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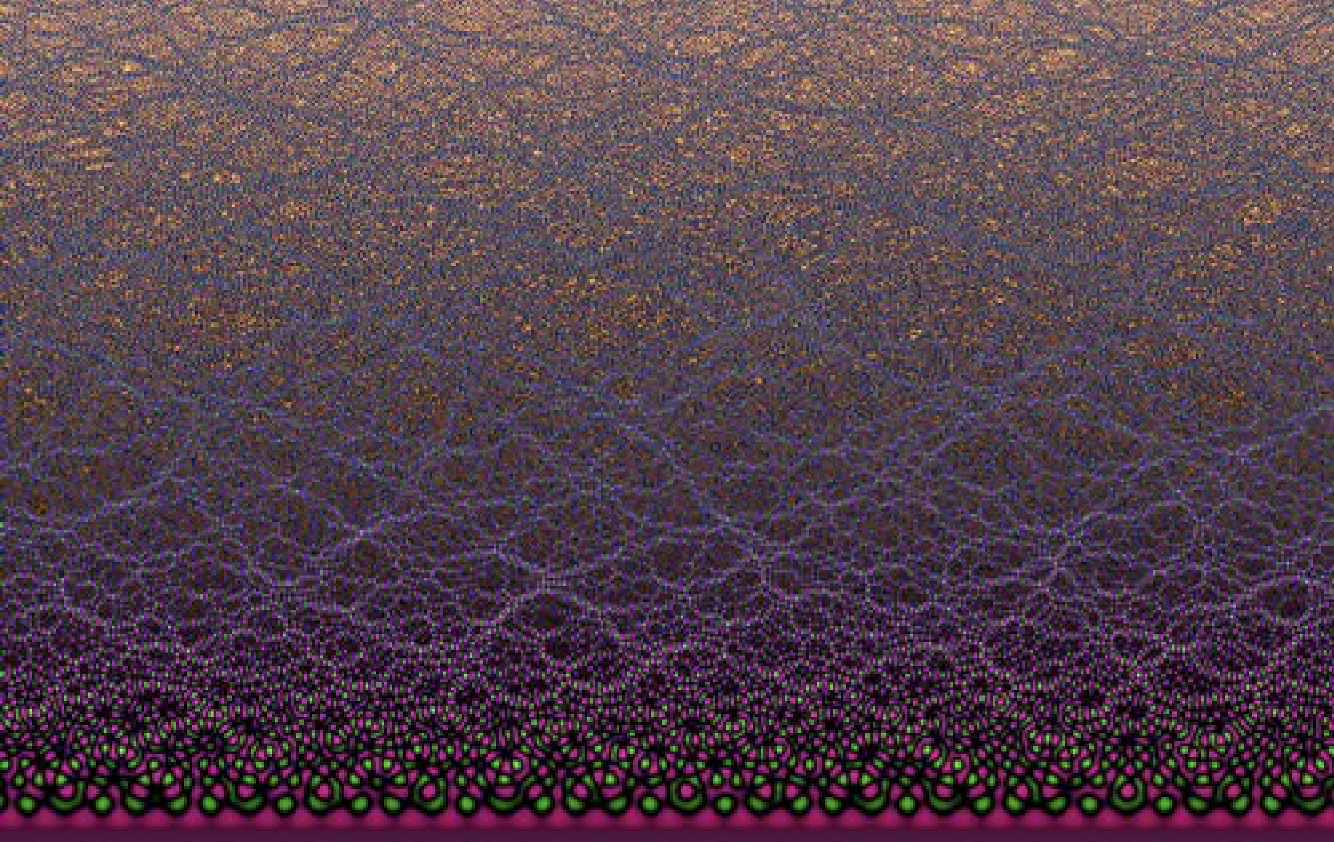

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A LONG $\mathbb{C}^2$ WITHOUT HOLOMORPHIC FUNCTIONS

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Dedicated to John Erik Fornæss

We construct for every integer $n > 1$ a complex manifold of dimension n which is exhausted by an increasing sequence of biholomorphic images of $\mathbb{C}^n$ (i.e., a *long* $\mathbb{C}^n$), but does not admit any nonconstant holomorphic or plurisubharmonic functions. Furthermore, we introduce new holomorphic invariants of a complex manifold X, the *stable core* and the *strongly stable core*, which are based on the long-term behavior of hulls of compact sets with respect to an exhaustion of X. We show that every compact polynomially convex set $B \subset \mathbb{C}^n$ such that $B = \overline{B^\circ}$ is the strongly stable core of a long $\mathbb{C}^n$; in particular, holomorphically nonequivalent sets give rise to nonequivalent long $\mathbb{C}^n$'s. Furthermore, for every open set $U \subset \mathbb{C}^n$ there exists a long $\mathbb{C}^n$ whose stable core is dense in U. It follows that for any $n > 1$ there is a continuum of pairwise nonequivalent long $\mathbb{C}^n$'s with no nonconstant plurisubharmonic functions and no nontrivial holomorphic automorphisms. These results answer several long-standing open problems.

1. Introduction

A complex manifold X of dimension n is said to be a *long* $\mathbb{C}^n$ if it is the union of an increasing sequence of domains $X_1 \subset X_2 \subset X_3 \subset \cdots \subset \bigcup_{j=1}^\infty X_j = X$ such that each X_j is biholomorphic to the complex Euclidean space $\mathbb{C}^n$. It is immediate that any long $\mathbb{C}$ is biholomorphic to $\mathbb{C}$. However, for $n > 1$, this class of complex manifolds is still very mysterious. The long-standing question, whether there exists a long $\mathbb{C}^n$ which is not biholomorphic to $\mathbb{C}^n$, was answered in 2010 by E. F. Wold [2010], who constructed a long $\mathbb{C}^n$ that is not holomorphically convex, hence not a Stein manifold. Wold's construction is based on his examples of non-Runge Fatou–Bieberbach domains in $\mathbb{C}^n$ (see [Wold 2008]; an exposition of both results can be found in [Forstnerič 2011, Section 4.20]). In spite of these interesting examples, the theory has not been developed since. In particular, it remained unknown whether there exist long $\mathbb{C}^2$'s without nonconstant holomorphic functions, and whether there exist at least two nonequivalent non-Stein long $\mathbb{C}^2$'s.

We begin with the following result, which answers the first question affirmatively.

Theorem 1.1. *For every integer $n > 1$ there exists a long $\mathbb{C}^n$ without any nonconstant holomorphic or plurisubharmonic functions.*

Theorem 1.1 is proved in Section 3. It contributes to the line of counterexamples to the classical union problem for Stein manifolds: *is an increasing union of Stein manifolds always Stein?* For domains in $\mathbb{C}^n$ this question was raised by Behnke and Thullen [1934], and an affirmative answer was given in [Behnke and Stein 1939]. Some progress on the general question was made by Stein [1956] and Docquier and

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Grauert [1960]. The first counterexample was given in any dimension $n \geq 3$ by J. E. Fornæss [1976]; he found an increasing union of balls that is not holomorphically convex, hence not Stein. The key ingredient in his proof is a construction of a biholomorphic map $\Phi : \Omega \to \Phi(\Omega) \subset \mathbb{C}^3$ on a bounded neighborhood $\Omega \subset \mathbb{C}^3$ of any compact set $K \subset \mathbb{C}^3$ with nonempty interior such that the polynomial hull of $\Phi(K)$ is not contained in $\Phi(\Omega)$. (A phenomenon of this type was first described by Wermer [1959].) Fornæss and Stout [1977] constructed an increasing union of three-dimensional polydiscs without nonconstant holomorphic functions. Fornæss [1978] gave a counterexample to the union problem in dimension 2. Increasing unions of hyperbolic Stein manifolds were studied further by Fornæss and Sibony [1981] and Fornæss [2004]. Wold [2010] constructed the first example of a non-Stein long $\mathbb{C}^2$. For the connection with Bedford's conjecture, see the survey [Abbondandolo et al. 2013].

Another question that has been asked repeatedly over a long period of time is whether there exist infinitely many nonequivalent long $\mathbb{C}^n$'s for any or all $n > 1$. Up to now, only two different long $\mathbb{C}^2$'s have been known, namely the standard $\mathbb{C}^2$ and a non-Stein long $\mathbb{C}^2$ constructed by Wold [2010]. In dimension $n > 2$ one can get a few more examples by considering Cartesian products of long $\mathbb{C}^k$'s for different values of k. In this paper, we introduce new biholomorphic invariants of a complex manifold, the *stable core* and the *strongly stable core* (see Definition 1.5), which allow us to distinguish certain long $\mathbb{C}^n$'s from one another. In our opinion, this is the main new contribution of the paper from the conceptual point of view. With the help of these invariants, we answer the above mentioned question affirmatively by proving the following result.

Recall that a compact subset B of a topological space X is said to be *regular* if it is the closure of its interior, $B = \overline{B^\circ}$.

Theorem 1.2. *Let $n > 1$. To every regular compact polynomially convex set $B \subset \mathbb{C}^n$ we can associate a complex manifold $X(B)$, which is a long $\mathbb{C}^n$ containing a biholomorphic copy of B, such that every biholomorphic map $\Phi : X(B) \to X(C)$ between two such manifolds takes B onto C. In particular, for every holomorphic automorphism $\Phi \in \mathrm{Aut}(X(B))$, the restriction $\Phi|_B$ is an automorphism of B. We can choose $X(B)$ such that it has no nonconstant holomorphic or plurisubharmonic functions.*

It follows that the manifold $X(B)$ can be biholomorphic to $X(C)$ only if B is biholomorphic to C. Our construction likely gives many nonequivalent long $\mathbb{C}^n$'s associated to the same set B. A more precise result is given by Theorem 1.6 below.

