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ANISOTROPIC ORNSTEIN NONINEQUALITIES

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We investigate the existence of a priori estimates for differential operators in the L^1 norm: for anisotropic homogeneous differential operators $T_1, \ldots, T_\ell$, we study the conditions under which the inequality

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true. Properties of homogeneous rank-one convex functions play the major role in the subject. We generalize the notions of quasi- and rank-one convexity to fit the anisotropic situation. We also discuss a similar problem for martingale transforms and provide various conjectures.

1. Introduction

In his seminal paper, Ornstein [1962] proved the following: let $\{T_j\}_{j=1}^{\ell}$ be homogeneous differential operators of the same order in d variables (with constant coefficients); if the inequality

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j. Here and in what follows "$a \lesssim b$" means "there exists a constant c such that $a \leqslant cb$ uniformly"; the meaning of the word "uniformly" will be clear from the context. For example, in the statement above, the constant should be uniform with respect to all functions f. The aim of the present paper is to extend this theorem to the case where the differential operators are anisotropic homogeneous; see also [Kazaniecki and Wojciechowski 2014], where partial progress in this direction was obtained by a simple Riesz product technique.

To formulate the results, we have to introduce a few notions. Each differential polynomial $P(\partial)$ in d variables has a Newton diagram which matches a set of integral points in $\mathbb{R}^d$ to each such polynomial. The monomial $a\partial_1^{m_1}\partial_2^{m_2}\cdots\partial_d^{m_d}$ corresponds to the point $m = (m_1, m_2, \ldots, m_d)$; for an arbitrary polynomial, its Newton diagram is the union of the Newton diagrams of its monomials.

Let Λ be an affine hyperplane in $\mathbb{R}^d$ that intersects all the positive semiaxes. We call such a plane a *pattern of homogeneity*. We say that a differential polynomial is homogeneous with respect to Λ (or simply Λ-homogeneous) if its Newton diagram lies on Λ.

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Conjecture 1. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$ and let $\{T_j\}_{j=1}^{\ell}$ be a collection of Λ-homogeneous differential operators. If the inequality*

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j.

This conjecture may seem to be a simple generalization of Ornstein's theorem. We warn the reader that sometimes the anisotropic character of homogeneity brings new difficulties to inequalities for differential operators (the main one being the lack of geometric tools such as the isoperimetric inequality, or the coarea formula, etc.). For example, the classical embedding

$$W_1^1(\mathbb{R}^d) \hookrightarrow L_{d/(d-1)}$$

due to Gagliardo and Nirenberg was generalized to the anisotropic case only in [Solonnikov 1972] and additionally considered in [Kolyada 1993]; if one deals with similar embeddings for vector fields, the isotropic case was successfully considered in [Van Schaftingen 2013] (see also the survey [Van Schaftingen 2014]), but there is almost no progress for the anisotropic case (however, see [Kislyakov et al. 2013; 2015]).

The method we use to attack the conjecture differs from that of Ornstein (though there are some similarities). However, it is not new. It was noticed in [Conti et al. 2005] that Ornstein's theorem is related to the behavior of certain rank-one convex functions (for some special operators this link had already been known; see [Iwaniec 2002]). The case $d = 2$ was considered there. As for the general case of Ornstein's (isotropic) theorem, its proof via rank-one convexity was announced in [Kirchheim and Kristensen 2011] (and the proof is now available in the very recent preprint [Kirchheim and Kristensen 2016]). In a sense, we follow the plan suggested in [Kirchheim and Kristensen 2011]. However, the notions of quasiconvexity, rank-one convexity and others should be properly adjusted to the anisotropic world; we have not seen such an adjustment anywhere. For all these notions in the classical setting of the first gradient, their relationship with each other, properties, etc., we refer the reader to the book [Dacorogna 2008]. There are certain problems in the general anisotropic case that are not present in the classical setting. For example, the existence of the elementary laminate is not quite clear; at least, the classical reasoning does not work. Quasiconvexity still implies the rank-one convexity, but this requires a new proof. The approach of rank-one convexity reduces Conjecture 1 to a certain geometric problem about separately convex functions (Theorem 14) that is covered by Theorem 1 in [Kirchheim and Kristensen 2011] (Theorem 1.1 in [Kirchheim and Kristensen 2016]). We give a simple proof of this fact, which is the second advantage of our paper (though our proof does not give the more advanced Theorem 1 of [Kirchheim and Kristensen 2011]). We did not know of the preprint [Kirchheim and Kristensen 2016] until shortly before the publication of the present text, and did our work independently. Discussion with the authors of this preprint has shown that though the spirit of our approach in the geometric part is similar to theirs, the presentation and details appear to be different.

We will prove a particular case of Conjecture 1, which still seems to be rather general (in particular, it covers the classical isotropic case).

Theorem 2. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$ and let $\{T_j\}_{j=1}^{\ell}$ be Λ-homogeneous differential operators. Suppose all the monomials present in the T_j have the same parity of degree. If the inequality*

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)} \tag{1}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j.

We note that the differential operators here are not necessarily scalar; i.e., one can prove the same theorem for the case where operators act on vector fields. It is one of the advantages of the general rank-one convexity approach. However, to facilitate the notation, we work on the scalar case.

We outline the structure of the paper. We begin with restating inequality (1) as an extremal problem described by a certain Bellman function (if inequality (1) holds, then the corresponding Bellman function is nonnegative). We also study the properties of our Bellman function (they are gathered in Theorem 6), the most important of which is the quasiconvexity. All this material constitutes Section 2. It turns out that quasiconvexity leads to a softer, but easier to work with, property of rank-one convexity. The proof of this fact is given in Section 3; see Theorem 9. So, the Bellman function in question is rank-one convex. In Section 4, we prove that rank-one convex functions homogeneous of order one are nonnegative, which gives us Theorem 2. In fact, it suffices to show a similar principle for separately convex functions on $\mathbb{R}^d$, which is formalized in Theorem 14. This theorem is purely convex geometric. Finally, we discuss related questions in Section 5.

