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EXACT CONTROLLABILITY FOR QUASILINEAR PERTURBATIONS OF KDV

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We prove that the KdV equation on the circle remains exactly controllable in arbitrary time with localized control, for sufficiently small data, also in the presence of quasilinear perturbations, namely nonlinearities containing up to three space derivatives, having a Hamiltonian structure at the highest orders. We use a procedure of reduction to constant coefficients up to order zero (adapting a result of Baldi, Berti and Montalto (2014)), the classical Ingham inequality and the Hilbert uniqueness method to prove the controllability of the linearized operator. Then we prove and apply a modified version of the Nash–Moser implicit function theorems by Hörmander (1976, 1985).

1. Introduction

A question in control theory for PDEs regards the persistence of controllability under perturbations. In this paper we study the effect of *quasilinear* perturbations (namely nonlinearities containing derivatives of the highest order) on the controllability of the KdV equation. We consider equations of the form

$$u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0 \tag{1-1}$$

on the circle $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, with $t \in \mathbb{R}$, where $u = u(t, x)$ is real-valued, and $\mathcal{N}$ is a given real-valued nonlinear function which is at least quadratic around $u = 0$. For solutions of small amplitude, (1-1) is a quasilinear perturbation of the Airy equation $u_t + u_{xxx} = 0$, which is the linear part of KdV; then the KdV nonlinear term uu_x can be included in $\mathcal{N}$.

Motivated by a question, which was posed in [Kappeler and Pöschel 2003], about the possibility of including the dependence on higher derivatives in nonlinear perturbations of KdV, equations of the form

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(1-1) have recently been studied in [Baldi, Berti, and Montalto 2014; 2016a; 2016b] in the context of KAM theory. In this paper we study (1-1) from the point of view of control theory, proving its exact controllability by means of an internal control, in arbitrary time, for sufficiently small data (Theorem 1.1).

Most of the known results about controllability of quasilinear PDEs deal with first-order quasilinear hyperbolic systems of the form $u_t + A(u)u_x = 0$ (including quasilinear wave, shallow water, and Euler equations); see, for example, [Li and Zhang 1998; Coron 2007, Chapter 6.2; Li and Rao 2003; Coron, Glass, and Wang 2010; Alabau-Boussouira, Coron and Olive 2015]. Recent results for different kinds of quasilinear PDEs are contained in [Alazard, Baldi, and Han-Kwan 2015] about the internal controllability of 2-dimensional gravity-capillary water waves equations, and in [Alazard 2015] about the boundary observability of 2- and 3-dimensional (fully nonlinear) gravity water waves. For a general introduction to the theory of control for PDEs, see, for example, [Lions 1988; Micu and Zuazua 2005; Coron 2007], while for important results in control for hyperbolic PDEs, see, for example, [Bardos, Lebeau, and Rauch 1992; Burq and Gérard 1997; Burq and Zworski 2012].

Regarding the KdV equation, the first controllability results are due to Zhang [1990] and Russell [1991]. Among recent results, we mention the work by Laurent, Rosier and Zhang [2010] for large data. A beautiful review on the literature on control for KdV can be found in [Rosier and Zhang 2009]. For more on KdV, see the rich survey [Guan and Kuksin 2014], and the many references therein.

1A. ***Main result.*** We assume that the nonlinearity $\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx})$ is at least quadratic around $u = 0$; namely the real-valued function $\mathcal{N} : \mathbb{T} \times \mathbb{R}^4 \to \mathbb{R}$ satisfies

$$|\mathcal{N}(x, z_0, z_1, z_2, z_3)| \le C|z|^2 \quad \forall z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4,\ |z| \le 1. \tag{1-2}$$

We assume that the dependence of $\mathcal{N}$ on u_{xx}, u_{xxx} is Hamiltonian, while no structure is required on its dependence on u, u_x. More precisely, we assume that

$$\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = \mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) + \mathcal{N}_0(x, u, u_x), \tag{1-3}$$

where

$$\mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) = \partial_x\{(\partial_u \mathcal{F})(x, u, u_x)\} - \partial_{xx}\{(\partial_{u_x} \mathcal{F})(x, u, u_x)\} \quad \text{for some function } \mathcal{F} : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}. \tag{1-4}$$

Note that the case $\mathcal{N}=\mathcal{N}_1$, $\mathcal{N}_0=0$ corresponds to the Hamiltonian equation $\partial_t u = \partial_x \nabla H(u)$, where the Hamiltonian is

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} u_x^2\, dx + \int_{\mathbb{T}} \mathcal{F}(x, u, u_x)\, dx \tag{1-5}$$

and ∇ denotes the $L^2(\mathbb{T})$-gradient. The unperturbed KdV is the case $\mathcal{F} = -\frac{1}{6}u^3$.

Notation. For periodic functions $u(x)$, $x \in \mathbb{T}$, we expand $u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx}$, and, for $s \in \mathbb{R}$, we consider the standard Sobolev space of periodic functions

$$H_x^s := H^s(\mathbb{T}, \mathbb{R}) := \{u : \mathbb{T} \to \mathbb{R} : \|u\|_s < \infty\}, \quad \|u\|_s^2 := \sum_{n\in\mathbb{Z}} |u_n|^2 \langle n\rangle^{2s}, \tag{1-6}$$

where $\langle n\rangle := (1+n^2)^{1/2}$. We consider the space $C([0,T], H^s_x)$ of functions $u(t,x)$ that are continuous in time with values in H^s_x. We will use the following notation for the standard norm in $C([0,T], H^s_x)$:

$$\|u\|_{T,s} := \|u\|_{C([0,T],H^s_x)} := \sup_{t\in[0,T]} \|u(t)\|_s. \tag{1-7}$$

For continuous functions $a : [0,T] \to \mathbb{R}$, we will define

$$|a|_T := \sup\{|a(t)| : t \in [0,T]\}. \tag{1-8}$$

Theorem 1.1 (exact controllability). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be a nonempty open set. There exist positive universal constants r, s_1 such that, if $\mathcal{N}$ in (1-1) is of class C^r in its arguments and satisfies (1-2), (1-3), (1-4), then there exists a positive constant δ_* depending on T, ω, $\mathcal{N}$ with the following property.*

Let $u_{\text{in}}, u_{\text{end}} \in H^{s_1}(\mathbb{T}, \mathbb{R})$ with

$$\|u_{\text{in}}\|_{s_1} + \|u_{\text{end}}\|_{s_1} \le \delta_*.$$

Then there exists a function $f(t,x)$ satisfying

$$f(t,x) = 0 \quad \text{for all } x \notin \omega, \quad \text{for all } t \in [0,T],$$

belonging to $C([0,T], H^s_x) \cap C^1([0,T], H^{s-3}_x) \cap C^2([0,T], H^{s-6}_x)$ for all $s < s_1$, such that the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = f & \forall (t,x) \in [0,T] \times \mathbb{T}, \\ u(0,x) = u_{\text{in}}(x) \end{cases} \tag{1-9}$$

has a unique solution $u(t,x)$ belonging to $C([0,T], H^s_x) \cap C^1([0,T], H^{s-3}_x) \cap C^2([0,T], H^{s-6}_x)$ for all $s < s_1$ which satisfies

$$u(T,x) = u_{\text{end}}(x). \tag{1-10}$$

Moreover, for all $s < s_1$,

$$\|u, f\|_{C([0,T],H^s_x)} + \|\partial_t u, \partial_t f\|_{C([0,T],H^{s-3}_x)} + \|\partial_{tt} u, \partial_{tt} f\|_{C([0,T],H^{s-6}_x)} \le C_s(\|u_{\text{in}}\|_{s_1} + \|u_{\text{end}}\|_{s_1}) \tag{1-11}$$

for some $C_s > 0$ depending on s, T, ω, $\mathcal{N}$.

Remark 1.2. In Theorem 1.1 there is an arbitrarily small loss of regularity: if the initial and final data $u_{\text{in}}, u_{\text{end}}$ have Sobolev regularity $H^{s_1}_x$, then the control f and the solution u are continuous in time with values in H^s_x for all $s < s_1$. Such loss of regularity is in some sense fictitious: it is due to our choice of working with standard Sobolev spaces, but it could be avoided by working with the (slightly "worse-looking") weak spaces E'_a introduced by Hörmander [1985] (see Appendix B). What we actually prove is that, if the initial and final data are in the weak space $(H^{s_1}_x)'$ (i.e., the weak version à la Hörmander [1985] of the Sobolev space $H^{s_1}_x$), then f and u are continuous in time with values in *the same* space $(H^{s_1}_x)'$. □

Remark 1.3. Our proof of Theorem 1.1 does not use results of existence and uniqueness for the Cauchy problem (1-9). On the contrary, our method directly proves local existence and uniqueness for (1-9) (see Theorem 1.4). This situation occurs quite often in control problems (see Remark 4.12 of [Coron 2007]). □

1B. ***Description of the proof.*** It would be natural to try to solve the control problem (1-9)–(1-10) using a fixed point argument or the usual implicit function theorem. However, this seems to be impossible because of the presence of three derivatives in the nonlinear term. A similar difficulty was overcome in [Alazard, Baldi, and Han-Kwan 2015] by using a suitable nonlinear iteration scheme adapted to quasilinear problems. Such a nonlinear scheme requires solving a linear control problem with variable coefficients at each step of the iteration, with no loss of regularity with respect to the coefficients (i.e., the solution must have the same regularity as the coefficients). In [Alazard, Baldi, and Han-Kwan 2015] this is achieved by means of paradifferential calculus, together with linear transformations, Ingham-type inequalities and the Hilbert uniqueness method.

As an alternative method, in this paper we use a Nash–Moser implicit function theorem. The Nash–Moser approach also demands the solving of a linear control problem with variable coefficients, but it has the advantage of requiring weaker estimates, allowing losses of regularity. The proof of such weaker estimates is easier to obtain, and it does not require the use of powerful techniques like paradifferential calculus. In this sense our Nash–Moser method is alternative to the method in [Alazard, Baldi, and Han-Kwan 2015] (for a discussion about pseudo- and paradifferential calculus in connection with the Nash–Moser theorem, see, for example, [Hörmander 1990; Alinhac and Gérard 2007]). On the other hand, the result that we obtain with the Nash–Moser method is slightly weaker than the one in [Alazard, Baldi, and Han-Kwan 2015] regarding the regularity of the solution of the nonlinear control problem with respect to the regularity of the data: the arbitrarily small loss of regularity in Theorem 1.1 is discussed in Remark 1.2, while Theorem 1.1 of [Alazard, Baldi, and Han-Kwan 2015] has no loss of regularity also in the standard Sobolev spaces.

Nash–Moser schemes in control problems for PDEs have been used in [Beauchard 2005; 2008; Beauchard and Coron 2006; Alabau-Boussouira, Coron and Olive 2015]. A discussion about Nash–Moser as a method to overcome the problem of the loss of derivatives in the context of controllability for PDEs can be found in [Coron 2007, Section 4.2.2]. Beauchard and Laurent [2010] were able to avoid the use of the Nash–Moser theorem in semilinear control problems thanks to a regularizing effect. We remark that Theorem 1.1 could also be proved without Nash–Moser (for example, by adapting the method of [Alazard, Baldi, and Han-Kwan 2015]).

Now we describe our method in more detail. Given a nonempty open set $\omega \subset \mathbb{T}$, we first fix a C^∞ function $\chi_\omega(x)$ with values in the interval $[0, 1]$ which vanishes outside ω, and takes value $\chi_\omega = 1$ on a nonempty open subset of ω. Thus, given initial and final data $u_{\text{in}}, u_{\text{end}}$, we look for u, f that solve

$$\begin{cases} P(u) = \chi_\omega f, \\ u(0) = u_{\text{in}}, \\ u(T) = u_{\text{end}}, \end{cases} \tag{1-12}$$

where

$$P(u) := u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}). \tag{1-13}$$

We define

$$\Phi(u, f) := \begin{pmatrix} P(u) - \chi_\omega f \\ u(0) \\ u(T) \end{pmatrix} \tag{1-14}$$

so that problem (1-12) is written as

$$\Phi(u, f) = (0, u_{\text{in}}, u_{\text{end}}).$$

The crucial assumption to verify in order to apply any Nash–Moser theorem is the existence of a right inverse of the linearized operator. The linearized operator $\Phi'(u, f)[h, \varphi]$ at the point (u, f) in the direction (h, φ) is

$$\Phi'(u, f)[h, \varphi] := \begin{pmatrix} P'(u)[h] - \chi_\omega \varphi \\ h(0) \\ h(T) \end{pmatrix}. \tag{1-15}$$

Thus we have to prove that, given any (u, f) and any $g := (g_1, g_2, g_3)$ in suitable function spaces, there exists (h, φ) such that

$$\Phi'(u, f)[h, \varphi] = g. \tag{1-16}$$

Moreover we have to estimate (h, φ) in terms of u, f, g in a "tame" way (an estimate is said to be tame when it is linear in the highest norms; see (B-13) and (4-41)).

Problem (1-16) is a linear control problem. We observe that the linearized operator $P'(u)[h]$ is a differential operator having variable coefficients also at the highest order (which is a consequence of linearizing a *quasilinear* PDE). Explicitly, it has the form

$$P'(u)[h] = \partial_t h + (1 + a_3(t, x))\,\partial_{xxx} h + a_2(t, x)\,\partial_{xx} h + a_1(t, x)\,\partial_x h + a_0(t, x) h.$$

We solve (1-16) in Theorem 4.5. Note that the choice of the function spaces is not given a priori: to fix a suitable functional setting is part of the problem.

Theorem 4.5 is proved by adapting a procedure of reduction to constant coefficients developed in [Baldi, Berti, and Montalto 2014; 2016a]. Such a procedure conjugates $P'(u)$ to an operator $\mathcal{L}_5$ (see (2-57)) having constant coefficients up to a bounded remainder. This conjugation is achieved by means of changes of the space variable, reparametrization of time, multiplication operators, and Fourier multipliers. Using the Ingham inequality and a perturbation argument we prove the observability of $\mathcal{L}_5$. Then we prove the observability of $P'(u)$, exploiting the explicit formulas of the transformations that conjugate $P'(u)$ to $\mathcal{L}_5$. The linear control problem (1-16) is solved in L^2_x by the HUM (Hilbert uniqueness method). Then further regularity of the solution (h, φ) of (1-16) is proved by adapting an argument used by Dehman and Lebeau [2009], Laurent [2010], and Alazard, Baldi, and Han-Kwan [2015].

To conclude the proof of Theorem 1.1 we apply Theorem B.1, which is a modified version of two Nash–Moser implicit function theorems (Theorem 2.2.2 in [Hörmander 1976] and the main theorem in [Hörmander 1985]; see also [Alinhac and Gérard 2007]). With respect to the abstract theorem in [Hörmander 1985], our Theorem B.1 assumes slightly stronger hypotheses on the nonlinear operator, and it removes two conditions that are assumed in [Hörmander 1985], which are the compact embeddings in the codomain scale of Banach spaces and the continuity of the approximate right inverse of the linearized operator with respect to the approximate linearization point. This improvement is obtained by adapting the iteration scheme introduced in [Hörmander 1976]. On the other hand, the Nash–Moser implicit function theorem in that work holds for Hölder spaces with noninteger indices, and it does not apply to Sobolev spaces (in particular, Theorem A.11 of [Hörmander 1976] does not hold for Sobolev spaces).

This method is not confined to KdV, and it could be applied to prove controllability of other quasilinear evolution PDEs.

The use of Ingham-type inequalities and the HUM is classical in control theory (see, for example, [Haraux 1989; Micu and Zuazua 2005; Komornik and Loreti 2005; Kahane 1962] for Ingham-type inequalities and [Lions 1988; Micu and Zuazua 2005; Coron 2007; Komornik 1994] for the HUM). As mentioned above, the Nash–Moser theorem has also been used in control theory (see, for example, [Beauchard 2005; 2008; Beauchard and Coron 2006; Alabau-Boussouira, Coron and Olive 2015]). It was first introduced by Nash [1956], and then several refinements were developed afterwards; see, for example, [Moser 1961; Zehnder 1975; 1976; Hamilton 1982; Gromov 1972; Hörmander 1976; 1985; 1990; Berti, Bolle, and Procesi 2010; Berti, Corsi, and Procesi 2015; Ekeland 2011; Ekeland and Séré 2015]. For our problem, Hörmander's versions [1976; 1985] seem to be the best ones concerning the loss of regularity of the solution with respect to the regularity of the data (see also Remark 1.2). As already said, the theorems in [Hörmander 1976; 1985] cannot be applied directly, but they can be adapted to our goal. This is the content of Appendix B.

1C. ***Byproduct: a local existence and uniqueness result.*** As a byproduct, with the same technique and no extra work, we have the following existence and uniqueness theorem for the Cauchy problem of the quasilinear PDE (1-1).

Theorem 1.4 (local existence and uniqueness). *There exist positive universal constants r, s_0 such that, if $\mathcal{N}$ in (1-1) is of class C^r in its arguments and satisfies (1-2), (1-3), (1-4), then the following property holds. For all $T > 0$ there exists $\delta_* > 0$ such that for all $u_{\text{in}} \in H_x^{s_0}$ and $f \in C([0,T], H_x^{s_0}) \cap C^1([0,T], H_x^{s_0-6})$ (possibly $f = 0$) satisfying*

$$\|u_{\text{in}}\|_{s_0} + \|f\|_{T,s_0} + \|\partial_t f\|_{T,s_0-6} \le \delta_*, \tag{1-17}$$

the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x,u,u_x,u_{xx},u_{xxx}) = f, & (t,x) \in [0,T] \times \mathbb{T}, \\ u(0,x) = u_{\text{in}}(x) \end{cases} \tag{1-18}$$

has one and only one solution $u \in C([0,T], H_x^s) \cap C^1([0,T], H_x^{s-3}) \cap C^2([0,T], H_x^{s-6})$ for all $s < s_0$. Moreover, for all $s < s_0$,

$$\|u\|_{C([0,T],H_x^s)} + \|\partial_t u\|_{C([0,T],H_x^{s-3})} + \|\partial_{tt} u\|_{C([0,T],H_x^{s-6})} \\ \le C_s \big(\|u_{\text{in}}\|_{s_0} + \|f\|_{C([0,T],H_x^{s_0})} + \|\partial_t f\|_{C([0,T],H_x^{s_0-6})} \big) \tag{1-19}$$

for some $C_s > 0$ depending on s, T, $\mathcal{N}$.

Remark 1.5. Theorem 1.4 is not sharp: we expect that better results for the Cauchy problem (1-18) can be proved by using a paradifferential approach. □

Remark 1.6. The loss of regularity in Theorem 1.4 is of the same type as the one in Theorem 1.1; see the discussion in Remark 1.2. □

1D. *Organization of the paper.* In Section 2 we describe the transformations that conjugate the linearized operator $P'(u)$ to constant coefficients up to a bounded remainder, and we give quantitative estimates on these transformations. In Section 3 we exploit these results to prove the observability of $P'(u)$. In Section 4 we use observability to solve the linear control problem (1-16) via the HUM (Theorem 4.5) and we fix suitable function spaces (4-36)–(4-37). In Section 5 we prove Theorems 1.1 and 1.4 by applying Theorem B.1. In Appendix A we prove well-posedness with tame estimates for all the linear operators involved in the reduction procedure. These well-posedness results are used many times in Sections 3, 4, and 5. In Appendix B we prove our Nash–Moser theorem, Theorem B.1. In Appendix C we recall standard tame estimates that are used in the rest of the paper.

2. Reduction of the linearized operator to constant coefficients

In this section we consider some changes of variables that conjugate the linearized operator to constant coefficients up to a bounded remainder. This reduction procedure closely follows the analysis in [Baldi, Berti, and Montalto 2014; 2016a], with some adaptations.

