

ANALYSIS & PDE

Volume 10 No. 2 2017

MARCELO F. DE ALMEIDA AND LUCAS C. F. FERREIRA

**TIME-WEIGHTED ESTIMATES IN LORENTZ SPACES AND
SELF-SIMILARITY FOR WAVE EQUATIONS WITH SINGULAR
POTENTIALS**

TIME-WEIGHTED ESTIMATES IN LORENTZ SPACES AND SELF-SIMILARITY FOR WAVE EQUATIONS WITH SINGULAR POTENTIALS

MARCELO F. DE ALMEIDA AND LUCAS C. F. FERREIRA

We show time-weighted estimates in Lorentz spaces for the linear wave equation with singular potential. As a consequence, assuming radial symmetry on initial data and potentials, we obtain well-posedness of global solutions in critical weak- L^p spaces for semilinear wave equations. In particular, we can consider the Hardy potential $V(x) = c|x|^{-2}$ for small $|c|$. Self-similar solutions are obtained for potentials and initial data with the right homogeneity. Our approach relies on performing estimates in the predual of weak- L^p , i.e., the Lorentz space $L^{(p',1)}$.

1. Introduction

We are concerned with the linear wave equation with potential

$$\begin{cases} \square u + Vu = f(x, t), & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ \vec{u}(0) = (u(0, x), \partial_t u(0, x)) = (0, 0), & x \in \mathbb{R}^n, \end{cases} \quad (1-1)$$

and the semilinear wave equation

$$\begin{cases} \square u + Vu = \mu|u|^{p-1}u, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ \vec{u}(0) = (u_0, u_1), & x \in \mathbb{R}^n, \end{cases} \quad (1-2)$$

where $\square = \partial_t^2 - \Delta_x$, $n \geq 5$ odd, $\mu \in \{+1, -1\}$ (focusing or defocusing case) and $p > (n^2 + n - 4)/(n(n - 3))$. The problems (1-1) and (1-2) are addressed in the radial setting.

The semilinear wave equation (1-2) with $V = 0$ has three notions of critical nonlinearity, namely the Strauss critical power $p = p_{\text{str}}$, conformal critical power $p = p_{\text{conf}}$ and energy critical power $p = p_e$. The former p_{str} is the positive root of

$$(n - 1)p^2 - (n + 1)p - 2 = 0.$$

Strauss [1981] conjectured about the existence for $p > p_{\text{str}}$ or nonexistence for $1 < p \leq p_{\text{str}}$ of global solutions for (1-2) with small compact support initial data. The conjecture of Strauss has a nice history (see, e.g., [Wang and Yu 2012]) and was completed by [Yordanov and Zhang 2006; Zhou 2007] (see also [Lai and Zhou 2014]). The conformal power p_{conf} is linked to the conformal symmetry map

$$u(x, t) \mapsto u_{\text{conf}}(x, t) = (t^2 - |x|^2)^{-\frac{n-1}{2}} u\left(\frac{x}{t^2 - |x|^2}, \frac{t}{t^2 - |x|^2}\right) \quad \text{for } |x| < |t|.$$

Ferreira was supported by FAPESP-SP and CNPQ, Brazil .

MSC2010: primary 35L05, 35L71, 35L15, 35A01, 35B06; secondary 35C06, 42B35.

Keywords: wave equations, singular potentials, self-similarity, radial symmetry, Lorentz spaces.

More precisely, u_{conf} solves (1-2) with $V = 0$ if u does and $p = p_{\text{conf}} = (n + 3)/(n - 1)$ for $n \geq 2$. The power $p_e = (n + 2)/(n - 2)$ ($p_e = \infty$ if $n = 2$) is connected to the scaling invariance of the conserved energy. In fact, for $p = p_e$ and $V = 0$, the conserved energy

$$E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla_x u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^n} |\partial_t u|^2 dx + \frac{\mu}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx$$

is invariant by the scaling map

$$u(x, t) \mapsto u_\gamma(x, t) := \gamma^{\frac{2}{p-1}} u(\gamma x, \gamma t), \quad \gamma > 0, \quad (1-3)$$

namely

$$E(u_\gamma, \partial_t u_\gamma) = \gamma^{\frac{4}{p-1} + 2 - n} E(u, \partial_t u) = E(u, \partial_t u).$$

We refer the reader to the classical papers [Grillakis 1990; 1992; Shatah and Struwe 1993; Struwe 1999] for results about solutions with finite energy.

A solution is called self-similar when it is invariant by (1-3), that is, $u(x, t) = u_\gamma(x, t)$. For a homogeneous function V of degree -2 , equation (1-2) presents the same scaling as in the case $V = 0$. Taking $t = 0$, the map (1-3) induces the scaling for the initial data:

$$(u_0(x), u_1(x)) \mapsto (\gamma^{\frac{2}{p-1}} u_0(\gamma x), \gamma^{\frac{p+1}{p-1}} u_1(\gamma x)). \quad (1-4)$$

In other words, self-similar solutions of (1-2) are associated to initial data u_0 and u_1 homogeneous of degrees $-\frac{2}{p-1}$ and $-\frac{p+1}{p-1}$, respectively, that is, homogeneous functions of the form

$$u_0(x) = \varepsilon c_1 |x|^{-\frac{2}{p-1}} \quad \text{and} \quad u_1(x) = \varepsilon c_2 |x|^{-\frac{p+1}{p-1}}, \quad (1-5)$$

where $c_1, c_2 \in \mathbb{R}$ and $\varepsilon > 0$.

For $V = 0$, there are a number of results about self-similar solutions in different frameworks. The first work is due to Kavian and Weissler [1990], where the authors proved the nonexistence of radially symmetric self-similar solutions with finite energy $E(u, \partial_t u)$ for $n \geq 3$ and $p_e \leq p < \infty$. Working in the infinite energy space of all Bochner-measurable functions $u : (0, \infty) \rightarrow L^r(\mathbb{R}^n)$ such that

$$\sup_{t>0} t^\beta \|u(\cdot, t)\|_{L^r(\mathbb{R}^n)} < \infty, \quad (1-6)$$

Pecher [2000a] showed the existence of self-similar solutions for $\varepsilon > 0$ in (1-5) sufficiently small by considering $n = 3$ and $p_1 < p \leq p_{\text{conf}}$, where p_1 is the larger positive root of

$$(n^2 - n)p^2 - (n^2 + 3n - 2)p + 2 = 0.$$

The parameters $\beta > 0$ and $r > 2$ are taken in such a way that the norm (1-6) is scaling invariant. The approach in [Pecher 2000a] is based on $L^q - L^r$ dispersive estimates for the wave group

$$\omega(t) = (-\Delta)^{-\frac{1}{2}} \sin(t(-\Delta)^{\frac{1}{2}}). \quad (1-7)$$

In fact, the case of nonradial homogeneous data also was considered in [Pecher 2000a]. Moreover, replacing L^r by suitable homogeneous Sobolev spaces $\dot{H}^{k,l}$ with $k > 0$, the upper condition $p \leq p_{\text{conf}}$

was removed by him. Still for $n = 3$, Pecher [2000b] obtained self-similar solutions (not necessarily radial) in the range $p_{\text{str}} < p < p_{\text{conf}}$ and showed that the lower bound p_{str} is sharp in the sense that in general no nontrivial self-similar solution exists even in the radial case when $p \leq p_{\text{str}}$. Unlike [Pecher 2000a], the paper [Pecher 2000b] developed pointwise estimates related to the weights $|x| \pm t$ and a norm due to F. John and did not employ L^p , Sobolev or Besov spaces. Hidano [2002] complemented these results by showing scattering and existence of self-similar solutions for (1-2) when $n = 2, 3$ and $p_{\text{str}} < p < p_{\text{conf}}$. The result of [Pecher 2000a] was proved to be true for $n = 2, 3, 4, 5$ by Ribaud and Youssefi [2002], recovering in particular $n = 2, 3$. Moreover, for $n \geq 6$ they considered $p \in (p_1, p_{\text{conf}}] \cup [2, \infty)$ or $p \in (p_1, p_{\text{conf}}] \cup (p_2, \infty)$, where p_2 is the larger positive root of

$$2(n + 1)p^2 - (n^2 + 3n + 4)p + (n^2 + 5n + 2) = 0.$$

Note that $p_{\text{str}} < p_1 < p_{\text{conf}} < p_2$ for all dimensions in which these parameters are defined.

