Appendices to "A note on stability shifting for the Muskat problem, II: From stable to unstable and back to stable"

Diego Córdoba, Javier Gómez-Serrano, Andrej Zlatoš

Appendix A: Integrals needed for the calculation of $\partial_{tx} z^1(0,0)$

We start with

$$\partial_t z^1(x,t) = \int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}dy$$

After taking a derivative in space:

$$\begin{aligned}
\partial_{tx} z^1(x,0) &= \int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y))(z^1_{xx}(x)-z^1_{xx}(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}dy \\
&+ \int_{\mathbb{T}} \frac{\cos(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))^2}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}dy \\
&- \int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
&\times (\sinh(z^2(x)-z^2(x-y))(z^2_x(x)-z^2_x(x-y)) + \sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y)))dy
\end{aligned}$$

Evaluating at $x=0$ and exploiting the symmetry of the integral:

$$\begin{aligned}
\partial_{tx} z^1(0,0) &= \int_{\mathbb{T}} \frac{\sin(z^1(y))(z^1_{xx}(0)+z^1_{xx}(y))}{\cosh(z^2(y))-\cos(z^1(y))}dy \\
&+ \int_{\mathbb{T}} \frac{\cos(z^1(y))(z^1_x(0)-z^1_x(y))^2}{\cosh(z^2(y))-\cos(z^1(y))}dy \\
&- \int_{\mathbb{T}} \frac{\sin(z^1(y))(z^1_x(0)-z^1_x(y))}{(\cosh(z^2(y))-\cos(z^1(y)))^2}(\sinh(z^2(y))(z^2_x(0)-z^2_x(y)) + \sin(z^1(y))(z^1_x(0)-z^1_x(y)))dy \\
&= A_1 + A_2 + A_3
\end{aligned}$$

$$\begin{aligned}
A_1 &= 2\int_0^\pi \frac{\sin(z^1(y))(z^1_{xx}(y))}{\cosh(z^2(y))-\cos(z^1(y))}dy \\
A_2 &= 2\int_0^\pi \frac{\cos(z^1(y))(z^1_x(0)-z^1_x(y))^2}{\cosh(z^2(y))-\cos(z^1(y))}dy \\
A_3 &= -2\int_0^\pi \frac{\sin(z^1(y))(z^1_x(0)-z^1_x(y))}{(\cosh(z^2(y))-\cos(z^1(y)))^2}(\sinh(z^2(y))(z^2_x(0)-z^2_x(y)) + \sin(z^1(y))(z^1_x(0)-z^1_x(y)))dy
\end{aligned}$$

Appendix B: Integrals needed for the calculation of $\partial_{ttx} z^1(0,0)$

After taking a derivative in time:

$$\begin{aligned}
\partial_{tt} z^1(x,t) &= \int_{\mathbb{T}} \frac{\cos(z^1(x)-z^1(x-y))(z^1_t(x)-z^1_t(x-y))(z^1_x(x)-z^1_x(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}dy \\
&+ \int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y))(z^1_{tx}(x)-z^1_{tx}(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}dy \\
&- \int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\sinh(z^2(x)-z^2(x-y))(z^2_t(x)-z^2_t(x-y)}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2}dy \\
&- \int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\sin(z^1(x)-z^1(x-y))(z^1_t(x)-z^1_t(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2}dy \\
&= I_1(x) + I_2(x) + I_3(x) + I_4(x)
\end{aligned}$$

We can further develop the terms of the second derivative:

$$I_1(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\cos(z^1(x)-z^1(x-y)))(z^1_x(x)-z^1_x(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \times \left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

$$I_2(x) = I_{21}(x)+I_{22}(x)+I_{23}(x)+I_{24}(x),$$

where

$$I_{21}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \times \left(\frac{\cos(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} - \frac{\cos(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

$$I_{22}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \times \left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_{xx}(x)-z^1_{xx}(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_{xx}(x-y)-z^1_{xx}(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

$$I_{23}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \times \left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\right) dydz$$

$$I_{24}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \times \left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\right) dydz$$

$$I_3(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\sinh(z^2(x)-z^2(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \times \left(\frac{\sin(z^1(x)-z^1(x-z))(z^2_x(x)-z^2_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

$$I_4(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\sin(z^1(x)-z^1(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \times \left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