2. Bellman function and its properties

Inequality (1) can be rewritten as

$$\inf_{\varphi \in C_0^\infty([0,1]^d)} \left(\sum_{j=2}^{\ell} \|T_j \varphi\|_{L_1(\mathbb{R}^d)} - c\|T_1 \varphi\|_{L_1(\mathbb{R}^d)} \right) = 0, \tag{2}$$

where c is a sufficiently small positive constant.

Definition 3. Suppose ∂^α, $\alpha \in A$, are all the partial derivatives that are present in the T_j (thus A is a subset of $\Lambda \cap \mathbb{Z}^d$). Consider the Hilbert space E with an orthonormal basis e_α indexed with the set A. For each function φ and each point x, we have a mapping

$$[0,1]^d \ni x \mapsto \nabla[\varphi](x) = \sum_{\alpha \in A} \partial^\alpha[\varphi](x) e_\alpha \in E.$$

We call the function $\nabla[\varphi]$ the generalized gradient of φ.

The operator $\nabla[\,\cdot\,]$ is an analogue of the usual gradient suitable for our problem.

Example 4. Let $T_j = \partial_{x_j}$ for $j = 1, \dots, d$. In this case the generalized gradient turns out to be the usual gradient on the Euclidean space $\mathbb{R}^d$.

Example 5. Let us take the differential operators

$$T_1[\varphi] = \partial^{(2,0,1)}[\varphi] - \partial^{(0,3,1)}[\varphi], \quad T_2[\varphi] = \partial^{(4,0,0)}\varphi, \quad T_3[\varphi] = \partial^{(0,6,0)}[\varphi], \quad T_4[\varphi] = \partial^{(0,0,2)}[\varphi]. \quad (3)$$

We can list all the partial derivatives present in the operators:

$$A = \left\{\partial^{(0,0,2)}, \partial^{(0,6,0)}, \partial^{(4,0,0)}, \partial^{(0,3,1)}, \partial^{(2,0,1)}\right\}.$$

All the operators T_j are Λ-homogeneous, where $\Lambda = \{x \in R^3 : \langle x, (3, 2, 6)\rangle = 12\}$. In this case the generalized gradient is of the form

$$\nabla[\varphi] = \left(\partial^{(0,0,2)}[\varphi], \partial^{(0,6,0)}[\varphi], \partial^{(4,0,0)}[\varphi], \partial^{(0,3,1)}[\varphi], \partial^{(2,0,1)}[\varphi]\right) \in \mathbb{R}^5.$$

We also consider the function $V : E \to \mathbb{R}$ given by the rule

$$V(e) = \left(\sum_{j=2}^{\ell} |\tilde{T}_j e| - c|\tilde{T}_1 e|\right), \quad (4)$$

where the $\tilde{T}_j$ are the linear functionals on E such that $\tilde{T}_j(e) = \sum_A c_{\alpha,j} e_\alpha$ if $T_j = \sum_A c_{\alpha,j}\partial^\alpha$. With this bit of abstract linear algebra, we rewrite formula (2) as

$$\inf_{\varphi \in C_0^\infty([0,1]^d)} \int_{[0,1]^d} V(\nabla[\varphi](x))\, dx = 0.$$

The main idea is to consider a perturbation of this extremal problem, i.e., the function $\boldsymbol{B} : E \to \mathbb{R}$ given by the formula

$$\boldsymbol{B}(e) = \inf_{\varphi \in C_0^\infty([0,1]^d)} \int_{[0,1]^d} V(e + \nabla[\varphi](x))\, dx. \quad (5)$$

Theorem 6. *Suppose that inequality (2) holds true. Then, the function $\boldsymbol{B}$ possesses the properties listed below.*

(1) *It satisfies the inequalities $-\|e\| \lesssim \boldsymbol{B}(e) \lesssim \|e\|$ and $\boldsymbol{B} \leqslant V$.*

(2) *It is one-homogeneous; i.e., $\boldsymbol{B}(\lambda e) = |\lambda| \boldsymbol{B}(e)$.*

(3) *It is a Lipschitz function.*

(4) *It is a generalized quasiconvex function; i.e., for any $\varphi \in C_0^\infty([0, 1]^d)$ and any $e \in E$ the inequality*

$$\boldsymbol{B}(e) \leqslant \int_{[0,1]^d} \boldsymbol{B}(e + \nabla[\varphi](x))\, dx \quad (6)$$

holds true.

Proof. (1) We get the upper estimates on the function $\boldsymbol{B}$ by plugging $\varphi \equiv 0$ in the formula for it:

$$\boldsymbol{B}(e) \leqslant \int_{[0,1]^d} V(e + \nabla[\varphi]) = V(e) \lesssim \|e\|.$$

We obtain the lower bounds on the function $\boldsymbol{B}$ from inequality (2) and the triangle inequality:

$$\begin{aligned}\int_{[0,1]^d}\Big(\sum_{j=2}^{\ell}|\tilde{T}_j(e+\nabla[\varphi])|-c|\tilde{T}_1(e+\nabla[\varphi])|\Big) &\geqslant \int_{[0,1]^d}\Big(\sum_{j=2}^{\ell}|\tilde{T}_j(e+\nabla[\varphi])|-c|\tilde{T}_1(\nabla[\varphi])|-c|\tilde{T}_1 e|\Big)\\ &\geqslant \int_{[0,1]^d}\Big(\sum_{j=2}^{\ell}|\tilde{T}_j(e+\nabla[\varphi])|-\sum_{j=2}^{\ell}|\tilde{T}_j(\nabla[\varphi])|-c|\tilde{T}_1 e|\Big)\\ &= \int_{[0,1]^d}\Big(\sum_{j=2}^{\ell}\big(|\tilde{T}_j(e+\nabla[\varphi])|-|\tilde{T}_j(\nabla[\varphi])|\big)-c|\tilde{T}_1 e|\Big)\\ &\geqslant -\sum_{j=2}^{\ell}|\tilde{T}_j e|-c|\tilde{T}_1 e|,\end{aligned}$$

where $\varphi \in C_0^\infty([0,1]^d)$ is an arbitrary function. We take infimum of the above inequality over all admissible φ:

$$-\|e\| \lesssim -\sum_{j=2}^{\ell}|\tilde{T}_j e| - c|\tilde{T}_1 e| \leqslant \boldsymbol{B}(e).$$