By considering the special case when B is the closure of a strongly pseudoconvex domain, Theorem 1.2 shows that the moduli space of long $\mathbb{C}^n$'s contains the moduli space of germs of smooth strongly pseudoconvex real hypersurfaces in $\mathbb{C}^n$. This establishes a surprising connection between long $\mathbb{C}^n$'s and the Cauchy–Riemann geometry. It has been known since Poincaré's paper [1907] that most pairs of smoothly bounded strongly pseudoconvex domains in $\mathbb{C}^n$ are not biholomorphic to each other, at least not by maps extending smoothly to the closed domains. It was shown much later by C. Fefferman [1974] that the latter condition is automatically fulfilled. (For elementary proofs of Fefferman's theorem, see [Pinchuk and Khasanov 1987; Forstnerič 1992].) A complete set of local holomorphic invariants of a strongly pseudoconvex real-analytic hypersurface is provided by the *Chern–Moser normal form*; see [Chern and Moser 1974]. Most such domains have no holomorphic automorphisms other than the identity map. (For surveys of this topic, see, e.g., [Baouendi et al. 1999; Forstnerič 1993].) Hence, Theorem 1.2 implies the following corollary.

Corollary 1.3. *For every $n > 1$ there is a continuum of pairwise nonequivalent long $\mathbb{C}^n$'s with no nonconstant holomorphic or plurisubharmonic functions and no nontrivial holomorphic automorphisms.*

We now describe the new biholomorphic invariants alluded to above, the *stable core* and the *strongly stable core* of a complex manifold. Their definition is based on the stable hull property defined below, which a compact set in a complex manifold may or may not have. Given a pair of compact sets $K \subset L$ in a complex manifold X, we write

$$\widehat{K}_{\mathscr{O}(L)} = \{x \in L : |f(x)| \le \sup_K |f| \text{ for all } f \in \mathscr{O}(L)\}, \tag{1-1}$$

where $\mathscr{O}(L)$ is the algebra of holomorphic functions on neighborhoods of L.

Definition 1.4 (the stable hull property). A compact set K in a complex manifold X has the *stable hull property* (SHP) if there exists an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^\infty K_j = X$ by compact sets such that $K \subset K_1$, $K_j \subset K_{j+1}^\circ$ for every $j \in \mathbb{N}$, and the increasing sequence of hulls $\widehat{K}_{\mathscr{O}(K_j)}$ stabilizes, i.e., there is a $j_0 \in \mathbb{N}$ such that

$$\widehat{K}_{\mathscr{O}(K_j)} = \widehat{K}_{\mathscr{O}(K_{j_0})} \quad \text{for all } j \ge j_0. \tag{1-2}$$

Obviously, SHP is a biholomorphically invariant property: if a compact set $K \subset X$ satisfies condition (1-2) with respect to some exhaustion $(K_j)_{j\in\mathbb{N}}$ of X, then for any biholomorphic map $F : X \to Y$ the set $F(K) \subset Y$ satisfies (1-2) with respect to the exhaustion $L_j = F(K_j)$ of Y. What is less obvious, but needed to make this condition useful, is its independence of the choice of the exhaustion; see Lemma 4.1.

Definition 1.5. Let X be a complex manifold.

(i) The *stable core* of X, denoted $\mathrm{SC}(X)$, is the open set consisting of all points $x \in X$ which admit a compact neighborhood $K \subset X$ with the stable hull property.

(ii) A regular compact set $B \subset X$ is called the *strongly stable core* of X, denoted $\mathrm{SSC}(X)$, if B has the stable hull property, but no compact set $K \subset X$ with $K^\circ \setminus B \neq \varnothing$ has the stable hull property.

Clearly, the stable core always exists and is a biholomorphic invariant, in the sense that any biholomorphic map $X \to Y$ maps $\mathrm{SC}(X)$ onto $\mathrm{SC}(Y)$. In particular, every holomorphic automorphism of X maps the stable core $\mathrm{SC}(X)$ onto itself. The strongly stable core $\mathrm{SSC}(X)$ need not exist in general; if it does, then its interior equals the stable core $\mathrm{SC}(X)$ and $\mathrm{SSC}(X) = \overline{\mathrm{SC}(X)}$. In (ii), we must restrict attention to *regular* compact sets since otherwise the definition would be ambiguous.

Theorem 1.6. *Let $n > 1$.*

(a) *For every regular compact polynomially convex set $B \subset \mathbb{C}^n$ (i.e., $B = \overline{B^\circ}$) there exists a long $\mathbb{C}^n$, $X(B)$, which admits no nonconstant plurisubharmonic functions and whose strongly stable core equals B: $\mathrm{SSC}(X(B)) = B$.*

(b) *For every open set $U \subset \mathbb{C}^n$ there exists a long $\mathbb{C}^n$, X, which admits no nonconstant holomorphic functions and satisfies $\mathrm{SC}(X) \subset U$ and $\overline{U} = \overline{\mathrm{SC}(X)}$.*

In Theorem 1.6 we have identified the sets $B, U \subset \mathbb{C}^n$ with their images in the long $\mathbb{C}^n$, $X = \bigcup_{k=1}^{\infty} X_k$, by identifying $\mathbb{C}^n$ with the first domain $X_1 \subset X$.

Assuming Theorem 1.6, we now prove Theorem 1.2.

Proof of Theorem 1.2. Let B be a regular compact polynomially convex set in $\mathbb{C}^n$ for some $n > 1$. By Theorem 1.6 there exists a long $\mathbb{C}^n$, $X = X(B)$, whose strongly stable core is B. Assume that $F \in \mathrm{Aut}(X)$. Then $F(B)$ has SHP (see Definition 1.4). Since B is the biggest regular compact subset of X with SHP (see (ii) in Definition 1.5), we have that $\Phi(B) \subset B$. Applying the same argument to the inverse automorphism $\Phi^{-1} \in \mathrm{Aut}(X)$ gives $\Phi^{-1}(B) \subset B$, and hence $B \subset \Phi(B)$. Both properties together imply that $\Phi(B) = B$, and hence $\Phi|_B \in \mathrm{Aut}(B)$.

In the same way, we see that a biholomorphic map $X(B) \to X(C)$ between two long $\mathbb{C}^n$'s, furnished by part (a) in Theorem 1.6, maps B biholomorphically onto C. Hence, if B is not biholomorphic to C, then $X(B)$ is not biholomorphic to $X(C)$. □

Theorem 1.6 is proved in Section 4. We construct manifolds with these properties by improving the recursive procedure devised by Wold [2008; 2010]. The following key ingredient was introduced in [Wold 2008]; it will henceforth be called the *Wold process* (see Remark 3.2).

Given a compact holomorphically convex set $L \subset \mathbb{C}^* \times \mathbb{C}^{n-1}$ with nonempty interior, there is a holomorphic automorphism $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^{n-1})$ such that the polynomial hull $\widehat{\psi(L)}$ of the set $\psi(L)$ intersects the hyperplane $\{0\} \times \mathbb{C}^{n-1}$. By precomposing ψ with a suitably chosen Fatou–Bieberbach map $\theta : \mathbb{C}^n \hookrightarrow \mathbb{C}^* \times \mathbb{C}^{n-1}$, we obtain a Fatou–Bieberbach map $\phi = \psi \circ \theta : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that, for a given polynomially convex set $K \subset \mathbb{C}^n$ with nonempty interior, we have that $\widehat{\phi(K)} \not\subset \phi(\mathbb{C}^n)$.

At every step of the recursion we perform the Wold process simultaneously on finitely many pairwise disjoint compact sets $K_1, \dots, K_m$ in the complement of the given regular polynomially convex set $B \subset \mathbb{C}^n$, chosen such that $\bigcup_{j=1}^{m} K_j \cup B$ is polynomially convex, thereby ensuring that polynomial hulls of their images $\phi(K_j)$ escape from the range of the injective holomorphic map $\phi : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ constructed in the recursive step. At the same time, we ensure that ϕ is close to the identity map on a neighborhood of B, and hence the image $\phi(B)$ remains polynomially convex. In practice, the sets K_j will be small pairwise disjoint closed balls in the complement of B whose number will increase during the process. We devise the process so that every point in a certain countable dense set $A = \{a_j\}_{j=1}^{\infty} \subset X \setminus B$ is the center of a decreasing sequence of balls whose $\mathscr{O}(X_k)$-hulls escape from each compact set in X; hence none of these balls has the stable hull property. This implies that B is the strongly stable core of X.

To prove part (b), we modify the recursion by introducing a new small ball $B' \subset U \setminus B$ at every stage. Thus, the set B acquires additional connected components during the recursive process. The sequence of added balls B_l is chosen such that their union is dense in the given open subset $U \subset \mathbb{C}^n$, while the sequence of sets K_j on which the Wold process is performed densely fills the complement $X \setminus \overline{U}$. It follows that the stable core of the limit manifold $X = \bigcup_{k=1}^{\infty} X_k$ is contained in U and is everywhere dense in U.

By combining the technique used in the proof of Theorem 1.1 (see Section 3) with those in [Forstnerič 2012, proof of Theorem 1.1], one can easily obtain the following result for holomorphic families of long $\mathbb{C}^n$'s. (Compare with [Forstnerič 2012, Theorem 1.1].) We leave out the details.