We now compute ∂_x of the integrals:

$$\partial_x I_1(x)|_{x=0} = B_{11}(x) + B_{12}(x) + B_{13}(x) + B_{14}(x) + B_{15}(x) + B_{16}(x)|_{x=0}$$

We have:

$$\begin{aligned} B_{11}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))(z^1_x(x)-z^1_x(x-y))^2}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\ & \times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right. \\ & \left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz \end{aligned}$$

$$\begin{aligned} B_{12}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\cos(z^1(x)-z^1(x-y)))(z^1_{xx}(x)-z^1_{xx}(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\ & \times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right. \\ & \left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz \end{aligned}$$

$$\begin{aligned} B_{13}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\cos(z^1(x)-z^1(x-y)))(z^1_x(x)-z^1_x(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\ & \times\left(\sinh(z^2(x)-z^2(x-y))(z^2_x(x)-z^2_x(x-y))+\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\right) \\ & \times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right. \\ & \left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz \end{aligned}$$

$$\begin{aligned} B_{14}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\cos(z^1(x)-z^1(x-y)))(z^1_x(x)-z^1_x(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\ & \times\left(\frac{\cos(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))^2}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right. \\ & \left.-\frac{\cos(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))^2}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz \end{aligned}$$

$$\begin{aligned} B_{15}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\cos(z^1(x)-z^1(x-y)))(z^1_x(x)-z^1_x(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\ & \times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_{xx}(x)-z^1_{xx}(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right. \\ & \left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_{xx}(x-y)-z^1_{xx}(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz \end{aligned}$$

$$\begin{aligned} B_{16}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\cos(z^1(x)-z^1(x-y)))(z^1_x(x)-z^1_x(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\ & \times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2}\right. \\ & \times(\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))+\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))) \\ & -\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2} \\ & \left.\times(\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))+\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)))\right)dydz \end{aligned}$$

We move on to $I_{21}(x)$. Taking a derivative yields:

$$\partial_x I_{21}(x) = B_{21}(x) + B_{22}(x) + B_{23}(x) + B_{24}(x) + B_{25}(x),$$

where

$$B_{21}(x)=\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\cos(z^1(x)-z^1(x-y))(z_x^1(x)-z_x^1(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}$$
$$\times\left(\frac{\cos(z^1(x)-z^1(x-z))(z_x^1(x)-z_x^1(x-z))^2}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\cos(z^1(x-y)-z^1(x-y-z))(z_x^1(x-y)-z_x^1(x-y-z))^2}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz$$

$$B_{22}(x)=-\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\sin(z^1(x)-z^1(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2}$$
$$\times\left(\sinh(z^2(x)-z^2(x-y))(z_x^2(x)-z_x^2(x-y))+\sin(z^1(x)-z^1(x-y))(z_x^1(x)-z_x^1(x-y))\right)$$
$$\times\left(\frac{\cos(z^1(x)-z^1(x-z))(z_x^1(x)-z_x^1(x-z))^2}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\cos(z^1(x-y)-z^1(x-y-z))(z_x^1(x-y)-z_x^1(x-y-z))^2}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz$$

$$B_{23}(x)=-\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}$$
$$\times\left(\frac{\sin(z^1(x)-z^1(x-z))(z_x^1(x)-z_x^1(x-z))^3}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z_x^1(x-y)-z_x^1(x-y-z))^3}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz$$

$$B_{24}(x)=2\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}$$
$$\times\left(\frac{\cos(z^1(x)-z^1(x-z))(z_{xx}^1(x)-z_{xx}^1(x-z))(z_x^1(x)-z_x^1(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\cos(z^1(x-y)-z^1(x-y-z))(z_{xx}^1(x-y)-z_{xx}^1(x-y-z))(z_x^1(x-y)-z_x^1(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz$$