(2) Since V is a one-homogeneous function, the following equality holds for every $\lambda \neq 0$:

$$\boldsymbol{B}(\lambda e) = \inf_{\varphi\in C_0^\infty([0,1]^d)} \int_{[0,1]^d} V(\lambda e+\nabla[\varphi]) = \inf_{\varphi\in C_0^\infty([0,1]^d)} \int_{[0,1]^d} |\lambda|\, V(e+\nabla[\lambda^{-1}\varphi]).$$

We know that $\lambda^{-1}C_0^\infty([0,1]^d) = C_0^\infty([0,1]^d)$ for every $\lambda \neq 0$; therefore

$$\boldsymbol{B}(\lambda e) = \inf_{\varphi\in C_0^\infty([0,1]^d)} \int_{[0,1]^d} |\lambda|\, V(e+\nabla[\lambda^{-1}\varphi]) = |\lambda| \inf_{\varphi\in C_0^\infty([0,1]^d)} \int_{[0,1]^d} V(e+\nabla[\varphi]) = |\lambda|\boldsymbol{B}(e).$$

(3) In order to get the Lipschitz continuity of $\boldsymbol{B}$, we rewrite the formula for it:

$$\text{for all } e \in E, \qquad \boldsymbol{B}(e) = \inf_{\varphi\in C_0^\infty([0,1]^d)} V_\varphi(e),$$

where

$$V_\varphi(e) = \int_{[0,1]^d} V(e+\nabla[\varphi](x))\,dx.$$

It follows from the Lipschitz continuity of V that every function V_φ is a Lipschitz function with the Lipschitz constant bounded by L, where L is the Lipschitz constant of the function V. For every two points $v_1, v_2 \in E$, we can find a sequence of functions V_{φ_n} such that $\boldsymbol{B}(v_j) = \inf_{n\in\mathbb{N}} V_{\varphi_n}(v_j)$ for $j \in \{1,2\}$. We define

$$f_k(e) = \min_{n=1,2,\ldots,k} V_{\varphi_n}(e).$$

For every $k \in \mathbb{N}$ the function f_k is the Lipschitz function with the Lipschitz constant bounded by L. Hence

$$|\boldsymbol{B}(v_1) - \boldsymbol{B}(v_2)| = \lim_{k\to\infty} |f_k(v_1) - f_k(v_2)| \leqslant L\|v_1 - v_2\|.$$

(4) Before we prove the generalized quasiconvexity of this function, we need to introduce some notation. We know that all $\alpha \in A$ have common pattern of homogeneity Λ; thus we can find a vector $\gamma \in \mathbb{N}^d$ and a number $k \in \mathbb{N}$ such that $\langle \alpha, \gamma \rangle = k$ for every $\alpha \in A$.

For every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$, we define

$$x_\lambda = (\lambda^{\gamma_1} x_1, \lambda^{\gamma_2} x_2, \ldots, \lambda^{\gamma_d} x_d).$$

For every $\lambda \in \mathbb{N}$ we define the partition of the unit cube $[0, 1]^d$ into small parallelepipeds:

$$Q_y = y + \prod_{j=1}^{d} [0, \lambda^{-\gamma_j}] \quad \text{for every } y \in Y,$$

where

$$Y = \left\{ y \in [0, 1]^d : y = \left(\frac{\kappa_1}{\lambda^{\gamma_1}}, \frac{\kappa_2}{\lambda^{\gamma_2}}, \ldots, \frac{\kappa_d}{\lambda^{\gamma_d}} \right) \text{ for } \kappa_j \in \mathbb{N} \cup \{0\} \text{ and } \kappa_j < \lambda^{\gamma_j} \right\}.$$

Here Y is the set of "leftmost lowest" vertices of the parallelepipeds Q_y. The parallelepipeds Q_y are disjoint up to sets of measure zero and $\bigcup_{y \in Y} Q_y = [0, 1]^d$. Let us fix $\varphi \in C_0^\infty([0, 1]^d)$. Since $\nabla[\varphi]$ is a uniformly continuous function on $[0, 1]^d$ and the diameter of the parallelepipeds Q_y tends to zero uniformly with the growth of λ, we can choose λ sufficiently large to obtain

$$\text{for all } y \in Y, \text{ for all } z, v \in Q_y, \quad \big|\nabla[\varphi](z) - \nabla[\varphi](v)\big| \leqslant \frac{\varepsilon}{L}, \tag{7}$$

where L is the Lipschitz constant of the function V. Let $\{\psi_y\}_{y \in Y}$ be a family of functions in $C_0^\infty([0, 1]^d)$. For these functions, we use the rescaling

$$\psi_{y,\lambda}(x) = \lambda^{-k} \psi_y((x - y)_\lambda).$$

Let us observe that the rescaling $(x - y)_\lambda$ transforms the cube $[0, 1]^d$ into Q_y; thus $\operatorname{supp} \psi_{y,\lambda} \subset Q_y$. Moreover, we know that

$$\partial^\alpha[\psi_{y,\lambda}](x) = \lambda^{-k} \lambda^{\left(\sum_{j=1}^{d} \alpha_j \gamma_j\right)} \partial^\alpha[\psi_y]((x - y)_\lambda) = \partial^\alpha[\psi_y]((x - y)_\lambda)$$

for every $\alpha \in A$. By (5), we have

$$\boldsymbol{B}(e) \leqslant \int_{[0,1]^d} V\Big(e + \sum_{y \in Y} \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\Big)\, dx = \sum_{y \in Y} \int_{Q_y} V\big(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\big)\, dx.$$

We assumed that (7) holds; therefore, for arbitrary $v_y \in Q_y$ we have the estimate

$$\begin{aligned} \int_{Q_y} V\big(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\big)\, dx &\leqslant \int_{Q_y} V\big(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](v_y)\big)\, dx + \varepsilon |Q_y| \\ &= \int_{Q_y} V\big(e + \nabla[\psi_y]((x - y)_\lambda) + \nabla[\varphi](v_y)\big)\, dx + \varepsilon |Q_y|. \end{aligned}$$