The weighted Strichartz estimate in $L^{(r,\infty)}(\mathbb{R}_+^{1+n})$

$$\| |t^2 - |x|^2|^a u \|_{L^{(r,\infty)}(\mathbb{R}_+^{1+n})} \leq C \| |t^2 - |x|^2|^b f \|_{L^{(r',\infty)}(\mathbb{R}_+^{1+n})} \tag{1-8}$$

was obtained by Kato and Ozawa [2003] for f radially symmetric in x -variables, $2 < r < 2(n + 1)/(n - 1)$ and suitable powers a, b . By using (1-8) and assuming $n \geq 3$ odd, they proved existence and uniqueness of radially symmetric self-similar solutions for (1-2) with initial data (1-5) provided that $p_{\text{str}} < p < p_{\text{conf}}$ and $\varepsilon > 0$ is small enough. In [Kato and Ozawa 2004], they extended their results to the case $n \geq 2$ even. By employing spherical harmonics and Sobolev spaces over the unit sphere, the condition of radial symmetry on u and f was removed in [Kato et al. 2007] for $2 \leq n \leq 5$. In the case $p \in \mathbb{N}$, $p > p_{\text{conf}}$ and $V = 0$, Planchon [2000] showed global well-posedness and existence of self-similar solutions for (1-2) in $L^\infty((0, \infty); \dot{B}_{2,\infty}^{s_p})$ for small data $(u_0, u_1) \in \dot{B}_{2,\infty}^{s_p} \times \dot{B}_{2,\infty}^{s_p-1}$ with $s_p = \frac{n}{2} - \frac{2}{p-1}$. Notice that the above results do not contradict the nonexistence result in [Kavian and Weissler 1990] because the obtained self-similar solutions have infinite energy.

Wave equations with singular potential arise in the study of stability of stationary solutions for a number of systems of PDEs, for example, wave-Schrödinger and Maxwell-Schrödinger ones (see, e.g., [D’Ancona and Pierfelice 2005]). Unlike the case $p > p_{\text{str}}$ and $V = 0$, where no blow-up occurs for (1-2), Strauss and Tsutaya [1997] proved blow-up of solutions when $n = 3$, $p > 1$ and $V \in C^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ decays like $c/|x|^{(2-\varepsilon)}$ as $|x| \rightarrow \infty$ for $0 < \varepsilon < 2$. Also, they showed global existence for $\varepsilon < 0$, $p > p_{\text{str}}$ and

$$\left\| (1 + |x|)^{2-\varepsilon} \sum_{|\alpha| \leq 2} |\partial_x^\alpha V(x)| \right\|_{L^\infty}$$

small enough. Still considering small, smooth and rapidly decaying potentials, Yajima [1995] obtained $L^p - L^q$ dispersive estimates for the linear wave equation (1-1). The borderline case $\varepsilon = 0$ corresponds to V homogeneous of degree $\sigma = -2$. In this case, the perturbation Vu has the same scaling of Δu and cannot be dealt with as a simple perturbation of lower order because it does not belong to the Kato class

$$\mathcal{K} = \left\{ V \in L^1_{\text{loc}} : \|V\|_{\mathcal{K}} = \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^{2-n} |V(y)| dy < \infty, n \geq 3 \right\}, \tag{1-9}$$

where $\|\cdot\|_{\mathcal{K}}$ is called the global Kato norm. Taking $n = 3$, $\vec{u}(0) = (0, u_1)$ and $f = 0$ in (1-1), Georgiev and Visciglia [2003] proved the dispersive estimate

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{n-1}{2}} \|u_1\|_{\dot{B}_{1,1}^1(\mathbb{R}^n)} \quad (1-10)$$

for potentials V that are Hölder continuous in $\mathbb{R}^3 \setminus \{0\}$ satisfying

$$0 \leq V(x) \leq \frac{C}{|x|^{2-\varepsilon} + |x|^{2+\varepsilon}} \quad \text{for all } x \in \mathbb{R}^3.$$

D'ancona and Pierfelice [2005] improved the class of potentials to nonresonant $V \in \mathcal{K}$ and obtained, in particular, the estimate (1-10) for $V \in L^{\frac{n}{2}-\delta} \cap L^{\frac{n}{2}+\delta} \subset L^{(\frac{n}{2}, 1)} \subset \mathcal{K}$ with small $\delta > 0$. Planchon et al. [2003a] proved (1-10) in the radial case for the critical potential $V(x) = c/|x|^2$ with $c \geq 0$. In the same work, they also proved a modified version of (1-10) for negative potentials $-((n-2)/2)^2 < c < 0$. Moreover, they showed that the classical $L^\infty - L^1$ estimate

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{n-1}{2}} \|(-\Delta)^{\frac{n-1}{4}} u_1\|_{L^1(\mathbb{R}^n)}$$

does not hold when $c < 0$ and $V(x) = c/|x|^2$. In particular, this estimate is false for general $V \in L^{(\frac{n}{2}, \infty)}$.

Burq et al. [2003] considered Strichartz estimates for (1-1) and showed

$$\|(-\Delta)^\sigma u\|_{L_t^p L_x^q} \leq C(\|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-1}}) \quad (1-11)$$

for σ, p, q, γ satisfying suitable conditions. See also [Planchon et al. 2003b] for the radial case and [Burq et al. 2004] for a more general class of potentials satisfying $V \in C^1(\mathbb{R}^n \setminus \{0\})$ and $\sup_{x \in \mathbb{R}^n} |x|^2 |V(x)| < \infty$, among some other conditions. Using (1-11), for $n \geq 2$, $p \geq p_{\text{conf}}$, $s_p = \frac{n}{2} - \frac{2}{p-1}$ and

$$\sqrt{c + \frac{(n-2)^2}{4}} > \frac{n-2}{2} - \frac{2}{p-1} + \max\left\{\frac{1}{2p}, \frac{2}{(n+1)(p-1)}\right\},$$

the authors of [Burq et al. 2003] also showed global well-posedness of (1-2) provided that $(u_0, u_1) \in H^{s_p} \times H^{s_p-1}$ is small enough. This result has been extended to the range

$$1 + \frac{4n}{(n-1)(n+1)} < p < p_{\text{conf}}$$

in [Miao et al. 2013] for $n \geq 3$ and small radial initial data.

In this paper we obtain estimates for solutions of (1-1) in weak- L^r ($L^{(r, \infty)}$) spaces for the case of small radial singular potentials $V \in L^{(\frac{n}{2}, \infty)}$. Examples of those are Hardy potentials $V = c|x|^{-2}$ with $|c|$ small enough (see Remark 3.2(II)). More precisely, for certain conditions on r, s , we prove the estimate

$$\|u\|_{L^\infty(\mathbb{R}; L^{(r, \infty)})} \leq \frac{K}{1 - KC_0 \|V\|_{L^{(n/2, \infty)}}} \|f\|_{L^\infty(\mathbb{R}; L^{(s, \infty)})}, \quad (1-12)$$

where C_0 and K are positive constants and V, f and u are radially symmetric in the x -variable. In our results, the potential V can have indefinite sign. Notice that taking $V = 0$ in (1-12), one also obtains, in particular, an estimate for the linear wave equation $\square u = f$. The estimate (1-12) can be regarded as an endpoint-inhomogeneous Strichartz-type estimate in weak- L^p spaces, specifically, from $L_t^1 L_x^{(l_2, \infty)}$ to

$L_t^{m_1} L_x^{(m_2, \infty)}$ in the case $(l_1, m_1) = (\infty, \infty)$, which is important because it corresponds to the natural persistence space in existence results. Even when $V = 0$, notice that (1-12) cannot be obtained as a consequence of the inhomogeneous Strichartz estimates by Keel and Tao [1998] and Taggart [2010].