$$B_{25}(x)=-\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}$$
$$\times\left(\frac{\cos(z^1(x)-z^1(x-z))(z_x^1(x)-z_x^1(x-z))^2}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2}\right.$$
$$\times(\sinh(z^2(x)-z^2(x-z))(z_x^2(x)-z_x^2(x-z))+\sin(z^1(x)-z^1(x-z))(z_x^1(x)-z_x^1(x-z)))$$
$$-\frac{\cos(z^1(x-y)-z^1(x-y-z))(z_x^1(x-y)-z_x^1(x-y-z))^2}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}$$
$$\left.\times(\sinh(z^2(x-y)-z^2(x-y-z))(z_x^2(x-y)-z_x^2(x-y-z))+\sin(z^1(x-y)-z^1(x-y-z))(z_x^1(x-y)-z_x^1(x-y-z)))\right)dydz$$

Next we differentiate $I_{22}(x)$:

$$\partial_x I_{22}(x)=B_{31}(x)+B_{32}(x)+B_{33}(x)+B_{34}(x)+B_{35}(x),$$

where

$$B_{31}(x)=\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\cos(z^1(x)-z^1(x-y))(z_x^1(x)-z_x^1(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))}$$
$$\times\left(\frac{\sin(z^1(x)-z^1(x-z))(z_{xx}^1(x)-z_{xx}^1(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z_{xx}^1(x-y)-z_{xx}^1(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right)dydz$$

$$
\begin{aligned}
B_{32}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} &\frac{\sin(z^1(x)-z^1(x-y)))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
&\times \big(\sinh(z^2(x)-z^2(x-y))(z^2_x(x)-z^2_x(x-y))+\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\big) \\
&\times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_{xx}(x)-z^1_{xx}(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \\
&-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_{xx}(x-y)-z^1_{xx}(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{33}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} &\frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
&\times \Bigg(\frac{\cos(z^1(x)-z^1(x-z))(z^1_{xx}(x)-z^1_{xx}(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \\
&-\frac{\cos(z^1(x-y)-z^1(x-y-z))(z^1_{xx}(x-y)-z^1_{xx}(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{34}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} &\frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
&\times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_{xxx}(x)-z^1_{xxx}(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \\
&-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_{xxx}(x-y)-z^1_{xxx}(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{35}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} &\frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
&\times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_{xx}(x)-z^1_{xx}(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
&\times (\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))+\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))) \\
&-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_{xx}(x-y)-z^1_{xx}(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2} \\
&\times(\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))+\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)))\Bigg)\,dydz
\end{aligned}
$$

The differentiation of $I_{23}(x)$ follows:

$$
\partial_x I_{23}(x) = B_{41}(x)+B_{42}(x)+B_{43}(x)+B_{44}(x)+B_{45}(x)+B_{46}(x)+B_{47}(x),
$$

where

$$
\begin{aligned}
B_{41}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} &\frac{\cos(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
&\times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
&-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{42}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} &\frac{\sin(z^1(x)-z^1(x-y)))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
&\times \big(\sinh(z^2(x)-z^2(x-y))(z^2_x(x)-z^2_x(x-y))+\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\big) \\
&\times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
&-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{43}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \Bigg(\frac{\cos(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))^2 \sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
& - \frac{\cos(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))^2 \sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\Bigg) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{44}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_{xx}(x)-z^1_{xx}(x-z)) \sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
& - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_{xx}(x-y)-z^1_{xx}(x-y-z)) \sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\Bigg) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{45}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z)) \cosh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))^2}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
& - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)) \cosh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))^2}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\Bigg) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{46}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z)) \sinh(z^2(x)-z^2(x-z))(z^2_{xx}(x)-z^2_{xx}(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
& - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)) \sinh(z^2(x-y)-z^2(x-y-z))(z^2_{xx}(x-y)-z^2_{xx}(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\Bigg) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{47}(x) = 2\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z)) \sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^3} \\
& \times (\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z)) + \sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))) \\
& - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)) \sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^3} \\
& \times (\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z)) + \sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)))\Bigg) dydz
\end{aligned}
$$