Theorem 1.7. *Let Y be a Stein manifold, and let A and B be disjoint finite or countable sets in Y. For every integer $n > 1$ there exists a complex manifold X of dimension* $\dim Y + n$ *and a surjective holomorphic submersion $\pi : X \to Y$ with the following properties*:

- *the fiber $X_y = \pi^{-1}(y)$ over any point $y \in Y$ is a long $\mathbb{C}^n$;*
- *X_y is biholomorphic to $\mathbb{C}^n$ if $y \in A$;*
- *X_y does not admit any nonconstant plurisubharmonic function if $y \in B$.*

If the base Y is $\mathbb{C}^p$, then X may be chosen to be a long $\mathbb{C}^{p+n}$.

Note that one or both of the sets A and B in Theorem 1.7 may be chosen everywhere dense in Y. The same result holds if A is a union of at most countably many closed complex subvarieties of Y and the set B is countable.

Several interesting questions on long $\mathbb{C}^n$'s remain open; we record some of them.

Problem 1.8. (A) Does there exist a long $\mathbb{C}^2$ which admits a nonconstant holomorphic function, but is not Stein?

(B) To what extent is it possible to prescribe the algebra $\mathscr{O}(X)$ of a long $\mathbb{C}^n$?

(C) Does there exist a long $\mathbb{C}^n$ for any $n > 1$ which is a Stein manifold different from $\mathbb{C}^n$?

(D) Does there exist a long $\mathbb{C}^n$ without nonconstant meromorphic functions?

(E) What can be said about the (non)existence of complex analytic subvarieties of positive dimension in non-Stein long $\mathbb{C}^n$'s?

In dimensions $n > 2$, an affirmative answer to problem (A) is provided by the product $X = \mathbb{C}^p \times X^{n-p}$ for any $p = 1, \ldots, n-2$, where X^{n-p} is a long $\mathbb{C}^{n-p}$ without nonconstant holomorphic functions, furnished by Theorem 1.1. Note that $\mathscr{O}(\mathbb{C}^p \times X^{n-p}) \cong \mathscr{O}(\mathbb{C}^p)$ is the algebra of functions coming from the base. Indeed, any example furnished by Theorem 1.7, with $Y = \mathbb{C}^p$ as base ($p \geq 1$) and B dense in $\mathbb{C}^p$, is of this kind.

Regarding question (D), note that the Fatou–Bieberbach maps $\phi_k : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ used in our constructions have rationally convex images, in the sense that for any compact polynomially convex set $K \subset \mathbb{C}^n$ its image $\phi_k(K)$ is a rationally convex set in $\mathbb{C}^n$; this gives rise to nontrivial meromorphic functions on the resulting long $\mathbb{C}^n$'s.

Since every long $\mathbb{C}^n$ is an *Oka manifold* [Lárusson 2010; Forstnerič 2011, Proposition 5.5.6, p. 200], the results of this paper also contribute to our understanding of the class of Oka manifolds, that is, manifolds which are the most natural targets for holomorphic maps from Stein manifolds and reduced Stein spaces.

Note that every long $\mathbb{C}^n$ is a topological cell according to a theorem of Brown [1961]. Furthermore, it was shown by Wold [2010, Theorem 1.2] that, if $X = \bigcup_{k=1}^{\infty} X_k$ is a long $\mathbb{C}^n$ and (X_k, X_{k+1}) is a Runge pair for every $k \in \mathbb{N}$, then X is biholomorphic to $\mathbb{C}^n$. Since the Runge property always holds in the $\mathscr{C}^\infty$ category, i.e., for smooth diffeomorphisms of Euclidean spaces, his proof can be adjusted to show that every long $\mathbb{C}^n$ is also diffeomorphic to $\mathbb{R}^{2n}$. Hence, Theorems 1.2 and 1.6 imply the following corollary.

Corollary 1.9. *For every $n > 1$ there exists a continuum of pairwise nonequivalent Oka manifolds of complex dimension n which are all diffeomorphic to $\mathbb{R}^{2n}$.*

In Section 5 we show that $\mathbb{C}^n$ for any $n > 1$ can also be represented as an increasing union of non-Runge Fatou–Bieberbach domains.

2. Preliminaries

In this section, we introduce the notation and recall the basic ingredients.

We denote by $\mathscr{O}(X)$ the algebra of all holomorphic functions on a complex manifold X. For a compact set $K \subset X$, $\mathscr{O}(K)$ stands for the algebra of functions holomorphic in open neighborhoods of K (in the sense of germs on K). The $\mathscr{O}(X)$*-convex hull* of K is

$$\widehat{K}_{\mathscr{O}(X)} = \{x \in X : |f(x)| \le \sup_K |f| \text{ for all } f \in \mathscr{O}(X)\}.$$

When $X = \mathbb{C}^n$, the set $\widehat{K} = \widehat{K}_{\mathscr{O}(\mathbb{C}^n)}$ is the *polynomial hull* of X. If $\widehat{K}_{\mathscr{O}(X)} = K$, we say that K is *holomorphically convex in* X; if $X = \mathbb{C}^n$ then K is polynomially convex. More generally, if $K \subset L$ are compact sets in X, we define the hull $\widehat{K}_{\mathscr{O}(L)}$ by (1-1).

Given a point $p \in \mathbb{C}^n$, we denote by $\mathbb{B}(p; r)$ the closed ball of radius r centered at p.

We shall frequently use the following basic result; see, e.g., [Stout 1971; 2007] for the first part (which is a simple application of E. Kallin's lemma) and [Forstnerič 1986] for the second part.

Lemma 2.1. *Assume that $B \subset \mathbb{C}^n$ is a compact polynomially convex set. For any $p_1, \dots, p_m \in \mathbb{C}^n \setminus B$ and for all sufficiently small numbers $r_1 > 0, \dots, r_m > 0$, the set $\bigcup_{j=1}^m \mathbb{B}(p_j, r_j) \cup B$ is polynomially convex. Furthermore, if B is the closure of a bounded strongly pseudoconvex domain with $\mathscr{C}^2$ boundary, then any sufficiently $\mathscr{C}^2$-small deformation of B in $\mathbb{C}^n$ is still polynomially convex.*

The key ingredient in our proofs is the main result of the Andersén–Lempert theory as formulated by Forstnerič and Rosay [1993, Theorem 1.1]; see Theorem 2.3 below. We use it not only for $\mathbb{C}^n$, but also for $X = \mathbb{C}^* \times \mathbb{C}^{n-1}$. The result holds for any Stein manifold which enjoys the following *density property* introduced by Varolin [2001]. (See also [Forstnerič 2011, Definition 4.10.1].)

Definition 2.2. A complex manifold X enjoys the (holomorphic) *density property* if every holomorphic vector field on X can be approximated, uniformly on compacts, by Lie combinations (sums and commutators) of $\mathbb{C}$-complete holomorphic vector fields on X.

By [Andersén 1990; Andersén and Lempert 1992], the complex Euclidean space $\mathbb{C}^n$ for $n > 1$ enjoys the density property. More generally, Varolin proved that any complex manifold $X = (\mathbb{C}^*)^k \times \mathbb{C}^l$ with $k + l \ge 2$ and $l \ge 1$ enjoys the density property [Varolin 2001]. For surveys of this subject, see for instance [Forstnerič 2011, Chapter 4; Kaliman and Kutzschebauch 2011].

Theorem 2.3. *Let X be a Stein manifold with the density property, and let*

$$\Phi_t : \Omega_0 \to \Omega_t = \Phi_t(\Omega_0) \subset X, \quad t \in [0, 1]$$

be a smooth isotopy of biholomorphic maps of Ω_0 *onto Runge domains* $\Omega_t \subset X$ *such that* $\Phi_0 = \mathrm{Id}_{\Omega_0}$. *Then, the map* $\Phi_1 : \Omega_0 \to \Omega_1$ *can be approximated uniformly on compacts in* Ω_0 *by holomorphic automorphisms of* X.

This is a version of [Forstnerič and Rosay 1993, Theorem 1.1] in which $\mathbb{C}^n$ is replaced by an arbitrary Stein manifold with the density property; see also [Forstnerič 2011, Theorem 4.10.6]. For a detailed proof of Theorem 2.3, see [Forstnerič and Rosay 1993, Theorem 1.1] for the case $X = \mathbb{C}^n$ and [Ritter 2013, Theorem 8] for the general case (which follows the one in [Forstnerič and Rosay 1993] essentially verbatim).

3. Construction of a long $\mathbb{C}^n$ without holomorphic functions

In this section, we prove Theorem 1.1. We begin by recalling the general construction of a long $\mathbb{C}^n$; see [Wold 2010] or [Forstnerič 2011, Section 4.20].

Recall that a *Fatou–Bieberbach map* is an injective holomorphic map $\phi : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that $\phi(\mathbb{C}^n) \subsetneq \mathbb{C}^n$; the image $\phi(\mathbb{C}^n)$ of such a map is called a *Fatou–Bieberbach domain*. Every complex manifold X which is a long $\mathbb{C}^n$ is determined by a sequence of Fatou–Bieberbach maps $\phi_k : \mathbb{C}^n \to \mathbb{C}^n$ $(k = 1, 2, 3, \ldots)$. The elements of X are represented by infinite strings $x = (x_i, x_{i+1}, \ldots)$, where $i \in \mathbb{N}$ and for every $k = i, i+1, \ldots$ we have $x_k \in \mathbb{C}^n$ and $x_{k+1} = \phi_k(x_k)$. Another string $y = (y_j, y_{j+1}, \ldots)$ determines the same element of X if and only if one of the following possibilities holds:

- $i = j$ and $x_i = y_i$ (and hence $x_k = y_k$ for all $k > i$);
- $i < j$ and $y_j = \phi_{j-1} \circ \cdots \circ \phi_i(x_i)$;
- $j < i$ and $x_i = \phi_{i-1} \circ \cdots \circ \phi_j(y_j)$.

