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
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THE WEAK- A_∞ PROPERTY OF HARMONIC AND p -HARMONIC MEASURES IMPLIES UNIFORM RECTIFIABILITY

STEVE HOFMANN, PHI LE, JOSÉ MARÍA MARTELL AND KAJ NYSTRÖM

Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an Ahlfors–David regular set of dimension n . We show that the weak- A_∞ property of harmonic measure, for the open set $\Omega := \mathbb{R}^{n+1} \setminus E$, implies uniform rectifiability of E . More generally, we establish a similar result for the Riesz measure, p -harmonic measure, associated to the p -Laplace operator, $1 < p < \infty$.

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1. Introduction

In this paper we prove quantitative, scale invariant results of free boundary type, for harmonic measure and, more generally, for p -harmonic measure. More precisely, let $\Omega \subset \mathbb{R}^{n+1}$ be an open set (not necessarily connected nor bounded) satisfying an interior corkscrew condition, whose boundary is n -dimensional Ahlfors–David regular (ADR) (see Definition 2.1). Given these background hypotheses we prove that if ω , the harmonic measure for Ω , is absolutely continuous with respect to σ , and if the Poisson kernel $k = d\omega/d\sigma$ verifies an appropriate scale invariant higher integrability estimate (in particular, if ω belongs to weak- A_∞ with respect to σ), then $\partial\Omega$ is uniformly rectifiable in the sense of [David and Semmes 1991; 1993]; see Theorem 1.1 and Corollary 1.5 below. In particular, our background hypotheses hold in the case that $\Omega := \mathbb{R}^{n+1} \setminus E$ is the complement of an ADR set of codimension 1, as in that case it is well known that the corkscrew condition is verified automatically in Ω , i.e., in every ball $B = B(x, r)$ centered on E , there is some component of $\Omega \cap B$ that contains a point Y with $\text{dist}(Y, E) \approx r$. Furthermore, our argument is general enough to allow us to establish a nonlinear version of Theorem 1.1 (see Theorem 1.12 below) involving the p -Laplace operator, p -harmonic functions, and p -harmonic measure.

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To briefly outline previous work, in [Hofmann et al. 2014] the first and third authors, together with I. Uriarte-Tuero, proved the same result (cf. Theorem 1.1 and Corollary 1.5) under the additional strong hypothesis that Ω is a connected domain, satisfying an interior Harnack chain condition. In hindsight, under that extra assumption, one obtains the stronger conclusion that the exterior domain $\mathbb{R}^{n+1} \setminus \bar{\Omega}$ in fact also satisfies a corkscrew condition, and hence that Ω is an NTA domain in the sense of [Jerison and Kenig 1982]; see [Azzam et al. 2014] for the details. Compared to [Hofmann et al. 2014] the main new advances in the present paper are two. First, we remove any connectivity hypothesis; in particular, we avoid the Harnack chain condition. Second, we are able to establish a version of our results also in the nonlinear case $1 < p < \infty$. Our main results — Theorem 1.1, Corollary 1.5, and Theorem 1.12 — are new even in the linear case $p = 2$.

Our approach is decidedly influenced by prior work of Lewis and Vogel [2006; 2007]. In particular, a version of Theorem 1.12 and Theorem 1.1 was proved in [Lewis and Vogel 2007], under the stronger hypothesis that p -harmonic measure μ itself is an Ahlfors–David regular measure, which in the linear case $p = 2$ implies that the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. However, to weaken the hypotheses on ω and μ , as we have done here, requires further considerations, which we discuss below in Section 1B.

To provide some additional context, we mention that our results here may be viewed as “large constant” analogues of results of Kenig and Toro [2003] in the linear case $p = 2$, and of J. Lewis and Nyström [2012], in the general p -harmonic case $1 < p < \infty$. These authors show that in the presence of a Reifenberg flatness condition and Ahlfors–David regularity, $\log k \in \text{VMO}$ implies that the unit normal ν to the boundary belongs to VMO, where k is either the Poisson kernel with pole at some fixed point or the density of p -harmonic Riesz measure associated to a particular ball $B(x, r)$. Moreover, under the same background hypotheses, the condition $\nu \in \text{VMO}$ is equivalent to a uniform rectifiability (UR) condition with vanishing trace. Thus $\log k \in \text{VMO} \implies$ vanishing UR, given sufficient Reifenberg flatness. On the other hand, our large constant version “almost” says “ $\log k \in \text{BMO} \implies \text{UR}$ ”. Indeed, it is well known that the A_∞ condition, i.e., weak- A_∞ plus the doubling property, implies that $\log k \in \text{BMO}$, while if $\log k \in \text{BMO}$ with small norm, then $k \in A_\infty$. We further note that, in turn, the results of [Kenig and Toro 2003] may be viewed as an “endpoint” version of the free boundary results of [Alt and Caffarelli 1981; Jerison 1990], which establish, again in the presence of Reifenberg flatness, that Hölder continuity of $\log k$ implies that of the unit normal ν (and indeed, that $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha > 0$).

1A. Statement of main results. Given an open set $\Omega \subset \mathbb{R}^{n+1}$, and a Euclidean ball $B = B(x, r) \subset \mathbb{R}^{n+1}$ centered on $\partial\Omega$, we let $\Delta = \Delta(x, r) := B \cap \partial\Omega$ denote the corresponding surface ball. For $X \in \Omega$, let ω^X be harmonic measure for Ω , with pole at X . As mentioned above, all other terminology and notation will be defined below.

Concerning the Laplace operator and harmonic measure we prove the following results.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension n (see Definition 2.1). Suppose that there are positive constants C_0 and c_0 , and an exponent $q > 1$, such*

that for every surface ball $\Delta = \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, there exists $X_\Delta \in B(x, r) \cap \Omega$, with $\text{dist}(X_\Delta, \partial\Omega) \geq c_0 r$, satisfying

(\star) **scale-invariant higher integrability:** $\omega^{X_\Delta} \ll \sigma$ in 2Δ , and $k^{X_\Delta} := d\omega^{X_\Delta}/d\sigma$ satisfies

$$\int_{2\Delta} k^{X_\Delta}(y)^q d\sigma(y) \leq C_0 \sigma(\Delta)^{1-q}. \tag{1.2}$$

Then $\partial\Omega$ is uniformly rectifiable and moreover the “UR character” (see Definition 2.4) depends only on n , the ADR constants, q , c_0 , and C_0 .

The point X_Δ in Theorem 1.1 is a “corkscrew point” for Ω , relative to Δ . An open set Ω for which there is such a point relative to every surface ball $\Delta(x, r)$, $x \in \partial\Omega$, $0 < r < \text{diam}(\partial\Omega)$, with a uniform constant c_0 , is said to satisfy the “corkscrew condition” (see Definition 2.5 below).

Remark 1.3. We note that, in lieu of absolute continuity and (\star), only the following apparently weaker condition is actually used in the proof of Theorem 1.1:

($\star\star$) **local nondegeneracy:** there exist uniform constants $\eta, \beta > 0$ such that if $A \subset \Delta$ is Borel measurable, then

$$\sigma(A) \geq (1 - \eta)\sigma(\Delta) \implies \omega^{X_\Delta}(A) \geq \beta\omega^{X_\Delta}(\Delta).^1 \tag{1.4}$$

Here $\Delta = \Delta(x, r)$ for $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, and $X_\Delta \in B(x, r/2) \cap \Omega$ with $\text{dist}(X_\Delta, \partial\Omega) \geq c_0 r/2$.² We observe that there turns out to be some flexibility in the choice of X_Δ (see the discussion at the beginning of Section 4), and consequently it is not hard to see that (\star) implies ($\star\star$); see Lemma 4.3.

We also have the following easy corollary of Theorem 1.1 (we shall give the short proof of the corollary in Section 5D).

Corollary 1.5. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set satisfying the corkscrew condition, whose boundary is Ahlfors–David regular of dimension n . Suppose further that for every ball $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, and every $Y \in \Omega \setminus B(x, 2r)$, harmonic measure ω^Y belongs to weak- $A_\infty(\Delta(x, r))$, i.e., there is a constant $C_0 \geq 1$ and an exponent $q > 1$, each of which is uniform with respect to x, r , and Y , such that $\omega^Y \ll \sigma$ in $\Delta(x, r)$, and $k^Y = d\omega^Y/d\sigma$ satisfies

$$\left(\int_{\Delta'} k^Y(z)^q d\sigma(z) \right)^{1/q} \leq C_0 \int_{2\Delta'} k^Y(z) d\sigma(z) \tag{1.6}$$

for every surface ball centered on the boundary $\Delta' = B' \cap \partial\Omega$ with $2B' \subset B(x, r)$. Then $\partial\Omega$ is uniformly rectifiable, and moreover, the “UR character” (see Definition 2.4) depends only on n , the ADR constant of $\partial\Omega$, q , C_0 , and the corkscrew constant.

Remark 1.7. As mentioned above, the corkscrew condition is automatically satisfied in the case that E is an n -dimensional ADR set (hence closed, see Definition 2.1 below), and $\Omega = \mathbb{R}^{n+1} \setminus E$ is its complement, with the corkscrew constant for Ω depending only on n and the ADR constant of E . Thus, in particular,

¹This formulation is adapted from [Mourgoglou and Tolsa 2015]; see the discussion in Section 1D.

²For aesthetic reasons, and for convenience in the sequel, in contrast to condition (\star), we prefer to state condition ($\star\star$) in terms of Δ rather than 2Δ , and with $X_\Delta \in B(x, r/2)$ rather than $B(x, r)$.

Corollary 1.5 applies in that setting, so in the presence of the weak reverse Hölder condition (1.6), we deduce that E is uniformly rectifiable.

Combining Theorem 1.1 with the results in [Bortz and Hofmann 2015], we obtain as an immediate consequence a “big pieces” characterization of uniformly rectifiable sets of codimension 1, in terms of harmonic measure. Here and in the sequel, given an ADR set E , Q denotes a “dyadic cube” on E in the sense of [David and Semmes 1991; 1993; Christ 1990], and $\mathbb{D}(E)$ denotes the collection of all such cubes; see Lemma 2.6 below.

Theorem 1.8. *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an n -dimensional ADR set. Let $\Omega := \mathbb{R}^{n+1} \setminus E$. Then E is uniformly rectifiable if and only if it has “big pieces of good harmonic measure estimates” in the following sense: for each $Q \in \mathbb{D}(E)$ there exists an open set $\tilde{\Omega} = \tilde{\Omega}_Q$ with the following properties, with uniform control of the various implicit constants:*

- $\partial\tilde{\Omega}$ is ADR;
- the interior corkscrew condition holds in $\tilde{\Omega}$;
- $\partial\tilde{\Omega}$ has a “big pieces” overlap with E , in the sense that $\sigma(Q \cap \partial\tilde{\Omega}) \gtrsim \sigma(Q)$;
- for each surface ball $\Delta = \Delta(x, r) := B(x, r) \cap \partial\tilde{\Omega}$ with $x \in \partial\tilde{\Omega}$ and $r \in (0, \text{diam}(\tilde{\Omega}))$, there is an interior corkscrew point $X_\Delta \in \tilde{\Omega}$ such that $\omega_{\tilde{\Omega}}^{X_\Delta}$, the harmonic measure for $\tilde{\Omega}$ with pole at X_Δ , satisfies $\omega_{\tilde{\Omega}}^{X_\Delta}(\Delta) \gtrsim 1$, and belongs to weak- $A_\infty(\Delta)$.

The “only if” direction is proved in [Bortz and Hofmann 2015], and the open sets $\tilde{\Omega}$ constructed in [Bortz and Hofmann 2015] even satisfy a 2-sided corkscrew condition, and moreover, $\tilde{\Omega} \subset \Omega$ with $\text{diam}(\tilde{\Omega}) \approx \text{diam}(Q)$. To obtain the converse direction, we simply observe that by Theorem 1.1, the subdomains $\tilde{\Omega}$ have uniformly rectifiable boundaries, with uniform control of the “UR character” of each $\partial\tilde{\Omega}$, and thus, by [David and Semmes 1993], E is uniformly rectifiable.

To formulate our main result in the nonlinear setting we first need to introduce some notation. If $O \subset \mathbb{R}^{n+1}$ is an open set and $1 \leq p \leq \infty$, then by $W^{1,p}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_{n+1}})$, both of which are q -th power integrable on O . Let $\|f\|_{1,p} = \|f\|_p + \|\nabla f\|_p$ be the norm in $W^{1,p}(O)$, where $\|\cdot\|_q$ denotes the usual Lebesgue q norm in O . Next, let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O , and let $W_0^{1,p}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,p}(O)$. We let $W_{\text{loc}}^{1,p}(O)$ be the set of all functions u such that $u\Theta \in W_0^{1,p}(O)$ whenever $\Theta \in C_0^\infty(O)$.

Given an open set O and $1 < p < \infty$, we say that u is p -harmonic in O provided $u \in W_{\text{loc}}^{1,p}(O)$ and

$$\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla \Theta \, dX = 0, \quad \forall \Theta \in C_0^\infty(O). \quad (1.9)$$

Observe that if u is smooth and $\nabla u \neq 0$ in O , then

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \quad \text{in } O, \quad (1.10)$$

and u is a classical solution in O to the p -Laplace partial differential equation. Here, as in the sequel, $\nabla \cdot$ is the divergence operator.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with n -dimensional ADR boundary. Let $p \in (1, \infty)$. Given $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, let u be a nonnegative p -harmonic function in $\Omega \cap B(x, r)$ which vanishes continuously on $\Delta(x, r) := B(x, r) \cap \partial\Omega$. Extend u to all of $B(x, r)$ by putting $u \equiv 0$ on $B(x, r) \setminus \bar{\Omega}$. Then there exists a unique nonnegative finite Borel measure μ on \mathbb{R}^{n+1} , with support contained in $\Delta(x, r)$, such that

$$-\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dX = \int_{\partial\Omega} \phi \, d\mu, \quad \forall \phi \in C_0^\infty(B(x, r)); \tag{1.11}$$

see [Heinonen et al. 2006, Chapter 21] and Lemma 3.43 below. We refer to μ as the p -harmonic measure associated to u . In the case $p = 2$, and if u is the Green function for Ω with pole at $X \in \Omega$, then the measure μ coincides with harmonic measure at X , $\omega = \omega^X$.

Concerning the p -Laplace operator, p -harmonic functions, and p -harmonic measure, we prove the following theorem.

Theorem 1.12. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension n . Let p , $1 < p < \infty$, be given. Let C be a sufficiently large constant (to be specified), depending only on n and the ADR constant, and suppose that there exist $q > 1$ and a positive constant C_0 for which the following holds: for each $x \in \partial\Omega$ and each $0 < r < \text{diam}(\partial\Omega)$, there is a nontrivial, nonnegative p -harmonic function $u = u_{x,r}$ in $\Omega \cap B(x, Cr)$, and corresponding p -harmonic measure $\mu = \mu_{x,r}$, such that $\mu \ll \sigma$ in $\Delta(x, Cr)$, and such that $k := d\mu/d\sigma$ satisfies*

$$\left(\int_{\Delta(x, Cr)} k(y)^q \, d\sigma(y) \right)^{1/q} \leq C_0 \frac{\mu(\Delta(x, r))}{\sigma(\Delta(x, r))}. \tag{1.13}$$

Then $\partial\Omega$ is uniformly rectifiable, and moreover the “UR character” (see Definition 2.4) depends only on n , the ADR constant, p , q , and C_0 .

Some remarks are in order concerning the hypotheses of Theorem 1.12. Let us observe that, in particular, Ahlfors–David regularity and (1.13) imply that

$$\mu(\Delta(x, Cr)) \leq C_1 \mu(\Delta(x, r)), \tag{1.14}$$

with $C_1 \approx C_0$. In the linear case, the latter estimate follows automatically, with $\mu = \omega^Y$ for some $Y \in B(x, r)$ such that $\text{dist}(Y, E) \approx r$, and with C_1 depending only on n and the ADR constant, by Bourgain’s Lemma 3.1 below, even though ω^Y need not be a doubling measure (i.e., (1.14) says nothing about points other than x nor about scales other than r). In the nonlinear case, it seems that we must impose condition (1.14) by hypothesis. We also observe that (1.13) holds in particular if $\mu \in \text{weak-}A_\infty(\Delta(x, 2Cr))$ and satisfies (1.14) (with radius $2C$ in place of C). Of course, (1.14) holds trivially if μ is a doubling measure, but we do not assume doubling.

Remark 1.15. We note that, as in Remark 1.3, the proof of Theorem 1.12 will in fact use, in lieu of absolute continuity and (1.13), only the apparently weaker condition that there exist uniform constants $\eta, \beta \in (0, 1)$ such that for all $\Delta = \Delta(x, r)$, and for all Borel sets $A \subset \Delta$,

$$\sigma(A) \geq (1 - \eta)\sigma(\Delta) \implies \mu(A) \geq \beta\mu(\Delta). \tag{1.16}$$

1B. Brief outline of the proofs of the main results. As mentioned, the approach in the present paper is strongly influenced by prior work due to Lewis and Vogel [2006; 2007], who in the latter paper proved a version of Theorem 1.12, and Theorem 1.1, under the stronger hypothesis that p -harmonic measure μ itself is an Ahlfors–David regular measure. In the linear case $p = 2$, this implies that the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. Assuming that p -harmonic measure μ is an Ahlfors–David regular measure, Lewis and Vogel were able to show that E satisfies the so-called weak exterior convexity (WEC) condition, which characterizes uniform rectifiability [David and Semmes 1993]. To weaken the hypotheses on ω and μ , as we have done here, requires two further considerations. The first is quite natural in this context: a stopping time argument, in the spirit of the proofs of the Kato square root conjecture [Hofmann and McIntosh 2002; Hofmann et al. 2002; Auscher et al. 2002a] (and of local Tb theorems [Christ 1990; Auscher et al. 2002b; Hofmann 2006]), by means of which we extract ample dyadic sawtooth regimes on which averages of harmonic measure and p -harmonic measure are bounded and accretive; see Lemma 4.12 below. This allows us to use the arguments of [Lewis and Vogel 2007] within these good sawtooth regions. The second new consideration is necessitated by the fact that in our setting, the doubling property may fail for harmonic and p -harmonic measure. In the absence of doubling, we are unable to obtain the WEC condition directly. Nonetheless, we are able to follow the arguments of [Lewis and Vogel 2007] very closely up to a point, to obtain a condition on $\partial\Omega$ which we call the “weak half space approximation” (WHSA) property (see Definition 2.19). Indeed, extracting the essence of the argument of [Lewis and Vogel 2007], while dispensing with the doubling property, one realizes that the WHSA is precisely what one obtains. In the sequel, we present the argument of [Lewis and Vogel 2007] as Lemma 5.10. Finally, having obtained that $\partial\Omega$ satisfies the WHSA property, we are able to prove the following proposition stating that WHSA implies uniform rectifiability.

Proposition 1.17. *An n -dimensional ADR set $E \subset \mathbb{R}^{n+1}$ is uniformly rectifiable if and only if it satisfies the WHSA property.*

While the WHSA condition, per se, is new, our proof of Proposition 1.17 is based on a modified version of part of the argument in [Lewis and Vogel 2007].

1C. Organization of the paper. The paper is organized as follows. In Section 2, we state several definitions, including definitions of ADR, UR, and dyadic grids, and introduce further notions and notation. In Section 3, we state, and either prove or give references for, the PDE estimates needed in the proofs of our main results. In Section 4, we begin the (simultaneous) proofs of Theorem 1.1 and Theorem 1.12 by giving some preliminary arguments. In Section 5, following [Lewis and Vogel 2006; 2007], we complete the proofs of Theorem 1.1 and Theorem 1.12, modulo Proposition 1.17. At the end of Section 5 we also give the (very short) proof of Corollary 1.5. In Section 6, we give the proof of Proposition 1.17, i.e., the proof of the fact that the WHSA condition implies uniform rectifiability.

1D. Discussion of recent related work. We note that some related work has recently appeared, or been carried out, while this manuscript was in preparation. In the setting of uniform domains with lower ADR boundary with locally finite n -dimensional Hausdorff measure, Mourougolou [2015] has shown that

rectifiability of the boundary implies absolute continuity of surface measure with respect to harmonic measure (for the Laplacian). Akman, Badger, Hofmann, and Martell [Akman et al. 2015], in the setting of uniform domains with ADR boundary, have characterized the rectifiability of the boundary in terms of the absolute continuity of harmonic measure and some elliptic measures and surface measure or in terms of some qualitative A_∞ condition. Also, Azzam, Mourgoglou, and Tolsa [Azzam et al. 2015] have obtained that absolute continuity of harmonic measure with respect to surface measure on an H^n -finite piece of the boundary implies that harmonic measure is rectifiable in that piece. The setting is very general as they only assume a “porosity” (i.e., corkscrew) condition in the complement of $\partial\Omega$. In [Hofmann et al. 2015], Hofmann, Martell, Mayboroda, Tolsa, and Volberg prove the same result removing the porosity assumption. Both [Azzam et al. 2015] and the follow-up version [Hofmann et al. 2015] (which will be combined in the forthcoming paper [Azzam et al. 2016]) rely on recent deep results of [Nazarov et al. 2014a; 2014b], concerning connections between rectifiability and the behavior of Riesz transforms.

Finally, we discuss two closely related papers treating the case $p = 2$. First, we mention that a preliminary version of our results, treating only the linear harmonic case (i.e., Theorem 1.1 of the present paper) under hypothesis (\star) , appeared earlier in the unpublished preprint [Hofmann and Martell 2015]. That result, again in the case $p = 2$, was then essentially reproved, by a different method, in [Mourgoglou and Tolsa 2015], but assuming condition $(\star\star)$ in place of (\star) . While the present paper was in preparation, we learned of the work in [Mourgoglou and Tolsa 2015], and we realized that our arguments (and those of [Hofmann and Martell 2015]), almost unchanged, also allow (\star) to be replaced by $(\star\star)$ or its p -harmonic equivalent. The current version of this manuscript incorporates this observation.³ Let us mention also that the approach in [Mourgoglou and Tolsa 2015] is based on showing that $(\star\star)$ for harmonic measure implies L^2 -boundedness of the Riesz transforms, and thus it is a quantitative version of the method of [Azzam et al. 2016]. An interesting feature of the proof in [Mourgoglou and Tolsa 2015] is that it works even without the lower bound in the Ahlfors–David condition; in that case, one may deduce rectifiability, as opposed to uniform rectifiability, of the underlying measure on $\partial\Omega$. On the other hand, it seems difficult to generalize the approach of [Mourgoglou and Tolsa 2015] to the p -Laplace setting, since it is based on Riesz transforms, which are tied to the linear harmonic case.

2. ADR, UR, and dyadic grids

Definition 2.1 (Ahlfors–David regular (ADR)). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n , is ADR if it is closed and if there is some uniform constant C such that

$$C^{-1}r^n \leq \sigma(\Delta(x, r)) \leq Cr^n, \quad \forall r \in (0, \text{diam}(E)), \quad x \in E, \quad (2.2)$$

where $\text{diam}(E)$ may be infinite. Here, $\Delta(x, r) := E \cap B(x, r)$ is the “surface ball” of radius r , and $\sigma := H^n|_E$ is the “surface measure” on E , where H^n denotes n -dimensional Hausdorff measure.

Definition 2.3 (uniformly rectifiable (UR)). An n -dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains “big pieces of Lipschitz images” of \mathbb{R}^n (BPLI). This means that there are positive

³We thank Mourgoglou and Tolsa for making their preprint available to us while our manuscript was in preparation.

constants θ and M_0 , such that for each $x \in E$ and each $r \in (0, \text{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$, with Lipschitz constant no larger than M_0 , such that

$$H^n(E \cap B(x, r) \cap \rho(\{z \in \mathbb{R}^n : |z| < r\})) \geq \theta r^n.$$

We recall that n -dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of \mathbb{R}^n ; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all “sufficiently nice” singular integrals are L^2 -bounded [David and Semmes 1991]. In fact, for n -dimensional ADR sets in \mathbb{R}^{n+1} , the L^2 -boundedness of certain special singular integral operators (the “Riesz transforms”) suffices to characterize uniform rectifiability (see [Mattila et al. 1996] for the case $n = 1$, and [Nazarov et al. 2014a] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior corkscrew and Harnack chain conditions), but that are totally nonrectifiable (e.g., see the construction of Garnett’s “4-corners Cantor set” in [David and Semmes 1993, Chapter 1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher codimensions); see [David and Semmes 1991; 1993], and in particular Theorem 2.14 below. In this paper, we also present a new characterization of UR sets of codimension 1 (see Proposition 1.17 below), which will be very useful in the proof of Theorem 1.1.

Definition 2.4 (UR character). Given a UR set $E \subset \mathbb{R}^{n+1}$, its “UR character” is just the pair of constants (θ, M_0) involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

Definition 2.5 (corkscrew condition). Following [Jerison and Kenig 1982], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the “corkscrew condition” if for some uniform constant $c_0 > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, there is a point $X_\Delta \in B(x, r) \cap \Omega$ such that $\text{dist}(X_\Delta, \partial\Omega) \geq c_0 r$. The point $X_\Delta \subset \Omega$ is called a “corkscrew point” relative to Δ .

Lemma 2.6 (existence and properties of the “dyadic grid” [David and Semmes 1991; 1993; Christ 1990]). *Suppose that $E \subset \mathbb{R}^{n+1}$ is a closed n -dimensional ADR set. Then there exist constants $a_0 > 0$, $\gamma > 0$, and $C_* < \infty$, depending only on n and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathcal{J}_k\}$$

of Borel sets (“cubes”), where \mathcal{J}_k denotes some (possibly finite) index set depending on k , satisfying

- (i) $E = \bigcup_j Q_j^k$ for each $k \in \mathbb{Z}$;
- (ii) if $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$;
- (iii) for each (j, k) and each $m < k$, there is a unique i such that $Q_j^k \subset Q_i^m$;
- (iv) $\text{diam}(Q_j^k) \leq C_* 2^{-k}$;
- (v) each Q_j^k contains some “surface ball” $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$;
- (vi) $H^n(\{x \in Q_j^k : \text{dist}(x, E \setminus Q_j^k) \leq \varrho 2^{-k}\}) \leq C_* \varrho^\gamma H^n(Q_j^k)$ for all k, j and for all $\varrho \in (0, a_0)$.

Let us make a few remarks concerning this lemma, and discuss some related notation and terminology.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [1990], with the dyadic parameter $\frac{1}{2}$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = \frac{1}{2}$ (cf. [Hofmann et al. 2017, Proof of Proposition 2.12]). In the presence of the Ahlfors–David property (2.2), the result already appears in [David and Semmes 1991; 1993].
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \text{diam}(E)$, in the case that the latter is finite.
- We denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k , i.e.,

$$\mathbb{D} := \bigcup_k \mathbb{D}_k,$$

where, if $\text{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \text{diam}(E)$.

- Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$, and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $r \approx 2^{-k} \approx \text{diam}(Q)$ and

$$\Delta(x_Q, r) \subset Q \subset \Delta(x_Q, Cr) \tag{2.7}$$

for some uniform constant C . We denote this ball and surface ball by

$$B_Q := B(x_Q, r), \quad \Delta_Q := \Delta(x_Q, r), \tag{2.8}$$

and we refer to the point x_Q as the “center” of Q .

- Given a dyadic cube $Q \in \mathbb{D}$, we define its “ κ -dilate” by

$$\kappa Q := E \cap B(x_Q, \kappa \text{diam}(Q)). \tag{2.9}$$

- For a dyadic cube $Q \in \mathbb{D}_k$, we set $\ell(Q) = 2^{-k}$, and we refer to this quantity as the “length” of Q . Clearly, $\ell(Q) \approx \text{diam}(Q)$.
- For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the “dyadic generation” to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.
- For any $Q \in \mathbb{D}(E)$, we set $\mathbb{D}_Q := \{Q' \in \mathbb{D} : Q' \subset Q\}$.
- Given $Q_0 \in \mathbb{D}(E)$ and a family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}$ of pairwise disjoint cubes, we set

$$\mathbb{D}_{\mathcal{F}, Q_0} := \{Q \in \mathbb{D}_{Q_0} : Q \text{ is not contained in any } Q_j \in \mathcal{F}\} = \mathbb{D}_{Q_0} \setminus \left(\bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} \right). \tag{2.10}$$

Definition 2.11 (ε -local BAUP). Given $\varepsilon > 0$, we say that $Q \in \mathbb{D}(E)$ satisfies the ε -local BAUP condition if there is a family \mathcal{P} of hyperplanes (depending on Q) such that every point in $10Q$ is at a distance at most $\varepsilon \ell(Q)$ from $\bigcup_{P \in \mathcal{P}} P$, and every point in $(\bigcup_{P \in \mathcal{P}} P) \cap B(x_Q, 10 \text{diam}(Q))$ is at a distance at most $\varepsilon \ell(Q)$ from E .

Definition 2.12 (BAUP). We say that an n -dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the condition of bilateral approximation by unions of planes (BAUP) if for some $\varepsilon_0 > 0$, and for every positive $\varepsilon < \varepsilon_0$,

there is a constant C_ε such that the set \mathcal{B} of bad cubes in $\mathbb{D}(E)$, for which the ε -local BAUP condition fails, satisfies the packing condition

$$\sum_{Q' \subset Q, Q' \in \mathcal{B}} \sigma(Q') \leq C_\varepsilon \sigma(Q), \quad \forall Q \in \mathbb{D}(E). \quad (2.13)$$

For future reference, we recall the following result of David and Semmes.

Theorem 2.14 [David and Semmes 1993, Theorem I.2.18, p. 36]. *Let $E \subset \mathbb{R}^{n+1}$ be an n -dimensional ADR set. Then E is uniformly rectifiable if and only if it satisfies BAUP.*

We remark that the definition of BAUP in [David and Semmes 1993] is slightly different in superficial appearance, but it is not hard to verify that the dyadic version stated here is equivalent to their condition. We note that we shall not need the full strength of this equivalence here, but only the fact that our version of BAUP implies the version in [David and Semmes 1993], and hence implies UR.

We also require a new characterization of UR sets of codimension 1, which is related to the BAUP and its variants. For a sufficiently large constant K_0 to be chosen (see Lemma 4.24 below), we set

$$B_Q^* := B(x_Q, K_0^2 \ell(Q)), \quad \Delta_Q^* := B_Q^* \cap E. \quad (2.15)$$

Given a small positive number ε , which we typically assume to be much smaller than K_0^{-6} , we also set

$$B_Q^{**} = B_Q^{**}(\varepsilon) := B(x_Q, \varepsilon^{-2} \ell(Q)), \quad B_Q^{***} = B_Q^{***}(\varepsilon) := B(x_Q, \varepsilon^{-5} \ell(Q)). \quad (2.16)$$

Definition 2.17 (ε -local WHSA). Given $\varepsilon > 0$, we say that $Q \in \mathbb{D}(E)$ satisfies the ε -local WHSA condition (or more precisely, the “ ε -local WHSA with parameter K_0 ”) if there is a half-space $H = H(Q)$, a hyperplane $P = P(Q) = \partial H$, and a fixed positive number K_0 satisfying

- (1) $\text{dist}(Z, E) \leq \varepsilon \ell(Q)$ for every $Z \in P \cap B_Q^{**}(\varepsilon)$,
- (2) $\text{dist}(Q, P) \leq K_0^{3/2} \ell(Q)$, and
- (3) $H \cap B_Q^{**}(\varepsilon) \cap E = \emptyset$.

Note that part (2) of the previous definition says that the hyperplane P has an “ample” intersection with the ball $B_Q^{**}(\varepsilon)$. Indeed,

$$\text{dist}(x_Q, P) \lesssim K_0^{3/2} \ell(Q) \ll \varepsilon^{-2} \ell(Q). \quad (2.18)$$

Definition 2.19 (WHSA). We say that an n -dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the *weak half-space approximation* property (WHSA) if for some pair of positive constants ε_0 and K_0 , and for every positive $\varepsilon < \varepsilon_0$, there is a constant C_ε such that the set \mathcal{B} of bad cubes in $\mathbb{D}(E)$, for which the ε -local WHSA condition with parameter K_0 fails, satisfies the packing condition

$$\sum_{Q \subset Q_0, Q \in \mathcal{B}} \sigma(Q) \leq C_\varepsilon \sigma(Q_0), \quad \forall Q_0 \in \mathbb{D}(E). \quad (2.20)$$

Next, we develop some further notation and terminology. Given a closed set E , set $\delta_E(Y) := \text{dist}(Y, E)$, simply writing $\delta(Y)$ when the set has been fixed.

Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω , so that the cubes in \mathcal{W} form a covering of Ω with nonoverlapping interiors, and which satisfy

$$4 \operatorname{diam}(I) \leq \operatorname{dist}(4I, \partial\Omega) \leq \operatorname{dist}(I, \partial\Omega) \leq 40 \operatorname{diam}(I) \tag{2.21}$$

and

$$\operatorname{diam}(I_1) \approx \operatorname{diam}(I_2), \quad \text{whenever } I_1 \text{ and } I_2 \text{ touch.} \tag{2.22}$$

Assuming that $E = \partial\Omega$ is ADR and given $Q \in \mathbb{D}(E)$, for the same constant K_0 as in (2.15) we set

$$\mathcal{W}_Q := \{I \in \mathcal{W} : K_0^{-1} \ell(Q) \leq \ell(I) \leq K_0 \ell(Q), \text{ and } \operatorname{dist}(I, Q) \leq K_0 \ell(Q)\}. \tag{2.23}$$

Fix a small, positive parameter τ , to be chosen momentarily, and given $I \in \mathcal{W}$, let

$$I^* = I^*(\tau) := (1 + \tau)I \tag{2.24}$$

denote the corresponding ‘‘fattened’’ Whitney cube. We now choose τ sufficiently small that the cubes I^* retain the usual properties of Whitney cubes, in particular that

$$\operatorname{diam}(I) \approx \operatorname{diam}(I^*) \approx \operatorname{dist}(I^*, E) \approx \operatorname{dist}(I, E).$$

We then define Whitney regions with respect to Q by setting

$$U_Q := \bigcup_{I \in \mathcal{W}_Q} I^*. \tag{2.25}$$

We observe that these Whitney regions may have more than one connected component, but that the number of distinct components is uniformly bounded, depending only upon K_0 and dimension. We enumerate the components of U_Q as $\{U_Q^i\}_i$. Moreover, we enlarge the Whitney regions as follows.

Definition 2.26. For $\varepsilon > 0$, and given $Q \in \mathbb{D}(E)$, we write $X \approx_{\varepsilon, Q} Y$ if X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3 \ell(Q) \leq \delta(Y_k) \leq \varepsilon^{-3} \ell(Q)$. Given a sufficiently small parameter $\varepsilon > 0$, we then set

$$\tilde{U}_Q^i := \{X \in \mathbb{R}^{n+1} \setminus E : X \approx_{\varepsilon, Q} Y, \text{ for some } Y \in U_Q^i\}. \tag{2.27}$$

Remark 2.28. Since \tilde{U}_Q^i is an enlarged version of U_Q , it may be that there are some $i \neq j$ for which \tilde{U}_Q^i meets \tilde{U}_Q^j . This overlap will be harmless.

3. PDE estimates

In this section we recall several estimates for harmonic measure and harmonic functions, and also for p -harmonic measure and p -harmonic functions. Although some of the PDE results in the harmonic case $p = 2$ can be subsumed into the general p -harmonic theory, we choose to present some aspects of the harmonic theory separately, in part for the convenience of those readers who are more familiar with the case $p = 2$, and in part because the presence of the Green function is unique to that case.

3A. PDE estimates: the harmonic case. Next, we recall several facts concerning harmonic measure and Green's functions. Let Ω be an open set, not necessarily connected, and set $\delta(X) = \delta_{\partial\Omega}(X) = \text{dist}(X, \partial\Omega)$.

Lemma 3.1 [Bourgain 1987]. *Suppose that $\partial\Omega$ is n -dimensional ADR. Then there are uniform constants $c \in (0, 1)$ and $C \in (1, \infty)$, depending only on n and ADR, such that for every $x \in \partial\Omega$ and every $r \in (0, \text{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$ then*

$$\omega^Y(\Delta(x, r)) \geq \frac{1}{C} > 0. \quad (3.2)$$

We refer the reader to [Bourgain 1987, Lemma 1] for the proof. We note for future reference that in particular, given $X \in \Omega$, if $\hat{x} \in \partial\Omega$ satisfies $|X - \hat{x}| = \delta(X)$ and $\Delta_X := \partial\Omega \cap B(\hat{x}, 10\delta(X))$, then for a slightly different uniform constant $C > 0$,

$$\omega^X(\Delta_X) \geq \frac{1}{C}. \quad (3.3)$$

Indeed, the latter bound follows immediately from (3.2), and the fact that we can form a Harnack chain connecting X to a point Y that lies on the line segment from X to \hat{x} and satisfies $|Y - \hat{x}| = c\delta(X)$.

A proof of the next lemma may be found, e.g., in [Hofmann et al. \geq 2017]. We note that, in particular, the ADR hypothesis implies that $\partial\Omega$ is Wiener regular at every point (see Lemma 3.27 below).

Lemma 3.4. *Let Ω be an open set with n -dimensional ADR boundary. There exist positive, finite constants C , depending only on dimension, and c_θ , depending on dimension and $\theta \in (0, 1)$, such that the Green function satisfies*

$$G(X, Y) \leq C|X - Y|^{1-n}; \quad (3.5)$$

$$c_\theta|X - Y|^{1-n} \leq G(X, Y), \quad \text{if } |X - Y| \leq \theta\delta(X), \theta \in (0, 1); \quad (3.6)$$

$$G(X, \cdot) \in C(\bar{\Omega} \setminus \{X\}) \quad \text{and} \quad G(X, \cdot)|_{\partial\Omega} \equiv 0, \quad \forall X \in \Omega; \quad (3.7)$$

$$G(X, Y) \geq 0, \quad \forall X, Y \in \Omega, X \neq Y; \quad (3.8)$$

$$G(X, Y) = G(Y, X), \quad \forall X, Y \in \Omega, X \neq Y; \quad (3.9)$$

and for every $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\int_{\partial\Omega} \Phi d\omega^X - \Phi(X) = - \iint_{\Omega} \nabla_Y G(Y, X) \cdot \nabla \Phi(Y) dY, \quad \forall X \in \Omega. \quad (3.10)$$

Next we present a version of one of the estimates obtained by Caffarelli, Fabes, Mortola, and Salsa in [Caffarelli et al. 1981], which remains true even in the absence of connectivity.

Lemma 3.11 (“CFMS” estimates). *Suppose that $\partial\Omega$ is n -dimensional ADR. For every $Y \in \Omega$ and $X \in \Omega$ such that $|X - Y| \geq \delta(Y)/2$, we have*

$$\frac{G(Y, X)}{\delta(Y)} \leq C \frac{\omega^X(\Delta_Y)}{\sigma(\Delta_Y)}, \quad (3.12)$$

where $\Delta_Y = B(\hat{y}, 10\delta(Y)) \cap E$, with $\hat{y} \in \partial\Omega$ such that $|Y - \hat{y}| = \delta(Y)$.

For future use, we note that as a consequence of (3.12), it follows directly that for every $Q \in \mathbb{D}(\partial\Omega)$, if $Y \in B(x_Q, C\ell(Q))$ with $\delta(Y) \geq c\ell(Q)$, then there exists $\kappa = \kappa(C, c)$ such that

$$\frac{G(Y, X)}{\ell(Q)} \lesssim \frac{\omega^X(\kappa Q)}{\sigma(Q)} \lesssim \kappa^n \left(\int_Q (\mathcal{M}\omega^X)^{1/2} d\sigma \right)^2, \quad \forall X \notin B(x_Q, \kappa\ell(Q)), \tag{3.13}$$

where κQ is defined in (2.9), and \mathcal{M} is the usual Hardy–Littlewood maximal operator on $\partial\Omega$.

Proof of Lemma 3.11. We follow the well known argument of [Caffarelli et al. 1981] (see also [Kenig 1994, Lemma 1.3.3]). Fix $Y \in \Omega$ and write $B^Y = \overline{B(Y, \delta(Y)/2)}$. Consider the open set $\widehat{\Omega} = \Omega \setminus B^Y$ for which clearly $\partial\widehat{\Omega} = \partial\Omega \cup \partial B^Y$. Set

$$u(X) := G(Y, X)/\delta(Y), \quad v(X) := \omega^X(\Delta_Y)/\sigma(\Delta_Y),$$

for every $X \in \widehat{\Omega}$. Note that both u and v are nonnegative harmonic functions in $\widehat{\Omega}$. If $X \in \partial\Omega$ then $u(X) = 0 \leq v(X)$. Take now $X \in \partial B^Y$, so that $u(X) \lesssim \delta(Y)^{-n}$ by (3.5). On the other hand, if we fix $X_0 \in \partial B^Y$ with X_0 on the line segment that joints Y and \hat{y} , then $2\Delta_{X_0} = \Delta_Y$, so that $v(X_0) \gtrsim \delta(Y)^{-n}$, by (3.3). By Harnack’s inequality, we then obtain $v(X) \gtrsim \delta(Y)^{-n}$ for all $X \in \partial B^Y$. Thus, $u \lesssim v$ in $\partial\widehat{\Omega}$ and by the maximum principle this immediately extends to $\widehat{\Omega}$ as desired. \square

Lemma 3.14. *Let $\partial\Omega$ be n -dimensional ADR. Let $B = B(x, r)$ with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, and set $\Delta = B \cap \partial\Omega$. There exist constants $\kappa_0 > 2$, $C > 1$, and $M_1 > 1$, depending only on n and the ADR constant of $\partial\Omega$, such that for $X \in \Omega \setminus B(x, \kappa_0 r)$, we have*

$$\sup_{\frac{1}{2}B} G(\cdot, X) \lesssim \frac{1}{|B|} \iint_B G(Y, X) dY \leq Cr \frac{\omega^X(\Delta(x, M_1 r))}{\sigma(\Delta)}. \tag{3.15}$$

Moreover, for each $\gamma \in (0, 1]$,

$$\frac{1}{|B|} \iint_{B \cap \{Y: \delta(Y) < \gamma r\}} G(Y, X) dY \leq C\gamma^2 r \frac{\omega^X(\Delta(x, M_1 r))}{\sigma(\Delta)}, \tag{3.16}$$

where C depends on n and the ADR constant of $\partial\Omega$.