We keep on differentiating, this time $I_{24}(x)$:

$$
\partial_x I_{24}(x) = B_{51}(x) + B_{52}(x) + B_{53}(x) + B_{54}(x) + B_{55}(x),
$$

which have the following expressions:

$$
\begin{aligned}
B_{51}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\cos(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \Bigg(\frac{(\sin(z^1(x)-z^1(x-z)))^2(z^1_x(x)-z^1_x(x-z))^2}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
& - \frac{(\sin(z^1(x-y)-z^1(x-y-z)))^2(z^1_x(x-y)-z^1_x(x-y-z))^2}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}\Bigg) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{52}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
& \times \left(\sinh(z^2(x)-z^2(x-y))(z^2_x(x)-z^2_x(x-y))+\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\right) \\
& \times \left(\frac{(\sin(z^1(x)-z^1(x-z)))^2(z^1_x(x)-z^1_x(x-z))^2}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \right. \\
& \left. - \frac{(\sin(z^1(x-y)-z^1(x-y-z)))^2(z^1_x(x-y)-z^1_x(x-y-z))^2}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2} \right) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{53}(x) = -2\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \left(\frac{\sin(z^1(x)-z^1(x-z))\cos(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))^3}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \right. \\
& \left. - \frac{\sin(z^1(x-y)-z^1(x-y-z))\cos(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))^3}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2} \right) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{54}(x) = -2\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \left(\frac{(\sin(z^1(x)-z^1(x-z)))^2(z^1_x(x)-z^1_x(x-z))(z^1_{xx}(x)-z^1_{xx}(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \right. \\
& \left. - \frac{(\sin(z^1(x-y)-z^1(x-y-z)))^2(z^1_x(x-y)-z^1_x(x-y-z))(z^1_{xx}(x-y)-z^1_{xx}(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2} \right) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{55}(x) = 2\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y)))}{\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y))} \\
& \times \left(\frac{(\sin(z^1(x)-z^1(x-z)))^2(z^1_x(x)-z^1_x(x-z))^2}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^3} \right. \\
& \times (\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))+\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))) \\
& - \frac{(\sin(z^1(x-y)-z^1(x-y-z)))^2(z^1_x(x-y)-z^1_x(x-y-z))^2}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^3} \\
& \left. \times(\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))+\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)))\right) dydz
\end{aligned}
$$

After that, we differentiate $I_3(x)$, resulting in:

$$
\partial_x I_3(x) = B_{61}(x)+B_{62}(x)+B_{63}(x)+B_{64}(x)+B_{65}(x)+B_{66}(x)+B_{67}(x)
$$

with

$$
\begin{aligned}
B_{61}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\cos(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))^2\sinh(z^2(x)-z^2(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
& \times \left(\frac{\sin(z^1(x)-z^1(x-z))(z^2_x(x)-z^2_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \right. \\
& \left. - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))} \right) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{62}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y))(z^1_{xx}(x)-z^1_{xx}(x-y))\sinh(z^2(x)-z^2(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
& \times \left(\frac{\sin(z^1(x)-z^1(x-z))(z^2_x(x)-z^2_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \right. \\
& \left. - \frac{\sin(z^1(x-y)-z^1(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))} \right) dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{63}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\cosh(z^2(x)-z^2(x-y))(z^2_x(x)-z^2_x(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
& \times\Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^2_x(x)-z^2_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \\
& -\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{64}(x) = 2\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\sinh(z^2(x)-z^2(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^3} \\
& \times\big(\sinh(z^2(x)-z^2(x-y))(z^2_x(x)-z^2_x(x-y))+\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\big) \\
& \times\Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^2_x(x)-z^2_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \\
& -\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{65}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\sinh(z^2(x)-z^2(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
& \times\Bigg(\frac{\cos(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))(z^2_x(x)-z^2_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \\
& -\frac{\cos(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{66}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\sinh(z^2(x)-z^2(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
& \times\Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^2_{xx}(x)-z^2_{xx}(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \\
& -\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^2_{xx}(x-y)-z^2_{xx}(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\Bigg)\,dydz
\end{aligned}
$$

$$
\begin{aligned}
B_{67}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\sinh(z^2(x)-z^2(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
& \times\Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^2_x(x)-z^2_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2} \\
& \times(\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))+\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))) \\
& -\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2} \\
& \times(\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))+\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)))\Bigg)\,dydz
\end{aligned}
$$