The linearized operator $P'(u)$ is

$$P'(u)[h] = \partial_t h + (1+a_3)\,\partial_{xxx} h + a_2\,\partial_{xx} h + a_1\,\partial_x h + a_0 h, \tag{2-1}$$

where the coefficients $a_i = a_i(t,x)$, $i = 0,\dots,3$, are real-valued functions of $(t,x) \in [0,T]\times\mathbb{T}$, depending on u by

$$a_i = a_i(u) := (\partial_{z_i}\mathcal{N})(x, u, u_x, u_{xx}, u_{xxx}), \quad i = 0,\dots,3 \tag{2-2}$$

(recall the notation $\mathcal{N} = \mathcal{N}(x, z_0, z_1, z_2, z_3)$). Note that $a_2 = 2\partial_x a_3$ because of the Hamiltonian structure of the component $\mathcal{N}_1$ of the nonlinearity; see (1-3)–(1-4).

Lemma 2.1. *Let $\mathcal{N} \in C^r(\mathbb{T}\times\mathbb{R}^4, \mathbb{R})$ satisfy (1-2). For all $1 \le s \le r-3$, and for all $u \in C^2([0,T], H_x^{s+3})$ such that $\|u, \partial_t u, \partial_{tt} u\|_{T,4} \le 1$, the coefficients $a_i(u)$ satisfy*

$$\|a_i(u), \partial_t a_i(u), \partial_{tt} a_i(u)\|_{T,s} \le C\|u, \partial_t u, \partial_{tt} u\|_{T,s+3}, \qquad i = 0,1,2,3. \tag{2-3}$$

Proof. Apply standard tame estimates for composition of functions; see Lemma C.2. □

Now we apply the reduction procedure to any linear operator of the form (2-1) where

$$a_2(t,x) = c\,\partial_x a_3(t,x) \tag{2-4}$$

for some constant $c \in \mathbb{R}$ (note that $P'(u)$ has $c = 2$ because of the Hamiltonian structure of $\mathcal{N}_1$). Regarding the loss of regularity with respect to the space variable x, the estimates in the sequel will be not sharp. In the whole section we consider $T > 0$ fixed, and, unless otherwise specified, all the constants may depend on T.

Remark 2.2. Given a linear operator $\mathcal{L}_0$ of the form (2-1), define the operator $\mathcal{L}_0^*$ as

$$\mathcal{L}_0^* h := -\partial_t h - \partial_{xxx}\{(1+a_3)h\} + \partial_{xx}(a_2 h) - \partial_x(a_1 h) + a_0 h. \tag{2-5}$$

Note that $-\mathcal{L}_0^*$ is still an operator of the form (2-1), namely

$$-\mathcal{L}_0^* = \partial_t + (1 + a_3^*)\,\partial_{xxx} + a_2^*\,\partial_{xx} + a_1^*\,\partial_x + a_0^*, \tag{2-6}$$

with

$$\begin{aligned} a_3^* &:= a_3, & a_2^* &:= 3(a_3)_x - a_2, \\ a_1^* &:= 3(a_3)_{xx} - 2(a_2)_x + a_1, & a_0^* &:= (a_3)_{xxx} - (a_2)_{xx} + (a_1)_x - a_0. \end{aligned} \tag{2-7}$$

It follows from (2-6), (2-7) that if $\mathcal{L}_0$ satisfies (2-4), then also $-\mathcal{L}_0^*$ satisfies (2-4) (with a different constant), namely $a_2^* = (3-c)\,\partial_x a_3^*$. In particular, if $\mathcal{L}_0$ satisfies (2-4) with $c = 2$ (which is the case if $\mathcal{L}_0 = P'(u)$), then $-\mathcal{L}_0^*$ satisfies (2-4) with $c = 1$. □

2A. ***Step 1: change of the space variable.*** We consider a t-dependent family of diffeomorphisms of the circle $\mathbb{T}$ of the form

$$y = x + \beta(t, x), \tag{2-8}$$

where β is a real-valued function, 2π periodic in x, defined for $t \in [0, T]$, with $|\beta_x(t, x)| \le \frac{1}{2}$ for all $(t, x) \in [0, T] \times \mathbb{T}$. We define the linear operator

$$(\mathcal{A}h)(t, x) := h(t, x + \beta(t, x)). \tag{2-9}$$

The operator $\mathcal{A}$ is invertible, with inverse $\mathcal{A}^{-1}$, transpose $\mathcal{A}^T$ (transpose with respect to the usual L^2_x-scalar product) and inverse transpose $\mathcal{A}^{-T}$ given by

$$\begin{aligned} (\mathcal{A}^{-1}v)(t, y) &= v(t, y + \tilde\beta(t, y)), \\ (\mathcal{A}^T v)(t, y) &= (1 + \tilde\beta_y(t, y))\, v(t, y + \tilde\beta(t, y)), \\ (\mathcal{A}^{-T} h)(t, x) &= (1 + \beta_x(t, x))\, h(t, x + \beta(t, x)), \end{aligned} \tag{2-10}$$

where $y \mapsto y + \tilde\beta(t, y)$ is the inverse diffeomorphism of (2-8), namely

$$x = y + \tilde\beta(t, y) \iff y = x + \beta(t, x). \tag{2-11}$$

Given the operator

$$\mathcal{L}_0 := \partial_t + (1 + a_3(t, x))\,\partial_{xxx} + a_2(t, x)\,\partial_{xx} + a_1(t, x)\,\partial_x + a_0(t, x), \tag{2-12}$$

with $a_2(t, x) = c\,\partial_x a_3(t, x)$, we calculate the conjugate $\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A}$. The conjugate $\mathcal{A}^{-1}a\mathcal{A}$ of any multiplication operator $a : h(t, x) \mapsto a(t, x)h(t, x)$ is the multiplication operator $(\mathcal{A}^{-1}a)$ that maps $v(t, y)$ to $(\mathcal{A}^{-1}a)(t, y)\, v(t, y)$. By conjugation, the differential operators become

$$\mathcal{A}^{-1}\,\partial_t\mathcal{A} = \partial_t + (\mathcal{A}^{-1}\beta_t)\,\partial_y, \quad \mathcal{A}^{-1}\,\partial_x\mathcal{A} = \{\mathcal{A}^{-1}(1 + \beta_x)\}\,\partial_y.$$

Then $\mathcal{A}^{-1}\,\partial_{xx}\mathcal{A} = (\mathcal{A}^{-1}\,\partial_x\mathcal{A})(\mathcal{A}^{-1}\,\partial_x\mathcal{A})$, and similarly for the conjugate of ∂_{xxx}. We calculate

$$\mathcal{L}_1 := \mathcal{A}^{-1}\mathcal{L}_0\mathcal{A} = \partial_t + a_4(t, y)\,\partial_{yyy} + a_5(t, y)\,\partial_{yy} + a_6(t, y)\,\partial_y + a_7(t, y), \tag{2-13}$$

where

$$\begin{aligned} a_4 &= \mathcal{A}^{-1}\{(1+a_3)(1+\beta_x)^3\}, & a_5 &= \mathcal{A}^{-1}\{a_2(1+\beta_x)^2 + 3(1+a_3)\beta_{xx}(1+\beta_x)\}, \\ a_6 &= \mathcal{A}^{-1}\{\beta_t + (1+a_3)\beta_{xxx} + a_2\beta_{xx} + a_1(1+\beta_x)\}, & a_7 &= \mathcal{A}^{-1}a_0. \end{aligned} \tag{2-14}$$

We look for $\beta(t,x)$ such that the coefficient $a_4(t,y)$ of the highest-order derivative ∂_{yyy} in (2-13) does not depend on y; namely $a_4(t,y)=b(t)$ for some function $b(t)$ of t only. This is equivalent to

$$(1+a_3(t,x))(1+\beta_x(t,x))^3=b(t); \tag{2-15}$$

namely

$$\beta_x=\rho_0, \quad \rho_0(t,x):=b(t)^{1/3}(1+a_3(t,x))^{-1/3}-1. \tag{2-16}$$

Equation (2-16) has a solution β, periodic in x, if and only if $\int_{\mathbb{T}}\rho_0(t,x)\,dx=0$ for all t. This condition uniquely determines

$$b(t)=\left(\frac{1}{2\pi}\int_{\mathbb{T}}(1+a_3(t,x))^{-1/3}\,dx\right)^{-3}. \tag{2-17}$$

Then we fix the solution (with zero average) of (2-16),

$$\beta(t,x):=(\partial_x^{-1}\rho_0)(t,x), \tag{2-18}$$

where $\partial_x^{-1}h$ is the primitive of h with zero average in x (defined in Fourier). We have conjugated $\mathcal{L}_0$ to

$$\mathcal{L}_1=\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A}=\partial_t+a_4(t)\,\partial_{yyy}+a_5(t,y)\,\partial_{yy}+a_6(t,y)\,\partial_y+a_7(t,y), \tag{2-19}$$

where $a_4(t):=b(t)$ is defined in (2-17).

We prove here some bounds that will be used later.

Lemma 2.3. *There exist positive constants σ, δ_* with the following properties. Let $s\geq 0$, and let $a_3(t,x), a_2(t,x), a_1(t,x), a_0(t,x)$ be four functions with $a_2=c\,\partial_x a_3$ for some $c\in\mathbb{R}$. Moreover, assume $\partial_{tt}a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0\in C([0,T],H_x^{s+\sigma})$. Let*

$$\delta(\mu):=\|\partial_{tt}a_3,\partial_t a_3,a_3,\partial_t a_1,a_1,a_0\|_{T,\mu+\sigma} \quad \forall\mu\in[0,s]. \tag{2-20}$$

If $\delta(0)\leq\delta_*$, then the operator $\mathcal{A}$ defined in (2-9), (2-18), (2-16), (2-17) belongs to $C([0,T],\mathcal{L}(H_x^\mu))$ for all $\mu\in[0,s]$ and satisfies

$$\|\mathcal{A}h\|_{T,\mu}\leq C_\mu\big(\|h\|_{T,\mu}+\delta(\mu)\|h\|_{T,0}\big) \quad \forall h\in C([0,T],H_x^\mu), \tag{2-21}$$

for some positive C_μ depending on μ. The inverse operator $\mathcal{A}^{-1}$, the transpose $\mathcal{A}^T$ and the inverse transpose $\mathcal{A}^{-T}$ all satisfy the same estimate (2-21) as $\mathcal{A}$.

The functions $a_4(t)=b(t)$, $a_5(t,y)$, $a_6(t,y)$, $a_7(t,y)$, $\beta(t,x)$, $\tilde\beta(t,y)$ defined in (2-17), (2-16), (2-18), (2-14), (2-11) belong to $C([0,T],H_x^\mu)$ for all $\mu\in[0,s]$ and satisfy

$$\|\beta,\tilde\beta,a_5,\partial_t a_5,a_6,\partial_t a_6,a_7\|_{T,\mu}+|a_4-1,a_4'|_T\leq C_\mu\delta(\mu). \tag{2-22}$$

Finally, the coefficient $a_5(t,y)$ satisfies

$$\int_{\mathbb{T}}a_5(t,y)\,dy=0 \quad \forall t\in[0,T]. \tag{2-23}$$

Proof. The proof of (2-21) and (2-22) is a straightforward application of the standard tame estimates for products, composition of functions and changes of variable; see Appendix C.

To prove (2-23), we use the definition of $b(t)$ in (2-17), the equality $a_2 = c\,\partial_x a_3$, and the change of variables (2-11), and we compute

$$\begin{aligned}\int_{\mathbb{T}} a_5(t,y)\,dy &= \int_{\mathbb{T}} \big[a_2(1+\beta_x)^2 + 3(1+a_3)\beta_{xx}(1+\beta_x)\big](1+\beta_x)\,dx \\ &= b(t)\Big\{c\int_{\mathbb{T}} \frac{\partial_x a_3(t,x)}{1+a_3(t,x)}\,dx + 3\int_{\mathbb{T}} \frac{\beta_{xx}(t,x)}{1+\beta_x(t,x)}\,dx\Big\} \\ &= b(t)\Big\{c\int_{\mathbb{T}} \partial_x \log(1+a_3(t,x))\,dx + 3\int_{\mathbb{T}} \partial_x \log(1+\beta_x(t,x))\,dx\Big\} = 0. \qquad \square\end{aligned}$$

2B. *Step 2: time reparametrization.* The goal of this section is to obtain a constant coefficient instead of $a_4(t)$. We consider a diffeomorphism $\psi : [0,T] \to [0,T]$ which gives the change of the time variable

$$\psi(t) = \tau \quad \Longleftrightarrow \quad t = \psi^{-1}(\tau), \tag{2-24}$$

with $\psi(0) = 0$ and $\psi(T) = T$. We define

$$(\mathcal{B}h)(t,y) := h(\psi(t),y), \quad (\mathcal{B}^{-1}v)(\tau,y) := v(\psi^{-1}(\tau),y). \tag{2-25}$$

By conjugation, the differential operators become

$$\mathcal{B}^{-1}\,\partial_t\mathcal{B} = \rho(\tau)\partial_\tau, \quad \mathcal{B}^{-1}\,\partial_y\mathcal{B} = \partial_y, \quad \rho := \mathcal{B}^{-1}(\psi'), \tag{2-26}$$

and therefore (2-19) is conjugated to

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\,\partial_\tau + (\mathcal{B}^{-1}a_4)\,\partial_{yyy} + (\mathcal{B}^{-1}a_5)\,\partial_{yy} + (\mathcal{B}^{-1}a_6)\,\partial_y + (\mathcal{B}^{-1}a_7). \tag{2-27}$$

We look for ψ such that the (variable) coefficients of the highest-order derivatives (∂_τ and ∂_{yyy}) are proportional; namely

$$(\mathcal{B}^{-1}a_4)(\tau) = m\rho(\tau) = m(\mathcal{B}^{-1}(\psi'))(\tau) \tag{2-28}$$

for some constant $m \in \mathbb{R}$. Since $\mathcal{B}$ is invertible, this is equivalent to requiring that

$$a_4(t) = m\psi'(t). \tag{2-29}$$

Integrating on $[0,T]$ determines the value of the constant m, and then we fix ψ:

$$m := \frac{1}{T}\int_0^T a_4(t)\,dt, \quad \psi(t) := \frac{1}{m}\int_0^t a_4(s)\,ds. \tag{2-30}$$

With this choice of ψ we get

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\,\mathcal{L}_2, \quad \mathcal{L}_2 := \partial_\tau + m\partial_{yyy} + a_8(\tau,y)\,\partial_{yy} + a_9(\tau,y)\,\partial_y + a_{10}(\tau,y), \tag{2-31}$$

where

$$a_8(\tau, y) := \frac{1}{\rho(\tau)} (\mathcal{B}^{-1} a_5)(\tau, y), \quad a_9(\tau, y) := \frac{1}{\rho(\tau)} (\mathcal{B}^{-1} a_6)(\tau, y), \quad a_{10}(\tau, y) := \frac{1}{\rho(\tau)} (\mathcal{B}^{-1} a_7)(\tau, y). \tag{2-32}$$

Note that for all $\tau \in [0, T]$ one has

$$\int_{\mathbb{T}} a_8(\tau, y)\, dy = \frac{1}{(\mathcal{B}^{-1}\psi')(\tau)} \int_{\mathbb{T}} (\mathcal{B}^{-1} a_5)(\tau, y)\, dy = \frac{1}{\psi'(t)} \int_{\mathbb{T}} a_5(t, y)\, dy = 0. \tag{2-33}$$

By straightforward calculations, we prove the following lemma.

Lemma 2.4. *There exists $\delta_* > 0$ with the following properties. Let $a_4 \in C([0, T], \mathbb{R})$ with $|a_4(t) - 1| \le \delta_*$ for all $t \in [0, T]$. Then the operator $\mathcal{B}$ defined in (2-25), (2-30) is an invertible isometry of $C([0, T], H^s_x)$ for all $s \ge 0$; namely,*

$$\|\mathcal{B}h\|_{T,s} = \|h\|_{T,s} \quad \forall h \in C([0, T], H^s_x),\ s \ge 0. \tag{2-34}$$

Moreover there exists a positive constant σ with the following property. Let $a_4 \in C^1([0, T], \mathbb{R})$, with $|a_4(t) - 1| \le \delta_*$ and $|a_4'(t)| \le 1$ for all $t \in [0, T]$. Let $s \ge 0$, and $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0, T], H^s_x)$ with $\int_{\mathbb{T}} a_5(t, y)\, dy = 0$ for all $t \in [0, T]$. Then the functions $a_8(t, x), a_9(t, x), a_{10}(t, x), \psi(t), \rho(t)$ and the constant m defined in (2-32), (2-30), (2-26) satisfy

$$|m - 1| + |\psi' - 1, \rho - 1|_T + \|a_8, \partial_\tau a_8, a_9, \partial_\tau a_9, a_{10}\|_{T,s} \le C \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,s}, \tag{2-35}$$

where C is independent of s. Moreover one has

$$\int_{\mathbb{T}} a_8(\tau, y)\, dy = 0 \quad \forall \tau \in [0, T]. \tag{2-36}$$

2C. *Step 3: multiplication.* In this section we eliminate the term $a_8(\tau, y)\, \partial_{yy}$ from the operator $\mathcal{L}_2$ defined in (2-31). To this end, we consider the multiplication operator $\mathcal{M}$ defined as

$$\mathcal{M}h(\tau, y) := q(\tau, y) h(\tau, y), \tag{2-37}$$

with $q : [0, T] \times \mathbb{T} \to \mathbb{R}$. We compute

$$\mathcal{M}^{-1} \mathcal{L}_2 \mathcal{M} = \partial_\tau + m\, \partial_{yyy} + a_{11}(\tau, y)\, \partial_{yy} + a_{12}(\tau, y)\, \partial_y + a_{13}(\tau, y), \tag{2-38}$$

with

$$a_{11} := a_8 + \frac{3mq_y}{q}, \quad a_{12} := a_9 + \frac{2a_8 q_y + 3mq_{yy}}{q}, \quad a_{13} := \frac{\mathcal{L}_2 q}{q}. \tag{2-39}$$

We want to choose q such that $a_{11} = 0$, which is equivalent to

$$3mq_y + a_8 q = 0. \tag{2-40}$$

Thanks to (2-36), equation (2-40) admits the space-periodic solution

$$q(\tau, y) := \exp\Big\{-\frac{1}{3m} (\partial_y^{-1} a_8)(\tau, y)\Big\}. \tag{2-41}$$

As a consequence, we get

$$\mathcal{L}_3 := \mathcal{M}^{-1} \mathcal{L}_2 \mathcal{M} = \partial_\tau + m\, \partial_{yyy} + a_{12}(\tau, y)\, \partial_y + a_{13}(\tau, y). \tag{2-42}$$

The proof of the following lemma is straightforward.