We note that in the previous estimates it is implicitly understood that $G(\cdot, X)$ is extended to be 0 outside of Ω .

Proof. Extending $G(\cdot, X)$ to be 0 outside of Ω , we obtain a subharmonic function in B . The first inequality in (3.15) follows immediately. The second inequality in (3.15) is just the special case $\gamma = 1$ of (3.16), so it suffices to prove the latter. Set $\Sigma_\gamma = \{I \in \mathcal{W} : I \cap B \neq \emptyset, \text{dist}(I, \partial\Omega) < \gamma r\}$, and note that if $I \in \Sigma_\gamma$ then by (2.21),

$$40^{-1} \text{dist}(I, \partial\Omega) \leq \text{diam}(I) \leq \text{dist}(I, \partial\Omega) < \gamma r \leq r, \quad \text{dist}(I, x) \leq r.$$

In particular, $I \subset B(x, 2r)$. Furthermore, we can find κ_0 , depending only on dimension, such that $\text{dist}(X, 4I) \geq 4r$ for every $I \in \Sigma_\gamma$ and $X \in \Omega \setminus B(x, \kappa_0 r)$. Let $Q_I \in \mathbb{D}$ be such that $\ell(Q_I) = \ell(I)$ and $\text{dist}(I, \partial\Omega) = \text{dist}(I, Q_I)$. Then $\ell(Q_I) \leq \gamma r$, and $Y(I)$, the center of I , satisfies $Y(I) \in B(x_{Q_I}, C\ell(Q_I))$

and $\delta(Y(I)) \approx \ell(I) = \ell(Q_I)$. Hence we can invoke (3.13) (taking κ_0 larger if needed) and obtain that for every $Y \in I$,

$$G(Y, X) \approx G(Y(I), X) \lesssim \ell(I) \frac{\omega^X(\kappa Q_I)}{\sigma(Q_I)},$$

where the first estimate uses Harnack's inequality in $2I \subset \Omega$. Hence,

$$\begin{aligned} \iint_{B \cap \{Y: \delta(Y) < \gamma r\}} G(Y, X) dY &\leq \sum_{I \in \Sigma_\gamma} \iint_I G(Y, X) dY \lesssim \sum_{I \in \Sigma_\gamma} \ell(I)^2 \omega^X(\kappa Q_I) \\ &\leq \sum_{k: 2^{-k} \lesssim \gamma r} 2^{-2k} \sum_{I \in \Sigma_\gamma: \ell(I) = 2^{-k}} \omega^X(\kappa Q_I) \lesssim (\gamma r)^2 \omega^X(\Delta(x, M_1 r)), \end{aligned}$$

where in the last step we have used that for each fixed k , the cubes κQ_I with $\ell(I) = 2^{-k}$ have uniformly bounded overlaps, and are all contained in $\Delta(x, M_1 r)$ for M_1 large enough. Dividing by $|B| \approx r^{n+1}$ and using the ADR property, we obtain the desired estimate. \square

3B. PDE estimates: the p -harmonic case. We now recall several fundamental estimates for p -harmonic functions and p -harmonic measure, some of which generalize certain of the preceding estimates that we have stated in the harmonic case. We ask the reader to forgive a moderate amount of redundancy. Given a closed set E , as above we set $\delta(Y) := \text{dist}(Y, E)$.

Lemma 3.17. *Let $p, 1 < p < \infty$, be given. Let u be a positive p -harmonic function in $B(X, 2r)$. Then*

$$\left(\frac{1}{|B(X, r/2)|} \iint_{B(X, r/2)} |\nabla u|^p dy \right)^{1/p} \leq \frac{C}{r} \max_{B(X, r)} u, \quad (3.18)$$

$$\max_{B(X, r)} u \leq C \min_{B(X, r)} u. \quad (3.19)$$

Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $Y, Y' \in B(X, r)$, then

$$|u(Y) - u(Y')| \leq C \left(\frac{|Y - Y'|}{r} \right)^\alpha \max_{B(X, 2r)} u. \quad (3.20)$$

Proof. The inequality (3.18) is a standard energy estimate, (3.19) is the well known Harnack inequality for positive solutions to the p -Laplace operator, and (3.20) is a well known interior Hölder continuity estimate for solutions to equations of p -Laplace type. We refer to [Serrin 1964] for these results. \square

Definition 3.21. Let $O \subset \mathbb{R}^{n+1}$ be open and let K be a compact subset of O . Given $p, 1 < p < \infty$, we let

$$\text{Cap}_p(K, O) = \inf \left\{ \iint_O |\nabla \phi|^p dY : \phi \in C_0^\infty(O), \phi \geq 1 \text{ in } K \right\}.$$

$\text{Cap}_p(K, O)$ is referred to as the p -capacity of K relative to O . The p -capacity of an arbitrary set $E \subset O$ is defined by

$$\text{Cap}_p(E, O) = \inf_{\substack{E \subset G \subset O \\ G \text{ open}}} \sup_{\substack{K \subset G \\ K \text{ compact}}} \text{Cap}_p(K, O). \quad (3.22)$$

Definition 3.23. Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $x \in E$, $0 < r < \text{diam}(E)$. Given p , $1 < p < \infty$, we say that $E \cap B(x, 4r)$ is p -thick if for every $x \in E \cap B(x, 4r)$ there exists $r_x > 0$ such that

$$\int_0^{r_x} \left[\frac{\text{Cap}_p(E \cap B(x, \rho), B(x, 2\rho))}{\text{Cap}_p(B(x, \rho), B(x, 2\rho))} \right]^{1/(p-1)} \frac{d\rho}{\rho} = \infty.$$

We note that this definition is just the Wiener criterion in the p -harmonic case. As it can be seen in [Heinonen et al. 2006, Chapter 6], p -thickness implies that all points on $E \cap B(x, 4r)$ are regular for the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$.

Definition 3.24. Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $x \in E$, $0 < r < \text{diam}(E)$. Given p , $1 < p < \infty$, and $\eta > 0$ we say that $E \cap B(x, 4r)$ is uniformly p -thick with constant η if

$$\frac{\text{Cap}_p(E \cap B(\hat{x}, \hat{r}), B(\hat{x}, 2\hat{r}))}{\text{Cap}_p(B(\hat{x}, \hat{r}), B(\hat{x}, 2\hat{r}))} \geq \eta \tag{3.25}$$

whenever $\hat{x} \in E \cap B(x, 4r)$ and $B(\hat{x}, 2\hat{r}) \subset B(x, 4r)$.

Remark 3.26. In the case $p = 2$, the condition defined in Definition 3.24 is sometimes called the capacity density condition (CDC); see for instance [Aikawa 2004]. Note that uniform p -thickness is a strong quantitative version of the p -thickness defined above and hence of the Wiener regularity for the Laplace and the p -Laplace operator.

Lemma 3.27. Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be Ahlfors–David regular of dimension n . Let p , $1 < p < \infty$, be given. Then $E \cap B(x, 4r)$ is uniformly p -thick for some constant η , depending only on p , n , and the ADR constant, whenever $x \in E$, $0 < r < \frac{1}{4} \text{diam } E$.

Proof. We first observe that since the ADR condition is scale-invariant we may translate and rescale to prove (3.25) only for $\hat{x} = 0$ and $\hat{r} = 1$ (we would also need to rescale E , but abusing the notation we still call it E). Write $B = B(0, 1)$ and observe that, for every $1 < p < \infty$, [Heinonen et al. 2006, Example 2.12] gives

$$\text{Cap}_p(B, 2B) = C(n, p). \tag{3.28}$$

The desired bound from below follows at once if $p > n + 1$ from the estimate in [Heinonen et al. 2006, Example 2.12]:

$$\text{Cap}_p(E \cap B, 2B) \geq \text{Cap}_p(\{0\}, 2B) = C(n, p)'$$

Let us now consider the case $1 < p \leq n + 1$. Write $K = E \cap \frac{1}{2}B$. Combining [Heinonen et al. 2006, Theorem 2.38; Adams and Hedberg 1999, Theorems 2.2.7 and 4.5.2] we have that

$$\text{Cap}_p(E \cap B, 2B) \gtrsim \widetilde{\text{Cap}}_p(K) \gtrsim \sup_{\mu} \left(\frac{\mu(K)}{\|W_p(\mu)\|_{L^1(\mu)}^{1/p'}} \right)^p. \tag{3.29}$$

In the previous expression the implicit constants depend only on p and n ; $\widetilde{\text{Cap}}_p$ stands for the inhomogeneous p -capacity, that is,

$$\widetilde{\text{Cap}}_p(K) = \inf \left\{ \iint_{\mathbb{R}^{n+1}} (|\phi|^p + |\nabla \phi|^p) dY : \phi \in C_0^\infty(\mathbb{R}^n), \phi \geq 1 \text{ in } K \right\};$$

the sup runs over all Radon positive measures supported on K ; and

$$W_p(\mu)(y) := \int_0^1 \left(\frac{\mu(B(y, t))}{t^{n+1-p}} \right)^{p'-1} \frac{dt}{t}, \quad x \in \text{supp } \mu.$$

We choose $\mu = H^n|_K$ and observe that, if $y \in \text{supp } \mu \subset K \subset E$ and $0 < t < 1$, then, by ADR, $\mu(B(y, t)) = \sigma(B(y, t) \cap B(0, \frac{1}{2})) \lesssim t^n$. This easily gives $W_p(\mu)(y) \lesssim 1$ for every $y \in \text{supp } \mu$ and, by ADR,

$$\int_K W_p(\mu)(y) d\mu(y) \leq \mu(K) \leq \sigma(B) \lesssim 1.$$

We can now use (3.29) and ADR again to conclude that

$$\text{Cap}_p(E \cap B, 2B) \gtrsim \mu(K) \geq \sigma(B(0, \frac{1}{2}))^p \gtrsim 1.$$

Combining this with (3.28) we readily obtain (3.25). \square

Lemma 3.30. *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be Ahlfors–David regular of dimension n . Let p , $1 < p < \infty$, be given. Let $x \in E$ and $0 < r < \text{diam}(E)$. Then, given $f \in W^{1,p}(B(x, 4r))$ there exists a unique p -harmonic function $u \in W^{1,p}(B(x, 4r) \setminus E)$ such that $u - f \in W_0^{1,p}(B(x, 4r) \setminus E)$. Furthermore, let $u, v \in W_{\text{loc}}^{1,p}(B(x, 4r) \setminus E)$ be a p -superharmonic function and a p -subharmonic function in Ω , respectively. If $\inf\{u - v, 0\} \in W_0^{1,p}(B(x, 4r) \setminus E)$, then $u \geq v$ a.e. in $B(x, 4r) \setminus E$. Finally, every point $\hat{x} \in E \cap B(x, 4r)$ is regular for the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$.*

Proof. The first part of the lemma is a standard maximum principle. The fact that every $\hat{x} \in E \cap B(x, 4r)$ is regular in the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$ follows from the fact that Lemma 3.27 implies that $E \cap B(x, 4r)$ is uniformly p -thick for every $1 < p < \infty$, and hence we can invoke [Heinonen et al. 2006, Chapter 6]. \square

Lemma 3.31. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension n . Let p , $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \text{diam}(\partial\Omega)$. Assume also that u is nonnegative and p -harmonic in $B(x, 4r) \cap \Omega$, continuous on $B(x, 4r) \cap \bar{\Omega}$, and that $u = 0$ on $\partial\Omega \cap B(x, 4r)$. Then, extending u to be 0 in $B(x, 4r) \setminus \bar{\Omega}$, we have*

$$\left(\frac{1}{|B(x, r/2)|} \iint_{B(x, r/2)} |\nabla u|^p dy \right)^{1/p} \leq \frac{C}{r} \left(\frac{1}{|B(x, r)|} \iint_{B(x, r)} u^{p-1} \right)^{1/(p-1)}. \quad (3.32)$$

Furthermore, there exists $\alpha \in (0, 1)$, depending only on p , n , and the ADR constant, such that if $Y, Y' \in B(x, r)$, then

$$|u(Y) - u(Y')| \leq C \left(\frac{|Y - Y'|}{r} \right)^\alpha \max_{B(x, 2r)} u. \quad (3.33)$$

Proof. Since u , extended as above to all of $B(x, 4r)$, is a nonnegative p -subsolution in $B(x, 4r)$, (3.32) is just a standard energy or Caccioppoli estimate plus a standard interior estimate. Thus, we only prove (3.33). Since $E \cap B(x, 4r)$ is uniformly p -thick as seen in Lemma 3.27, we can invoke [Heinonen et al. 2006, Theorem 6.38] to obtain that there exist $C \geq 1$ and $\alpha = \alpha \in (0, 1)$, depending only on n , p , and the ADR

constant, such that

$$\max_{B(x,\rho)} u \leq C \left(\frac{\rho}{r}\right)^\alpha \max_{B(x,r)} u, \quad \text{whenever } 0 < \rho \leq r. \tag{3.34}$$

This, the triangle inequality, and elementary arguments give (3.33). \square

Lemma 3.35. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension n . Let p , $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \text{diam}(\partial\Omega)$. Assume also that u is nonnegative and p -harmonic in $B(x, 4r) \cap \Omega$, continuous on $B(x, 4r) \cap \bar{\Omega}$, and that $u = 0$ on $\partial\Omega \cap B(x, 4r)$. Then, extending u to be 0 in $B(x, 4r) \setminus \bar{\Omega}$, there exists $\alpha > 0$ such that*

$$u(Y) \leq C \left(\frac{\delta(Y)}{r}\right)^\alpha \left(\frac{1}{|B(x, 2r)|} \iint_{B(x, 2r)} u^{p-1}(Z) dZ\right)^{1/(p-1)} \tag{3.36}$$

for all $Y \in B(x, r)$, where the constants C and α depend only on n , p , and the ADR constant of $\partial\Omega$.

Proof. This follows from Lemma 3.31 and standard estimates for p -subsolutions. Let us note that in the linear case (i.e, $p = 2$) one can give an alternative proof based on Bourgain’s Lemma 3.1 and an iteration argument (see [Hofmann et al. \geq 2017] for details). \square

Lemma 3.37. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension n . Let p , $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \text{diam}(\partial\Omega)$. Assume also that u is nonnegative and p -harmonic in $B(x, 4r) \cap \Omega$, continuous on $B(x, 4r) \cap \bar{\Omega}$, that $u = 0$ on $\partial\Omega \cap B(x, 4r)$, and that u is extended to be 0 in $B(x, 4r) \setminus \bar{\Omega}$. Then u has a representative in $W^{1,p}(B(x, 4r))$ with Hölder continuous partial derivatives in $B(x, 4r) \setminus \partial\Omega$. Furthermore, there exists $\beta \in (0, 1]$ such that if $Y, Y' \in B(X, \hat{r}/2)$, with $B(X, 4\hat{r}) \subset B(x, 4r) \setminus \partial\Omega$, then*

$$|\nabla u(Y) - \nabla u(Y')| \lesssim \left(\frac{|Y - Y'|}{\hat{r}}\right)^\beta \max_{B(X, \hat{r})} |\nabla u| \lesssim \frac{1}{\hat{r}} \left(\frac{|Y - Y'|}{\hat{r}}\right)^\beta \max_{B(X, 2\hat{r})} u, \tag{3.38}$$

where β and the implicit constants depend only on p and n . Furthermore, if

$$\frac{u(Y)}{\delta(Y)} \approx |\nabla u(Y)|, \quad Y \in B(X, 3\hat{r}), \tag{3.39}$$

then u has continuous second derivatives in $B(X, 3\hat{r})$, and there exists $C \geq 1$, depending only on n , p , and the implicit constants in (3.39), such that

$$\max_{B(X, \hat{r}/2)} |\nabla^2 u| \leq C \left(\frac{1}{|B(X, \hat{r})|} \iint_{B(X, \hat{r})} |\nabla^2 u(Y)|^2 dY\right)^{1/2} \leq C^2 \frac{u(X)}{\delta(X)^2}. \tag{3.40}$$

Proof. For (3.38) we refer, for example, to [Tolsdorf 1984]; (3.40) is a consequence of (3.38), (3.39), and Schauder type estimates, see [Gilbarg and Trudinger 1983]. For a more detailed proof of (3.40), see [Lewis and Vogel 2006, Lemma 2.4(d)] for example. \square

Remark 3.41. We note that the second inequality in (3.38) and (3.19) give

$$|\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)}, \quad Y \in B(x, 2r) \setminus \partial\Omega. \tag{3.42}$$

Lemma 3.43. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and assume that $\partial\Omega$ is Ahlfors–David regular of dimension n . Let p , $1 < p < \infty$, be given. Let $x \in \partial\Omega$, $0 < r < \text{diam}(\partial\Omega)$, and suppose that u is nonnegative and p -harmonic in $B(x, 4r) \cap \Omega$, vanishing continuously on $B(x, 4r) \cap \Omega$ (hence u is continuous in $B(x, 4r)$ after being extended by 0 in $B(x, 4r) \setminus \bar{\Omega}$). There exists a unique finite positive Borel measure μ on \mathbb{R}^{n+1} , with support in $\partial\Omega \cap B(x, 4r)$, such that*

$$-\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dY = \int \phi \, d\mu \tag{3.44}$$

whenever $\phi \in C_0^\infty(B(x, 4r))$. Furthermore, there exists $C < \infty$, depending only on p, n , and the ADR constant, such that

$$\left(\frac{\max_{B(x,r)} u}{r}\right)^{p-1} \leq C \frac{\mu(\Delta(x, 2r))}{\sigma(\Delta(x, 2r))}. \tag{3.45}$$

Note that (3.45) is the p -harmonic analogue of Lemma 3.11.

Proof. For the proof of (3.44), see [Heinonen et al. 2006, Chapter 21]. Using Lemma 3.27 and Lemma 3.31, (3.45) follows directly from [Kilpeläinen and Zhong 2003, Lemma 3.1]; see also [Eremenko and Lewis 1991]. □

The following lemma generalizes Lemma 3.14 to the case $1 < p < \infty$.

Lemma 3.46. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and assume that $\partial\Omega$ is Ahlfors–David regular of dimension n . Let p , $1 < p < \infty$, be given. Let $x \in \partial\Omega$, $0 < r < \text{diam}(\partial\Omega)$, and suppose that u and μ are as in Lemma 3.43. Then there exist constants C and M_1 , depending only on n and the ADR constant, such that if $B(y, M_1s) \subset B(x, 2r)$ with $y \in \partial\Omega$, then*

$$\max_{B(y,s/2)} u^{p-1} \lesssim \frac{1}{|B(y,s)|} \iint_{B(y,s)} u^{p-1}(Z) \, dZ \leq C s^{p-1} \frac{\mu(\Delta(y, M_1s))}{\sigma(\Delta(y, s))}.$$

Moreover, for all $\gamma \in (0, 1]$,

$$\frac{1}{|B(y, s)|} \iint_{B(y,s) \cap \{Y: \delta(Y) \leq \gamma s\}} u^{p-1}(Z) \, dZ \leq C \gamma^p s^{p-1} \frac{\mu(\Delta(y, M_1s))}{\sigma(\Delta(y, s))}.$$

We note that in the previous estimates it is implicitly understood that u is extended to be 0 on $B(x, 4r) \setminus \bar{\Omega}$.

Proof. Using (3.45), the proof of Lemma 3.46 is the same mutatis mutandi as that of Lemma 3.14. We omit further details. □

4. Proofs of Theorem 1.1 and Theorem 1.12: preliminary arguments

We start the proofs of Theorem 1.1 and Theorem 1.12 by giving some preliminary arguments. We first show that (1.2) implies (1.4). To this end, we claim that, without loss of generality, we may suppose that for a surface ball $\Delta = \Delta(x, r)$, the point X_Δ in the statement of Theorem 1.1 satisfies (3.2), i.e., there is some $c_1 = c_1(n, \text{ADR}) > 0$ such that

$$\omega^{X_\Delta}(\Delta) \geq c_1. \tag{4.1}$$

The only price to be paid is that the constants c_0, C_0 may now be slightly different (depending only on n and ADR), and that (1.2) now holds with Δ in place of 2Δ , i.e., for the (possibly) new point X_Δ , we have

$$\int_{\Delta} k^{X_\Delta}(y)^q d\sigma(y) \leq C_0 \sigma(\Delta)^{1-q}. \tag{4.2}$$

Indeed, set $\Delta' := \Delta(x, r/2)$, and let $X' := X_{\Delta'} \in B(x, r/2) \cap \Omega$ be the point such that (1.2) holds for Δ' . Fix $\hat{x} \in \partial\Omega$ such that $\delta(X') = |X' - \hat{x}|$. Suppose first that $\delta(X') \leq r/4$, in which case $\Delta(\hat{x}, r/4) \subset \Delta$. Thus, if in addition $\delta(X') < cr/4$, where $c \in (0, 1)$ is the constant in Lemma 3.1, then we set $X_\Delta := X'$, and (4.1) holds by Lemma 3.1. On the other hand, if $cr/4 \leq \delta(X_\Delta) \leq r/4$, we select X_Δ along the line segment joining X' to \hat{x} , such that $\delta(X_\Delta) = |X_\Delta - \hat{x}| = cr/8$, and (4.1) holds exactly as before. Moreover, (4.2) holds for this new X_Δ , in the first case, immediately by (1.2) applied to $X' = X_{\Delta'}$, and in the second case, by moving from X' to X_Δ via Harnack's inequality (which may be used within the touching ball $B(X', \delta(X'))$). Let us finally consider the case $\delta(X') > r/4$. Then we can use Harnack within the ball $B(X', r/4)$ to pass to a point X'' on the line segment joining X' to x such that $|X' - X''| = r/8$, and consequently $\delta(X'') \leq |X'' - x| < 3r/8$ (since $X' \in B(x, r/2)$). Hence (1.2) holds (with different constant) for Δ' with X'' in place of $X_{\Delta'}$. Now take $\hat{x} \in \partial\Omega$ such that $\delta(X'') = |X'' - \hat{x}|$ and note that $\Delta(\hat{x}, r/4) \subset \Delta$. We can now repeat the previous argument with X'' in place of X' . Details are left to the interested reader.

Similarly, if (1.4) holds for $\Delta = \Delta(x, r)$, with $X_\Delta \in B(x, r/2) \cap \Omega$, then again without loss of generality we may suppose that (4.1) holds, possibly for a new $X_\Delta \in B(x, r) \cap \Omega$. Indeed if we let $X' \in B(x, r/2) \cap \Omega$ be the original point X_Δ for which (1.4) holds, we may then follow the argument in the previous paragraph, mutatis mutandi. We choose $\hat{x} \in \partial\Omega$ such that $\delta(X') = |X' - \hat{x}|$ and suppose first that $\delta(X') \leq r/4$, so that $\Delta(\hat{x}, r/4) \subset \Delta$. Considering the same two cases as before we pick X_Δ and in either case (4.1) holds by Lemma 3.1 applied to the surface ball $\Delta(\hat{x}, r/4)$. Note that in the second case, (1.4) continues to hold for X_Δ , with a different but still uniform β , using Harnack's inequality within the touching ball $B(X', \delta(X'))$ to move from X' to X_Δ . When $r/4 < \delta(X')$ we choose X'' as before, and by Harnack's inequality, (1.4) holds with X'' in place of X' , for a different but still uniform β . Again, if we let $\hat{x} \in \partial\Omega$ with $\delta(X'') = |X'' - \hat{x}|$, then $\Delta(\hat{x}, r/4) \subset \Delta$, and we may now repeat the previous argument with X'' in place of X' .

We are now ready to show that (1.2) implies (1.4).

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n -dimensional ADR boundary, and let $\Delta = \Delta(x, r)$ be a surface ball on $\partial\Omega$. Let μ be a measure on $\partial\Omega$ such that $\mu|_\Delta \ll \sigma$, and such that for some $q > 1$ and $\Lambda < \infty$,*

$$\int_{\Delta} k^q d\sigma \leq \Lambda, \tag{4.4}$$

where $k := d\mu/d\sigma$ on Δ . Suppose also that

$$\frac{\mu(\Delta)}{\sigma(\Delta)} \geq 1. \tag{4.5}$$

Then there are constants $\eta, \beta \in (0, 1)$, depending only on n, q, Λ , and ADR, such that for any Borel set $A \subset \Delta$,

$$\sigma(A) \geq (1 - \eta)\sigma(\Delta) \implies \mu(A) \geq \beta\mu(\Delta). \tag{4.6}$$

Remark 4.7. Let k be a normalized version of harmonic measure: $k = c_1^{-1}\sigma(\Delta)k^{X_\Delta}$, with X_Δ a point for which (4.1) and (4.2) hold. Then clearly (4.4) and (4.5) hold for k , and the conclusion (4.6) is just a reformulation of (1.4). We note that in the sequel, we actually use only (4.6) or (1.4), rather than condition (4.4) or (4.2). Thus, Theorem 1.1 could just as well have been stated with condition $(\star\star)$ (see Remark 1.3) in place of (\star) .

Proof of Lemma 4.3. Set $F := \Delta \setminus A$, so $\sigma(F) \leq \eta\sigma(\Delta)$. Then

$$\begin{aligned} \mu(F) &= \int_F k \, d\sigma \leq \sigma(F)^{1/q'} \left(\int_\Delta k^q \, d\sigma \right)^{1/q} \\ &\leq \Lambda^{1/q} \sigma(F)^{1/q'} \sigma(\Delta)^{1/q} \leq \Lambda^{1/q} \eta^{1/q'} \sigma(\Delta) \leq \Lambda^{1/q} \eta^{1/q'} \mu(\Delta), \end{aligned}$$

where in the last step we have used (4.5). Thus,

$$\mu(A) \geq (1 - \Lambda^{1/q} \eta^{1/q'}) \mu(\Delta) \geq \frac{1}{2} \mu(\Delta)$$

for η small enough. This completes the proof. □

Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. As in (2.8), we set $B_{Q_0} = B(x_{Q_0}, r_0)$, with $r_0 := r_{Q_0} \approx \ell(Q_0)$, so that $\Delta_{Q_0} = B_{Q_0} \cap \partial\Omega \subset Q_0$.

Proceeding first in the setting of Theorem 1.1, let $X_0 := X_{\Delta_{Q_0}}$ be the point relative to $\Delta = \Delta_{Q_0}$ such that (4.1) and (4.2) hold. Note that (4.1) trivially implies that

$$\omega^{X_0}(Q_0) \geq c_1.$$

With the pole X_0 fixed, we define the normalized harmonic measure and the normalized Green’s function, respectively, by

$$\mu := \frac{1}{c_1} \sigma(Q_0) \omega^{X_0}, \quad u(Y) := \frac{1}{c_1} \sigma(Q_0) G(X_0, Y). \tag{4.8}$$

Then under this normalization, setting $\|\mu\| = \mu(\partial\Omega)$, we have

$$1 \leq \frac{\mu(Q_0)}{\sigma(Q_0)} \leq \frac{\|\mu\|}{\sigma(Q_0)} \leq C_1, \tag{4.9}$$

with $C_1 = 1/c_1$. Furthermore, we may apply Lemma 4.3 (using (4.1) and with $\Lambda \approx C_0/c_1$) to obtain (4.6) for μ , with $\Delta = \Delta_{Q_0}$. In turn, the latter bound, in conjunction with (4.1) and ADR, clearly implies an analogous estimate for Q_0 , namely that there are constants that we again call $\eta, \beta \in (0, 1)$ such that for any Borel set $A \subset Q_0$,

$$\sigma(A) \geq (1 - \eta)\sigma(Q_0) \implies \mu(A) \geq \beta\mu(Q_0). \tag{4.10}$$

Here, of course, we may have different values of the parameters η and β , but these have the same dependence as the original values, so for convenience we maintain the same notation.

In the p -harmonic case, proceeding under the setup of Theorem 1.12, we let u and μ be the p -harmonic function and its associated p -harmonic measure, corresponding to the point $x = x_{Q_0}$ and the radius $r = Cr_0 := Cr_{Q_0}$, satisfying the hypotheses of Theorem 1.12, where we choose the constant C depending only on n and ADR, such that $Q_0 \subset \Delta(x_{Q_0}, Cr_0)$ (thus, in particular, μ is defined on Q_0). Since we assume

that u is nontrivial and nonnegative, we can apply Lemma 3.43 in $B(x_{Q_0}, Cr_0)$ and use (1.14) to conclude that $\mu(\Delta_{Q_0}) > 0$. We can therefore normalize u and μ (abusing the notation we call the normalizations u and μ) so that $\mu(\Delta_{Q_0})/\sigma(Q_0) = 1$, and since $\Delta_{Q_0} \subset Q_0 \subset \Delta(x_{Q_0}, Cr_0)$ by (1.14), we also have $\mu(\Delta(x_{Q_0}, Cr_0))/\sigma(\Delta(x_{Q_0}, Cr_0)) \approx \mu(Q_0)/\sigma(Q_0) \approx 1$. Set $k := d\mu/d\sigma$. As above, by (1.13) and (1.14), we may then use Lemma 4.3 to see that again μ satisfies both (4.9), now with $\|\mu\| := \mu(\Delta(x_{Q_0}, Cr_0))$, and (4.10). The constants C_1, η , and β depend on C, n , the ADR constant, C_0 , and q .

Remark 4.11. Under the assumptions of Theorems 1.1 and 1.12 and throughout this section and Section 6, for $Q_0 \in \mathbb{D}(E)$ fixed, u and μ will continue to denote the normalized Green function and harmonic measure or the normalized nonnegative p -harmonic solution and p -harmonic Riesz measure, as defined above. In particular, (4.9) and (4.10) hold for all $1 < p < \infty$.

As above, let \mathcal{M} denote the usual Hardy–Littlewood maximal operator on $\partial\Omega$ and recall the definition of $\mathbb{D}_{\mathcal{F}, Q_0}$ in (2.10).

Lemma 4.12. *Let $Q_0 \in \mathbb{D}$, and suppose that μ satisfies (4.9) and (4.10). Then there is a pairwise disjoint family $\mathcal{F} = \{Q_j\}_{j \geq 1} \subset \mathbb{D}_{Q_0}$ such that*

$$\sigma(Q_0 \setminus (\bigcup_j Q_j)) \geq \frac{1}{C} \sigma(Q_0) \tag{4.13}$$

and

$$\frac{\beta}{2} \leq \frac{\mu(Q)}{\sigma(Q)} \leq \left(\int_Q (\mathcal{M}\mu)^{1/2} d\sigma \right)^2 \leq C, \quad \forall Q \in \mathbb{D}_{\mathcal{F}, Q_0}, \tag{4.14}$$

where $C > 1$ depends only on η, β, C_1, n , and ADR.

Proof. The proof is based on a stopping time argument similar to those used in the proof of the Kato square root conjecture [Hofmann and McIntosh 2002; Hofmann et al. 2002; Auscher et al. 2002a], and in local Tb theorems. We begin by noting that

$$\|\mathcal{M}\mu\|_{L^{1,\infty}(\sigma)} := \sup_{\lambda > 0} \lambda \sigma\{\mathcal{M}\mu > \lambda\} \lesssim \|\mu\| \lesssim \sigma(Q_0) \tag{4.15}$$

by the Hardy–Littlewood theorem and (4.9). Consequently, by Kolmogorov’s criterion,

$$\int_{Q_0} (\mathcal{M}\mu)^{1/2} d\sigma \leq C = C(n, \text{ADR}, C_1). \tag{4.16}$$

We now perform a stopping time argument to extract a family $\mathcal{F} = \{Q_j\}$ of dyadic subcubes of Q_0 that are maximal with respect to the property that either

$$\frac{\mu(Q_j)}{\sigma(Q_j)} < \frac{\beta}{2} \tag{4.17}$$

and/or

$$\int_{Q_j} (\mathcal{M}\mu)^{1/2} d\sigma > K, \tag{4.18}$$

where $K \geq 1$ is a sufficiently large number to be chosen momentarily. Note that $Q_0 \notin \mathcal{F}$, by (4.9) and (4.16). We say that Q_j is of “type I” if (4.17) holds, and of “type II” if (4.18) holds but (4.17) does not. Set $A := Q_0 \setminus (\bigcup_j Q_j)$, and $F := \bigcup_{Q_j \text{ type II}} Q_j$. Then by (4.9),

$$\sigma(Q_0) \leq \mu(Q_0) = \sum_{Q_j \text{ type I}} \mu(Q_j) + \mu(F) + \mu(A). \quad (4.19)$$

By definition of the type I cubes,

$$\sum_{Q_j \text{ type I}} \mu(Q_j) \leq \frac{\beta}{2} \sum_j \sigma(Q_j) \leq \frac{\beta}{2} \sigma(Q_0). \quad (4.20)$$

To handle the remaining terms, observe that

$$\begin{aligned} \sigma(F) &= \sum_{Q_j \text{ type II}} \sigma(Q_j) \leq \frac{1}{K} \sum_j \int_{Q_j} (\mathcal{M}\mu)^{1/2} d\sigma \\ &\leq \frac{1}{K} \int_{Q_0} (\mathcal{M}\mu)^{1/2} d\sigma \leq \eta \sigma(Q_0), \end{aligned} \quad (4.21)$$

by the definition of the type II cubes, (4.16), and the choice of $K = C\eta^{-1}$. By (4.10) and complementation, we therefore find that

$$\mu(F) \leq (1 - \beta)\mu(Q_0). \quad (4.22)$$

Next, if $x \in A$, then every $Q \in \mathbb{D}_{Q_0}$ that contains x must satisfy the opposite inequality to (4.18), and therefore, by Lebesgue’s differentiation theorem,

$$\mathcal{M}\mu(x) \leq K^2, \quad \text{for } \sigma\text{-a.e. } x \in A.$$

Thus $\mu|_A \ll \sigma$, with $d\mu|_A/d\sigma \leq K^2$, and thus,

$$\mu(A) \leq K^2 \sigma(A).$$

Combining the latter estimate with (4.19), (4.20), and (4.22), we obtain

$$\beta \mu(Q_0) \leq \frac{\beta}{2} \sigma(Q_0) + K^2 \sigma(A).$$

Using (4.9), we then find that

$$\beta \sigma(Q_0) \leq \beta \mu(Q_0) \leq \frac{\beta}{2} \sigma(Q_0) + K^2 \sigma(A).$$

The conclusion of the lemma now follows readily. \square

For future reference, let us note an easy consequence of the last inequality in (4.14) and the ADR property: for all $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, and for any constant $b > 1$, we have

$$\mu(\Delta(x_Q, b \operatorname{diam}(Q))) \lesssim b^n \sigma(Q) \left(\int_Q (\mathcal{M}\mu)^{1/2} d\sigma \right)^2 \lesssim b^n \sigma(Q). \quad (4.23)$$

Recall that the ball B_Q^* and surface ball Δ_Q^* are defined in (2.15).

Lemma 4.24. *Let u, μ be as in Remark 4.11. If the constant K_0 in (2.15) and (2.23) is chosen sufficiently large, then for each $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ with $\ell(Q) \leq K_0^{-1} \ell(Q_0)$, there exists $Y_Q \in U_Q$ with*

$$\delta(Y_Q) \leq |Y_Q - x_Q| \lesssim \ell(Q),$$

where the implicit constant is independent of K_0 , such that

$$\frac{\mu(Q)}{\sigma(Q)} \leq C |\nabla u(Y_Q)|^{p-1}, \tag{4.25}$$

where C depends on K_0 and the implicit constants in the hypotheses of Theorems 1.1 and 1.12.

Remark 4.26. Recalling the construction at the beginning of Section 4, and the fact that we have defined $X_0 := X_{\Delta_{Q_0}}$, we see that $\ell(Q_0) \approx \delta(X_0) \geq K_0^{-1/2} \ell(Q_0)$, for K_0 chosen large enough. We note further that the point Y_Q whose existence is guaranteed by Lemma 4.24 is essentially a corkscrew point relative to Q . Indeed, $\delta(Y_Q) \gtrsim K_0^{-1} \ell(Q)$ (since $Y \in U_Q$), and also $|Y_Q - x_Q| \lesssim \ell(Q)$ (with constant independent of K_0). With a slight abuse of terminology, we shall refer to Y_Q as a corkscrew point relative to Q , with corkscrew constant depending on K_0 .

Proof of Lemma 4.24. Fix $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, with $\ell(Q) \leq K_0^{-1} \ell(Q_0)$, where, as in Remark 4.26, we have chosen K_0 large enough that $\ell(Q_0) \approx \delta(X_0) \geq K_0^{-1/2} \ell(Q_0)$. Recall (2.7) and (2.8), and set $\hat{B}_Q = B(x_Q, \hat{r}_Q)$, $\hat{\Delta}_Q = \hat{B}_Q \cap \partial\Omega$, with $\hat{r}_Q \approx \ell(Q)$ and $Q \subset \hat{\Delta}_Q$. Let $0 \leq \phi_Q \in C_0^\infty(2\hat{B}_Q)$, such that $\phi_Q \equiv 1$ in \hat{B}_Q and $\|\nabla \phi_Q\| \lesssim \ell(Q)^{-1}$. Note that

$$K_0^{1/2} \ell(Q) \leq K_0^{-1/2} \ell(Q_0) \leq \delta(X_0) \leq |X_0 - x_Q|,$$

which implies that $X_0 \notin 4\hat{B}_Q$ provided K_0 is large enough. Thus, by (3.10) in the linear case, or (3.44) in general,

$$\begin{aligned} \ell(Q)\mu(Q) &\leq \ell(Q) \int_{\partial\Omega} \phi_Q d\mu \lesssim \iint_{\hat{B}_Q \cap \Omega} |\nabla u(Y)|^{p-1} dY \\ &\leq \iint_{\hat{B}_Q \cap U_Q} |\nabla u(Y)|^{p-1} dY + \iint_{(\hat{B}_Q \cap \Omega) \setminus U_Q} |\nabla u(Y)|^{p-1} dY \\ &=: \mathcal{I} + \mathcal{II}. \end{aligned} \tag{4.27}$$

Notice that by construction,

$$(\hat{B}_Q \cap \Omega) \setminus U_Q \subset \{Y \in \hat{B}_Q : \delta(Y) \leq CK_0^{-1} \ell(Q)\}.$$

We may therefore cover the latter region by a family of balls $\{B_k\}_k$, centered on $\partial\Omega$, of radius $CK_0^{-1} \ell(Q)$, such that their doubles $\{2B_k\}$ have bounded overlaps and satisfy

$$\bigcup_k 2B_k \subset \{Y \in 2\hat{B}_Q : \delta(Y) \leq 2CK_0^{-1} \ell(Q)\} =: \Sigma(K_0).$$

By the boundary Cacciopoli estimate in Lemma 3.31, plus Hölder's inequality, we obtain

$$\begin{aligned}
\mathcal{I}\mathcal{I} &\leq \sum_k \iint_{B_k} |\nabla u(Y)|^{p-1} dY \lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} \sum_k \iint_{2B_k} |u(Y)|^{p-1} dY \\
&\lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} \iint_{\Sigma(K_0)} |u(Y)|^{p-1} dY \\
&\lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} K_0^{-p} \ell(Q)^p \mu(\Delta(x_Q, 2M_1 \hat{r}_Q)) \\
&\lesssim K_0^{-1} \ell(Q) \sigma(Q) \leq \frac{1}{2} \ell(Q) \mu(Q),
\end{aligned}$$

where in the last three steps we have used (3.16) (when $p = 2$) or Lemma 3.46 ($1 < p < \infty$), (4.23), and finally the choice of K_0 large enough. We can then hide this term on the left-hand side of (4.27), so that

$$\begin{aligned}
\ell(Q) \mu(Q) &\lesssim \mathcal{I} = \iint_{\hat{B}_Q \cap U_Q} |\nabla u(Y)|^{p-1} dY = \sum_i \iint_{\hat{B}_Q \cap U_Q^i} |\nabla u(Y)|^{p-1} dY \\
&\lesssim \ell(Q)^{n+1} \max_i \sup_{Y \in \hat{B}_Q \cap U_Q^i} |\nabla u(Y)|^{p-1} \\
&\approx \ell(Q) \sigma(Q) \max_i \sup_{Y \in \hat{B}_Q \cap U_Q^i} |\nabla u(Y)|^{p-1},
\end{aligned}$$

and we recall that $\{U_Q^i\}_i$ is an enumeration of the connected components of U_Q , and that the number of these components is uniformly bounded. Thus, for some i , there is a point $Y_Q \in \hat{B}_Q \cap U_Q^i$ such that $\mu(Q)/\sigma(Q) \lesssim |\nabla u(Y_Q)|^{p-1}$. To complete the proof, we simply observe that by construction, $\delta(Y_Q) \leq |Y_Q - x_Q| \leq \hat{r}_Q \lesssim \ell(Q)$. \square

5. Proof of Theorem 1.1, Corollary 1.5, and Theorem 1.12

In this section we complete the proofs of Theorem 1.1 and Theorem 1.12 by proving that $E := \partial\Omega$ satisfies WHSA, and hence, by Proposition 1.17, E is UR. The proof of Corollary 1.5 follows almost immediately from Theorem 1.1 and we supply the proof at the end of the section. Our approach to the proofs of Theorems 1.1 and 1.12 is a refinement and extension of the arguments in [Lewis and Vogel 2007], who, as mentioned in the introduction, treated the special case that $k \approx 1$.

We fix $Q_0 \in \mathbb{D}(E)$ and we let u and μ be as in Remark 4.11. We recall that by (4.9),

$$\frac{\mu(Q_0)}{\sigma(Q_0)} \approx 1. \tag{5.1}$$

Let $\mathcal{F} = \{Q_j\}_j$ be the family of maximal stopping time cubes constructed in Lemma 4.12. Combining (4.25) and (4.14), we see that

$$|\nabla u(Y_Q)| \gtrsim 1, \quad \forall Q \in \mathbb{D}_{\mathcal{F}, Q_0}^* := \{Q \in \mathbb{D}_{\mathcal{F}, Q_0} : \ell(Q) \leq K_0^{-1} \ell(Q_0)\}, \tag{5.2}$$

where $Y_Q \in U_Q$ is the point constructed in Lemma 4.24. We recall that the Whitney region U_Q has a uniformly bounded number of connected components, which we have enumerated as $\{U_Q^i\}_i$. We now fix

the particular i such that $Y_Q \in U_Q^i \subset \tilde{U}_Q^i$, where the latter is the enlarged Whitney region constructed in Definition 2.26.