Since $\lambda^{-\left(\sum_{j=1}^{d} \gamma_j\right)} = |Q_y|$, we have

$$\int_{Q_y} V\big(e + \nabla[\psi_y]((x - y)_\lambda) + \nabla[\varphi](v_y)\big)\, dx = |Q_y| \int_{[0,1]^d} V\big(e + \nabla[\psi_y](z) + \nabla[\varphi](v_y)\big)\, dz$$

for $z = (x-y)_\lambda$. Now for every $y \in Y$ and $v_y \in Q_y$ we can choose ψ_y such that

$$\int_{[0,1]^d} V\big(e + \nabla[\psi_y](z) + \nabla[\varphi](v_y)\big)\,dz \leqslant \boldsymbol{B}(e + \nabla[\varphi](v_y)) + \varepsilon$$

(this choice depends on v_y, however, we treat v_y as of a fixed parameter). We obtain

$$\boldsymbol{B}(e) \leqslant \sum_{y\in Y} |Q_y|\,\boldsymbol{B}(e + \nabla[\varphi](v_y)) + 2\varepsilon$$

from the above inequalities. We take mean integrals of this inequality over each cube Q_y with respect to v_y, which gives us

$$\boldsymbol{B}(e) \leqslant \sum_{y\in Y} \int_{Q_y} \boldsymbol{B}(e + \nabla[\varphi](v_y))\,dv_y + 2\varepsilon = \int_{[0,1]^d} \boldsymbol{B}(e + \nabla[\varphi](x))\,dx + 2\varepsilon.$$

Since ε was an arbitrary positive number, we have proved the generalized quasiconvexity of $\boldsymbol{B}$. □

The proof of the fourth point seems very similar to the standard *Bellman induction step* (see [Nazarov et al. 2001; Osękowski 2012; Stolyarov and Zatitskiy 2016; Volberg 2011] or any other paper on the Bellman function method in probability or harmonic analysis); moreover, the function $\boldsymbol{B}$ itself is, in a sense, a Bellman function and inequality (6) is a Bellman inequality. We suspect that this "similarity" should be more well-studied.

3. Rank-one convexity

Inequality (6) looks like a convexity inequality. Sometimes that is really the case.

Definition 7. We call a vector $e_x \in E$ a generalized rank-one vector if it is of the form

$$\sum_{\alpha\in A} i^{|\alpha|+|\alpha_0|} x^\alpha e_\alpha, \qquad x \in \mathbb{R}^d,\ \alpha_0 \in A.$$

Remark 8. In Theorem 2, we only consider the case where every $\alpha \in A$ has the same parity as the other elements of A. Therefore, $i^{|\alpha|+|\alpha_0|} \in \mathbb{R}$ for every $\alpha_0, \alpha \in A$. Hence the coefficients of the generalized rank-one vector are real.

Theorem 9. *The function $\boldsymbol{B}$ is a generalized rank-one convex function; i.e., it is convex in the directions of generalized rank-one vectors.*

To prove the theorem, we need two auxiliary lemmas.

Lemma 10. *For every $x \in \mathbb{R}^d$ and every $\varepsilon, \delta > 0$, there exists a function $l_{x,\varepsilon,\delta} \in C_0^\infty([0,1]^d)$ and a set $B \subset [0,1]^d$ such that the following hold.*

(1) $\|\nabla[l_{x,\varepsilon,\delta}]\| \leqslant \|e_x\| + \varepsilon$.

(2) $|B| \geqslant 1 - \delta$.

(3) *The function* $\nabla[l_{x,\varepsilon,\delta}]|_B$ *with respect to the measure* $\mu = |B|^{-1}\,dx|_B$ *is equimeasurable with the function* $\cos(2\pi t)e_x,\ t \in [0,1]$; *i.e.*,

$$\mu\big(\{\nabla[l_{x,\varepsilon,\delta}] \in W\}\big) = \Big|\Big\{t \in [0,1] : \cos(2\pi t)e_x \in W\Big\}\Big|$$

for every Borel set W *in* E.

Proof. For a given $x \in \mathbb{R}^d$ we take the same γ and k as in the proof of the fourth point of Theorem 6. We consider the function

$$l_{x,\varepsilon,\delta}(\xi) = t^{-k}\cos\Big(\sum_{j=1}^{d} t^{\gamma_j} x_j \xi_j\Big)\Phi(\xi),$$

where Φ is the smooth hat function:

$$\Phi(\xi) = \begin{cases} 1, & \xi \in [2\delta', 1-2\delta']^d, \\ 0, & \xi \in [0,1]^d \setminus [\delta', 1-\delta']^d, \\ \Theta(\xi) \in [0,1], & \text{otherwise} \end{cases}$$

for δ' sufficiently small (in particular, we need $2(2\delta')^d < \delta$). Similarly to the fourth point of Theorem 6, we define the set of proper parallelepipeds

$$Y_t = \Big\{Q\colon\ Q = (k_j v_j)_{j=1,\dots,d} + \prod_{j=1}^{d}[0, w_j],\ \ k_j \in \{1\} \cup \Big\{k_j \in \mathbb{N}\colon k_j < \frac{t^{\gamma_j} x_j}{2\pi} - 1\Big\}\Big\},$$

where $v_j = w_j = 2\pi t^{-\gamma_j} x_j^{-1}$ if $x_j \neq 0$ and $v_j = \delta',\ w_j = (1-2\delta')$ otherwise. For any δ', we can choose t to be so large that

$$\Big|\bigcup_{\substack{Q \in Y_t \\ Q \subset [2\delta', 1-2\delta']^d}} Q\Big| \geqslant 1 - \delta.$$

We put B to be this union, i.e., the union of the parallelepipeds Q from the family Y_t that belong to $[2\delta', 1-2\delta']^d$ entirely.