Lemma 2.5. *Let $s \geq 0$ and let $a_8 \in C([0,T], H^s_x)$ with $\int_{\mathbb{T}} a_8(\tau, y)\, dy = 0$ for all $\tau \in [0,T]$. Then for all $\mu \in [0,s]$, the operator $\mathcal{M}$ defined in (2-37), (2-41) and its inverse $\mathcal{M}^{-1}$ belong to $C([0,T], \mathcal{L}(H^\mu_x))$. Note that $\mathcal{M} = \mathcal{M}^T$.*

Furthermore, there exist two positive constants δ_*, σ with the following properties. Assume that $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H^{s+\sigma}_x)$ and let

$$\delta(\mu) := \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\mu+\sigma}. \tag{2-43}$$

Then if $\delta(0) \leq \delta_*$, for all $\mu \in [0,s]$ the operator $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$ satisfy

$$\|\mathcal{M}^{\pm 1} h\|_{T,\mu} \leq C_\mu \big(\|h\|_{T,\mu} + \delta(\mu) \|h\|_{T,0} \big) \quad \forall h \in C([0,T], H^\mu_x), \tag{2-44}$$

for some positive C_μ depending on μ. Moreover, the functions $a_{12}(\tau, y), a_{13}(\tau, y), q(\tau, y)$ defined in (2-39), (2-41) satisfy

$$\|q-1, a_{12}, \partial_t a_{12}, a_{13}\|_{T,\mu} \leq C_\mu \delta(\mu). \tag{2-45}$$

2D. *Step 4: translation of the space variable.* We consider the change of the space variable $z = y + p(\tau)$ and the operators

$$\mathcal{T} h(\tau, y) := h(\tau, y + p(\tau)), \quad \mathcal{T}^{-1} v(\tau, z) := v(\tau, z - p(\tau)), \tag{2-46}$$

where p is a function $p : [0,T] \to \mathbb{R}$. The differential operators become $\mathcal{T}^{-1} \partial_y \mathcal{T} = \partial_z$ and $\mathcal{T}^{-1} \partial_\tau \mathcal{T} = \partial_\tau + p'(\tau) \partial_z$. This is a special, simple case of the transformation $\mathcal{A}$ of Section 2A. Thus

$$\mathcal{L}_4 := \mathcal{T}^{-1} \mathcal{L}_3 \mathcal{T} = \partial_\tau + m\, \partial_{zzz} + a_{14}(\tau, z)\, \partial_z + a_{15}(\tau, z), \tag{2-47}$$

where

$$a_{14}(\tau, z) := p'(\tau) + (\mathcal{T}^{-1} a_{12})(\tau, z), \quad a_{15}(\tau, z) := (\mathcal{T}^{-1} a_{13})(\tau, z). \tag{2-48}$$

Now we look for $p(\tau)$ such that a_{14} has zero space average. We fix

$$p(\tau) := -\frac{1}{2\pi} \int_0^\tau \int_{\mathbb{T}} a_{12}(s, y)\, dy\, ds. \tag{2-49}$$

With this choice of p, after renaming the space-time variables $z = x$ and $\tau = t$, we have

$$\mathcal{L}_4 = \partial_t + m\, \partial_{xxx} + a_{14}(t, x)\, \partial_x + a_{15}(t, x), \qquad \int_{\mathbb{T}} a_{14}(t, x)\, dx = 0 \quad \forall t \in [0,T]. \tag{2-50}$$

With direct calculations we prove the following estimates.

Lemma 2.6. *Let $a_{12} \in C([0,T], L^2_x)$. Then the operator $\mathcal{T}$ defined in (2-46) and (2-49) belongs to $C([0,T], \mathcal{L}(H^s_x))$ for all $s \in [0, +\infty)$. In fact $\mathcal{T}$ is an isometry, namely*

$$\|\mathcal{T} h\|_{T,s} = \|h\|_{T,s} \quad \forall h \in C([0,T], H^s_x). \tag{2-51}$$

Moreover, $\mathcal{T}$ is invertible and its transpose is $\mathcal{T}^T = \mathcal{T}^{-1}$.

Let $s \geq 0$, and let $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H^{s+1}_x)$ with $\|a_{12}\|_{T,0} \leq 1$. Then the functions a_{14}, a_{15}, p defined in (2-48) and (2-49) satisfy

$$\sup_{t\in[0,T]} |p(t)| + \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,s} \leq C \|a_{12}, \partial_t a_{12}, a_{13}\|_{T,s+1}, \tag{2-52}$$

where C is independent of s.

2E. *Step 5: elimination of the order one.* The goal of this section is to eliminate the term $a_{14}(t,x)\,\partial_x$. Consider an operator $\mathcal{S}$ of the form

$$\mathcal{S}h := h + \gamma(t,x)\,\partial_x^{-1} h, \tag{2-53}$$

where $\gamma(t,x)$ is a function to be determined. Note $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = \pi_0$, where $\pi_0 h := h - \frac{1}{2\pi}\int_{\mathbb{T}} h\,dx$. We directly calculate

$$\mathcal{L}_4\mathcal{S} - \mathcal{S}(\partial_t + m\partial_{xxx}) = a_{16}\,\partial_x + a_{17} + a_{18}\,\partial_x^{-1}, \tag{2-54}$$

where

$$a_{16} := 3m\gamma_x + a_{14}, \quad a_{17} := a_{15} + (3m\gamma_{xx} + a_{14}\gamma)\pi_0, \quad a_{18} := \gamma_t + m\gamma_{xxx} + a_{14}\gamma_x + a_{15}\gamma. \tag{2-55}$$

We fix γ as

$$\gamma := -\frac{1}{3m}\,\partial_x^{-1} a_{14}, \tag{2-56}$$

so that $a_{16} = 0$. By the following Lemma 2.7, $\mathcal{S}$ is invertible, and we obtain

$$\mathcal{L}_5 := \mathcal{S}^{-1}\mathcal{L}_4\mathcal{S} = \partial_t + m\,\partial_{xxx} + \mathcal{R}, \quad \mathcal{R} := \mathcal{S}^{-1}(a_{17} + a_{18}\,\partial_x^{-1}). \tag{2-57}$$

Lemma 2.7. *There exist positive constants σ, δ_* with the following properties. Let $s \geq 0$, let a_{14}, a_{15} be two functions with $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_{14}(t,x)\,dx = 0$. Let*

$$\delta(\mu) := \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{2-58}$$

If $\delta(0) \leq \delta_*$, then the operator $\mathcal{S}$ defined in (2-53), (2-56) belongs to $C([0,T], \mathcal{L}(H^\mu_x))$ for all $\mu \in [0,s]$ and satisfies

$$\|\mathcal{S}h\|_{T,\mu} \leq C_\mu\big(\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0}\big) \quad \forall h \in C([0,T], H^\mu_x), \tag{2-59}$$

for some positive C_μ depending on μ. The operator $\mathcal{S}$ is invertible, and its inverse $\mathcal{S}^{-1}$, its transpose $\mathcal{S}^T$ and its inverse transpose $\mathcal{S}^{-T}$ all satisfy the same estimate (2-59) as $\mathcal{S}$.

The operator $\mathcal{R}$ defined in (2-57) belongs to $C([0,T], \mathcal{L}(H^\mu_x))$ for all $\mu \in [0,s]$ and it satisfies

$$\|\mathcal{R}h\|_{T,\mu} \leq C_\mu\big(\delta(0)\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0}\big) \quad \forall h \in C([0,T], H^\mu_x). \tag{2-60}$$

The transpose $\mathcal{R}^T$ belongs to $C([0,T], \mathcal{L}(H^\mu_x))$ and satisfies the same estimate (2-60) as $\mathcal{R}$.

Proof. Estimate $\|\gamma\,\partial_x^{-1}h\|_{T,\mu}$ by the usual tame estimates for the product of two functions (Lemma C.1), then use Neumann series in its tame version. $\square$

3. Observability

In this section we prove the observability of linear operators of the form (2-12). Such an observability property will be used in Section 4 in order to prove controllability of the linearized problem. We split the proof into several simple lemmas, starting with a direct consequence of the Ingham inequality. Since we actually need observability of a Cauchy problem flowing backwards in time (see Lemma 4.2) with datum at time T, we will accordingly state our lemmas.

Lemma 3.1 (Ingham inequality for $\partial_t + m\,\partial_{xxx}$). *For every $T > 0$ there exists a positive constant $C_1(T)$ such that, for all $(w_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C})$, all $m \geq \frac{1}{2}$,*

$$\int_0^T \Big| \sum_{n\in\mathbb{Z}} w_n e^{imn^3 t} \Big|^2 dt \geq C_1(T) \sum_{n\in\mathbb{Z}} |w_n|^2.$$

Proof. See, for example, Theorem 4.3 in Section 4.1 of [Micu and Zuazua 2005]. The fact that the constant $C_1(T)$ does not depend on m is obtained by closely following the proof in the above-mentioned work, and taking into account the lower bound for the distance between two different eigenvalues $|mn^3 - mk^3| \geq m \geq \frac{1}{2}$ for all $n, k \in \mathbb{Z}$, $n \neq k$. □

The following observability result is classical (see, e.g., [Russell and Zhang 1993] for a closely related result); for completeness, we also give here its proof.

Lemma 3.2 (observability for $\partial_t + m\,\partial_{xxx}$). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $v_T \in L^2(\mathbb{T})$, $m \geq \frac{1}{2}$, and let v satisfy*

$$\partial_t v + m\,\partial_{xxx} v = 0, \quad v(T) = v_T. \tag{3-1}$$

Then

$$\int_0^T \int_\omega |v(t,x)|^2\,dx\,dt \geq C_2 \|v_T\|^2_{L^2_x}, \tag{3-2}$$

with $C_2 := C_1(T)|\omega|$, where $C_1(T)$ is the constant of Lemma 3.1, and $|\omega|$ is the Lebesgue measure of ω.

Proof. Let $v_T(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, so that $v(t,x) = \sum_{n\in\mathbb{Z}} w_n(x) e^{imn^3 t}$, where $w_n(x) := a_n e^{i(nx - mn^3 T)}$. By Lemma 3.1, for each $x \in \mathbb{T}$ we have

$$\int_0^T \Big| \sum_{n\in\mathbb{Z}} w_n(x) e^{imn^3 t} \Big|^2 dt \geq C_1(T) \sum_{n\in\mathbb{Z}} |w_n(x)|^2 = C_1(T) \sum_{n\in\mathbb{Z}} |a_n|^2 = C_1(T) \|v_T\|^2_{L^2(\mathbb{T})}.$$

Then we integrate over $x \in \omega$. □

Lemma 3.3 (observability of $\mathcal{L}_5 := \partial_t + m\,\partial_{xxx} + \mathcal{R}$). *Let $T > 0$, let $\omega \subset \mathbb{T}$ be an open set and let $m \geq \frac{1}{2}$. Let $\mathcal{R} \in C([0,T], \mathcal{L}(L^2_x))$, with $\|\mathcal{R}(t)h\|_0 \leq r_0 \|h\|_0$ for all $h \in L^2_x$, all $t \in [0,T]$, where r_0 is a positive constant. Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem*

$$\partial_t v + m\,\partial_{xxx} v + \mathcal{R} v = 0, \quad v(T) = v_T, \tag{3-3}$$

which is globally well-posed by Lemma A.2(iii). Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \ge C_3 \|v_T\|_{L_x^2}^2,$$

with $C_3 := \frac{1}{4} C_2$*, provided that* r_0 *is small enough (more precisely,* r_0 *is smaller than a constant depending only on* T, C_2*, where* C_2 *is the constant in Lemma 3.2).*

Proof. Let v_1 be the solution of $\partial_t v_1 + m\,\partial_{xxx} v_1 = 0$, $v_1(T) = v_T$, and let $v_2 := v - v_1$. Then v_2 solves

$$(\partial_t + m\,\partial_{xxx} + \mathcal{R}) v_2 = -\mathcal{R} v_1, \qquad v_2(T) = 0. \tag{3-4}$$

By (A-10), applied for $s = 0$, $\alpha = 0$, $f = -\mathcal{R} v_1$, we get

$$\|v_2\|_{T,0} \le 2^{4Tr_0} 4T \|\mathcal{R} v_1\|_{T,0} \le 2^{4Tr_0} 4T r_0 \|v_T\|_0. \tag{3-5}$$

Using the elementary inequality $(a+b)^2 \ge \frac{1}{2} a^2 - b^2$ for all $a, b \in \mathbb{R}$,

$$\int_0^T \int_\omega |v|^2 \, dx \, dt \ge \frac{1}{2} \int_0^T \int_\omega |v_1|^2 \, dx \, dt - \int_0^T \int_\omega |v_2|^2 \, dx \, dt.$$

The integral of $|v_1|^2$ is estimated from below by (3-2). The integral of $|v_2|^2$ is bounded by $T \|v_2\|_{T,0}^2$; then use (3-5). □

Lemma 3.4 (observability of $\mathcal{L}_4 := \partial_t + m\,\partial_{xxx} + a_{14}(t,x)\,\partial_x + a_{15}(t,x)$, a_{14} with zero mean). *There exists a universal constant* $\sigma > 0$ *with the following property. Let* $T > 0$*, and let* $\omega \subset \mathbb{T}$ *be an open set. Let* $m \ge \frac{1}{2}$ *and let* $a_{14}(t,x)$, $a_{15}(t,x)$ *be two functions, with* $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H_x^\sigma)$,

$$\int_{\mathbb{T}} a_{14}(t,x) \, dx = 0 \quad \forall t \in [0,T], \qquad \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\sigma} \le \delta. \tag{3-6}$$

Let $v_T \in L^2(\mathbb{T})$ *and let* $v \in C([0,T], L_x^2)$ *be the solution of the Cauchy problem*

$$\mathcal{L}_4 v = 0, \qquad v(T) = v_T, \tag{3-7}$$

which is globally well-posed by Lemma A.3. Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \ge C_4 \|v_T\|_{L_x^2}^2,$$

with $C_4 := \frac{1}{16} C_3$*, provided that* δ *is small enough (more precisely,* δ *is smaller than a constant depending only on* T, C_3*).*

Proof. Following the procedure of Section 2E, we consider the transformation $\mathcal{S}$ in (2-53), (2-56), which conjugates $\mathcal{L}_4$ to

$$\mathcal{L}_5 := \mathcal{S}^{-1} \mathcal{L}_4 \mathcal{S} = \partial_t + m\,\partial_{xxx} + \mathcal{R},$$

where the operator $\mathcal{R}$ is defined in (2-57), (2-55); it belongs to $C([0,T], \mathcal{L}(L_x^2))$, and satisfies the bounds in Lemma 2.7. Let v be the solution of (3-7), and define $\tilde{v} := \mathcal{S}^{-1} v$. Then $\tilde{v}$ solves $\mathcal{L}_5 \tilde{v} = 0$, $\tilde{v}(T) = \tilde{v}_T$, where $\tilde{v}_T := \mathcal{S}^{-1}(T)v_T$, and therefore Lemma 3.3 applies to $\tilde{v}$ if δ is sufficiently small. By Lemmas 2.7 and A.3 and Remark A.8 we get

$$\int_0^T \int_\omega |(\mathcal{S}^{-1} - I)v|^2 \, dx \, dt \le T \|(\mathcal{S}^{-1} - I)v\|_{T,0}^2 \le C\delta^2 \|v\|_{T,0}^2 \le C'\delta^2 \|v_T\|_0^2$$

for some constant C' depending on T. We split $\tilde{v} = v + (\mathcal{S}^{-1} - I)v$, and we get

$$\int_0^T \int_\omega |\tilde{v}|^2 \, dx \, dt \le 2 \int_0^T \int_\omega |v|^2 \, dx \, dt + 2C'\delta^2 \|v_T\|_0^2.$$

Moreover $\|v_T\|_0 = \|\mathcal{S}(T)v_T\|_0 \le 2\|\tilde{v}_T\|_0$, and the thesis follows for δ small enough. □

Lemma 3.5 (observability of $\mathcal{L}_3 := \partial_t + m\,\partial_{xxx} + a_{12}(t,x)\,\partial_x + a_{13}(t,x)$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set and let $m \ge \frac{1}{2}$. Let $a_{12}(t,x)$, $a_{13}(t,x)$ be two functions, with $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H_x^\sigma)$,*

$$\|a_{12}, \partial_t a_{12}, a_{13}\|_{T,\sigma} \le \delta. \tag{3-8}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L_x^2)$ be the solution of the Cauchy problem

$$\mathcal{L}_3 v = 0, \qquad v(T) = v_T, \tag{3-9}$$

which is globally well-posed by Lemma A.4. Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \ge C_5 \|v_T\|_{L_x^2}^2 \tag{3-10}$$

for some $C_5 > 0$ depending on T, ω, provided that δ in (3-8) is sufficiently small (more precisely, δ is smaller than a constant depending on T, ω, C_4).

Proof. Following the procedure of Section 2D, we consider the transformation $\mathcal{T}$ defined in (2-46), (2-49), which conjugates $\mathcal{L}_3$ to

$$\mathcal{L}_4 := \mathcal{T}^{-1}\mathcal{L}_3\mathcal{T} = \partial_t + m\,\partial_{xxx} + a_{14}(t,x)\,\partial_x + a_{15}(t,x),$$

where a_{14}, a_{15} are defined in (2-48), and $\int_{\mathbb{T}} a_{14}(t,x)\,dx = 0$. By (2-52), the function p defined in (2-49) satisfies $|p(t)| \le C\delta$ for all $t \in [0,T]$. Let v be the solution of the Cauchy problem (3-9). Then $\tilde{v} := \mathcal{T}^{-1}v$ solves $\mathcal{L}_4\tilde{v} = 0$, $\tilde{v}(T) = \mathcal{T}^{-1}(T)v_T$. Let $\omega_1 = [\alpha_1, \beta_1]$ be an interval contained in ω. For δ small enough, one has

$$[\alpha_1 - p(t), \beta_1 - p(t)] \subseteq [\alpha_1 - \delta, \beta_1 + \delta] \subset \omega \quad \forall t \in [0,T].$$

The change of variable $x - p(t) = y$, $dx = dy$ gives

$$\int_0^T \int_{\omega_1} |\tilde{v}(t,x)|^2 \, dx \, dt = \int_0^T \int_{\alpha_1 - p(t)}^{\beta_1 - p(t)} |v(t,y)|^2 \, dy \, dt \le \int_0^T \int_\omega |v(t,y)|^2 \, dy \, dt.$$

By (2-52), for δ small enough, Lemma 3.4 can be applied to $\tilde{v}$ on the interval ω_1 and the thesis follows, since $\|\tilde{v}(T)\|_0 = \|\mathcal{T}^{-1}(T)v_T\|_0 = \|v_T\|_0$. □

Lemma 3.6 (observability of $\mathcal{L}_2 := \partial_t + m\,\partial_{xxx} + a_8(t,x)\,\partial_{xx} + a_9(t,x)\,\partial_x + a_{10}(t,x)$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set and let $m \geq \frac{1}{2}$. Let $a_8(t,x)$, $a_9(t,x)$, $a_{10}(t,x)$ be three functions, with $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H^\sigma_x)$,*

$$\int_{\mathbb{T}} a_8(t,x)\,dx = 0 \quad \forall t \in [0,T], \qquad \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\sigma} \leq \delta. \tag{3-11}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_2 v = 0, \quad v(T) = v_T, \tag{3-12}$$

which is globally well-posed by Lemma A.5. Then

$$\int_0^T \int_\omega |v(t,x)|^2\,dx\,dt \geq C_6 \|v_T\|^2_{L^2_x} \tag{3-13}$$

for some $C_6 > 0$ depending on T, ω, provided that δ in (3-11) is sufficiently small (more precisely, δ is smaller than a constant depending on T, ω, C_5).