For a suitably small ε_0 , say $\varepsilon_0 \ll K_0^{-6}$, we fix an arbitrary positive $\varepsilon < \varepsilon_0$, and we fix also a large positive number M to be chosen. For each point $Y \in \Omega$, we set

$$B_Y := \overline{B(Y, (1 - \varepsilon^{2M/\alpha})\delta(Y))}, \quad \tilde{B}_Y := \overline{B(Y, \delta(Y))}, \tag{5.3}$$

where $0 < \alpha < 1$ is the exponent appearing in Lemma 3.35. For $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, we consider three cases.

Case 0: $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, with $\ell(Q) > \varepsilon^{10}\ell(Q_0)$.

Case 1: $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, with $\ell(Q) \leq \varepsilon^{10}\ell(Q_0)$ and

$$\sup_{X \in \tilde{U}_Q^i} \sup_{Z \in B_X} |\nabla u(Z) - \nabla u(Y_Q)| > \varepsilon^{2M}. \tag{5.4}$$

Case 2: $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, with $\ell(Q) \leq \varepsilon^{10}\ell(Q_0)$ and

$$\sup_{X \in \tilde{U}_Q^i} \sup_{Z \in B_X} |\nabla u(Z) - \nabla u(Y_Q)| \leq \varepsilon^{2M}. \tag{5.5}$$

We trivially see that the cubes in Case 0 satisfy a packing condition:

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \text{Case 0 holds}}} \sigma(Q) \leq \sum_{\substack{Q \in \mathbb{D}_{Q_0} \\ \ell(Q) > \varepsilon^{10}\ell(Q_0)}} \sigma(Q) \lesssim (\log \varepsilon^{-1})\sigma(Q_0). \tag{5.6}$$

Note that in Case 1 and Case 2 we have $Q \in \mathbb{D}_{\mathcal{F}, Q_0}^*$ (see (5.2)). Furthermore, if $\ell(Q) \leq \varepsilon^{10}\ell(Q_0)$, then by (5.2), (3.42), and either (3.13) (which we apply in the case $p = 2$, with $X = X_0$, since $\ell(Q) \ll \ell(Q_0)$) or (3.45) (for general p , $1 < p < \infty$), and (4.14), we have

$$1 \lesssim |\nabla u(Y_Q)| \lesssim \frac{u(Y_Q)}{\delta(Y_Q)} \lesssim 1. \tag{5.7}$$

Regarding Case 1 we obtain the following packing condition.

Lemma 5.8. *Under the previous assumptions, the following packing condition holds:*

$$\frac{1}{\sigma(Q_0)} \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \leq C(\varepsilon, K_0, M, \eta). \tag{5.9}$$

On the other hand, we show that the cubes in Case 2 satisfy the ε -local WHSA property. Given $\varepsilon > 0$, recall that $B_Q^{***}(\varepsilon) = B(x_Q, \varepsilon^{-5}\ell(Q))$ (see (2.16)). We also introduce

$$B_Q^{\text{big}} = B_Q^{\text{big}}(\varepsilon) := B(x_Q, \varepsilon^{-8}\ell(Q)), \quad \Delta_Q^{\text{big}} := B_Q^{\text{big}} \cap E.$$

Lemma 5.10. *Fix $\varepsilon \in (0, K_0^{-6})$, and let $1 < p < \infty$. Suppose that u is nonnegative and p -harmonic in $\Omega_Q := \Omega \cap B_Q^{\text{big}}$, $u \in C(\overline{\Omega_Q})$, $u \equiv 0$ on Δ_Q^{big} . Suppose also that for some i , there exists a point $Y_Q \in U_Q^i$ such that*

$$|\nabla u(Y_Q)| \approx 1, \tag{5.11}$$

and furthermore, that

$$\sup_{B_Q^{***}} u \lesssim \varepsilon^{-5} \ell(Q) \tag{5.12}$$

and

$$\sup_{X, Y \in \tilde{U}_Q^i} \sup_{Z_1 \in B_Y, Z_2 \in B_X} |\nabla u(Z_1) - \nabla u(Z_2)| \leq 2\varepsilon^{2M}. \tag{5.13}$$

Then Q satisfies the ε -local WHSA, provided that M is large enough, depending only on dimension and on the implicit constants in the stated hypotheses.

Assuming these results momentarily, we can complete the proofs of Theorem 1.1 and Theorem 1.12 as follows. First we see that we can apply Lemma 5.10 to the cubes in Case 2. Indeed, let Q be a cube such that $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, $\ell(Q) \leq \varepsilon^{10} \ell(Q_0)$, and (5.5) holds. Hence (5.11) follows by virtue of (5.7), while (5.12) holds by Lemma 3.14 applied with $B = 2B_Q^{***}$ (or Lemma 3.46, with $B(y, s) = 2B_Q^{***}$), and (4.23). Moreover, (5.13) follows trivially from (5.5). Thus, the hypotheses of Lemma 5.10 are all verified and hence Q satisfies the ε -local WHSA condition. In particular, the cubes $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, which belong to the bad collection \mathcal{B} of cubes in $\mathbb{D}(E)$ for which the ε -local WHSA condition fails, must be as in Case 0 or Case 1. By (5.6) and (5.9) these cubes satisfy the packing estimate

$$\sum_{Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}, Q_0}} \sigma(Q) \leq C_\varepsilon \sigma(Q_0). \tag{5.14}$$

For each $Q_0 \in \mathbb{D}(E)$, there is a family $\mathcal{F} \subset \mathbb{D}_{Q_0}$ for which (5.14), and also the ‘‘ampleness’’ condition (4.13), hold uniformly. We may therefore invoke a well known lemma of John–Nirenberg type to deduce that (2.20) holds for all $\varepsilon \in (0, \varepsilon_0)$, and therefore to conclude that E satisfies the WHSA condition, Definition 2.19. Hence E is UR by Proposition 1.17.

The rest of the section is devoted to the proof of Lemmas 5.8 and 5.10. We shall first prove Lemma 5.8 in the relatively simpler linear case $p = 2$ (see Section 5A). The proof of Lemma 5.8 in the general case $1 < p < \infty$ is a bit more delicate and given in Section 5B. Lemma 5.10 is proved in Section 5C. Finally, the proof of Corollary 1.5 is given in Section 5D.

Before passing to the subsections we first introduce some additional notation to be used in the sequel. We augment \tilde{U}_Q^i as follows. Set

$$\mathcal{W}_Q^{i,*} := \left\{ I \in \mathcal{W} : I^* \text{ meets } B_Y \text{ for some } Y \in \left(\bigcup_{X \in \tilde{U}_Q^i} B_X \right) \right\} \tag{5.15}$$

(and define $\mathcal{W}_Q^{j,*}$ analogously for all other \tilde{U}_Q^j), and set

$$U_Q^{i,*} := \bigcup_{I \in \mathcal{W}_Q^{i,*}} I^{**}, \quad U_Q^* := \bigcup_j U_Q^{j,*}, \tag{5.16}$$

where $I^{**} = (1 + 2\tau)I$ is a suitably fattened Whitney cube, with τ fixed as above. By construction,

$$\tilde{U}_Q^i \subset \bigcup_{X \in \tilde{U}_Q^i} B_X \subset \bigcup_{Y \in \bigcup_{X \in \tilde{U}_Q^i} B_X} B_Y \subset U_Q^{i,*},$$

and for all $Y \in U_Q^{i,*}$, we have that $\delta(Y) \approx \ell(Q)$ (depending of course on ε). Moreover, also by construction, there is a Harnack path connecting any pair of points in $U_Q^{i,*}$ (depending again on ε), and furthermore, for every $I \in \mathcal{W}_Q^{i,*}$ (or for that matter for every $I \in \mathcal{W}_Q^{j,*}$, $j \neq i$),

$$\varepsilon^s \ell(Q) \lesssim \ell(I) \lesssim \varepsilon^{-3} \ell(Q), \quad \text{dist}(I, Q) \lesssim \varepsilon^{-4} \ell(Q),$$

where $0 < s = s(M, \alpha)$. Thus, by Harnack's inequality and (5.7),

$$C^{-1} \delta(Y) \leq u(Y) \leq C \delta(Y), \quad \forall Y \in U_Q^{i,*}, \tag{5.17}$$

with $C = C(K_0, \varepsilon, M)$. Moreover, for future reference, we note that the upper bound for u holds in all of U_Q^* , i.e.,

$$u(Y) \leq C \delta(Y), \quad \forall Y \in U_Q^*, \tag{5.18}$$

by (3.12) or (3.45) and (4.14), where again $C = C(K_0, \varepsilon, M)$.

5A. Proof of Lemma 5.8 in the linear case ($p = 2$). Here we complete the proof of estimate (5.9) in the relatively simpler linear case $p = 2$. To start the proof of (5.9), we fix $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ so that Case 1 holds. We see that if we choose Z as in (5.4), and use the mean value property of harmonic functions, then

$$\varepsilon^{2M} \leq C_\varepsilon (\ell(Q))^{-(n+1)} \iint_{B_Z \cup B_{Y_Q}} |\nabla u(Y) - \vec{\beta}| dY,$$

where $\vec{\beta}$ is a constant vector at our disposal. By Poincaré's inequality (see, e.g., [Hofmann and Martell 2014, Section 4] in this context), we obtain that

$$\sigma(Q) \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 \delta(Y) dY \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 u(Y) dY,$$

where the implicit constants depend on ε , and in the last step we have used (5.17). Consequently,

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \ell(Q) \leq \varepsilon^{10} \ell(Q_0)}} \iint_{U_Q^*} |\nabla^2 u(Y)|^2 u(Y) dY \lesssim \iint_{\Omega_{\mathcal{F}, Q_0}^*} |\nabla^2 u(Y)|^2 u(Y) dY, \tag{5.19}$$

where

$$\Omega_{\mathcal{F}, Q_0}^* := \text{int} \left(\bigcup_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \ell(Q) \leq \varepsilon^{10} \ell(Q_0)}} U_Q^* \right), \tag{5.20}$$

and where we have used that the enlarged Whitney regions U_Q^* have bounded overlaps.

Take an arbitrary $N > 1/\varepsilon$ (eventually $N \rightarrow \infty$), and augment \mathcal{F} by adding to it all subcubes $Q \subset Q_0$ with $\ell(Q) \leq 2^{-N} \ell(Q_0)$. Let $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$ denote the collection of maximal cubes of this augmented family. Thus, $Q \in \mathbb{D}_{\mathcal{F}_N, Q_0}$ if and only if $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ and $\ell(Q) > 2^{-N} \ell(Q_0)$. Clearly, $\mathbb{D}_{\mathcal{F}_N, Q_0} \subset \mathbb{D}_{\mathcal{F}_{N'}, Q_0}$ if $N \leq N'$, and therefore $\Omega_{\mathcal{F}_N, Q_0}^* \subset \Omega_{\mathcal{F}_{N'}, Q_0}^*$ (where $\Omega_{\mathcal{F}_{N'}, Q_0}^*$ is defined as in (5.20) with \mathcal{F}_N replacing \mathcal{F}).

By monotone convergence and (5.19), we have that

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \lesssim \limsup_{N \rightarrow \infty} \iint_{\Omega_{\mathcal{F}_N, Q_0}^*} |\nabla^2 u(Y)|^2 u(Y) dY. \quad (5.21)$$

It therefore suffices to establish bounds for the latter integral that are uniform in N , with N large.

Let us then fix $N > 1/\varepsilon$. Since $\Omega_{\mathcal{F}_N, Q_0}^*$ is a finite union of fattened Whitney boxes, we may now integrate by parts, using the identity $2|\nabla \partial_k u|^2 = \operatorname{div} \nabla (\partial_k u)^2$ for harmonic functions, to obtain that

$$\iint_{\Omega_{\mathcal{F}_N, Q_0}^*} |\nabla^2 u(Y)|^2 u(Y) dY \lesssim \int_{\partial \Omega_{\mathcal{F}_N, Q_0}^*} (|\nabla^2 u| |\nabla u| u + |\nabla u|^3) dH^n \leq C_\varepsilon H^n(\partial \Omega_{\mathcal{F}_N, Q_0}^*), \quad (5.22)$$

where in the second inequality we have used the standard estimates

$$\delta(Y) |\nabla^2 u(Y)|, |\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)},$$

along with (5.18). We observe that $\Omega_{\mathcal{F}_N, Q_0}^*$ is a sawtooth domain in the sense of [Hofmann et al. 2016], or to be more precise, it is a union of a bounded number, depending on ε , of such sawtooths, one for each maximal subcube of Q_0 with length on the order of $\varepsilon^{10} \ell(Q_0)$. By [Hofmann et al. 2016, Appendix A] each of the previous sawtooth domains is ADR uniformly in N . Hence, its union is upper ADR uniformly in N with constant depending on the number of sawtooth domains in the union, which ultimately depends on ε . Therefore,

$$H^n(\partial \Omega_{\mathcal{F}_N, Q_0}^*) \leq C_\varepsilon (\operatorname{diam}(\partial \Omega_{\mathcal{F}_N, Q_0}^*))^n \leq C_\varepsilon \sigma(Q_0).$$

Combining the latter estimate with (5.21) and (5.22), we obtain (5.9), as desired, in the case $p = 2$.

5B. Proof of Lemma 5.8 in the general case ($1 < p < \infty$). Here we prove (5.9) for general p , $1 < p < \infty$, by proceeding along the lines of the proof of Lemma 2.5 in [Lewis and Vogel 2006]. We fix $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$ so that Case 1, and hence (5.4), holds. Let us recall that we have verified estimates (5.7), (5.17), and (5.18) for all p , $1 < p < \infty$.

Recall that if $X \in \tilde{U}_Q^i$, then by definition X can be connected to some $\tilde{Y} \in U_Q^i$, and then to $Y_Q \in U_Q^i$, by a chain of at most $C\varepsilon^{-1}$ balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3 \ell(Q) \leq \delta(Y_k) \leq \varepsilon^{-3} \ell(Q)$. Note that using the triangle inequality and the definition of \tilde{U}_Q^i , we may suppose that $Y_{k+1} \in B(Y_k, 3\delta(Y_k)/4) \subset B_{Y_k}$; otherwise we increase the chain by introducing some intermediate points and the new chain will have essentially the same length. Fix now Q , a cube in Case 1, and by (5.4) we can pick $X \in \tilde{U}_Q^i$ so that

$$\sup_{Y \in B_X} |\nabla u(Y) - \nabla u(Y_Q)| > \varepsilon^{2M}.$$

As observed previously, we can form a Harnack chain connecting X and Y_Q so that $Y_1 = Y_Q$ and $Y_l = X$ and $l \leq C\varepsilon^{-1}$. Then the previous expression can be written as

$$\sup_{Y \in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_1)| > \varepsilon^{2M}. \quad (5.23)$$

Obviously we may assume that

$$\sup_{Y \in B_{Y_j}} |\nabla u(Y) - \nabla u(Y_1)| \leq \varepsilon^{2M} \tag{5.24}$$

whenever $1 < j \leq l - 1$, and $l > 1$, since otherwise we shorten the chain (and work with the first Y_j for which (5.23) holds). This and the fact that $Y_{j+1} \in B_{Y_j}$ for every $1 \leq j \leq l - 1$ imply that

$$|\nabla u(Y_j)| \geq |\nabla u(Y_1)| - \varepsilon^{2M}, \quad \text{for } 1 \leq j \leq l. \tag{5.25}$$

Furthermore, using the triangle inequality,

$$\varepsilon^{2M} \leq \sup_{Y \in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_l)| + \sum_{j=1}^{l-1} |\nabla u(Y_{j+1}) - \nabla u(Y_j)|. \tag{5.26}$$

Hence, using this and the fact that $l \lesssim \varepsilon^{-1}$ we have that either

$$\begin{aligned} \text{(i)} \quad & \sup_{Y \in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_l)| \geq \varepsilon^{2M+2}, \quad \text{or} \\ \text{(ii)} \quad & |\nabla u(Y_{j+1}) - \nabla u(Y_j)| \geq \varepsilon^{2M+2}, \quad \text{for some } 1 \leq j \leq l - 1. \end{aligned} \tag{5.27}$$

By (5.18) and (3.42) we have

$$|\nabla u(Y)| \leq C_\varepsilon, \quad \forall Y \in U_Q^*. \tag{5.28}$$

In scenario (i) of (5.27) we take Y , a point where the sup is attained. This choice, (5.28), and the first inequality in (3.38) imply that $|Y - Y_l| \approx_\varepsilon \ell(Q)$. We then construct $\Gamma_0(Q)$ a (possibly rotated) rectangle as follows. The base and the top are two n -dimensional cubes of side length $c_\varepsilon \ell(Q)$, with c_ε chosen sufficiently small, centered respectively at the points Y and Y_l , and lying in the two parallel hyperplanes passing through the points Y and Y_l and perpendicular to the vector joining these two points. Note that for this rectangle, all side lengths are of the order of $\ell(Q)$ with implicit constants possibly depending on ε . In scenario (ii) of (5.27) we do the same construction with Y_{j+1} and Y_j in place of Y and Y_l and define $\Gamma_0(Q)$ which verifies the same properties. Note that in either case, (5.28) and the first inequality in (3.38) give the property that

$$|\nabla u(Y) - \nabla u(W)| \geq \varepsilon^{2M+4} \tag{5.29}$$

whenever W and Y are in the base and top of the parallelepiped, respectively. By construction, at least the top, which we denote by $t(Q)$, is centered on Y_j , for some $1 \leq j \leq l$. We observe that by (5.25) and (5.7), since $Y_1 := Y_Q$, and since ε is very small, we have for each Y_j , $1 \leq j \leq l$,

$$|\nabla u(Y_j)| \geq a, \tag{5.30}$$

for some uniform constant a independent of ε . Therefore, by (3.38), we also have

$$|\nabla u(Y)| \geq \frac{a}{2}, \quad \forall Y \in t(Q), \tag{5.31}$$

provided that we take c_ε small enough, since $\text{diam}(t(Q)) \approx c_\varepsilon \ell(Q)$. Moving downward, that is, from top to base, through $\Gamma_0(Q)$ along slices parallel to $t(Q)$, we stop the first time that we reach a slice $b(Q)$

which contains a point Z with $|\nabla u(Z)| \leq a/4$. If there is such a slice, we form a new rectangle $\Gamma(Q)$ with base $b(Q)$ and top $t(Q)$; otherwise, we set $\Gamma(Q) := \Gamma_0(Q)$, and let $b(Q)$ denote the base in this case as well. In either case, $\text{dist}(b(Q), t(Q)) \approx \ell(Q)$, with implicit constants possibly depending on ε , by (3.38) and (5.31). Note that by construction and the continuity of ∇u ,

$$|\nabla u(Y)| \geq \frac{a}{4}, \quad \forall Y \in \Gamma(Q), \quad (5.32)$$

and that $|\Gamma(Q)| \approx \ell(Q)^{n+1}$, again with implicit constants which may depend on ε . Furthermore, if $\Gamma(Q) = \Gamma_0(Q)$, then (5.29) holds for all $W \in b(Q)$ and $Y \in t(Q)$. Otherwise, if $\Gamma(Q)$ is strictly contained in $\Gamma_0(Q)$, then, since $\text{diam}(b(Q)) \approx c_\varepsilon \ell(Q)$ with c_ε small, and since by construction $b(Q)$ contains a point Z with $|\nabla u(Z)| = a/4$, it follows that $|\nabla u(W)| \leq 3a/8$ for all $W \in b(Q)$, by (3.38). Hence, in either situation, since $a/8 \gg \varepsilon^{2M+4}$, we have

$$|\nabla u(Y) - \nabla u(W)| \geq \varepsilon^{2M+4}, \quad \forall W \in b(Q), Y \in t(Q). \quad (5.33)$$

We let $\gamma = a/8$ and set

$$F_\gamma(|\nabla u|) := \max(|\nabla u|^2 - \gamma^2, 0).$$

Then by (5.32) we see that

$$F_\gamma(|\nabla u|) \geq \frac{a^2}{64}, \quad \forall Y \in \Gamma(Q). \quad (5.34)$$

Furthermore, by (5.33), the fundamental theorem of calculus, (5.17), (5.32), and (5.34), we have

$$\ell(Q)^n \lesssim \iint_{\Gamma(Q)} u |\nabla^2 u|^2 dX \lesssim \iint_{\Gamma(Q)} u F_\gamma(|\nabla u|) |\nabla u|^{p-2} |\nabla^2 u|^2 dY,$$

where the implicit constants depend on ε . In particular, since $\Gamma(Q) \subset U_Q^{i,*} \subset U_Q^*$, by ADR we obtain

$$\sigma(Q) \lesssim \iint_{U_Q^*} u F_\gamma(|\nabla u|) |\nabla u|^{p-2} |\nabla^2 u|^2 dY,$$

where the implicit constants still depend on ε , and this estimate holds for all cubes $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, so that Case 1 holds. Hence,

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}, Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \lesssim \iint_{\Omega_{\mathcal{F}, Q_0}^*} u F_\gamma(|\nabla u|) |\nabla u|^{p-2} |\nabla^2 u|^2 dY, \quad (5.35)$$

where $\Omega_{\mathcal{F}, Q_0}^*$ was defined in (5.20) and where we have used that the enlarged Whitney regions U_Q^* have bounded overlaps. To prove (5.9) in the general case $1 < p < \infty$, it therefore suffices to establish the local square function bound

$$\iint_{\Omega_{\mathcal{F}, Q_0}^*} u F_\gamma(|\nabla u|) |\nabla u|^{p-2} |\nabla^2 u|^2 dY \lesssim \sigma(Q_0), \quad (5.36)$$

where, as we recall, u is a nonnegative p -harmonic function in the open set $\Omega_0 := \Omega \cap B(x_{Q_0}, Cr_{Q_0})$, vanishing on $\Delta(x_{Q_0}, Cr_{Q_0})$.

To start the proof of (5.36), for each $Q \in \mathbb{D}(E)$, we define a further fattening of U_Q^* as follows. Set

$$\begin{aligned} U_Q^{i,**} &:= \bigcup_{I \in \mathcal{W}_Q^{i,*}} I^{***}, & U_Q^{**} &:= \bigcup_i U_Q^{i,**}, \\ U_Q^{i,***} &:= \bigcup_{I \in \mathcal{W}_Q^{i,*}} I^{****}, & U_Q^{***} &:= \bigcup_i U_Q^{i,***}, \end{aligned}$$

where $I^{***} = (1 + 3\tau)I$ and $I^{****} = (1 + 4\tau)I$ are fattened Whitney regions, for some fixed small τ as above; see (5.15)–(5.16). Notice that $I^{**} \subset I^{***} \subset I^{****}$. We observe that the fattened Whitney regions U_Q^{***} have bounded overlaps, say

$$\sum_{Q \in \mathbb{D}(E)} 1_{U_Q^{***}}(Y) \leq M_0, \tag{5.37}$$

where $M_0 < \infty$ is a uniform constant depending on $K_0, \varepsilon, \tau,$ and n . Next, let $\{\eta_Q\}_Q$ be a partition of unity adapted to U_Q^{**} . That is,

- (1) $\sum_Q \eta_Q(Y) \equiv 1$ whenever $Y \in \Omega$,
- (2) $\text{supp } \eta_Q \subset U_Q^{**}$, and
- (3) $\eta_Q \in C_0^\infty(\mathbb{R}^{n+1})$ with $0 \leq \eta_Q \leq 1$, $\eta_Q \geq c$ on U_Q^* , and $|\nabla \eta_Q| \leq C\ell(Q)^{-1}$.

Set

$$\mathbb{D}_{\mathcal{F}, Q_0, \varepsilon} := \{Q \in \mathbb{D}_{\mathcal{F}, Q_0} : \ell(Q) \leq \varepsilon^{10} \ell(Q_0)\},$$

and recall from (5.20) that

$$\Omega_{\mathcal{F}, Q_0}^* := \text{int} \left(\bigcup_{Q \in \mathbb{D}_{\mathcal{F}, Q_0, \varepsilon}} U_Q^* \right).$$

Given a large number $N \gg \varepsilon^{-10}$, set

$$\Lambda = \Lambda(N) = \{Q \in \mathbb{D}(E) : U_Q^{**} \cap \Omega_{\mathcal{F}, Q_0}^* \neq \emptyset \text{ and } \ell(Q) \geq N^{-1} \ell(Q_0)\}.$$

Eventually, we shall let $N \rightarrow \infty$. Let

$$I_1(N) := \sum_{Q \in \Lambda(N)} \iint u F_\gamma(|\nabla u|) \left(\sum_{i,j=1}^{n+1} u_{y_i y_j}^2 \right) \eta_Q dY$$

and note, by positivity of u and the properties of η_Q , that we then have

$$\iint_{\Omega_{\mathcal{F}, Q_0}^*} u F_\gamma(|\nabla u|) |\nabla^2 u|^2 dY \lesssim \lim_{N \rightarrow \infty} I_1(N).$$

We now fix N . We intend to perform integration by parts and in this argument, we exploit that $|\nabla u|^2$ is a subsolution to a certain linear PDE defined based on u . To describe this in detail, let $Q \in \Lambda(N)$ be such that $F_\gamma(|\nabla u(Y)|) \neq 0$ for some $Y \in U_Q^{**}$. Then $|\nabla u(Y)| \geq \gamma$ and there exists $C = C(\gamma) \geq 1$ such that

$$C^{-1} \leq |\nabla u(X)| \lesssim 1 \quad \text{whenever } X \in B(Y, \delta(Y)/C), \tag{5.38}$$

and where the upper bound follows from (5.18) and the lower bound uses also (3.38). Let $\zeta = \nabla u \cdot \xi$, for some $\xi \in \mathbb{R}^{n+1}$. Then ζ satisfies, at $X \in B(Y, \delta(Y)/C)$, the partial differential equation

$$L\zeta = \nabla \cdot [(p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla \zeta)\nabla u + |\nabla u|^{p-2}\nabla \zeta] = 0, \quad (5.39)$$

as is seen by a straightforward calculation from differentiating the p -Laplace partial differential equation for u with respect to ξ . Note that (5.39) can be written in the form

$$L\zeta = \sum_{i,j=1}^{n+1} \frac{\partial}{\partial y_i} [b_{ij}(\cdot)\zeta_{y_j}(\cdot)] = 0, \quad (5.40)$$

where

$$b_{ij}(Y) = |\nabla u|^{p-4}[(p-2)u_{y_i}u_{y_j} + \delta_{ij}|\nabla u|^2](Y), \quad 1 \leq i, j \leq n+1, \quad (5.41)$$

and δ_{ij} is the Kronecker δ . Clearly we also have

$$Lu(Y) = (p-1)\nabla \cdot [|\nabla u|^{p-2}\nabla u](Y) = 0. \quad (5.42)$$

In particular, u and $(\nabla u \cdot \xi)$ for each $\xi \in \mathbb{R}^{n+1}$ all satisfy the divergence form partial differential equation (5.40).

It is easy to see that $(b_{ij})_{ij}$ satisfies the following degenerate ellipticity condition: for every $\xi \in \mathbb{R}^{n+1}$ one has

$$\begin{aligned} \sum_{i,j=1}^{n+1} b_{ij}\xi_i\xi_j &= (p-2)|\nabla u|^{p-4} \sum_{i,j=1}^{n+1} u_{y_i}u_{y_j}\xi_i\xi_j + |\nabla u|^{p-2} \sum_{i,j=1}^{n+1} \delta_{ij}\xi_i\xi_j \\ &= (p-2)|\nabla u|^{p-4}(\nabla u \cdot \xi)^2 + |\nabla u|^{p-2}|\xi|^2 \geq \min\{1, p-1\}|\nabla u|^{p-2}|\xi|^2, \end{aligned} \quad (5.43)$$

where the last inequality is immediate when $p \geq 2$ and uses the Cauchy–Schwarz inequality when $1 < p < 2$. Hence, $|\nabla u|^2$ is a subsolution to the PDE defined in (5.40), (5.41), as seen from the calculation

$$L(|\nabla u|^2) = 2 \sum_{i,j,k=1}^{n+1} b_{ij}u_{y_i}u_{y_k}u_{y_j}u_{y_k} \gtrsim |\nabla u|^{p-2} \left(\sum_{i,j=1}^{n+1} u_{y_i}^2 u_{y_j}^2 \right). \quad (5.44)$$

Now, using (5.44) and the fact that (5.38) holds for every Y such that $F_\gamma(|\nabla u(Y)|) \neq 0$, we see that $I_1(N) \lesssim J_1(N)$, where

$$J_1(N) := \sum_{Q \in \Lambda(N)} \iint u F_\gamma(|\nabla u|) L(|\nabla u|^2) \eta_Q dY.$$

Hence it suffices to establish bounds for the integral $J_1 := J_1(N)$ that are uniform in N , with N large. In the following we let $v = F_\gamma(|\nabla u|)$ and note that $\nabla v = \nabla(|\nabla u|^2)$ whenever $v > 0$. Using this and integration by parts we see that

$$J_1 = -J_2 - J_3 - J_4,$$

where

$$\begin{aligned}
 J_2 &= \sum_{Q \in \Lambda(N)} \iint v \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} v_{y_j} \eta_Q dY, \\
 J_3 &= \sum_{Q \in \Lambda(N)} \iint u \sum_{i,j=1}^{n+1} b_{ij} v_{y_i} v_{y_j} \eta_Q dY, \\
 J_4 &= \sum_{Q \in \Lambda(N)} \iint uv \sum_{i,j=1}^{n+1} b_{ij} v_{y_j} (\eta_Q)_{y_i} dY.
 \end{aligned}$$

We estimate J_4 first. Set $\Lambda_1 = \Lambda_{11} \cup \Lambda_{12}$, where

$$\Lambda_{11} := \{Q \in \Lambda : U_Q^{**} \text{ meets } \Omega \setminus \Omega_{\mathcal{F}, Q_0}\},$$

and

$$\Lambda_{12} := \{Q \in \Lambda : U_Q^{**} \text{ meets } U_{Q'}^{**} \text{ such that } \ell(Q') < N^{-1}\ell(Q_0)\}.$$

From the definition of η_Q , we obtain

$$|J_4| \lesssim \sum_{Q \in \Lambda_{11}} \iint uv \sum_{i,j=1}^{n+1} |u_{ij}| |u_i| |(\eta_Q)_j| dY + \sum_{Q \in \Lambda_{12}} \iint uv \sum_{i,j=1}^{n+1} |u_{ij}| |u_i| |(\eta_Q)_j| dY =: J_{51} + J_{52}.$$

Notice that, equivalently, Λ_{11} is the subcollection of $Q \in \Lambda_1$ such that U_Q^{**} meets $\partial\Omega_{\mathcal{F}, Q_0}^*$. We start with J_{51} . Note that by (3.38), (5.18), and Harnack’s inequality,

$$\delta(Y)|\nabla u(Y)| \lesssim u(Y) \lesssim \delta(Y) \approx \ell(Q) \tag{5.45}$$

whenever $Y \in U_Q^{***}$. Furthermore, if $v \neq 0$ for some $Y \in U_Q^{***}$, then using (5.38) and (3.40), we also have

$$(\delta(Y))^2 |\nabla^2 u(Y)| \lesssim u(Y) \lesssim \delta(Y) \approx \ell(Q). \tag{5.46}$$

In particular, $u|\nabla\eta_Q| \lesssim 1$ by construction of η_Q , $|\nabla u(Y)| \lesssim 1$ whenever $Y \in U_Q^{***}$, and $\delta(Y)|\nabla^2 u(Y)| \lesssim 1$ whenever $Y \in U_Q^{***}$ and $v \neq 0$. Thus,

$$J_{51} \lesssim \sum_{Q \in \Lambda_{11}} \ell(Q)^n \lesssim \sum_{Q \in \Lambda_{11}} H^n(U_Q^{***} \cap \partial\Omega_{\mathcal{F}, Q_0}^*) \lesssim \sum_{Q \in \Lambda_{11}} H^n(\partial\Omega_{\mathcal{F}, Q_0}^*) \lesssim \sigma(Q_0),$$

where we have used that $\partial\Omega_{\mathcal{F}, Q_0}^*$ is ADR (see [Hofmann et al. 2016]), and the bounded overlap property (5.37). To estimate J_{52} , observe that for each $Q \in \Lambda_{12}$, we have $\ell(Q) \approx N^{-1}\ell(Q_0)$ by properties of Whitney regions. Hence, by a slightly simpler version of the argument used for J_{51} , we obtain

$$J_{52} \lesssim \sum_{Q \in \Lambda_{12}} \sigma(Q) \lesssim \sigma(Q_0).$$

Therefore, $|J_4| \lesssim J_{51} + J_{52} \lesssim \sigma(Q_0)$.

To handle J_2 we use the fact that u is a solution to (5.40). Indeed, by integration by parts, using the identity $2vv_{y_j} = (v^2)_{y_j}$ we see that

$$2J_2 = \sum_{Q \in \Lambda(N)} \iint \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} (v^2)_{y_j} \eta_Q dY = - \sum_{Q \in \Lambda(N)} \iint \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} v^2 (\eta_Q)_{y_j} dY,$$

and by the same argument as in the estimate of J_4 we obtain $|J_2| \lesssim \sigma(Q_0)$.

To conclude, we collect the estimates for J_2 and J_4 , and use the fact that J_3 is nonnegative by (5.43) to obtain $J_1(N) \lesssim \sigma(Q_0)$, with constants independent of N . The proof of (5.9) in the general case $1 < p < \infty$ is then complete.

5C. Proof of Lemma 5.10. To prove Lemma 5.10, we follow the corresponding argument in [Lewis and Vogel 2007] closely, but with some modifications due to the fact that in contrast to the situation in that paper, our solution u need not be Lipschitz up to the boundary, and our harmonic/ p -harmonic measures need not be doubling. It is the latter obstacle that has forced us to introduce the WHSA condition, rather than to work with the weak exterior convexity condition used by Lewis and Vogel. Lemma 5.10 is essentially a distillation of the main argument of the corresponding part of [Lewis and Vogel 2007], but with the doubling hypothesis removed.

In the remainder of this section, for convenience we use the notational convention that implicit and generic constants are allowed to depend upon K_0 , but not on ε or M . Dependence on the latter is stated explicitly. We first prove the following lemma. Recall that the balls B_Y and \tilde{B}_Y are defined in (5.3).

Lemma 5.47. *Let $Y \in U_Q^i$, $X \in \tilde{U}_Q^i$. Suppose first that $w \in \partial\tilde{B}_Y \cap E$, and let W be the radial projection of w onto ∂B_Y . Then*

$$u(W) \lesssim \varepsilon^{2M-5} \delta(Y). \tag{5.48}$$

If $w \in \partial\tilde{B}_X \cap E$, and W now is the radial projection of w onto ∂B_X , then

$$u(W) \lesssim \varepsilon^{2M-5} \ell(Q). \tag{5.49}$$

Proof. Since $K_0^{-1} \ell(Q) \lesssim \delta(Y) \lesssim K_0 \ell(Q)$ for $Y \in U_Q^i$, it is enough to prove (5.49). To prove (5.49), we first note that

$$|W - w| = \varepsilon^{2M/\alpha} \delta(X) \lesssim \varepsilon^{2M/\alpha} \varepsilon^{-3} \ell(Q),$$

by definition of B_X , \tilde{B}_X and the fact that by construction of \tilde{U}_Q^i ,

$$\varepsilon^3 \ell(Q) \lesssim \delta(X) \lesssim \varepsilon^{-3} \ell(Q), \quad \forall X \in \tilde{U}_Q^i. \tag{5.50}$$

In addition, again by construction of \tilde{U}_Q^i ,

$$\text{diam}(\tilde{U}_Q^i) \lesssim \varepsilon^{-4} \ell(Q). \tag{5.51}$$

Consequently, $W \in \frac{1}{2} B_Q^{***} = B(x_Q, \frac{1}{2} \varepsilon^{-5} \ell(Q))$, so by Lemma 3.35 and (5.12),

$$u(W) \lesssim \left(\frac{\varepsilon^{2M/\alpha} \varepsilon^{-3} \ell(Q)}{\varepsilon^{-5} \ell(Q)} \right)^\alpha \frac{1}{|B_Q^{***}|} \iint_{B_Q^{***}} u \lesssim \varepsilon^{2M+2\alpha-5} \ell(Q) \leq \varepsilon^{2M-5} \ell(Q). \quad \square$$

Claim 5.52. Let $Y \in U_Q^i$. For all $W \in B_Y$,

$$|u(W) - u(Y) - \nabla u(Y) \cdot (W - Y)| \lesssim \varepsilon^{2M} \delta(Y). \quad (5.53)$$

Proof of Claim 5.52. Let $W \in B_Y$. Then for some $\tilde{W} \in B_Y$,

$$u(W) - u(Y) = \nabla u(\tilde{W}) \cdot (W - Y).$$

We may then invoke (5.13), with $X = Y$, $Z_1 = \tilde{W}$, and $Z_2 = Y$, to obtain (5.53). \square

Claim 5.54. Let $Y \in U_Q^i$. Suppose that $w \in \partial \tilde{B}_Y \cap E$. Then

$$|u(Y) - \nabla u(Y) \cdot (Y - w)| = |u(w) - u(Y) - \nabla u(Y) \cdot (w - Y)| \lesssim \varepsilon^{2M-5} \delta(Y). \quad (5.55)$$

Proof of Claim 5.54. Given $w \in \partial \tilde{B}_Y \cap E$, let W be the radial projection of w onto ∂B_Y , so that $|W - w| = \varepsilon^{2M/\alpha} \delta(Y)$. Since $u(w) = 0$, by (5.48) we have

$$|u(W) - u(w)| = u(W) \lesssim \varepsilon^{2M-5} \delta(Y).$$

Since (5.53) holds for W , we obtain (5.55) by (5.11) and (5.13). \square

To simplify notation, we now set $Y := Y_Q$, the point in U_Q^i satisfying (5.11). By (5.11) and (5.13), for $\varepsilon < \frac{1}{2}$, and M chosen large enough, we have that

$$|\nabla u(Z)| \approx 1, \quad \forall Z \in \tilde{U}_Q^i. \quad (5.56)$$

By translation and rotation, we assume that $0 \in \partial \tilde{B}_Y \cap E$ and that $Y = \delta(Y)e_{n+1}$, where as usual $e_{n+1} := (0, \dots, 0, 1)$.