The last term we differentiate is $I_4(x)$, which yields

$$
\partial_x I_4(x) = B_{71}(x)+B_{72}(x)+B_{73}(x)+B_{74}(x)+B_{75}(x)+B_{76}(x)
$$

$$
\begin{aligned}
B_{71}(x) = -2\int_{\mathbb{T}}\int_{\mathbb{T}} & \frac{(\sin(z^1(x)-z^1(x-y)))(\cos(z^1(x)-z^1(x-y)))(z^1_x(x)-z^1_x(x-y))^2}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2} \\
& \times\Bigg(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))} \\
& -\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\Bigg)\,dydz
\end{aligned}
$$

$$B_{72}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(\sin(z^1(x)-z^1(x-y)))^2(z^1_{xx}(x)-z^1_{xx}(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2}$$
$$\times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

$$B_{73}(x) = 2\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(\sin(z^1(x)-z^1(x-y)))^2(z^1_x(x)-z^1_x(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^3}$$
$$\times\big(\sinh(z^2(x)-z^2(x-y))(z^2_x(x)-z^2_x(x-y))+\sin(z^1(x)-z^1(x-y))(z^1_x(x)-z^1_x(x-y))\big)$$
$$\times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

$$B_{74}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(\sin(z^1(x)-z^1(x-y)))^2(z^1_x(x)-z^1_x(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2}$$
$$\times\left(\frac{\cos(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))^2}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\cos(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))^2}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

$$B_{75}(x) = -\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(\sin(z^1(x)-z^1(x-y)))^2(z^1_x(x)-z^1_x(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2}$$
$$\times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_{xx}(x)-z^1_{xx}(x-z))}{\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z))}\right.$$
$$\left.-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_{xx}(x-y)-z^1_{xx}(x-y-z))}{\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z))}\right) dydz$$

$$B_{76}(x) = \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(\sin(z^1(x)-z^1(x-y)))^2(z^1_x(x)-z^1_x(x-y))}{(\cosh(z^2(x)-z^2(x-y))-\cos(z^1(x)-z^1(x-y)))^2}$$
$$\times\left(\frac{\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z))}{(\cosh(z^2(x)-z^2(x-z))-\cos(z^1(x)-z^1(x-z)))^2}\right.$$
$$\times(\sinh(z^2(x)-z^2(x-z))(z^2_x(x)-z^2_x(x-z))+\sin(z^1(x)-z^1(x-z))(z^1_x(x)-z^1_x(x-z)))$$
$$-\frac{\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z))}{(\cosh(z^2(x-y)-z^2(x-y-z))-\cos(z^1(x-y)-z^1(x-y-z)))^2}$$
$$\left.\times(\sinh(z^2(x-y)-z^2(x-y-z))(z^2_x(x-y)-z^2_x(x-y-z))+\sin(z^1(x-y)-z^1(x-y-z))(z^1_x(x-y)-z^1_x(x-y-z)))\right) dydz$$

Appendix C: Auxiliary tables

Integral	Degree Num. in y	Degree Num. in z	Degree Den. in y	Degree Den. in z
B_{11}	6	4	2	4
B_{12}	2	4	2	4
B_{13}	6	4	4	4
B_{14}	3	4	2	4
B_{15}	3	4	2	4
B_{16}	3	8	2	8
B_{21}	3	4	2	4
B_{22}	5	4	4	4
B_{23}	2	6	2	4
B_{24}	2	4	2	4
B_{25}	2	8	2	8
B_{31}	3	4	2	4
B_{32}	5	4	4	4
B_{33}	2	4	2	4
B_{34}	2	4	2	4
B_{35}	2	8	2	8
B_{41}	3	8	2	8
B_{42}	5	8	4	8
B_{43}	2	8	2	8
B_{44}	2	8	2	8
B_{45}	2	8	2	8
B_{46}	2	8	2	8
B_{47}	2	12	2	12
B_{51}	3	8	2	8
B_{52}	5	8	4	8
B_{53}	2	8	2	8
B_{54}	2	8	2	8
B_{55}	2	12	2	12
B_{61}	6	4	4	4
B_{62}	4	4	4	4
B_{63}	6	4	4	4
B_{64}	8	4	6	4
B_{65}	5	4	4	4
B_{66}	5	4	4	4
B_{67}	5	8	4	8
B_{71}	6	4	4	4
B_{72}	4	4	4	4
B_{73}	8	4	6	4
B_{74}	5	4	4	4
B_{75}	5	4	4	4
B_{76}	5	8	4	8