Proof. Following the procedure of Section 2C, we consider the multiplication operator $\mathcal{M}$ defined in (2-37), (2-41), which conjugates $\mathcal{L}_2$ to

$$\mathcal{M}^{-1}\mathcal{L}_2\mathcal{M} = \mathcal{L}_3, \quad \mathcal{L}_3 = \partial_t + m\,\partial_{xxx} + a_{12}(t,x)\,\partial_x + a_{13}(t,x),$$

where a_{12}, a_{13} are defined in (2-39). Let v be the solution of the Cauchy problem (3-12). Then $\tilde{v} := \mathcal{M}^{-1}v$ solves $\mathcal{L}_3\tilde{v} = 0$, $\tilde{v}(T) = \mathcal{M}^{-1}(T)v_T$. Using (2-45), we have

$$\int_0^T \int_\omega |v(t,x)|^2\,dx\,dt = \int_0^T \int_\omega |\tilde{v}|^2\,dx\,dt + \int_0^T \int_\omega |\tilde{v}|^2(|q|^2 - 1)\,dx\,dt \geq (C_5 - C\delta)\|v_T\|_0^2.$$

The first of the two integrals has been estimated from below by applying Lemma 3.5 to $\mathcal{L}_3$ (by Lemma 2.5, this can be done provided that δ is sufficiently small). The second integral has been estimated using the bound (2-45), since $|q(t) - 1| \leq C\|q - 1\|_{T,1} \leq C'\delta$. Moreover, we have used the inequality $\|\tilde{v}\|_{T,0} \leq C\|\tilde{v}_T\|_0$ from Lemma A.4. The thesis follows with $C_6 := \frac{1}{2}C_5$ by choosing δ small enough. ☐

Lemma 3.7 (observability of $\mathcal{L}_1 := \partial_t + a_4(t)\,\partial_{xxx} + a_5(t,x)\,\partial_{xx} + a_6(t,x)\,\partial_x + a_7(t,x)$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let a_4, a_5, a_6, a_7 be four functions, with $a_4 \in C^1([0,T], \mathbb{R})$ and $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0,T], H^\sigma_x)$, satisfying*

$$\int_{\mathbb{T}} a_5(t,x)\,dx = 0 \quad \forall t \in [0,T], \qquad \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,\sigma} + |a_4 - 1, a_4'|_T \leq \delta. \tag{3-14}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_1 v = 0, \quad v(T) = v_T, \tag{3-15}$$

which is globally well-posed by Lemma A.6. Then

$$\int_0^T \int_\omega |v(t,x)|^2\,dx\,dt \geq C_7 \|v_T\|^2_{L^2_x} \tag{3-16}$$

for some $C_7 > 0$ depending on T, ω, provided that δ in (3-14) is sufficiently small (more precisely, δ is smaller than a constant depending on T, ω, C_6).

Proof. Following the procedure of Section 2B, we consider the reparametrization of time $\mathcal{B}$ defined in (2-25), (2-30), which conjugates $\mathcal{L}_1$ to

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\mathcal{L}_2, \quad \mathcal{L}_2 = \partial_\tau + m\,\partial_{xxx} + a_8(\tau,x)\,\partial_{xx} + a_9(\tau,x)\,\partial_x + a_{10}(\tau,x),$$

where ρ, a_8, a_9, a_{10} are defined in (2-28), (2-32) and $\int_{\mathbb{T}} a_8(\tau,x) = 0$ for all $\tau \in [0,T]$. Let v be the solution of the Cauchy problem (3-15). Then $\tilde v := \mathcal{B}^{-1}v$ solves $\mathcal{L}_2\tilde v = 0$, $\tilde v(T) = \mathcal{B}^{-1}(T)v_T$. Using (2-35), we have

$$\begin{aligned}\int_0^T\int_\omega |v(t,x)|^2\,dx\,dt &= \int_0^T\int_\omega |\tilde v(\psi(t),x)|^2\,dx\,dt\\ &= \int_0^T\int_\omega |\tilde v(\psi(t),x)|^2[\psi'(t) + (1-\psi'(t))]\,dx\,dt\\ &= \int_0^T\int_\omega |\tilde v(\tau,x)|^2\,dx\,d\tau + \int_0^T\int_\omega |\tilde v(\psi(t),x)|^2(1-\psi'(t))\,dx\,dt\\ &\geq (C_6 - C\delta)\|v_T\|_0^2.\end{aligned}$$

The first of the two integrals has been estimated from below by applying Lemma 3.6 to $\mathcal{L}_2$ (by Lemma 2.4, this can be done provided that δ is sufficiently small). The second integral has been estimated using the bound (2-35) for $|\psi'(t)-1|$ and also the inequality $\|\tilde v\|_{T,0} \leq C\|\tilde v_T\|_0$ from Lemma A.5. The thesis follows with $C_7 := \frac{1}{2}C_6$ by choosing δ small enough, since $\|\tilde v_T\|_0 = \|\mathcal{B}^{-1}(T)v_T\|_0 = \|v_T\|_0$. □

Lemma 3.8 (observability of $\mathcal{L}_0 := \partial_t + (1+a_3)\,\partial_{xxx} + a_2\,\partial_{xx} + a_1\,\partial_x + a_0$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $c \in \mathbb{R}$ and $a_3(t,x)$, $a_2(t,x)$, $a_1(t,x)$, $a_0(t,x)$ be four functions with $a_2 = c\,\partial_x a_3$,*

$$\|\partial_{tt}a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0\|_{T,\sigma} \leq \delta. \tag{3-17}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_0 v = 0, \quad v(T) = v_T, \tag{3-18}$$

which is globally well-posed by Lemma A.7. Then

$$\int_0^T\int_\omega |v(t,x)|^2\,dx\,dt \geq C_8\|v_T\|_{L^2_x}^2 \tag{3-19}$$

for some $C_8 > 0$ depending on T, ω, provided that δ in (3-17) is sufficiently small (more precisely, δ is smaller than a constant depending on T, ω, C_7).

Proof. Following the procedure of Section 2A, we consider the transformation $\mathcal{A}$ defined in (2-9), (2-16), (2-17), (2-18), which conjugates $\mathcal{L}_0$ to

$$\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A} = \mathcal{L}_1 = \partial_t + a_4(t)\,\partial_{xxx} + a_5(t,x)\,\partial_{xx} + a_6(t,x)\,\partial_x + a_7(t,x)$$

(see (2-19)), where a_4, a_5, a_6, a_7 are defined in (2-14) and $\int_{\mathbb{T}} a_5(t,x) = 0$ for all $t \in [0,T]$. Let v be the solution of the Cauchy problem (3-18). Then $\tilde{v} := \mathcal{A}^{-1} v$ solves $\mathcal{L}_1 \tilde{v} = 0$, $\tilde{v}(T) = \tilde{v}_T$, where $\tilde{v}_0 := \mathcal{A}^{-1}(0) v_0$. Let $\omega_1 = [\alpha_1, \beta_1] \subset \omega$. By (2-22) in Lemma 2.3, for δ sufficiently small Lemma 3.7 applies to $\tilde{v}$ on ω_1, and

$$\int_0^T \int_{\omega_1} |\tilde{v}|^2 \, dy \, dt \geq C_7 \|\tilde{v}_T\|_0^2.$$

By Lemma 2.3, $\|v_T\|_0 = \|\mathcal{A}(T)\tilde{v}_T\|_0 \leq C\|\tilde{v}_T\|_0$. The change of integration variable $y = x + \beta(t,x)$, $dy = (1+\beta_x(t,x))dx$ gives

$$\begin{aligned}\int_0^T \int_{\omega_1} |\tilde{v}|^2 \, dy \, dt &= \int_0^T \int_{\omega_1} |(\mathcal{A}^{-1} v)(t,y)|^2 \, dy \, dt \\ &= \int_0^T \int_{\omega_2(t)} \frac{|v(t,x)|^2}{1+\beta_x(t,x)} \, dx \, dt \leq 2 \int_0^T \int_{\omega} |v(t,x)|^2 \, dx \, dt,\end{aligned}$$

where $\omega_2(t) := \{x : x + \beta(t,x) \in \omega_1\}$. We have used the fact that, for δ small enough, $\omega_2(t) \subset \omega$, and the bound (2-22) for $|\beta_x(t,x)| \leq C\|\beta\|_{T,2} \leq C'\delta$. □

4. Controllability

In this section we prove the controllability of the linearized operator $\mathcal{L}_0$, using its observability (Lemma 3.8), by means of the HUM. We also prove higher regularity of the control.

Lemma 4.1 (controllability of $\mathcal{L}_0$). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let a_3, a_2, a_1, a_0 be four functions of (t,x) with $a_2 = 2\partial_x a_3$ satisfying (3-17). Let $\mathcal{L}_0$ be the linear operator*

$$\mathcal{L}_0 := \partial_t + (1+a_3)\,\partial_{xxx} + a_2\,\partial_{xx} + a_1\,\partial_x + a_0. \tag{4-1}$$

(i) Existence. *There exist constants δ_0, C such that, if δ in (3-17) is smaller than δ_0, then the following property holds. Given any three functions $g_1(t,x), g_2(x), g_3(x)$, with $g_1 \in C([0,T], L^2_x)$ and $g_2, g_3 \in L^2_x$, there exists a function $\varphi \in C([0,T], L^2_x)$ such that the solution h of the Cauchy problem*

$$\mathcal{L}_0 h = g_1 + \chi_\omega \varphi, \qquad h(0) = g_2 \tag{4-2}$$

satisfies $h(T) = g_3$. (Note that the Cauchy problem (4-2) is globally well-posed by Lemma A.7). Moreover

$$\|\varphi\|_{T,0} \leq C(\|g_1\|_{T,0} + \|g_2\|_0 + \|g_3\|_0). \tag{4-3}$$

(ii) Uniqueness. *Let $\mathcal{L}_0^*$ be the linear operator*

$$\mathcal{L}_0^* \psi := -\partial_t \psi - \partial_{xxx}\{(1+a_3)\psi\} + \partial_{xx}(a_2\psi) - \partial_x(a_1\psi) + a_0\psi. \tag{4-4}$$

The control φ in (i) *is the unique solution of the equation $\mathcal{L}_0^*\varphi = 0$ such that the solution h of the Cauchy problem (4-2) satisfies $h(T) = g_3$.*

The proof of Lemma 4.1 is given below, and it is based on the following classical lemma. In this section we use the standard notation $\langle u, v\rangle := \int_{\mathbb{T}} uv\,dx$.

Lemma 4.2. *Let a_3, a_2, a_1, a_0 be functions satisfying (3-17) and $a_2 = 2\partial_x a_3$. Let $\mathcal{L}_0^*$ be the operator defined in (4-4). For every (g_1, g_2, g_3), with $g_1 \in C([0,T], L^2_x)$ and $g_2, g_3 \in L^2_x$, there exists a unique $\varphi_1 \in L^2_x$ such that for all $\psi_1 \in L^2_x$, the solutions $\varphi, \psi \in C([0,T], L^2_x)$ of the Cauchy problems*

$$\begin{cases} \mathcal{L}_0^* \varphi = 0, \\ \varphi(T) = \varphi_1 \end{cases} \qquad \text{and} \qquad \begin{cases} \mathcal{L}_0^* \psi = 0, \\ \psi(T) = \psi_1 \end{cases} \tag{4-5}$$

satisfy

$$\int_0^T \langle g_1 + \chi_\omega \varphi, \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle = 0 \tag{4-6}$$

(*note that the global well-posedness of the Cauchy problems (4-5) follows by Lemma A.7 and Remark A.8*). *Moreover φ satisfies (4-3).*

Proof. Given $\varphi_1, \psi_1 \in L^2_x$, let φ, ψ be the solutions of the Cauchy problems (4-5), and define

$$B(\varphi_1, \psi_1) := \int_0^T \langle \chi_\omega \varphi, \psi \rangle \, dt, \quad \Lambda(\psi_1) := \langle g_3, \psi(T) \rangle - \langle g_2, \psi(0) \rangle - \int_0^T \langle g_1, \psi \rangle \, dt. \tag{4-7}$$

The bilinear map $B : L^2_x \times L^2_x \to \mathbb{R}$ is well-defined and continuous because $|\chi_\omega(x)| \le 1$ and, by Lemma A.7 and Remark A.8, $\|\varphi\|_{T,0} \le C \|\varphi_1\|_0$, and similarly for ψ. Moreover B is coercive by Lemma 3.8 and Remark 2.2. The linear functional Λ is bounded, with

$$|\Lambda(\psi_1)| \le C \|g\|_{T,0} \|\psi_1\|_0 \quad \forall \psi_1 \in L^2_x, \qquad \|g\|_{T,0} := \|g_1\|_{T,0} + \|g_2\|_0 + \|g_3\|_0.$$

Thus, by Riesz representation theorem (or Lax–Milgram), there exists a unique $\varphi_1 \in L^2_x$ such that

$$B(\varphi_1, \psi_1) = \Lambda(\psi_1) \quad \forall \psi_1 \in L^2_x. \tag{4-8}$$

Moreover $\|\varphi_1\|_0 \le C \|\Lambda\|_{\mathcal{L}(L^2_x, \mathbb{R})} \le C' \|g\|_{T,0}$. Since $\|\varphi\|_{T,0} \le C \|\varphi_1\|_0$, we get (4-3). □

Proof of Lemma 4.1. (i) Let $\varphi_1 \in L^2_x$ be the unique solution of (4-8) given by Lemma 4.2. Consider any $\psi_1 \in L^2_x$, and let $\varphi, \psi \in C([0,T], L^2_x)$ be the unique solutions of the Cauchy problems (4-5). Recalling (4-6), (4-2) and integrating by parts, we have

$$\begin{aligned} 0 &= \int_0^T \langle g_1 + \chi_\omega \varphi, \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\ &= \int_0^T \langle \mathcal{L}_0 h, \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\ &= \langle h(T), \psi(T) \rangle - \langle h(0), \psi(0) \rangle + \int_0^T \langle h, \mathcal{L}_0^* \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\ &= \langle h(T), \psi(T) \rangle - \langle g_3, \psi(T) \rangle \\ &= \langle h(T) - g_3, \psi_1 \rangle, \end{aligned}$$

from which it follows that $h(T) = g_3$.

(ii) Assume that $\tilde{\varphi} \in C([0,T], L^2_x)$ satisfies $\mathcal{L}_0^* \tilde{\varphi} = 0$ and it has the property that the solution h of the Cauchy problem (4-2) satisfies $h(T) = g_3$. Let $\tilde{\varphi}_1 := \tilde{\varphi}(T)$. The same integration by parts as above shows that $B(\tilde{\varphi}_1, \psi_1) = \Lambda(\psi_1)$ for all $\psi_1 \in L^2_x$. By the uniqueness in Lemma 4.2, $\tilde{\varphi}_1 = \varphi_1$. □

Lemma 4.3 (higher regularity). *Let T, ω, a_3, a_2, a_1, a_0, $\mathcal{L}_0$, g_1, g_2, g_3 be as in Lemma 4.1. There exist two positive constants δ_*, σ with the following property. Let $s > 0$ be given. Assume that $a_0, a_1, a_2, a_3 \in C^2([0,T], H^{s+\sigma}_x)$. Let*

$$\delta(\mu) := \sum_{k=0,1,2,\ i=0,1,2,3} \|\partial_t^k a_i\|_{T,\mu+\sigma}, \qquad \mu \in [0,s].$$

Let $\|g\|_{T,s} := \|g_1\|_{T,s} + \|g_2\|_s + \|g_3\|_s < \infty$. If $\delta(0) \le \delta_*$, then the control φ constructed in Lemma 4.1 and the solution h of (4-2) satisfy

$$\|\varphi, h\|_{T,s} \le C_s(\|g\|_{T,s} + \delta(s)\|g\|_{T,0}) \tag{4-9}$$

for some positive C_s depending on s, T, ω. Moreover, if $g_1 \in C^1([0,T], H^s_x)$, then

$$\|\partial_t\varphi, \partial_t h\|_{T,s+3} + \|\partial_{tt}\varphi, \partial_{tt} h\|_{T,s} \le C_s\big\{\|g\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \delta(s)\|g\|_{T,6}\big\}. \tag{4-10}$$

Proof. Let $g_1 \in C([0,T], H^s_x)$ and $g_2, g_3 \in H^s_x$. Let $\varphi, h \in C([0,T], L^2_x)$ be the solution of the control problem constructed in Lemma 4.1, namely

$$\mathcal{L}_0^*\varphi = 0, \quad \mathcal{L}_0 h = \chi_\omega \varphi + g_1, \quad h(0) = g_2, \quad h(T) = g_3. \tag{4-11}$$

To prove that $h, \varphi \in C([0,T], H^s_x)$, it is convenient to use the transformations of Section 2, to prove higher regularity for the solution $\tilde{h}$, $\tilde{\varphi}$ of the transformed control problem, and then to go back to h, φ proving their higher regularity. Recall that

$$\mathcal{L}_0 = \mathcal{A}\mathcal{B}\rho\mathcal{M}\mathcal{T}\mathcal{S}\mathcal{L}_5\mathcal{S}^{-1}\mathcal{T}^{-1}\mathcal{M}^{-1}\mathcal{B}^{-1}\mathcal{A}^{-1}, \tag{4-12}$$

where $\mathcal{L}_5 = \partial_t + m\,\partial_{xxx} + \mathcal{R}$ and $\mathcal{A}$, $\mathcal{B}$, ρ, $\mathcal{M}$, $\mathcal{T}$, $\mathcal{S}$ are defined in Section 2. In particular,

- $\mathcal{A}$ is the change of the space variable $(\mathcal{A}h)(t,x) = h(t, x + \beta(t,x))$ (see (2-9)), where β is defined in (2-18), (2-16), (2-17);
- $\mathcal{B}$ is the reparametrization of time $(\mathcal{B}h)(t,x) = h(\psi(t), x)$ (see (2-25)), where ψ is defined in (2-30);
- $\rho(t)$ is the function defined in (2-26);
- $\mathcal{M}$ is the multiplication operator $(\mathcal{M}h)(t,x) = q(t,x)h(t,x)$ (see (2-37)), where q is defined in (2-41);
- $\mathcal{T}$ is the translation of the space variable $(\mathcal{T}h)(t,x) = h(t, x + p(t))$ (see (2-46)), where p is defined in (2-49);
- $\mathcal{S}$ is the pseudodifferential operator $(\mathcal{S}h)(t,x) = h(t,x) + \gamma(t,x)\partial_x^{-1}h(t,x)$ (see (2-53)), where γ is defined in (2-56) and $\partial_x^{-1}h$ is the primitive of h with zero average in x (defined in Fourier);
- $\mathcal{R}$ is the bounded operator defined in (2-57).