Claim 5.57. We claim that

$$|\nabla u(Y) \cdot e_{n+1} - |\nabla u(Y)|| \lesssim \varepsilon^{2M-5}. \quad (5.58)$$

Proof of Claim 5.57. We apply (5.55), with $w = 0$, to obtain

$$|u(Y) - \nabla u(Y) \cdot Y| \lesssim \varepsilon^{2M-5} \delta(Y).$$

Combining the latter bound with (5.53), we find that

$$|u(W) - \nabla u(Y) \cdot W| = |u(W) - \nabla u(Y) \cdot Y - \nabla u(Y) \cdot (W - Y)| \lesssim \varepsilon^{2M-5} \delta(Y), \quad \forall W \in B_Y. \quad (5.59)$$

Fix $W \in \partial B_Y$ so that $\nabla u(Y) \cdot \frac{W - Y}{|W - Y|} = -|\nabla u(Y)|$. Since $|W - Y| = (1 - \varepsilon^{2M/\alpha})\delta(Y)$, and since $u \geq 0$, we have

$$\begin{aligned} 0 &\leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} \leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} + \frac{u(W)}{\delta(Y)} \\ &\leq \frac{1}{\delta(Y)} \left(-\nabla u(Y) \cdot \frac{(W - Y)}{1 - \varepsilon^{2M/\alpha}} - \nabla u(Y) \cdot Y + u(W) \right) \\ &\lesssim (\varepsilon^{2M-5} + \varepsilon^{2M/\alpha}) \approx \varepsilon^{2M-5}, \end{aligned} \quad (5.60)$$

by (5.59) and (5.11). \square

Claim 5.61. *Suppose that $M > 5$. Then*

$$\left| |\nabla u(Y)|e_{n+1} - \nabla u(Y) \right| \lesssim \varepsilon^{M-3}. \quad (5.62)$$

Proof of Claim 5.61. By Claim 5.57,

$$\left| |\nabla u(Y)|e_{n+1} - (\nabla u(Y) \cdot e_{n+1})e_{n+1} \right| \lesssim \varepsilon^{2M-5}.$$

Therefore, it is enough to consider $\nabla_{\parallel} u := \nabla u - (\nabla u(Y) \cdot e_{n+1})e_{n+1}$. Observe that

$$\begin{aligned} |\nabla_{\parallel} u(Y)|^2 &= |\nabla u(Y)|^2 - (\nabla u(Y) \cdot e_{n+1})^2 \\ &= (|\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1})(|\nabla u(Y)| + \nabla u(Y) \cdot e_{n+1}) \lesssim \varepsilon^{2M-5}, \end{aligned}$$

by (5.58) and (5.11). □

Now for $Y = \delta(Y)e_{n+1} \in U_Q^i$ fixed as above, we consider another point $X \in \tilde{U}_Q^i$. By definition of \tilde{U}_Q^i , there is a polygonal path in \tilde{U}_Q^i , joining Y to X , with vertices

$$Y_0 := Y, Y_1, Y_2, \dots, Y_N := X, \quad N \lesssim \varepsilon^{-4},$$

such that $Y_{k+1} \in B_{Y_k} \cap B(Y_k, \ell(Q))$, $0 \leq k \leq N-1$, and such that the distance between consecutive vertices is at most $C\ell(Q)$. Indeed, by definition of \tilde{U}_Q^i , we may connect Y to X by a polygonal path connecting the centers of at most ε^{-1} balls, such that the distance between consecutive vertices is between $\varepsilon^3\ell(Q)/2$ and $\varepsilon^{-3}\ell(Q)/2$. If any such distance is greater than $\ell(Q)$, we take at most $C\varepsilon^{-3}$ intermediate vertices with distances on the order of $\ell(Q)$. The total length of the path is thus on the order of $N\ell(Q)$ with $N \lesssim \varepsilon^{-4}$. Furthermore, by (5.13) and (5.62),

$$\begin{aligned} \left| \nabla u(W) - |\nabla u(Y)|e_{n+1} \right| &\leq |\nabla u(W) - \nabla u(Y)| + \left| \nabla u(Y) - |\nabla u(Y)|e_{n+1} \right| \\ &\lesssim \varepsilon^{2M} + \varepsilon^{M-3} \lesssim \varepsilon^{M-3}, \quad \forall W \in B_Z, \forall Z \in \tilde{U}_Q^i. \end{aligned} \quad (5.63)$$

Claim 5.64. *Assume $M > 7$. Then for each $k = 1, 2, \dots, N$,*

$$\left| u(Y_k) - |\nabla u(Y)|Y_k \cdot e_{n+1} \right| \lesssim k\varepsilon^{M-3}\ell(Q). \quad (5.65)$$

Moreover,

$$\left| u(W) - |\nabla u(Y)|W_{n+1} \right| \lesssim \varepsilon^{M-7}\ell(Q), \quad \forall W \in B_X, \forall X \in \tilde{U}_Q^i. \quad (5.66)$$

Proof of Claim 5.64. By (5.59) and (5.62), we have

$$\begin{aligned} \left| u(W) - |\nabla u(Y)|W_{n+1} \right| &\lesssim |u(W) - \nabla u(Y) \cdot W| + \left| (\nabla u(Y) - |\nabla u(Y)|e_{n+1}) \cdot W \right| \\ &\lesssim \varepsilon^{2M-5}\delta(Y) + \varepsilon^{M-3}|W| \lesssim \varepsilon^{M-3}\ell(Q), \quad \forall W \in B_Y, \end{aligned} \quad (5.67)$$

since $\delta(Z) \approx \ell(Q)$, for all $Z \in U_Q^i$ (so in particular, for $Z = Y$), and since $|W| \leq 2\delta(Y) \lesssim \ell(Q)$, for all $W \in B_Y$. Thus, (5.65) holds with $k = 1$, since $Y_1 \in B_Y$, by construction. Now suppose that (5.65) holds for all $1 \leq i \leq k$, with $k \leq N$. Let $W \in B_{Y_k}$, so that W may be joined to Y_k by a line segment of

length less than $\delta(Y_k) \lesssim \varepsilon^{-3}\ell(Q)$ (the latter bound holds by (5.50)). We note also that if $k \leq N - 1$, and if $W = Y_{k+1}$, then this line segment has length at most $\ell(Q)$, by construction. Then

$$\begin{aligned} |u(W) - |\nabla u(Y)|W_{n+1}| &\leq |u(W) - u(Y_k) + |\nabla u(Y)|(Y_k - W) \cdot e_{n+1}| + |u(Y_k) - |\nabla u(Y)|Y_k \cdot e_{n+1}| \\ &= |(W - Y_k) \cdot \nabla u(W_1) + |\nabla u(Y)|(Y_k - W) \cdot e_{n+1}| + O(k\varepsilon^{M-3}\ell(Q)), \end{aligned}$$

where W_1 is an appropriate point on the line segment joining W and Y_k , and where we have used that Y_k satisfies (5.65). By (5.63), applied to W_1 , we find in turn that

$$|u(W) - |\nabla u(Y)|W_{n+1}| \lesssim \varepsilon^{M-3}|W - Y_k| + k\varepsilon^{M-3}\ell(Q), \tag{5.68}$$

which, by our previous observations, is bounded by $C(k + 1)\varepsilon^{M-3}\ell(Q)$ if $W = Y_{k+1}$, or by $(\varepsilon^{M-6} + k\varepsilon^{M-3})\ell(Q)$ in general. In the former case, we find that (5.65) holds for all $k = 1, 2, \dots, N$, and in the latter case, taking $k = N \lesssim \varepsilon^{-4}$, we obtain (5.66). \square

Claim 5.69. *Let $X \in \tilde{U}_Q^i$, and let $w \in E \cap \partial\tilde{B}_X$. Then*

$$|\nabla u(Y)||w_{n+1}| \lesssim \varepsilon^{M/2}\ell(Q). \tag{5.70}$$

Proof of Claim 5.69. Let W be the radial projection of w onto ∂B_X , so that

$$|W - w| = \varepsilon^{2M/\alpha}\delta(X) \lesssim \varepsilon^{(2M/\alpha)-3}\ell(Q), \tag{5.71}$$

by (5.50). We write

$$|\nabla u(Y)||w_{n+1}| \leq |\nabla u(Y)||W - w| + |u(W) - |\nabla u(Y)|W_{n+1}| + u(W) =: I + II + u(W).$$

Note that $I \lesssim \varepsilon^{(2M/\alpha)-3}\ell(Q)$ by (5.71) and (5.11) (recall that $Y = Y_Q$), and that $II \lesssim \varepsilon^{M-7}\ell(Q)$ by (5.66). Furthermore, $u(W) \lesssim \varepsilon^{2M-5}\ell(Q)$, by (5.49). For M chosen large enough, we obtain (5.70). \square

We note that since we have fixed $Y = Y_Q$, it then follows from (5.70) and (5.11) that

$$|w_{n+1}| \lesssim \varepsilon^{M/2}\ell(Q), \quad \forall w \in E \cap \partial\tilde{B}_X, \quad \forall X \in \tilde{U}_Q^i. \tag{5.72}$$

Recall that x_Q denotes the ‘‘center’’ of Q (see (2.7)–(2.8)). Set

$$\mathcal{O} := B(x_Q, 2\varepsilon^{-2}\ell(Q)) \cap \{W : W_{n+1} > \varepsilon^2\ell(Q)\}. \tag{5.73}$$

Claim 5.74. *For every point $X \in \mathcal{O}$, we have $X \approx_{\varepsilon, Q} Y$ (see Definition 2.26). Thus, in particular, $\mathcal{O} \subset \tilde{U}_Q^i$.*

Proof of Claim 5.74. Let $X \in \mathcal{O}$. We need to show that X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \leq \delta(Y_k) \leq \varepsilon^{-3}\ell(Q)$ (for convenience, we shall refer to such balls as ‘‘admissible’’). We first observe that if $X = te_{n+1}$, with $\varepsilon^3\ell(Q) \leq t \leq \varepsilon^{-3}\ell(Q)$, then by an iteration argument using (5.72) (with M chosen large enough), we may join X to Y by at most $C \log(1/\varepsilon)$ admissible balls. The point $(2\varepsilon)^{-3}\ell(Q)e_{n+1}$ may then be joined to any point of the form $(X', (2\varepsilon)^{-3}\ell(Q))$ by a chain of at most C admissible balls, whenever $X' \in \mathbb{R}^n$ with $|X'| \leq \varepsilon^{-3}\ell(Q)$. In turn, the latter point may then be joined to $(X', \varepsilon^3\ell(Q))$ by at most $C \log(1/\varepsilon)$ admissible balls. \square

We note that Claim 5.74 implies that

$$E \cap O = \emptyset. \tag{5.75}$$

Indeed, $O \subset \tilde{U}_Q^i \subset \Omega$. Let P_0 denote the hyperplane

$$P_0 := \{Z : Z_{n+1} = 0\}.$$

Claim 5.76. *If $Z \in P_0$, with $|Z - x_Q| \leq \frac{3}{2}\varepsilon^{-2}\ell(Q)$, then*

$$\delta(Z) = \text{dist}(Z, E) \leq 16\varepsilon^2\ell(Q). \tag{5.77}$$

Proof of Claim 5.76. Observe that $B(Z, 2\varepsilon^2\ell(Q))$ meets O . Then by Claim 5.74, there is a point $X \in \tilde{U}_Q^i \cap B(Z, 2\varepsilon^2\ell(Q))$. Suppose that (5.77) is false, which in particular implies that $\delta(X) \geq 14\varepsilon^2\ell(Q)$. Then $B(Z, 4\varepsilon^2\ell(Q)) \subset B_X$, so by (5.66), we have

$$|u(W) - |\nabla u(Y)||_{W_{n+1}}| \leq C\varepsilon^{M-7}\ell(Q), \quad \forall W \in B(Z, 4\varepsilon^2\ell(Q)). \tag{5.78}$$

In particular, since $Z_{n+1} = 0$, we may choose W such that $W_{n+1} = -\varepsilon^2\ell(Q)$, to obtain that

$$|\nabla u(Y)|\varepsilon^2\ell(Q) \leq C\varepsilon^{M-7}\ell(Q),$$

since $u \geq 0$. But for $\varepsilon < \frac{1}{2}$, and M large enough, this is a contradiction, by (5.11) (recall that we have fixed $Y = Y_Q$). □

It now follows by Definition 2.17 that Q satisfies the ε -local WHSA condition, with

$$P = P(Q) := \{Z : Z_{n+1} = \varepsilon^2\ell(Q)\}, \quad H = H(Q) := \{Z : Z_{n+1} > \varepsilon^2\ell(Q)\}.$$

This concludes the proof of Lemma 5.10.

5D. Proof of Corollary 1.5. Now Corollary 1.5 follows almost immediately from Theorem 1.1. Let $B = B(x, r)$ and $\Delta = B \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$. Let c be the constant in Lemma 3.1. By hypothesis, there is a point $X_\Delta \in B \cap \Omega$ which is a corkscrew point relative to Δ , that is, there is a uniform constant $c_0 > 0$ such that $\delta(X_\Delta) \geq c_0r$. Thus, to apply Theorem 1.1, it remains only to verify hypothesis (\star) . For a sufficiently large constant C_1 , set $\Delta^{\text{fat}} = \Delta(x, C_1r)$. Cover Δ^{fat} by a collection of surface balls $\{\Delta_i\}_{i=1}^N$ with $\Delta_i = B_i \cap \partial\Omega$ and $B_i := B(x_i, c_0r/4)$, where $x_i \in \Delta^{\text{fat}}$ and where N is uniformly bounded, depending only on n, c_0, C_1 , and ADR. By construction, $X_\Delta \in \Omega \setminus 4B_i$, so by hypothesis, $\omega^{X_\Delta} \in \text{weak-}A_\infty(2\Delta_i)$. Hence, $\omega^{X_\Delta} \ll \sigma$ in $2\Delta_i$, and (1.6) holds with $Y = X_\Delta$, and with $\Delta' = \Delta_i$. Consequently, $\omega^{X_\Delta} \ll \sigma$ in Δ^{fat} , and if we write $k^{X_\Delta} = d\omega^{X_\Delta}/d\sigma$, we obtain

$$\begin{aligned} \int_{\Delta^{\text{fat}}} k^{X_\Delta}(z)^q d\sigma(z) &\leq \sum_{i=1}^N \int_{\Delta_i} k^{X_\Delta}(z)^q d\sigma(z) \lesssim \sum_{i=1}^N \sigma(\Delta_i) \left(\int_{2\Delta_i} k^{X_\Delta}(z) d\sigma(z) \right)^q \\ &\lesssim \sum_{i=1}^N \sigma(2\Delta_i)^{1-q} \omega^{X_\Delta}(2\Delta_i) \lesssim \sigma(\Delta^{\text{fat}})^{1-q}, \end{aligned}$$

where in the last estimate we have used the ADR property, the uniform boundedness of N , and the fact that $\omega^{X_\Delta}(2\Delta_i) \leq 1$. By Theorem 1.1, it then follows that $\partial\Omega$ is UR as desired. □

6. Proof of Proposition 1.17

Here we prove Proposition 1.17. We first observe that if E is UR then it satisfies the so-called “bilateral weak geometric lemma” (BWGL); see [David and Semmes 1991, Theorem I.2.4, p. 32]. In turn, in [David and Semmes 1991, Section II.2.1, p. 97], one can find a dyadic formulation of the BWGL as follows. Given ε small enough and $k > 1$ large to be chosen, $\mathbb{D}(E)$ can be split in two collections, one of “bad cubes” and another of “good cubes”, so that the “bad cubes” satisfy a packing condition and each “good cube” Q verifies the following: there is a hyperplane $P = P(Q)$ such that $\text{dist}(Z, E) \leq \varepsilon \ell(Q)$ for every $Z \in P \cap B(x_Q, k\ell(Q))$, and $\text{dist}(Z, P) \leq \varepsilon \ell(Q)$ for every $Z \in B(x_Q, k\ell(Q)) \cap E$. In turn, this implies that $B(x_Q, k\ell(Q)) \cap E$ is sandwiched between two planes parallel to P at distance $\varepsilon \ell(Q)$. Hence, at that scale, we have a half-space (indeed we have two) free of E , and clearly the 2ε -local WHSA holds provided K is taken of the order of ε^{-2} or larger. Further details are left to the interested reader. Thus we obtain the easy implication $\text{UR} \implies \text{WHSA}$.

The main part of the proof is to establish the opposite implication. To this end, we assume that E satisfies the WHSA property and show that E is UR. Given a positive $\varepsilon < \varepsilon_0 \ll K_0^{-6}$, we let \mathcal{B}_0 denote the collection of bad cubes for which ε -local WHSA fails. By Definition 2.19, \mathcal{B}_0 satisfies the Carleson packing condition (2.20). We now introduce a variant of the packing measure for \mathcal{B}_0 . We recall that $B_Q^* = B(x_Q, K_0^2 \ell(Q))$, and given $Q \in \mathbb{D}(E)$, we set

$$\mathbb{D}_\varepsilon(Q) := \{Q' \in \mathbb{D}(E) : \varepsilon^{3/2} \ell(Q) \leq \ell(Q') \leq \ell(Q), Q' \text{ meets } B_Q^*\}. \tag{6.1}$$

Set

$$\alpha_Q := \begin{cases} \sigma(Q) & \text{if } \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{6.2}$$

and define

$$m(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q, \quad \mathbb{D}' \subset \mathbb{D}(E). \tag{6.3}$$

Then m is a discrete Carleson measure, with

$$m(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0} \alpha_Q \leq C_\varepsilon \sigma(Q_0), \quad Q_0 \in \mathbb{D}(E). \tag{6.4}$$

Indeed, note that for any Q' , the cardinality of $\{Q : Q' \in \mathbb{D}_\varepsilon(Q)\}$ is uniformly bounded, depending on n , ε , and ADR, and that $\sigma(Q) \leq C_\varepsilon \sigma(Q')$ if $Q' \in \mathbb{D}_\varepsilon(Q)$. Then given any $Q_0 \in \mathbb{D}(E)$,

$$\begin{aligned} m(\mathbb{D}_{Q_0}) &= \sum_{Q \subset Q_0 : \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \neq \emptyset} \sigma(Q) \leq \sum_{Q' \in \mathcal{B}_0} \sum_{Q \subset Q_0 : Q' \in \mathbb{D}_\varepsilon(Q)} \sigma(Q) \\ &\leq C_\varepsilon \sum_{Q' \in \mathcal{B}_0 : Q' \subset 2B_{Q_0}^*} \sigma(Q') \leq C_\varepsilon \sigma(Q_0), \end{aligned}$$

by (2.20) and ADR.

To prove Proposition 1.17, we are required to show that the collection \mathcal{B} of bad cubes for which the $\sqrt{\varepsilon}$ -local BAUP condition fails satisfies a packing condition. That is, we establish the discrete Carleson

measure estimate

$$\tilde{\mathfrak{m}}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0: Q \in \mathcal{B}} \sigma(Q) \leq C_\varepsilon \sigma(Q_0), \quad Q_0 \in \mathbb{D}(E). \quad (6.5)$$

To this end, by (6.4), it suffices to show that if $Q \in \mathcal{B}$, then $\alpha_Q \neq 0$ (and thus $\alpha_Q = \sigma(Q)$, by definition). In fact, we prove the contrapositive statement.

Claim 6.6. *Suppose that $\alpha_Q = 0$. Then the $\sqrt{\varepsilon}$ -local BAUP condition holds for Q .*

Proof of Claim 6.6. We first note that since $\alpha_Q = 0$, then by definition of α_Q ,

$$B_0 \cap \mathbb{D}_\varepsilon(Q) = \emptyset. \quad (6.7)$$

Thus, the ε -local WHSA condition (Definition 2.17) holds for every $Q' \in \mathbb{D}_\varepsilon(Q)$ (in particular, for Q itself). By rotation and translation, we may suppose that the hyperplane $P = P(Q)$ in Definition 2.17 is

$$P = \{Z \in \mathbb{R}^{n+1} : Z_{n+1} = 0\},$$

and that the half-space $H = H(Q)$ is the upper half-space $\mathbb{R}_+^{n+1} = \{Z : Z_{n+1} > 0\}$. We recall that by Definition 2.17, P and H satisfy

$$\text{dist}(Z, E) \leq \varepsilon \ell(Q), \quad \forall Z \in P \cap B_Q^{**}(\varepsilon), \quad (6.8)$$

$$\text{dist}(P, Q) \leq K_0^{3/2} \ell(Q), \quad (6.9)$$

and

$$H \cap B_Q^{**}(\varepsilon) \cap E = \emptyset. \quad (6.10)$$

The proof now follows by a construction similar to that in [Lewis and Vogel 2007], used to establish the weak exterior convexity condition. By (6.10), there are two cases.

Case 1: $10Q \subset \{Z : -\sqrt{\varepsilon} \ell(Q) \leq Z_{n+1} \leq 0\}$. In this case, the $\sqrt{\varepsilon}$ -local BAUP condition holds trivially for Q , with $\mathcal{P} = \{P\}$.

Case 2: There is a point $x \in 10Q$ such that $x_{n+1} < -\sqrt{\varepsilon} \ell(Q)$. In this case, we choose $Q' \ni x$ with $\varepsilon^{3/4} \ell(Q) \leq \ell(Q') < 2\varepsilon^{3/4} \ell(Q)$. Thus,

$$Q' \subset \{Z : Z_{n+1} \leq -\frac{1}{2}\sqrt{\varepsilon} \ell(Q)\}. \quad (6.11)$$

Moreover, $Q' \in \mathbb{D}_\varepsilon(Q)$, so by (6.7), $Q' \notin B_0$, i.e., Q' satisfies the ε -local WHSA. Let $P' = P(Q')$ and $H' = H(Q')$ denote the hyperplane and half-space corresponding to Q' in Definition 2.17, so that

$$\text{dist}(Z, E) \leq \varepsilon \ell(Q') \leq 2\varepsilon^{7/4} \ell(Q), \quad \forall Z \in P' \cap B_{Q'}^{**}(\varepsilon), \quad (6.12)$$

$$\text{dist}(P', Q') \leq K_0^{3/2} \ell(Q') \approx K_0^{3/2} \varepsilon^{3/4} \ell(Q) \ll \varepsilon^{1/2} \ell(Q) \quad (6.13)$$

(where the last inequality holds since $\varepsilon \ll K_0^{-6}$), and

$$H' \cap B_{Q'}^{**}(\varepsilon) \cap E = \emptyset, \quad (6.14)$$

where we recall that $B_{Q'}^{**}(\varepsilon) := B(x_{Q'}, \varepsilon^{-2}\ell(Q'))$ (see (2.16)). We note that

$$B_Q^* \subset \tilde{B}_Q(\varepsilon) := B(x_Q, \varepsilon^{-1}\ell(Q)) \subset B_{Q'}^{**}(\varepsilon) \cap B_Q^{**}(\varepsilon), \tag{6.15}$$

by construction, since $\varepsilon \ll K_0^{-6}$. Let ν' denote the unit normal vector to P' , pointing into H' . Note that by (6.10), (6.12), and the definition of H ,

$$P' \cap \tilde{B}_Q(\varepsilon) \cap \{Z : Z_{n+1} > 2\varepsilon^{7/4}\ell(Q)\} = \emptyset. \tag{6.16}$$

Moreover, ν' points “downward”, i.e., $\nu' \cdot e_{n+1} < 0$, as otherwise, $H' \cap \tilde{B}_Q(\varepsilon)$ would meet E by (6.8), (6.11), and (6.13). More precisely, we have the following.

Claim 6.17. *The angle θ between ν' and $-e_{n+1}$ satisfies $0 \leq \theta \approx \sin \theta \lesssim \varepsilon$.*

Indeed, since Q' meets $10Q$, (6.9) and (6.13) imply that $\text{dist}(P, P') \lesssim K_0^{3/2}\ell(Q)$, and that the latter estimate is attained near Q . By (6.16) and a trigonometric argument, one then obtains Claim 6.17 (more precisely, one obtains $\theta \lesssim K_0^{3/2}\varepsilon$, but in this section, we continue to use the notational convention that implicit constants may depend upon K_0 , but K_0 is fixed, and $\varepsilon \ll K_0^{-6}$). The interested reader could probably supply the remaining details of the argument that we have just sketched, but for the sake of completeness, we give the full proof at the end of this section.

We therefore take Claim 6.17 for granted, and proceed with the argument. We note first that every point in $(P \cup P') \cap B_Q^*$ is at a distance at most $\varepsilon\ell(Q)$ from E by (6.8), (6.12), and (6.15). To complete the proof of Claim 6.6, it therefore remains only to verify the following. As with the previous claim, we provide a condensed proof immediately, and present a more detailed argument at the end of the section.

Claim 6.18. *Every point in $10Q$ lies within $\sqrt{\varepsilon}\ell(Q)$ of a point in $P \cup P'$.*

Suppose not. We could then repeat the previous argument, to construct a cube Q'' , a hyperplane P'' , a unit vector ν'' forming a small angle with $-e_{n+1}$, and a half-space H'' with boundary P'' , with the same properties as Q' , P' , ν' , and H' . In particular, we have the respective analogues of (6.13) and (6.14), namely

$$\text{dist}(P'', Q'') \leq K_0^{3/2}\ell(Q') \approx K_0^{3/2}\varepsilon^{3/4}\ell(Q) \ll \varepsilon^{1/2}\ell(Q) \tag{6.19}$$

and

$$H'' \cap B_{Q''}^{**}(\varepsilon) \cap E = \emptyset, \tag{6.20}$$

Also, we have the analogue of (6.11), with Q'', P'' in place of Q', P' . Thus

$$\text{dist}(Q'', P'') \geq \frac{1}{2}\sqrt{\varepsilon}\ell(Q) \quad \text{and} \quad Q'' \cap H' = \emptyset. \tag{6.21}$$

In addition, as in (6.15), we also have $B_Q^* \subset B_{Q''}^{**}(\varepsilon)$. On the other hand, the angle between ν' and ν'' is very small. Thus, combining (6.12), (6.19), and (6.21), we see that $H'' \cap B_Q^*$ captures points in E , which contradicts (6.20).

Claim 6.6 therefore holds (in fact, with a union of at most 2 planes), and thus we obtain the conclusion of Proposition 1.17. □

We now provide detailed proofs of Claims 6.17 and 6.18.

Proof of Claim 6.17. By (6.13) we can pick $x' \in Q'$, $y' \in P'$ such that $|y' - x'| \ll \varepsilon^{1/2} \ell(Q)$, and therefore $y' \in 11Q$. Also, from (6.9) and (6.10) we can find $\bar{x} \in Q$ such that $-K_0^{3/2} \ell(Q) < \bar{x}_{n+1} \leq 0$. This and (6.11) yield

$$-2K_0^{3/2} \ell(Q) < y'_{n+1} < -\frac{1}{4} \sqrt{\varepsilon} \ell(Q). \quad (6.22)$$

Let π be the orthogonal projection onto P . Let $Z \in P$ (i.e., $Z_{n+1} = 0$) be such that $|Z - \pi(y')| \leq K_0^{3/2} \ell(Q)$. Then $Z \in B(x_Q, 4K_0^{3/2} \ell(Q)) \subset B_Q^*$. Hence $Z \in P \cap B_Q^{**}(\varepsilon)$ and by (6.8), $\text{dist}(Z, E) \leq \varepsilon \ell(Q)$. Then there exists $x_Z \in E$ with $|Z - x_Z| \leq \varepsilon \ell(Q)$, which in turn implies that $|(x_Z)_{n+1}| \leq \varepsilon \ell(Q)$. Note that $x_Z \in B(x_Q, 5K_0^{3/2} \ell(Q)) \subset B_Q^*$ and by (6.15), $x_Z \in E \cap B_Q^{**}(\varepsilon) \cap B_{Q'}^{**}(\varepsilon)$. This, (6.10), and (6.14) imply that $x_Z \notin H \cup H'$. Hence, $(x_Z)_{n+1} \leq 0$ and $(x_Z - y') \cdot \nu' \leq 0$, since $y' \in P'$ and ν' denote the unit normal vector to P' pointing into H' . Using (6.22) we observe that

$$\frac{1}{8} \sqrt{\varepsilon} \ell(Q) < -\varepsilon \ell(Q) + \frac{1}{4} \sqrt{\varepsilon} \ell(Q) < (x_Z - y')_{n+1} < 2K_0^{3/2} \ell(Q), \quad (6.23)$$

and that

$$\begin{aligned} (x_Z - y')_{n+1} \nu'_{n+1} &\leq -\pi(x_Z - y') \cdot \pi(\nu') \\ &\leq |x_Z - z| - \pi(Z - y') \cdot \pi(\nu') \leq \varepsilon \ell(Q) - \pi(Z - y') \cdot \pi(\nu'). \end{aligned} \quad (6.24)$$

We prove that $\nu'_{n+1} < -\frac{1}{8} < 0$ by considering two cases.

Case 1: $|\pi(\nu')| \geq \frac{1}{2}$. We pick

$$Z_1 = \pi(y') + K_0^{3/2} \ell(Q) \frac{\pi(\nu')}{|\pi(\nu')|}.$$

By construction, $Z_1 \in P$ and $|Z_1 - \pi(y')| \leq K_0^{3/2} \ell(Q)$. Hence, we can use (6.24) with Z_1 :

$$\begin{aligned} (x_{Z_1} - y')_{n+1} \nu'_{n+1} &\leq \varepsilon \ell(Q) - \pi(Z_1 - y') \cdot \pi(\nu') \\ &= \varepsilon \ell(Q) - K_0^{3/2} \ell(Q) |\pi(\nu')| \leq -\frac{1}{4} K_0^{3/2} \ell(Q). \end{aligned}$$

This together with (6.23) give that $\nu'_{n+1} < -\frac{1}{8} < 0$.

Case 2: $|\pi(\nu')| < \frac{1}{2}$. This case is much simpler. Note first that $|\nu'_{n+1}|^2 = 1 - |\pi(\nu')|^2 > \frac{3}{4}$, and thus either $\nu'_{n+1} < -\frac{1}{2} \sqrt{3}$ or $\nu'_{n+1} > \frac{1}{2} \sqrt{3}$. We see that the second scenario leads to a contradiction. Assume then that $\nu'_{n+1} > \frac{1}{2} \sqrt{3}$. We take $Z_2 = \pi(y') \in P$, which clearly satisfies $|Z_2 - \pi(y')| \leq K_0^{3/2} \ell(Q)$. Again (6.24) and (6.23) are applicable with Z_2 :

$$\frac{1}{8} \sqrt{\varepsilon} \ell(Q) \frac{\sqrt{3}}{2} < (x_{Z_2} - y')_{n+1} \nu'_{n+1} \leq \varepsilon \ell(Q) \ll \sqrt{\varepsilon} \ell(Q),$$

and we get a contradiction. Hence necessarily $\nu'_{n+1} \leq -\frac{1}{2} \sqrt{3} < -\frac{1}{8} < 0$.

Having proved that $\nu'_{n+1} < -\frac{1}{8} < 0$, we estimate θ , the angle between ν' and $-e_{n+1}$. Note first $\cos \theta = -\nu'_{n+1} > \frac{1}{8}$. If $\cos \theta = 1$ (which occurs if $\nu' = -e_{n+1}$), then $\theta = \sin \theta = 0$ and the proof is complete. Assume then that $\cos \theta \neq 1$, in which case $\frac{1}{8} < -\nu'_{n+1} < 1$ and hence $|\pi(\nu')| \neq 0$. Pick

$$Z_3 = y' + \frac{\ell(Q)}{2\varepsilon} \hat{\nu}', \quad \hat{\nu}' = \frac{e_{n+1} - \nu'_{n+1} \nu'}{|\pi(\nu')|}.$$

Then $\hat{v}' \cdot v' = 0$ and hence $Z_3 \in P'$ as $y' \in P'$. Also, $|\hat{v}'| = 1$ and therefore $|Z_3 - y'| = \ell(Q)/(2\varepsilon)$. This in turn gives that $Z_3 \in \tilde{B}_Q(\varepsilon)$. We have obtained that $Z_3 \in P' \cap \tilde{B}_Q(\varepsilon)$, and hence $(Z_3)_{n+1} \leq 2\varepsilon^{7/4}\ell(Q)$ by (6.16). This and (6.23) applied to Z_3 easily give

$$\begin{aligned} 4K_0^{3/2}\ell(Q) &\geq 2\varepsilon^{7/4}\ell(Q) \geq (Z_3)_{n+1} = y'_{n+1} + \frac{\ell(Q)}{2\varepsilon} \frac{1 - (v'_{n+1})^2}{|\pi(v')|} \\ &= y'_{n+1} + \frac{\ell(Q)}{2\varepsilon} |\pi(v')| \geq -2K_0^{3/2}\ell(Q) + \frac{\ell(Q)}{2\varepsilon} |\pi(v')|. \end{aligned}$$

This readily yields $|\sin \theta| = |\pi(v')| \leq 8K_0^{3/2}\varepsilon$, and the proof is complete. □

Proof of Claim 6.18. We want to prove that every point in $10Q$ lies within $\sqrt{\varepsilon}\ell(Q)$ of a point in $P \cup P'$. We argue by contradiction and hence we assume that there exists $x' \in 10Q$ with $\text{dist}(x', P \cup P') > \sqrt{\varepsilon}\ell(Q)$. In particular, $x'_{n+1} < -\sqrt{\varepsilon}\ell(Q)$, and as observed above, we may repeat the previous argument to construct a cube Q'' , a hyperplane P'' , a unit vector v'' forming a small angle with $-e_{n+1}$, and a half-space H'' with boundary P'' , with the same properties as Q' , P' , v' , and H' , namely (6.19), (6.21), and (6.20). Also,

$$\sqrt{\varepsilon}\ell(Q) \leq \text{dist}(x', P') \leq \text{diam}(Q'') + \text{dist}(Q'', P') \leq \frac{1}{2}\sqrt{\varepsilon}\ell(Q) + \text{dist}(Q'', P'),$$

and, in addition, as in (6.15), we have $B_Q^* \subset B_{Q''}^{**}(\varepsilon)$.

By (6.19) there is $y'' \in Q''$ and $z'' \in P''$ such that $|y'' - z''| \ll \varepsilon^{1/2}\ell(Q)$. By (6.20) $y'' \notin H'$. Write π' to denote the orthogonal projection onto P' and note that (6.21) gives $\text{dist}(y'', P') = |y'' - \pi'(y'')| \geq \frac{1}{2}\sqrt{\varepsilon}\ell(Q)$. Note also that

$$\begin{aligned} |y'' - \pi'(y'')| &= \text{dist}(y'', P') \\ &\leq |y'' - x'| + |x' - x| + \text{diam}(Q') + \text{dist}(Q', P') \leq 11 \text{diam}(Q) \end{aligned}$$

and that

$$|\pi'(y'') - x_Q| \leq |\pi'(y'') - y''| + |y'' - x'| + |x' - x_Q| < 22 \text{diam}(Q) < K_0^2\ell(Q).$$

Hence $\pi'(y'') \in B_Q^* \subset \tilde{B}_Q(\varepsilon)$, and since $\pi'(y'') \in P'$, (6.12) gives $\tilde{y} \in E$ with $|\pi'(y'') - \tilde{y}| \leq 2\varepsilon^{7/4}\ell(Q)$. Then $\tilde{y} \in 23Q \subset B_Q^* \cap E$ and $|\tilde{y} - z''| < 12 \text{diam}(Q)$. To complete our proof we just need to show that $\tilde{y} \in H''$, which contradicts (6.20).

Write v'' to denote the unit normal vector to P'' pointing into H'' , and let us momentarily assume that

$$|v' - v''| \leq 16\sqrt{2}K_0^{2/3}\varepsilon. \tag{6.25}$$

Recalling that $y'' \notin H'$, we then obtain that

$$\begin{aligned} \frac{1}{2}\sqrt{\varepsilon}\ell(Q) &\leq |y'' - \pi'(y'')| = (\pi'(y'') - y'') \cdot v' \\ &\leq |\pi'(y'') - \tilde{y}| + |\tilde{y} - z''| |v' - v''| + (\tilde{y} - z'') \cdot v'' + |z'' - y''| \\ &< \frac{1}{4}\sqrt{\varepsilon}\ell(Q) + (\tilde{y} - z'') \cdot v''. \end{aligned}$$

This immediately gives that $(\tilde{y} - z'') \cdot v'' > \frac{1}{4} \sqrt{\varepsilon} \ell(Q) > 0$, and hence $\tilde{y} \in H''$ as desired. Thus, to complete the proof we have to prove (6.25). We first note that if $|\alpha| < \frac{\pi}{4}$, then

$$1 - \cos \alpha = 1 - \sqrt{1 - \sin^2 \alpha} \leq \sin^2 \alpha.$$

In particular, we can apply this to θ (resp. θ'), which is the angle between v' (resp. v'') and $-e_{n+1}$, and as we showed that $|\sin \theta|, |\sin \theta'| \leq 8K_0^{3/2} \varepsilon$, we see that

$$\sqrt{1 - \cos \theta} + \sqrt{1 - \cos \theta'} \leq 16K_0^{3/2} \varepsilon.$$

Using the trivial formula

$$|a - b|^2 = 2(1 - a \cdot b), \quad \forall a, b \in \mathbb{R}^{n+1}, \quad |a| = |b| = 1,$$

we conclude that

$$\begin{aligned} |v' - v''| &\leq |v' - (-e_{n+1})| + |(-e_{n+1}) - v''| \\ &= \sqrt{2(1 + v' \cdot e_{n+1})} + \sqrt{2(1 + v'' \cdot e_{n+1})} \\ &= \sqrt{2(1 - \cos \theta)} + \sqrt{2(1 - \cos \theta')} \leq 16\sqrt{2}K_0^{3/2} \varepsilon. \end{aligned}$$

This proves (6.25), and hence the proof of Claim 6.18 is complete. \square

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THE ONE-PHASE PROBLEM FOR HARMONIC MEASURE IN TWO-SIDED NTA DOMAINS

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We show that if $\Omega \subset \mathbb{R}^3$ is a two-sided NTA domain with AD-regular boundary such that the logarithm of the Poisson kernel belongs to $VMO(\sigma)$, where σ is the surface measure of Ω , then the outer unit normal to $\partial\Omega$ belongs to $VMO(\sigma)$ too. The analogous result fails for dimensions larger than 3. This answers a question posed by Kenig and Toro and also by Bortz and Hofmann.

1. Introduction

In this paper we study a one-phase free boundary problem in connection with the Poisson kernel. The study of this type of problems was initiated in the pioneering work [Alt and Caffarelli 1981], where they showed that for a Reifenberg flat domain with n -AD-regular boundary in \mathbb{R}^{n+1} , if the logarithm of the Poisson kernel is in C^α for some $\alpha > 0$, then the domain is of class $C^{1,\beta}$ for some $\beta > 0$. Later on, Jerison [1990] showed that, in fact, one can take $\beta = \alpha$. Kenig and Toro [1997; 1999; 2003] considered the endpoint case of the logarithm of the Poisson kernel being in VMO , and they obtained the following remarkable result:

Theorem A [Kenig and Toro 2003]. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a δ -Reifenberg flat chord-arc domain for some $\delta > 0$ small enough. Denote by σ the surface measure of Ω and by h the Poisson kernel with a pole in Ω if Ω is bounded or with the pole at infinity if Ω is unbounded. Then $\log h \in VMO(\sigma)$ if and only if the outer unit normal \vec{n} to $\partial\Omega$ is in $VMO(\sigma)$.*

A domain $\Omega \subset \mathbb{R}^{n+1}$ is called *chord-arc* if it is an NTA domain with n -AD-regular boundary. Its Poisson kernel with pole at $p \in \Omega$ equals $h = d\omega^p/d\sigma$, where ω^p stands for the harmonic measure of Ω with pole at p . For the definitions of Reifenberg flatness, NTA, and VMO , we refer the reader to Section 2.

We also remark that, in fact, Kenig and Toro [2003] proved a slightly weaker statement than the one in Theorem A. Indeed, instead of showing that when $\log h \in VMO(\sigma)$, the outer unit normal \vec{n} to $\partial\Omega$ is in $VMO(\sigma)$, they proved that \vec{n} belongs to $VMO_{\text{loc}}(\sigma)$ (which coincides with $VMO(\sigma)$ when Ω is bounded). However, as we explain in Remark 9.1, a minor modification of their arguments in [Kenig and Toro 2003] proves the full statement above in Theorem A.

Without the Reifenberg flatness assumption and just assuming the NTA condition, the conclusion of the theorem above need not hold: Kenig and Toro [1999, Proposition 3.1] showed that for the Kowalski–Preiss

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cone $\Omega = \{(x, y, z, w) : x^2 + y^2 + z^2 > w^2\} \subset \mathbb{R}^4$, the harmonic measure with pole at infinity coincides with the surface measure modulo a constant factor, and thus has $\log h \in \text{VMO}(\sigma)$, even though the outer unit normal is not in $\text{VMO}(\sigma)$. In fact, a similar conical example in \mathbb{R}^3 was shown previously by Alt and Caffarelli [1981, Section 2.7].

It was conjectured by Kenig and Toro [2006] and Bortz and Hofmann [2016] that, instead of the Reifenberg flatness assumption, being a two-sided chord-arc domain should be enough for the implication $\log h \in \text{VMO}(\sigma) \implies \vec{n} \in \text{VMO}(\sigma)$. By a two-sided chord-arc domain we mean a chord-arc domain such that its exterior is also connected and chord-arc. Kenig and Toro showed that this holds under the additional assumption that the logarithm of the Poisson kernel of the exterior domain is also in $\text{VMO}(\sigma)$. Their precise result reads as follows:

Theorem B [Kenig and Toro 2006, Corollary 5.2]. *Let Ω be a two-sided chord-arc domain in \mathbb{R}^{n+1} . Assume further that $\log(d\omega/d\sigma), \log(d\omega_{\text{ext}}/d\sigma) \in \text{VMO}_{\text{loc}}(\sigma)$. Then $\vec{n} \in \text{VMO}_{\text{loc}}(\sigma)$.*

Bortz and Hofmann [2016] showed that this same result holds under the assumption that $\partial\Omega$ is uniformly n -rectifiable, so that the measure theoretic boundary has full surface measure, instead of the two-sided chord-arc condition above. The boundary of any two-sided chord-arc domain is always uniformly n -rectifiable by results due to David and Jerison [1990], and thus this is more general than Theorem B. We also note that, by Proposition 4.10 in [Hofmann et al. 2010], such domains with $\vec{n} \in \text{VMO}_{\text{loc}}(\sigma)$ are also vanishing Reifenberg flat. It is also worth mentioning that the arguments in [Bortz and Hofmann 2016] are very different from the ones in [Kenig and Toro 2006]: while the latter uses blow-up techniques, the former relies on the relationship between the Riesz transform and harmonic measure and exploit the jump relations for the gradient of the single layer potential.

In this paper we resolve the conjecture mentioned above:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a two-sided chord-arc domain. Denote by σ the surface measure of Ω and by h the Poisson kernel with a pole in Ω if Ω is bounded or with the pole at infinity if Ω is unbounded. If $\log h \in \text{VMO}(\sigma)$, then the outer unit normal of Ω also belongs to $\text{VMO}(\sigma)$.*

On the other hand, for $d \geq 4$, there are two-sided chord-arc domains $\Omega \subset \mathbb{R}^d$ satisfying $h \equiv 1$ and such that the outer unit normal of Ω does not belong to $\text{VMO}(\sigma)$.

Most of the paper is devoted to proving the positive result stated in the theorem for \mathbb{R}^3 . Our arguments use the powerful blow-up techniques developed by Kenig and Toro [2003]. Indeed, by arguments analogous to the ones of Kenig and Toro, we reduce our problem to the study of the case when Ω_∞ is an unbounded two-sided chord-arc domain such that its Poisson kernel with pole at infinity is constantly equal to 1. By combining a monotonicity formula due to Weiss [1998] and some topological arguments inspired by a work from Caffarelli, Jerison and Kenig [Caffarelli et al. 2004], we then show that for such domains all blow-downs are flat. This is probably one of the main novelties in our paper. Then an application of a variant of a well-known theorem of Alt and Caffarelli [1981] shows that Ω_∞ must be a half-space.

The aforementioned reduction of the problem to the case when the Poisson kernel with pole at infinity is constantly equal to 1 requires estimating from above the gradient of the Green function. This estimate is obtained in [Kenig and Toro 2003] under the assumption that the domain Ω is Reifenberg flat, and this

is one of the main technical difficulties of that paper. In [Kenig and Toro 2006] it is shown how these estimates can be extended to the case when Ω is not Reifenberg flat. In our present paper we provide some alternative arguments to estimate the gradient of the Green function. The main difference with respect to the ones in [Kenig and Toro 2003; 2006] is that in the present paper we use the jump relations for the gradient of single layer potentials, instead of the (perhaps) less standard approach in the aforementioned works. We think that our approach has some independent interest (especially because of the connection between harmonic measure with pole at infinity and the Riesz transform that we describe in Section 3).

Concerning the negative result for dimensions $d \geq 4$ in Theorem 1.1, basically we recall in the last section of the paper an example of a conical domain in \mathbb{R}^4 by Guanghao Hong¹[2015] such that the harmonic measure with pole at infinity coincides with surface measure, and so that the outer unit normal does not belong to $VMO(\sigma)$. One can check easily that such domain is two-sided NTA. Probably, this example was unnoticed in some recent works in this area.

2. Preliminaries

For $a, b \geq 0$, we will write $a \lesssim b$ if there is $C > 0$ so that $a \leq Cb$ and $a \lesssim_t b$ if the constant C depends on the parameter t . We write $a \approx b$ to mean $a \lesssim b \lesssim a$ and define $a \approx_t b$ similarly.