Table 1: Degree of the Taylor expansions in y and z of the different integrands written down as fractions $\frac{\text{numerator}}{\text{denominator}}$.

Integral	Bounded Region	Singularity Center	Singularity y Axis	Singularity z Axis
B_{11}	-21.93^{58}_{09}	$[-4.9\cdot10^{-13}, 4.9\cdot10^{-13}]$	$[-6.3\cdot10^{-7}, 6.3\cdot10^{-7}]$	-0.20^{71}_{50}
B_{12}	19.1^{19}_{32}	$[8.9\cdot10^{-8}, 3.7\cdot10^{-7}]$	$[-2.3\cdot10^{-3}, 2.5\cdot10^{-3}]$	0.2^{76}_{82}
B_{13}	$-2.^{13}_{03}$	$[-4.2\cdot10^{-8}, 5.6\cdot10^{-8}]$	$[-3.2\cdot10^{-5}, 3.2\cdot10^{-5}]$	$0.^{38}_{41}$
B_{14}	$4.^{39}_{41}$	$[7.8\cdot10^{-8}, 1.9\cdot10^{-7}]$	$[-1.1\cdot10^{-4}, 1.0\cdot10^{-4}]$	0.2^{76}_{82}
B_{15}	8.5^{35}_{43}	$[-9.0\cdot10^{-10}, 1.3\cdot10^{-7}]$	$[-7.7\cdot10^{-5}, 6.1\cdot10^{-5}]$	0.0^{88}_{91}
B_{16}	$1^{4.9}_{5.1}$	$[-7.3\cdot10^{-8}, 7.4\cdot10^{-8}]$	$[-7.0\cdot10^{-4}, 8.8\cdot10^{-4}]$	$-0.^{42}_{28}$
B_{21}	$4.^{39}_{41}$	$[7.8\cdot10^{-8}, 1.9\cdot10^{-7}]$	$[-1.1\cdot10^{-4}, 1.0\cdot10^{-4}]$	0.2^{76}_{82}
B_{22}	$14.^{55}_{62}$	$[-6.0\cdot10^{-8}, 6.0\cdot10^{-8}]$	$[-5.9\cdot10^{-6}, 5.9\cdot10^{-6}]$	0.1^{77}_{91}
B_{23}	-13.35^{59}_{12}	$[-9.2\cdot10^{-10}, 9.2\cdot10^{-10}]$	$[-4.1\cdot10^{-5}, 6.6\cdot10^{-5}]$	-0.0000^{46}_{33}
B_{24}	$3^{0.9}_{1.1}$	$[-2.5\cdot10^{-8}, 1.9\cdot10^{-7}]$	$[-1.5\cdot10^{-4}, 1.9\cdot10^{-4}]$	0.04^{22}_{52}
B_{25}	$-29.^{79}_{66}$	$[-3.5\cdot10^{-7}, 3.5\cdot10^{-7}]$	$[-1.7\cdot10^{-3}, 1.6\cdot10^{-3}]$	$-0.^{28}_{19}$
B_{31}	8.5^{35}_{43}	$[-9.0\cdot10^{-10}, 1.3\cdot10^{-7}]$	$[-7.7\cdot10^{-5}, 6.1\cdot10^{-5}]$	0.0^{88}_{91}
B_{32}	$10.^{09}_{14}$	$[-6.2\cdot10^{-8}, 6.3\cdot10^{-8}]$	$[-5.8\cdot10^{-6}, 5.8\cdot10^{-6}]$	0.28^{21}_{74}
B_{33}	$15.^{49}_{51}$	$[-3.6\cdot10^{-8}, 1.2\cdot10^{-7}]$	$[-7.2\cdot10^{-5}, 9.5\cdot10^{-5}]$	0.02^{11}_{26}
B_{34}	-14.94^{63}_{18}	$[-5.5\cdot10^{-8}, 5.4\cdot10^{-8}]$	$[-5.0\cdot10^{-5}, 4.9\cdot10^{-5}]$	-0.14^{52}_{36}