Let

$$\mathcal{L}_5^* := -\partial_t - m\,\partial_{xxx} + \mathcal{R}^T, \tag{4-13}$$

where $\mathcal{R}^T$ is the L^2_x-adjoint of $\mathcal{R}$. Let

$$\begin{aligned} \tilde h &:= (\mathcal{ABMTS})^{-1} h, & \tilde g_1 &:= (\mathcal{AB}\rho\mathcal{MTS})^{-1} g_1, \\ \tilde g_2 &:= (\mathcal{ABMTS})^{-1}|_{t=0}\, g_2, & \tilde g_3 &:= (\mathcal{ABMTS})^{-1}|_{t=T}\, g_3, \\ \tilde\varphi &:= \mathcal{S}^T\mathcal{T}^T\mathcal{M}^T\mathcal{B}^{-1}\mathcal{A}^T\varphi, & K\tilde\varphi &:= (\mathcal{AB}\rho\mathcal{MTS})^{-1}(\chi_\omega(\mathcal{S}^T\mathcal{T}^T\mathcal{M}^T\mathcal{B}^{-1}\mathcal{A}^T)^{-1}\tilde\varphi). \end{aligned} \tag{4-14}$$

Note that, except for $\mathcal{S}^{-1}$, $\mathcal{S}^{-T}$, the operator K is a multiplication operator; namely

$$K\tilde\varphi = \mathcal{S}^{-1}(\zeta\mathcal{S}^{-T}\tilde\varphi), \quad \text{where } \zeta(t,x) := \rho^{-1}\mathcal{T}^{-1}\mathcal{M}^{-2}\mathcal{B}^{-1}\mathcal{A}^{-1}[(1+\beta_x)\chi_\omega]. \tag{4-15}$$

Since $h, \varphi \in C([0,T], L^2_x)$, and $g_1 \in C([0,T], H^s_x)$ and $g_2, g_3 \in H^s_x$, by (4-14) and the estimates for $\mathcal{A}, \mathcal{B}, \rho, \mathcal{M}, \mathcal{T}, \mathcal{S}$ in Section 2, one has

$$\tilde h, \tilde\varphi, K\tilde\varphi \in C([0,T], L^2_x), \quad \tilde g_1 \in C([0,T], H^s_x), \quad \tilde g_2, \tilde g_3 \in H^s_x.$$

Since h, φ satisfy (4-11), one proves that $\tilde h, \tilde\varphi$ satisfy

$$\mathcal{L}_5^*\tilde\varphi = 0, \quad \mathcal{L}_5\tilde h = K\tilde\varphi + \tilde g_1, \quad \tilde h(0) = \tilde g_2, \quad \tilde h(T) = \tilde g_3. \tag{4-16}$$

The last three equations in (4-16) are straightforward. To prove that $\mathcal{L}_5^*\tilde\varphi = 0$, we start from the equality

$$\langle\varphi(T), v(T)\rangle - \langle\varphi(0), v(0)\rangle = \int_0^T \langle\varphi, \mathcal{L}_0 v\rangle\,dt \quad \forall v \in C^\infty([0,T]\times\mathbb{T})$$

(which is a weak form of $\mathcal{L}_0^*\varphi = 0$), we recall (4-12), and apply all the changes of variables $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{T}, \mathcal{S}$ in the integral. Thus $\tilde h, \tilde\varphi$ solve this control problem:

> Given $\tilde g_1, \tilde g_2, \tilde g_3$, find $\tilde\varphi$ such that the solution $\tilde h$ of the Cauchy problem $\mathcal{L}_5\tilde h = K\tilde\varphi + \tilde g_1$, $\tilde h(0) = \tilde g_2$ satisfies $\tilde h(T) = \tilde g_3$, and moreover $\tilde\varphi$ solves $\mathcal{L}_5^*\tilde\varphi = 0$. (4-17)

The function $\tilde\varphi$ is the unique solution of (4-17). To prove it, assume that $\tilde\varphi_{\text{bis}} \in C([0,T], L^2_x)$ solves (4-17), and let $\tilde h_{\text{bis}}$ be the solution of the corresponding Cauchy problem $\mathcal{L}_5\tilde h_{\text{bis}} = K\tilde\varphi_{\text{bis}} + \tilde g_1$, $\tilde h_{\text{bis}}(0) = \tilde g_2$. Define

$$h_{\text{bis}} := \mathcal{ABMTS}\tilde h_{\text{bis}}, \quad \varphi_{\text{bis}} := \mathcal{A}^{-T}\mathcal{B}\mathcal{M}^{-T}\mathcal{T}^{-T}\mathcal{S}^{-T}\tilde\varphi_{\text{bis}}.$$

Then $h_{\text{bis}}, \varphi_{\text{bis}}$ solve (4-11). By the uniqueness in Lemma 4.1(ii) it follows that $\varphi_{\text{bis}} = \varphi$, $h_{\text{bis}} = h$. Therefore $\tilde\varphi_{\text{bis}} = \tilde\varphi$ and $\tilde h_{\text{bis}} = \tilde h$.

Now we prove that $\tilde h, \tilde\varphi \in C([0,T], H^s_x)$. We follow an argument used by Dehman and Lebeau [2009, Lemma 4.2], Laurent [2010, Lemma 3.1], and Alazard, Baldi, and Han-Kwan [2015, Proposition 8.1]. First, we prove the thesis for $\tilde g_1 = 0$, $\tilde g_3 = 0$. Consider the map

$$S\colon L^2_x \to L^2_x, \quad S\tilde\varphi_1 = \tilde h(0), \tag{4-18}$$

obtained by the composition $\tilde\varphi_1 \mapsto \tilde\varphi \mapsto \tilde h \mapsto \tilde h(0)$, where $\tilde\varphi, \tilde h$ are the solutions of the Cauchy problems

$$\begin{cases} \mathcal{L}_5^* \tilde\varphi = 0, \\ \tilde\varphi(T) = \tilde\varphi_1, \end{cases} \qquad \begin{cases} \mathcal{L}_5 \tilde h = K\tilde\varphi, \\ \tilde h(T) = 0. \end{cases} \tag{4-19}$$

From the existence and uniqueness of $\tilde\varphi_1 \in L^2_x$ such that $\tilde\varphi$ solves (4-17), it follows that S is an isomorphism of L^2_x. The initial datum $\tilde g_2$ is given, so we fix $\tilde\varphi_1 \in L^2_x$ such that $S\tilde\varphi_1 = \tilde g_2$. We have to estimate $\|\Lambda^s\tilde\varphi_1\|_0 \le C\|S\Lambda^s\tilde\varphi_1\|_0$, where Λ^s is the Fourier multiplier of symbol $\langle\xi\rangle^s := (1+\xi^2)^{s/2}$, $s>0$. To study the commutator $[S, \Lambda^s]$, we compare $(\Lambda^s\tilde\varphi, \Lambda^s\tilde h)$ with $(\bar\varphi, \bar h)$ defined by

$$\begin{cases} \mathcal{L}_5^* \bar\varphi = 0, \\ \bar\varphi(T) = \Lambda^s\varphi_1, \end{cases} \qquad \begin{cases} \mathcal{L}_5 \bar h = K\bar\varphi, \\ \bar h(T) = 0. \end{cases} \tag{4-20}$$

The difference $\Lambda^s\tilde\varphi - \bar\varphi$ satisfies

$$\begin{cases} \mathcal{L}_5^*(\Lambda^s\tilde\varphi - \bar\varphi) = \mathcal{F}_1, \\ (\Lambda^s\tilde\varphi - \bar\varphi)(T) = 0, \end{cases} \qquad \text{where } \mathcal{F}_1 := [\mathcal{L}_5^*, \Lambda^s]\tilde\varphi = [\mathcal{R}^T, \Lambda^s]\tilde\varphi. \tag{4-21}$$

From Lemma A.2 and Remark A.8, $\|\Lambda^s\tilde\varphi - \bar\varphi\|_{T,0} \le C\|\mathcal{F}_1\|_{T,0}$. We recall the classical estimate for the commutator of Λ^s and any multiplication operator $h \mapsto ah$:

$$\|[\Lambda^s, a]h\|_0 \le C_s\big(\|a\|_2\|h\|_{s-1} + \|a\|_{s+1}\|h\|_0\big). \tag{4-22}$$

By (4-22) and formulas (2-53), (2-56), (2-57), the commutator $\mathcal{F}_1 = [\mathcal{R}^T, \Lambda^s]\tilde\varphi$ satisfies

$$\begin{aligned} \|\mathcal{F}_1\|_{T,0} &\le C_s\big(\|a_{14}, a_{17}, a_{18}\|_{T,\sigma}\|\tilde\varphi\|_{T,s-1} + \|a_{14}, a_{17}, a_{18}\|_{T,s+\sigma}\|\tilde\varphi\|_{T,0}\big) \\ &\le C_s\big(\delta(0)\|\tilde\varphi\|_{T,s-1} + \delta(s)\|\tilde\varphi\|_{T,0}\big). \end{aligned} \tag{4-23}$$

The difference $\Lambda^s\tilde h - \bar h$ satisfies

$$\begin{cases} \mathcal{L}_5(\Lambda^s\tilde h - \bar h) = K(\Lambda^s\tilde\varphi - \bar\varphi) + \mathcal{F}_2, \\ (\Lambda^s\tilde h - \bar h)(T) = 0, \end{cases} \qquad \text{where } \mathcal{F}_2 := [\mathcal{R}^T, \Lambda^s]\tilde h + [\Lambda^s, K]\tilde\varphi. \tag{4-24}$$

We have $\|K(\Lambda^s\tilde\varphi - \bar\varphi)\|_{T,0} \le C\|\Lambda^s\tilde\varphi - \bar\varphi\|_{T,0} \le C\|\mathcal{F}_1\|_{T,0}$, and therefore, by Lemma A.2,

$$\|\Lambda^s\tilde h - \bar h\|_{T,0} \le C\big(\|\mathcal{F}_1\|_{T,0} + \|\mathcal{F}_2\|_{T,0}\big). \tag{4-25}$$

Using (4-22) and (4-15), we get

$$\|\mathcal{F}_2\|_{T,0} \le C_s\big(\|\tilde h, \tilde\varphi\|_{T,s-1} + \delta(s)\|\tilde h, \tilde\varphi\|_{T,0}\big). \tag{4-26}$$

By (4-23), (4-25) and (4-26) we deduce that

$$\|\Lambda^s\tilde h - \bar h\|_{T,0} \le C_s\big(\|\tilde h, \tilde\varphi\|_{T,s-1} + \delta(s)\|\tilde h, \tilde\varphi\|_{T,0}\big).$$

By (4-19), Lemma A.2 and Remark A.8,

$$\|\tilde h, \tilde\varphi\|_{T,\mu} \le C_\mu\big(\|\tilde\varphi\|_{T,\mu} + \delta(\mu)\|\tilde\varphi\|_{T,0}\big) \le C_\mu\big(\|\tilde\varphi_1\|_\mu + \delta(\mu)\|\tilde\varphi_1\|_0\big), \quad \mu \ge 0. \tag{4-27}$$

Therefore

$$\|(\Lambda^s\tilde h - \bar h)(0)\|_0 \le \|\Lambda^s\tilde h - \bar h\|_{T,0} \le C_s\big(\|\tilde\varphi_1\|_{s-1} + \delta(s)\|\tilde\varphi_1\|_0\big). \tag{4-28}$$

Since $S\tilde{\varphi}_1 = \tilde{h}(0) = \tilde{g}_2$, we have $\Lambda^s \tilde{h}(0) = \Lambda^s g_2$. Moreover, by the definition of S in (4-18)–(4-19), $\bar{h}(0) = S\Lambda^s \tilde{\varphi}_1$. Thus

$$\|S\Lambda^s\tilde{\varphi}_1\|_0 \le \|(\Lambda^s\tilde{h} - \bar{h})(0)\|_0 + \|\Lambda^s\tilde{h}(0)\|_0 \le C_s\big(\|\tilde{\varphi}_1\|_{s-1} + \delta(s)\|\tilde{\varphi}_1\|_0\big) + \|\tilde{g}_2\|_s. \tag{4-29}$$

Since S is an isomorphism of L^2_x, we have $\|\Lambda^s\tilde{\varphi}_1\|_0 \le C\|S\Lambda^s\tilde{\varphi}_1\|_0$, whence

$$\|\tilde{\varphi}_1\|_s \le C_s\big(\|\tilde{g}_2\|_s + \|\tilde{\varphi}_1\|_{s-1} + \delta(s)\|\tilde{\varphi}_1\|_0\big). \tag{4-30}$$

Since $\|\tilde{\varphi}_1\|_0 \le C\|\tilde{g}_2\|_0$, by induction we deduce that

$$\|\tilde{\varphi}_1\|_s \le C_s\big(\|\tilde{g}_2\|_s + \delta(s)\|\tilde{g}_2\|_0\big). \tag{4-31}$$

By (4-27), we obtain

$$\|\tilde{h}, \tilde{\varphi}\|_{T,s} \le C_s\big(\|\tilde{g}_2\|_s + \delta(s)\|\tilde{g}_2\|_0\big), \tag{4-32}$$

which is the thesis in the case $\tilde{g}_1 = 0,\ \tilde{g}_3 = 0$.

Now we prove the higher regularity of $\tilde{h}, \tilde{\varphi}$ removing the assumption $\tilde{g}_1 = 0,\ \tilde{g}_3 = 0$. Let $\tilde{g}_1 \in C([0,T], H^s_x)$ and $\tilde{g}_2, \tilde{g}_3 \in H^s_x$, and let $\tilde{h}, \tilde{\varphi}$ be the solution of (4-17). Let w be the solution of the problem

$$\mathcal{L}_5 w = \tilde{g}_1, \quad w(T) = \tilde{g}_3.$$

By Lemma A.2, $w \in C([0,T], H^s_x)$, with

$$\|w\|_{T,s} \le C_s\big\{\|\tilde{g}_1\|_{T,s} + \|\tilde{g}_3\|_s + \delta(s)(\|\tilde{g}_1\|_{T,0} + \|\tilde{g}_3\|_0)\big\}. \tag{4-33}$$

Let $v := \tilde{h} - w$. Then

$$\mathcal{L}_5 v = K\tilde{\varphi}, \quad v(0) = \tilde{g}_2 - w(0), \quad v(T) = 0.$$

This means that $v, \tilde{\varphi}$ solve (4-17) where $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$ are replaced by $(0, \tilde{g}_2 - w(0), 0)$. Hence (4-32) applies to $v, \tilde{\varphi}$, and we get

$$\|v, \tilde{\varphi}\|_{T,s} \le C_s\big(\|\tilde{g}_2 - w(0)\|_s + \delta(s)\|\tilde{g}_2 - w(0)\|_0\big). \tag{4-34}$$

We estimate $\|\tilde{g}_2 - w(0)\|_s \le \|\tilde{g}_2\|_s + \|w\|_{T,s}$; we use (4-33) and $\|\tilde{h}\|_{T,s} \le \|v\|_{T,s} + \|w\|_{T,s}$ to conclude

$$\|\tilde{h}, \tilde{\varphi}\|_{T,s} \le C_s\big\{\|\tilde{g}\|_{T,s} + \delta(s)\|\tilde{g}\|_{T,0}\big\}, \tag{4-35}$$

where we have denoted, in short, $\|\tilde{g}\|_{T,s} := \|\tilde{g}_1\|_{T,s} + \|\tilde{g}_2\|_s + \|\tilde{g}_3\|_s$. This proves the higher regularity for the transformed control problem (4-17). By the definitions in (4-14),

$$\|\varphi\|_{T,s} \le C_s\big(\|\tilde{\varphi}\|_{T,s} + \delta(s)\|\tilde{\varphi}\|_{T,0}\big), \quad \|h\|_{T,s} \le C_s\big(\|\tilde{h}\|_{T,s} + \delta(s)\|\tilde{h}\|_{T,0}\big),$$
$$\|\tilde{g}\|_{T,s} \le C_s\big(\|g\|_{T,s} + \delta(s)\|g\|_{T,0}\big),$$

and the proof of (4-9) is complete.

The bound (4-10) is deduced in a classical way from the fact that h, φ solve the equations $\mathcal{L}_0^*\varphi = 0$, $\mathcal{L}_0 h = \chi_\omega\varphi + g_1$. □

Remark 4.4. Another possible way to prove higher regularity for h, φ is to apply the argument of [Dehman and Lebeau 2009; Laurent 2010; Alazard, Baldi, and Han-Kwan 2015] directly to the control problem for $\mathcal{L}_0$, instead of passing to the transformed problem (4-17), applying that argument, and then going back to h, φ. Such a more direct method adapted to the present case would require the construction of two operators A_s, B_s such that

(1) $C_1\|v\|_s \le \|A_s v\|_0 \le C_2\|v\|_s$ (equivalent norm in H^s),

(2) the commutator $[\mathcal{L}_0, A_s]$ is an operator of order $s-1$,

(3) the difference $B_s\mathcal{L}_0^* - \mathcal{L}_0^* A_s$ is also of order $s-1$.

The construction of such A_s, B_s is possible, but probably the proof given above is more straightforward, and it fully exploits the advantages of conjugating $\mathcal{L}_0$ to $\mathcal{L}_5$ (Section 2). The main point is that the commutator $[\mathcal{L}_5, \Lambda^s]$ is of order $s-1$ (because $\mathcal{L}_5$ has constant coefficients up to a *bounded* remainder), while $[\mathcal{L}_0, \Lambda^s]$ is of order $s+2$ (because $\mathcal{L}_0$, which was obtained by linearizing a *quasilinear* PDE, has variable coefficients also at the highest order), so that a modified version A_s of Λ^s is needed. □

In view of the application of the Nash–Moser theorem in Section 5, we define the spaces

$$E_s := X_s \times X_s, \quad X_s := C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^{s+3}) \cap C^2([0,T], H_x^s), \tag{4-36}$$

$$F_s := \{g = (g_1, g_2, g_3) : g_1 \in C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^s),\, g_2, g_3 \in H_x^{s+6}\} \tag{4-37}$$

equipped with the norms

$$\|u, f\|_{E_s} := \|u\|_{X_s} + \|f\|_{X_s}, \quad \|u\|_{X_s} := \|u\|_{T,s+6} + \|\partial_t u\|_{T,s+3} + \|\partial_{tt} u\|_{T,s}, \tag{4-38}$$

$$\|g\|_{F_s} := \|g_1\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \|g_2, g_3\|_{s+6}. \tag{4-39}$$

With this notation, we have proved the following linear inversion result.

Theorem 4.5 (right inverse of the linearized operator). *Let $T > 0$ and $\omega \subset \mathbb{T}$ be an open set. There exist two universal constants $\tau, \sigma \ge 3$ and a positive constant δ_* depending on T, ω with the following property.*

Let $s \in [0, r-\tau]$, where r is the regularity of the nonlinearity $\mathcal{N}$ (see Lemma 2.1). *Let $g = (g_1, g_2, g_3) \in F_s$ and let $(u, f) \in E_{s+\sigma}$, with $\|u\|_{X_\sigma} \le \delta_*$. Then there exists $(h, \varphi) := \Psi(u, f)[g] \in E_s$ such that*

$$P'(u)[h] - \chi_\omega \varphi = g_1, \quad h(0) = g_2, \quad h(T) = g_3, \tag{4-40}$$

and

$$\|h, \varphi\|_{E_s} \le C_s\big(\|g\|_{F_s} + \|u\|_{X_{s+\sigma}}\|g\|_{F_0}\big), \tag{4-41}$$

where C_s depends on s, T, ω.

5. Proofs

In this section we prove Theorems 1.1 and 1.4.

5A. ***Proof of Theorem 1.1.*** The spaces defined in (4-36)–(4-39), with $s \ge 0$, form scales of Banach spaces. We define smoothing operators S_θ in the following way. We fix a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ with $0 \le \varphi \le 1$,

$$\varphi(\xi) = 1 \quad \forall |\xi| \le 1 \qquad \text{and} \qquad \varphi(\xi) = 0 \quad \forall |\xi| \ge 2.$$

For any real number $\theta \geq 1$, let S_θ be the Fourier multiplier with symbol $\varphi(\xi/\theta)$, namely

$$S_\theta u(x) := \sum_{k\in\mathbb{Z}} \hat{u}_k\, \varphi(k/\theta)\, e^{ikx}, \quad \text{where } u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx} \in L^2(\mathbb{T}). \tag{5-1}$$

The definition of S_θ extends to functions $u(t,x) = \sum_{k\in\mathbb{Z}} \hat{u}_k(t)\, e^{ikx}$ depending on time in the obvious way. Since S_θ and ∂_t commute, the smoothing operators S_θ are defined on the spaces E_s, F_s defined in (4-36)–(4-37) by setting $S_\theta(u, f) := (S_\theta u, S_\theta f)$ and similarly on $g = (g_1, g_2, g_3)$. One easily verifies that S_θ satisfies (B-1)–(B-4) on E_s and F_s. We define the spaces E'_a with norm $\|\cdot\|'_a$ and F'_b with $\|\cdot\|'_b$ as constructed in Appendix B.