Definition 2.1. Given a closed set E , $x \in \mathbb{R}^d$, $r > 0$, and P a d -plane, we set

$$\Theta_E(x, r, P) = r^{-1} \max \left\{ \sup_{y \in E \cap B(x, r)} \text{dist}(y, P), \sup_{y \in P \cap B(x, r)} \text{dist}(y, E) \right\}.$$

Also define

$$\Theta_E(x, r) = \inf_P \Theta_E(x, r, P),$$

where the infimum is over all d -planes P . A set E is δ -Reifenberg flat if $\Theta_E(x, r) < \delta$ for all $x \in E$ and $r > 0$, and is *vanishing Reifenberg flat* if

$$\limsup_{r \rightarrow 0} \sup_{x \in E} \Theta_E(x, r) = 0.$$

Definition 2.2. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and let $0 < \delta < \frac{1}{2}$. We say that Ω is a δ -Reifenberg flat domain if it satisfies the following conditions:

- (a) $\partial\Omega$ is δ -Reifenberg flat.
- (b) For every $x \in \partial\Omega$ and $r > 0$, denote by $\mathcal{P}(x, r)$ an n -plane that minimizes $\Theta_E(x, r)$. Then one of the connected components of

$$B(x, r) \cap \{x \in \mathbb{R}^{n+1} : \text{dist}(x, \mathcal{P}(x, r)) \geq 2\delta r\}$$

is contained in Ω and the other is contained in $\mathbb{R}^{n+1} \setminus \Omega$.

If, additionally, $\partial\Omega$ is vanishing Reifenberg flat, then Ω is said to be vanishing Reifenberg flat, too.

Definition 2.3. Let $\Omega \subset \mathbb{R}^{n+1}$. We say that Ω satisfies the *Harnack chain condition* if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geq 1$, and every pair of points $x, y \in \Omega$ with $\text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega) \geq \rho$ and $|x - y| < \Lambda\rho$, there is a chain of open balls $B_1, \dots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $x \in B_1$, $y \in B_N$,

¹ So the statement in the theorem referring to the case $d \geq 4$ should not be attributed to us.

$B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1} \text{diam}(B_k) \leq \text{dist}(B_k, \partial\Omega) \leq C \text{diam}(B_k)$. The chain of balls is called a *Harnack chain*. Note that if such a chain exists, then

$$u(x) \approx_N u(y).$$

For $C \geq 2$, the set Ω is a *C-corkscrew domain* if for all $\xi \in \partial\Omega$ and $r > 0$ there are two balls of radius r/C contained in $B(\xi, r) \cap \Omega$ and $B(\xi, r) \setminus \Omega$ respectively. If

$$B(x, r/C) \subseteq B(\xi, r) \cap \Omega,$$

we call x a *corkscrew point* for the ball $B(\xi, r)$. Finally, we say Ω is *C-nontangentially accessible* (or *C-NTA*, or just *NTA*) if it satisfies the Harnack chain condition and is a *C-corkscrew domain*. We say Ω is *two-sided C-NTA* if both Ω and $\Omega_{\text{ext}} := (\bar{\Omega})^c$ are *C-NTA*. Finally, it is *chord-arc* if, additionally, $\partial\Omega$ is *n-AD-regular*, meaning there is $C > 0$ so that, if σ denotes surface measure, then

$$C^{-1}r^n < \sigma(B(x, r)) < Cr^n \quad \text{for all } x \in \partial\Omega, 0 < r \leq \text{diam}(\Omega).$$

Any measure σ that satisfies the preceding estimate for all $x \in \text{supp } \sigma$ and $0 < r \leq \text{diam}(\text{supp } \sigma)$ is called *n-AD-regular*.

Definition 2.4. Let σ be an *n-AD-regular* measure in \mathbb{R}^n and f a locally integrable function with respect to σ . We say $f \in \text{VMO}(\sigma)$ if

$$\lim_{r \rightarrow 0} \sup_{x \in \text{supp } \sigma} \int_{B(x, r)} \left| f - \int_{B(x, r)} f d\sigma \right|^2 d\sigma = 0. \quad (2-1)$$

We say $f \in \text{VMO}_{\text{loc}}(\sigma)$ if, for any compact set K ,

$$\lim_{r \rightarrow 0} \sup_{x \in \text{supp } \sigma \cap K} \int_{B(x, r)} \left| f - \int_{B(x, r)} f d\sigma \right|^2 d\sigma = 0.$$

It is well known that the space *VMO* coincides with the closure of the set of bounded uniformly continuous functions on $\text{supp } \sigma$ in the *BMO* norm.

We also remark that one can find slightly different definitions of *VMO* in the literature. For example, in some references besides (2-1) the additional condition that

$$\lim_{r \rightarrow \infty} \sup_{x \in \text{supp } \sigma} \int_{B(x, r)} \left| f - \int_{B(x, r)} f d\sigma \right|^2 d\sigma = 0$$

is required. In this case, it turns out that *VMO* coincides with the closure of the set of *compactly supported* continuous functions on $\text{supp } \sigma$ in the *BMO* norm. However, the definition we will use in our paper is Definition 2.4 (as in other works like [Kenig and Toro 1999; 2003]).

3. The Riesz transform of the harmonic measure with pole at infinity

Readers that are familiar with the results in [Kenig and Toro 2003; 2006] may skip this section, as well as Sections 4 and 5, and go directly to Section 6 without much harm. In fact, in Sections 3–5 we provide the alternative arguments to estimate the gradient of the Green function that we mentioned in the Introduction. Our approach uses the jump relations for the gradient of the single layer potential (derived by Hofmann, Mitrea, and Taylor [Hofmann et al. 2010] in the case of chord-arc domains and somewhat more general

settings). Modulo these standard relations, our arguments are reasonably self-contained and shorter than the ones in [Kenig and Toro 2003; 2006].

Recall from [Kenig and Toro 1999, Lemma 3.7] that if $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded NTA domain, then there exist a function $u \in C(\bar{\Omega})$ and a measure ω in $\partial\Omega$ such that

$$\Delta u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega, \quad u > 0 \quad \text{in } \Omega, \tag{3-1}$$

and

$$\int_{\Omega} u \Delta \phi \, dm = \int_{\partial\Omega} \phi \, d\omega \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^{n+1}). \tag{3-2}$$

The function u and the measure ω are unique modulo constant factors, and u is the so-called Green function with pole at infinity and ω the harmonic measure with pole at infinity.

From now on, we will assume that u is also defined in $\mathbb{R}^{n+1} \setminus \Omega$ and vanishes identically here, so that $u \in C(\mathbb{R}^{n+1})$.

Given a Radon measure μ in \mathbb{R}^{n+1} , its n -dimensional Riesz transform is defined by

$$\mathcal{R}\mu(x) = c_n \int \frac{x - y}{|x - y|^{n+1}} \, d\mu(y),$$

whenever the integral makes sense. We assume that the constant c_n is chosen so that

$$K(x) := c_n \frac{x}{|x|^{n+1}}$$

coincides with the gradient of the fundamental solution of the Laplacian.

The main result of this section is the following.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded NTA domain, and let u and ω be the Green function and the associated harmonic measure with pole at infinity, respectively. Suppose that for all $x \in \partial\Omega$ there exist some constants $0 < \delta < 1$ and $C > 0$ (both possibly depending on x) such that*

$$\omega(B(x, r)) \leq C r^{n+\delta} \quad \text{for all } r \geq 1. \tag{3-3}$$

Then we have

$$\mathcal{R}\omega(x) - \mathcal{R}\omega(y) = \nabla u(y) - \nabla u(x) \quad \text{for all } x, y \in \mathbb{R}^{n+1} \setminus \partial\Omega. \tag{3-4}$$

Some remarks are in order. First, it is easy to check that if the condition (3-3) holds for all $x \in \partial\Omega$, then it also holds for all $x \in \mathbb{R}^{n+1}$ (with some constants C, δ depending also on x). For the identity (3-4) to be true, it is important to define the Riesz transform so that its kernel is the gradient of the fundamental solution of the Laplacian, as we did above. On the other hand, the function $\mathcal{R}\omega$ is defined modulo a constant term (i.e., in a BMO sense). So for all $x, y \in \mathbb{R}^{n+1} \setminus \partial\Omega$, by definition we write

$$\mathcal{R}\omega(x) - \mathcal{R}\omega(y) = \int (K(x - z) - K(y - z)) \, d\omega(z).$$

Then it turns out that the integral on the right-hand side above is absolutely convergent. Indeed, defining $d = \max(2|x - y|, 1)$, we have

$$\begin{aligned} \int_{|x-z|\geq d} |K(x-z) - K(y-z)| d\omega(z) &\lesssim \int_{|x-z|\geq d} \frac{|x-y|}{|x-z|^{n+1}} d\omega(z) \\ &\lesssim \sum_{k\geq 0} \frac{|x-y|}{(2^k d)^{n+1}} \omega(B(x, 2^k d)) \lesssim_x \sum_{k\geq 0} \frac{|x-y|}{(2^k d)^{n+1}} (2^k d)^{k(n+\delta)} < \infty, \end{aligned} \tag{3-5}$$

which implies

$$\int |K(x - z) - K(y - z)| d\omega(z) < \infty,$$

since $x, y \notin \text{supp } \omega = \partial\Omega$.

Before proceeding with the proof of Proposition 3.1, we recall a few lemmas about NTA domains. These lemmas were originally shown in [Jerison and Kenig 1982] for bounded NTA domains, but as the arguments for these results are purely local, they also hold for unbounded NTA domains.

Lemma 3.2 [Jerison and Kenig 1982, Lemma 4.4]. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be NTA and B a ball centered on $\partial\Omega$ with $0 < r(B) < \text{diam } \partial\Omega$. Let x_B be a corkscrew point for B in Ω and let g be the Green function for Ω . Then*

$$\omega^z(B) \approx g(x_B, z)r^{1-n} \quad \text{for all } z \in \Omega \setminus 2B. \tag{3-6}$$

Lemma 3.3 [Jerison and Kenig 1982, Lemma 4.10]. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an NTA domain and B a ball centered on $\partial\Omega$ with $0 < Mr(B) < \text{diam } \partial\Omega$, where M depends on the NTA character of Ω . Suppose u, v are two positive harmonic functions in Ω vanishing continuously on $MB \cap \partial\Omega$ and let x_B be a corkscrew point for B in Ω . Then*

$$\frac{u(z)}{v(z)} \approx \frac{u(x_B)}{v(x_B)} \quad \text{for all } z \in B \cap \Omega. \tag{3-7}$$

Proof of Proposition 3.1. As shown in [Kenig and Toro 1999, Section 3], the Green function u and the harmonic measure ω with pole at infinity can be constructed as follows. Given a fixed point $a \in \Omega$ and a sequence of points $p_j \in \Omega$ such that $p_j \rightarrow \infty$, we consider the function

$$u_j(x) = \begin{cases} g(x, p_j)/g(a, p_j) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \bar{\Omega}, \end{cases}$$

and the measure

$$\omega_j = \frac{1}{g(a, p_j)} \omega^{p_j}.$$

Passing to a subsequence and relabeling if necessary, we may assume that u_j is locally uniformly convergent and that ω_j is weakly convergent. Then it turns out that u is the weak limit of the sequence u_j and ω is the weak limit of ω_j . For simplicity, we choose points p_j such that $|p_j - a| \approx \text{dist}(p_j, \partial\Omega) \rightarrow \infty$. Observe that by (3-6) and our definitions of u and ω , it follows that for all balls B centered on $\partial\Omega$, if x_B is a corkscrew point for B in Ω , then

$$\omega(B)r^{1-n} \approx u(x_B). \tag{3-8}$$

It is well known that the Green function $g(\cdot, \cdot)$ equals

$$g(x, p) = \mathcal{E}(x - p) - \int \mathcal{E}(x - z) d\omega^p(z) \quad \text{for } x, p \in \Omega,$$

where \mathcal{E} stands for the fundamental solution of the Laplacian. On the other hand, the right-hand side above vanishes if $x \in \mathbb{R}^{n+1} \setminus \bar{\Omega}$, $p \in \Omega$. So we deduce that for all $x \notin \partial\Omega$,

$$\nabla u_j(x) = \frac{1}{g(a, p_j)} K(x - p_j) - \mathcal{R}\omega_j(x).$$

Thus, for all $x, y \notin \partial\Omega$,

$$\nabla u_j(y) - \nabla u_j(x) = \frac{1}{g(a, p_j)} (K(y - p_j) - K(x - p_j)) + \mathcal{R}\omega_j(x) - \mathcal{R}\omega_j(y).$$

Since u_j is harmonic outside of $\partial\Omega$ and u_j converges locally uniformly to u , it turns out that ∇u_j converges also locally uniformly to ∇u outside of $\partial\Omega$. Hence, to prove the proposition, it suffices to show that

$$\lim_{j \rightarrow \infty} \frac{1}{g(a, p_j)} (K(y - p_j) - K(x - p_j)) = 0, \tag{3-9}$$

$$\lim_{j \rightarrow \infty} (\mathcal{R}\omega_j(x) - \mathcal{R}\omega_j(y)) = \mathcal{R}\omega(x) - \mathcal{R}\omega(y). \tag{3-10}$$

To prove the above identities, first we will estimate $g(a, p_j)$ in terms of u and ω . To this end, we will apply the boundary Harnack principle.

Let $\xi_j \in \partial\Omega$ be such that $|\xi_j - p_j| = \text{dist}(p_j, \partial\Omega)$, and consider the ball $B(p_j) = B(\xi_j, |\xi_j - p_j|)$. Suppose that $|\xi_j - p_j| \gg \text{dist}(a, \partial\Omega)$. Consider a corkscrew point $\tilde{p}_j \in \frac{1}{2}B(p_j) \cap \Omega$, so that $\text{dist}(\tilde{p}_j, \partial\Omega) \approx r(B(p_j))$. Since u and $g(\cdot, p_j)$ are harmonic in $\Omega \cap B(p_j)$ and vanish identically in $\partial\Omega$, we deduce from (3-7) that

$$\frac{g(\tilde{p}_j, p_j)}{g(a, p_j)} \approx \frac{u(\tilde{p}_j)}{u(a)},$$

since a belongs to $CB(p_j)$ for some fixed constant C , and $\text{dist}(a, \partial\Omega) \ll r(B(p_j))$ by assumption. Taking into account that by (3-8)

$$u(\tilde{p}_j) \approx u(p_j) \approx \omega(B(p_j)) |p_j - \xi_j|^{1-n} \approx \omega(B(p_j)) |p_j - a|^{1-n}$$

and

$$g(\tilde{p}_j, p_j) \approx \frac{1}{|\tilde{p}_j - p_j|^{n-1}} \approx \frac{1}{|p_j - a|^{n-1}},$$

we infer that

$$g(a, p_j) \approx \frac{u(a)}{\omega(B(p_j))}. \tag{3-11}$$

With (3-11) at hand, we are ready to prove (3-9):

$$\frac{1}{g(a, p_j)} |K(y - p_j) - K(x - p_j)| \lesssim \frac{\omega(B(p_j))}{u(a)} \frac{|x - y|}{|x - p_j|^{n+1}}.$$

For j big enough, we have $r(B(p_j)) \approx |x - p_j|$, and then we derive

$$\frac{\omega(B(p_j))}{|x - p_j|^{n+1}} \lesssim_x \frac{|x - p_j|^{n+\delta}}{|x - p_j|^{n+1}} = \frac{1}{|x - p_j|^{1-\delta}},$$

and thus

$$\frac{1}{g(a, p_j)} |K(y - p_j) - K(x - p_j)| \lesssim_x \frac{|x - y|}{u(a)} \frac{1}{|x - p_j|^{1-\delta}} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

We turn our attention to the identity (3-10) now. Take an auxiliary radial C^∞ function $\phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ such that $\chi_{B(0,1)} \leq \phi \leq \chi_{B(0,2)}$ and define $\phi_\varepsilon(z) = \phi(z/\varepsilon)$. For $\varepsilon > 0$, we define

$$K_\varepsilon = (1 - \phi_\varepsilon) K \quad \text{and} \quad \tilde{K}_\varepsilon = \phi_\varepsilon K.$$

Notice that K_ε and \tilde{K}_ε are standard Calderón–Zygmund kernels. We denote by \mathcal{R}_ε and $\tilde{\mathcal{R}}_\varepsilon$ the respective associated operators, so that, at least formally, $\tilde{\mathcal{R}}_\varepsilon$ tends to \mathcal{R} as $\varepsilon \rightarrow \infty$. Then we write

$$\begin{aligned} & |(\mathcal{R}\omega_j(x) - \mathcal{R}\omega_j(y)) - (\mathcal{R}\omega(x) - \mathcal{R}\omega(y))| \\ & \leq |(\tilde{\mathcal{R}}_\varepsilon\omega_j(x) - \tilde{\mathcal{R}}_\varepsilon\omega_j(y)) - (\tilde{\mathcal{R}}_\varepsilon\omega(x) - \tilde{\mathcal{R}}_\varepsilon\omega(y))| + |\mathcal{R}_\varepsilon\omega_j(x) - \mathcal{R}_\varepsilon\omega_j(y)| + |\mathcal{R}_\varepsilon\omega(x) - \mathcal{R}_\varepsilon\omega(y)|. \end{aligned} \quad (3-12)$$

Since the function $\tilde{K}_\varepsilon(x - \cdot) - \tilde{K}_\varepsilon(y - \cdot)$ is continuous on $\partial\Omega$ (recall that $x, y \in \mathbb{R}^{n+1} \setminus \partial\Omega$) and has compact support, we infer that

$$|(\tilde{\mathcal{R}}_\varepsilon\omega_j(x) - \tilde{\mathcal{R}}_\varepsilon\omega_j(y)) - (\tilde{\mathcal{R}}_\varepsilon\omega(x) - \tilde{\mathcal{R}}_\varepsilon\omega(y))| \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad (3-13)$$

by the weak convergence of ω_j to ω .

Concerning the second term on the right-hand side of (3-12), we will show below that

$$|\mathcal{R}_\varepsilon\omega_j(x) - \mathcal{R}_\varepsilon\omega_j(y)| \lesssim_x \frac{|x - y|}{u(a)} \left(\frac{1}{\varepsilon^{1-\delta}} + \frac{1}{|x - p_j|^{1-\delta}} \right). \quad (3-14)$$

The last term in (3-12) is estimated as in (3-5). Indeed, for $\varepsilon \gg |x - y|$,

$$\begin{aligned} |\mathcal{R}_\varepsilon\omega(x) - \mathcal{R}_\varepsilon\omega(y)| & \lesssim \int |K_\varepsilon(x - z) - K_\varepsilon(y - z)| d\omega(z) \lesssim \int_{|x-z| \geq \varepsilon/2} \frac{|x - y|}{|x - z|^{n+1}} d\omega(z) \\ & \lesssim \sum_{k \geq 0} \frac{|x - y|}{(2^k \varepsilon)^{n+1}} \omega(B(x, 2^k \varepsilon)) \lesssim_x \sum_{k \geq 0} \frac{|x - y|}{(2^k \varepsilon)^{n+1}} (2^k \varepsilon)^{n+\delta} \approx \frac{|x - y|}{\varepsilon^{1-\delta}}. \end{aligned} \quad (3-15)$$

From (3-12), (3-13), (3-14) and (3-15) we deduce that

$$\limsup_{j \rightarrow \infty} |(\mathcal{R}\omega_j(x) - \mathcal{R}\omega_j(y)) - (\mathcal{R}\omega(x) - \mathcal{R}\omega(y))| \lesssim_x \frac{|x - y|}{u(a) \varepsilon^{1-\delta}} + \frac{|x - y|}{\varepsilon^{1-\delta}}.$$

Since this holds for any arbitrarily big $\varepsilon > 0$, the limit vanishes and this concludes the proof of (3-4).

Finally we deal with the estimate (3-14). Arguing as in (3-15), with ω replaced by ω_j , we obtain

$$|\mathcal{R}_\varepsilon\omega_j(x) - \mathcal{R}_\varepsilon\omega_j(y)| \lesssim \sum_{k \geq 0} \frac{|x - y|}{(2^k \varepsilon)^{n+1}} \omega_j(B(x, 2^k \varepsilon)).$$

We split the last sum according to whether $2^k \varepsilon \leq |p_j - x|$ or not, so that

$$|\mathcal{R}_\varepsilon\omega_j(x) - \mathcal{R}_\varepsilon\omega_j(y)| \leq S_1 + S_2,$$

where

$$S_1 = \sum_{\substack{k \geq 0 \\ 2^k \varepsilon \leq |p_j - x|}} \frac{|x - y|}{(2^k \varepsilon)^{n+1}} \omega_j(B(x, 2^k \varepsilon)) \quad \text{and} \quad S_2 = \sum_{\substack{k \geq 0 \\ 2^k \varepsilon > |p_j - x|}} \frac{|x - y|}{(2^k \varepsilon)^{n+1}} \omega_j(B(x, 2^k \varepsilon)).$$

To estimate S_1 we use the fact that, for $2^k \varepsilon \leq |p_j - x|$,

$$\omega_j(B(x, 2^k \varepsilon)) = \frac{1}{g(a, p_j)} \omega^{p_j}(B(x, 2^k \varepsilon)) \approx \frac{1}{g(a, p_j)} \frac{\omega(B(x, 2^k \varepsilon))}{\omega(B(p_j))}.$$

Hence, by (3-11),

$$\omega_j(B(x, 2^k \varepsilon)) \approx \frac{\omega(B(x, 2^k \varepsilon))}{u(a)},$$

and so

$$S_1 \lesssim \sum_{k \geq 0} \frac{|x - y|}{(2^k \varepsilon)^{n+1}} \frac{\omega(B(x, 2^k \varepsilon))}{u(a)} \lesssim_x \sum_{k \geq 0} \frac{|x - y|}{(2^k \varepsilon)^{n+1}} \frac{(2^k \varepsilon)^{n+\delta}}{u(a)} \lesssim \frac{|x - y|}{u(a) \varepsilon^{1-\delta}}.$$

To estimate S_2 we use the trivial estimate

$$\omega_j(B(x, 2^k \varepsilon)) = \frac{1}{g(a, p_j)} \omega^{p_j}(B(x, 2^k \varepsilon)) \leq \frac{1}{g(a, p_j)} \approx \frac{\omega(B(p_j))}{u(a)}.$$

Therefore,

$$S_2 \approx \sum_{\substack{k \geq 0 \\ 2^k \varepsilon > |p_j - x|}} \frac{|x - y|}{(2^k \varepsilon)^{n+1}} \frac{\omega(B(p_j))}{u(a)} \lesssim \frac{|x - y|}{|p_j - x|^{n+1}} \frac{\omega(B(p_j))}{u(a)}.$$

Assuming that $|p_j - x| \geq 1$, we have

$$\omega(B(p_j)) \lesssim_x r(B(p_j))^{n+\delta} \approx |p_j - x|^{n+\delta},$$

and thus

$$S_2 \lesssim_x \frac{|x - y|}{|p_j - x|^{1-\delta}} \frac{\omega(B(p_j))}{u(a)}.$$

From this estimate and the one for S_1 , we obtain (3-14), as wished. □

We recall now the following version of the jump equations for the gradient of the single layer potential due to Hofmann, Mitrea and Taylor [Hofmann et al. 2010]:

Proposition 3.4 [Hofmann et al. 2010, Proposition 3.30]. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain in \mathbb{R}^{n+1} with uniformly rectifiable boundary such that $\sigma(\partial\Omega \setminus \partial_*\Omega) = 0$, where $\partial_*\Omega$ stands for the measure theoretic boundary and σ for the surface measure of Ω . Let $f \in L^p(\sigma|_{\partial\Omega})$ for $1 \leq p < \infty$. Then, for σ -a.e. $x \in \partial\Omega$,*

$$\lim_{\Gamma^-(x) \ni z \rightarrow x} \mathcal{R}(f\sigma)(z) = -\frac{1}{2} \vec{n}(x) f(x) + \text{pv} \mathcal{R}(f\sigma)(x), \tag{3-16}$$

and

$$\lim_{\Gamma^+(x) \ni z \rightarrow x} \mathcal{R}(f\sigma)(z) = \frac{1}{2} \vec{n}(x) f(x) + \text{pv} \mathcal{R}(f\sigma)(x), \tag{3-17}$$

where $\Gamma^+(x)$ is a nontangential cone at x relative to Ω , (that is,

$$\Gamma^+(x) = \{y \in \Omega : \text{dist}(y, \Omega^c) > t|y - x|\}$$

for some $t > 0$), $\Gamma^-(x)$ is a nontangential cone at x relative to $\mathbb{R}^{n+1} \setminus \bar{\Omega}$, and $\vec{n}(x)$ is the outer normal to Ω at x .

In particular, if Ω is a chord-arc domain in \mathbb{R}^{n+1} , then $\partial\Omega$ is uniformly rectifiable (see [David and Jerison 1990]) and $\sigma(\partial\Omega \setminus \partial_*\Omega) = 0$; thus the preceding proposition can be applied.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain in \mathbb{R}^{n+1} . Let ω and u be the harmonic measure and the Green function with a pole either at infinity or at some point $p \in \Omega$. Suppose that for each $x \in \partial\Omega$ there exist some constants $0 < \delta < 1$ and $C > 0$ such that*

$$\omega(B(x, r)) \leq C r^{n+\delta} \quad \text{for all } r \geq 1. \quad (3-18)$$

Suppose $h := d\omega/d\sigma \in L^p_{\text{loc}}(\sigma)$ for some $p \geq 1$. Then $\lim_{\Gamma^+(x) \ni z \rightarrow x} \nabla u(z)$ exists for σ -a.e. $x \in \partial\Omega$ and

$$\lim_{\Gamma^+(x) \ni z \rightarrow x} \nabla u(z) = -h(x) \vec{n}(x). \quad (3-19)$$

Proof. Assume that the pole for ω and u is at infinity (the arguments for the case when the pole is finite are analogous). Let B be a ball centered at $\partial\Omega$. By Proposition 3.4, for σ -a.e. $x \in B$,

$$\lim_{\Gamma^-(x) \ni z \rightarrow x} \mathcal{R}(\chi_{2B}\omega)(z) = -\frac{1}{2} \vec{n}(x) h(x) + \text{pv} \mathcal{R}(\chi_{2B}\omega),$$

and

$$\lim_{\Gamma^+(x) \ni z \rightarrow x} \mathcal{R}(\chi_{2B}\omega)(z) = \frac{1}{2} \vec{n}(x) h(x) + \text{pv} \mathcal{R}(\chi_{2B}\omega)$$

In particular,

$$\lim_{\Gamma^+(x) \ni z \rightarrow x} \mathcal{R}(\chi_{2B}\omega)(z) - \lim_{\Gamma^-(x) \ni z \rightarrow x} \mathcal{R}(\chi_{2B}\omega)(z) = \vec{n}(x) h(x).$$

Using the condition (3-18), by estimates analogous to the ones in (3-5), it is immediate to check that

$$\lim_{\Gamma^+(x) \ni z \rightarrow x} \mathcal{R}(\chi_{2B}\omega)(z) - \lim_{\Gamma^-(x) \ni z \rightarrow x} \mathcal{R}(\chi_{2B}\omega)(z) = \lim_{\Gamma^+(x) \ni z \rightarrow x} \mathcal{R}\omega(z) - \lim_{\Gamma^-(x) \ni z \rightarrow x} \mathcal{R}\omega(z).$$

Then, by Proposition 3.1 we infer that

$$\lim_{\Gamma^-(x) \ni z \rightarrow x} \nabla u(z) - \lim_{\Gamma^+(x) \ni z \rightarrow x} \nabla u(z) = \vec{n}(x) h(x).$$

Since $u \equiv 0$ in $\mathbb{R}^{n+1} \setminus \bar{\Omega}$, we have $\lim_{\Gamma^-(x) \ni z \rightarrow x} \nabla u(z) = 0$ and so

$$-\lim_{\Gamma^+(x) \ni z \rightarrow x} \nabla u(z) = \vec{n}(x) h(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega \cap B. \quad \square$$

4. Some technical lemmas

From now on, given a domain $\Omega \subset \mathbb{R}^{n+1}$ and $x \in \mathbb{R}^{n+1}$, we define

$$d_\Omega(x) = \text{dist}(x, \Omega^c).$$

The following is a well-known result. See, for example, [Jerison and Kenig 1982, Section 4].

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain and let B be a ball centered at $\partial\Omega$. There exist some constants $C, \alpha > 0$ depending on the NTA character of Ω such that the following holds. If u is a nonnegative harmonic function on $\Omega \cap 2B$ which vanishes continuously on $\partial\Omega \cap 2B$, then*

$$u(x) \leq C \left(\frac{d_\Omega(x)}{r(B)} \right)^\alpha \sup_{y \in \partial(2B) \cap \Omega} u(y) \quad \text{for all } x \in B \cap \Omega.$$

If x_B is a corkscrew point for B , then

$$\sup_{y \in B \cap \Omega} u(y) \leq Cu(x_B).$$

We will also need the next auxiliary result.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain. There exist some constants $C, \alpha > 0$ depending on the NTA character of Ω such that the Green function of Ω satisfies*

$$g(x, y) \leq C \frac{1}{|x - y|^{n-1}} \left(\frac{\min(d_\Omega(x), d_\Omega(y))}{|x - y|} \right)^\alpha \quad \text{for all } x, y \in \Omega. \tag{4-1}$$

Proof. It is enough to show that, for some $C, \alpha' > 0$,

$$g(x, y) \leq C \frac{1}{|x - y|^{n-1}} \left(\frac{d_\Omega(x)}{|x - y|} \right)^\alpha \quad \text{for all } x, y \in \Omega, \tag{4-2}$$

because then the analogous inequality interchanging x by y also holds, by symmetry.

Because of the trivial estimate $g(x, y) \lesssim 1/|x - y|^{n-1}$, to prove (4-2) we may assume $|x - y| > 10 d_\Omega(x)$. Let $\xi_x \in \partial\Omega$ be such that $|\xi_x - x| = d_\Omega(x)$ and consider the ball $B := B(\xi_x, |x - y|/8)$. Clearly $x \in B$, as

$$|x - \xi_x| = d_\Omega(x) \leq \frac{1}{10}|x - y| = \frac{8}{10}r(B).$$

Note also that, for all $z \in \partial(2B)$,

$$|y - z| \geq |x - y| - |x - z| \geq 8r(B) - 4r(B) = 4r(B) \approx |x - y|.$$

Hence $g(z, y) \lesssim 1/|y - z|^{n-1} \lesssim 1/|x - y|^{n-1}$ for all $z \in \partial(2B)$. Thus, (4-2) follows from Lemma 4.1 applied to the function $g(\cdot, y)$. □

The following rather standard result is shown in [Kenig and Toro 2003, Theorem 2.1].

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain, $f \in \text{VMO}(\sigma)$, and $h = e^f$. Then, for all $x \in \partial\Omega$, $0 < r \leq \text{diam}(\Omega)$ and $1 < p < \infty$,*

$$\left(\int_{B(x,r)} h^p d\sigma \right)^{1/p} \leq C_p \int_{B(x,r)} h d\sigma \quad \text{and} \quad \left(\int_{B(x,r)} h^{-p} d\sigma \right)^{1/p} \leq C_p \int_{B(x,r)} h^{-1} d\sigma.$$

The next lemma is proven in [Jerison and Kenig 1982, Lemma 4.11]:

Lemma 4.4. *Let Ω be an NTA domain, B a ball centered on $\partial\Omega$ with $0 < r(B) < \text{diam} \partial\Omega$, and let $E \subseteq B \cap \partial\Omega$ be Borel. If x_B is a corkscrew point for B in Ω , then*

$$\frac{\omega^z(E)}{\omega^z(B)} \approx \omega^{x_B}(E) \quad \text{for } z \in \Omega \setminus 2B. \tag{4-3}$$

Note that this implies that if ω is the harmonic measure with pole at infinity, we also have

$$\frac{\omega(E)}{\omega(B)} \approx \omega^{x_B}(E). \tag{4-4}$$

The next corollary is an easy consequence of the preceding lemma, as shown in [Kenig and Toro 2003, Corollary 2.4].

Corollary 4.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. If the harmonic measure ω in Ω is such that $d\omega/d\sigma \in \text{VMO}(\sigma)$, then, for all $\varepsilon > 0$, $x \in \partial\Omega$, $0 < r \leq \text{diam}(\Omega)$ and $E \subset B(x, r) \cap \partial\Omega$,*

$$C(\varepsilon)^{-1} \left(\frac{\sigma(E)}{\sigma(B(x, r))} \right)^{1+\varepsilon} \leq \frac{\omega(E)}{\omega(B(x, r))} \leq C(\varepsilon) \left(\frac{\sigma(E)}{\sigma(B(x, r))} \right)^{1-\varepsilon}.$$

Let us remark that the pole of harmonic measure above can be either a point $p \in \Omega$ (in which case the constants also depend on p) or infinity in the case Ω is unbounded.

Another easy consequence of Lemma 4.3 is the following.

Corollary 4.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. Suppose that the harmonic measure ω in Ω with pole at infinity is such that $\log(d\omega/d\sigma) \in \text{VMO}(\sigma)$. For $z \in \Omega$, let $K_z = d\omega^z/d\sigma$ (i.e., K_z is the Poisson kernel). For $1 < p < \infty$, if $x \in \partial\Omega$, $0 < r \leq \text{diam}(\Omega)$, and $z \in \Omega \setminus B(x, 2r)$, then*

$$\left(\int_{B(x,r)} (K_z)^p d\sigma \right)^{1/p} \leq C(p) \int_{B(x,r)} K_z d\sigma.$$

For this corollary to hold we assume either the pole of ω is at ∞ if Ω is unbounded, or it is in Ω .

Proof. Since $z \in \Omega \setminus B(x, 2r)$, if z_0 is a corkscrew point for $B(x, r)$, then whenever $B(y, s) \subset B(x, r)$ and all $0 < s < r/10$, by (4-3) and (4-4),

$$\frac{\omega(B(y, s))}{\omega(B(x, r))} \approx \omega^{z_0}(B(y, s)) \approx \frac{\omega^z(B(y, s))}{\omega^z(B(x, r))}.$$

Hence, by the Lebesgue differentiation theorem, if we define $h = d\omega/d\sigma$ for σ -a.e. $y \in B(x, r) \cap \partial\Omega$,

$$K_z(y) = \frac{d\omega^z}{d\sigma}(y) = \lim_{s \rightarrow 0} \frac{\omega^z(B(y, s))}{\sigma(B(y, s))} \approx \frac{\omega^z(B(x, r))}{\omega(B(x, r))} \lim_{s \rightarrow 0} \frac{\omega(B(y, s))}{\sigma(B(y, s))} = \frac{\omega^z(B(x, r))}{\omega(B(x, r))} h(y).$$

Therefore, by Lemma 4.3, since $\log h \in \text{VMO}(\sigma)$,

$$\begin{aligned} \left(\int_{B(x,r)} K_z(y)^p d\sigma(y) \right)^{1/p} &\approx \frac{\omega^z(B(x,r))}{\omega(B(x,r))} \left(\int_{B(x,r)} h(y)^p d\sigma(y) \right)^{1/p} \\ &\lesssim \frac{\omega^z(B(x,r))}{\omega(B(x,r))} \int_{B(x,r)} h(y) d\sigma(y) \approx \int_{B(x,r)} K_z(y) d\sigma(y). \quad \square \end{aligned}$$

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. Suppose that the harmonic measure ω in Ω with pole either at infinity or at some fixed point $p \in \Omega$ is such that $\log(d\omega/d\sigma) \in \text{VMO}(\sigma)$. Denote by u the associated Green function. Then, for σ -a.e. $x \in \partial\Omega$, we have $\nabla u(z)$ converges to $-\vec{n}(x)(d\omega/d\sigma)(x)$ as $\Omega \ni z \rightarrow x$ nontangentially, where \vec{n} is the outer unit normal of Ω .*

This lemma is proved in [Kenig and Toro 2003] under the additional assumption that Ω is Reifenberg flat. In [Kenig and Toro 2006] it is shown how to prove this without the Reifenberg flatness assumption. The delicate arguments involved in [Kenig and Toro 2003; 2006] do not use the connection between harmonic measure and the Riesz transform and instead are of a more geometric nature.

Proof. This is an immediate consequence of Proposition 3.5 and Corollary 4.5. Indeed, this corollary, implies that for all $x \in \partial\Omega$ and all $0 < r_0 \leq r \leq \text{diam}(\Omega)$,

$$\left(\frac{\sigma(B(x, r_0))}{\sigma(B(x, r))} \right)^{1+\varepsilon} \leq C(\varepsilon) \frac{\omega(B(x, r_0))}{\omega(B(x, r))}.$$

Hence, using also the AD-regularity of σ we get

$$\omega(B(x, r)) \leq C(\varepsilon) \omega(B(x, r_0)) \left(\frac{\sigma(B(x, r))}{\sigma(B(x, r_0))} \right)^{1+\varepsilon} \approx \frac{\omega(B(x, r_0))}{\sigma(B(x, r_0))^{1+\varepsilon}} r^{n(1+\varepsilon)}.$$

Therefore, choosing $\varepsilon = 1/(2n)$,

$$\omega(B(x, r)) \leq C(x) r^{n+1/2} \quad \text{for } r \geq r_0.$$

So the assumption (3-18) in Proposition 3.5 holds and thus

$$\lim_{\Gamma^+(x) \ni z \rightarrow x} \nabla u(z) = -\frac{d\omega}{d\sigma}(x) \vec{n}(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \quad \square$$

The next result is an auxiliary calculation which will be used several times in the next section. The arguments for the proof are quite standard. Similar calculations appear, for example, in the proofs of Lemma 5.2 of [Kenig and Toro 2006], Lemma 3.3 of [Kenig and Toro 2003] or Lemma 3.30 of [Hofmann and Martell 2014].

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain, and let ω be the harmonic measure in Ω with pole either at infinity or at some fixed point $p \in \Omega$. Let $B \subset \mathbb{R}^{n+1}$ be a ball centered at $\partial\Omega$ such that $p \notin 10B$. Then for any constant $\varepsilon > 0$,*

$$\int_{B \cap \Omega} \left(\frac{d_\Omega(y)}{r(B)} \right)^\varepsilon \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}} dy \leq C(\varepsilon) \omega(B).$$

Proof. We write

$$\int_{B \cap \Omega} \left(\frac{d_\Omega(y)}{r(B)} \right)^\varepsilon \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}} dy \lesssim \sum_{j \geq 0} 2^{-j\varepsilon} \int_{\substack{y \in B \cap \Omega \\ 2^{-j-1}r(B) < d_\Omega(y) \leq 2^{-j}r(B)}} \frac{\omega(B(y, 2^{-j+1}r(B)))}{(2^{-j}r(B))^{n+1}} dy. \quad (4-5)$$

We define $A_j := \{y \in B \cap \Omega : 2^{-j-1}r(B) < d_\Omega(y) \leq 2^{-j}r(B)\}$. For each $y \in A_j$ consider a ball B_y^j with radius $2^{-j+1}r(B)$ centered at a point $\xi_y \in \partial\Omega$ such that $|y - \xi_y| = d_\Omega(y)$. Clearly $y \in B_y^j$ for each $y \in A_j$, and thus we can extract a subfamily of pairwise disjoint balls $\{B_k^j\}_k \subset \{B_y^j\}_{y \in A_j}$ so that

$$A_j \subset \bigcup_k 3B_k^j.$$

Notice that for each $y \in B_k^j$, since ω is doubling,

$$\omega(B(y, 2^{-j+1}r(B))) \leq \omega(6B_k^j) \lesssim \omega(B_k^j).$$

Therefore, taking also into account that the balls B_k^j are contained in $6B$,

$$\begin{aligned} \int_{\substack{y \in B \cap \Omega: \\ 2^{-j-1}r(B) < d_\Omega(y) \leq 2^{-j}r(B)}} \frac{\omega(B(y, 2^{-j+1}r(B)))}{(2^{-j}r(B))^{n+1}} dy &\lesssim \sum_k \int_{B_k^j} \frac{\omega(B_k^j)}{(2^{-j}r(B))^{n+1}} dy \\ &= C \sum_k \omega(B_k^j) \lesssim \omega(6B) \lesssim \omega(B). \end{aligned}$$

Plugging this estimate into (4-5), the lemma follows. \square

5. Estimates for the gradient of Green's function

The reader should compare the arguments in this section to the ones in Section 3 of [Kenig and Toro 2003] and Section 2 of [Kenig and Toro 2006], which in turn rely on the results in the Appendices A1 and A2 of [Kenig and Toro 2003].

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded chord-arc domain. Suppose that the harmonic measure ω in Ω with pole at infinity satisfies $\log(d\omega/d\sigma) \in \text{VMO}(\sigma)$. Denote by u the associated Green function. Then*

$$|\nabla u(x)| \leq \int_{\partial\Omega} \frac{d\omega}{d\sigma}(y) d\omega^x(y) \quad \text{for all } x \in \Omega. \quad (5-1)$$

The proof of this lemma would be quite immediate if the function $d\omega/d\sigma$ inside the integral in (5-1) were compactly supported, taking into account that ∇u is harmonic. However, this is not the case and so the arguments are more delicate. The next auxiliary lemma will be used to take care of this question by a localization of singularities technique.

Lemma 5.2. *Under the assumptions of Lemma 5.1, suppose that $0 \in \partial\Omega$. Fix $R > 1$ large and let $\phi_R \in C_c^\infty(\mathbb{R}^{n+1})$ such that $\chi_{B(0,R)} \leq \phi_R \leq \chi_{B(0,2R)}$, $|\nabla^j \phi_R| \lesssim 1/R^j$ for $j = 1, 2$. For $x \in \Omega$, define*

$$w_R(x) = \int_{\Omega} g(x, y) \Delta[\phi_R \nabla u](y) dy.$$

Then $w_R \in C^{\alpha/2}(\bar{\Omega})$ for some $\alpha > 0$, $w_R|_{\partial\Omega} \equiv 0$, and the following estimates hold for $x \in \Omega$:

- (a) $|w_R(x)| \lesssim \frac{\omega(B(0, R))}{R^n} \left(\frac{d_\Omega(x)}{R}\right)^{\alpha/2}$ if $|x| \leq 4R$.
- (b) $|w_R(x)| \lesssim \frac{\omega(B(0, R))}{|x|^{n-1+\alpha/2} R^{1-\alpha/2}} \left(\frac{d_\Omega(x)}{|x|}\right)^{\alpha/2}$ if $|x| > 4R$.