In order to obtain (1-12), we need to show a time-weighted estimate for the wave group (1-7) in the predual of $L^{(r, \infty)}$, i.e., the Lorentz space $L^{(r', 1)}$, which is of its own interest (see Lemma 4.1). As will be seen below, this estimate will lead us to global well-posedness results for (1-2) in critical spaces. We denote the solution of the Cauchy problem for the linear homogeneous wave equation by

$$L_{\tilde{u}(0)}(t) = \omega(t)u_1 + \dot{\omega}(t)u_0, \quad \text{where } \dot{\omega}(t) = \partial_t \omega(t), \tag{1-13}$$

and consider the space of initial data

$$\mathcal{I}_{\text{rad}} = \{(u_0, u_1) \in \mathcal{S}'_{\text{rad}} \times \mathcal{S}'_{\text{rad}} : L_{\tilde{u}(0)}(t) \in L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0, \infty)}(\mathbb{R}^n))\}, \tag{1-14}$$

where $r_0 = \frac{n(p-1)}{2}$ and the subindex “rad” means space of radial distributions. The norm $\|\cdot\|_{\mathcal{I}_{\text{rad}}}$ is defined as

$$\|(u_0, u_1)\|_{\mathcal{I}_{\text{rad}}} = \sup_{t \in \mathbb{R}} \|L_{\tilde{u}(0)}(t)\|_{L_{\text{rad}}^{(r_0, \infty)}}. \tag{1-15}$$

Applying the estimate (1-12), we obtain global well-posedness for (1-2) in the scaling-invariant space $E = L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0, \infty)}(\mathbb{R}^n))$ provided that $n \geq 5$ odd, $p > (n^2 + n - 4)/(n(n - 3))$ and $\|(u_0, u_1)\|_{\mathcal{I}_{\text{rad}}}$ is small enough (see Theorem 3.3(I)). The continuous inclusion $(\dot{B}_{2, \infty}^{sp} \times \dot{B}_{2, \infty}^{sp-1})_{\text{rad}} \subset \mathcal{I}_{\text{rad}}$ holds true and so, in the radial case, our result extends the initial data class in [Planchon 2000]. In fact, we have $\dot{B}_{2, \infty}^{sp} \subset L^{(\frac{n(p-1)}{2}, \infty)}$ (see Remark 3.4(I)) and

$$\sup_{t \in \mathbb{R}} \|L_{\tilde{u}(0)}(t)\|_{L^{(\frac{n(p-1)}{2}, \infty)}} \leq C \sup_{t \in \mathbb{R}} \|L_{\tilde{u}(0)}(t)\|_{\dot{B}_{2, \infty}^{sp}} \leq C \|(u_0, u_1)\|_{\dot{B}_{2, \infty}^{sp} \times \dot{B}_{2, \infty}^{sp-1}}, \tag{1-16}$$

where the second inequality in (1-16) can be found in [Planchon 2000, estimate (29), p. 815]. Also, we have $\mathcal{K} \not\subset L^{(\frac{n}{2}, \infty)}$ and then our class of potentials is larger than the Kato one in the radial setting (see Remark 3.4(II)). Note that

$$p_{\text{str}} < p_{\text{conf}} < p_e < \frac{n^2 + n - 4}{n(n - 3)}$$

and our range of admissible powers p differs from those of [Kato and Ozawa 2003; 2004; Planchon 2000; Ribaud and Youssfi 2002]. Finally, as a byproduct, we obtain the existence of radially symmetric self-similar solutions when u_0, u_1 and V are homogeneous of degrees $-\frac{2}{p-1}$, $-\frac{p+1}{p-1}$ and -2 , respectively (see Theorem 3.3(II)).

This paper is organized as follows. In Section 2, we recall the definition of Lorentz spaces and some of their properties. Our results are stated in Section 3 and proved in Section 4.

2. Lorentz spaces

We start by recalling the decreasing rearrangement of a measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\} \quad \text{for } t > 0,$$

where $d_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|$ is the distribution function of f . The Lorentz space $L^{(p,z)} = L^{(p,z)}(\mathbb{R}^n)$ is the vector space of all measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\|f\|_{(p,z)}^* = \begin{cases} [\int_0^\infty (t^{\frac{1}{p}} [f^*(t)])^z \frac{dt}{t}]^{\frac{1}{z}} < \infty & \text{for } 0 < p \leq \infty, 1 \leq z < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} [f^*(t)] < \infty & \text{for } 0 < p \leq \infty, z = \infty. \end{cases} \quad (2-1)$$

The space $L^{(\infty,z)}$ is trivial for $1 \leq z < \infty$. Also, $L^{(p,p)}$ is the Lebesgue space L^p with $\|f\|_{(p,p)}^* = \|\cdot\|_{L^p}$ and $L^{(p,\infty)}$ is the so-called weak- L^p . The quantity $\|f\|_{(p,z)}^*$ defines a complete quasinorm on $L^{(p,z)}$ that in general is not a norm. Considering the double rearrangement

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds,$$

one can define the norm $\|f\|_{(p,z)}$ on $L^{(p,z)}$ by replacing f^* by f^{**} in (2-1). For $1 < p \leq \infty$, we have the inequality

$$\|\cdot\|_{(p,z)}^* \leq \|\cdot\|_{(p,z)} \leq \frac{p}{p-1} \|\cdot\|_{(p,z)}^*,$$

and then $\|\cdot\|_{(p,z)}$ and $\|\cdot\|_{(p,z)}^*$ are topologically equivalent. The pair $(L^{(p,z)}, \|\cdot\|_{(p,z)})$ is a Banach space. From now on, we consider $L^{(p,z)}$ endowed with $\|\cdot\|_{(p,z)}$ and $\|\cdot\|_{(p,z)}^*$ when $1 < p \leq \infty$ and $0 < p \leq 1$, respectively. The continuous inclusions

$$L^{(p,1)} \subset L^{(p,z_1)} \subset L^p \subset L^{(p,z_2)} \subset L^{(p,\infty)} \quad (2-2)$$

hold true for $1 \leq z_1 \leq p \leq z_2 \leq \infty$ and $1 \leq p \leq \infty$. Lorentz spaces have the same scaling as L^p -spaces, namely

$$\|\delta_c(f)\|_{(p,z)} = c^{-\frac{n}{p}} \|f\|_{(p,z)},$$

where δ_λ stands for the operator $\delta_c(f)(x) = f(cx)$.

Let $0 < \theta < 1$ and $1 \leq z \leq \infty$. Consider the interpolation functor $(\cdot, \cdot)_{\theta,z}$ constructed via the $K_{\theta,z}$ -method and defined on the categories of quasinormed and normed spaces. For $0 < p_1 < p_2 \leq \infty$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $1 \leq z_1, z_2 \leq \infty$, we have (see [Bergh and Löfström 1976, Theorems 5.3.1 and 5.3.2])

$$(L^{(p_1,z_1)}, L^{(p_2,z_2)})_{\theta,z} = L^{(p,z)}. \quad (2-3)$$

Moreover, $(\cdot, \cdot)_{\theta,z}$ is exact of exponent θ .

The pointwise product operator works well in Lorentz spaces; i.e., Hölder inequality is verified in this setting (see [Hunt 1966; O'Neil 1963]). Let $1 < p_1, p_2, p_3 \leq \infty$ and $1 \leq z_1, z_2, z_3 \leq \infty$ be such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{z_1} + \frac{1}{z_2} \geq \frac{1}{z_3}$. Then

$$\|fg\|_{(p_3,z_3)} \leq C \|f\|_{(p_1,z_1)} \|g\|_{(p_2,z_2)}, \quad (2-4)$$

where the constant $C > 0$ is independent of f and g .

Finally, we recall that the dual space of $L^{(p,z)}$ is $L^{(p',z')}$ for $1 \leq p, z < \infty$ (see [Grafakos 2004, p. 52]). Taking $z = 1$, we have $(L^{(p,1)})' = L^{(p',\infty)}$ for $1 \leq p < \infty$.

3. Main results

Throughout the paper, the subindex “rad” means space of radial functions or distributions. For instance,

$$L^{(r,z)}_{\text{rad}} = \{u \in L^{(r,z)} : u \text{ is radially symmetric}\}. \tag{3-1}$$

We define the open triangles $\Delta_{P_1 P_2 P_3}$ and $\Delta_{P_2 P_4 P_5}$ whose vertices P_i are

$$\begin{aligned} P_1 &= \left(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} - \frac{1}{n+1}\right), & P_2 &= \left(\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2} - \frac{1}{n-1}\right), \\ P_3 &= \left(\frac{1}{2} + \frac{1}{n-1}, \frac{1}{2} + \frac{1}{n-1}\right), & P_4 &= \left(1, \frac{n-1}{2n}\right) \quad \text{and} \quad P_5 = (1, 1) \end{aligned} \tag{3-2}$$

(see Figure 1). The vertices P_2 and P_3 are defined as $(0, 0)$ and $(1, 1)$, respectively, when $n = 1, 2$.