We observe that $\Phi(u, f) := (P(u) - \chi_\omega f,\ u(0),\ u(T))$ defined in (1-13)–(1-14) belongs to F_s when $(u, f) \in E_{s+3}$, $s \in [0, r-6]$, with $\|u\|_{T,4} \leq 1$. Its second derivative is

$$\Phi''(u, f)[(h_1, \varphi_1), (h_2, \varphi_2)] = \begin{pmatrix} P''(u)[h_1, h_2] \\ 0 \\ 0 \end{pmatrix}.$$

For u in a fixed ball $\|u\|_{X_1} \leq \delta_0$, with δ_0 small enough, we estimate

$$\|P''(u)[h, w]\|_{F_s} \leq C_s\big(\|h\|_{X_1}\|w\|_{X_{s+3}} + \|h\|_{X_{s+3}}\|w\|_{X_1} + \|u\|_{X_{s+3}}\|h\|_{X_1}\|w\|_{X_1}\big) \tag{5-2}$$

for all $s \in [0, r-6]$. We fix $V = \{(u, f) \in E_3 : \|(u, f)\|_{E_3} \leq \delta_0\}$, $\delta_1 = \delta_*$,

$$a_0 = 1, \quad \mu = 3, \quad a_1 = \sigma, \quad \alpha = \beta = 2\sigma, \quad a_2 \in (3\sigma, r - \tau], \tag{5-3}$$

where δ_*, σ, τ are given by Theorem 4.5, and r is the regularity of $\mathcal{N}$ in Theorem 1.1. The right inverse Ψ in Theorem 4.5 satisfies the assumptions of Theorem B.1. Thus by Theorem B.1 we obtain that, if $g = (0, u_{\text{in}}, u_{\text{end}}) \in F'_\beta$ with $\|g\|'_{F_\beta} \leq \delta$, then there exists a solution $(u, f) \in E'_\alpha$ of the equation $\Phi(u, f) = g$, with $\|u, f\|'_{E_\alpha} \leq C\|g\|'_{F_\beta}$ (and recall that $\beta = \alpha$). We fix $s_1 := \alpha + 6$, and (1-11) is proved. In fact, we have proved slightly more than (1-11), because $\|g\|'_{F_\beta} \leq C\|g\|_{F_\beta}$ and $\|u, f\|_{E_a} \leq C_a\|u, f\|'_{E_\alpha}$ for all $a < \alpha$.

We have found a solution (u, f) of the control problem (1-9)–(1-10). Now we prove that u is the unique solution of the Cauchy problem (1-9), with that given f. Let u, v be two solutions of (1-9) in E_{s-6} for all $s < s_1$. We calculate

$$P(u) - P(v) = \int_0^1 P'(v + \lambda(u - v))[u - v]\, d\lambda =: \widetilde{\mathcal{L}}_0[u - v],$$

where

$$\widetilde{\mathcal{L}}_0 := \partial_t + (1 + \tilde{a}_3(t,x))\partial_{xxx} + \tilde{a}_2(t,x)\partial_{xx} + \tilde{a}_1(t,x)\partial_x + \tilde{a}_0(t,x),$$

$$\tilde{a}_i(t,x) := \int_0^1 a_i(v + \lambda(u - v))(t,x)\, d\lambda, \quad i = 0, 1, 2, 3,$$

and $a_i(u)$ is defined in (2-2). Note that $\tilde{a}_2 = 2\partial_x \tilde{a}_3$ because $a_2(v + \lambda(u - v)) = 2\partial_x a_3(v + \lambda(u - v))$ for all $\lambda \in [0, 1]$. The difference $u - v$ satisfies $\widetilde{\mathcal{L}}_0(u - v) = 0$, $(u - v)(0) = 0$. Hence, by Lemma A.7, $u - v = 0$. The proof of Theorem 1.1 is complete. $\square$

5B. ***Proof of Theorem 1.4.*** We define

$$E_s := C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^{s+3}) \cap C^2([0,T], H_x^s), \tag{5-4}$$

$$F_s := \big\{g = (g_1, g_2) : g_1 \in C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^s),\ g_2 \in H_x^{s+6}\big\} \tag{5-5}$$

equipped with norms

$$\|u\|_{E_s} := \|u\|_{T,s+6} + \|\partial_t u\|_{T,s+3} + \|\partial_{tt} u\|_{T,s}, \tag{5-6}$$

$$\|g\|_{F_s} := \|g_1\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \|g_2\|_{s+6}, \tag{5-7}$$

and $\Phi(u) := (P(u), u(0))$. Given $g := (f, u_{\text{in}}) \in F_{s_0}$, the Cauchy problem (1-18) becomes $\Phi(u) = g$. We fix $V, \delta_1, a_0, \mu, a_1, \alpha, \beta, a_2$ as in (5-3), where the constants σ, δ_* are now given in Lemma A.7 and $\tau = \sigma + 9$ by Lemma 2.1 combined with Lemma A.7 and the definition of the spaces E_s, F_s. Assumption (B-13) about the right inverse of the linearized operator is satisfied by Lemmas A.7 and 2.1. We fix $s_0 := \alpha + 6$. Then Theorem B.1 applies, giving the existence part of Theorem 1.4. The uniqueness of the solution is proved exactly as in the proof of Theorem 1.1. □

Appendix A: Well-posedness of linear operators

Lemma A.1. *Let $T > 0$, $m \in \mathbb{R}$, $s \in \mathbb{R}$, $f \in C([0,T], H_x^s)$, with $f(t,x) = \sum_{n\in\mathbb{Z}} f_n(t) e^{inx}$. Let A be the linear operator defined by $Af := v$, where v is the solution of*

$$\begin{cases} \partial_t v + m\partial_{xxx} v = f & \forall (t,x) \in [0,T] \times \mathbb{T}, \\ v(0,x) = 0. \end{cases} \tag{A-1}$$

Then

$$Af(t,x) = \sum_{n\in\mathbb{Z}} (Af)_n(t) e^{inx}, \qquad (Af)_n(t) = \int_0^t e^{imn^3(\tau - t)} f_n(\tau)\, d\tau, \tag{A-2}$$

Af belongs to $C([0,T], H_x^s) \cap C^1([0,T], H_x^{s-3})$, and

$$\|Af\|_{T,s} \le T \|f\|_{T,s}. \tag{A-3}$$

Proof. Formula (A-2) simply comes from variation of constants. By Hölder's inequality,

$$|(Af)_n(t)| \le \sqrt{t} \bigg(\int_0^t |f_n(\tau)|^2\, d\tau\bigg)^{1/2} \qquad \forall t \in [0,T]$$

and therefore, for each $t \in [0,T]$,

$$\begin{aligned} \|Af(t)\|_{H_x^s}^2 &= \sum_{n\in\mathbb{Z}} |(Af)_n(t)|^2 \langle n\rangle^{2s} \le \sum_{n\in\mathbb{Z}} t \int_0^t |f_n(\tau)|^2\, d\tau\, \langle n\rangle^{2s} \\ &\le t \int_0^t \sum_{n\in\mathbb{Z}} |f_n(\tau)|^2 \langle n\rangle^{2s}\, d\tau = t \int_0^t \|f(\tau)\|_{H_x^s}^2\, d\tau \le t^2 \|f\|_{C([0,t],H_x^s)}^2. \end{aligned}$$

Taking the sup over $t \in [0,T]$ we get the thesis. □

We remark that for $s \leq 3$ the operator A is well-defined in the sense of distributions. We also recall that $\mathcal{L}(H_x^s)$ is the space of linear bounded operators of H_x^s into itself, with operator norm $\|L\|_{\mathcal{L}(H_x^s)} := \sup\{\|Lh\|_s : h \in H_x^s,\ \|h\|_s = 1\}$.

Lemma A.2. (i) (LWP). *Let $T > 0$, $s \in \mathbb{R}$, $\mathcal{R} \in C([0,T], \mathcal{L}(H_x^s))$, and let*

$$r_s := \|\mathcal{R}\|_{C([0,T],\mathcal{L}(H_x^s))} = \sup_{t\in[0,T]} \|\mathcal{R}(t)\|_{\mathcal{L}(H_x^s)}, \qquad \mathcal{L}_5 := \partial_t + m\partial_{xxx} + \mathcal{R}. \tag{A-4}$$

Let $\alpha \in H_x^s$ and $f \in C([0,T], H_x^s)$. If $Tr_s \leq \frac{1}{2}$, then the Cauchy problem

$$\begin{cases} \mathcal{L}_5 u = f, \\ u(0,x) = \alpha(x) \end{cases} \tag{A-5}$$

has a unique solution $u \in C([0,T], H_x^s)$. The solution u satisfies

$$\|u\|_{T,s} \leq (1 + 2Tr_s)\|\alpha\|_s + 2T\|f\|_{T,s} \leq 2(\|\alpha\|_s + T\|f\|_{T,s}). \tag{A-6}$$

(ii) (tame LWP). *Let $T > 0$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ with $s \geq s_1$, and let $\mathcal{R} \in C([0,T], \mathcal{L}(H_x^s)) \cap C([0,T], \mathcal{L}(H_x^{s_1}))$. Assume that*

$$\|\mathcal{R}(t)h\|_s \leq c_1\|h\|_s + c_s\|h\|_{s_1}, \qquad \|\mathcal{R}(t)h\|_{s_1} \leq c_1\|h\|_{s_1} \quad \forall h \in H_x^s, \tag{A-7}$$

for all $t \in [0,T]$, where c_1, c_s are positive constants. Let $\alpha \in H_x^s$. If

$$Tc_1 \leq \tfrac{1}{2}, \tag{A-8}$$

then the solution $u \in C([0,T], H_x^{s_1})$ of the Cauchy problem (A-5) given in (i) *belongs to $C([0,T], H_x^s)$, with*

$$\|u\|_{T,s} \leq 2T\|f\|_{T,s} + (1 + 2Tc_1)\|\alpha\|_s + 4Tc_s(T\|f\|_{T,s_1} + \|\alpha\|_{s_1}). \tag{A-9}$$

(iii) (GWP). *Let $T > 0$, $s \in \mathbb{R}$, $\mathcal{R} \in C([0,T], \mathcal{L}(H_x^s))$, and let r_s be defined in (A-4). Let $\alpha \in H_x^s$. Then the Cauchy problem (A-5) has a unique global solution $u \in C([0,T], H_x^s)$, with*

$$\|u\|_{T,s} \leq 2^{4Tr_s}(\|\alpha\|_s + 4T\|f\|_{T,s}). \tag{A-10}$$

(iv) (tame GWP). *Let $T > 0$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ with $s \geq s_1$, and let $\mathcal{R} \in C([0,T], \mathcal{L}(H_x^s)) \cap C([0,T], \mathcal{L}(H_x^{s_1}))$. Assume that (A-7) holds for all $t \in [0,T]$, where c_1, c_s are positive constants. Let $\alpha \in H_x^s$. Then the global solution $u \in C([0,T], H_x^s)$ of the Cauchy problem (A-5) given in* (iii) *satisfies*

$$\|u\|_{T,s} \leq 2^{4Tc_1}\big(\|\alpha\|_s + 4Tc_s\|\alpha\|_{s_1} + 2T\|f\|_{T,s} + 4T^2c_s\|f\|_{T,s_1}\big). \tag{A-11}$$

Proof. (i) Write $u = v + w$, where $v(t,x)$ is the solution of

$$\partial_t v + m\,\partial_{xxx} v = 0, \qquad v(0,x) = \alpha(x). \tag{A-12}$$

Hence u solves (A-5) if and only if $w(t,x)$ solves

$$\partial_t w + m\,\partial_{xxx} w + \mathcal{R}w = -\mathcal{R}v + f, \qquad w(0,x) = 0. \tag{A-13}$$

By Lemma A.1, (A-13) is the fixed point problem

$$w = \Psi(w), \tag{A-14}$$

where $\Psi(w) := A[f - \mathcal{R}(v + w)]$. Let $B_\rho := \{w \in C([0, T], H^s_x) : \|u\|_{T,s} \le \rho\}$, $\rho \ge 0$. Then

$$\|\Psi(w)\|_{T,s} \le T(\|f\|_{T,s} + r_s\|\alpha\|_s + r_s\rho), \qquad \|\Psi(w_1) - \Psi(w_2)\|_{T,s} \le T\, r_s\|w_1 - w_2\|_{T,s} \tag{A-15}$$

for all $w, w_1, w_2 \in B_\rho$. By assumption, $Tr_s \le \frac{1}{2}$. Therefore, for any $\rho \ge 2T(\|f\|_{T,s} + r_s\|\alpha\|_s)$, Ψ is a contraction in B_ρ. In particular, we fix $\rho = \rho_0 := 2T(\|f\|_{T,s} + r_s\|\alpha\|_s)$. Hence there exists a fixed point $w \in B_{\rho_0}$ of Ψ, with $\|w\|_{T,s} \le \rho_0 \le 2T\|f\|_{T,s} + \|\alpha\|_s$. As a consequence, there exists a solution $u \in C([0, T], H^s_x)$ of (A-5) with $\|u\|_{T,s} \le 2(T\|f\|_{T,s} + \|\alpha\|_s)$. By the contraction lemma, the solution u is unique in any ball B_ρ, $\rho \ge \rho_0$, and therefore it is unique in $C([0, T], H^s_x)$.

(ii) By assumption, $Tc_1 \le \frac{1}{2}$, and therefore, by (i), there exists a unique solution $u \in C([0, T], H^{s_1}_x)$. It remains to prove that u satisfies (A-9). By construction, $u = v + w$, where $v \in C([0, T], H^s_x)$ is the solution of (A-12), with $\|v(t)\|_s = \|\alpha\|_s$ for all $t \in [0, T]$, and $w \in C([0, T], H^{s_1}_x)$ solves (A-14). By the iterative scheme of the contraction lemma, w is the limit in $C([0, T], H^{s_1}_x)$ of the sequence (w_n), where $w_0 := 0$, and $w_{n+1} := \Psi(w_n)$ for all $n \in \mathbb{N}$. By (A-7) and (A-3), Ψ maps $C([0, T], H^s_x)$ into itself; therefore $w_n \in C([0, T], H^s_x)$ for all $n \ge 0$. Let $h_n := w_n - w_{n-1}$, $n \ge 1$, so that $w_n = \sum_{k=1}^n h_k$. One has $h_{n+1} = -A\mathcal{R}h_n$ for all $n \ge 1$, and

$$\|h_{n+1}\|_{T,s} \le Tc_1\|h_n\|_{T,s} + Tc_s\|h_n\|_{T,s_1}, \qquad \|h_{n+1}\|_{T,s_1} \le Tc_1\|h_n\|_{T,s_1} \quad \forall n \ge 1.$$

Hence, by induction, for all $n \ge 1$ we have

$$\|h_n\|_{T,s} \le (Tc_1)^{n-1}\|h_1\|_{T,s} + (n-1)(Tc_1)^{n-2}Tc_s\|h_1\|_{T,s_1}, \qquad \|h_n\|_{T,s_1} \le (Tc_1)^{n-1}\|h_1\|_{T,s_1}. \tag{A-16}$$

Also, $\|h_1\|_{T,s} \le T\|f\|_{T,s} + Tc_1\|\alpha\|_s + Tc_s\|\alpha\|_{s_1}$ and $\|h_1\|_{T,s_1} \le T\|f\|_{T,s_1} + Tc_1\|\alpha\|_{s_1}$. Therefore

$$\begin{aligned} \|h_n\|_{T,s} &\le (Tc_1)^{n-1}T\|f\|_{T,s} + (Tc_1)^n\|\alpha\|_s + (n-1)(Tc_1)^{n-2}Tc_sT\|f\|_{T,s_1} \\ &\qquad\qquad + n(Tc_1)^{n-1}Tc_s\|\alpha\|_{s_1}, \\ \|h_n\|_{T,s_1} &\le (Tc_1)^{n-1}T\|f\|_{T,s_1} + (Tc_1)^n\|\alpha\|_{s_1} \qquad \forall n \ge 1. \end{aligned} \tag{A-17}$$

Since $Tc_1 \le \frac{1}{2}$, the sequence $w_n = \sum_{k=1}^n h_k$ converges in $C([0, T], H^s_x)$ to some limit $\tilde{w} \in C([0, T], H^s_x)$. Since w_n converges to w in $C([0, T], H^{s_1}_x)$, the two limits coincide, and $w \in C([0, T], H^s_x)$. Since $\|w\|_{T,s} \le \sum_{k=1}^\infty \|h_k\|_{T,s}$, we get

$$\|w\|_{T,s} \le 2T(\|f\|_{T,s} + c_1\|\alpha\|_s) + 4Tc_s(T\|f\|_{T,s_1} + \|\alpha\|_{s_1}). \tag{A-18}$$

Since $u = v + w$, we deduce (A-9).

(iii) If $Tr_s \le \frac{1}{2}$, the result is given by (i). Let $Tr_s > \frac{1}{2}$, and fix $N \in \mathbb{N}$ such that $2Tr_s \le N \le 4Tr_s$. Let $T_0 := T/N$, so that $\frac{1}{4} \le T_0 r_s \le \frac{1}{2}$. Divide the interval $[0, T]$ into the union $I_1 \cup \cdots \cup I_N$, where $I_n := [(n-1)T_0, nT_0]$. Applying (i) on the time interval $I_1 = [0, T_0]$ gives the solution $u_1 \in C(I_1, H^s_x)$, with $\|u_1\|_{C(I_1, H^s_x)} \le b\|\alpha\|_s + 2T_0\|f\|_{T,s}$, where $b := 1 + 2T_0 r_s$. Now consider the Cauchy problem on I_2

with initial datum $u(T_0) = u_1(T_0)$. Applying (i) on I_2 gives the solution $u_2 \in C(I_2, H^s_x)$, with

$$\|u_2\|_{C(I_2,H^s_x)} \le b\|u_1(T_0)\|_s + 2T_0\|f\|_{T,s} \le b^2\|\alpha\|_s + (1+b)2T_0\|f\|_{T,s}.$$

We iterate the procedure N times. At the last step, we find the solution u_N defined on I_N, with

$$\|u_N\|_{C(I_N,H^s_x)} \le b^N\|\alpha\|_s + (b^N - 1)\frac{1}{b-1}2T_0\|f\|_{T,s}.$$

We define $u(t) := u_n(t)$ for $t \in I_n$, and the thesis follows, using that $b \le 2$.