For each $k \in \mathbb{N}$, let $\psi_k : \mathbb{C}^n \hookrightarrow X$ be the injective map sending $z \in \mathbb{C}^n$ to the equivalence class of the string $(z, \phi_k(z), \phi_{k+1}(\phi_k(z)), \ldots)$. Set $X_k = \psi_k(\mathbb{C}^n)$ and let $\iota_k : X_k \hookrightarrow X_{k+1}$ be the inclusion map induced by left shift $(x_k, x_{k+1}, x_{k+2}, \ldots) \mapsto (x_{k+1}, x_{k+2}, \ldots)$. Then

$$\iota_k \circ \psi_k = \psi_{k+1} \circ \phi_k, \qquad k = 1, 2, \ldots. \tag{3-1}$$

Recall that a compact set L in a complex manifold X is said to be *holomorphically contractible* if there exist a neighborhood $U \subset X$ of L and a smooth 1-parameter family of injective holomorphic maps $F_t : U \to U$ $(t \in [0, 1))$ such that F_0 is the identity map on U, $F_t(L) \subset L$ for every $t \in [0, 1]$, and $\lim_{t \to 1} F_t$ is a constant map $L \mapsto p \in L$.

The first part of the following lemma is the key ingredient in the construction of the sequence $(\phi_k)_{k \in \mathbb{N}}$ determining a long $\mathbb{C}^n$ as in Theorem 1.1. The same construction gives the second part, which we include for future applications. We write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Lemma 3.1. *Let* K *be a compact set with nonempty interior in* $\mathbb{C}^n$ *for some* $n > 1$. *For every point* $a \in \mathbb{C}^n$ *there exists an injective holomorphic map* $\phi : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ *such that the polynomial hull of the set* $\phi(K)$ *contains the point* $\phi(a)$. *More generally, if* $L \subset \mathbb{C}^n$ *is a compact holomorphically contractible set* *disjoint from K such that $K \cup L$ is polynomially convex, then there exists an injective holomorphic map $\phi : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that $\phi(L) \subset \widehat{\phi(K)}$ and $\widehat{\phi(K)} \setminus \phi(\mathbb{C}^n) \neq \varnothing$.*

Proof. To simplify the notation, we consider the case $n = 2$; it will be obvious that the same proof applies in any dimension $n \geq 2$. We follow Wold's construction [2008; 2010] up to a certain point, adding a new twist at the end.

Let M be a compact set in $\mathbb{C}^* \times \mathbb{C}$ enjoying the following properties:

(1) M is a disjoint union of two smooth, embedded, totally real discs.

(2) M is holomorphically convex in $\mathbb{C}^* \times \mathbb{C}$.

(3) the polynomial hull $\widehat{M}$ of M contains the origin $(0, 0) \in \mathbb{C}^2$.

A set M with these properties was constructed by Stolzenberg [1966]; it has been reproduced in [Stout 1971, pp. 392–396; Wold 2008, Section 2; Forstnerič 2011, Section 4.20].

Choose a Fatou–Bieberbach map $\theta : \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ whose image $\theta(\mathbb{C}^2)$ is Runge in $\mathbb{C}^2$. For example, we may take the basin of an attracting fixed point of a holomorphic automorphism of $\mathbb{C}^2$ which fixes $\{0\} \times \mathbb{C}$; see [Rosay and Rudin 1988] for explicit examples. Replacing the set K by its polynomial hull $\widehat{K}$, we may assume that K is polynomially convex. Since $\theta(\mathbb{C}^2)$ is Runge in $\mathbb{C}^2$, the set $\theta(K)$ is also polynomially convex, and hence $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex. By [Wold 2008, Lemma 3.2], there exists a holomorphic automorphism $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that

$$\psi(M) \subset \theta(K^\circ).$$

The construction of such an automorphism ψ uses Theorem 2.3 applied to the manifold $X = \mathbb{C}^* \times \mathbb{C}$. We include a brief outline.

By shrinking each of the two discs in M within themselves until they become very small and then translating them into K° within $\mathbb{C}^* \times \mathbb{C}$, we find an isotopy of diffeomorphisms $h_t : M = M_0 \to M_t \subset \mathbb{C}^* \times \mathbb{C}$ ($t \in [0, 1]$), where each $M_t = h_t(M)$ is a totally real $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex submanifold of $\mathbb{C}^* \times \mathbb{C}$, such that $M_1 \subset K^\circ$. Since $\mathbb{C}^* \times \mathbb{C}$ has the holomorphic density property (see [Varolin 2001]), each diffeomorphism h_t can be approximated uniformly on M (and even in the smooth topology on M) by holomorphic automorphisms of $\mathbb{C}^* \times \mathbb{C}$. This is done in two steps. First, we approximate h_t by a smooth isotopy of biholomorphic maps $f_t : U_0 \to U_t$ from a neighborhood U_0 of M_0 onto a neighborhood U_t of M_t; this is done as in [Forstnerič and Løw 1997]. Since the submanifold M_t is totally real and $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex for each $t \in [0, 1]$, we can arrange that the neighborhood U_t is Runge in $\mathbb{C}^* \times \mathbb{C}$ for each $t \in [0, 1]$. Hence, Theorem 2.3 furnishes an automorphism $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ which approximates the diffeomorphism $h_1 : M \to M_1$ sufficiently closely such that $\psi(M) \subset B$. It follows that the injective holomorphic map $\tilde{\phi} = \psi^{-1} \circ \theta : \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ satisfies $M \subset \tilde{\phi}(K^\circ)$. Note that $K' := \tilde{\phi}(K)$ is a compact $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex set which contains M in its interior. Therefore, its polynomial hull $\widehat{K}'$ contains a neighborhood of $\widehat{M}$, and hence a neighborhood $V \subset \mathbb{C}^2$ of the origin $(0, 0) \in \mathbb{C}^2$. We may assume that $\overline{V} \cap K' = \varnothing$.

Let $a \in \mathbb{C}^2$. If $\tilde{\phi}(a) \in \widehat{K}'$, then we take $\phi = \tilde{\phi}$ and we are done. If this is not the case, we choose a point $a' \in V \cap (\mathbb{C}^* \times \mathbb{C})$ and apply Theorem 2.3 to find a holomorphic automorphism $\tau \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ which is close to the identity map on K' and satisfies $\tau(\tilde{\phi}(a)) = a'$. Such τ exists since the union of K' with a single point of $\mathbb{C}^* \times \mathbb{C}$ is $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex, so it suffices to apply the cited result to an isotopy of injective holomorphic maps which is the identity near K' and which moves $\tilde{\phi}(a)$ to a' in $\mathbb{C}^* \times \mathbb{C} \setminus K'$. Assuming that τ is sufficiently close to the identity map on K', we have $M \subset \tau(K')$, and hence $a' \in \widehat{M} \subset \widehat{\tau(K')}$. Clearly, the map $\phi = \tau \circ \tilde{\phi} : \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$ satisfies $\phi(a) \in \widehat{\phi(K)}$. This proves the first part of the lemma.

The second part is proved similarly. Since the set $L' := \theta(L) \subset \mathbb{C}^* \times \mathbb{C}$ is holomorphically contractible and $K' \cup L'$ is $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex, there exists an automorphism $\tau \in \operatorname{Aut}(\mathbb{C}^* \times \mathbb{C})$ which approximates the identity map on K' and satisfies $\tau(L') \subset V \cap (\mathbb{C}^* \times \mathbb{C})$. (To find such τ, we apply Theorem 2.3 to a smooth isotopy $h_t : U \to h_t(U) \subset \mathbb{C}^* \times \mathbb{C}$ ($t \in [0, 1]$) of injective holomorphic maps on a small neighborhood $U \subset \mathbb{C}^* \times \mathbb{C}$ of $K' \cup L'$ such that h_0 is the identity on U, h_t is the identity near K' for every $t \in [0, 1]$, and $h_1(L') \subset V$. On the set L', h_t first squeezes L' within itself almost to a point and then moves it to a position within V. Clearly, such an isotopy can be found such that $h_t(K' \cup L') = K_t \cup h_t(L')$ is $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex for all $t \in [0, 1]$.) If τ is sufficiently close to the identity on K', then the polynomial hull $\widehat{\tau(K')}$ still contains V, and hence $\tau(L') \subset V \subset \widehat{\tau(K')}$. The map $\phi = \tau \circ \tilde{\phi} : \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$ then satisfies the desired conclusion. □

Proof of Theorem 1.1. Pick a compact set $K \subset \mathbb{C}^n$ with nonempty interior and a countable dense sequence $\{a_j\}_{j \in \mathbb{N}}$ in $\mathbb{C}^n$. Set $K_1 = \widehat{K}$. Lemma 3.1 furnishes an injective holomorphic map $\phi_1 : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\phi_1(a_1) \in \widehat{\phi_1(K_1)} =: K_2. \tag{3-2}$$

Applying Lemma 3.1 to the set K_2 and the point $\phi_1(a_2) \in \mathbb{C}^n$ gives an injective holomorphic map $\phi_2 : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that

$$\phi_2(\phi_1(a_2)) \in \widehat{\phi_2(K_2)} =: K_3.$$

From the first step we also have $\phi_1(a_1) \in K_2$, and hence $\phi_2(\phi_1(a_1)) \in K_3$.