If t is sufficiently large, then for every $\beta \in \mathbb{N}^d$ satisfying $0 \leqslant \langle \beta, \gamma\rangle < k$, we have

$$\sup_{\xi \in [0,1]^d} |t^{-1}\partial^\beta[\Phi](\xi)| \leqslant \varepsilon'. \tag{8}$$

For any $\beta \in \mathbb{N}^d$,

$$\partial^\beta\Big[\cos\Big(\sum_{j=1}^{d} t^{\gamma_j} x_j \xi_j\Big)\Big] = t^{\langle\beta,\gamma\rangle} x^\beta \partial^\beta[\cos]\Big(\sum_{j=1}^{d} t^{\gamma_j} x_j \xi_j\Big).$$

Since all $\alpha \in A$ have the same parity, we either have $\partial^\alpha[\cos](\xi) = (-1)^{|\alpha|/2}\cos(\xi)$ for every $\alpha \in A$ or $\partial^\alpha[\cos](x) = (-1)^{(|\alpha|+1)/2}\sin(\xi)$ for every $\alpha \in A$. Without loss of generality we may assume $2 \,\big|\, |\alpha|$, because the functions sine and cosine are equimeasurable on their periodic domains. Therefore, for every

$\xi \in [0, 1]^d$ and $\alpha \in A$ we have

$$\partial^\alpha[l_{\xi,\varepsilon,\delta}](\xi) = \Phi(\xi)\partial^\alpha\Big[t^{-k}\cos\Big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Big)\Big] + \sum_{\substack{\alpha'+\beta=\alpha\\ \beta\neq(0,0,\dots,0)}} c_{\alpha',\beta}\, t^{-k}\partial^{\alpha'}\Big[\cos\Big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j(x)\Big)\Big]\partial^\beta[\Phi]$$

$$= \Phi(\xi)x^\alpha\partial^\alpha[\cos]\Big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Big) + \sum_{\substack{\alpha'+\beta=\alpha\\ \beta\neq(0,0,\dots,0)}} c_{\alpha',\beta}\, t^{\langle\alpha',\gamma\rangle-k}\partial^{\alpha'}[\cos]\Big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j(x)\Big)\partial^\beta[\Phi]$$

$$= (-1)^{|\alpha|/2}x^\alpha\cos\Big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Big) + \text{error}, \tag{9}$$

where the coefficients $c_{\alpha',\beta}$ come from the Leibniz formula. The error is $O(\varepsilon')$ in absolute value by (8) and is equal to zero on the set $[2\delta', 1-2\delta']^d$ (because the function Φ is constant there). For every $\xi \in [0, 1]^d$ we have

$$\nabla[l_{\xi,\varepsilon,\delta}](\xi) = \sum_{\alpha\in A}\partial^\alpha[l_{\xi,\varepsilon,\delta}](\xi)e_\alpha = \sum_{\alpha\in A}\Big((-1)^{|\alpha|/2}x^\alpha\cos\Big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Big) + \text{error}\Big)e_\alpha$$

$$= e_x\cos\Big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Big) + \text{error}.$$

Thus, for every $\xi \in [0, 1]^d$ and ε' sufficiently small, we obtain

$$\|\nabla[l_{\xi,\varepsilon,\delta}](\xi)\| \leqslant \|e_x\| + \|\text{error}\| \leqslant \|e_x\| + \varepsilon.$$

Since the error is equal to zero on the set $[2\delta', 1-2\delta']^d$, it follows from (9) that for every $\xi \in B$ we have

$$\nabla[l_{\xi,\varepsilon,\delta}](\xi) = \cos\Big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Big)e_x.$$

We note that the function $\cos\big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\big)e_x$ restricted to any $Q \in Y_t$ is equimeasurable (with respect to the measure $dx/|Q|$ on Q) with the function $\cos(2\pi t)e_x$, $t\in[0, 1]$ (one can verify this fact using an appropriate dilation). Since B is a union of several parallelepipeds Q, the same holds with Q replaced by B. $\square$

Lemma 11. *Suppose $v : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that*

$$v(x) \leqslant \int_0^1 v(x + \lambda\cos(2\pi t))\,dt \tag{10}$$

for any $x, \lambda \in \mathbb{R}$. Then v is convex.

Proof. We are going to verify that v is convex as a distribution, or equivalently, that the distribution v'' is nonnegative. For that, we multiply inequality (10) by a positive function $\varphi \in C_0^\infty(\mathbb{R})$. Since v is a

Lipschitz function, we can integrate it over $\mathbb{R}$:

$$\begin{aligned}\int_{\mathbb{R}} v(x)\varphi(x)\,dx &\leqslant \int_{\mathbb{R}}\int_0^1 v(x+\lambda\cos(2\pi t))\varphi(x)\,dt\,dx = \int_{\mathbb{R}}\int_0^1 v(x)\varphi(x-\lambda\cos(2\pi t))\,dt\,dx\\ &= \int_{\mathbb{R}} v(x)\int_0^1 \big(\varphi(x)-\lambda\cos(2\pi t)\varphi'(x)+\tfrac{1}{2}\lambda^2\cos^2(2\pi t)\varphi''(x)+o(\lambda^2)\big)\,dt\,dx\\ &= \int_{\mathbb{R}}\bigg(v(x)\varphi(x)+v(x)\varphi''(x)\tfrac{1}{2}\lambda^2\bigg(\int_0^1\cos^2(2\pi t)\bigg)+o(\lambda^2)\bigg)\,dx.\end{aligned}$$

Therefore,

$$0\leqslant \frac{1}{2}\bigg(\int_0^1 \cos^2(2\pi t)\,dt\bigg)\int_{\mathbb{R}} v(x)\varphi''(x)\,dx+\frac{o(\lambda^2)}{\lambda^2}.$$

Letting $\lambda\to 0$, we show that v'' as a distribution satisfies $v''(\varphi)\geqslant 0$ for all $\varphi\in C_0^\infty(\mathbb{R})$ and $\varphi\geqslant 0$. From the Schwartz theorem it follows that v'' is a nonnegative measure of locally finite variation. Thus v' is an increasing function and therefore v is convex. □