(iv) If $Tc_1 \le \frac{1}{2}$, the result is given by (ii). Let $Tc_1 > \frac{1}{2}$, and fix $N \in \mathbb{N}$ such that $2Tc_1 \le N \le 4Tc_1$. Let $T_0 := T/N$, so that $\frac{1}{4} \le T_0c_1 \le \frac{1}{2}$. Split $[0, T] = I_1 \cup \cdots \cup I_N$, where $I_n := [(n-1)T_0, nT_0]$. Perform the same procedure as above. Using (A-9), and $1 + 2T_0c_1 \le 2$, by induction we get

$$\|u_n\|_{C(I_n,H^s_x)} \le 2^n\|\alpha\|_s + (2^n-1)2T_0\|f\|_{T,s} + n2^{n-1}4T_0c_s\|\alpha\|_{s_1} + [2^n(n-1)+1]4T_0c_sT_0\|f\|_{T,s_1},$$
$$\|u_n\|_{C(I_n,H^{s_1}_x)} \le 2^n\|\alpha\|_{s_1} + (2^n-1)2T_0\|f\|_{T,s_1}.$$

This implies (A-11), recalling that $T_0c_1 \le \frac{1}{2}$ and also $NT_0 = T,\ N \ge 1$. □

Lemma A.3. *There exist universal positive constants σ, δ_* with the following properties. Let $s \ge 0$, let $m \ge \frac{1}{2}$, and let $a_{14}(t,x)$, $a_{15}(t,x)$ be two functions with $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_{14}(t,x)\,dx = 0$, and let $\mathcal{L}_4 := \partial_t + m\,\partial_{xxx} + a_{14}\,\partial_x + a_{15}$. Let*

$$\delta(\mu) := \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \le \delta_*$. Let $f \in C([0,T], H^s_x)$ and $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_4 u = f, \quad u(0) = \alpha \tag{A-19}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \le C_s\big\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\big\}. \tag{A-20}$$

Proof. Following the procedure given in Section 2E, we define $\mathcal{S} := I + \gamma(t,x)\partial_x^{-1}$ (see (2-53)) with $\gamma(t,x) := -\frac{1}{3m}\partial_x^{-1}a_{14}(t,x)$. We have that u solves (A-19) if and only if $\tilde u := \mathcal{S}^{-1}u$ satisfies

$$\mathcal{L}_5\tilde u = \tilde f, \quad \tilde u(0) = \tilde\alpha,$$

where $\tilde f := \mathcal{S}^{-1}f$, $\tilde\alpha := \mathcal{S}^{-1}(0)\alpha$ and $\mathcal{L}_5 = \partial_t + m\,\partial_{xxx} + \mathcal{R}$, with

$$\mathcal{R} = \mathcal{S}^{-1}\big\{a_{15} + (a_{14}\gamma - (a_{14})_x)\pi_0 + (\mathcal{L}_4\gamma)\,\partial_x^{-1}\big\}.$$

Then the thesis follows by Lemmas A.2 and 2.7. □

Lemma A.4. *There exist universal positive constants σ, δ_* with the following properties. Let $s \ge 0$, let $m \ge \frac{1}{2}$, and let $a_{12}(t,x)$, $a_{13}(t,x)$ be two functions with $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H^{s+\sigma}_x)$, and let $\mathcal{L}_3 := \partial_t + m\,\partial_{xxx} + a_{12}\,\partial_x + a_{13}$. Let*

$$\delta(\mu) := \|a_{12}, \partial_t a_{12}, a_{13}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \le \delta_*$. Let $f \in C([0,T], H^s_x)$ and $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_3 u = f, \quad u(0) = \alpha \tag{A-21}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \le C_s\big\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\big\}. \tag{A-22}$$

Proof. Following the procedure given in Section 2D, we define $\mathcal{T}h(t,x) := h(t, x + p(t))$ (see (2-46)), with $p(t) := -\frac{1}{2\pi}\int_0^t \int_{\mathbb{T}} a_{12}(s,x)\,dx\,ds$. We have that u solves (A-21) if and only if $\tilde u := \mathcal{T}^{-1}u$ satisfies

$$\mathcal{L}_4 \tilde u = \tilde f, \quad \tilde u(0) = \alpha$$

(note that $\mathcal{T}(0)$ is the identity), where $\tilde f := \mathcal{T}^{-1} f$, and $\mathcal{L}_4 = \partial_t + m\,\partial_{xxx} + a_{14}\,\partial_x + a_{15}$, with a_{14}, a_{15} given by formula (2-48). Then the thesis follows by Lemmas A.3 and 2.6. □

Lemma A.5. *There exist universal positive constants σ, δ_* with the following properties. Let $s \ge 0$, let $m \ge \frac{1}{2}$, and let $a_8(t,x), a_9(t,x), a_{10}(t,x)$ be three functions with $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_8(t,x)\,dx = 0$, and let $\mathcal{L}_2 := \partial_t + m\,\partial_{xxx} + a_8\,\partial_{xx} + a_9\,\partial_x + a_{10}$. Let*

$$\delta(\mu) := \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \le \delta_*$. Let $f \in C([0,T], H^s_x)$ and $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_2 u = f, \quad u(0) = \alpha \tag{A-23}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \le C_s\big\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\big\}. \tag{A-24}$$

Proof. Following the procedure given in Section 2C, we define $\mathcal{M}h(t,x) := q(t,x)h(t,x)$ (see (2-37)), with $q(t,x) := \exp\{-\frac{1}{3m}(\partial_x^{-1} a_8)(t,x)\}$. We have that u solves (A-23) if and only if $\tilde u := \mathcal{M}^{-1}u$ satisfies

$$\mathcal{L}_3 \tilde u = \tilde f, \quad \tilde u(0) = \tilde \alpha,$$

where $\tilde f := \mathcal{M}^{-1} f$, $\tilde\alpha := \mathcal{M}^{-1}(0)\alpha$, and $\mathcal{L}_3 = \partial_t + m\,\partial_{xxx} + a_{12}\,\partial_x + a_{13}$, with a_{12}, a_{13} given by formula (2-39). Then the thesis follows by Lemmas A.4 and 2.5. □

Lemma A.6. *There exist universal positive constants σ, δ_* with the following properties. Let $s \ge 0$ and let $a_4(t), a_5(t,x), a_6(t,x), a_7(t,x)$ be four functions with $a_4 \in C^1([0,T], \mathbb{R})$ and $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_5(t,x)\,dx = 0$, and let $\mathcal{L}_1 := \partial_t + a_4\partial_{xxx} + a_5\partial_{xx} + a_6\partial_x + a_7$. Let*

$$\delta(\mu) := \sup_{t\in[0,T]} |a_4(t) - 1| + \sup_{t\in(0,T)} |a_4'(t)| + \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{A-25}$$

Assume $\delta(0) \le \delta_*$. Let $f \in C([0,T], H^s_x)$ and $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_1 u = f, \quad u(0) = \alpha \tag{A-26}$$

admits a unique solution $u \in C([0,T], H^s_x)$, *with*

$$\|u\|_{T,s} \le C_s\big\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\big\}. \tag{A-27}$$

Proof. Following the procedure given in Section 2B, we define $\mathcal{B}h(t,x) := h(\psi(t),x)$ (see (2-25)), with $\psi(t) := \frac{1}{m}\int_0^t a_4(s)\,ds$, where $m := \frac{1}{T}\int_0^T a_4(t)\,dt$. We have that u solves (A-26) if and only if $\tilde u := \mathcal{B}^{-1}u$ satisfies

$$\mathcal{L}_2\tilde u = \tilde f, \quad \tilde u(0) = \alpha$$

(note that $\mathcal{B}(0)$ is the identity), where $\tilde f := \mathcal{B}^{-1} f$, and $\mathcal{L}_2 = \partial_t + m\,\partial_{xxx} + a_8\,\partial_{xx} + a_9\,\partial_x + a_{10}$, with a_8, a_9, a_{10} given by formula (2-32) (see also (2-26)). Then the thesis follows by Lemma A.5 and 2.4. □

Lemma A.7. *There exist universal positive constants* σ, δ_* *with the following properties. Let* $s \ge 0$ *and let* $a_3(t,x)$, $a_2(t,x)$, $a_1(t,x)$, $a_0(t,x)$ *be four functions with* $a_3, \partial_t a_3, \partial_{tt}a_3, a_1, \partial_t a_1, a_0 \in C([0,T], H^{s+\sigma}_x)$ *and* $a_2 = c\,\partial_x a_3$ *for some* $c \in \mathbb{R}$. *Let*

$$\delta(\mu) := \|a_3, \partial_t a_3, \partial_{tt}a_3, a_1, \partial_t a_1, a_0\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{A-28}$$

Assume $\delta(0) \le \delta_*$. *Let* $\mathcal{L}_0 := \partial_t + (1+a_3)\,\partial_{xxx} + a_2\,\partial_{xx} + a_1\,\partial_x + a_0$. *Let* $f \in C([0,T], H^s_x)$ *and* $\alpha \in H^s_x$. *Then the Cauchy problem*

$$\mathcal{L}_0 u = f, \quad u(0) = \alpha \tag{A-29}$$

admits a unique solution $u \in C([0,T], H^s_x)$, *with*

$$\|u\|_{T,s} \le C_s\big\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\big\}. \tag{A-30}$$

Proof. Following the procedure given in Section 2A, we define $(\mathcal{A}h)(t,x) := h(t, x+\beta(t,x))$ (see (2-9)), with $\beta(t,x) := (\partial_x^{-1}\rho_0)(t,x)$, where ρ_0 is defined in (2-16)–(2-17). We have that u solves (A-29) if and only if $\tilde u := \mathcal{A}^{-1}u$ satisfies

$$\mathcal{L}_1\tilde u = \tilde f, \quad \tilde u(0) = \tilde\alpha,$$

where $\tilde f := \mathcal{A}^{-1} f$, $\tilde\alpha := \mathcal{A}^{-1}(0)\alpha$, and $\mathcal{L}_1 = \partial_t + a_4\,\partial_{xxx} + a_5\,\partial_{xx} + a_6\,\partial_x + a_7$, with a_4 not depending on the space variable x and with a_4, a_5, a_6, a_7 given by formula (2-14). Then the thesis follows by Lemmas A.6 and 2.3. □

Remark A.8. Consider the operators $\mathcal{L}_0, \dots, \mathcal{L}_5$ defined in Lemmas A.2–A.7. Define

$$\begin{aligned}
\mathcal{L}_0^* h &:= -\partial_t h - \partial_{xxx}[(1+a_3)h] + \partial_{xx}(a_2 h) - \partial_x(a_1 h) + a_0 h,\\
\mathcal{L}_1^* h &:= -\partial_t h - a_4\partial_{xxx}h + \partial_{xx}(a_5 h) - \partial_x(a_6 h) + a_7 h,\\
\mathcal{L}_2^* h &:= -\partial_t h - m\partial_{xxx}h + \partial_{xx}(a_8 h) - \partial_x(a_9 h) + a_{10} h,\\
\mathcal{L}_3^* h &:= -\partial_t h - m\partial_{xxx}h - \partial_x(a_{12} h) + a_{13} h,\\
\mathcal{L}_4^* h &:= -\partial_t h - m\partial_{xxx}h - \partial_x(a_{14} h) + a_{15} h,\\
\mathcal{L}_5^* h &:= -\partial_t h - m\partial_{xxx}h + \mathcal{R}^T h.
\end{aligned}$$

It is straightforward to check that Lemmas A.2–A.7 also hold when the operator $\mathcal{L}_k$ $(k = 0, \dots, 5)$ is replaced by $\mathcal{L}_k^*$. The crucial observation is that for all $k = 0, \dots, 5$ (see Remark 2.2 for the case $k = 0$) the operator $-\mathcal{L}_k^*$ has the same structure as $\mathcal{L}_k$ (one might need to worsen the constants σ since the coefficients of $-\mathcal{L}_k^*$ involve space derivatives of the coefficients of $\mathcal{L}_k$). It is also immediate to verify that the same estimates also hold for the backward Cauchy problems

$$\begin{cases} \mathcal{L}_k u = f, \\ u(T) = \alpha, \end{cases} \qquad \begin{cases} \mathcal{L}_k^* u = f, \\ u(T) = \alpha, \end{cases} \qquad k = 0, \dots, 5. \tag{A-31}$$

Appendix B: Nash–Moser theorem

In this section we prove a Nash–Moser implicit function theorem that is a modified version of the theorem in [Hörmander 1985]. With respect to that paper, here (Theorem B.1) we assume slightly stronger hypotheses on the nonlinear operator Φ and its second derivative. These hypotheses are naturally verified in applications to PDEs. We use the iteration scheme of [Hörmander 1976] (called the *discrete Nash method* by Hörmander), which is neither the Newton scheme with smoothings used in [Berti, Bolle, and Procesi 2010; Berti, Corsi, and Procesi 2015; Baldi, Berti, and Montalto 2016a], nor the scheme in [Hörmander 1985; Alinhac and Gérard 2007]. The scheme of [Hörmander 1976] is based on a telescoping series like in [Hörmander 1985], but some corrections y_n (see (B-15)) are also introduced. In this way the scheme converges directly to a solution of the equation $\Phi(u) = \Phi(0) + g$, avoiding the intermediate step in [Hörmander 1985] where the Leray–Schauder theorem is applied. This makes it possible to remove two assumptions of Hörmander's theorem [1985], which are the compact embeddings $F_b \hookrightarrow F_a$ in the codomain scale of Banach spaces $(F_a)_{a \geq 0}$, and the continuity of the approximate right inverse $\Psi(v)$ with respect to the approximate linearization point v. We point out that, unlike Theorem 2.2.2 of [Hörmander 1976], our Theorem B.1 also applies to the case of Sobolev spaces.

Let us begin with recalling the construction of "weak" spaces in [Hörmander 1985].

Let E_a, $a \geq 0$, be a decreasing family of Banach spaces with injections $E_b \hookrightarrow E_a$ of norm ≤ 1 when $b \geq a$. Set $E_\infty = \bigcap_{a \geq 0} E_a$ with the weakest topology making the injections $E_\infty \hookrightarrow E_a$ continuous. Assume that $S_\theta : E_0 \to E_\infty$ for $\theta \geq 1$ are linear operators such that, with constants C bounded when a and b are bounded,

$$\|S_\theta u\|_b \leq C \|u\|_a \qquad \text{if } b \leq a, \tag{B-1}$$

$$\|S_\theta u\|_b \leq C \theta^{b-a} \|u\|_a \qquad \text{if } a < b, \tag{B-2}$$

$$\|u - S_\theta u\|_b \leq C \theta^{b-a} \|u\|_a \qquad \text{if } a > b, \tag{B-3}$$

$$\left\| \frac{d}{d\theta} S_\theta u \right\|_b \leq C \theta^{b-a-1} \|u\|_a. \tag{B-4}$$

From (B-2)–(B-3) one can obtain the logarithmic convexity of the norms

$$\|u\|_{\lambda a + (1-\lambda) b} \leq C \|u\|_a^\lambda \|u\|_b^{1-\lambda} \qquad \text{if } 0 < \lambda < 1. \tag{B-5}$$

Consider the sequence $\{\theta_j\}_{j\in\mathbb{N}}$, with $1=\theta_0<\theta_1<\cdots\to\infty$, such that θ_{j+1}/θ_j is bounded. Set $\Delta_j:=\theta_{j+1}-\theta_j$ and

$$R_0 u := \frac{S_{\theta_1}u}{\Delta_0}, \qquad R_j u := \frac{S_{\theta_{j+1}}u - S_{\theta_j}u}{\Delta_j}, \qquad j\ge 1. \tag{B-6}$$

By (B-3) we deduce that, if $u\in E_b$ for some $b>a$, then

$$u=\sum_{j=0}^{\infty}\Delta_j R_j u \tag{B-7}$$

with convergence in E_a. Moreover, (B-4) implies that, for all b,

$$\|R_j u\|_b \le C_{a,b}\theta_j^{b-a-1}\|u\|_a. \tag{B-8}$$

Conversely, assume that $a_1<a<a_2$, that $u_j\in E_{a_2}$ and that

$$\|u_j\|_b \le M\theta_j^{b-a-1} \quad \text{if } b=a_1 \text{ or } b=a_2. \tag{B-9}$$

By (B-5) this remains true with a constant factor on the right-hand side if $a_1<b<a_2$, so that $u=\sum\Delta_j u_j$ converges in E_b if $b<a$.

Let E'_a be the set of all sums $u=\sum\Delta_j u_j$ with u_j satisfying (B-9) and introduce the norm $\|u\|'_a$ as the infimum of M over all such decompositions. It follows that $\|\cdot\|'_a$ is stronger than $\|\cdot\|_b$ if $a>b$, while (B-7) and (B-8) show that $\|\cdot\|'_a$ is weaker than $\|\cdot\|_a$. Moreover (i) the space E'_a and, up to equivalence, its norm are independent of the choice of a_1 and a_2; (ii) E'_a is defined by (B-8) for any values of b to the left and to the right of a; (iii) E'_a does not depend on the smoothing operators; (iv) in (B-3) we can replace $\|u\|_a$ by $\|u\|'_a$, namely,

$$\|u-S_\theta u\|_b \le C'_{a,b}\theta^{b-a}\|u\|'_a \quad \text{if } a>b, \tag{B-10}$$

if we take another constant $C'_{a,b}$, which may tend to ∞ as b approaches a. These four statements (i)–(iv) are proved in [Hörmander 1985].

Now let us suppose that we have another family F_a of decreasing Banach spaces with smoothing operators having the same properties as above. We use the same notation also for the smoothing operators. Unlike [Hörmander 1985], here we do not need to assume that the embedding $F_b\hookrightarrow F_a$ is compact for $b>a$.

Theorem B.1. *Let $a_1, a_2, \alpha, \beta, a_0, \mu$ be real numbers with*

$$0\le a_0\le\mu\le a_1, \quad a_1+\tfrac12\beta\le\alpha<a_1+\beta\le a_2, \quad 2\alpha<a_1+a_2. \tag{B-11}$$

Let V be a convex neighborhood of 0 in E_μ. Let Φ be a map from V to F_0 such that $\Phi: V\cap E_{a+\mu}\to F_a$ is of class C^2 for all $a\in[0,a_2-\mu]$, with

$$\|\Phi''(u)[v,w]\|_a \le C\big(\|v\|_{a+\mu}\|w\|_{a_0}+\|v\|_{a_0}\|w\|_{a+\mu}+\|u\|_{a+\mu}\|v\|_{a_0}\|w\|_{a_0}\big) \tag{B-12}$$

for all $u \in V \cap E_{a+\mu}$, $v, w \in E_{a+\mu}$. Also assume that $\Phi'(v)$ for $v \in E_\infty \cap V$ belonging to some ball $\|v\|_{a_1} \le \delta_1$ has a right inverse $\Psi(v)$ mapping F_∞ to E_{a_2}, and that

$$\|\Psi(v)g\|_a \le C\big(\|g\|_{a+\beta-\alpha} + \|g\|_0\|v\|_{a+\beta}\big) \quad \forall a \in [a_1, a_2]. \tag{B-13}$$

There exists $\delta > 0$ such that, for every $g \in F'_\beta$ in the ball $\|g\|'_\beta \le \delta$, there exists $u \in E'_\alpha$, with $\|u\|'_\alpha \le C\|g\|'_\beta$, solving $\Phi(u) = \Phi(0) + g$.

Proof. We follow the proof in [Hörmander 1985] where possible, but we use a different iteration scheme. Let $\theta_j := j + 1$, so that $\Delta_j = 1$ for all j. Let $g \in F'_\beta$ and $g_j := R_j g$. Thus

$$g = \sum_{j=0}^{\infty} g_j, \quad \|g_j\|_b \le C_b \theta_j^{b-\beta-1}\|g\|'_\beta \quad \forall b \in [0, +\infty). \tag{B-14}$$

We claim that if $\|g\|'_\beta$ is small enough, then we can define a sequence $u_j \in V \cap E_{a_2}$ with $u_0 := 0$ by the recursion formula

$$u_{j+1} := u_j + h_j, \quad v_j := S_{\theta_j} u_j, \quad h_j := \Psi(v_j)(g_j + y_j) \quad \forall j \ge 0, \tag{B-15}$$

where $y_0 := 0$,

$$y_1 := -S_{\theta_1} e_0, \qquad y_j := -S_{\theta_j} e_{j-1} - R_{j-1}\sum_{i=0}^{j-2} e_i \quad \forall j \ge 2, \tag{B-16}$$

and $e_j := e'_j + e''_j$,

$$e'_j := \Phi(u_j + h_j) - \Phi(u_j) - \Phi'(u_j)h_j, \quad e''_j := (\Phi'(u_j) - \Phi'(v_j))h_j. \tag{B-17}$$

We prove that for all $j \ge 0$,

$$\|h_j\|_a \le K_1\|g\|'_\beta\, \theta_j^{a-\alpha-1} \quad \forall a \in [a_1, a_2], \tag{B-18}$$

$$\|v_j\|_a \le K_2\|g\|'_\beta\, \theta_j^{a-\alpha} \quad \forall a \in [a_1+\beta, a_2+\beta], \tag{B-19}$$

$$\|u_j - v_j\|_a \le K_3\|g\|'_\beta\, \theta_j^{a-\alpha} \quad \forall a \in [0, a_2]. \tag{B-20}$$

For $j = 0$, (B-19) and (B-20) are trivially satisfied, and (B-18) follows from (B-14) because $h_0 = \Psi(0)g_0$ and $\theta_0 = 1$.