Proof. By the relationship between Green’s function and harmonic measure, for all $y \in \Omega$ we have

$$u(y) \approx \frac{1}{d_\Omega(y)^{n-1}} \omega(B(y, 2d_\Omega(y))),$$

and by standard estimates for positive harmonic functions we derive

$$|\nabla u(y)| \lesssim \frac{u(y)}{d_\Omega(y)} \approx \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^n} \quad \text{and} \quad |\nabla^2 u(y)| \lesssim \frac{u(y)}{d_\Omega(y)^2} \approx \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}}.$$

Thus,

$$\begin{aligned} |w_R(x)| &= \left| \int_{\Omega} g(x, y) (\Delta\phi_R(y) \nabla u(y) + 2\nabla\phi_R(y) \cdot \nabla^2 u(y)) dy \right| \\ &\lesssim \int_{A(0, R, 2R) \cap \Omega} g(x, y) \left(\frac{\omega(B(y, 2d_\Omega(y)))}{R^2 d_\Omega(y)^n} + \frac{\omega(B(y, 2d_\Omega(y)))}{R d_\Omega(y)^{n+1}} \right) dy \\ &\lesssim \int_{B(0, 2R) \cap \Omega} g(x, y) \frac{\omega(B(y, 2d_\Omega(y)))}{R d_\Omega(y)^{n+1}} dy. \end{aligned} \tag{5-2}$$

Case 1: $|x| \leq 4R$.

We split the integral on the right-hand side of (5-2) as follows:

$$\begin{aligned} |w_R(x)| &\lesssim \int_{|y-x| \leq d_\Omega(x)/2} g(x, y) \frac{\omega(B(y, 2d_\Omega(y)))}{R d_\Omega(y)^{n+1}} dy + \int_{\substack{y \in B(0, 2R) \cap \Omega \\ |y-x| > d_\Omega(x)/2}} g(x, y) \frac{\omega(B(y, 2d_\Omega(y)))}{R d_\Omega(y)^{n+1}} dy \\ &=: I_1 + I_2. \end{aligned} \tag{5-3}$$

First we will deal with I_1 . In the domain of integration of I_1 we have $d_\Omega(y) \approx d_\Omega(x)$. Taking into account that ω is doubling, in this case we derive $\omega(B(y, 2d_\Omega(y))) \approx \omega(B(x, 2d_\Omega(x)))$. Then using also the trivial estimate $g(x, y) \lesssim 1/|x - y|^{n-1}$, we get

$$I_1 \lesssim \int_{|y-x| \leq d_\Omega(x)/2} \frac{1}{|x - y|^{n-1}} \frac{\omega(B(x, 2d_\Omega(x)))}{R d_\Omega(x)^{n+1}} dy \approx \frac{\omega(B(x, 2d_\Omega(x)))}{R d_\Omega(x)^{n-1}}.$$

Notice that, by Lemma 4.1,

$$u(x) \lesssim \left(\frac{d_\Omega(x)}{R}\right)^\alpha \sup_{y \in \partial B(0, 8R) \cap \Omega} u(y) \lesssim \left(\frac{d_\Omega(x)}{R}\right)^\alpha u(x_R), \tag{5-4}$$

where x_R is a corkscrew point for $B(0, R)$. That is, $x_R \in B(0, R) \cap \Omega$ and $d_\Omega(x_R) \approx R$. Hence using that $\omega(B(z, 2d_\Omega(z))) \approx u(z) d_\Omega(z)^{n-1}$ both for $z = x$ and $z = x_R$, we deduce that

$$I_1 \lesssim \frac{\omega(B(x, 2d_\Omega(x)))}{R d_\Omega(x)^{n-1}} \lesssim \left(\frac{d_\Omega(x)}{R}\right)^\alpha \frac{\omega(B(x_R, 2d_\Omega(x_R)))}{R d_\Omega(x_R)^{n-1}} \approx \left(\frac{d_\Omega(x)}{R}\right)^\alpha \frac{\omega(B(0, R))}{R^n}. \tag{5-5}$$

We consider now the integral I_2 in (5-3). To estimate this we use the inequality

$$g(x, y) \lesssim \frac{1}{|x - y|^{n-1}} \left(\frac{d_\Omega(x)}{|x - y|} \right)^{\alpha/2} \left(\frac{d_\Omega(y)}{|x - y|} \right)^{\alpha/2}, \quad (5-6)$$

which is an immediate consequence of (4-1). To shorten notation, for each integer $j \geq 0$ we write $r_j := 2^j d_\Omega(x)$. Denote by j_{\max} the least integer such that $B(0, 2R) \subset B(x, r_{j_{\max}})$, so that $r_{j_{\max}} \approx R$. Then plugging the estimate (5-6) into I_2 and splitting, we obtain

$$I_2 \lesssim \sum_{0 \leq j \leq j_{\max}} \frac{1}{R r_j^{n-1}} \left(\frac{d_\Omega(x)}{r_j} \right)^{\alpha/2} \int_{\substack{y \in \Omega \\ r_{j-1} < |y-x| \leq r_j}} \left(\frac{d_\Omega(y)}{r_j} \right)^{\alpha/2} \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}} dy.$$

Let $\xi_x \in \partial\Omega$ be such that $|x - \xi_x| = d_\Omega(x)$. It is immediate to check that if $|y - x| \leq r_j = 2^j d_\Omega(x)$, then $y \in \bar{B}(\xi_x, 2r_j)$. So the last integral is bounded above by

$$\int_{\Omega \cap \bar{B}(\xi_x, 2r_j)} \left(\frac{d_\Omega(y)}{r_j} \right)^{\alpha/2} \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}} dy,$$

and then, by Lemma 4.8, this does not exceed $C \omega(B(\xi_x, r_j))$. Hence,

$$I_2 \lesssim \sum_{0 \leq j \leq j_{\max}} \frac{1}{R r_j^{n-1}} \left(\frac{d_\Omega(x)}{r_j} \right)^{\alpha/2} \omega(B(\xi_x, r_j)). \quad (5-7)$$

To estimate the right-hand side in the inequality above, we argue as in (5-4). We consider a corkscrew point x_j in each ball $B(\xi_x, r_j)$, and then since $\text{dist}(x_j, \partial\Omega) \approx r_j$, we deduce

$$u(x_j) \lesssim \left(\frac{r_j}{R} \right)^\alpha u(x_R)$$

(recall that x_R is a corkscrew point for $B(0, R)$). Thus,

$$\frac{\omega(B(\xi_x, r_j))}{r_j^{n-1}} \lesssim \left(\frac{r_j}{R} \right)^\alpha \frac{\omega(B(0, R))}{R^{n-1}}.$$

Plugging this estimate into (5-7) we obtain

$$I_2 \lesssim \sum_{0 \leq j \leq j_{\max}} \left(\frac{d_\Omega(x)}{r_j} \right)^{\alpha/2} \left(\frac{r_j}{R} \right)^\alpha \frac{\omega(B(0, R))}{R^n} = \frac{d_\Omega(x)^{\alpha/2}}{R^\alpha} \frac{\omega(B(0, R))}{R^n} \sum_{0 \leq j \leq j_{\max}} r_j^{\alpha/2}.$$

Since the last sum is geometric, it turns out that

$$\sum_{0 \leq j \leq j_{\max}} r_j^{\alpha/2} \approx r_{j_{\max}}^{\alpha/2} \approx R^{\alpha/2}.$$

Therefore,

$$I_2 \lesssim \frac{d_\Omega(x)^{\alpha/2}}{R^{\alpha/2}} \frac{\omega(B(0, R))}{R^n}.$$

Together with the estimate for I_1 in (5-5), this yields the inequality (a) in the lemma.

Case 2: $|x| > 4R$.

To estimate the integral on the right-hand side of (5-2) we use the fact that, for $y \in B(0, 2R)$, by (4-1),

$$g(x, y) \lesssim \frac{1}{|x|^{n-1}} \left(\frac{d_\Omega(x)}{|x|} \right)^{\alpha/2} \left(\frac{d_\Omega(y)}{|x|} \right)^{\alpha/2},$$

taking into account that $|x - y| \approx |x|$. Then we get

$$\begin{aligned} |w_R(x)| &\lesssim \frac{1}{|x|^{n-1}} \left(\frac{d_\Omega(x)}{|x|} \right)^{\alpha/2} \int_{B(0, 2R) \cap \Omega} \left(\frac{d_\Omega(y)}{|x|} \right)^{\alpha/2} \frac{\omega(B(y, 2d_\Omega(y)))}{R d_\Omega(y)^{n+1}} dy \\ &= \frac{1}{R|x|^{n-1}} \left(\frac{d_\Omega(x)}{|x|} \right)^{\alpha/2} \left(\frac{R}{|x|} \right)^{\alpha/2} \int_{B(0, 2R) \cap \Omega} \left(\frac{d_\Omega(y)}{R} \right)^{\alpha/2} \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}} dy. \end{aligned}$$

By Lemma 4.8, the last integral above does not exceed $C\omega(B(0, R))$, and so we deduce that

$$|w_R(x)| \lesssim \frac{1}{R|x|^{n-1}} \left(\frac{d_\Omega(x)}{|x|} \right)^{\alpha/2} \left(\frac{R}{|x|} \right)^{\alpha/2} \omega(B(0, R)),$$

which gives the inequality (b) in the lemma. □

Proof of Lemma 5.1. The arguments are similar to the ones for [Kenig and Toro 2003, Theorem 3.1]. For the reader's convenience, we show the details below.

Suppose that $0 \in \partial\Omega$ and, for $R \geq 1$, let ϕ_R and w_R be the functions introduced in Lemma 5.2. For $x \in \Omega$, we define

$$h_R(x) = \phi_R(x) \nabla u(x) - w_R(x).$$

Since $\Delta w_R = \Delta[\phi_R \nabla u]$ in Ω , it turns out that h_R is harmonic in Ω . Further, the estimates (a) and (b) in Lemma 5.2, in particular, ensure that w_R vanishes continuously at $\partial\Omega$. Thus h_R vanishes on $\partial\Omega \setminus B(0, 2R)$.

By Lemma 4.7 it follows that $\nabla u(z)$ converges nontangentially to $-(d\omega/d\sigma)(y)\vec{n}(y)$ as $\Omega \ni z \rightarrow y$ for σ -a.e. $y \in \partial\Omega$. Also, as mentioned above, $w_R(z) \rightarrow 0$ as $z \rightarrow y$. Therefore, if we define

$$h(y) = \frac{d\omega}{d\sigma}(y),$$

we have

$$\lim_{\Gamma^+(y) \ni z \rightarrow y} h_R(z) = -\phi_R(y) h(y) \vec{n}(y) \quad \text{for } \sigma\text{-a.e. } y \in \partial\Omega.$$

We claim that for all $x \in \Omega$,

$$h_R(x) = - \int \phi_R(y) h(y) \vec{n}(y) d\omega^x(y). \tag{5-8}$$

To prove this, recalling that h_R vanishes at ∞ , by Theorem 5.8 and Lemma 8.3 in [Jerison and Kenig 1982] it suffices to show that $\mathcal{N}_1 h_R \in L^1(\omega^x)$ for all $x \in \Omega$, where \mathcal{N}_1 stands for the operator defined by

$$\mathcal{N}_1 h_R(y) = \sup_{z \in \Gamma_1^+(y)} h_R(z),$$

with $\Gamma_1^+(y) = \Gamma^+(y) \cap \bar{B}(y, 1)$. By Lemma 5.2, w_R is bounded, and thus $\mathcal{N}_1 w_R \in L^1(\omega^x)$. Hence it is enough to prove that $\mathcal{N}_1(\phi_R \nabla u) \in L^1(\omega^x)$. To this end, notice that if $z \in \Gamma_1^+(y)$, then

$$|\nabla u(z)| \lesssim \frac{u(z)}{d_\Omega(z)} \approx \frac{\omega(B(y, d_\Omega(z)))}{d_\Omega(z)^n}.$$

Thus,

$$\mathcal{N}_1(\phi_R \nabla u)(y) \lesssim \sup_{0 < r \leq 1} \frac{\omega(B(y, r))}{r^n} = \sup_{0 < r \leq 1} \frac{1}{r^n} \int_{B(y, r)} |h| d\sigma =: \mathcal{M}_1 h(y).$$

Also, $\mathcal{N}_1(\phi_R \nabla u)(y)$ vanishes outside of $B' := \bar{B}(0, 2R + 1)$ because in this case $\phi_R(z) = 0$ whenever $z \in \Gamma_1^+(y)$. Therefore,

$$\int \mathcal{N}_1(\phi_R \nabla u) d\omega^x = \int_{B'} \mathcal{N}_1(\phi_R \nabla u) K_x d\sigma \lesssim \left(\int_{B'} |\mathcal{M}_1 h|^2 d\sigma \right)^{1/2} \left(\int_{B'} (K_x)^2 d\sigma \right)^{1/2}.$$

By the $L^2(\sigma)$ -boundedness of \mathcal{M}_1 , it follows that

$$\int_{B'} |\mathcal{M}_1 h|^2 d\sigma = \int_{B'} |\mathcal{M}_1(\chi_{B''} h)|^2 d\sigma < \infty,$$

where $B'' = \bar{B}(0, 2R + 2)$. Also, by Corollary 4.6,

$$\int_{B'} (K_x)^2 d\sigma < \infty,$$

and so $\mathcal{N}_1(\phi_R \nabla u) \in L^1(\omega^x)$ and (5-8) holds.

From the definition of h_R and (5-8) we deduce that

$$\phi_R(x) \nabla u(x) = - \int \phi_R(y) h(y) \vec{n}(y) d\omega^x(y) + w_R(x). \tag{5-9}$$

Hence, letting $R \rightarrow \infty$,

$$|\nabla u(x)| \leq \int |h(y)| d\omega^x(y) + \liminf_{R \rightarrow \infty} |w_R(x)|.$$

By Lemma 5.2(a) and Corollary 4.5 (with ε small enough), we deduce easily that $w_R(x) \rightarrow 0$ as $R \rightarrow \infty$, for any fixed $x \in \Omega$, and then the lemma follows. \square

Now we wish to obtain a variant of Lemma 5.1 suitable for the case when the pole for harmonic measure is finite. This is what we do in the next lemma.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. Suppose that the harmonic measure ω^p in Ω with pole at $p \in \Omega$ satisfies $\log(d\omega^p/d\sigma) \in \text{VMO}(\sigma)$. Then, for all $x \in \Omega$ such that $d_\Omega(x) \leq d_\Omega(p)/8$ and all $q_x \in \partial\Omega$ such that $|x - q_x| \leq d_\Omega(p)/8$,*

$$|\nabla g(x, p)| \leq \int_{\partial\Omega} K_p(y) d\omega^x(y) + C \frac{\omega^p(B(q_x, d_\Omega(p)))}{d_\Omega(p)^n} \left(\frac{d_\Omega(x)}{d_\Omega(p)} \right)^{\alpha/2}. \tag{5-10}$$

Proof. Let $\xi \in \partial\Omega$ and take a C^∞ function ϕ compactly supported in $B(\xi, d_\Omega(p)/4)$ which is identically 1 on $B(\xi, d_\Omega(p)/8)$, so that $|\nabla^j \phi| \lesssim 1/d_\Omega(p)^j$ for $j = 1, 2$. Note that, in particular, ϕ vanishes on $B(p, d_\Omega(p)/4)$. We consider the function

$$w_0(x) = \int_\Omega g(x, y) \Delta[\phi \nabla g(\cdot, p)](y) dy \quad \text{for } x \in \Omega.$$

We claim that

$$|w_0(x)| \lesssim \frac{\omega(B(\xi, d_\Omega(p)/8))}{d_\Omega(p)^n} \left(\frac{d_\Omega(x)}{d_\Omega(p)}\right)^{\alpha/2} \quad \text{if } |x - \xi| \leq \frac{d_\Omega(p)}{4}. \tag{5-11}$$

The arguments to prove (5-11) are quite similar to the ones in Lemma 5.2. By the relationship between Green’s function and harmonic measure and by standard estimates for positive harmonic functions, for all $y \in B(\xi, d_\Omega(p)/4) \cap \Omega$ we have

$$|\nabla g(y, p)| \lesssim \frac{g(y, p)}{d_\Omega(y)} \approx \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(y)^n} \quad \text{and} \quad |\nabla^2 g(y, p)| \lesssim \frac{g(y, p)}{d_\Omega(y)^2} \approx \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(y)^{n+1}}.$$

Thus,

$$\begin{aligned} |w_0(x)| &= \left| \int_\Omega g(x, y) (\Delta\phi(y) \nabla g(y, p) + 2\nabla\phi(y) \cdot \nabla^2 g(y, p)) dy \right| \\ &\lesssim \int_{A(\xi, d_\Omega(p)/8, d_\Omega(p)/4) \cap \Omega} g(x, y) \left(\frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(p)^2 d_\Omega(y)^n} + \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(p) d_\Omega(y)^{n+1}} \right) dy \\ &\lesssim \int_{B(\xi, d_\Omega(p)/4) \cap \Omega} g(x, y) \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(p) d_\Omega(y)^{n+1}} dy. \end{aligned}$$

Notice that the integral on the right-hand side above is very similar to the one on the right-hand side of (5-2). The reader can check that exactly the same arguments and estimates used to prove Lemma 5.2(a) yield (5-11), with ξ instead of 0, $d_\Omega(p)/8$ instead of R , ω^p instead of ω , and $g(y, p)$ instead of $u(y)$. We leave the details for the reader.

From (5-11) it follows that $w_0 \in C^{\alpha/2}(\bar{\Omega})$ and it vanishes at $\partial\Omega$. Further, the function defined by

$$h_0(x) = \phi(x) \nabla g(x, p) - w_0(x), \quad x \in \Omega,$$

is harmonic in Ω , because $\Delta w_0 = \phi \nabla g(\cdot, p)$. Hence, arguing as in (5-9), we derive

$$\phi(x) \nabla g(x, p) = - \int \phi(y) K_p(y) \vec{n}(y) d\omega^x(y) + w_0(x).$$

If $|x - \xi| \leq d_\Omega(p)/8$, then $\phi(x) = 1$ and from the last identity and the inequality (5-11) with $\xi = q_x$, we deduce (5-10). □

6. The pseudo-blow-up of harmonic measure is surface measure

Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. We recall that harmonic measure with either a finite pole $p \in \Omega$ or pole at infinity is in the $A_\infty(\sigma)$ class of weights by [David and Jerison 1990] or [Semmes 1990] and thus, the Poisson kernel $d\omega/d\sigma$ exists and is positive and finite. We denote by u either the Green’s function with pole at $p \in \Omega$ or with pole at infinity and by h the corresponding Poisson kernel (see (3-2) for pole at infinity).

6A. Pseudo-blow-ups of chord-arc domains. Here we introduce the notion of *pseudo-blow-ups* from [Kenig and Toro 2003], but with a slight modification. Let $x_i \in \partial\Omega$ and let $\{r_i\}_{i \geq 1}$ be a sequence of positive numbers so that $\lim_{i \rightarrow \infty} r_i = 0$. Consider now the domains

$$\Omega_i = \frac{1}{r_i}(\Omega - x_i),$$

so that $\partial\Omega_i = (1/r_i)(\partial\Omega - x_i)$, and the functions u_i in Ω_i defined by

$$u_i(x) = \frac{g(r_i x + x_i, p_i)}{r_i \omega^{p_i}(B(x_i, r_i))} \sigma(B(x_i, r_i)),$$

where either $p_i = \infty$ or $p_i \in \Omega \setminus \{x_i\}$ satisfies

$$\frac{p_i - x_i}{r_i} \rightarrow \infty \quad \text{as } i \rightarrow \infty.$$

Note that u_i vanishes at $\partial\Omega_i$ and is harmonic in $\Omega_i \setminus \{(p_i - x_i)/r_i\}$. We denote by $d\omega_i = h_i d\sigma_i$ the harmonic measure of Ω_i with pole at infinity or $(p_i - x_i)/r_i$ depending on the pole of u , where $\sigma_i = \mathcal{H}^n|_{\partial\Omega_i}$. Moreover, the corresponding Poisson kernel² h_i satisfies

$$h_i(x) = \frac{h(r_i x + x_i)}{\omega^{p_i}(B(x_i, r_i))} \sigma(B(x_i, r_i)).$$

Theorem 6.1 [Kenig and Toro 2003, Theorem 4.1]. *If $\Omega \subset \mathbb{R}^{n+1}$ is a chord-arc domain, then there exists a subsequence satisfying*

$$\begin{aligned} \Omega_i &\rightarrow \Omega_\infty && \text{in the Hausdorff metric, uniformly on compact sets,} \\ \partial\Omega_i &\rightarrow \partial\Omega_\infty && \text{in the Hausdorff metric, uniformly on compact sets,} \end{aligned}$$

where Ω_∞ is a chord-arc domain. Moreover, there exists $u_\infty \in C(\bar{\Omega}_\infty)$ such that $u_i \rightarrow u_\infty$ uniformly on compact sets which satisfies (3-1) for $\Omega = \Omega_\infty$. Furthermore, $\omega_i \rightarrow \omega_\infty$ weakly as Radon measures and ω_∞ is the harmonic measure of Ω_∞ with pole at infinity (corresponding to u_∞).

This was originally shown in [Kenig and Toro 2003] under the assumption that p_i is a fixed point and x_i converges to some point in $\partial\Omega$. However, the same proof gives the result above.

Theorem 6.2. *If $\Omega_\infty \subset \mathbb{R}^{n+1}$ and u_∞ are as in Theorem 6.1, then*

$$\sup_{z \in \Omega_\infty} |\nabla u_\infty(z)| \leq 1. \tag{6-1}$$

Theorem 6.3. *If $\Omega_\infty \subset \mathbb{R}^{n+1}$ and u_∞ and ω_∞ are as in Theorem 6.1, then*

$$\frac{d\omega_\infty}{d\sigma_\infty} \geq 1, \quad \mathcal{H}^n\text{-a.e. on } \partial\Omega_\infty, \tag{6-2}$$

where $\sigma_\infty = \mathcal{H}^n|_{\partial\Omega_\infty}$.

²In fact, this is the Poisson kernel of Ω_i with pole at p_i modulo a constant factor.

Both theorems were proved in [Kenig and Toro 2003, Theorems 4.2 and 4.3] for Reifenberg flat domains with n -AD regular boundary, although, an inspection of the proofs shows that the same arguments, with very minor changes, work also for NTA domains with n -AD regular boundary, i.e., for chord-arc domains.

Corollary 6.4. *If $\Omega_\infty \subset \mathbb{R}^{n+1}$ and u_∞ and ω_∞ are as in Theorem 6.1, then*

$$|\nabla u_\infty| = \frac{d\omega_\infty}{d\sigma_\infty} = 1 \quad \mathcal{H}^n\text{-a.e. on } \partial\Omega_\infty. \tag{6-3}$$

Proof. Combining (3-19) and (6-1) we get that $d\omega_\infty/d\sigma_\infty \leq 1$ for \mathcal{H}^n -a.e. on $\partial\Omega_\infty$. Then (6-3) follows from (6-2). □

Lemma 6.5. *The subsequence introduced in Theorem 6.1 satisfies $\sigma_i \rightharpoonup \sigma_\infty$ weakly as Radon measures.*

Proof. This was essentially proved in Theorem 4.4 in [Kenig and Toro 2003]. The only difference is that instead of invoking [Kenig and Toro 2003, Theorem 2] in the proof, which is particular to the Reifenberg flat case, we just use Corollary 6.4. □

6B. Blow-downs of unbounded chord-arc domains. In the course of proving our main result we will need to construct the *blow-down* domain with respect to a fixed point $x_0 \in \partial\Omega$ of an unbounded chord-arc domain Ω such that $d\omega/d\sigma = 1$ σ -a.e. on $\partial\Omega$ (i.e., $\omega = \sigma$). To do so, we let $x_i = x_0$ for all $i \geq 1$ and a sequence of positive numbers r_i such that $\lim_{i \rightarrow \infty} r_i = \infty$. Now we take Ω_i and u_i as in the construction of pseudo-blow-ups in Section 6A and $p = p_i = \infty$. Then similar (but easier) arguments show that there exists a chord-arc domain $\tilde{\Omega}$ such that

$$\begin{aligned} \Omega_i &\rightarrow \tilde{\Omega} && \text{in the Hausdorff metric, uniformly on compact sets,} \\ \partial\Omega_i &\rightarrow \partial\tilde{\Omega} && \text{in the Hausdorff metric, uniformly on compact sets.} \end{aligned}$$

Moreover, there exists $\tilde{u} \in C(\tilde{\Omega})$ such that $u_i \rightarrow u_0$ uniformly on compact sets which satisfies

$$\Delta\tilde{u} = 0 \quad \text{in } \tilde{\Omega}, \quad \tilde{u} > 0 \quad \text{in } \tilde{\Omega}, \quad \tilde{u} = 0 \quad \text{in } \partial\tilde{\Omega}.$$

7. Application of the monotonicity formula of Weiss: blow-downs are planes in \mathbb{R}^3

We first introduce the notion of a variational solution of the one-phase free boundary problem in an open ball $B \subset \mathbb{R}^{n+1}$,

$$\begin{cases} u \geq 0 & \text{in } B, \\ \Delta u = 0 & \text{in } B^+(u) := B \cap \{u > 0\}, \\ |\nabla u| = 1 & \text{on } F(u) := \partial B^+(u) \cap B. \end{cases} \tag{7-1}$$

Definition 7.1. We define $u \in W_{loc}^{1,2}(B)$ to be a *variational solution* of (7-1) if

- (1) $u \in C(B) \cap C^2(B^+(u))$,
- (2) $\chi_{\{u>0\}} \in L^1_{loc}(B)$ and
- (3) the first variation with respect to the functional

$$F(v) := \int_B (|\nabla v|^2 + \chi_{\{v>0\}}) dm \tag{7-2}$$

vanishes at $v = u$; i.e.,

$$0 = -\frac{d}{d\varepsilon} F(u(x + \varepsilon\phi(x)))|_{\varepsilon=0} = \int_B [(|\nabla u|^2 + \chi_{\{u>0\}}) \operatorname{div} \phi - 2\nabla u \cdot D\phi(\nabla u)^T] dm \quad (7-3)$$

for any $\phi \in C_c^\infty(B; \mathbb{R}^{n+1})$.

Definition 7.2. We say that u is a *weak solution* of $\Delta u = \mathcal{H}^n(\partial\{u > 0\} \cap \cdot)$ in B if the following are satisfied:

- (1) $u \in W_{\text{loc}}^{1,2}(B) \cap C(B^+(u))$, $u \geq 0$ in B , and u is harmonic in the open set $\{u > 0\}$.
- (2) *Nondegeneracy and regularity*: for any open $D \Subset B$ there exist $0 < c_D \leq C_D < \infty$ such that for any $B(x, r) \subset D$ satisfying $x \in \partial\{u > 0\}$ we have

$$c_D \leq r^{-n-1} \int_{\partial B(x,r)} u d\mathcal{H}^n \leq C_D. \quad (7-4)$$

- (3) $\{u > 0\}$ is locally in B a set of finite perimeter and

$$-\int \nabla u \cdot \nabla \zeta dm = \int_{\partial^*\{u>0\}} \zeta d\mathcal{H}^n \quad (7-5)$$

for any $\zeta \in C_c^\infty(B)$, where $\partial^*\{u > 0\}$ stands for the reduced boundary of $\{u > 0\}$.

Let us now record a useful lemma whose proof is contained in the one of [Weiss 1998, Theorem 5.1].

Lemma 7.3. *If u is a weak solution of $\Delta u = \mathcal{H}^n(\partial\{u > 0\} \cap \cdot)$ in a ball B in the sense of Definition 7.2, then it is also a variational solution in the ball B in the sense of Definition 7.1.*

Lemma 7.4. *Assume that Ω_∞ is the blow-up domain and u_∞ is the blow-up Green's function constructed in Theorem 6.1. If B is a ball centered on $\partial\{u_\infty > 0\} = \partial\Omega_\infty$, then the extension by zero of u_∞ outside $\{u_\infty > 0\}$ is a weak solution of $\Delta u = \mathcal{H}^n(\partial\{u > 0\} \cap \cdot)$ in B .*

Proof. By construction, $\Omega_\infty = \{u_\infty > 0\}$, $u_\infty > 0$ in Ω_∞ , $u_\infty = 0$ in $\partial\Omega_\infty$, u_∞ is harmonic in Ω_∞ , $u_\infty \in C(\bar{\Omega}_\infty)$, and $|\nabla u_\infty| \leq 1$ in Ω_∞ . Therefore, it is trivial to see that its extension by zero in the complement of Ω_∞ satisfies the condition (1) in Definition 7.2 for the ball B . Notice also that by Harnack's inequality at the boundary, if x_r is a corkscrew point in $B(x, r) \cap \Omega_\infty$, it holds that

$$\max_{z \in \partial B(x,r) \cap \Omega_\infty} u_\infty(z) = \max_{z \in B(x,r) \cap \Omega_\infty} u_\infty(z) \approx u_\infty(x_r).$$

Therefore, we have that by (3-8) and Corollary 6.4,

$$r^{-n-1} \int_{\partial B(x,r)} u_\infty d\mathcal{H}^n \approx \frac{\mathcal{H}^n(\partial B(x, r))}{r^{n+1}} u_\infty(x_r) \approx \frac{\omega_\infty(B(x, r))}{\sigma_\infty(B(x, r))} = 1.$$

Since $\partial\Omega_\infty$ is n -AD regular, we have that $\mathcal{H}^n|_{\partial\Omega_\infty}$ is locally finite, and thus Ω_∞ is of locally finite perimeter in \mathbb{R}^{n+1} . By the generalized Gauss–Green formula for sets of locally finite perimeter, we infer that

$$\begin{aligned} \int_{\partial\Omega_\infty} \zeta d\mathcal{H}^n &= \int_{\partial\Omega_\infty} \zeta d\omega_\infty = \int_{\Omega_\infty} u_\infty \Delta \zeta dm \\ &= \int_{\Omega_\infty} \operatorname{div}(u_\infty \nabla \zeta) dm - \int_{\Omega_\infty} \nabla u_\infty \cdot \nabla \zeta dm = 0 - \int_{\Omega_\infty} \nabla u_\infty \cdot \nabla \zeta dm \end{aligned}$$

for any $\zeta \in C_c^\infty(\mathbb{R}^n)$. Note that $\mathcal{H}^n(\partial\Omega_\infty \setminus \partial^*\Omega_\infty) = 0$ in any NTA domain and thus, condition (3) in Definition 7.2 is satisfied. \square

We state without proof a lemma from [Jerison and Kamburov 2016] which allows us to conclude that any blow-down domain of Ω_∞ is in fact a cone.

Lemma 7.5 [Jerison and Kamburov 2016, Lemma 5.2]. *Let u be a variational solution of (7-1) in \mathbb{R}^{n+1} which is globally Lipschitz. Assume that $0 \in F(u)$ and consider a sequence $R_j \rightarrow \infty$. If the sequence*

$$v_j(x) = R_j^{-1}u(R_jx)$$

converges uniformly on compact sets as $j \rightarrow \infty$, its limit is Lipschitz continuous and homogeneous of degree 1.

Lemma 7.6. *Assume that $\Omega_\infty \subset \mathbb{R}^{n+1}$ is the blow-up domain and u_∞ is the blow-up Green’s function constructed in Theorem 6.1. If $x \in \partial\Omega_\infty$, then any blow-down domain of Ω_∞ at x is a cone.*

By a cone we mean a set $F \subset \mathbb{R}^{n+1}$ such that if $x \in F$, then $\lambda x \in F$ for all $\lambda > 0$. A conical domain is a domain which is a cone.

Proof. It follows from Lemmas 7.3, 7.4 and 7.5 in view of Section 6B. \square

Lemma 7.7. *If $\Omega_0 \subset \mathbb{R}^3$ is a conical two-sided NTA domain in \mathbb{R}^3 with 2-AD-regular boundary such that $d\omega_0/d\sigma_0 = 1$ σ_0 -a.e. in $\partial\Omega_0$, then Ω_0 is a half-space.*

Proof. Since Ω_0 is a conical two-sided NTA domain, the intersection of Ω_0 with the sphere S^2 is an open connected subset of S^2 , and the interior of its complement should be another open connected set of S^2 . Further, as shown in [Caffarelli et al. 2004, Remark 2 and p. 92] by studying the mean curvature of $\partial\Omega_0 \cap S^2$, one deduces that $\partial\Omega_0 \cap S^2$ is a convex curve and Ω_0^c is a convex cone. One can check that a convex cone in \mathbb{R}^3 is a Lipschitz domain, and also its exterior domain. Hence, by the results of Farina and Valdinoci [2010] (or by arguments analogous to the ones in [Caffarelli et al. 2004, p. 92]), Ω_0 is a half-space. \square

Corollary 7.8. *Suppose that Ω_0 is a two-sided NTA domain in \mathbb{R}^3 with 2-AD-regular boundary such that $d\omega_0/d\sigma_0 = 1$ σ_0 -a.e. in $\partial\Omega_0$. Then, for any $x \in \partial\Omega_0$,*

$$\lim_{r \rightarrow \infty} \Theta_{\partial\Omega_0}(x, r) = 0.$$

Proof. This is an immediate consequence of Lemmas 7.6 and 7.7. \square

8. The Alt–Caffarelli theorem

The objective of this section is to explain how to prove the following lemma.

Lemma 8.1. *Let Ω_0 be an NTA domain in \mathbb{R}^{n+1} with n -AD-regular boundary with constant C_0 . Suppose $0 \in \partial\Omega_0$ and*

$$\frac{d\omega_0}{d\sigma_0} \equiv 1 \quad \sigma_0\text{-a.e. in } \partial\Omega_0. \tag{8-1}$$

There exists $\delta_0 > 0$ small enough depending on n , the NTA character of Ω_0 , and C_0 such that if $\mathbb{B} = B(0, 1)$ satisfies

$$\Theta_{\partial\Omega_0}(\lambda\mathbb{B}) \leq \delta_0 \quad \text{for all } \lambda > 1, \quad (8-2)$$

then Ω_0 is a half-space.

Before turning to the proof of this lemma, notice that an immediate consequence of this and Corollary 7.8 is the following.

Corollary 8.2. *Suppose that Ω_0 is a two-sided chord-arc in \mathbb{R}^3 such that $d\omega_0/d\sigma_0 = 1$ σ_0 -a.e. in $\partial\Omega_0$. Then, Ω_0 is a half-space.*

Lemma 8.1 is essentially proven in [Kenig and Toro 2004], which assumes that the domain is Reifenberg flat. This is a variant of some of the results by Alt and Caffarelli [1981]. In [Kenig and Toro 2004] the authors also assume in the statement of their theorem that

$$|\nabla u_0| \leq \chi_\Omega, \quad (8-3)$$

where u_0 is its Green function with pole at infinity. However, this estimate is an immediate consequence of the assumptions of Lemma 8.1, especially (8-1), and Lemma 5.1. Thus, we will only explain how to read and adjust the proof in [Kenig and Toro 2004] in order to obtain the lemma, adding details where necessary.

Lemma 8.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a two-sided C -corkscrew domain so that Ω_{ext} is also connected. Then whenever $\xi \in \partial\Omega$, $r > 0$, and $\beta_{\partial\Omega}(\xi, r, P) < 1/(2C)$ for some n -plane P ,*

$$\Theta_{\partial\Omega}(\xi, r/2, P) \leq 2\beta_{\partial\Omega}(\xi, r, P) \quad (8-4)$$

and there are half-spaces H^\pm such that

$$\begin{aligned} H^+ \cup H^- &= \{y : \text{dist}(y, P) > \beta_{\partial\Omega}(\xi, r, P)\}, \\ H^+ \cap B(\xi, r) &\subset \Omega \quad \text{and} \quad H^- \cap B(\xi, r) \subset \Omega_{\text{ext}}. \end{aligned}$$

In particular, if π_P is the projection onto P , then $\pi_P(\partial\Omega \cap B(\xi, r)) \supseteq \pi_P(B(\xi, r/2))$.

Proof. Without loss of generality, we assume $\xi = 0, r = 1$, so $B(\xi, r) = \mathbb{B} = B(0, 1)$. Let $\varepsilon = \beta_{\partial\Omega}(\xi, r, P)$. If $(H^+ \cup H^-) \cap \mathbb{B} \subset \Omega$, then

$$\Omega_{\text{ext}} \cap \mathbb{B} \subset \{y : \text{dist}(y, P) \leq \varepsilon\},$$

but since Ω has exterior corkscrews, there must be

$$B(y, 1/C) \subset \mathbb{B} \cap \Omega_{\text{ext}} \subset \{y : \text{dist}(y, P) \leq \varepsilon\},$$

which is a contradiction for $\varepsilon < 1/(2C)$. We also get a contradiction if $(H^+ \cup H^-) \cap \mathbb{B} \subset \Omega_{\text{ext}}$, and so $H^\pm \cap \mathbb{B}$ must be in two different components. Assume $H^+ \cap \mathbb{B} \subset \Omega$ and $H^- \subset \Omega_{\text{ext}}$. The last part of the lemma now follows from this, since for any $y \in \pi_P(B(\xi, r))$, the line $\pi_P^{-1}(y)$ must pass through both H^+ and H^- , and thus it must intersect $\partial\Omega$.

To prove (8-4) it suffices to show that if $x \in \frac{1}{2}\mathbb{B} \cap P$, then $\text{dist}(x, \partial\Omega) \leq 2\varepsilon$. Suppose there is $x \in \frac{1}{2}\mathbb{B} \cap P$ so that $B(x, 2\varepsilon) \subset (\partial\Omega)^c$. Then the set

$$U = \mathbb{B} \cap B(x, 2\varepsilon)^c \cap \{y : \text{dist}(y, P) > \varepsilon\}$$

is a connected open subset of $(\partial\Omega)^c$, and hence $U \subset \Omega$ or $U \subset \Omega_{\text{ext}}$. Without loss of generality, we can assume the former case. Then

$$\Omega_{\text{ext}} \cap \mathbb{B} \subset \{y : \text{dist}(y, P) \leq \varepsilon\} \cup B(x, 2\varepsilon).$$

But by the exterior corkscrew condition, $B(y, 1/C) \subset \mathbb{B} \cap \Omega_{\text{ext}}$, which is impossible if $\varepsilon < 1/(2C)$. \square

The following definition comes from [Kenig and Toro 2004], and it is a variant of one that appears in [Alt and Caffarelli 1981].