Continuing inductively, we obtain a sequence $\phi_j : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ of injective holomorphic maps for $j = 1, 2, \ldots$ such that, setting $\Phi_k = \phi_k \circ \cdots \circ \phi_1 : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$, we have

$$\Phi_k(a_j) \in \widehat{\Phi_k(K)} \quad \text{for all } j = 1, \ldots, k. \tag{3-3}$$

In the limit manifold $X = \bigcup_{k=1}^{\infty} X_k$ (the long $\mathbb{C}^n$) determined by the sequence $(\phi_k)_{k=1}^{\infty}$, the $\mathscr{O}(X)$-hull of the initial set $K \subset \mathbb{C}^n = X_1 \subset X$ clearly contains the set $\Phi_k(K) \subset X_{k+1}$ for each $k = 1, 2, \ldots$. (We have identified the k-th copy of $\mathbb{C}^n$ in the sequence with its image $\psi_k(\mathbb{C}^n) = X_k \subset X$.) It follows from (3-3) that the hull $\widehat{K}_{\mathscr{O}(X)}$ contains the set $\{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C}^n = X_1$. Since this set is everywhere dense in $\mathbb{C}^n$, every holomorphic function on X is bounded on $X_1 = \mathbb{C}^n$, and hence constant. By the identity principle, it follows that the function is constant on all of X.

The same argument shows that the plurisubharmonic hull $\widehat{K}_{\mathrm{Psh}(X)}$ of K contains the set $A_1 := \{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C}^n \cong X_1$, and hence every plurisubharmonic function $u \in \mathrm{Psh}(X)$ is bounded from above on A_1. Since A_1 is dense in X_1, it follows that u is bounded from above on X_1. (This is obvious if u is continuous; the general case follows by observing that u can be approximated from above, uniformly on compacts in $X_1 \cong \mathbb{C}^n$, by continuous plurisubharmonic functions.) It follows from Liouville's theorem for plurisubharmonic functions that u is constant on X_1.

In order to ensure that u is constant on each copy $X_k \cong \mathbb{C}^n$ $(k \in \mathbb{N})$ in the given exhaustion of X, we modify the construction as follows. After choosing the first Fatou–Bieberbach map $\phi_1 : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that $\phi_1(a_1) \in \widehat{\phi_1(K)}$ (see (3-2)), we choose a countable dense set $A'_2 = \{a'_{2,1}, a'_{2,2}, \ldots\}$ in $\mathbb{C}^n \setminus \phi_1(\mathbb{C}^n)$ and set $A_2 = \phi_1(A_1) \cup A'_2$ to get a countable dense set in $X_2 \cong \mathbb{C}^n$. Next, we find a Fatou–Bieberbach map $\phi_2 : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that the first two points $\phi_1(a_1), \phi_1(a_2)$ of the set $\phi_1(A_1)$, and also the first point $a'_{2,1}$ of A'_2, are mapped by ϕ_2 into the polynomial hull of $\phi_2(\phi_1(K))$. We continue inductively. At the k-th stage of the construction we have chosen a Fatou–Bieberbach map $\phi_k : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$, and we take $A_{k+1} = \phi_k(A_k) \cup A'_{k+1}$, where A'_{k+1} is a countable dense set in $\mathbb{C}^n \setminus \phi_k(A_k)$. In the manifold X we thus get an increasing sequence $A_1 \subset A_2 \subset \cdots$ whose union $A := \bigcup_{k=1}^\infty A_k$ is dense in X and such that every point of A ends up in the hull $\widehat{K}_{\mathscr{O}(X_k)} = \widehat{K}_{\mathrm{Psh}(X_k)}$ for all sufficiently big $k \in \mathbb{N}$. (See the proof of Theorem 1.6 for more details in a related context.) Hence, the plurisubharmonic hull $\widehat{K}_{\mathrm{Psh}(X)}$ contains the countable dense subset A of X. We conclude as before that any plurisubharmonic function on X is bounded on every $X_k \cong \mathbb{C}^n$, and hence constant. □

Remark 3.2 (Wold process). The key ingredient in the proof of Lemma 3.1 is the method, introduced by E. F. Wold [2008], of stretching the image of a compact set in $\mathbb{C}^* \times \mathbb{C}^{n-1}$ by an automorphism of $\mathbb{C}^* \times \mathbb{C}^{n-1}$ so that its image swallows a compact set M whose polynomial hull in $\mathbb{C}^n$ intersects the hyperplane $\{0\} \times \mathbb{C}^{n-1}$. This is called the *Wold process*. A recursive application of this method, possibly at several places simultaneously and with additional approximation of the identity map on a certain other compact polynomially convex set (see Lemma 4.3), causes the hulls of the respective sets to reach out of all domains $X_k \cong \mathbb{C}^n$ in the exhaustion of X.

4. Construction of manifolds $X(B)$

In this section, we construct long $\mathbb{C}^n$'s satisfying Theorems 1.2 and 1.6.

We begin by showing that the stable hull property of a compact set in a complex manifold X (see Definition 1.4) is independent of the choice of exhaustion of X by compact sets.

Lemma 4.1. *Let $X = \bigcup_{j=1}^\infty K_j$, where $K_j \subset K^\circ_{j+1}$ is a sequence of compact sets. Let B be a compact set in X. Assume that there exists an integer $j_0 \in \mathbb{N}$ such that $B \subset K_{j_0}$ and*

$$\widehat{B}_{\mathscr{O}(K_j)} = \widehat{B}_{\mathscr{O}(K_{j_0})} \quad \textit{for all } j \ge j_0. \tag{4-1}$$

Then B satisfies the same condition with respect to any exhaustion of X by an increasing sequence of compact sets.

Proof. Set $C := \widehat{B}_{\mathscr{O}(K_{j_0})}$. Let $(L_l)_{l\in\mathbb{N}}$ be another exhaustion of X by compact sets satisfying $L_l \subset L^\circ_{l+1}$ for all $l \in \mathbb{N}$. Pick an integer $l_0 \in \mathbb{N}$ such that $C \subset L_{l_0}$. Since both sequences K°_j and L°_l exhaust X, we can find sequences of integers $j_1 < j_2 < j_3 < \cdots$ and $l_1 < l_2 < l_3 < \cdots$ such that $j_0 \le j_1$, $l_0 \le l_1$, and

$$K_{j_0} \subset L_{l_1} \subset K_{j_1} \subset L_{l_2} \subset K_{j_2} \subset L_{l_3} \subset \cdots.$$

From this and (4-1) we obtain

$$C = \widehat{B}_{\mathscr{O}(K_{j_0})} \subset \widehat{B}_{\mathscr{O}(L_{l_1})} \subset \widehat{B}_{\mathscr{O}(K_{j_1})} = C \subset \widehat{B}_{\mathscr{O}(L_{l_2})} \subset \widehat{B}_{\mathscr{O}(K_{j_2})} = C \subset \cdots.$$

It follows that $\widehat{B}_{\mathscr{O}(L_{l_j})} = C$ for all $j \in \mathbb{N}$. Since the sequence of hulls $\widehat{B}_{\mathscr{O}(L_l)}$ is increasing with l, we conclude that

$$\widehat{B}_{\mathscr{O}(L_l)} = C \quad \text{for all } l \geq l_1.$$

Hence, B has the stable hull property with respect to the exhaustion $(L_l)_{l\in\mathbb{N}}$ of X. □

Remark 4.2. If a complex manifold X is exhausted by an increasing sequence of Stein domains $X_1 \subset X_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} X_j = X$ (this holds for example if X is a long $\mathbb{C}^n$ or a short $\mathbb{C}^n$, where the latter term refers to a manifold exhausted by biholomorphic copies of the ball), then we can choose an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = X$ such that K_j is a compact set contained in X_j and $\widehat{(K_j)}_{\mathscr{O}(X_j)} = K_j$ for every $j \in \mathbb{N}$. If K is a compact set contained in some K_{j_0}, then clearly $\widehat{K}_{\mathscr{O}(K_j)} = \widehat{K}_{\mathscr{O}(X_j)}$ for all $j \geq j_0$. In such case, K has the stable hull property if and only if the sequence of hulls $\widehat{K}_{\mathscr{O}(X_j)}$ stabilizes. This notion is especially interesting for a long $\mathbb{C}^n$. Imagining the exhaustion $X_j \cong \mathbb{C}^n$ of X as an increasing sequence of universes, the stable hull property means that K only influences finitely many of these universes in a nontrivial way, while a set without SHP has nontrivial influence on all subsequent universes. □

We shall need the following lemma, which generalizes [Wold 2008, Lemma 3.2].

Lemma 4.3. *Let $n > 1$. Assume that B is a compact polynomially convex set in $\mathbb{C}^n$, $K_1, \ldots, K_m$ are pairwise disjoint compact sets with nonempty interiors in $\mathbb{C}^n \setminus B$ such that $B \cup \bigl(\bigcup_{j=1}^m K_j\bigr)$ is polynomially convex, and $\beta \subset \mathbb{C}^n \setminus \bigl(B \cup \bigl(\bigcup_{j=1}^m K_j\bigr)\bigr)$ is a finite set. Then there exists a Fatou–Bieberbach map $\phi : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ satisfying the following conditions*:

(i) $\widehat{\phi(B)} = \phi(B)$;

(ii) $\widehat{\phi(K_j)} \not\subset \phi(\mathbb{C}^n)$ *for all* $j = 1, \ldots, m$;

(iii) $\phi(\beta) \subset \widehat{\phi(K_1)}$.

Furthermore, we can choose ϕ such that $\phi|_B$ is as close as desired to the identity map.

Proof. For simplicity of notation we give the proof for $n = 2$; the same argument applies for any $n \geq 2$.