Our first result consists in linear estimates in weak- L^p for the linear wave equation with singular potential.

Theorem 3.1. *Let $n \geq 5$ be odd and $\Delta_{P_2 P_4 P_5}$ be the open triangle defined by the points P_2, P_4 and P_5 in (3-2). If*

$$1 < r', s' < \frac{2(n-1)}{n-3} \quad \text{with} \quad \left(1 - \frac{1}{r}, 1 - \frac{1}{s}\right) \in \Delta_{P_2 P_4 P_5} \quad \text{and} \quad \frac{1}{s} - \frac{1}{r} = \frac{2}{n}, \tag{3-3}$$

then there are $K, C_0 > 0$ such that the solution u of (1-1) satisfies

$$\sup_{t \in \mathbb{R}} \|u(\cdot, t)\|_{(r, \infty)} \leq \frac{K}{1 - C_0 K \|V\|_{(\frac{n}{2}, \infty)}} \sup_{t \in \mathbb{R}} \|f(\cdot, t)\|_{(s, \infty)} \tag{3-4}$$

for all $f \in L^\infty(\mathbb{R}; L^{(s, \infty)}_{\text{rad}}(\mathbb{R}^n))$, provided that $V \in L^{(\frac{n}{2}, \infty)}_{\text{rad}}(\mathbb{R}^n)$ and $C_0 K \|V\|_{(\frac{n}{2}, \infty)} < 1$. The supremum in (3-4) is taken in the essential sense.

Some comments on Theorem 3.1 are in order.

Remark 3.2. (I) Let us point out that the range in Theorem 3.1 is not empty. In order to see this, set $w = 1 - \frac{1}{r}$ and $h = \frac{1}{s} - \frac{1}{r}$. Now notice that $(1 - \frac{1}{r}, 1 - \frac{1}{s}) \in \Delta_{P_2 P_4 P_5}$ is equivalent to

$$\left(1 - \frac{1}{r}, 1 - \frac{1}{s}\right) = (w, w - h) \in \Delta_{P_2 P_4 P_5}. \tag{3-5}$$

In turn, for (3-5) we need only that $0 < h < \frac{n+1}{2n}$ holds true when $h = \frac{2}{n}$ and $n \geq 5$.

(II) The critical Hardy potential $V(x) = c_0|x|^{-2} \in L^{(\frac{n}{2}, \infty)}_{\text{rad}}(\mathbb{R}^n)$ is covered by Theorem 3.1 with $|c_0| < (C_0 K \| |x|^{-2} \|_{(\frac{n}{2}, \infty)})^{-1}$. The constant C_0 in (3-4) is that of the Hölder inequality

$$\|Vu\|_{(s, \infty)} \leq C_0 \|V\|_{(\frac{n}{2}, \infty)} \|u\|_{(r, \infty)} \quad \text{with} \quad \frac{1}{s} = \frac{2}{n} + \frac{1}{r}.$$

(III) Taking $V = 0$, Theorem 3.1 also provides an estimate for the linear wave equation $\square u = f$.

Let $\{\omega(t)\}_{t \in \mathbb{R}}$ be the wave group $\omega(t) = (-\Delta)^{-\frac{1}{2}} \sin(t(-\Delta)^{\frac{1}{2}})$ and define $\zeta(f)$ by

$$\zeta(f)(x, t) = \int_0^t \omega(t-s) f(s) ds. \tag{3-6}$$

Formally, the IVP (1-2) is equivalent to the integral equation

$$u = L_{\tilde{u}(0)}(t) + \mathcal{N}(u) + \mathcal{T}(u), \tag{3-7}$$

where

$$\mathcal{N}(u) = \mu\zeta(|u|^{p-1}u) \quad \text{and} \quad \mathcal{T}(u) = -\zeta(Vu).$$

Solutions of (3-7) are called mild solutions for the Cauchy problem (1-2).

We will look for solutions of (3-7) in the Banach space $E = L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0, \infty)})$ whose norm is

$$\|u\|_E = \sup_{t \in \mathbb{R}} \|u(\cdot, t)\|_{(r_0, \infty)}. \tag{3-8}$$

The supremum in (3-8) is taken in the essential sense. This space is invariant by the scaling (1-3) and allows the existence of self-similar solutions (i.e., $u = u_\gamma$).

Consider

$$A_1 = \left(\frac{n+1}{2(n-1)}, \frac{n+1}{2(n-1)} - \frac{2}{n} \right) \quad \text{and} \quad A_2 = \left(1, \frac{n-2}{n} \right).$$

Let $]A_1, A_2[$ be the open segment line. Notice that $]A_1, A_2[\in \Delta_{P_2 P_4 P_5} \setminus \Delta_{P_1 P_2 P_3}$ for all $n \geq 4$.

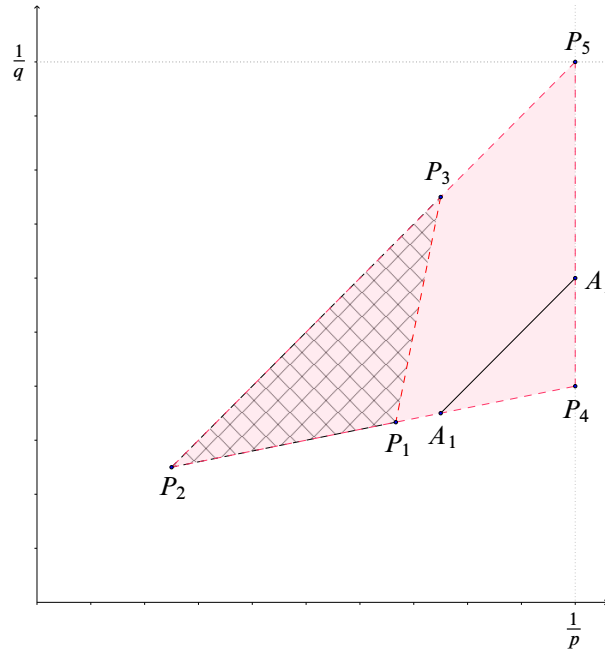


Figure 1. $]A_1, A_2[\in \Delta_{P_2 P_4 P_5} \setminus \Delta_{P_1 P_2 P_3}$.

Observe that $p > (n^2 + n - 4)/(n(n - 3))$ is equivalent to

$$\left(1 - \frac{2}{n(p-1)}, 1 - \frac{2p}{n(p-1)} \right) \in]A_1, A_2[. \tag{3-9}$$

Our well-posedness and self-similarity results for (1-2) are stated below.

Theorem 3.3. *Let $n \geq 5$ be odd, $p > (n^2 + n - 4)/(n(n - 3))$ and $r_0 = n(p - 1)/2$. Suppose $(u_0, u_1) \in \mathcal{I}_{\text{rad}}$ and $V \in L_{\text{rad}}^{(\frac{n}{2}, \infty)}$.*

(I) (**Global well-posedness**) *There are $\varepsilon, C_1 > 0$ such that if $\|(u_0, u_1)\|_{\mathcal{I}_{\text{rad}}} \leq \varepsilon$, then the IVP (1-2) has a unique mild solution $u \in L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0, \infty)})$ satisfying*

$$\sup_{t \in \mathbb{R}} \|u(\cdot, t)\|_{(r_0, \infty)} \leq \frac{2\varepsilon}{1 - \eta}$$

provided that $\eta = C_1 \|V\|_{(\frac{n}{2}, \infty)} < 1$. Moreover, the solution u depends continuously on data (u_0, u_1) and potential V .

(II) (**Self-similarity**) *Under the hypotheses of item (I), the solution u is self-similar provided that u_0, u_1, V are homogeneous of degrees $-\frac{2}{p-1}, -\frac{p+1}{p-1}$ and -2 , respectively.*

In what follows, we make some comments on Theorem 3.3.