Definition 8.4. Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain. Let $x_0 \in \partial\Omega$, $\rho > 0$, $\sigma_+, \sigma \in (0, 1)$, $\nu \in \mathbb{S}^n$, and v be the Green function with pole at infinity. We say $v \in F(\sigma_+, \sigma)$ in $B(x_0, \rho)$ in the direction $\nu \in \mathbb{S}^n$ if, for all $x \in B(x_0, \rho)$,

$$v(x) = 0 \quad \text{if } (x - x_0) \cdot \nu \geq \sigma_+ \rho \tag{8-5}$$

and

$$v(x) \geq -(x - x_0) \cdot \nu - \sigma \rho \quad \text{if } (x - x_0) \cdot \nu \leq -\sigma \rho. \tag{8-6}$$

Observe that $v \equiv 0$ exactly on Ω^c and $v > 0$ exactly on Ω , and so

$$v \in F(\sigma, \sigma) \text{ in direction } \nu \text{ in } B(x_0, \rho) \text{ implies } \beta_{\partial\Omega}(x_0, \rho) \leq \sigma. \tag{8-7}$$

Indeed, assume $x_0 = 0$, $\rho = 1$, and note that by (8-5), since $v = 0$ only when Ω^c , we have that for $x \in B(x_0, \rho)$,

$$\{x \in \mathbb{B} : x \cdot \nu \geq \sigma\} \subseteq \Omega^c.$$

By (8-6), if $x \cdot \nu < -\sigma\rho$, then

$$v(x) \geq -x \cdot \nu - \sigma > 0$$

and since $v(x) > 0$ only when $x \in \Omega$,

$$\{x \in \mathbb{B} : x \cdot \nu < -\sigma\} \subseteq \Omega.$$

Since v is continuous, we thus have

$$\beta_{\partial\Omega}(0, 1) < \sigma.$$

Lemma 8.5. *Let Ω be a two-sided NTA domain and v the Green function with pole at infinity. Let $x_0 \in \partial\Omega$, $\rho, \sigma > 0$, and $\nu \in \mathbb{S}^n$. If $v \in F(\sigma, 1)$ in $B(x_0, \rho)$ in the direction ν , then $v \in F(2\sigma, C\sigma)$ in $B(x_0, \rho/2)$ in the same direction, where $C = C(n)$.*

Proof. The proof is exactly the same as in Lemma 0.4 in [Kenig and Toro 2004]. Its proof and that of Lemma 0.3 in the same paper, upon which it depends, do not require the Reifenberg flat assumption and the proofs are identical. \square

Lemma 8.6. *Let Ω be a two-sided NTA domain and v the Green function with pole at infinity. There is some ε_0 small enough so that the following holds. Let $x_0 \in \partial\Omega$, $\rho > 0$, and $v \in \mathbb{S}^n$. Given $\theta \in (0, 1)$, there is $\sigma_\theta > 0$ and $\eta \in (0, 1)$ so that if $0 < \sigma < \sigma_\theta$ and $v \in F(\sigma, \sigma)$ in $B(x_0, \rho)$ in the direction v and $\beta_{\partial\Omega}(x_0, 2\rho) < \varepsilon_0$, then $v \in F(\theta\sigma, 1)$ in $B(x_0, \eta\rho)$ in some direction v' such that $|v - v'| < C\sigma$.*

Proof. Again, the proof is exactly the same as that of Lemma 0.5 in [Kenig and Toro 2004]. The only time Kenig and Toro use the Reifenberg flatness assumption is to show that the intersection of a cylinder C with the boundary (with axis passing through Q_0) has projection in the direction of the cylinder equal to the base of the cylinder (i.e., a ball); see right below equation (0.69) in [Kenig and Toro 2004]. However, we can just replace this with the assumption that $\beta_{\partial\Omega}(x_0, 2\rho) < \varepsilon_0$ is small and then apply Lemma 8.3. \square

Proof of Lemma 8.1. Let $\theta' \in (0, 1/2)$ and $\delta_0 \in (0, \sigma_{n, \theta'} / (8 + 2C))$. Note that (8-2) implies that for $r > 1$, there is a plane P_r so that

$$\beta_{\partial\Omega}(0, r, P_r) \leq \Theta_{\partial\Omega}(0, r, P_r) \leq \delta_0. \quad (8-8)$$

Let $L_r = P_{2r} - \pi_{P_{2r}}(0)$ and let $v_r \in \mathbb{S}^n$ be a unit vector orthogonal to L_r so that $rv_r/2 \in \Omega^c$. Then

$$\{x \in B(0, r) : x \cdot v_r > \delta_0 r\} \subseteq \{x \in B(0, r) : \text{dist}(x, L_r) > \delta_0 r\} \subseteq (\partial\Omega)^c.$$

Since $\{x \in B(0, r) : x \cdot v_r > \delta_0 r\}$ and Ω^c are connected and $rv_r/2$ is in their intersection, we actually have

$$\{x \in B(0, r) : x \cdot v_r > \delta_0 r\} \subseteq \Omega^c.$$

Hence, $v(x) = 0$ for $x \in B(0, r)$ such that $x \cdot v > \delta_0 r$. Furthermore, we trivially have

$$\{x \in B(0, r) : x \cdot v_r > r\} = \emptyset$$

and thus $v \in F(\delta_0, 1)$. Lemma 8.5 implies $v \in F(2\delta_0, C\delta_0)$ in $\frac{1}{2}r\mathbb{B}$ in the same direction, and so $v \in F(\delta, \delta)$ in $\frac{1}{2}r\mathbb{B}$, where $\delta = \max\{2, C\}$. Let $\theta' \in (0, 1)$. By Lemma 8.6 and (8-8), there is $\eta' \in (0, 1)$ (depending only on θ') so that $v \in F(\theta'\delta, 1)$ in $\frac{1}{2}(\eta'r)\mathbb{B}$. Again, by Lemma 8.5, we have $v \in F(2\theta'\delta, C\theta'\delta)$ in $\frac{1}{4}(\eta'r)\mathbb{B}$, and hence $v \in F(\theta\delta, \theta\delta)$ in $\eta r\mathbb{B}$, where $\theta = \max\{2\theta', C\theta'\}$ and $\eta = \frac{1}{4}\eta'$ in the direction of some vector $v \in \mathbb{S}^n$. By (8-7), we have

$$\beta_{\partial\Omega}(0, \eta r) < \theta\delta.$$

Iterating, we get that for all $m \in \mathbb{N}$,

$$v \in F(\theta^m \delta, \theta^m \delta) \quad \text{in } \eta^m r \mathbb{B} \quad (8-9)$$

and

$$\Theta_{\partial\Omega}(0, \eta^m r/2) \leq 2\beta_{\partial\Omega}(0, \eta^m r) \leq \theta^m \delta.$$

Let $1 < s \ll r$ and pick m so that $\eta^{m+1}r \leq s < \eta^m r$. Then this implies

$$\Theta_{\partial\Omega}(0, s/2) \leq 2\Theta_{\partial\Omega}(0, \eta^m r/2) \leq 2\theta^m \delta = 2\eta^{\log \theta / \log \eta^m} \delta \leq 2(\eta^{-1} s r^{-1})^{\log \theta / \log \eta} \delta.$$

Thus, by sending $r \rightarrow \infty$, we get $\Theta_{\partial\Omega}(0, s/2) = 0$. Since this holds for every $s > 1$, we have that $\partial\Omega$ is equal to an n -plane, and since Ω is connected, it must be a half-space. \square

9. The proof of Theorem 1.1

Our arguments are very similar to the ones in [Kenig and Toro 2003]. The only difference is that in our pseudo-blow-ups we allow the points x_i to escape to ∞ . In this way, we are able to show that the outer unit normal \vec{n} belongs to $VMO(\sigma)$, not only to $VMO_{loc}(\sigma)$. For the reader's convenience, we replicate the arguments of [Kenig and Toro 2003] here.

Let

$$\ell = \lim_{r \rightarrow 0} \sup_{x \in \partial\Omega} \|\vec{n}\|_*(B(x, r)).$$

We will show $\ell = 0$. Let $x_i \in \partial\Omega$ and $r_i \downarrow 0$ be such that

$$\lim_{i \rightarrow \infty} \left(\int_{B(x_i, r_i)} |\vec{n} - \vec{n}_{B(x_i, r_i)}|^2 d\sigma \right)^{1/2} = \ell.$$

Let $\Omega_i = (1/r_i)(\Omega - x_i)$ and $u_i^{p_i}, \omega_i^{p_i}$ be as in Theorem 6.1. By this theorem, we can pass to a subsequence so that all these quantities converge to some Ω_∞, u_∞ , and ω_∞ . By Lemma 6.5, σ_i also converges to $\sigma_\infty = \mathcal{H}^n|_{\partial\Omega_\infty}$. By Lemma 7.7, Ω_∞ is a half-space (suppose it is \mathbb{R}_+^{n+1}) and $\omega_\infty = \mathcal{H}^n|_{\mathbb{R}^n}$. For ϕ a smooth, nonnegative, and compactly supported function with $\phi \geq \chi_{\mathbb{B}}$, and \vec{n}_i the outer unit normal to $\partial\Omega_i$, we thus have

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{\partial\Omega_i \cap \mathbb{B}} |\vec{n}_i + e_{n+1}|^2 d\sigma_i &\leq \lim_{i \rightarrow \infty} \int_{\partial\Omega_i} \phi |\vec{n}_i + e_{n+1}|^2 d\sigma_i \\ &= \lim_{i \rightarrow \infty} \left(2 \int_{\partial\Omega_i} \phi d\sigma_i + 2 \int_{\partial\Omega_i} \phi \vec{n}_i \cdot e_{n+1} d\sigma_i \right) \\ &= 2 \int_{\mathbb{R}^n} \phi d\sigma_\infty + 2 \lim_{i \rightarrow \infty} \int_{\Omega_i} \operatorname{div}(\phi e_{n+1}) dm \\ &= 2 \int_{\mathbb{R}^n} \phi d\sigma_\infty + 2 \int_{\mathbb{R}_+^{n+1}} \operatorname{div}(\phi e_{n+1}) dm \\ &= 2 \int_{\mathbb{R}^n} \phi d\sigma_\infty - 2 \int_{\mathbb{R}^n} \phi e_{n+1} \cdot e_{n+1} d\sigma_\infty = 0 \end{aligned}$$

and hence

$$\ell = \lim_{i \rightarrow \infty} \left(\int_{B(x_i, r_i)} |\vec{n} - \vec{n}_{B(x_i, r_i)}|^2 d\sigma \right)^{1/2} \leq 2 \lim_{i \rightarrow \infty} \left(\int_{B(x_i, r_i)} |\vec{n} + e_{n+1}|^2 d\sigma \right)^{1/2} = 0. \quad \square$$

Remark 9.1. The same arguments as above show that Theorem A by Kenig and Toro is valid as stated in the Introduction. That is, under the assumptions of Theorem A, one deduces that $\vec{n} \in VMO(\sigma)$, instead of the weaker statement $\vec{n} \in VMO_{loc}(\sigma)$ proven in [Kenig and Toro 2003].

10. Counterexample for $\mathbb{R}^d, d \geq 4$

In this section we show that, for all $d \geq 4$, there exists a two-sided chord-arc unbounded domain $\Omega \subset \mathbb{R}^d$ for which the Poisson kernel with pole at infinity is constant and such that the outer unit normal is not in $VMO(\sigma)$. Indeed, Hong [2015, Example 1] constructed $u \in C(\mathbb{R}^d)$ such that $u \geq 0; u(rx) = ru(x)$,

$r > 0$; $\Delta u = 0$ in $\Gamma = \{u > 0\}$; $\partial\Gamma \setminus \{0\}$ is smooth; $\partial u / \partial \vec{n} = -1$, where \vec{n} is the outward unit normal on Γ and u is singular, i.e., $u \notin x_1^+$ (modulo rotations). We describe his example in some detail below.

Since u is homogeneous of degree 1, it is determined by its values on the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. Further, u solves the following overdetermined first eigenvalue problem on \mathbb{S}^{d-1} for $d = 4$:

$$\begin{cases} \Delta_{\mathbb{S}^{d-1}} u + (d - 1)u = 0 & \text{and } u > 0 & \text{in } \Omega := \Gamma \cap \mathbb{S}^{d-1}, \\ \frac{\partial u}{\partial \vec{n}} = -1 & \text{and } u = 0 & \text{in } \partial\Omega := \partial\Gamma \cap \mathbb{S}^{d-1}, \\ u \equiv 0 & & \text{in } \Omega^c. \end{cases} \tag{10-1}$$

To be more precise, let us consider in $\mathbb{S}^3 \subset \mathbb{R}^4$ the coordinates

$$\begin{aligned} x_1 &= \cos \theta \cos \phi, & x_2 &= \cos \theta \sin \phi, \\ x_3 &= \sin \theta \cos \psi, & x_4 &= \sin \theta \sin \psi, \end{aligned} \tag{10-2}$$

where $\theta \in [0, \pi/2]$ and $\phi, \psi \in [0, 2\pi]$. Let $u(\theta, \phi, \psi) = \tau f(\theta)$, where $\tau > 0$ and f is a sufficiently nice function. To find u that satisfies (10-1), it is enough to solve the ODE

$$\begin{cases} (\sin \theta \cos \theta f')' + \sin \theta \cos \theta f = 0, & \theta \in (0, \pi/2), \\ f(0) = 1, f'(\pi/2) = 0. \end{cases}$$

Then it is shown in [Hong 2015] that there exists $\theta_0 \in (0, \pi/2)$ such that $f(\theta_0) = 0$, $f'(\theta_0) < 0$ and $f'(\theta) > 0$ for all $\theta \in (0, \theta_0)$. If u is defined on \mathbb{S}^3 by $u(\theta, \phi, \psi) = (-1/f'(\theta_0))f(\theta)$ for all $\theta \in [0, \theta_0]$ and $u \equiv 0$ in $[\theta_0, \pi/2]$, then $v(x) = v(r\xi) = ru(\xi)$, for $r > 0$ and $\xi \in \mathbb{S}^3$, is the solution to the one-phase free boundary problem we are after.

The above-mentioned construction provides us with a domain for which Theorem 1.1 does not hold. Indeed, let

$$\Omega := \{x \in \mathbb{R}^4 : x = r\xi \text{ for some } \xi \in \mathbb{S}^3 \text{ satisfying (10-2) for } \theta \in [0, \theta_0]\} = \{v > 0\},$$

whose boundary is given by all points $x \in \mathbb{R}^4$ so that $x = r\xi$ for some $r > 0$ and $\xi \in \mathbb{S}^3$ that satisfies (10-2) for $\theta = \theta_0$. Remark here that as v is a homogeneous, degree-1 function and $v \notin x_1^+$ (under rotation), Ω is a cone in \mathbb{R}^4 but not a half-space. Thus, Ω is not a Reifenberg flat domain with vanishing constant, which infers that the outward unit normal \vec{n} is not in $\text{VMO}(\partial\Omega)$. Moreover, as the Poisson kernel $h = -\partial u / \partial \vec{n} = 1$, it is clear that $\log h \in \text{VMO}$. Therefore, it is enough to show that Ω is a two-sided chord-arc domain.

To this end, notice that every $x \in \partial\Omega$ satisfies the equation $x_1^2 + x_2^2 = \cos^2 \theta_0$, $x_3^2 + x_4^2 = \sin^2 \theta_0$ while, for $x \in \Omega$,

$$x_1^2 + x_2^2 = \cos^2 \theta > \cos^2 \theta_0 \quad \text{and} \quad x_3^2 + x_4^2 = \sin^2 \theta < \sin^2 \theta_0.$$

So Ω coincides with the set of those points $x \in \mathbb{R}^4$ such that

$$x_1^2 + x_2^2 > (x_3^2 + x_4^2) \cot^2 \theta_0. \tag{10-3}$$

Therefore, Ω is bi-Lipschitz equivalent to the domain $\{x \in \mathbb{R}^4 : x_1^2 + x_2^2 > x_3^2 + x_4^2\}$, which is a well-known two-sided chord-arc domain. The AD-regularity is easier to see as the boundary is locally a Lipschitz

graph away from the origin by the implicit function theorem, so it is locally AD-regular, and the fact that it is a cone easily gives that it is globally Ahlfors regular. Hence, Ω is also a two-sided chord-arc domain, which finishes our proof in \mathbb{R}^4 .

If we set $D := \Omega \otimes \mathbb{R}^{d-4} \subset \mathbb{R}^d$, where $\Omega \subset \mathbb{R}^4$ is the domain just constructed, then D is a two-sided chord-arc domain in \mathbb{R}^d for which the Poisson kernel is constant and such that the outer unit normal is not in $VMO(\sigma)$.

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FOCUSING QUANTUM MANY-BODY DYNAMICS, II: THE RIGOROUS DERIVATION OF THE 1D FOCUSING CUBIC NONLINEAR SCHRÖDINGER EQUATION FROM 3D

XUWEN CHEN AND JUSTIN HOLMER

We consider the focusing 3D quantum many-body dynamic which models a dilute Bose gas strongly confined in two spatial directions. We assume that the microscopic pair interaction is attractive and given by $a^{3\beta-1}V(a^\beta \cdot)$, where $\int V \leq 0$ and a matches the Gross–Pitaevskii scaling condition. We carefully examine the effects of the fine interplay between the strength of the confining potential and the number of particles on the 3D N -body dynamic. We overcome the difficulties generated by the attractive interaction in 3D and establish new focusing energy estimates. We study the corresponding BBGKY hierarchy, which contains a diverging coefficient as the strength of the confining potential tends to ∞ . We prove that the limiting structure of the density matrices counterbalances this diverging coefficient. We establish the convergence of the BBGKY sequence and hence the propagation of chaos for the focusing quantum many-body system. We derive rigorously the 1D focusing cubic NLS as the mean-field limit of this 3D focusing quantum many-body dynamic and obtain the exact 3D-to-1D coupling constant.

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1. Introduction

Since the experimental achievement of Bose–Einstein condensates (BEC) was reported in [Anderson et al. 1995; Davis et al. 1995] — a feat for which Cornell, Wieman and Ketterle won the 2001 Nobel Prize in Physics — the investigation of this new state of matter has become one of the most active areas of contemporary research. A BEC, first predicted theoretically by Einstein for noninteracting particles in 1925, is a peculiar gaseous state at which particles of integer spin (bosons) occupy a macroscopic quantum state.

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Let $t \in \mathbb{R}$ be the time variable and $\mathbf{r}_N = (r_1, r_2, \dots, r_N) \in \mathbb{R}^{nN}$ be the position vector of N particles in \mathbb{R}^n . Then, naively, BEC means that, up to a phase factor solely depending on t , the N -body wave function $\psi_N(t, \mathbf{r}_N)$ satisfies

$$\psi_N(t, \mathbf{r}_N) \sim \prod_{j=1}^N \varphi(t, r_j)$$

for some one-particle state φ . That is, every particle takes the same quantum state. Equivalently, there is the Penrose–Onsager formulation of BEC: if we let $\gamma_N^{(k)}$ be the k -particle marginal densities associated with ψ_N by

$$\gamma_N^{(k)}(t, \mathbf{r}_k; \mathbf{r}'_k) = \int \psi_N(t, \mathbf{r}_k, \mathbf{r}_{N-k}) \bar{\psi}_N(t, \mathbf{r}'_k, \mathbf{r}_{N-k}) d\mathbf{r}_{N-k}, \quad \mathbf{r}_k, \mathbf{r}'_k \in \mathbb{R}^{nk}, \quad (1)$$

then BEC equivalently means

$$\gamma_N^{(k)}(t, \mathbf{r}_k; \mathbf{r}'_k) \sim \prod_{j=1}^k \varphi(t, r_j) \bar{\varphi}(t, r'_j). \quad (2)$$

It is widely believed that the cubic nonlinear Schrödinger equation (NLS)

$$i \partial_t \phi = L\phi + \mu |\phi|^2 \phi,$$

where L is the Laplacian $-\Delta$ or the Hermite operator $-\Delta + \omega^2 |x|^2$, fully describes the one-particle state φ in (2), also called the condensate wave function since it characterizes the whole condensate. Such a belief is one of the main motivations for studying the cubic NLS. Here, the nonlinear term $\mu |\phi|^2 \phi$ represents a strong on-site interaction taken as a mean-field approximation of the pair interactions between the particles: a repelling interaction gives a positive μ , while an attractive interaction yields a $\mu < 0$. Gross and Pitaevskii proposed such a description of the many-body effect. Thus the cubic NLS is also called the Gross–Pitaevskii equation. Because the cubic NLS is a phenomenological equation of mean-field type, naturally, its validity has to be established rigorously from the many-body system which it is supposed to characterize.

In a series of works [Lieb et al. 2005; Adami et al. 2007; Elgart et al. 2006; Erdős et al. 2006; 2007; 2009; 2010; T. Chen and Pavlović 2011; 2014; X. Chen 2012a; 2013; Benedikter et al. 2015; X. Chen and Holmer 2013; Grillakis and Machedon 2013; Sohinger 2015], it has been proven rigorously that, for a repelling interaction potential with suitable assumptions, relation (2) holds; moreover, the one-particle state φ solves the defocusing cubic NLS ($\mu > 0$).

It is then natural to ask if BEC happens (whether relation (2) holds) when we have attractive interparticle interactions and if the condensate wave function φ satisfies a focusing cubic NLS ($\mu < 0$) if relation (2) does hold. In contemporary experiments, both positive [Khaykovich et al. 2002; Strecker et al. 2002] and negative [Cornish et al. 2000; Donley et al. 2001] results exist. To present the mathematical interpretations of the experiments, we adopt the notation

$$r_i = (x_i, z_i) \in \mathbb{R}^{2+1}$$

and investigate the procedure of laboratory experiments of BEC subject to attractive interactions according to [Cornish et al. 2000; Donley et al. 2001; Khaykovich et al. 2002; Strecker et al. 2002].

Step A. Confine a large number of bosons, whose interactions are originally *repelling*, inside a trap. Reduce the temperature of the system so that the many-body system reaches its ground state. It is expected that this ground state is a BEC state/factorized state. This step then corresponds to the following mathematical problem:

Problem 1. Show that if $\psi_{N,0}$ is the ground state of the N -body Hamiltonian $H_{N,0}$ defined by

$$H_{N,0} = \sum_{j=1}^N (-\Delta_{r_j} + \omega_{0,x}^2 |x_j|^2 + \omega_{0,z}^2 z_j^2) + \sum_{1 \leq i < j \leq N} \frac{1}{a^{3\beta-1}} V_0 \left(\frac{r_i - r_j}{a\beta} \right), \quad (3)$$

where $V_0 \geq 0$, then the marginal densities $\{\gamma_{N,0}^{(k)}\}$ associated with $\psi_{N,0}$, defined in (1), satisfy relation (2).

Here, the quadratic potential $\omega^2 |\cdot|^2$ stands for the trapping since [Cornish et al. 2000; Donley et al. 2001; Khaykovich et al. 2002; Strecker et al. 2002] and many other experiments of BEC use the harmonic trap and measure the strength of the trap with ω . We use $\omega_{0,x}$ to denote the trapping strength in the x -direction and $\omega_{0,z}$ to denote the trapping strength in the z -direction, as we will explain later that in order to have a BEC with attractive interaction, either experimentally or mathematically, it is important to have $\omega_{0,x} \neq \omega_{0,z}$. Moreover, we define

$$\frac{1}{a} V_{0,a}(r) = \frac{1}{a^{3\beta-1}} V_0 \left(\frac{r}{a\beta} \right), \quad \beta > 0,$$

to be the interaction potential.¹ On the one hand, $V_{0,a}$ is an approximation of the identity as $a \rightarrow 0$ and hence matches the Gross–Pitaevskii description that the many-body effect should be modeled by an on-site strong self-interaction. On the other hand, the extra $1/a$ is to make sure that the Gross–Pitaevskii scaling condition is satisfied. This step is exactly the same as the preparation of the experiments with repelling interactions, and satisfactory answers to Problem 1 have been given in [Lieb et al. 2004].

Step B. Use the property of Feshbach resonance, strengthen the trap (increase $\omega_{0,x}$ or $\omega_{0,z}$) to make the interaction attractive and observe the evolution of the many-body system. This technique continuously controls the sign and the size of the interaction in a certain range.² The system is then time-dependent. In order to observe BEC, the factorized structure obtained in Step A must be preserved in time. Assuming this to be the case, we then reset the time so that $t = 0$ represents the point at which this Feshbach-resonance phase is complete. The subsequent evolution should then be governed by a focusing time-dependent N -body Schrödinger equation with an attractive-pair interaction V subject to an asymptotically factorized initial datum. The confining strengths are different from Step A as well and we denote them by ω_x and ω_z . A mathematically precise statement is the following:

¹ From here on, we consider the $\beta > 0$ case solely. For $\beta = 0$ (the Hartree dynamic), see [Fröhlich et al. 2009; Erdős and Yau 2001; Knowles and Pickl 2010; Rodnianski and Schlein 2009; Michelangeli and Schlein 2012; Grillakis et al. 2010; 2011; X. Chen 2012b; Ammari and Nier 2011; 2008; L. Chen et al. 2011].

² See [Cornish et al. 2000, Figure 1; Khaykovich et al. 2002, Figure 2; Strecker et al. 2002, Figure 1] for graphs of the relationship between ω and V .

Problem 2. Let $\psi_N(t, \mathbf{x}_N)$ be the solution to the N -body Schrödinger equation

$$i \partial_t \psi_N = \sum_{j=1}^N (-\Delta_{r_j} + \omega_x^2 |x_j|^2 + \omega_z^2 z_j^2) \psi_N + \sum_{1 \leq i < j \leq N} \frac{1}{a^{3\beta-1}} V \left(\frac{r_i - r_j}{a^\beta} \right) \psi_N, \quad (4)$$

where $V \leq 0$, with $\psi_{N,0}$ from Step A as initial datum. Prove that the marginal densities $\{\gamma_N^{(k)}(t)\}$ associated with $\psi_N(t, \mathbf{x}_N)$ satisfy relation (2).³

In the experiment [Cornish et al. 2000] by Cornell and Wieman's group (the JILA group), once the interaction is turned attractive, the condensate suddenly shrinks to below the resolution limit; then after ~ 5 ms, the many-body system blows up. That is, there is no BEC once the interaction becomes attractive. Moreover, there is no condensate wave function due to the absence of the condensate. Hence, the current NLS theory, which is about the condensate wave function when there is a condensate, cannot explain this 5 ms of time or the blow up. This is currently an open problem in the study of quantum many-body systems. The JILA group later conducted finer experiments and remarked in [Donley et al. 2001, p. 299] that these are simple systems with dramatic behavior, and this behavior provides puzzling results when mean-field theory is tested against them.

In [Khaykovich et al. 2002; Strecker et al. 2002], the particles are confined in a strongly anisotropic cigar-shape trap to simulate a 1D system. That is, $\omega_x \gg \omega_z$. In this case, the experiment is a success in the sense that one obtains a persistent BEC after the interaction is switched to attractive. Moreover, a soliton is observed in [Khaykovich et al. 2002] and a soliton train is observed in [Strecker et al. 2002]. The solitons in these two works have different motion patterns.

In [X. Chen and Holmer 2016b], we have studied the simplified 1D version of (4) as a model case and derived the 1D focusing cubic NLS from it. In the present paper, we consider the full 3D problem of (4), as in the experiments [Khaykovich et al. 2002; Strecker et al. 2002]: We take $\omega_z = 0$ and let $\omega_x \rightarrow \infty$ in (4). We derive rigorously the 1D cubic focusing NLS directly from a real 3D quantum many-body system. Here, "directly" means that we are not passing through any 3D cubic NLS. On the one hand, one infers from the experiment [Cornish et al. 2000] that not only it is very difficult to prove the 3D focusing NLS as the mean-field limit of a 3D focusing quantum many-body dynamic, but such a limit also may not be true. On the other hand, the route which first derives

$$i \partial_t \varphi = -\Delta_x + \omega^2 |x|^2 \varphi - \partial_z^2 \varphi - |\varphi|^2 \varphi \quad (5)$$

as an $N \rightarrow \infty$ limit, from the 3D N -body dynamic, and then considers the $\omega \rightarrow \infty$ limit of (5), corresponds to the iterated limit ($\lim_{\omega \rightarrow \infty} \lim_{N \rightarrow \infty}$) of the N -body dynamic; i.e., the 1D focusing cubic NLS coming from such a path approximates the 3D focusing N -body dynamic when ω is large and N is infinity (if not substantially larger than ω). In experiments, it is fully possible to have N and ω comparable to each other. In fact, N is about 10^4 and ω is about 10^3 in [Görlitz et al. 2001; Stock et al. 2005; Hadzibabic et al. 2006; Desbuquois et al. 2012]. Moreover, as seen in the experiment [Donley et al. 2001], even if ω_x is one digit larger than ω_z , negative result persists if N is three digits larger than ω_x . Thus, in this

³ Since $\omega \neq \omega_0$, $V \neq V_0$, one could not expect that $\psi_{N,0}$, the ground state of (3), is close to the ground state of (4).

paper, we derive rigorously the 1D focusing cubic NLS as the double limit ($\lim_{N,\omega \rightarrow \infty}$) of a real focusing 3D quantum N -body dynamic directly, without passing through any 3D cubic NLS. Furthermore, the interaction between the two parameters N and ω plays a central role. To be specific, we establish the following theorem.

Theorem 1.1 (main theorem). *Assume that the pair interaction V is an even Schwartz class function which has a nonpositive integral, i.e., $\int_{\mathbb{R}^3} V(r) dr \leq 0$, but may not be negative everywhere. Let $\psi_{N,\omega}(t, \mathbf{r}_N)$ be the N -body Hamiltonian evolution $e^{itH_{N,\omega}}\psi_{N,\omega}(0)$ with the focusing N -body Hamiltonian $H_{N,\omega}$ given by*

$$H_{N,\omega} = \sum_{j=1}^N (-\Delta_{r_j} + \omega^2|x_j|^2) + \sum_{1 \leq i < j \leq N} (N\omega)^{3\beta-1} V((N\omega)^\beta(r_i - r_j)) \tag{6}$$

for some $\beta \in (0, \frac{3}{7})$. Let $\{\gamma_{N,\omega}^{(k)}\}$ be the family of marginal densities associated with $\psi_{N,\omega}$. Suppose that the initial datum $\psi_{N,\omega}(0)$ verifies the following conditions:

- (a) $\psi_{N,\omega}(0)$ is normalized; that is, $\|\psi_{N,\omega}(0)\|_{L^2} = 1$,
- (b) $\psi_{N,\omega}(0)$ is asymptotically factorized in the sense that

$$\lim_{N,\omega \rightarrow \infty} \text{Tr} \left| \frac{1}{\omega} \gamma_{N,\omega}^{(1)} \left(0, \frac{x_1}{\sqrt{\omega}}, z_1; \frac{x'_1}{\sqrt{\omega}}, z'_1 \right) - h(x_1)h(x'_1)\phi_0(z_1)\bar{\phi}_0(z'_1) \right| = 0 \tag{7}$$

for some one-particle state $\phi_0 \in H^1(\mathbb{R})$ and h is the normalized ground state for the 2D Hermite operator $-\Delta_x + |x|^2$, i.e., $h(x) = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}|x|^2}$.

- (c) Away from the x -directional ground-state energy, $\psi_{N,\omega}(0)$ has finite energy per particle:

$$\sup_{\omega,N} \frac{1}{N} \langle \psi_{N,\omega}(0), (H_{N,\omega} - 2N\omega)\psi_{N,\omega}(0) \rangle \leq C.$$

Then there exist C_1 and C_2 which depend solely on V such that $\forall k \geq 1, t \geq 0$, and $\varepsilon > 0$, we have the convergence in trace norm (propagation of chaos)

$$\lim_{\substack{N,\omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \text{Tr} \left| \frac{1}{\omega^k} \gamma_{N,\omega}^{(k)} \left(t, \frac{\mathbf{x}_k}{\sqrt{\omega}}, \mathbf{z}_k; \frac{\mathbf{x}'_k}{\sqrt{\omega}}, \mathbf{z}'_k \right) - \prod_{j=1}^k h(x_j)h(x'_j)\phi(t, z_j)\bar{\phi}(t, z'_j) \right| = 0, \tag{8}$$

where $v_1(\beta)$ and $v_2(\beta)$ are defined by

$$v_1(\beta) = \frac{\beta}{1-\beta}, \tag{9}$$

$$v_2(\beta) = \min \left(\frac{1-\beta}{\beta}, \frac{\frac{3}{5}-\beta}{\beta-\frac{1}{5}} \mathbf{1}_{\beta \geq \frac{1}{5}} + \infty \cdot \mathbf{1}_{\beta < \frac{1}{5}}, \frac{2\beta}{1-2\beta}, \frac{\frac{7}{8}-\beta}{\beta} \right) \tag{10}$$

(see Figure 1) and $\phi(t, z)$ solves the 1D focusing cubic NLS with the 3D-to-1D coupling constant $b_0(\int |h(x)|^4 dx)$, that is,

$$i \partial_t \phi = -\partial_z^2 \phi - b_0 \left(\int |h(x)|^4 dx \right) |\phi|^2 \phi \quad \text{in } \mathbb{R} \tag{11}$$

with initial condition $\phi(0, z) = \phi_0(z)$ and $b_0 = |\int V(r) dr|$.

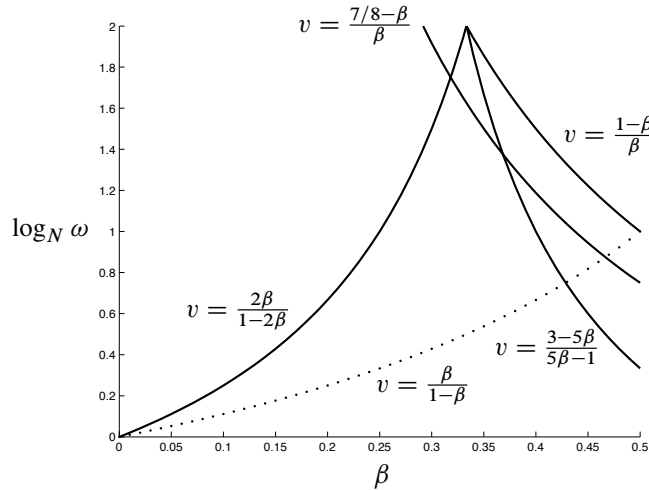


Figure 1. A graph of the various rational functions of β appearing in (9) and (10). In Theorems 1.1 and 1.2, the limit $(N, \omega) \rightarrow \infty$ is taken with $v_1(\beta) \leq \log_N \omega \leq v_2(\beta)$. The region of validity is above the dashed curve and below the solid curves. It is a nonempty region for $0 < \beta \leq \frac{3}{7}$. As shown here, there are values of β for which $v_1(\beta) \leq 1 \leq v_2(\beta)$, which allows $N \sim \omega$, as in [Cornish et al. 2000; Donley et al. 2001; Khaykovich et al. 2002; Strecker et al. 2002; Görlitz et al. 2001; Stock et al. 2005; Hadzibabic et al. 2006; Desbuquois et al. 2012]. Moreover, our result includes part of the $\beta > \frac{1}{3}$ self-interaction region. We will explain why we call the $\beta > \frac{1}{3}$ case self-interaction later in Introduction. We remark that it is not a coincidence that three restrictions intersect at $\beta = \frac{1}{3}$.

Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2 (main theorem). *Assume that the pair interaction V is an even Schwartz class function which has a nonpositive integral, i.e., $\int_{\mathbb{R}^3} V(r) dr \leq 0$, but may not be negative everywhere. Let $\psi_{N,\omega}(t, \mathbf{r}_N)$ be the N -body Hamiltonian evolution $e^{itH_{N,\omega}} \psi_{N,\omega}(0)$, where the focusing N -body Hamiltonian $H_{N,\omega}$ is given by (6) for some $\beta \in (0, \frac{3}{7})$. Let $\{\gamma_{N,\omega}^{(k)}\}$ be the family of marginal densities associated with $\psi_{N,\omega}$. Suppose that the initial datum $\psi_{N,\omega}(0)$ is normalized, asymptotically factorized in the sense of (a) and (b) of Theorem 1.1 and satisfies the energy condition that*

(c') *there is a $C > 0$ such that*

$$\langle \psi_{N,\omega}(0), (H_{N,\omega} - 2N\omega)^k \psi_{N,\omega}(0) \rangle \leq C^k N^k, \quad \forall k \geq 1. \tag{12}$$

Then there exist C_1, C_2 which depend solely on V such that $\forall k \geq 1, \forall t \geq 0$, we have the convergence in trace norm (propagation of chaos)

$$\lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \text{Tr} \left| \frac{1}{\omega^k} \gamma_{N,\omega}^{(k)} \left(t, \frac{\mathbf{x}_k}{\sqrt{\omega}}, \mathbf{z}_k; \frac{\mathbf{x}'_k}{\sqrt{\omega}}, \mathbf{z}'_k \right) - \prod_{j=1}^k h(x_j) h(x'_j) \phi(t, z_j) \bar{\phi}(t, z'_j) \right| = 0,$$

where $v_1(\beta)$ and $v_2(\beta)$ are given by (9) and (10) and $\phi(t, z)$ solves the 1D focusing cubic NLS (11).

We remark that the assumptions in Theorem 1.1 are reasonable assumptions on the initial datum coming from Step A. In [Lieb et al. 2004, (1.10)], a satisfying answer has been found by Lieb, Seiringer, and Yngvason for Step A (Problem 1) in the $\omega_{0,x} \gg \omega_{0,z}$ case. For convenience, set $\omega_{0,z} = 1$ in the defocusing N -body Hamiltonian (3) in Step A. Let $\text{scat}(W)$ denote the 3D scattering length of the potential W . By [Erdős et al. 2007, Lemma A.1], for $0 < \beta \leq 1$ and $a \ll 1$, we have

$$\text{scat}\left(a \cdot \frac{1}{a^{3\beta}} V\left(\frac{r}{a^\beta}\right)\right) \sim \begin{cases} a/(8\pi) \int_{\mathbb{R}^3} V & \text{if } 0 < \beta < 1, \\ a \text{scat}(V) & \text{if } \beta = 1. \end{cases}$$

Lieb et al. [2004, (1.10)] define the quantity $g = g(\omega_{0,x}, N, a)$ by

$$g := 8\pi a \omega_{0,x} \left(\int |h(x)|^4 dx \right).$$

Then if $Ng \sim 1$, they proved in Theorem 5.1 of the same work that BEC happens in Step A and the Gross–Pitaevskii limit holds.⁴ To be specific, they proved that

$$\lim_{N, \omega_{0,x} \rightarrow \infty} \text{Tr} \left| \frac{1}{\omega_{0,x}} \gamma_{N, \omega_{0,x}}^{(1)} \left(0, \frac{x_1}{\sqrt{\omega_{0,x}}}, z_1; \frac{x'_1}{\sqrt{\omega_{0,x}}}, z'_1 \right) - h(x_1)h(x'_1)\phi_0(z_1)\bar{\phi}_0(z'_1) \right| = 0$$

provided that ϕ_0 is the minimizer to the 1D defocusing NLS energy functional

$$E_{Ng} = \int_{\mathbb{R}} (|\partial_z \phi(z)|^2 + z^2 |\phi(z)|^2 + 4\pi Ng |\phi(z)|^4) dz \tag{13}$$

subject to the constraint $\|\phi\|_{L^2(\mathbb{R})} = 1$. Hence, the assumptions in Theorem 1.1 are reasonable assumptions on the initial datum drawn from Step A. To be specific, we have chosen $a = (N\omega)^{-1}$ in the interaction so that $Ng \sim 1$ and assumptions (a), (b) and (c) are the conclusions of [Lieb et al. 2004, Theorem 5.1].⁵

The equivalence of Theorems 1.1 and 1.2 for asymptotically factorized initial data is well known. In the main part of this paper, we prove Theorem 1.2 in full detail. For completeness, we discuss briefly how to deduce Theorem 1.1 from Theorem 1.2 in Appendix B.

To our knowledge, Theorems 1.1 and 1.2 offer the first rigorous derivation of the 1D focusing cubic NLS (11) from the 3D focusing quantum N -body dynamic (6). Moreover, our result covers part of the $\beta > \frac{1}{3}$ self-interaction region in 3D. As pointed out in [Elgart et al. 2006], the study of Step B is of particular interest when $\beta \in (\frac{1}{3}, 1]$ in 3D. The reason is the following. The initial datum coming from Step A is the ground state of (3) with $\omega_{0,x}, \omega_{0,z} \neq 0$ and hence is localized in space. We can assume all N particles are in a box of length 1. Let the effective radius of the pair interaction V be R_0 , then the effective radius of $V((N\omega)^\beta (r_i - r_j))$ is about $R_0/(N\omega)^\beta$. Thus every particle in the box interacts with $(R_0/(N\omega)^\beta)^3 \times N$ other particles. Thus, for $\beta > \frac{1}{3}$ and large N , every particle interacts with only itself. This exactly matches the Gross–Pitaevskii theory that the many-body effect should be modeled

⁴ This corresponds to Region 2 of [Lieb et al. 2004]. The other four regions are the ideal gas case, the 1D Thomas–Fermi case, the Lieb–Liniger case, and the Girardeau–Tonks case. As mentioned on page 388 of that work, BEC is not expected in the Lieb–Liniger and the Girardeau–Tonks cases, and is an open problem in the Thomas–Fermi case; we deal only with Region 2 in this paper.

⁵ We remark that the interaction potential $N^{3\beta-1}\omega^{3\beta}V((N\omega)^\beta(r_i - r_j))$, which looks like a “direct” extension of the interaction potential from the nD -to- nD work, does not satisfy $Ng \sim 1$ in the $N, \omega \rightarrow \infty$ process.

by a strong on-site self-interaction. Therefore, for the mathematical justification of the Gross–Pitaevskii theory, it is of particular interest to prove Theorems 1.1 and 1.2 for self-interaction ($\beta > \frac{1}{3}$).

A main tool used to prove Theorem 1.2 is the analysis of the BBGKY hierarchy of

$$\left\{ \tilde{\gamma}_{N,\omega}^{(k)}(t) = \frac{1}{\omega^k} \gamma_{N,\omega}^{(k)} \left(t, \frac{\mathbf{x}_k}{\sqrt{\omega}}, \mathbf{z}_k; \frac{\mathbf{x}'_k}{\sqrt{\omega}}, \mathbf{z}'_k \right) \right\}_{k=1}^N$$

as $N, \omega \rightarrow \infty$. In the classical setting, deriving equations of mean-field type by studying the limit of the BBGKY hierarchy was proposed by Kac and demonstrated by Lanford’s work on the Boltzmann equation. In the quantum setting, the usage of the BBGKY hierarchy was suggested by Spohn [1980] and was proven successful by Elgart, Erdős, Schlein, and Yau in their fundamental papers [Elgart et al. 2006; Erdős et al. 2006; 2007; 2009; 2010],⁶ which rigorously derive the 3D cubic defocusing NLS from a 3D quantum many-body dynamic with repulsive-pair interactions and no trapping. The Elgart–Erdős–Schlein–Yau program⁷ consists of two principal parts: in one part, they consider the sequence of the marginal densities $\{\gamma_N^{(k)}\}$ associated with the Hamiltonian evolution $e^{itH_N} \psi_N(0)$, where

$$H_N = \sum_{j=1}^N -\Delta_{r_j} + \frac{1}{N} \sum_{1 \leq i < j \leq N} N^{3\beta} V(N^\beta(r_i - r_j)),$$

and prove that an appropriate limit, as $N \rightarrow \infty$, solves the 3D Gross–Pitaevskii hierarchy

$$i \partial_t \gamma^{(k)} + \sum_{j=1}^k [\Delta_{r_j}, \gamma^{(k)}] = b_0 \sum_{j=1}^k \text{Tr}_{r_{k+1}} [\delta(r_j - r_{k+1}), \gamma^{(k+1)}] \quad \text{for all } k \geq 1. \tag{14}$$

In another part, they show that hierarchy (14) has a unique solution which is therefore a completely factorized state. However, the uniqueness theory for hierarchy (14) is surprisingly delicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. By assuming a space-time bound on the limit of $\{\gamma_N^{(k)}\}$, Klainerman and Machedon [2008] gave another uniqueness theorem regarding (14) through a collapsing estimate originating from the multilinear Strichartz estimates and a board game argument inspired by the Feynman graph argument in [Erdős et al. 2007].

The method by Klainerman and Machedon [2008] was taken up by Kirkpatrick, Schlein, and Staffilani [Kirkpatrick et al. 2011], who derived the 2D cubic defocusing NLS from the 2D quantum many-body dynamic; by T. Chen and Pavlović [2011], who considered the 1D and 2D three-body repelling interaction problem; by X. Chen [2012a; 2013], who investigated the defocusing problem with trapping in 2D and 3D; and by X. Chen and J. Homer [2013], who proved the effectiveness of the defocusing 3D to 2D reduction problem. Such a method has also inspired the study of the general existence theory of hierarchy (14); see [T. Chen et al. 2010; 2012; T. Chen and Pavlović 2010; Gressman et al. 2014; Sohinger and Staffilani 2015].

One main open problem in the uniqueness theory of Klainerman–Machedon type is the verification of the uniqueness condition in 3D, though it is fully solved in 1D and 2D using trace theorems in [Kirkpatrick et al. 2011]. For the 3D defocusing problem without traps, T. Chen and Pavlović [2014] showed that,

⁶ Around the same time, there was the 1D defocusing work [Adami et al. 2007].

⁷ See [Benedikter et al. 2015; Grillakis and Machedon 2013; Pickl 2011] for different approaches.

for $\beta \in (0, \frac{1}{4})$, the limit of the BBGKY sequence satisfies the uniqueness condition.⁸ X. Chen [2013] extended and simplified their method to study the 3D trapping problem for $\beta \in (0, \frac{2}{7}]$. The $\beta \in (0, \frac{2}{7}]$ result by X. Chen was then extended to $\beta \in (0, \frac{2}{3})$ using X_b spaces and Littlewood–Paley theory in [X. Chen and Holmer 2016c] and further to $\beta < 1$ in [X. Chen and Holmer 2016a] via correlation structures and many-body scattering process. The $\beta = 1$ case is still open.