By enlarging B slightly, we may assume that it is a compact strongly pseudoconvex and polynomially convex domain in $\mathbb{C}^n$. Choose a closed ball $\mathcal{B} \subset \mathbb{C}^2$ containing B in its interior. Let $\Lambda \subset \mathbb{C}^2 \setminus \mathcal{B}$ be an affine complex line. Up to an affine change of coordinates on $\mathbb{C}^2$ we may assume that $\Lambda = \{0\} \times \mathbb{C}$.

As in the proof of Lemma 3.1, we find an injective holomorphic map $\theta_1 : \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ whose image is Runge in $\mathbb{C}^2$, and hence the set $\theta_1(\mathcal{B})$ is polynomially convex. Since $\mathcal{B}$ is contractible, we can connect the identity map on $\mathcal{B}$ to $\theta_1|_{\mathcal{B}}$ by an isotopy of biholomorphic maps $h_t : \mathcal{B} \to \mathcal{B}_t$ ($t \in [0, 1]$) with Runge images in $\mathbb{C}^* \times \mathbb{C}$. Theorem 2.3 furnishes an automorphism $\theta_2 \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that θ_2 approximates θ_1^{-1} on $\theta_1(\mathcal{B})$. The composition $\theta = \theta_2 \circ \theta_1 : \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ is then an injective holomorphic map which is close to the identity on $\mathcal{B}$. Assuming that the approximation is close enough, the set $B' := \theta(B)$ is polynomially convex in view of Lemma 2.1.

Set $K = \bigcup_{j=1}^m K_j$, $K'_j = \theta(K_j)$ for $j = 1, \ldots, m$, and $K' = \theta(K) = \bigcup_{j=1}^m K'_j$. Note that the set $B' \cup K' = \theta(B \cup K) \subset \mathbb{C}^* \times \mathbb{C}$ is $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex.

Choose m pairwise disjoint copies $M_1, \ldots, M_m \subset (\mathbb{C}^* \times \mathbb{C}) \setminus B'$ of Stolzenberg's [1966] compact set M described in the proof of Lemma 3.1. Explicitly, each set M_j is $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex and its polynomial hull $\widehat{M}_j$ intersects the complex line $\{0\} \times \mathbb{C}$ (which lies in the complement of $\theta(\mathbb{C}^2)$). By placing the sets M_j sufficiently far apart and away from B', we may assume that the compact set $B' \cup \big(\bigcup_{j=1}^m M_j\big)$ is $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex. Pick a slightly bigger compact set $B'' \subset \theta(\mathbb{C}^2)$, containing B' in its interior, such that the sets $B'' \cup \big(\bigcup_{j=1}^m K_j'\big)$ and $B'' \cup \big(\bigcup_{j=1}^m M_j\big)$ are still $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex.

We claim that for every $\epsilon > 0$ there is an automorphism $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that

(a) $|\psi(z) - z| < \epsilon$ for all $z \in B''$, and

(b) $\psi(M_j) \subset K_j'$ for $j = 1, \dots, m$.

To obtain such a ψ, we apply the construction in the proof of Lemma 3.1 to find an isotopy of smooth diffeomorphisms

$$h_t : M = \bigcup_{j=1}^m M_j \to M^t = h_t(M) \subset \mathbb{C}^* \times \mathbb{C}, \quad t \in [0, 1],$$

such that $h_0 = \mathrm{Id}\,|_M$, the set $M^t = \bigcup_{j=1}^m h_t(M_j)$ consists of smooth totally real submanifolds, $B'' \cap M^t = \varnothing$ for all $t \in [0, 1]$, $B'' \cup M^t$ is $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex for all $t \in [0, 1]$, and $h_1(M_j) \subset K_j'^{\circ}$ for $j = 1, \dots, m$. It follows that h_1 can be approximated uniformly on M by a holomorphic automorphism $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ which at the same time approximates the identity map on B''. (For the details in a similar context, see [Forstnerič and Rosay 1993, proof of Theorem 2.3] or [Forstnerič 2011, proof of Corollary 4.12.4].) The injective holomorphic map

$$\phi := \psi^{-1} \circ \theta : \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$$

then approximates the identity map on a neighborhood of B and satisfies $M_j \subset \phi(K_j)$ for $j = 1, \dots, m$. It follows that

$$\widehat{\phi(K_j)} \cap (\{0\} \times \mathbb{C}) \neq \varnothing \quad \text{for all } j = 1, \dots, m.$$

If the approximation $\psi|_{B''} \approx \mathrm{Id}$ in (a) is close enough, then the set $\phi(B) = \psi^{-1}(B')$ is still polynomially convex by Lemma 2.1. Clearly, ϕ satisfies properties (i) and (ii), and property (iii) can be achieved by applying Lemma 3.1. □

Proof of Theorem 1.6(a). Let B be the given regular compact polynomially convex set in $\mathbb{C}^n$. To begin the induction, set $B_1 := B \subset \mathbb{C}^n = X_1$ and choose a pair of disjoint countable set

$$A_1 = \{a_1^l : l \in \mathbb{N}\} \subset \mathbb{C}^n \setminus B_1, \qquad \overline{A}_1 = \mathbb{C}^n \setminus B_1^{\circ},$$
$$\Gamma_1 = \{\gamma_1^l : l \in \mathbb{N}\} \subset \mathbb{C}^n \setminus (A_1 \cup B_1), \quad \overline{\Gamma}_1 = \mathbb{C}^n \setminus B_1^{\circ}.$$

Let $\mathbb{B}(a_1^1, r_1)$ denote the closed ball of radius r_1 centered at a_1^1. By choosing $r_1 > 0$ small enough we may ensure that $\mathbb{B}(a_1^1, r_1) \cap B_1 = \varnothing$, $\gamma_1^1 \notin \mathbb{B}(a_1^1, r_1) \cup B_1$, and the set $\mathbb{B}(a_1^1, r_1) \cup B_1$ is polynomially convex (see Lemma 2.1). Lemma 4.3 furnishes an injective holomorphic map $\phi_1 : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that the set $B_2 := \phi_1(B_1) \subset \mathbb{C}^n$ is polynomially convex, while the compact set

$$C_{1,1}^1 := \phi_1(\mathbb{B}(a_1^1, r_1)) \subset \mathbb{C}^n$$

satisfies

$$\widehat{C_{1,1}^1} \setminus \phi_1(\mathbb{C}^n) \neq \varnothing \quad \text{and} \quad \phi_1(\gamma_1^1) \in \widehat{C_{1,1}^1}.$$

We proceed recursively. Suppose that for some $k \in \mathbb{N}$ we have found

- injective holomorphic maps $\phi_1, \dots, \phi_k : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$,
- compact polynomially convex sets $B_1, B_2, \dots, B_{k+1}$ in $\mathbb{C}^n$ such that $B_{i+1} = \phi_i(B_i)$ for $i = 1, \dots, k$,
- countable sets $A_1, \dots, A_k \subset \mathbb{C}^n$ such that for every $i = 1, \dots, k$ we have
$$A_i \subset \mathbb{C}^n \setminus B_i, \quad \overline{A}_i = \mathbb{C}^n \setminus B_i^\circ, \quad A_i = \phi_{i-1}(A_{i-1}) \cup \{a_i^l : l \in \mathbb{N}\}$$
(where we set $A_0 = \varnothing$),
- countable sets $\Gamma_1, \dots, \Gamma_k \subset \mathbb{C}^n$ such that for every $i = 1, \dots, k$ we have
$$\Gamma_i \subset \mathbb{C}^n \setminus A_i \cup B_i, \quad \overline{\Gamma}_i = \mathbb{C}^n \setminus B_i^\circ, \quad \Gamma_i = \phi_{i-1}(\Gamma_{i-1}) \cup \{\gamma_i^l : l \in \mathbb{N}\}$$
(where we set $\Gamma_0 = \varnothing$), and
- numbers $r_1 > \cdots > r_k > 0$

such that, setting for all $(i, l) \in \mathbb{N}^2$ with $1 \le i + l \le k + 1$
$$\begin{aligned} b_{k,i}^l &:= \phi_{k-1} \circ \cdots \circ \phi_i(a_i^l) \in A_k \quad \text{if } (i, l) \neq (k, 1), \qquad b_{k,k}^1 := a_k^1, \\ \beta_{k,i}^l &:= \phi_{k-1} \circ \cdots \circ \phi_i(\gamma_i^l) \in \Gamma_k \quad \text{if } (i, l) \neq (k, 1), \qquad \beta_{k,k}^1 := \gamma_k^1, \end{aligned}$$
the following conditions hold for all pairs $(i, l) \in \mathbb{N}^2$ with $i + l \le k + 1$:

(1_k) the closed balls $\mathbb{B}(b_{k,i}^l, r_k)$ are pairwise disjoint and contained in $\mathbb{C}^n \setminus B_k$, and
$$\{\beta_{k,i}^l : i + l \le k + 1\} \cap \bigcup_{i+l \le k+1} \mathbb{B}(b_{k,i}^l, r_k) = \varnothing$$
(since $A_k \cap \Gamma_k = \varnothing$, the latter condition holds provided $r_k > 0$ is small enough);

(2_k) the set $\bigcup_{i+l \le k+1} \mathbb{B}(b_{k,i}^l, r_k) \cup B_k$ is polynomially convex;

(3_k) the set $(\phi_{k-1} \circ \cdots \circ \phi_i)^{-1}(\mathbb{B}(b_{k,i}^l, r_k))$ is contained in $\mathbb{B}(a_i^l, r_i/2^k)$;

(4_k) the set $C_{k,i}^l := \phi_k(\mathbb{B}(b_{k,i}^l, r_k))$ satisfies $\widehat{C_{k,i}^l} \setminus \phi_k(\mathbb{C}^n) \neq \varnothing$;

(5_k) $\{\phi_k(\beta_{k,i}^l) : i + l \le k + 1\} \subset \widehat{C_{k,1}^1}$.