Remark 3.4. (I) Taking $V = 0$, Theorem 3.3 provides a well-posedness result for semilinear wave equations in odd dimensions $n \geq 5$. Moreover, we have the continuous inclusions (see [Bergh and Löfström 1976, p. 154])

$$\dot{H}_{r_1}^s \hookrightarrow \dot{B}_{r_1, \infty}^s \hookrightarrow L^{(r_2, \infty)}, \tag{3-10}$$

where $\frac{1}{r_1} - \frac{s}{n} = \frac{1}{r_2}$ and $r_2 \geq r_1$. In particular, for $s_p = \frac{n}{2} - \frac{2}{p-1}$ we obtain $\dot{H}^{s_p} \hookrightarrow \dot{B}_{2, \infty}^{s_p} \hookrightarrow L^{(\frac{n(p-1)}{2}, \infty)}$. In fact, the inclusions in (3-10) are strict and then the space $L^{(\frac{n(p-1)}{2}, \infty)}$ is larger than $\dot{B}_{2, \infty}^{s_p}$, i.e., the one considered by Planchon [2000]. So, Theorem 3.3 extends the existence result of [Planchon 2000] in the case of radial solutions and $n \geq 5$ odd.

(II) Let \mathcal{K} be the Kato class of potentials defined in (1-9). In view of the continuous strict inclusions

$$L^{\frac{n}{2}-\delta} \cap L^{\frac{n}{2}+\delta} \hookrightarrow L^{(\frac{n}{2}, 1)} \hookrightarrow \mathcal{K} \hookrightarrow L^{(\frac{n}{2}, \infty)}, \quad \delta > 0, \tag{3-11}$$

our class for V is larger than \mathcal{K} in the radial setting. For $L^{(\frac{n}{2}, 1)} \hookrightarrow \mathcal{K}$, we can use Hölder inequality (2-4) to obtain

$$\|V\|_{\mathcal{K}} \leq C \left\| \frac{1}{|x - y|^{(n-2)}} \right\|_{(\frac{n}{n-2}, \infty)} \|V\|_{(\frac{n}{2}, 1)} \leq L \|V\|_{(\frac{n}{2}, 1)},$$

where

$$L = C \left\| |x - y|^{-(n-2)} \right\|_{(\frac{n}{n-2}, \infty)}$$

is a positive constant. Next recall that $f \in L^{(p, \infty)}$ if and only if there is a constant $C > 0$ such that

$$|E|^{\frac{1}{p}-1} \int_E |f(y)| dy \leq C \tag{3-12}$$

for every Borel set E . The supremum of the left-hand side of (3-12) over all Borel sets gives an equivalent norm in $L^{(p, \infty)}$. It follows from (3-12) that $\mathcal{K} \hookrightarrow L^{(\frac{n}{2}, \infty)}$. In fact, it is sufficient to check (3-12) for every open ball $E = \mathcal{B}_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$. For that, we estimate

$$\int_{\mathcal{B}_r(x)} |V(y)| dy \leq \int_{\mathcal{B}_r(x)} \frac{r^{n-2}}{|x - y|^{n-2}} |V(y)| dy \leq C \|V\|_{\mathcal{K}} |\mathcal{B}_r(x)|^{1-\frac{2}{n}},$$

where $|\mathcal{B}_r(x)| = (\pi^{\frac{n}{2}} / \Gamma(\frac{n}{2} + 1)) r^n$ is the volume of $\mathcal{B}_r(x)$.

4. Proofs

Collecting estimates in [Brenner 1975; Peral 1980; Strichartz 1970], we have that the wave group $\{\omega(t)\}_{t \in \mathbb{R}}$ is bounded from L^{l_1} to L^{l_2} at $t = 1$, i.e.,

$$\|\omega(1)h\|_{l_2} \leq M_1 \|h\|_{l_1}, \quad (4-1)$$

provided that $(\frac{1}{l_1}, \frac{1}{l_2}) \in \overline{\Delta_{P_1 P_2 P_3}}$, where \overline{X} stands for the closure of X . It follows from scaling properties of $\omega(t)$ and L^p -spaces that

$$\|\omega(t)h\|_{l_2} \leq M_1 |t|^{-n(\frac{1}{l_1} - \frac{1}{l_2})+1} \|h\|_{l_1}. \quad (4-2)$$

Interpolating the estimate (4-2) (see, e.g., [Bergh and Löfström 1976]), we get

$$\|\omega(t)h\|_{(l_2, z)} \leq M_2 |t|^{-n(\frac{1}{l_1} - \frac{1}{l_2})+1} \|h\|_{(l_1, z)}, \quad (4-3)$$

where $1 \leq z \leq \infty$. Assuming radial symmetry for h , the authors of [Ebert et al. 2016] extended the range of (4-2) to the closed triangle $\overline{\Delta_{P_2 P_4 P_5}}$ except for the semiopen segment line $]P_1, P_4]$ (see Figure 1). Thus, again by interpolation, for $(\frac{1}{l_1}, \frac{1}{l_2})$ belonging to the open triangle $\Delta_{P_2 P_4 P_5}$ and $1 \leq z \leq \infty$, we obtain the estimate

$$\|\omega(t)h\|_{(l_2, z)} \leq M_3 |t|^{-n(\frac{1}{l_1} - \frac{1}{l_2})+1} \|h\|_{(l_1, z)} \quad (4-4)$$

for all $h \in L_{\text{rad}}^{(l_1, z)}(\mathbb{R}^n)$.

Yamazaki [2000] dealt with Navier–Stokes equations and Stokes and heat semigroups in weak- L^p spaces. The next estimate could be seen as a version of the Yamazaki estimate [2000, Corollary 2.3] for the wave group $\{\omega(t)\}_{t \in \mathbb{R}}$. Notice that it consists in a time-weighted estimate in preduals of weak- L^p spaces.

Lemma 4.1. *Let f be radially symmetric, $n \geq 3$ odd, and let $\Delta_{P_2 P_4 P_5}$ be the open triangle defined by the points P_2, P_4, P_5 in (3-2). If $1 < d_1, d_2 < 2(n-1)/(n-3)$ (∞ if $n = 3$) with $(\frac{1}{d_1}, \frac{1}{d_2}) \in \Delta_{P_2 P_4 P_5}$ then $|t|^{n(\frac{1}{d_1} - \frac{1}{d_2})-2} \omega(t)f \in L^1(\mathbb{R}; L^{(d_2, 1)}(\mathbb{R}^n))$ and there is $C > 0$ such that*

$$\int_{\mathbb{R}} |t|^{\frac{n}{d_1} - \frac{n}{d_2} - 2} \|\omega(t)f\|_{(d_2, 1)} dt \leq C \|f\|_{(d_1, 1)} \quad (4-5)$$

for all $f \in L_{\text{rad}}^{(d_1, 1)}(\mathbb{R}^n)$.

Proof. Let p_1 and p_2 be such that $p_1 < d_1 < p_2$, $\frac{1}{d_1} - \frac{1}{p_2} < \frac{1}{n}$ and

$$\left(\frac{1}{p_j}, \frac{1}{d_2}\right) \in \Delta_{P_2 P_4 P_5} \quad \text{for } j = 1, 2.$$

Using the estimate (4-4) with $(l_1, l_2, z) = (p_1, d_2, 1)$ and $(l_1, l_2, z) = (p_2, d_2, 1)$, we obtain

$$\|\omega(t)f\|_{(d_2, 1)} \leq C_k |t|^{1 + \frac{n}{d_2} - \frac{n}{p_k}} \|f\|_{(p_k, 1)} \quad \text{for } k = 1, 2. \quad (4-6)$$

Next consider the sublinear operator Ξ as a map from $L_{\text{rad}}^{(p_1, 1)} \cap L_{\text{rad}}^{(p_2, 1)}$ to a function $\Xi(f)(t)$ in \mathbb{R} defined by

$$\Xi(f)(t) = |t|^{\frac{n}{d_1} - \frac{n}{d_2} - 2} \|\omega(t)f\|_{(d_2, 1)}.$$

In view of (4-6), we can estimate

$$\Xi(f)(t) \leq C_k |t|^{\frac{n}{d_1} - \frac{n}{d_2} - 2} |t|^{1 + \frac{n}{d_2} - \frac{n}{p_k}} \|f\|_{(p_k, 1)} = C_k |t|^{\frac{n}{d_1} - \frac{n}{p_k} - 1} \|f\|_{(p_k, 1)} \quad \text{for } k = 1, 2. \quad (4-7)$$

Hence, the operator Ξ is bounded from $L_{\text{rad}}^{(p_k, 1)}(\mathbb{R}^n)$ to $L^{(s_k, \infty)}(\mathbb{R})$, where $\frac{1}{s_k} = 1 - (\frac{n}{d_1} - \frac{n}{p_k})$. Indeed, it follows from (4-7) that

$$\|\Xi(f)(t)\|_{L^{(s_k, \infty)}(\mathbb{R})} \leq C_k \| |t|^{-\frac{1}{s_k}} \|_{L^{(s_k, \infty)}(\mathbb{R})} \|f\|_{(p_k, 1)} \leq L_k \|f\|_{(p_k, 1)}, \quad (4-8)$$

where $L_k = C_k \| |t|^{-1/s_k} \|_{L^{(s_k, \infty)}(\mathbb{R})}$, $k = 1, 2$.