Proof of Theorem 9. The function $\boldsymbol{B}$ is a generalized quasiconvex function; hence it satisfies (6) for every $\varphi\in C_0^\infty([0,1]^d)$. Let us fix $x\in\mathbb{R}^d$, $\lambda\in\mathbb{R}$. We plug $\lambda l_{x,\varepsilon,\delta}$ into (6). We get (for every $e\in E$)

$$\begin{aligned}\boldsymbol{B}(e) &\leqslant \int_{[0,1]^d}\boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}]) = \int_B \boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}])+\int_{[0,1]^d\setminus B}\boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}])\\ &\leqslant \int_B \boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}])+O\big(\lambda(\|e\|+\|e_x\|+\varepsilon)\delta\big)\end{aligned}$$

by Lemma 10. Since $\nabla[l_{x,\varepsilon,\delta}]|_B$ is equimeasurable (B equipped with the measure $dx/|B|$) with $\cos(2\pi t)e_x$,

$$\int_B \boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}])\,\frac{dx}{|B|}=\int_{[0,1]}\boldsymbol{B}(e+\lambda\cos(2\pi t)e_x)\,dt.$$

Therefore,

$$\boldsymbol{B}(e)\leqslant |B|\int_{[0,1]}\boldsymbol{B}(e+\lambda\cos(2\pi t)e_x)\,dt+O\big(\lambda(\|e\|+\|e_x\|+\varepsilon)\delta\big).$$

Since for $\delta\to 0$, we have $|B|\to 1$, we get

$$\boldsymbol{B}(e)\leqslant \int_{[0,1]}\boldsymbol{B}(e+\lambda\cos(2\pi t)e_x)\,dt. \tag{11}$$

For a fixed $e\in E$, consider the function $\mathbb{R}\ni s\mapsto \boldsymbol{B}(e+se_x)$. By (11),

$$\boldsymbol{B}(e+se_x)\leqslant \int_{[0,1]}\boldsymbol{B}(e+se_x+\lambda\cos(2\pi t)e_x)\,dt.$$

Thus, by Lemma 11, the function $\mathbb{R}\ni s\mapsto \boldsymbol{B}(e+se_x)$ is convex (one simply applies the lemma to this function). Since $e\in E$ and $x\in\mathbb{R}^d$, $\lambda\in\mathbb{R}$ were arbitrary, this proves the generalized rank-one convexity of the function $\boldsymbol{B}$. □

4. Separately convex homogeneous functions and proof of Theorem 2

Lemma 12. *Generalized rank-one vectors span E.*

Proof. Since E is a finite-dimensional Hilbert space, every functional on E is of the form $\varphi^*(\,\cdot\,) = \big\langle \sum_{\alpha\in A} a_\alpha e_\alpha, \cdot\big\rangle$. We get

$$\varphi^*(e_x) = \sum_{\alpha\in A} a_\alpha x^\alpha i^{|\alpha|+|\alpha_0|}$$

for every $x \in \mathbb{R}^d$. If E is not a span of generalized rank-one vectors, then there exists a nontrivial φ^* such that

$$0 = \varphi^*(e_x) = \sum_{\alpha\in A} a_\alpha x^\alpha i^{|\alpha|+|\alpha_0|}$$

for every $x \in \mathbb{R}^d$. However, x^α are linearly independent monomials. Therefore, $a_\alpha = 0$ for every $\alpha \in A$. Hence $\varphi^* \equiv 0$ and the generalized rank-one vectors span E. □

We recall that our aim was to show that T_1 is a linear combination of the other T_j. By comparing the kernels of the $\tilde{T}_j$, it is equivalent to the fact that $V \geqslant 0$ everywhere. By the evident inequality $B \leqslant V$, it suffices to prove that B is nonnegative. By Lemma 12 and Theorem 9, this will follow from the theorem below. Hence it suffices to prove Theorem 14 to get Theorem 2.

Definition 13. A function $F : \mathbb{R}^d \to \mathbb{R}$ is separately convex if it is convex with respect to each variable.

Theorem 14. *A function $F : \mathbb{R}^d \to \mathbb{R}$ that is separately convex and homogeneous of order one is nonnegative.*

Before moving to the proof, we cite [Dacorogna 2008, Theorem 2.31], which says that a separately convex function is continuous. This fact will be implicitly used several times in the reasoning below.

Proof. We proceed by induction. Suppose the statement of the theorem holds true for the dimension $d-1$. We then prove it for the dimension d. Construct the function $G : \mathbb{R}^{d-1} \to \mathbb{R}$ by the formula

$$G(x) = F(x, 1), \quad x \in \mathbb{R}^{d-1}.$$

This function is separately convex and convex with respect to radius, i.e., for every $x \in \mathbb{R}^{d-1}$ the function $\mathbb{R}_+ \ni t \mapsto G(tx)$ is a convex function. Indeed, the function F is one-homogeneous and separately convex; thus for $t, r > 0$ and $\tau \in (0, 1)$ we have

$$\begin{aligned}
\tau G(tx) + (1-\tau)G(rx) &= \tau F(tx, 1) + (1-\tau)F(rx, 1) \\
&= (\tau t + (1-\tau)r)\left(\frac{\tau t F\big(x, \frac{1}{t}\big) + (1-\tau)rF\big(x, \frac{1}{r}\big)}{\tau t + (1-\tau)r}\right) \\
&\geqslant (\tau t + (1-\tau)r)\, F\left(x, \frac{1}{\tau t + (1-\tau)r}\right) \\
&= F\big((\tau t + (1-\tau)r)x, 1\big) = G\big((\tau t + (1-\tau)r)x\big).
\end{aligned}$$

We claim that for each $x \in \mathbb{R}^{d-1}$, the function $\mathbb{R} \ni t \mapsto G(tx)$ is convex. Since the function G is continuous, it suffices to prove that $G(tx) + G(-tx) \geqslant G(0)$ for all $t \in \mathbb{R}$. Consider another function V:

$$V(x) = \lim_{t \to 0+} \frac{G(tx) + G(-tx) - 2G(0)}{t}, \quad x \in \mathbb{R}^{d-1}.$$

The limit exists due to the convexity with respect to radius. This function V is one-homogeneous and separately convex. However, it may have attained the value $-\infty$. Fortunately, this is not the case. If there exists $x \in \mathbb{R}^d$ such that $V(x) = -\infty$ then

$$2V(0, x_2, \dots, x_d) \leqslant V(x_1, \dots, x_d) + V(-x_1, \dots, x_d) = -\infty.$$

Therefore $V(0, x_2, \dots, x_d) = -\infty$. We repeat the above reasoning with $x_2, \dots, x_d$ instead of x_1 and we get $V(0) = -\infty$, but from the definition of V we know that

$$V(0) = \lim_{t \to 0+} \frac{G(0) + G(0) - 2G(0)}{t} = 0.$$

Hence $V(x)$ is finite for every $x \in \mathbb{R}^{d-1}$. Thus, by the induction hypothesis, V is nonnegative. So, $\mathbb{R} \ni t \mapsto G(tx)$ is a convex function.