Now assume that (B-18), (B-19), (B-20) hold for $j = 0, \dots, k$, for some $k \ge 0$. First we prove (B-20) for $j = k+1$. Since $u_{k+1} = \sum_{j=0}^{k} h_j$, the definition of the norm of E'_α and (B-18) for $j = 0, \dots, k$ imply that $\|u_{k+1}\|'_\alpha \le K_1\|g\|'_\beta$. By (B-10) one has

$$\|u_{k+1} - v_{k+1}\|_0 \le C K_1\|g\|'_\beta \theta_{k+1}^{-\alpha}, \tag{B-21}$$

where the constant C depends on α. From now until the end of this proof we denote by C any constant (possibly different from line to line) depending only on a_1, a_2, α, β, μ, a_0, which are fixed parameters.

From (B-18) with $j = 0, \dots, k$ we get

$$\|u_{k+1}\|_a \le K_1 \|g\|'_\beta \sum_{j=0}^{k} \theta_j^{a-\alpha-1} \quad \forall a \in [a_1, a_2]. \tag{B-22}$$

We note that

$$\sum_{j=0}^{k} \theta_j^{p-1} \le \frac{2}{p} \theta_{k+1}^p \quad \forall p > 0. \tag{B-23}$$

For $a = a_2$, by (B-1) one gets $\|v_{k+1}\|_{a_2} \le C \|u_{k+1}\|_{a_2}$. Thus, using (B-23) at $p = a_2 - \alpha$,

$$\|u_{k+1} - v_{k+1}\|_{a_2} \le C \|u_{k+1}\|_{a_2} \le C K_1 \|g\|'_\beta \theta_{k+1}^{a_2-\alpha}. \tag{B-24}$$

Using (B-5) to interpolate between (B-21) and (B-24), we get (B-20) for $j = k+1$, for all $a \in [0, a_2]$, provided that $K_3 \ge C K_1$.

To prove (B-19) for $j = k+1$, we use (B-2), (B-22) and (B-23) and we get

$$\|v_{k+1}\|_a \le C \theta_{k+1}^{a-a_1-\beta} \|u_{k+1}\|_{a_1+\beta} \le C \theta_{k+1}^{a-a_1-\beta} K_1 \|g\|'_\beta \sum_{j=0}^{k} \theta_j^{a_1+\beta-\alpha-1} \le C K_1 \|g\|'_\beta \, \theta_{k+1}^{a-\alpha}$$

for all $a \in [a_1 + \beta, a_2 + \beta]$. This gives (B-19) for $j = k+1$ provided that $K_2 \ge C K_1$.

To prove (B-18) for $j = k+1$, we begin with proving that

$$\|y_{k+1}\|_b \le C K_1 (K_1 + K_3) \|g\|'^2_\beta \theta_{k+1}^{b-\beta-1} \quad \forall b \in [0, a_2 + \beta - \alpha]. \tag{B-25}$$

Since $u_j, v_j, u_j + h_j$ belong to V for all $j = 0, \dots, k$, we use Taylor's formula and (B-12) to deduce that, for $j = 0, \dots, k$ and $a \in [0, a_2 - \mu]$,

$$\begin{aligned}\|e_j\|_a \le C\big(&\|h_j\|_{a_0} \|h_j\|_{a+\mu} + \|u_j\|_{a+\mu} \|h_j\|_{a_0}^2 + \|h_j\|_{a_0} \|v_j - u_j\|_{a+\mu} \\ &+ \|h_j\|_{a+\mu} \|v_j - u_j\|_{a_0} + \|u_j\|_{a+\mu} \|h_j\|_{a_0} \|v_j - u_j\|_{a_0}\big).\end{aligned} \tag{B-26}$$

Hence at $j = k$, using (B-2) and then (B-26), we have

$$\begin{aligned}\|S_{\theta_{k+1}} e_k\|_{a_2+\beta-\alpha} &\le C \theta_{k+1}^p \|e_k\|_{a_2+\beta-\alpha-p} \\ &\le C \theta_{k+1}^p \big(\|h_k\|_{a_0} \|h_k\|_q + \|u_k\|_q \|h_k\|_{a_0}^2 + \|h_k\|_{a_0} \|v_k - u_k\|_q \\ &\qquad + \|h_k\|_q \|v_k - u_k\|_{a_0} + \|u_k\|_q \|h_k\|_{a_0} \|v_k - u_k\|_{a_0}\big),\end{aligned} \tag{B-27}$$

where $p := \max\{0, \beta - \alpha + \mu\}$ and $q := a_2 + \beta - \alpha - p + \mu$. Note that $a_2 + \beta - \alpha - p \ge 0$ because $a_2 \ge \mu$. Since $q \le a_2$, using also (B-23) we have

$$\|u_k\|_q \le \|u_k\|_{a_2} \le \sum_{j=0}^{k-1} \|h_j\|_{a_2} \le K_1 \|g\|'_\beta \sum_{j=0}^{k-1} \theta_j^{a_2-\alpha-1} \le C K_1 \|g\|'_\beta \theta_k^{a_2-\alpha}. \tag{B-28}$$

By (B-28), (B-18), (B-20), and since $a_0 \le a_1$, the bound (B-27) implies that

$$\|S_{\theta_{k+1}} e_k\|_{a_2+\beta-\alpha} \le C K_1 (K_1 + K_3) \|g\|'^2_\beta \theta_{k+1}^p (\theta_k^{a_1+q-2\alpha-1} + \theta_k^{a_2+2a_1-3\alpha-1})$$

provided that $K_1\|g\|'_\beta \le 1$. We assume that

$$K_1\|g\|'_\beta \le 1. \tag{B-29}$$

Both the exponents $a_1+q-2\alpha-1$ and $a_2+2a_1-3\alpha-1$ are $\le a_2-\alpha-1-p$ because $a_1 < \alpha$ and $a_1+\beta+\mu \le 2\alpha$. Thus

$$\|S_{\theta_{k+1}}e_k\|_{a_2+\beta-\alpha} \le CK_1(K_1+K_3)\|g\|'^2_\beta \theta_{k+1}^{a_2-\alpha-1}. \tag{B-30}$$

Now we estimate $\|S_{\theta_{k+1}}e_k\|_0$. Since $a_0, \mu \le a_1$, by (B-1) and (B-26) we get

$$\|S_{\theta_{k+1}}e_k\|_0 \le C\|e_k\|_0 \le C(1+\|u_k\|_\mu)(\|h_k\|^2_{a_1} + \|h_k\|_{a_1}\|v_k-u_k\|_{a_1}). \tag{B-31}$$

By (B-18) and (B-29),

$$\|u_k\|_\mu \le \|u_k\|_{a_1} \le \sum_{j=0}^{k-1}\|h_j\|_{a_1} \le K_1\|g\|'_\beta \sum_{j=0}^{\infty}\theta_j^{a_1-\alpha-1} = CK_1\|g\|'_\beta \le C. \tag{B-32}$$

We use (B-18), (B-20) and (B-32) in (B-31), and the bound $\theta_{k+1}^{2a_1-2\alpha-1} \le \theta_{k+1}^{-\beta-1}$, to deduce that

$$\|S_{\theta_{k+1}}e_k\|_0 \le CK_1(K_1+K_3)\|g\|'^2_\beta \theta_{k+1}^{-\beta-1}. \tag{B-33}$$

Using (B-5) to interpolate between (B-30) and (B-33), we obtain

$$\|S_{\theta_{k+1}}e_k\|_b \le CK_1(K_1+K_3)\|g\|'^2_\beta \theta_{k+1}^{b-\beta-1} \quad \forall b \in [0, a_2+\beta-\alpha]. \tag{B-34}$$

Now we estimate the other terms in y_{k+1} (see (B-16)). By (B-8), (B-26), (B-18), (B-20) and (B-23),

$$\begin{aligned}\sum_{i=0}^{k-1}\|R_k e_i\|_b &\le \sum_{i=0}^{k-1} C\theta_k^{b-a_2+\mu-1}\|e_i\|_{a_2-\mu} \\ &\le CK_1(K_1+K_3)\|g\|'^2_\beta \theta_k^{b-a_2+\mu-1}\sum_{i=0}^{k-1}\theta_i^{a_1+a_2-2\alpha-1}\end{aligned} \tag{B-35}$$

for all $b \in [0, a_2+\beta-\alpha]$. Since $a_1+a_2-2\alpha > 0$, we apply (B-23) to the last sum in (B-35). Then, recalling that $\theta_k/\theta_{k+1} \in \left[\frac{1}{2}, 1\right]$, and using the bound $a_1+\beta+\mu \le 2\alpha$, we deduce that

$$\sum_{i=0}^{k-1}\|R_k e_i\|_b \le CK_1(K_1+K_3)\|g\|'^2_\beta \theta_{k+1}^{b-\beta-1} \quad \forall b \in [0, a_2+\beta-\alpha]. \tag{B-36}$$

The sum of (B-34) and (B-36) completes the proof of (B-25).

Now we are ready to prove (B-18) at $j = k+1$. By (B-1) and (B-22) we have $\|v_{k+1}\|_{a_1} \le C\|u_{k+1}\|_{a_1} \le CK_1\|g\|'_\beta$, and we assume that $CK_1\|g\|'_\beta \le \delta_1$, so that $\Psi(v_{k+1})$ is defined. By (B-15), (B-13), (B-14), (B-25), (B-19) one has, for all $a \in [a_1, a_2]$,

$$\|h_{k+1}\|_a \le C\|g\|'_\beta\left\{1+(K_1+K_3)K_1\|g\|'_\beta\right\}\theta_{k+1}^{a-\alpha-1} \tag{B-37}$$

provided that $K_2\|g\|'_\beta \le 1$. Bound (B-37) implies (B-18) provided that $C\left\{1+(K_1+K_3)K_1\|g\|'_\beta\right\} \le K_1$.

The induction proof of (B-18), (B-19), (B-20) is complete if K_1, K_2, K_3, $\|g\|'_\beta$ satisfy

$$K_3 \geq C_0 K_1, \quad K_2 \geq C_0 K_1, \quad C_0 K_1 \|g\|'_\beta \leq 1, \quad K_2 \|g\|'_\beta \leq 1, \quad C_0\{1 + (K_1 + K_3) K_1 \|g\|'_\beta\} \leq K_1,$$

where C_0 is the largest of the constants appearing above. First we fix $K_1 \geq 2C_0$. Then we fix K_2 and K_3 larger than $C_0 K_1$, and finally we fix $\delta_0 > 0$ such that the last three inequalities hold for all $\|g\|'_\beta \leq \delta_0$. This completes the proof of (B-18), (B-19), (B-20).

Bound (B-18) implies that the sequence (u_k) converges in E_a for all $a \in [0, \alpha)$. We call u its limit. Since $u = \sum_{j=0}^\infty h_j$ and each term h_j satisfies (B-18), it follows that $u \in E'_\alpha$ and $\|u\|'_\alpha \leq K_1 \|g\|'_\beta$ by the definition of the norm in E'_α.

Finally, we prove the convergence of the Nash–Moser scheme. By (B-16) and (B-6) one proves by induction that

$$\sum_{j=0}^{k} (e_j + y_j) = e_k + r_k, \quad \text{where } \; r_k := (I - S_{\theta_k}) \sum_{j=0}^{k-1} e_j, \quad \forall k \geq 1.$$

Hence, by (B-15) and (B-17), recalling that $\Phi'(v_j)\Psi(v_j)$ is the identity map, one has

$$\Phi(u_{k+1}) - \Phi(u_0) = \sum_{j=0}^{k} [\Phi(u_{j+1}) - \Phi(u_j)] = \sum_{j=0}^{k} (e_j + g_j + y_j) = G_k + e_k + r_k,$$

where $G_k := \sum_{j=0}^k g_j$. By (B-14), $\|G_k - g\|_b \to 0$ as $k \to \infty$ for all $b \in [0, \beta)$. Let $a \in [a_1 - \mu, \alpha - \mu)$. By (B-22) and (B-29) we get $\|u_j\|_{a+\mu} \leq C$. By (B-26), (B-18) and (B-20) we deduce that

$$\|e_j\|_a \leq C K_1 (K_1 + K_3) \|g\|'^2_\beta \, \theta_j^{a_1 + a + \mu - 2\alpha - 1}. \tag{B-38}$$

Hence $\|e_k\|_a \to 0$ as $k \to \infty$ because $a_1 + a + \mu - 2\alpha < 0$, and, moreover, $\sum_{j=0}^\infty \|e_j\|_a$ converges. By (B-3) and (B-38), for all $\rho \in [0, a)$ we have

$$\|r_k\|_\rho \leq \sum_{j=0}^{k-1} \|(I - S_{\theta_k}) e_j\|_\rho \leq C \sum_{j=0}^{k-1} \theta_k^{\rho - a} \|e_j\|_a \leq C \theta_k^{\rho - a}, \tag{B-39}$$

so that $\|r_k\|_\rho \to 0$ as $k \to \infty$. We have proved that $\|\Phi(u_k) - \Phi(u_0) - g\|_\rho \to 0$ as $k \to \infty$ for all ρ in the interval $0 \leq \rho < \min\{\alpha - \mu, \beta\}$. Since $u_k \to u$ in E_a for all $a \in [0, \alpha)$, it follows that $\Phi(u_k) \to \Phi(u)$ in F_b for all $b \in [0, \alpha - \mu)$. $\square$

Appendix C: Tame estimates

In this appendix we recall classical tame estimates for products, compositions of functions and changes of variables which are repeatedly used in the paper. Recall the notation (1-6) for functions $u(x)$, $x \in \mathbb{T}$, in the Sobolev space $H^s := H^s(\mathbb{T}, \mathbb{R})$.

Lemma C.1. *Let s_0, s_1, s_2, s denote nonnegative real numbers, with $s_0 > \frac{1}{2}$. There exist positive constants C_s, $s \geq s_0$, with the following properties.*

- (embedding and algebra) *For all* $u, v \in H^{s_0}$,

$$\|u\|_{L^\infty} \le C_{s_0}\|u\|_{s_0}, \qquad \|uv\|_{s_0} \le C_{s_0}\|u\|_{s_0}\|v\|_{s_0}. \tag{C-1}$$

- (interpolation) *For* $0 \le s_1 \le s \le s_2$ *and* $s = \lambda s_1 + (1-\lambda)s_2$, *for all* $u \in H^{s_2}$,

$$\|u\|_s \le \|u\|_{s_1}^{\lambda}\|u\|_{s_2}^{1-\lambda}. \tag{C-2}$$

- (tame product) *For* $s \ge s_0$, *for all* $u, v \in H^s$,

$$\|uv\|_s \le C_{s_0}\|u\|_s\|v\|_{s_0} + C_s\|u\|_{s_0}\|v\|_s, \tag{C-3}$$

and for $s \in [0, s_0]$, *for all* $u \in H^{s_0}$ *and* $v \in H^s$,

$$\|uv\|_s \le C_{s_0}\|u\|_{s_0}\|v\|_s. \tag{C-4}$$

Proof. The lemma can be proved by using Fourier series and the Hölder inequality. Otherwise, for (C-2) see, e.g., [Alinhac and Gérard 2007, p. 82] or [Moser 1966, p. 269]; for (C-3) adapt [Berti, Bolle, and Procesi 2010, Appendix] or [Alinhac and Gérard 2007, p. 84]. For (C-4) use the bound $\sum_{j\in\mathbb{Z}}\langle n\rangle^{2s}\langle j\rangle^{-2s}\langle n-j\rangle^{-2s_0} \le C_{s_0}$ for all $n \in \mathbb{Z}$, all $0 \le s \le s_0$, which can be proved by splitting the two cases $2|j| \le |n|$ and $2|j| > |n|$. □

A function $f : \mathbb{T} \times B \to \mathbb{R}$, where $B := \{y \in \mathbb{R}^{p+1} : |y| < R\}$, induces the composition operator

$$\tilde f(u)(x) := f\big(x, u(x), u'(x), u''(x), \dots, u^{(p)}(x)\big), \tag{C-5}$$

where $u^{(k)}(x)$ denotes the k-th derivative of $u(x)$. Let B_p be a ball in $W^{p,\infty}(\mathbb{T}, \mathbb{R})$ such that, if $u \in B_p$, then the vector $\big(u(x), u'(x), \dots, u^{(p)}(x)\big)$ belongs to B for all $x \in \mathbb{T}$.

Lemma C.2 (composition of functions). *Assume* $f \in C^r(\mathbb{T} \times B)$. *Then, for all* $u \in H^{s+p} \cap B_p$, $s \in [0, r]$, *the composition operator* (C-5) *is well-defined and*

$$\|\tilde f(u)\|_s \le C\|f\|_{C^r}(\|u\|_{s+p} + 1),$$

where C *depends on* r, p. *If, in addition,* $f \in C^{r+2}$, *then, for* $u, h \in H^{s+p}$ *with* $u, u+h \in B_p$, *one has*

$$\|\tilde f(u+h) - \tilde f(u)\|_s \le C\|f\|_{C^{r+1}}\big(\|h\|_{s+p} + \|h\|_{W^{p,\infty}}\|u\|_{s+p}\big),$$
$$\|\tilde f(u+h) - \tilde f(u) - \tilde f'(u)[h]\|_s \le C\|f\|_{C^{r+2}}\|h\|_{W^{p,\infty}}\big(\|h\|_{s+p} + \|h\|_{W^{p,\infty}}\|u\|_{s+p}\big).$$

Proof. For $s \in \mathbb{N}$ see [Moser 1966, pp. 272–275] and [Rabinowitz 1967, Lemma 7, pp. 202–203]. For $s \notin \mathbb{N}$ see [Alinhac and Gérard 2007, Proposition 2.2, p. 87]. □

Lemma C.3 (change of variable). *Let* $p \in W^{s,\infty}(\mathbb{T}, \mathbb{R})$, $s \ge 1$, *with* $\|p\|_{W^{1,\infty}} \le \frac{1}{2}$. *Let* $f(x) = x + p(x)$. *Then* f *is invertible, its inverse is* $f^{-1}(y) = g(y) = y + q(y)$, *where* q *is* 2π*-periodic,* $q \in W^{s,\infty}(\mathbb{T}, \mathbb{R})$, *and* $\|q\|_{W^{s,\infty}} \le C\|p\|_{W^{s,\infty}}$, *where* C *depends on* d, s.

Moreover, if $u \in H^s(\mathbb{T}, \mathbb{R})$, *then* $u \circ f(x) = u(x + p(x))$ *also belongs to* H^s, *and*

$$\|u \circ f\|_s + \|u \circ g\|_s \le C(\|u\|_s + \|p\|_{W^{s,\infty}}\|u\|_1). \tag{C-6}$$

Proof. For $s \in \mathbb{N}$ see, e.g., [Baldi 2013, Lemma B.4], where this lemma is proved by adapting [Hamilton 1982, Lemma 2.3.6, p. 149]. For $s \notin \mathbb{N}$ the lemma can be proved by studying the conjugate of the pseudodifferential operator $|D_x|^s$ by a change of variable, either by Egorov's theorem, see [Taylor 1981, Chapter VIII, Section 1, p. 150] and [Alazard, Baldi, and Han-Kwan 2015, Appendix C, Section C.1], or by an asymptotic formula, see [Alinhac and Gérard 2007, Proposition 7.1, p. 37]. □

Remark C.4. For time-dependent functions $u(t,x)$, $u \in C([0,T], H^s(\mathbb{T},\mathbb{R}))$, all the estimates of the present appendix hold with $\|u\|_s$ replaced by $\|u\|_{T,s} := \sup_{t\in[0,T]} \|u(t)\|_s$. □

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