Take now $\theta \in (0, 1)$ such that $\frac{1}{d_1} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. So, we have $\frac{1-\theta}{s_1} + \frac{\theta}{s_2} = 1$ and $0 < s_1 < 1 < s_2$. By interpolation, it follows that

$$L_{\text{rad}}^{(d_1, 1)} = (L_{\text{rad}}^{(p_1, 1)}, L_{\text{rad}}^{(p_2, 1)})_{\theta, 1} \quad \text{and} \quad L^1(\mathbb{R}) = (L^{(s_1, \infty)}(\mathbb{R}), L^{(s_2, \infty)}(\mathbb{R}))_{\theta, 1}$$

and then

$$\begin{aligned} \|\Xi(f)(t)\|_{L^1(\mathbb{R})} &\leq m_1^{1-\theta} m_2^\theta \|f\|_{(d_1, 1)} \\ &\leq L_1^{1-\theta} L_2^\theta \|f\|_{(d_1, 1)}, \end{aligned}$$

where $m_k = \|\Xi(f)\|_{L_{\text{rad}}^{(p_k, 1)} \rightarrow L^{(s_k, \infty)}} \leq L_k$. This gives us the estimate (4-5). □

Proof of Theorem 3.1. Let us rewrite $\zeta(f)$ in (3-6) as

$$\zeta(f)(x, t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} W(x - y, t - s) f(y, s) dy ds,$$

where the kernel W is given by

$$\widehat{W}(\xi, t - s) = \begin{cases} \sin((t - s)|\xi|)/|\xi|, & 0 < s < t, \\ 0, & \text{otherwise.} \end{cases}$$

Given a suitable function $\phi \in C^\infty(\mathbb{R}^n)$, we set

$$\langle \zeta(f), \phi \rangle = \int_{\mathbb{R}^n} \zeta(f)(x, t) \phi(x) dx.$$

Here all functions are considered to be radially symmetric. Using Tonelli's theorem and the Hölder inequality (2-4), we obtain

$$\begin{aligned} |\langle \zeta(f), \phi \rangle| &\leq \int_{-\infty}^{\infty} \langle |f(\cdot, \tau)|, |\omega(t - \tau)\phi| \rangle d\tau \\ &\leq C \int_{-\infty}^{\infty} \|f(\cdot, \tau)\|_{(s, \infty)} \|\omega(t - \tau)\phi\|_{(s', 1)} d\tau. \end{aligned} \quad (4-9)$$

In (4-9) we have proceeded somewhat formally but we are going to see that its right-hand side is indeed finite, which justifies the above computations. Take $(d_1, d_2) = (r', s')$ and note that

$$\frac{n}{d_1} - \frac{n}{d_2} - 2 = \frac{n}{r'} - \frac{n}{s'} - 2 = 0.$$

Using duality in $L^{(p,z)}$, the inequality (4-9) and Lemma 4.1 with $(d_1, d_2) = (s', r')$, it follows that

$$\begin{aligned}
\|\zeta(f)(\cdot, t)\|_{(r,\infty)} &= \sup_{\|\phi\|_{(r',1)}=1} |\langle \zeta(f), \phi \rangle| \\
&\leq C \sup_{t \in \mathbb{R}} \|f(\cdot, t)\|_{(s,\infty)} \sup_{\|\phi\|_{(r',1)}=1} \int_{-\infty}^{\infty} \|\omega(t-\tau)\phi\|_{(s',1)} d\tau \\
&\leq K \sup_{t \in \mathbb{R}} \|f(\cdot, t)\|_{(s,\infty)} \sup_{\|\phi\|_{(r',1)}=1} \{\|\phi\|_{(r',1)}\} \\
&\leq K \sup_{t \in \mathbb{R}} \|f(\cdot, t)\|_{(s,\infty)} \tag{4-10}
\end{aligned}$$

for a.e. $t \in \mathbb{R}$. Next let $\tilde{f} = f + Vu$ and $u = \zeta(\tilde{f})$ be the mild solution of (1-1). Since $\frac{1}{s} = \frac{1}{n/2} + \frac{1}{r}$, the Hölder inequality (2-4) gives

$$\|Vu\|_{(s,\infty)} \leq C_0 \|V\|_{(\frac{n}{2},\infty)} \|u(\cdot, t)\|_{(r,\infty)}. \tag{4-11}$$

Thus, in view of (4-10), we get

$$\begin{aligned}
\sup_{t \in \mathbb{R}} \|u(\cdot, t)\|_{(r,\infty)} &\leq K \sup_{t \in \mathbb{R}} \|\tilde{f}(\cdot, t)\|_{(s,\infty)} \\
&\leq K \sup_{t \in \mathbb{R}} \|f(\cdot, t)\|_{(s,\infty)} + KC_0 \|V\|_{(\frac{n}{2},\infty)} \sup_{t \in \mathbb{R}} \|u(\cdot, t)\|_{(r,\infty)},
\end{aligned}$$

which implies the desired estimate because $KC_0 \|V\|_{(\frac{n}{2},\infty)} < 1$. \square

Proof of Theorem 3.3. Part (I). (Well-posedness) Take $r = r_0 = \frac{n(p-1)}{2}$ and $s = \frac{r_0}{p}$, and note that

$$\frac{1}{s} - \frac{1}{r} = (p-1)\frac{1}{r_0} = \frac{2}{n}.$$

In view of (3-9), we have $(1 - \frac{1}{r}, 1 - \frac{1}{s}) \in \Delta_{P_2 P_4 P_5}$ and then we can employ Theorem 3.1 with $V = 0$ in order to obtain

$$\|\zeta(f)\|_E \leq K \|f\|_{L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0/p, \infty)})}. \tag{4-12}$$

Since $\frac{2}{n} + \frac{1}{r_0} = \frac{p}{r_0}$, it follows from (4-12) and the Hölder inequality (2-4) that

$$\begin{aligned}
\|\mathcal{T}(u) - \mathcal{T}(v)\|_E &= \|\mathcal{T}(u-v)\|_E = \sup_{t \in \mathbb{R}} \|\zeta(V(u-v))(\cdot, t)\|_{(r_0,\infty)} \\
&\leq K \sup_{t \in \mathbb{R}} \|V(u-v)(\cdot, t)\|_{(\frac{r_0}{p},\infty)} \\
&\leq KC_0 \|V\|_{(\frac{n}{2},\infty)} \sup_{t \in \mathbb{R}} \|u(\cdot, t) - v(\cdot, t)\|_{(r_0,\infty)} \\
&\leq \eta \|u-v\|_E, \tag{4-13}
\end{aligned}$$

where $\eta = C_1 \|V\|_{(\frac{n}{2},\infty)}$, $C_1 = KC_0$, and C_0 is the constant in the Hölder inequality $\|Vh\|_{(\frac{r_0}{p},\infty)} \leq C_0 \|V\|_{(\frac{n}{2},\infty)} \|h\|_{(r_0,\infty)}$. Next recall the inequality

$$||u|^{p-1}u - |v|^{p-1}v| \leq C|u-v|(|u|^{p-1} + |v|^{p-1})$$

and let $\frac{p}{r_0} = \frac{1}{r_0} + \frac{p-1}{r_0}$. Using the Hölder inequality (2-4), we can estimate

$$\begin{aligned} \| |u|^{p-1}u - |v|^{p-1}v \|_{(\frac{r_0}{p}, \infty)} &\leq C \| |u-v|(|u|^{p-1} + |v|^{p-1}) \|_{(\frac{r_0}{p}, \infty)} \\ &\leq C \| u-v \|_{(r_0, \infty)} \| (|u|^{p-1} + |v|^{p-1}) \|_{(\frac{r_0}{p-1}, \infty)} \\ &\leq C \| u-v \|_{(r_0, \infty)} (\|u\|_{(r_0, \infty)}^{p-1} + \|v\|_{(r_0, \infty)}^{p-1}). \end{aligned} \tag{4-14}$$