By symmetry, $G(x) + G(-x) \geqslant 2F(x, 0)$. On the other hand, $\lim_{t \to \pm\infty} G(tx)/t = F(x, 0)$. So, the convexity of $t \mapsto G(tx)$ gives the inequality $|G(x) - G(-x)| \leqslant 2F(x, 0)$. Adding these two inequalities, we get $F(x, 1) \geqslant 0$. □

Proof of Theorem 2. Assume that inequality (1) holds. Then, by Theorem 6, the function $\boldsymbol{B}$ given by (5) is Lipschitz, one-homogeneous, generalized quasiconvex, and satisfies the inequality $B \leqslant V$, where the function V is given by formula (4). Then, by Theorem 9, $\boldsymbol{B}$ is a generalized rank-one convex function.

Let $e \in E$ be an arbitrary point. By Lemma 12, e is a linear combination of generalized rank-one vectors $e_{x_1}, e_{x_2}, \dots, e_{x_k}$. We may assume that they are linearly independent. Consider the function $F : \mathbb{R}^k \to \mathbb{R}$ given by the rule

$$F(z_1, z_2, \dots, z_k) = \boldsymbol{B}(z_1 e_{x_1} + z_2 e_{x_2} + \dots + z_k e_{x_k}).$$

By the generalized rank-one convexity of $\boldsymbol{B}$, we see that F is separately convex. It is also one-homogeneous; thus $F \geqslant 0$ by Theorem 14. Therefore, $\boldsymbol{B}(e)$ is also nonnegative for arbitrary $e \in E$.

Since $\boldsymbol{B} \geqslant 0$, we have $V \geqslant 0$. In such a case, it follows from formula (4) that $\operatorname{Ker} \tilde{T}_1 \supset \bigcap_{j=2}^{\ell} \operatorname{Ker} \tilde{T}_j$. Therefore, T_1 is a linear combination of the other T_j. □

5. Related questions

5.1. ***Towards Conjecture 1.*** The following statement plays the same role in view of Conjecture 1 as Theorem 14 plays in the proof of Theorem 2.

Conjecture 15. *Let $F : \mathbb{R}^{2d} \to \mathbb{R}$ be a Lipschitz homogeneous function of order one. Suppose that for any $j = 1, 2, \dots, d$ the function F is subharmonic with respect to the variables (x_j, x_{j+d}). Then, F is nonnegative.*

Indeed, plugging the cosine function into (6) as we did in the proof of Theorem 9 leads to "subharmonicity"[1] of the function $\boldsymbol{B}$ in the directions of projections of a generalized rank-one vector onto subspaces generated by odd and even monomials in A correspondingly. Therefore, Conjecture 1 follows from Conjecture 15.

We are not able to prove Conjecture 15. However, we know the following: in the case $d = 1$, the function F is not only nonnegative, but, in fact, convex (i.e., a one-homogeneous subharmonic function is convex). On the other hand, there is not much hope for simplifications: a subharmonic one-homogeneous function in $\mathbb{R}^3$ (and thus in $\mathbb{R}^d$, $d \geqslant 3$) can attain negative values; e.g., in $\mathbb{R}^4$ one may take the function

$$\frac{x_1^2+x_2^2+x_3^2-x_4^2}{\sqrt{x_1^2+x_2^2+x_3^2}}.$$

There are also reasons that differ from the ones discussed in the present paper that may "break" inequality (1). One of them is a certain geometric property of the spaces generated by the operators T_j. Not stating any general theorem or conjecture, we treat an instructive example. Consider the noninequality

$$\|\partial_1^2\partial_2 f\|_{L_1} \lesssim \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}. \tag{12}$$

Conjecture 1 suggests that it cannot be true. We will disprove it on the torus $\mathbb{T}^2$ and leave to the reader the rigorous formulation and proof of the corresponding transference principle, whose heuristic form is "inequalities of the sort (1) are true or untrue simultaneously on the torus and the Euclidean space". Consider two anisotropic homogeneous Sobolev spaces W_1 and W_2, which are obtained from the set of trigonometric polynomials by completion and factorization over the null-space with respect to the seminorms

$$\|f\|_{W_1} = \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}, \quad \|f\|_{W_2} = \|\partial_1^2\partial_2 f\|_{L_1} + \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}.$$

If inequality (12) holds true, then these two spaces are, in fact, equal (the identity operator is a Banach-space isomorphism between these spaces). However, it follows from the results of [Pełczyński and Wojciechowski 1992] (see [Wojciechowski 1991; 1993] as well) that W_2 has a complemented translation-invariant Hilbert subspace,[2] whereas W_1 does not, a contradiction.

Martingale transforms. Let $S = \{S_n\}_n$, $n \in \{0\} \cup \mathbb{N}$, be an increasing filtration of finite algebras on the standard probability space. We suppose that it differentiates L_1 (i.e., for any $f \in L_1(\Omega)$ the sequence $\mathbb{E}(f \mid S_n)$ tends to f almost surely). We will be working with martingales adapted to this filtration.

Definition 16. Let $\alpha = \{\alpha_n\}_n$ be a bounded sequence. The linear operator

$$T_\alpha[f] = \sum_{j=1}^{\infty} \alpha_{j-1}(f_j - f_{j-1}), \quad f = \{f_n\}_n \text{ is an } L_1 \text{ martingale},$$

is called a martingale transform.

[1] The "subharmonicity" means that $D\boldsymbol{B} \geqslant 0$ as a distribution, where D is an elliptic symmetric differential operator of second order (with constant real coefficients); one can then pass to usual subharmonicity by an appropriate change of variable.

