. MR 1100809 Zbl 0738.46011

[Caffarelli 1990a] L. A. Caffarelli, "A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity", *Ann. of Math.* (2) **131**:1 (1990), 129–134. MR 1038359 Zbl 0704.35045

[Caffarelli 1990b] L. A. Caffarelli, "Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation", *Ann. of Math.* (2) **131**:1 (1990), 135–150. MR 1038360 Zbl 0704.35044

[Caffarelli 1991] L. A. Caffarelli, "Some regularity properties of solutions of Monge Ampère equation", *Comm. Pure Appl. Math.* **44**:8-9 (1991), 965–969. MR 1127042 Zbl 0761.35028

[Caffarelli 1992a] L. A. Caffarelli, "Boundary regularity of maps with convex potentials", *Comm. Pure Appl. Math.* **45**:9 (1992), 1141–1151. MR 1177479 Zbl 0778.35015

[Caffarelli 1992b] L. A. Caffarelli, "The regularity of mappings with a convex potential", *J. Amer. Math. Soc.* **5**:1 (1992), 99–104. MR 1124980 Zbl 0753.35031

[Caffarelli 1996] L. A. Caffarelli, "Boundary regularity of maps with convex potentials, II", *Ann. of Math.* (2) **144**:3 (1996), 453–496. MR 1426885 Zbl 0916.35016

[Caffarelli and McCann 2010] L. A. Caffarelli and R. J. McCann, "Free boundaries in optimal transport and Monge–Ampère obstacle problems", *Ann. of Math.* (2) **171**:2 (2010), 673–730. MR 2630054 Zbl 1196.35231

[Figalli and Loeper 2009] A. Figalli and G. Loeper, "C^1 regularity of solutions of the Monge–Ampère equation for optimal transport in dimension two", *Calc. Var. Partial Differential Equations* **35**:4 (2009), 537–550. MR 2496656 Zbl 1170.35400

[Figalli and Rifford 2009] A. Figalli and L. Rifford, "Continuity of optimal transport maps and convexity of injectivity domains on small deformations of $\mathbb{S}^2$", *Comm. Pure Appl. Math.* **62**:12 (2009), 1670–1706. MR 2569074 Zbl 1175.49040

[Figalli et al. 2010] A. Figalli, L. Rifford, and C. Villani, "On the Ma–Trudinger–Wang curvature on surfaces", *Calc. Var. Partial Differential Equations* **39**:3-4 (2010), 307–332. MR 2729302 Zbl 1203.53034

[Figalli et al. 2011] A. Figalli, L. Rifford, and C. Villani, "Necessary and sufficient conditions for continuity of optimal transport maps on Riemannian manifolds", *Tohoku Math. J.* (2) **63**:4 (2011), 855–876. MR 2872966 Zbl 1262.58013

[Figalli et al. 2012] A. Figalli, L. Rifford, and C. Villani, "Nearly round spheres look convex", *Amer. J. Math.* **134**:1 (2012), 109–139. MR 2876141 Zbl 1241.53031

[Figalli et al. 2013a] A. Figalli, Y.-H. Kim, and R. J. McCann, "Hölder continuity and injectivity of optimal maps", *Arch. Ration. Mech. Anal.* **209**:3 (2013), 747–795. MR 3067826 Zbl 1281.49037

[Figalli et al. 2013b] A. Figalli, Y.-H. Kim, and R. J. McCann, "Regularity of optimal transport maps on multiple products of spheres", *J. Eur. Math. Soc.* (*JEMS*) **15**:4 (2013), 1131–1166. MR 3055758 Zbl 1268.49053

[Kim and McCann 2010] Y.-H. Kim and R. J. McCann, "Continuity, curvature, and the general covariance of optimal transportation", *J. Eur. Math. Soc.* (*JEMS*) **12**:4 (2010), 1009–1040. MR 2654086 Zbl 1191.49046

[Liu 2009] J. Liu, "Hölder regularity of optimal mappings in optimal transportation", *Calc. Var. Partial Differential Equations* **34**:4 (2009), 435–451. MR 2476419 Zbl 1166.35331

[Liu and Wang 2015] J. Liu and X.-J. Wang, "Interior a priori estimates for the Monge–Ampère equation", pp. 151–177 in *Surveys in differential geometry 2014: regularity and evolution of nonlinear equations*, edited by H.-D. Cao et al., Surv. Differ. Geom. **19**, International Press, Somerville, MA, 2015. MR 3381500 Zbl 1326.35168

[Liu et al. 2010] J. Liu, N. S. Trudinger, and X.-J. Wang, "Interior $C^{2,\alpha}$ regularity for potential functions in optimal transportation", *Comm. Partial Differential Equations* **35**:1 (2010), 165–184. MR 2748621 Zbl 1189.35142

[Loeper 2011] G. Loeper, "Regularity of optimal maps on the sphere: the quadratic cost and the reflector antenna", *Arch. Ration. Mech. Anal.* **199**:1 (2011), 269–289. MR 2754343 Zbl 1231.35280

[Loeper and Villani 2010] G. Loeper and C. Villani, "Regularity of optimal transport in curved geometry: the nonfocal case", *Duke Math. J.* **151**:3 (2010), 431–485. MR 2605867 Zbl 1192.53041

[Savin 2005] O. Savin, "The obstacle problem for Monge Ampere equation", *Calc. Var. Partial Differential Equations* **22**:3 (2005), 303–320. MR 2118901 Zbl 1084.35026

[Sheng et al. 2004] W. Sheng, N. Trudinger, and X.-J. Wang, "Convex hypersurfaces of prescribed Weingarten curvatures", *Comm. Anal. Geom.* **12**:1-2 (2004), 213–232. MR 2074877 Zbl 1065.53008

[Trudinger and Wang 2009a] N. S. Trudinger and X.-J. Wang, "On strict convexity and continuous differentiability of potential functions in optimal transportation", *Arch. Ration. Mech. Anal.* **192**:3 (2009), 403–418. MR 2505359 Zbl 1214.49038

[Trudinger and Wang 2009b] N. S. Trudinger and X.-J. Wang, "On the second boundary value problem for Monge–Ampère type equations and optimal transportation", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **8**:1 (2009), 143–174. MR 2512204 Zbl 1182.35134

Received 20 Nov 2015. Revised 8 Mar 2016. Accepted 29 Apr 2016.

ELINA ANDRIYANOVA: elina.andriyanova@anu.edu.au
Mathematical Science Institute, The Australian National University, Canberra ACT 2601, Australia

SHIBING CHEN: shibing.chen@anu.edu.au
Mathematical Science Institute, The Australian National University, Canberra ACT 2601, Australia

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2016 is US $235/year for the electronic version, and $430/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 9 No. 6 2016

2157-5045(2016)9:6;1-6