Estimates (4-12) and (4-14) yield

$$\begin{aligned} \|\mathcal{N}(u) - \mathcal{N}(v)\|_E &= \|\zeta(|u|^{p-1}u - |v|^{p-1}v)(\cdot, t)\|_E \\ &\leq K \| (|u|^{p-1}u - |v|^{p-1}v)(\cdot, t) \|_{L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0/p, \infty)})} \\ &\leq C_2 \|u-v\|_E (\|u\|_E^{p-1} + \|v\|_E^{p-1}). \end{aligned} \tag{4-15}$$

Let $\Psi(u) = L_{\tilde{u}(0)}(t) + \mathcal{N}(u) + \mathcal{T}(u)$ be defined in the closed ball $B_\varepsilon = \{u \in E : \|u\|_E \leq 2\varepsilon/(1-\eta)\}$, where $\varepsilon > 0$ and $\eta = C_1 \|V\|_{(n/2, \infty)} < 1$. We are going to show that Ψ is a contraction in B_ε for ε small enough. For $u, v \in B_\varepsilon$, we have

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_E &\leq \|\mathcal{N}(u) - \mathcal{N}(v)\|_E + \|\mathcal{T}u - \mathcal{T}v\|_E \\ &\leq C_2 \|u-v\|_E (\|u\|_E^{p-1} + \|v\|_E^{p-1}) + \eta \|u-v\|_E \\ &\leq \|u-v\|_E \left(C_2 \left(\frac{2\varepsilon}{1-\eta} \right)^{p-1} + C_2 \left(\frac{2\varepsilon}{1-\eta} \right)^{p-1} + \eta \right) \\ &\leq \left(C_2 \frac{2^p \varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) \|u-v\|_E. \end{aligned} \tag{4-16}$$

Choose $\varepsilon > 0$ in such a way that

$$\left(C_2 \frac{2^p \varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) < 1. \tag{4-17}$$

Moreover, taking $v = 0$ in (4-16), we arrive at

$$\|\Psi(u)\|_E \leq \|L_{\tilde{u}(0)}\|_E + \|\Psi(u) - \Psi(0)\|_E \leq \varepsilon + \left(C_2 \frac{2^{p-1} \varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) \frac{2\varepsilon}{1-\eta} < \frac{2\varepsilon}{1-\eta}$$

for all $u \in B_\varepsilon$. Hence, the map $\Psi : B_\varepsilon \rightarrow B_\varepsilon$ is a contraction in E . It follows that its fixed point in B_ε is the unique solution for (3-7) such that

$$\|u\|_E \leq \frac{2\varepsilon}{1-\eta}.$$

The continuous dependence follows naturally from the above estimates and fixed point argument. We include its proof for the sake of completeness. Let $u, v \in B_\varepsilon$ be the unique mild solutions associated to data (u_0, u_1, V) and (v_0, v_1, U) , respectively. Then, defining $L_{\tilde{u}(0)}(t) = \omega(t)u_1 + \dot{\omega}(t)u_0$ and

$L_{\tilde{v}(0)}(t) = \omega(t)v_1 + \dot{\omega}(t)v_0$, we have

$$\begin{aligned} \|u - v\|_E &\leq \|L_{\tilde{u}(0)} - L_{\tilde{v}(0)}\|_E + \|\mathcal{N}(u) - \mathcal{N}(v)\|_E + \|\mathcal{T}(u) - \mathcal{T}(v)\|_E \\ &= \|L_{\tilde{u}(0)} - L_{\tilde{v}(0)}\|_E + \|\mathcal{N}(u) - \mathcal{N}(v)\|_E + \|\zeta((V - U)v + V(u - v))\|_E \\ &\leq \|L_{\tilde{u}(0)} - L_{\tilde{v}(0)}\|_E + C_2 \frac{2^p \varepsilon^{p-1}}{(1 - \eta)^{p-1}} \|u - v\|_E + KC_0 \|V - U\|_{(\frac{n}{2}, \infty)} \|v\|_E + \eta \|u - v\|_E \\ &\leq \|(u_0 - v_0, u_1 - v_1)\|_{\mathcal{I}_{\text{rad}}} + \left(C_2 \frac{2^p \varepsilon^{p-1}}{(1 - \eta)^{p-1}} + \eta \right) \|u - v\|_E + \frac{2KC_0 \varepsilon}{1 - \eta} \|V - U\|_{(\frac{n}{2}, \infty)}, \end{aligned}$$

which yields the desired continuity because of (4-17).

Part (II). (Self-similarity) First note that the homogeneous pair (u_0, u_1) is in \mathcal{I}_{rad} . Due to the fixed point argument, the solution u in item (I) is the limit in $E = L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0, \infty)})$ of the sequence

$$u^{(1)} = L_{\tilde{u}(0)}(t) \quad \text{and} \quad u^{(k+1)} = L_{\tilde{u}(0)}(t) + \mathcal{N}(u^{(k)}) + \mathcal{T}(u^{(k)}) \quad \text{for } k \in \mathbb{N}. \quad (4-18)$$

Using the homogeneity properties of u_0, u_1 and V , one can show that $u^{(k)}$ is invariant by (1-3), that is,

$$u^{(k)} = (u^{(k)})_\gamma := \gamma^{\frac{2}{p-1}} u^{(k)}(\gamma x, \gamma t).$$

Now, since $(E, \|\cdot\|_E)$ is invariant by (1-3), a change of variable gives

$$\|(u^{(k)})_\gamma - (u)_\gamma\|_E = \|(u^{(k)} - u)_\gamma\|_E = \|u^{(k)} - u\|_E \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (4-19)$$

Since $(u^{(k)})_\gamma = u^{(k)}$, it follows that $u^{(k)}$ also converges to $(u)_\gamma$. Then, $u \equiv (u)_\gamma$ for each $\gamma > 0$, as required. \square

References

- [Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer, Berlin, 1976. MR Zbl
- [Brenner 1975] P. Brenner, “On $L_p - L_{p'}$ estimates for the wave-equation”, *Math. Z.* **145**:3 (1975), 251–254. MR Zbl
- [Burq et al. 2003] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, “Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential”, *J. Funct. Anal.* **203**:2 (2003), 519–549. MR Zbl
- [Burq et al. 2004] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, “Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay”, *Indiana Univ. Math. J.* **53**:6 (2004), 1665–1680. MR Zbl
- [D’ancona and Pierfelice 2005] P. D’ancona and V. Pierfelice, “On the wave equation with a large rough potential”, *J. Funct. Anal.* **227**:1 (2005), 30–77. MR Zbl
- [Ebert et al. 2016] M. R. Ebert, R. A. Kapp, and T. Picon, “ $L^1 - L^p$ estimates for radial solutions of the wave equation and application”, *Ann. Mat. Pura Appl. (4)* **195**:4 (2016), 1081–1091. MR Zbl
- [Georgiev and Visciglia 2003] V. Georgiev and N. Visciglia, “Decay estimates for the wave equation with potential”, *Comm. Partial Differential Equations* **28**:7–8 (2003), 1325–1369. MR Zbl
- [Grafakos 2004] L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Upper Saddle River, NJ, 2004. MR Zbl
- [Grillakis 1990] M. G. Grillakis, “Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity”, *Ann. of Math. (2)* **132**:3 (1990), 485–509. MR Zbl