[2] That means that there exists a subspace $X \subset W_2$ such that $g \in X$ whenever $g(\cdot + t) \in X$, $t \in \mathbb{T}^2$, X is isomorphic to an infinite-dimensional Hilbert space, and there exists a continuous projector $P : W_2 \to X$.

Our definition is not as general as the usual one, and we refer the reader to the book [Osękowski 2012] for the information about such operators. We only mention that martingale transforms serve as a probabilistic analogue for the Calderón–Zygmund operators. Here is the probabilistic version of Conjecture 1.

Conjecture 17. *Suppose $\alpha^1, \alpha^2, \dots, \alpha^\ell$ are bounded sequences. Suppose that the algebras S_n uniformly grow; i.e., there exists $\gamma < 1$ such that each atom a of S_n is split in S_{n+1} into atoms of probability not greater than $\gamma|a|$ each. The inequality*

$$\|T_{\alpha^1} f\|_{L_1} \lesssim \sum_{j=2}^{\ell} \|T_{\alpha^j} f\|_{L_1} \tag{13}$$

holds for any martingale f adapted to $\{S_n\}_n$ if and only if α^1 is a sum of a linear combination of the α^j and an ℓ_1 sequence.

We do not know whether the condition of uniform growth fits this conjecture. Anyway, it is clear that one should require some condition of this sort (otherwise one may take $S_n = S_{n+1} = \dots = S_{n+k}$ very often and lose all the control of the sequences α^j on these time intervals). Again, we are not able to prove the conjecture in the full generality, but will deal with an important particular case.

Theorem 18. *Suppose $\alpha^1, \alpha^2, \dots, \alpha^\ell$ to be bounded periodic sequences. The inequality*

$$\|T_{\alpha^1} f\|_{L_1} \lesssim \sum_{j=2}^{\ell} \|T_{\alpha^j} f\|_{L_1}$$

holds if and only if α^1 is a linear combination of the other α^j.

Proof. To avoid technicalities, we will be working with finite martingales (denote the class of such martingales by $\mathcal{M}$). The general case can be derived by stopping time. Assume that inequality (13) holds true. Consider the Bellman function $\boldsymbol{B} : \mathbb{R}^\ell \to \mathbb{R}$ given by the formula

$$\boldsymbol{B}(x) = \inf_{f \in \mathcal{M}} \left(\sum_{j=2}^{\ell} \left\| x_j + T_{\alpha^j}[f] \right\|_{L_1} - c \left\| x_1 + T_{\alpha^1}[f] \right\|_{L_1} \right).$$

It is easy to verify that this function is one-homogeneous and Lipschitz. Moreover, $\boldsymbol{B}$ is convex in the direction of $(\alpha_n^1, \alpha_n^2, \dots, \alpha_n^\ell)$ for each n (by the assumption of periodicity, there is only a finite number of these vectors); the proof of this assertion is a simplification of Theorem 9 (here we do not have to make additional approximations; however, see [Stolyarov and Zatitskiy 2016, Lemma 2.17] for a very similar reasoning). Thus, by Theorem 14, $\boldsymbol{B}$ is nonnegative on the span of $\{(\alpha_n^1, \alpha_n^2, \dots, \alpha_n^\ell)\}_n$. Since $\boldsymbol{B}(x) \leqslant \sum_{j \geqslant 2} |x_j| - c|x_1|$, the aforementioned span does not contain the x_1-axis. Therefore, α^1 is a linear combination of the other α^j. □

Case $p > 1$. Inequality (1) may become valid provided one replaces the L_1 norm with the L_p one, $1 < p < \infty$. Let c_p be the best possible constant in the inequality

$$\|T_1 f\|_{L_p(\mathbb{R}^d)}^p \leqslant c_p \sum_{j=2}^{\ell} \|T_j f\|_{L_p(\mathbb{R}^d)}^p. \tag{14}$$

It is interesting to compute the asymptotics of c_p as $p \to 1$. Some particular cases have been considered in [Berkson et al. 2001]; we also refer the reader there for a discussion of similar questions.

Conjecture 19. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$ and let $\{T_j\}_{j=1}^{\ell}$ be a collection of Λ-homogeneous differential operators. If T_1 cannot be expressed as a linear combination of the other T_j, then $c_p \gtrsim \frac{1}{p-1}$.*

The conjecture claims that if there is no continuity at the endpoint, then the inequality behaves at least as if it had a weak type $(1, 1)$ there (it is also interesting to study when there is a weak type $(1, 1)$ indeed). First, we note that this question is interesting even when there are only two polynomials. Second, this is only a bound from below for c_p. Even in the case of two polynomials, c_p can be as big as $(p-1)^{1-d}$ (and thus the endpoint inequality may not be of weak type $(1, 1)$, at least when $d \geqslant 3$); see [Berkson et al. 2001] for the example.

Conjecture 19 will follow from the corresponding geometric statement in the spirit of Theorem 14.

Conjecture 20. *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a separately convex p-homogeneous function (i.e., $F(\lambda x) = |\lambda|^p F(x)$). Suppose $F(x) \leqslant |x|^p$. Then, $F(x) \gtrsim (1-p)|x|^p$.*

Conjecture 19 is derived from Conjecture 20 in the same way as Theorem 2 is derived from Theorem 14: one considers the Bellman function (5) with the function V given by the formula

$$V(e) = \left(c_p \sum_{j=2}^{\ell} |\tilde{T}_j e|^p - |\tilde{T}_1 e|^p\right),$$

proves its generalized quasiconvexity, which leads to the generalized rank-one convexity, and then uses Conjecture 20 to estimate c_p from below.

It is not difficult to verify the case $d = 2$ of Conjecture 20. Therefore, there exists a C_0^∞-function f_p such that

$$(p-1)\|\partial_1\partial_2 f_p\|_{L_p(\mathbb{R}^2)} \gtrsim \left(\|\partial_1^2 f_p\|_{L_p(\mathbb{R}^2)} + \|\partial_2^2 f_p\|_{L_p(\mathbb{R}^2)}\right).$$

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