We now explain the inductive step. We begin by adding to $\phi_k(A_k)$ countably many points in $\mathbb{C}^n \setminus (\phi_k(A_k) \cup B_{k+1})$ to get a countable set
$$A_{k+1} = \phi_k(A_k) \cup \{a_{k+1}^l : l \in \mathbb{N}\} \subset \mathbb{C}^n \setminus B_{k+1}$$
such that
$$\overline{A}_{k+1} = \mathbb{C}^n \setminus B_{k+1}^\circ.$$
In the same way, we find the next countable set
$$\Gamma_{k+1} = \phi_k(\Gamma_k) \cup \{\gamma_{k+1}^l : l \in \mathbb{N}\} \subset \mathbb{C}^n \setminus (A_{k+1} \cup B_{k+1})$$

such that

$$\overline{\Gamma}_{k+1} = \mathbb{C}^n \setminus B_{k+1}^\circ.$$

For every pair of indices $(i,l) \in \mathbb{N}^2$ with $i+l \le k+2$, we set

$$\begin{aligned} b_{k+1,i}^l &:= \phi_k \circ \cdots \circ \phi_i(a_i^l) \in A_{k+1} \quad \text{if } (i,l) \neq (k+1,1), \qquad b_{k+1,k+1}^1 := a_{k+1}^1, \\ \beta_{k+1,i}^l &:= \phi_k \circ \cdots \circ \phi_i(\gamma_i^l) \in \Gamma_{k+1} \quad \text{if } (i,l) \neq (k+1,1), \qquad \beta_{k+1,k+1}^1 := \gamma_{k+1}^1. \end{aligned}$$

Choose a number r_{k+1} with $0 < r_{k+1} < r_k$ and so small that the following conditions hold for all $(i,l) \in \mathbb{N}^2$ with $i+l \le k+2$:

(1_{k+1}) the closed balls $\mathbb{B}(b_{k+1,i}^l, r_{k+1})$ are pairwise disjoint and contained in $\mathbb{C}^n \setminus B_{k+1}$, and

$$\{\beta_{k+1,i}^l : i+l \le k+2\} \cap \Bigg(\bigcup_{i+l \le k+2} \mathbb{B}(b_{k+1,i}^l, r_{k+1}) \cup B_{k+1} \Bigg) = \varnothing;$$

(2_{k+1}) the set $\bigcup_{i+l \le k+2} \mathbb{B}(b_{k+1,i}^l, r_{k+1}) \cup B_{k+1}$ is polynomially convex;

(3_{k+1}) the set $(\phi_k \circ \cdots \circ \phi_i)^{-1}(\mathbb{B}(b_{k+1,i}^l, r_{k+1}))$ is contained in $\mathbb{B}(a_i^l, r_i/2^{k+1})$.

Lemma 4.3 gives a Fatou–Bieberbach map $\phi_{k+1} : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that the compact set $B_{k+2} := \phi_{k+1}(B_{k+1})$ is polynomially convex, while the compact sets

$$C_{k+1,i}^l := \phi_{k+1}(\mathbb{B}(b_{k+1,i}^l, r_{k+1})), \quad i+l \le k+2$$

satisfy the following conditions:

(4_{k+1}) $\widehat{C_{k+1,i}^l} \setminus \phi_{k+1}(\mathbb{C}^n) \neq \varnothing$ for all $(i,l) \in \mathbb{N}^2$ with $i+l \le k+2$;

(5_{k+1}) $\{\phi_{k+1}(\beta_{k+1,i}^l) : i+l \le k+2\} \subset \widehat{C_{k+1,1}^1}$.

This completes the induction step and the recursion may continue.

Let $X = \bigcup_{k=1}^\infty X_k$ be the long $\mathbb{C}^n$ determined by the sequence $(\phi_k)_{k=1}^\infty$. Since the set $B_k \subset \mathbb{C}^n$ is polynomially convex and $B_{k+1} = \phi_k(B_k)$ for all $k \in \mathbb{N}$, the sequence $(B_k)_{k \in \mathbb{N}}$ determines a subset $B = B_1 \subset X$ such that

$$\widehat{B}_{\mathscr{O}(X_k)} = B \quad \text{for all } k \in \mathbb{N}. \tag{4-2}$$

This means that the initial compact set $B \subset \mathbb{C}^n = X_1$ has the stable hull property in X.

By the construction, the countable sets $A_k \subset \mathbb{C}^n \setminus B_k$ satisfy $\phi_k(A_k) \subset A_{k+1}$ for each $k \in \mathbb{N}$, and hence they determine a countable set $A \subset X \setminus B$. Furthermore, since $\overline{A}_k = \mathbb{C}^n \setminus B_k^\circ$ for every $k \in \mathbb{N}$, it follows that $\overline{A} = X \setminus B^\circ$. Similarly, the family $(\Gamma_k)_{k \in \mathbb{N}}$ determines a countable set $\Gamma \subset X \setminus B$ such that $\overline{\Gamma} = X \setminus B^\circ$.

We now show that B is the biggest regular compact set in X with the stable hull property. Note that condition (4_k), together with the fact that each set $C_{k,i}^l$ contains one of the sets $C_{k+1,i'}^{l'}$ in the next generation according to condition (3_{k+1}) (and hence it contains one of the sets $C_{k+j,i'}^{l'}$ for every $j = 1, 2, \ldots$), implies

$$\widehat{(C_{k,i}^l)}_{\mathscr{O}(X_{k+j+1})} \setminus X_{k+j} \neq \varnothing \quad \text{for all } j = 0, 1, 2, \ldots. \tag{4-3}$$

Thus, none of the sets $C^l_{k,i}$ has the stable hull property. Our construction ensures that the centers of these sets form a dense sequence in $X \setminus B$, consisting of all points in the set A determined by the family $(A_k)_{k\in\mathbb{N}}$, in which every point appears infinitely often. Furthermore, condition (3_k) shows that every compact set $K \subset X$ with $K^\circ \setminus B \neq \varnothing$ contains one (in fact, infinitely many) of the sets $C^l_{k,i}$. In view of (4-3), it follows that there is an integer $k_0 \in \mathbb{N}$ (depending on K) such that

$$\widehat{K}_{\mathscr{O}(X_{k+1})} \not\subset X_k \quad \text{for all } k \geq k_0.$$

This means that K does not have the stable hull property. It follows that the set B is the strongly stable core of X.

Finally, condition (5_k) ensures that the $\mathscr{O}(X)$-hull of a compact ball centered at the point $a^1_1 \in A$ contains the countable set $\Gamma \subset X$ determined by the family $\{\Gamma_k\}_{k\in\mathbb{N}}$. Since Γ is dense in $X \setminus B$, it follows that the manifold X does not admit any nonconstant plurisubharmonic function. (See the proof of Theorem 1.1 for the details.)

This proves part (a) of Theorem 1.6. □

Proof of Theorem 1.6(b). Let $U \subset \mathbb{C}^n$ be an open set. Pick a regular compact polynomially convex set B contained in U. We modify the recursion in the proof of part (a) by adding to B a new small closed ball $B' \subset U \setminus B$ at every stage. In this way, we inductively build an increasing sequence $B = B^1 \subset B^2 \subset \cdots \subset U$ of compact polynomially convex sets whose union $\mathcal{B} := \bigcup_{k=1}^\infty B^k \subset U$ is everywhere dense in U, and a sequence of Fatou–Bieberbach maps $\phi_k : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that, writing

$$B^k_1 = B^k \quad \text{and} \quad B^k_{j+1} = \phi_j(B^k_j) \text{ for all } j, k \in \mathbb{N},$$

the following two conditions hold:

(i) $B^k = B^{k-1} \cup \mathcal{B}^k$ for all $k > 1$, where $\mathcal{B}^k$ is a small closed ball in $U \setminus B^{k-1}$;

(ii) the set B^k_j is polynomially convex for all $j, k \in \mathbb{N}$.

At the k-th stage of the construction we have already chosen Fatou–Bieberbach maps $\phi_1, \ldots, \phi_k$, but we can nevertheless achieve condition (ii) for all $j = 1, \ldots, k+1$ by choosing the ball $\mathcal{B}^k$ sufficiently small. Indeed, the image of a small ball by an injective holomorphic map is a small strongly convex domain, and hence the polynomial convexity of the set B^k_j for $j = 1, \ldots, k+1$ follows from Lemma 2.1. For values $j > k+1$, (ii) is achieved by the construction in the proof of Lemma 4.3; indeed, each of the subsequent maps $\phi_{k+1}, \phi_{k+2}, \ldots$ in the sequence preserves polynomial convexity of B^k_{k+1}.

By identifying the sets U and $B^k = B^k_1$ (considered as subsets of $\mathbb{C}^n = X_1$) with their images in the limit manifold $X = \bigcup_{k=1}^\infty X_k$, we thus obtain the following analogue of (4-2):

$$\widehat{(B^k)}_{\mathscr{O}(X_j)} = B^k \quad \text{for all } j, k \in \mathbb{N}.$$

This means that each set B^k ($k \in \mathbb{N}$) lies in the stable core $\mathrm{SC}(X)$. Since $\bigcup_{k=1}^\infty B^k$ is dense in U by the construction, we have that $\overline{U} \subset \overline{\mathrm{SC}(X)}$.