- [Grillakis 1992] M. G. Grillakis, “Regularity for the wave equation with a critical nonlinearity”, *Comm. Pure Appl. Math.* **45:6** (1992), 749–774. MR Zbl
- [Hidano 2002] K. Hidano, “Scattering and self-similar solutions for the nonlinear wave equation”, *Differential Integral Equations* **15:4** (2002), 405–462. MR Zbl
- [Hunt 1966] R. A. Hunt, “On $L(p, q)$ spaces”, *Enseignement Math. (2)* **12** (1966), 249–276. MR Zbl
- [Kato and Ozawa 2003] J. Kato and T. Ozawa, “Weighted Strichartz estimates and existence of self-similar solutions for semilinear wave equations”, *Indiana Univ. Math. J.* **52:6** (2003), 1615–1630. MR Zbl
- [Kato and Ozawa 2004] J. Kato and T. Ozawa, “Weighted Strichartz estimates for the wave equation in even space dimensions”, *Math. Z.* **247:4** (2004), 747–764. MR Zbl
- [Kato et al. 2007] J. Kato, M. Nakamura, and T. Ozawa, “A generalization of the weighted Strichartz estimates for wave equations and an application to self-similar solutions”, *Comm. Pure Appl. Math.* **60:2** (2007), 164–186. MR Zbl
- [Kavian and Weissler 1990] O. Kavian and F. B. Weissler, “Finite energy self-similar solutions of a nonlinear wave equation”, *Comm. Partial Differential Equations* **15:10** (1990), 1381–1420. MR Zbl
- [Keel and Tao 1998] M. Keel and T. Tao, “Endpoint Strichartz estimates”, *Amer. J. Math.* **120:5** (1998), 955–980. MR Zbl
- [Lai and Zhou 2014] N.-A. Lai and Y. Zhou, “An elementary proof of Strauss conjecture”, *J. Funct. Anal.* **267:5** (2014), 1364–1381. MR Zbl
- [Miao et al. 2013] C. Miao, J. Zhang, and J. Zheng, “Strichartz estimates for wave equation with inverse square potential”, *Commun. Contemp. Math.* **15:6** (2013), art. id. 1350026. MR Zbl
- [O’Neil 1963] R. O’Neil, “Convolution operators and $L(p, q)$ spaces”, *Duke Math. J.* **30** (1963), 129–142. MR Zbl
- [Pecher 2000a] H. Pecher, “Self-similar and asymptotically self-similar solutions of nonlinear wave equations”, *Math. Ann.* **316:2** (2000), 259–281. MR Zbl
- [Pecher 2000b] H. Pecher, “Sharp existence results for self-similar solutions of semilinear wave equations”, *NoDEA Nonlinear Differential Equations Appl.* **7:3** (2000), 323–341. MR Zbl
- [Peral 1980] J. C. Peral, “ L^p estimates for the wave equation”, *J. Funct. Anal.* **36:1** (1980), 114–145. MR Zbl
- [Planchon 2000] F. Planchon, “Self-similar solutions and semi-linear wave equations in Besov spaces”, *J. Math. Pures Appl. (9)* **79:8** (2000), 809–820. MR Zbl
- [Planchon et al. 2003a] F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, “Dispersive estimate for the wave equation with the inverse-square potential”, *Discrete Contin. Dyn. Syst.* **9:6** (2003), 1387–1400. MR Zbl
- [Planchon et al. 2003b] F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, “ L^p estimates for the wave equation with the inverse-square potential”, *Discrete Contin. Dyn. Syst.* **9:2** (2003), 427–442. MR Zbl
- [Ribaud and Youssfi 2002] F. Ribaud and A. Youssfi, “Global solutions and self-similar solutions of semilinear wave equation”, *Math. Z.* **239:2** (2002), 231–262. MR Zbl
- [Shatah and Struwe 1993] J. Shatah and M. Struwe, “Regularity results for nonlinear wave equations”, *Ann. of Math. (2)* **138:3** (1993), 503–518. MR Zbl
- [Strauss 1981] W. A. Strauss, “Nonlinear scattering theory at low energy”, *J. Funct. Anal.* **41:1** (1981), 110–133. MR Zbl
- [Strauss and Tsutaya 1997] W. A. Strauss and K. Tsutaya, “Existence and blow up of small amplitude nonlinear waves with a negative potential”, *Discrete Contin. Dynam. Systems* **3:2** (1997), 175–188. MR Zbl
- [Strichartz 1970] R. S. Strichartz, “Convolutions with kernels having singularities on a sphere”, *Trans. Amer. Math. Soc.* **148** (1970), 461–471. MR Zbl
- [Struwe 1999] M. Struwe, “Uniqueness for critical nonlinear wave equations and wave maps via the energy inequality”, *Comm. Pure Appl. Math.* **52:9** (1999), 1179–1188. MR Zbl
- [Taggart 2010] R. J. Taggart, “Inhomogeneous Strichartz estimates”, *Forum Math.* **22:5** (2010), 825–853. MR Zbl
- [Wang and Yu 2012] C. Wang and X. Yu, “Recent works on the Strauss conjecture”, pp. 235–256 in *Recent advances in harmonic analysis and partial differential equations*, edited by A. R. Nahmod et al., Contemp. Math. **581**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

- [Yajima 1995] K. Yajima, “The $W^{k,p}$ -continuity of wave operators for Schrödinger operators”, *J. Math. Soc. Japan* **47**:3 (1995), 551–581. MR Zbl
- [Yamazaki 2000] M. Yamazaki, “The Navier–Stokes equations in the weak- L^n space with time-dependent external force”, *Math. Ann.* **317**:4 (2000), 635–675. MR Zbl
- [Yordanov and Zhang 2006] B. T. Yordanov and Q. S. Zhang, “Finite time blow up for critical wave equations in high dimensions”, *J. Funct. Anal.* **231**:2 (2006), 361–374. MR Zbl
- [Zhou 2007] Y. Zhou, “Blow up of solutions to semilinear wave equations with critical exponent in high dimensions”, *Chin. Ann. Math. Ser. B* **28**:2 (2007), 205–212. MR Zbl

Received 2 Jul 2016. Accepted 12 Nov 2016.

MARCELO F. DE ALMEIDA: nucaltiado@gmail.com

Departamento de Matemática, Universidade Federal de Sergipe, CCET, Av. Marechal Rondon, s/n, 49100-000 Aracaju-SE, Brazil

LUCAS C. F. FERREIRA: lcff@ime.unicamp.br

Departamento de Matemática, Universidade Estadual de Campinas, IMECC, Rua Sérgio Buarque de Holanda, 651, 13083-859 Campinas-SP, Brazil

Analysis & PDE

msp.org/apde

EDITORS

EDITOR-IN-CHIEF

Patrick Gérard
patrick.gerard@math.u-psud.fr
Université Paris Sud XI
Orsay, France

BOARD OF EDITORS

Nicolas Burq	Université Paris-Sud 11, France nicolas.burq@math.u-psud.fr	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Sun-Yung Alice Chang	Princeton University, USA chang@math.princeton.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Wilhelm Schlag	University of Chicago, USA schlag@math.uchicago.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu
Vaughan Jones	U.C. Berkeley & Vanderbilt University vaughan.f.jones@vanderbilt.edu	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Gilles Lebeau	Université de Nice Sophia Antipolis, France lebeau@unice.fr	András Vasy	Stanford University, USA andras@math.stanford.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbm@math.mit.edu	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu
Clément Mouhot	Cambridge University, UK c.mouhot@dpms.cam.ac.uk		

PRODUCTION

production@msp.org
Silvio Levy, Scientific Editor


See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2017 is US \$265/year for the electronic version, and \$470/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2017 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 10 No. 2 2017

Some energy inequalities involving fractional GJMS operators JEFFREY S. CASE	253
Exact controllability for quasilinear perturbations of KdV PIETRO BALDI, GIUSEPPE FLORIDIA and EMANUELE HAUS	281
Operators of subprincipal type NILS DENCKER	323
Anisotropic Ornstein nonequalities KRYSTIAN KAZANIECKI, DMITRIY M. STOLYAROV and MICHAŁ WOJCIECHOWSKI	351
A note on stability shifting for the Muskat problem, II: From stable to unstable and back to stable DIEGO CÓRDOBA, JAVIER GÓMEZ-SERRANO and ANDREJ ZLATOŠ	367
Derivation of an effective evolution equation for a strongly coupled polaron RUPERT L. FRANK and ZHOU GANG	379
Time-weighted estimates in Lorentz spaces and self-similarity for wave equations with singular potentials MARCELO F. DE ALMEIDA and LUCAS C. F. FERREIRA	423
Optimal well-posedness for the inhomogeneous incompressible Navier–Stokes system with general viscosity COSMIN BURTEA	439
Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential MASAHIRO IKEDA and TAKAHISA INUI	481



2157-5045(2017)10:2;1-W