B_{35}	$-9.^{96}_{88}$	$[-6.3\cdot10^{-7}, 6.3\cdot10^{-7}]$	$[-1.2\cdot10^{-3}, 1.2\cdot10^{-3}]$	-0.0^{87}_{48}
B_{41}	$-6.^{62}_{47}$	$[-7.3\cdot10^{-8}, 7.4\cdot10^{-8}]$	$[-7.2\cdot10^{-4}, 8.0\cdot10^{-4}]$	$-0.^{21}_{08}$
B_{42}	$15.^{13}_{60}$	$[-5.5\cdot10^{-8}, 5.6\cdot10^{-8}]$	$[-2.5\cdot10^{-5}, 2.5\cdot10^{-5}]$	$-1.^{40}_{17}$
B_{43}	$-0.^{64}_{52}$	$[-3.5\cdot10^{-7}, 3.5\cdot10^{-7}]$	$[-1.7\cdot10^{-3}, 1.7\cdot10^{-3}]$	$-0.^{13}_{05}$
B_{44}	$9.^{07}_{15}$	$[-6.2\cdot10^{-7}, 6.3\cdot10^{-7}]$	$[-1.3\cdot10^{-3}, 1.2\cdot10^{-3}]$	-0.0^{73}_{36}
B_{45}	$-751.^{73}_{06}$	$[-3.0\cdot10^{-7}, 3.0\cdot10^{-7}]$	$[-1.4\cdot10^{-2}, 1.2\cdot10^{-2}]$	$[-4.2, -3.8]$
B_{46}	$50.^{27}_{33}$	$[-6.9\cdot10^{-7}, 6.9\cdot10^{-7}]$	$[-5.3\cdot10^{-4}, 6.1\cdot10^{-4}]$	0.6^{53}_{99}
B_{47}	$68^{5.1}_{7.2}$	$[-2.1\cdot10^{-6}, 2.1\cdot10^{-6}]$	$[-2.9\cdot10^{-2}, 3.2\cdot10^{-2}]$	$[3.7, 4.7]$
B_{51}	21.5^{63}_{84}	$[-6.8\cdot10^{-8}, 8.0\cdot10^{-8}]$	$[-2.1\cdot10^{-4}, 3.1\cdot10^{-4}]$	$-0.^{22}_{19}$
B_{52}	$-15.^{13}_{02}$	$[-8.9\cdot10^{-8}, 8.8\cdot10^{-8}]$	$[-1.1\cdot10^{-5}, 1.1\cdot10^{-5}]$	-0.0^{77}_{39}
B_{53}	$-58.^{31}_{27}$	$[-5.1\cdot10^{-8}, 5.1\cdot10^{-8}]$	$[-1.2\cdot10^{-3}, 9.7\cdot10^{-4}]$	$-0.^{31}_{28}$
B_{54}	-38.0^{75}_{55}	$[-5.5\cdot10^{-8}, 5.5\cdot10^{-8}]$	$[-6.8\cdot10^{-4}, 6.7\cdot10^{-4}]$	-0.0^{32}_{19}
B_{55}	$10^{3.5}_{4.1}$	$[-5.1\cdot10^{-8}, 5.1\cdot10^{-8}]$	$[-8.3\cdot10^{-3}, 8.5\cdot10^{-3}]$	$0.^{34}_{52}$
B_{61}	48.5^{07}_{82}	$[-4.9\cdot10^{-8}, 6.3\cdot10^{-8}]$	$[2.5\cdot10^{-5}, 4.1\cdot10^{-5}]$	0.5^{09}_{27}
B_{62}	33.6^{46}_{95}	$[-2.2\cdot10^{-7}, 2.2\cdot10^{-7}]$	$[-5.0\cdot10^{-4}, 5.4\cdot10^{-4}]$	0.4^{77}_{88}
B_{63}	$-61^{5.3}_{4.7}$	$[-4.7\cdot10^{-8}, 4.7\cdot10^{-8}]$	$[-5.4\cdot10^{-6}, 5.4\cdot10^{-6}]$	0.4^{18}_{96}
B_{64}	$49^{4.9}_{5.9}$	$[-1.1\cdot10^{-7}, 1.1\cdot10^{-7}]$	$[-4.7\cdot10^{-6}, 4.7\cdot10^{-6}]$	$-0.^{54}_{41}$