On the other hand, writing $U_1 = U$ and $U_{k+1} := \phi_k \circ \cdots \circ \phi_1(U)$ for $k = 1, 2, \ldots$, the balls $\mathbb{B}(b^l_{k,i}, r_k)$ chosen at the k-th stage of the construction (see the proof of part (a)) are contained in $\mathbb{C}^n \setminus \overline{U}$ and, as

k increases, they include more and more points from a countable dense set $A \subset X \setminus \overline{U}$, which is built inductively as in the proof of part (a). By performing the Wold process on each of the balls $\mathbb{B}(b^l_{k,i}, r_k)$ (see condition (4_k) above) at every stage, we can ensure that none of the points of A belongs to the stable core $\mathrm{SC}(X)$. Since $\mathrm{SC}(X)$ is an open set by the definition and $\overline{A} = X \setminus U$, we conclude that $\mathrm{SC}(X) \subset U$. We have seen above that $\overline{U} \subset \overline{\mathrm{SC}(X)}$, and hence $\overline{\mathrm{SC}(X)} = \overline{U}$.

It remains to show that X can be chosen such that it does not admit any nonconstant holomorphic function. By the same argument as in the proof of part (a), we can find a countable dense set $\Gamma \subset X \setminus (A \cup \overline{U})$ which is dense in $X \setminus U$ and is contained in the $\mathscr{O}(X)$-hull of a certain compact set in $X \setminus \overline{U}$. It follows that every plurisubharmonic function f on X is bounded above on Γ, and hence on $\overline{\Gamma} = X \setminus U$. If $\overline{U}$ is compact, the maximum principle implies that f is also bounded on U; hence it is bounded on X and therefore constant. If U is not relatively compact then we are unable to make this conclusion. However, we can easily ensure that $X \setminus \overline{U}$ contains a Fatou–Bieberbach domain; indeed, it suffices to choose the first Fatou–Bieberbach map $\phi_1 : \mathbb{C}^n \to \mathbb{C}^n$ in the sequence determining X such that $\mathbb{C}^n \setminus \phi_1(\mathbb{C}^n)$ contains a Fatou–Bieberbach domain Ω. In this case, every holomorphic function $f \in \mathscr{O}(X)$ is bounded on Γ, and hence on Ω, so it is constant on $\Omega \cong \mathbb{C}^n$. Therefore it is constant on X by the identity principle.

This proves part (b) and hence completes the proof of Theorem 1.6. □

5. An exhaustion of $\mathbb{C}^2$ by non-Runge Fatou–Bieberbach domains

In this section, we show the following result (see also [Boc Thaler 2016, Section 4.4]).

Proposition 5.1. *Let $n > 1$. There exists an increasing sequence $X_1 \subset X_2 \subset \cdots \subset \bigcup_{k=1}^{\infty} X_k = \mathbb{C}^n$ of Fatou–Bieberbach domains in $\mathbb{C}^n$ which are not Runge in $\mathbb{C}^n$.*

We shall construct such an example by ensuring that all Fatou–Bieberbach maps $\phi_k : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ in the sequence (see Section 3) have non-Runge images, but they approximate the identity map on increasingly large balls centered at the origin. For this purpose, we shall need the following lemma.

Lemma 5.2. *Let B and $\mathcal{B}$ be a pair of closed disjoint balls in $\mathbb{C}^n$ $(n > 1)$. For every $\epsilon > 0$ there exists a Fatou–Bieberbach map $\phi : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ satisfying the following conditions:*

(a) $\|\phi|_B - \mathrm{Id}\| < \varepsilon$;

(b) $\|\phi^{-1}|_B - \mathrm{Id}\| < \varepsilon$;

(c) *$\phi(\mathcal{B})$ is not polynomially convex.*

Proof. Pick a slightly bigger ball B' containing B in the interior such that $B' \cap \mathcal{B} = \varnothing$. By an affine linear change of coordinates, we may assume that $B' \subset \mathbb{C}^* \times \mathbb{C}^{n-1}$. Choose a Fatou–Bieberbach map $\theta : \mathbb{C}^n \hookrightarrow \mathbb{C}^* \times \mathbb{C}^{n-1}$ such that $\theta|_{B'}$ is close to the identity. (See the proof of Lemma 4.3.) Theorem 2.3 provides a $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^{n-1})$ which approximates the identity map on $\theta(B')$ and such that $\psi(\theta(\mathcal{B}))$ is not polynomially convex (in fact, its polynomial hull intersects the hyperplane $\{0\} \times \mathbb{C}^{n-1}$). The composition $\phi = \psi \circ \theta : \mathbb{C}^n \hookrightarrow \mathbb{C}^* \times \mathbb{C}^{n-1}$ then satisfies condition (a) on B', and condition (c). If ϕ is sufficiently close to the identity on B', then it also satisfies condition (b) since $B \subset B'^{\circ}$. □

Proof of Proposition 5.1. Let $B_k = \mathbb{B}(0, k) \subset \mathbb{C}^n$ denote the closed ball of radius k centered at the origin. Choose an integer $n_1 \in \mathbb{N}$ and a small ball $\mathcal{B}^1$ disjoint from B_{n_1}. Let $\phi_1 : \mathbb{C}^n \to \mathbb{C}^n$ be a Fatou–Bieberbach map satisfying the following conditions:

(1) $\|\phi_1 - \mathrm{Id}\| < \varepsilon_1$ on B_{n_1};

(2) $\|\phi_1^{-1} - \mathrm{Id}\| < \varepsilon_1$ on B_{n_1};

(3) $\phi(\mathcal{B}^1)$ is not polynomially convex.

Suppose inductively that for some $k \in \mathbb{N}$ we have already found Fatou–Bieberbach maps $\phi_1, \ldots, \phi_k$, integers $n_1 < n_2 < \cdots < n_k$, and balls $\mathcal{B}^j \subset \mathbb{C}^n \setminus B_{n_j}$ for $j = 1, \ldots, k$ such that the following conditions hold:

(1_k) $\|\phi_k - \mathrm{Id}\| < \varepsilon_k$ on B_{n_k};

(2_k) $\|\phi_k^{-1} - \mathrm{Id}\| < \varepsilon_k$ on B_{n_k};

(3_k) $\phi_k(\mathcal{B}^k)$ is not polynomially convex.

Choose an integer $n_{k+1} > n_k$ such that

$$\phi_k(B_{n_k+1}) \cup \phi_k(\phi_{k-1}(B_{n_{k-1}+2})) \cup \cdots \cup \phi_k\big(\cdots(\phi_1(B_{n_1+k}))\big) \cup \phi_k(\mathcal{B}^k) \subset B_{n_{k+1}},$$

and pick a ball $\mathcal{B}^{k+1} \subset \mathbb{C}^n \setminus B_{n_{k+1}}$. By Lemma 5.2, there exists a Fatou–Bieberbach map $\phi_{k+1} : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ satisfying the following conditions:

(1_{k+1}) $\|\phi_{k+1} - \mathrm{Id}\| < \varepsilon_{k+1}$ on $B_{n_{k+1}}$;

(2_{k+1}) $\|\phi_{k+1}^{-1} - \mathrm{Id}\| < \varepsilon_{k+1}$ on $B_{n_{k+1}}$;

(3_{k+1}) $\phi_{k+1}(\mathcal{B}^{k+1})$ is not polynomially convex.

This closes the induction step.

Let $X = \bigcup_{k=1}^{\infty} X_k$ be the long $\mathbb{C}^n$ determined by the sequence $(\phi_k)_k$, let $\iota_k : X_k \hookrightarrow X_{k+1}$ denote the inclusion map, and let $\psi_k : \mathbb{C}^n \to X_k \subset X$ denote the biholomorphic map from $\mathbb{C}^n$ onto the k-th element of the exhaustion such that

$$\iota_k \circ \psi_k = \psi_{k+1} \circ \phi_k, \quad k = 1, 2, \ldots.$$

(See Section 3, in particular (3-1).) By the construction, the sequence $\psi_k(B_{n_k})$ is a Runge exhaustion of X. If the sequence $\varepsilon_k > 0$ has been chosen to be summable, then the sequence ψ_k converges on every ball B_{n_j} and the limit map $\Psi = \lim_{k\to\infty} \psi_k : \mathbb{C}^n \to X$ is a biholomorphism (see [Forstnerič 2011, Corollary 4.4.2, p. 115]). In the terminology of Dixon and Esterle [1986, Theorem 5.2], we have that

$$(\psi_k, B_{n_k}) \to (\Psi, \mathbb{C}^n) \quad \text{as } k \to \infty,$$

where $\Psi(\mathbb{C}^n) = X$ and Ψ is biholomorphic. □

Remark 5.3. If we only assume that the images of Fatou–Bieberbach maps $\phi_k : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ contain large enough balls centered at the origin, we get an exhaustion of a long $\mathbb{C}^n$ with Runge images of balls. By [Arosio et al. 2013, Theorem 3.4], such a long $\mathbb{C}^n$ is biholomorphic to a Stein Runge domain in $\mathbb{C}^n$. Therefore, the following problem is closely related to problem (C) stated in the introduction.

(C') If a long $\mathbb{C}^n$ is exhausted by Runge images of balls, is it necessarily biholomorphic to $\mathbb{C}^n$?

In this connection, we mention that the first author proved in his thesis [Boc Thaler 2016, Theorem IV.15, p. 62] that $\mathbb{C}^n$ is the only Stein manifold with the density property (see Definition 2.2) having an exhaustion by Runge images of balls.

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