B_{65}	$49.^{09}_{16}$	$[-6.1\cdot10^{-8}, 6.1\cdot10^{-8}]$	$[-4.9\cdot10^{-6}, 4.9\cdot10^{-6}]$	0.4^{77}_{88}
B_{66}	7.4^{18}_{57}	$[-6.2\cdot10^{-8}, 6.2\cdot10^{-8}]$	$[-4.7\cdot10^{-6}, 4.7\cdot10^{-6}]$	-0.0^{85}_{63}
B_{67}	$-24.^{62}_{15}$	$[-5.7\cdot10^{-8}, 5.7\cdot10^{-8}]$	$[-1.1\cdot10^{-5}, 1.1\cdot10^{-5}]$	$-0.^{68}_{48}$
B_{71}	$-84.^{81}_{78}$	$[-8.1\cdot10^{-10}, 8.0\cdot10^{-10}]$	$[-3.5\cdot10^{-6}, 3.6\cdot10^{-6}]$	-0.62^{79}_{12}
B_{72}	-29.3^{69}_{59}	$[-1.4\cdot10^{-7}, 7.0\cdot10^{-8}]$	$[-4.6\cdot10^{-6}, 4.6\cdot10^{-6}]$	-0.22^{39}_{18}
B_{73}	$13^{7.8}_{8.1}$	$[-8.8\cdot10^{-10}, 5.4\cdot10^{-10}]$	$[-3.0\cdot10^{-6}, 2.8\cdot10^{-6}]$	0.8^{57}_{94}
B_{74}	7.0^{85}_{96}	$[-2.1\cdot10^{-9}, 2.4\cdot10^{-9}]$	$[-3.5\cdot10^{-6}, 3.3\cdot10^{-6}]$	-0.22^{39}_{17}
B_{75}	-2.8^{23}_{14}	$[-4.2\cdot10^{-9}, 4.5\cdot10^{-9}]$	$[-3.9\cdot10^{-6}, 3.7\cdot10^{-6}]$	0.04^{29}_{29}
B_{76}	$-35.^{63}_{52}$	$[-5.0\cdot10^{-8}, 5.2\cdot10^{-8}]$	$[-3.6\cdot10^{-6}, 3.6\cdot10^{-6}]$	0.0^{42}_{68}

Table 2: Detailed breakdown of the rigorous integration results.

Term and region	Number of integrals	Time (HH:MM)
B_{11}-B_{76} (nonsingular)	82	14:48
B_{11}-B_{76} (center-singular)	82	02:03
B_{11}-B_{76} (singular-first)	82	01:26
B_{11}-B_{16} (singular-second)	12	11:57
B_{21}-B_{25} (singular-second)	10	09:57
B_{31}-B_{35} (singular-second)	10	11:29
B_{41}-B_{46} (singular-second)	12	32:19
B_{51}-B_{54} (singular-second)	8	16:44
B_{61}-B_{67} (singular-second)	14	13:59
B_{71}-B_{76} (singular-second)	12	09:46
B_{47} (singular-second - subregions 1 and 2)	4	35:53
B_{47} (singular-second - subregions 3 and 4)	4	60:48
B_{47} (singular-second - subregions 5 and 6)	4	82:02
B_{55} (singular-second - subregions 1 and 2)	4	16:02
B_{55} (singular-second - subregions 3 and 4)	4	56:12
B_{55} (singular-second - subregions 5 and 6)	4	74:50

Table 3: Performance of the code in the different integrals and regions.