Using a version of the quantum de Finetti theorem from [Lewin et al. 2014], T. Chen, Hainzl, Pavlović, and Seiringer provided an alternative proof to the uniqueness theorem in [Erdős et al. 2007] and showed that it is an unconditional uniqueness result in the sense of NLS theory. With this method, Sohinger [2015] derived the 3D defocusing cubic NLS in the periodic case. See also [X. Chen and Smith 2014; Hong et al. 2015].

Recently, the first step in the mass critical focusing case has been taken in [X. Chen and Holmer 2016d].

Organization of the paper. We first outline the proof of our main theorem, Theorem 1.2, in Section 2. The components of the proof are in Sections 3, 4, and 5.

The first main part is the proof of the needed focusing energy estimate, stated and proved as Theorem 3.1 in Section 3. The main difficulty in establishing the energy estimate is understanding the interplay between two parameters N and ω . On the one hand, as suggested by the experiments [Cornish et al. 2000; Donley et al. 2001; Khaykovich et al. 2002; Strecker et al. 2002], in order to have to a tensor product state (BEC) in this focusing setting, one has to explore “the 1D feature” of the 3D focusing N -body Hamiltonian (6), which comes from a large ω . At the same time, an N too large would allow the 3D effect to dominate, and one has to avoid this. This suggests that an inequality of the form $N^{v_1(\beta)} \leq \omega$ is a natural requirement. On the other hand, according to the uncertainty principle, in 3D, as the x -component of the particles’ position becomes more and more determined to be 0, the x -component of the momentum and thus the energy must blow up. Hence the energy of the system is dominated by its x -directional part, which is in fact infinity as $\omega \rightarrow \infty$. Since the particles are interacting via 3D potential, to avoid the excessive x -directional energy being transferred to the z -direction, during the $N, \omega \rightarrow \infty$ process, ω cannot be too large either. Such a problem is totally new and does not exist in the 1D model [X. Chen and Holmer 2016b]. It suggests that an inequality of the form $\omega \leq N^{v_2(\beta)}$ is a natural requirement.

The second main part of the proof is the analysis of the focusing “ $\infty - \infty$ ” BBGKY hierarchy of

$$\left\{ \tilde{\gamma}_{N,\omega}^{(k)}(t) = \frac{1}{\omega^k} \gamma_{N,\omega}^{(k)} \left(t, \frac{\mathbf{x}_k}{\sqrt{\omega}}, \mathbf{z}_k; \frac{\mathbf{x}'_k}{\sqrt{\omega}}, \mathbf{z}'_k \right) \right\}_{k=1}^N$$

as $N, \omega \rightarrow \infty$. With our definition, the sequence of the marginal densities $\{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N$ satisfies the BBGKY hierarchy

$$\begin{aligned} i \partial_t \tilde{\gamma}_{N,\omega}^{(k)} &= \omega \sum_{j=1}^k [-\Delta_{x_j} + |x_j|^2, \tilde{\gamma}_{N,\omega}^{(k)}] + \sum_{j=1}^k [-\partial_{z_j}^2, \tilde{\gamma}_{N,\omega}^{(k)}] \\ &\quad + \frac{1}{N} \sum_{1 \leq i < j \leq k} [V_{N,\omega}(r_i - r_j), \tilde{\gamma}_{N,\omega}^{(k)}] + \frac{N-k}{N} \sum_{j=1}^k \text{Tr}_{r_{k+1}} [V_{N,\omega}(r_j - r_{k+1}), \tilde{\gamma}_{N,\omega}^{(k+1)}], \end{aligned}$$

⁸ See also [T. Chen and Taliaferro 2014].

where $V_{N,\omega}$ is defined in (17). We call it an “ $\infty - \infty$ ” BBGKY hierarchy because it is not clear whether the term

$$\omega[-\Delta_{x_j} + |x_j|^2, \tilde{\gamma}_{N,\omega}^{(k)}]$$

tends to a limit as $N, \omega \rightarrow \infty$. Since $\tilde{\gamma}_{N,\omega}^{(k)}$ is not a factorized state for $t > 0$, one cannot expect the commutator to be zero; though it is zero if $\tilde{\gamma}_{N,\omega}^{(k)}$ is exactly the limit in (8). This is in strong contrast with the nD -to- nD work⁹ [Adami et al. 2007; Elgart et al. 2006; Erdős et al. 2006; 2007; 2009; 2010; T. Chen and Pavlović 2011; 2014; X. Chen 2012a; 2013; Sohinger 2015] in which the formal limit of the corresponding BBGKY hierarchy is clear. With the aforementioned focusing energy estimate, we find that this diverging coefficient is counterbalanced by the limiting structure of the density matrices and establish the weak* compactness and convergence of this focusing BBGKY hierarchy in Sections 4 and 5.

2. Proof of the main theorem

We start by setting up some notation for the rest of the paper. Recall $h(x) = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}|x|^2}$, which is the ground state for the 2D Hermite operator $-\Delta_x + |x|^2$; i.e., it solves $(-2 - \Delta_x + |x|^2)h = 0$. Then the normalized ground-state eigenfunction $h_\omega(x)$ of $-\Delta_x + \omega^2|x|^2$ is given by $h_\omega(x) = \omega^{\frac{1}{2}} h(\omega^{\frac{1}{2}}x)$; i.e., it solves

$$(-2\omega - \Delta_x + \omega^2|x|^2)h_\omega = 0.$$

In particular, $h_1 = h$. Noticing that both of the convergences (7) and (8) involve scaling, we introduce the rescaled solution

$$\tilde{\psi}_{N,\omega}(t, \mathbf{r}_N) := \frac{1}{\omega^{\frac{1}{2}N}} \psi_{N,\omega}\left(t, \frac{\mathbf{x}_N}{\sqrt{\omega}}, \mathbf{z}_N\right) \tag{15}$$

and the rescaled Hamiltonian

$$\tilde{H}_{N,\omega} = \left[\sum_{j=1}^N -\partial_{z_j}^2 + \omega(-\Delta_x + |x|^2) \right] + \frac{1}{N} \sum_{1 \leq i < j \leq N} V_{N,\omega}(r_i - r_j), \tag{16}$$

where

$$V_{N,\omega}(r) = N^{3\beta} \omega^{3\beta-1} V\left(\frac{(N\omega)^\beta}{\sqrt{\omega}}x, (N\omega)^\beta z\right). \tag{17}$$

Then

$$(\tilde{H}_{N,\omega} \tilde{\psi}_{N,\omega})(t, \mathbf{x}_N, \mathbf{z}_N) = \frac{1}{\omega^{\frac{1}{2}N}} (H_{N,\omega} \psi_{N,\omega})\left(t, \frac{\mathbf{x}_N}{\sqrt{\omega}}, \mathbf{z}_N\right),$$

and hence when $\psi_{N,\omega}(t)$ is the Hamiltonian evolution given by (6) and $\tilde{\psi}_{N,\omega}$ is defined by (15), we have

$$\tilde{\psi}_{N,\omega}(t, \mathbf{r}_N) = e^{it\tilde{H}_{N,\omega}} \tilde{\psi}(0, \mathbf{r}_N).$$

⁹ Here, “ nD -to- nD ” means “deriving the nD NLS equation from the nD many-body evolution”.

If we let $\{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N$ be the marginal densities associated with $\tilde{\psi}_{N,\omega}$, then $\{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N$ satisfies the “ $\infty-\infty$ ” focusing BBGKY hierarchy

$$i \partial_t \tilde{\gamma}_{N,\omega}^{(k)} = \omega \sum_{j=1}^k [-\Delta_{x_j} + |x_j|^2, \tilde{\gamma}_{N,\omega}^{(k)}] + \sum_{j=1}^k [-\partial_{z_j}^2, \tilde{\gamma}_{N,\omega}^{(k)}] + \frac{1}{N} \sum_{1 \leq i < j \leq k} [V_{N,\omega}(r_i - r_j), \tilde{\gamma}_{N,\omega}^{(k)}] + \frac{N-k}{N} \sum_{j=1}^k \text{Tr}_{r_{k+1}} [V_{N,\omega}(r_j - r_{k+1}), \tilde{\gamma}_{N,\omega}^{(k+1)}]. \quad (18)$$

We will always take $\omega \geq 1$. For the rescaled marginals $\{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N$, we define

$$\tilde{S}_j := [1 - \partial_{z_j}^2 + \omega(-\Delta_{x_j} + |x_j|^2 - 2)]^{\frac{1}{2}}. \quad (19)$$

Two immediate properties of \tilde{S}_j are the following. On the one hand,

$$\tilde{S}_j^2 (h_1(x_j)\phi(z_j)) = h_1(x_j)(1 - \partial_{z_j}^2)\phi(z_j)$$

and thus the diverging parameter ω has no consequence when \tilde{S}_j is applied to a tensor product function $h_1(x_j)\phi(z_j)$ for which the x_j -component rests in the ground state. On the other hand, $\tilde{S}_j \geq 0$ as an operator because $-\Delta_{x_j} + |x_j|^2 - 2 \geq 0$.

Now, noticing that the eigenvalues of $-\Delta_x + \omega^2|x|^2$ in 2D are $\{2(l+1)\omega\}_{l=0}^\infty$, let $P_{l\omega}$ be the orthogonal projection onto the eigenspace associated with eigenvalue $2(l+1)\omega$. That is, $I = \sum_{l=0}^\infty P_{l\omega}$, where I is the identity operator on $L^2(\mathbb{R}^3)$. As a matter of notation for our multicoordinate problem, $P_{l\omega}^j$ will refer to the projection in x_j -coordinate at energy $2(l+1)\omega$; i.e.,

$$I = \prod_{j=1}^k \left(\sum_{l=0}^\infty P_{l\omega}^j \right). \quad (20)$$

In (20), I is the identity operator on $L^2(\mathbb{R}^{3k})$. In particular, when $\omega = 1$, we use simply P_l . That is, P_0 denotes the orthogonal projection onto the ground state of $-\Delta_x + |x|^2$ and $P_{\geq 1}$ means the orthogonal projection onto all higher-energy modes of $-\Delta_x + |x|^2$ so that $I = P_0 + P_{\geq 1}$, where $I : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$. Since we will only use P_0 and $P_{\geq 1}$ for the $\omega = 1$ case, we define \mathcal{P}_0^j and \mathcal{P}_1^j to be respectively P_0 and $P_{\geq 1}$ acting on the x_j -variable, and

$$\mathcal{P}_\alpha = \mathcal{P}_{\alpha_1}^1 \cdots \mathcal{P}_{\alpha_k}^k \quad (21)$$

for a k -tuple $\alpha = (\alpha_1, \dots, \alpha_k)$ with $\alpha_j \in \{0, 1\}$ and adopt the notation $|\alpha| = \alpha_1 + \dots + \alpha_k$. Then

$$I = \sum_{\alpha} \mathcal{P}_\alpha, \quad (22)$$

where $I : L^2(\mathbb{R}^{3k}) \rightarrow L^2(\mathbb{R}^{3k})$.

We next introduce an appropriate topology on the density matrices, as was previously done in [Elgart et al. 2006; Erdős and Yau 2001; Erdős et al. 2006; 2007; 2009; 2010; Kirkpatrick et al. 2011; T. Chen and Pavlović 2011; X. Chen 2012a; 2013; X. Chen and Holmer 2013; 2016b; 2016c; Sohinger 2015].

Denote the spaces of compact operators and trace class operators on $L^2(\mathbb{R}^{3k})$ as \mathcal{K}_k and \mathcal{L}_k^1 , respectively. Then $(\mathcal{K}_k)' = \mathcal{L}_k^1$. By the fact that \mathcal{K}_k is separable, we pick a dense countable subset

$$\{J_i^{(k)}\}_{i \geq 1} \subset \mathcal{K}_k$$

in the unit ball of \mathcal{K}_k (so $\|J_i^{(k)}\|_{\text{op}} \leq 1$, where $\|\cdot\|_{\text{op}}$ is the operator norm). For $\gamma_1^{(k)}, \gamma_2^{(k)} \in \mathcal{L}_k^1$, we then define a metric d_k on \mathcal{L}_k^1 by

$$d_k(\gamma_1^{(k)}, \gamma_2^{(k)}) = \sum_{i=1}^{\infty} 2^{-i} |\text{Tr } J_i^{(k)}(\gamma_1^{(k)} - \gamma_2^{(k)})|.$$

A uniformly bounded sequence $\tilde{\gamma}_{N,\omega}^{(k)} \in \mathcal{L}_k^1$ converges to $\tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$ with respect to the weak* topology if and only if

$$\lim_{N,\omega \rightarrow \infty} d_k(\tilde{\gamma}_{N,\omega}^{(k)}, \tilde{\gamma}^{(k)}) = 0.$$

For fixed $T > 0$, let $C([0, T], \mathcal{L}_k^1)$ be the space of functions of $t \in [0, T]$ with values in \mathcal{L}_k^1 which are continuous with respect to the metric d_k . On $C([0, T], \mathcal{L}_k^1)$, we define the metric

$$\hat{d}_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) = \sup_{t \in [0, T]} d_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)),$$

and denote by τ_{prod} the topology on the space $\bigoplus_{k \geq 1} C([0, T], \mathcal{L}_k^1)$ given by the product of topologies generated by the metrics \hat{d}_k on $C([0, T], \mathcal{L}_k^1)$.

With the above topology on the space of marginal densities, we prove Theorem 1.2. The proof is divided into five steps.

Step I (focusing energy estimate). We first establish, via an elaborate calculation in Theorem 3.1, that one can compensate for the negativity of the interaction in the focusing many-body Hamiltonian (6) by adding a product of N and some constant α depending on V , provided that $C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}$, where C_1 and C_2 depend solely on V . Henceforth, though $H_{N,\omega}$ is not positive-definite, we derive, from the energy condition (12), an H^1 -type energy bound:

$$\langle \psi_{N,\omega}, (\alpha + N^{-1} H_{N,\omega} - 2\omega)^k \psi_{N,\omega} \rangle \geq C^k \left\| \prod_{j=1}^k S_j \psi_{N,\omega} \right\|_{L^2(\mathbb{R}^{3N})}^2,$$

where

$$S_j := (1 - \Delta_{x_j} + \omega^2 |x_j|^2 - 2\omega - \partial_{z_j}^2)^{\frac{1}{2}}.$$

Since the quantity $\langle \psi_{N,\omega}, (H_{N,\omega} - 2N\omega)^k \psi_{N,\omega} \rangle$ is conserved by the evolution, via Corollary 3.2, we deduce the a priori bounds, crucial to the analysis of the “ $\infty - \infty$ ” BBGKY hierarchy (18), on the scaled marginal densities,

$$\begin{aligned} \sup_t \text{Tr} \left(\prod_{j=1}^k \tilde{S}_j \right) \tilde{\gamma}_{N,\omega}^{(k)} \left(\prod_{j=1}^k \tilde{S}_j \right) &\leq C^k, & \sup_t \text{Tr} \prod_{j=1}^k (1 - \Delta_{r_j}) \tilde{\gamma}_{N,\omega}^{(k)} &\leq C^k, \\ \sup_t \text{Tr} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta &\leq C^k \omega^{-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|}, \end{aligned}$$

where \mathcal{P}_α and \mathcal{P}_β are defined as in (21). We remark that the quantity

$$\mathrm{Tr}(1 - \Delta_{r_1})\tilde{\gamma}_{N,\omega}^{(1)}$$

is not the one-particle kinetic energy of the system; the one-particle kinetic energy of the system is $\mathrm{Tr}(1 - \omega \Delta_{x_1} - \partial_{z_1}^2)\tilde{\gamma}_{N,\omega}^{(1)}$ and grows like ω . This is also in contrast to the nD -to- nD work,

Step II (compactness of BBGKY). We fix $T > 0$ and work in the time interval $t \in [0, T]$. In Theorem 4.1, we establish the compactness of the BBGKY sequence $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\} \subset \bigoplus_{k \geq 1} C([0, T], \mathcal{L}_k^1)$ with respect to the product topology τ_{prod} even though hierarchy (18) contains attractive interactions and an indefinite $\infty - \infty$. Moreover, in Corollary 4.2, we prove that, to be compatible with the energy bound obtained in Step I, every limit point $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ must take the form

$$\tilde{\gamma}^{(k)}(t, (\mathbf{x}_k, \mathbf{z}_k); (\mathbf{x}'_k, \mathbf{z}'_k)) = \left(\prod_{j=1}^k h_1(x_j) h_1(x'_j) \right) \tilde{\gamma}_z^{(k)}(t, \mathbf{z}_k; \mathbf{z}'_k),$$

where $\tilde{\gamma}_z^{(k)} = \mathrm{Tr}_x \tilde{\gamma}^{(k)}$ is the z -component of $\tilde{\gamma}^{(k)}$.

Step III (limit points of BBGKY satisfy GP). In Theorem 5.1, we prove that if $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a $C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}$ limit point of $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\}$ with respect to the product topology τ_{prod} , then $\{\tilde{\gamma}_z^{(k)} = \mathrm{Tr}_x \tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a solution to the focusing coupled Gross–Pitaevskii (GP) hierarchy subject to initial data $\tilde{\gamma}_z^{(k)}(0) = |\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0 = |\int V(r) dr|$, which, written in differential form, is

$$i \partial_t \tilde{\gamma}_z^{(k)} = \sum_{j=1}^k [-\partial_{z_j}^2, \tilde{\gamma}_z^{(k)}] - b_0 \sum_{j=1}^k \mathrm{Tr}_{z_{k+1}} \mathrm{Tr}_x [\delta(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}]. \quad (23)$$

Together with the limiting structure concluded in Corollary 4.2, we can further deduce $\{\tilde{\gamma}_z^{(k)} = \mathrm{Tr}_x \tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a solution to the 1D focusing GP hierarchy subject to initial data $\tilde{\gamma}_z^{(k)}(0) = |\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0(\int |h_1(x)|^4 dx)$, which, written in differential form, is

$$i \partial_t \tilde{\gamma}_z^{(k)} = \sum_{j=1}^k [-\partial_{z_j}^2, \tilde{\gamma}_z^{(k)}] - b_0 \left(\int |h_1(x)|^4 dx \right) \sum_{j=1}^k \mathrm{Tr}_{z_{k+1}} [\delta(z_j - z_{k+1}), \tilde{\gamma}_z^{(k+1)}]. \quad (24)$$

Step IV (GP has a unique solution). When $\tilde{\gamma}_z^{(k)}(0) = |\phi_0\rangle\langle\phi_0|^{\otimes k}$, we know one solution to the 1D focusing GP hierarchy (24), namely $|\phi\rangle\langle\phi|^{\otimes k}$ if ϕ solves the 1D focusing NLS (11). Since we have proven the a priori bound,

$$\sup_t \mathrm{Tr} \left(\prod_{j=1}^k \langle \partial_{z_j} \rangle \right) \tilde{\gamma}_z^{(k)} \left(\prod_{j=1}^k \langle \partial_{z_j} \rangle \right) \leq C^k.$$

A trace theorem then shows that $\{\tilde{\gamma}_z^{(k)}\}$ verifies the requirement of the following uniqueness theorem and hence we conclude that $\tilde{\gamma}_z^{(k)} = |\phi\rangle\langle\phi|^{\otimes k}$.

Theorem 2.1 [X. Chen and Holmer 2016b, Theorem 1.3]. ¹⁰Let

$$B_{j,k+1}\gamma_z^{(k+1)} = \text{Tr}_{z_{k+1}}[\delta(z_j - z_{k+1}), \gamma_z^{(k+1)}].$$

If $\{\gamma_z^{(k)}\}_{k=1}^\infty$ solves the 1D focusing GP hierarchy (24) subject to zero initial data and the space-time bound¹¹

$$\int_0^T \left\| \left(\prod_{j=1}^k \langle \partial_{z_j} \rangle^\varepsilon \langle \partial_{z'_j} \rangle^\varepsilon \right) B_{j,k+1} \gamma_z^{(k+1)}(t, \cdot; \cdot) \right\|_{L^2_{z,z'}} dt \leq C^k \quad (25)$$

for some $\varepsilon, C > 0$ and all $1 \leq j \leq k$, then $\forall k, t \in [0, T]$, we have $\gamma_z^{(k+1)} = 0$.

Thus the compact sequence $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\}$ has only one $C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}$ limit point, namely

$$\tilde{\gamma}^{(k)} = \prod_{j=1}^k h_1(x_j) h_1(x'_j) \phi(t, z_j) \bar{\phi}(t, z'_j).$$

We then infer from the definition of the topology that as trace class operators

$$\tilde{\gamma}_{N,\omega}^{(k)} \rightarrow \prod_{j=1}^k h_1(x_j) h_1(x'_j) \phi(t, z_j) \bar{\phi}(t, z'_j) \quad \text{weak}^*.$$

Step V (weak* convergence upgraded to strong). Since the limit concluded in Step IV is an orthogonal projection, the well-known argument in [Erdős et al. 2010] upgrades the weak* convergence to strong. In fact, testing the sequence against the compact observable

$$J^{(k)} = \prod_{j=1}^k h_1(x_j) h_1(x'_j) \phi(t, z_j) \bar{\phi}(t, z'_j),$$

and noticing the fact that $(\tilde{\gamma}_{N,\omega}^{(k)})^2 \leq \tilde{\gamma}_{N,\omega}^{(k)}$ since the initial data is normalized, we see that as Hilbert–Schmidt operators,

$$\tilde{\gamma}_{N,\omega}^{(k)} \rightarrow \prod_{j=1}^k h_1(x_j) h_1(x'_j) \phi(t, z_j) \bar{\phi}(t, z'_j) \quad \text{strongly}.$$

Since $\text{Tr} \tilde{\gamma}_{N,\omega}^{(k)} = \text{Tr} \tilde{\gamma}^{(k)}$, we deduce the strong convergence

$$\lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \text{Tr} \left| \tilde{\gamma}_{N,\omega}^{(k)}(t, \mathbf{x}_k, \mathbf{z}_k; \mathbf{x}'_k, \mathbf{z}'_k) - \prod_{j=1}^k h_1(x_j) h_1(x'_j) \phi(t, z_j) \bar{\phi}(t, z'_j) \right| = 0$$

via Grümm’s convergence theorem [Simon 2005, Theorem 2.19].¹²

¹⁰ For other uniqueness theorems or related estimates regarding the GP hierarchies, see [Erdős et al. 2007; Klainerman and Machedon 2008; Kirkpatrick et al. 2011; Grillakis and Margetis 2008; X. Chen 2011; 2012a; Beckner 2014; Gressman et al. 2014; T. Chen et al. 2015; Hong et al. 2015; Sohinger 2015]

¹¹ Though the space-time bound (25) follows from a simple trace theorem here, verifying such a condition in 3D is highly nontrivial and is merely partially solved so far. See [T. Chen and Pavlović 2014; X. Chen 2013; X. Chen and Holmer 2016c].

¹² One can also use the argument in [X. Chen 2013, Appendix A] to conclude the convergence with general datum.

3. Focusing energy estimate

We find it more convenient to prove the energy estimate for $\psi_{N,\omega}$ and then convert it by scaling to an estimate for $\tilde{\psi}_{N,\omega}$; see (15). Note that, as an operator, we have the positivity

$$-\Delta_{x_j} + \omega^2|x_j|^2 - 2\omega \geq 0.$$

Define

$$S_j := (1 - \Delta_{x_j} + \omega^2|x_j|^2 - 2\omega - \partial_{z_j}^2)^{\frac{1}{2}} = (1 - 2\omega - \Delta_{r_j} + \omega^2|x_j|^2)^{\frac{1}{2}},$$

and write

$$S^{(k)} = \prod_{j=1}^k S_j.$$

Theorem 3.1 (energy estimate). *For $\beta \in (0, \frac{3}{7})$, let¹³*

$$v_E(\beta) = \min\left(\frac{1-\beta}{\beta}, \frac{\frac{3}{5}-\beta}{\beta-\frac{1}{5}} \mathbf{1}_{\beta \geq \frac{1}{5}} + \infty \cdot \mathbf{1}_{\beta < \frac{1}{5}}, \frac{\frac{7}{8}-\beta}{\beta}\right). \tag{26}$$

There are constants¹⁴ $C_1 = C_1(\|V\|_{L^1}, \|V\|_{L^\infty})$, $C_2 = C_2(\|V\|_{L^1}, \|V\|_{L^\infty})$, and absolute constant C_3 , and for each $k \in \mathbb{N}$, there is an integer $N_0(k)$, such that for any $k \in \mathbb{N}$, $N \geq N_0(k)$ and ω which satisfy

$$C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_E(\beta)}, \tag{27}$$

there holds

$$\langle (\alpha + N^{-1}H_{N,\omega} - 2\omega)^k \psi, \psi \rangle \geq \frac{1}{2k} (\|S^{(k)}\psi\|_{L^2}^2 + N^{-1}\|S_1 S^{(k-1)}\psi\|_{L^2}^2), \tag{28}$$

where

$$\alpha = C_3 \|V\|_{L^1}^2 + 1.$$

Proof. For smoothness of presentation, we postpone the proof to Section 3. □

Recall the rescaled operator (19),

$$\tilde{S}_j = [1 - \partial_{z_j}^2 + \omega(-\Delta_{x_j} + |x_j|^2 - 2)]^{\frac{1}{2}}.$$

We notice that

$$(S_j \psi)(t, \mathbf{x}_N, \mathbf{z}_N) = \omega^{N/2} (\tilde{S}_j \tilde{\psi})(t, \sqrt{\omega} \mathbf{x}_N, \mathbf{z}_N)$$

if $\tilde{\psi}_{N,\omega}$ is defined via (15). Thus we can convert the conclusion of Theorem 3.1 into statements about $\tilde{\psi}_{N,\omega}$, \tilde{S}_j , and $\tilde{\gamma}_{N,\omega}^{(k)}$, which we will utilize in the rest of the paper.

¹³ One notices that $v_E(\beta)$ is different from $v_2(\beta)$ in the sense that the term $2\beta/(1-2\beta)$ – is missing. That restriction comes from Theorem 5.1.

¹⁴ By *absolute* constant we mean a constant independent of V, N, ω , etc. Formulas for C_1, C_2 in terms of $\|V\|_{L^1}, \|V\|_{L^\infty}$ can, in principle, be extracted from the proof.

Corollary 3.2. *Define*

$$\tilde{S}^{(k)} = \prod_{j=1}^k \tilde{S}_j, \quad L^{(k)} = \prod_{j=1}^k \langle \nabla_{r_j} \rangle.$$

Assume $C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_E(\beta)}$. Let $\tilde{\psi}_{N,\omega}(t) = e^{it\tilde{H}_{N,\omega}} \tilde{\psi}_{N,\omega}(0)$ and $\{\tilde{\gamma}_{N,\omega}^{(k)}(t)\}$ be the associated marginal densities. Then for all $\omega \geq 1$, $k \geq 0$, N large enough, we have the uniform-in-time bound

$$\mathrm{Tr} \tilde{S}^{(k)} \tilde{\gamma}_{N,\omega}^{(k)} \tilde{S}^{(k)} = \|\tilde{S}^{(k)} \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})}^2 \leq C^k. \quad (29)$$

Consequently,

$$\mathrm{Tr} L^{(k)} \tilde{\gamma}_{N,\omega}^{(k)} L^{(k)} = \|L^{(k)} \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})}^2 \leq C^k, \quad (30)$$

and

$$\|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \leq C^k \omega^{-\frac{1}{2}|\alpha|}, \quad |\mathrm{Tr} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta| \leq C^k \omega^{-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|}, \quad (31)$$

where \mathcal{P}_α and \mathcal{P}_β are defined as in (21).

Proof. Substituting (15) into estimate (28) and rescaling, we obtain

$$\|\tilde{S}^{(k)} \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})}^2 \leq C^k \langle \tilde{\psi}_{N,\omega}(t), (\alpha + N^{-1} \tilde{H}_{N,\omega} - 2\omega)^k \tilde{\psi}_{N,\omega}(t) \rangle.$$

The quantity on the right-hand side is conserved; therefore

$$\|\tilde{S}^{(k)} \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})}^2 = C^k \langle \tilde{\psi}_{N,\omega}(0), (\alpha + N^{-1} \tilde{H}_{N,\omega} - 2\omega)^k \tilde{\psi}_{N,\omega}(0) \rangle.$$

Applying the binomial theorem twice,

$$\begin{aligned} \|\tilde{S}^{(k)} \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})}^2 &\leq C^k \sum_{j=0}^k \binom{k}{j} \alpha^j \langle \tilde{\psi}_{N,\omega}(0), (N^{-1} \tilde{H}_{N,\omega} - 2\omega)^{k-j} \tilde{\psi}_{N,\omega}(0) \rangle \\ &\leq C^k \sum_{j=0}^k \binom{k}{j} \alpha^j (C)^{k-j} \\ &= C^k (\alpha + C)^k \leq \tilde{C}^k, \end{aligned}$$

where we used condition (12) in the second-to-last line. So we have proved (29). Putting (29) and (70) together, estimate (30) then follows.¹⁵ The first inequality of (31) follows from (29) and (72). By Lemma A.5,

$$\mathrm{Tr} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta = \langle \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle,$$

so the second inequality of (31) follows by Cauchy–Schwarz. \square

¹⁵ We remark that, though $L^{(k)} \leq 3^k \tilde{S}^{(k)}$, it is not true that $L^{(k)} \leq C^k S^{(k)}$ for any C independent of ω because of the ground-state case.

Proof of the focusing energy estimate. Note that

$$N^{-1}H_{N,\omega} - 2\omega = N^{-1} \sum_{i=1}^N (-\Delta_{r_i} + \omega^2|x_i|^2 - 2\omega) + N^{-2}\omega^{-1} \sum_{1 \leq i < j \leq N} V_{N\omega}(r_i - r_j),$$

where we have used the notation¹⁶

$$V_{N\omega}(r) = (N\omega)^{3\beta} V((N\omega)^\beta r).$$

Define

$$H_{Kij} = (\alpha - \Delta_{r_i} + \omega^2|x_i|^2 - 2\omega) + (\alpha - \Delta_{r_j} + \omega^2|x_j|^2 - 2\omega),$$

where the K stands for “kinetic” and

$$H_{Iij} = \omega^{-1}V_{N\omega ij} = \omega^{-1}V_{N\omega}(r_i - r_j),$$

where the I is for “interaction”. If we write

$$H_{ij} = H_{Kij} + H_{Iij},$$

then

$$\alpha + N^{-1}H_{N,\omega} - 2\omega = \frac{1}{2}N^{-2} \sum_{1 \leq i \neq j \leq N} H_{ij} = N^{-2} \sum_{1 \leq i < j \leq N} H_{ij}. \tag{32}$$

We will first prove Theorem 3.1 for $k = 1$ and $k = 2$. Then, by a two-step induction (result known for k implies result for $k + 2$), we establish the general case. Before we proceed, we prove some estimates regarding the Hermite operator.

Estimates needed to prove Theorem 3.1.

Lemma 3.3. *Recall that $P_{l\omega}$ is the orthogonal projection onto the eigenspace of $-\Delta_x + \omega^2|x|^2$ associated with eigenvalue $2(l + 1)\omega$. There is a constant independent of l and ω such that*

$$\|P_{l\omega}f\|_{L_x^\infty} \leq C\omega^{\frac{1}{2}}\|f\|_{L_x^2}. \tag{33}$$

Proof. This estimate has more than one proof. It is a special result in 2D. It does not follow from the Strichartz estimates. For a modern argument which proves the estimate for general, at most quadratic potentials, see [Koch and Tataru 2005, Corollary 2.2]. In the special case of the quantum harmonic oscillator, one can also use a special property of 2D Hermite projection kernels to yield a direct proof without using Littlewood–Paley theory — see [Thangavelu 1993, Lemma 3.2.2; X. Chen 2011, Remark 8].

□

Lemma 3.4. *There is an absolute constant $C_3 > 0$ and a constant $C_1 = C(\|V\|_{L^1}, \|V\|_{L^\infty})$ such that if*

$$\omega \geq C_1 N^{\frac{\beta}{1-\beta}}$$

¹⁶ We remind the reader that this $V_{N\omega}$ is different from $V_{N,\omega}$ defined in (17).

then

$$\begin{aligned} & \frac{1}{\omega} \int |V_{N\omega}(r_1 - r_2)| |\psi(r_1, r_2)|^2 dr_1 \\ & \leq \frac{1}{100} \langle \psi(r_1, r_2), (-\Delta_{r_1} + \omega^2 |x_1|^2 - 2\omega) \psi(r_1, r_2) \rangle_{r_1} + C_3 \|V\|_{L^1}^2 \|\psi(r_1, r_2)\|_{L^2_{r_1}}^2. \end{aligned} \quad (34)$$

The above estimate is performed in one coordinate only (taken to be r_1), and the other coordinate r_2 is effectively “frozen”. In particular, let

$$f(r_2, \dots, r_N) = \int |V_{N\omega}(r_1 - r_2)| |\psi_1(r_1, \dots, r_N)| |\psi_2(r_1, \dots, r_N)| dr_1.$$

Then

$$f(r_2, \dots, r_N) \lesssim \omega \|S_1 \psi_1(r_1, \dots, r_N)\|_{L^2_{r_1}} \|S_1 \psi_2(r_1, \dots, r_N)\|_{L^2_{r_1}}. \quad (35)$$

The implicit constant in the \lesssim is an absolute constant times $\|V\|_{L^1} + \|V\|_{L^\infty}$.

Proof. By Cauchy–Schwarz,

$$\int |V_{N\omega}| |\psi_1| |\psi_2| dr_1 \leq \left(\int |V_{N\omega}| |\psi_1|^2 dr_1 \right)^{\frac{1}{2}} \left(\int |V_{N\omega}| |\psi_2|^2 dr_1 \right)^{\frac{1}{2}}.$$

Thus, assuming (34) and using the facts

$$S_1^2 \geq 1, \quad S_1^2 \geq (-\Delta_{r_1} + \omega^2 |x_1|^2 - 2\omega),$$

we obtain (35). So we only need to prove (34).

Taking $P_{l\omega}$ to be the projection onto the x_1 -component at the moment, we decompose ψ into ground state, middle energies, and high energies as follows:

$$\psi = P_{0\omega} \psi + \sum_{\ell=1}^{e-1} P_{l\omega} \psi + P_{\geq e\omega} \psi,$$

where e is an integer, and the optimal choice of e is determined below. It then suffices to bound

$$A_{\text{low}} := \frac{1}{\omega} \int |V_{N\omega}(r_1 - r_2)| |P_{0\omega} \psi(r_1, r_2)|^2 dr_1, \quad (36)$$

$$A_{\text{mid}} := \frac{1}{\omega} \int |V_{N\omega}(r_1 - r_2)| \left| \sum_{l=2}^{e-1} P_{l\omega} \psi(r_1, r_2) \right|^2 dr_1, \quad (37)$$

$$A_{\text{high}} := \frac{1}{\omega} \int |V_{N\omega}(r_1 - r_2)| |P_{\geq e\omega} \psi(r_1, r_2)|^2 dr_1. \quad (38)$$

For each estimate, we will only work in the $r_1 = (x_1, z_1)$ component, and thus will not even write the r_2 -variable. First we consider (36):

$$A_{\text{low}} \leq \frac{1}{\omega} \|V_{N\omega}\|_{L^1} \|P_{0\omega} \psi\|_{L^\infty_{x_1} L^\infty_{z_1}}^2.$$

By the standard 1D Sobolev-type estimate,

$$A_{\text{low}} \lesssim \frac{1}{\omega} \|V\|_{L^1} \|P_{0\omega} \partial_z \psi\|_{L_x^\infty L_z^2} \|P_{0\omega} \psi\|_{L_x^\infty L_z^2}.$$

Then using the estimate (33), we get

$$\begin{aligned} A_{\text{low}} &\lesssim \|V\|_{L^1} \|P_{0\omega} \partial_z \psi\|_{L_r^2} \|P_{0\omega} \psi\|_{L_r^2} \\ &\lesssim \|V\|_{L^1} \|\partial_z \psi\|_{L^2} \|\psi\|_{L^2} \\ &\lesssim \epsilon \|\partial_z \psi\|_{L^2}^2 + \frac{1}{\epsilon} \|V\|_{L^1}^2 \|\psi\|_{L^2}^2. \end{aligned}$$

Since $(-\Delta_r + \omega^2|x|^2 - 2\omega)$ is a sum of two positive operators, namely $-\Delta_x + \omega^2|x|^2 - 2\omega$ and $-\partial_z^2$, we conclude the estimate for A_{low} .

Now consider the middle harmonic energies given by (37). We aim to estimate A_{mid} . For any $l \geq 1$, we have

$$\|P_{l\omega} \psi\|_{L_z^\infty L_x^\infty} \leq \|P_{l\omega} \partial_z \psi\|_{L_z^2 L_x^\infty}^{\frac{1}{2}} \|P_{l\omega} \psi\|_{L_z^2 L_x^\infty}^{\frac{1}{2}}.$$

By (33),

$$\begin{aligned} \|P_{l\omega} \psi\|_{L_z^\infty L_x^\infty} &\lesssim \omega^{\frac{1}{2}} \|P_{l\omega} \partial_z \psi\|_{L_z^2 L_x^2}^{\frac{1}{2}} \|P_{l\omega} \psi\|_{L_z^2 L_x^2}^{\frac{1}{2}} \\ &= \omega^{\frac{1}{4}} \|P_{l\omega} \partial_z \psi\|_{L^2}^{\frac{1}{2}} \left(\|P_{l\omega} \psi\|_{L^2} l^{\frac{1}{2}} \omega^{\frac{1}{2}} \right)^{\frac{1}{2}} l^{-\frac{1}{4}} \\ &= \omega^{\frac{1}{4}} \|P_{l\omega} \partial_z \psi\|_{L_r^2}^{\frac{1}{2}} \|P_{l\omega} (-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}} \psi\|_{L^2}^{\frac{1}{2}} l^{-\frac{1}{4}}. \end{aligned}$$

Summing over $1 \leq l \leq e-1$, and using the Hölder inequality with exponents 4, 4, and 2, we get

$$\begin{aligned} \sum_{l=1}^{e-1} \|P_{l\omega} \psi\|_{L_z^\infty L_x^\infty} &\lesssim \omega^{\frac{1}{4}} \left(\sum_{l=1}^{e-1} \|P_{l\omega} \partial_z \psi\|_{L^2}^2 \right)^{\frac{1}{4}} \left(\sum_{l=1}^{e-1} \|P_{l\omega} (-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}} \psi\|_{L^2}^2 \right)^{\frac{1}{4}} \left(\sum_{l=1}^e l^{-\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\lesssim \omega^{\frac{1}{4}} e^{\frac{1}{4}} \|\partial_z \psi\|_{L^2}^{\frac{1}{2}} \|(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}} \psi\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

Applying this to estimate (37),

$$A_{\text{mid}} \lesssim \omega^{-\frac{1}{2}} e^{\frac{1}{2}} \|V\|_{L^1} \|\partial_z \psi\|_{L^2} \|(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}} \psi\|_{L^2}.$$

Take e to be the largest integer so that $\omega^{-\frac{1}{2}} e^{\frac{1}{2}} \|V\|_{L^1} \leq \epsilon$, i.e.,

$$e = \left\lfloor \frac{\epsilon^2}{\|V\|_{L^1}^2 \omega} \right\rfloor, \quad (39)$$

and then we have

$$A_{\text{mid}} \lesssim \epsilon \|\partial_z \psi\|_{L^2}^2 + \epsilon \|(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}} \psi\|_{L^2}^2.$$

For (38),

$$\begin{aligned} A_{\text{high}} &\lesssim \omega^{-1} \|V_{N\omega}\|_{L^\infty} \|P_{\geq e\omega}\psi\|_{L^2}^2 \\ &\lesssim \omega^{-2} e^{-1} \|V_{N\omega}\|_{L^\infty} \|e^{\frac{1}{2}} \omega^{\frac{1}{2}} P_{\geq e\omega}\psi\|_{L^2}^2 \\ &\lesssim \omega^{-2} e^{-1} (N\omega)^{3\beta} \|V\|_{L^\infty} \|(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}}\psi\|_{L^2}^2. \end{aligned}$$

We need

$$\omega^{-2} e^{-1} (N\omega)^{3\beta} \|V\|_{L^\infty} \leq \epsilon.$$

Substituting the specification of e given by (39), we obtain

$$\omega^{-2} (N\omega)^{3\beta} \leq \frac{e\epsilon}{\|V\|_{L^\infty}} \leq \frac{\epsilon^3}{\|V\|_{L^\infty} \|V\|_{L^1}^2} \omega,$$

which is

$$N^{3\beta} \omega^{3\beta-3} \leq \frac{\epsilon^3}{\|V\|_{L^1}^2 \|V\|_{L^\infty}}.$$

That is, $\omega \geq C_1 N^{\frac{\beta}{1-\beta}}$ as required in the statement of Lemma 3.4. \square

In the following lemma, we have excited-state estimates and ground-state estimates, and the ground-state estimates are weaker (they involve a loss of $\omega^{\frac{1}{2}}$).

Lemma 3.5. *Taking $\psi = \psi(r)$, we have the “excited-state” estimate*

$$\|\omega^{\frac{1}{2}} P_{\geq 1\omega}\psi\|_{L^2} + \|\omega|x|P_{\geq 1\omega}\psi\|_{L^2} + \|\nabla_r P_{\geq 1\omega}\psi\|_{L^2} \lesssim \|S\psi\|_{L^2}, \quad (40)$$

and the “ground-state” estimate

$$\|\omega^{\frac{1}{2}} P_{0\omega}\psi\|_{L^2} + \|\omega|x|P_{0\omega}\psi\|_{L^2} + \|\nabla_x P_{0\omega}\psi\|_{L^2} \lesssim \omega^{\frac{1}{2}} \|\psi\|_{L^2}. \quad (41)$$

We are, however, spared from the $\omega^{\frac{1}{2}}$ loss when working only with the z -derivative:

$$\|\partial_z P_{0\omega}\psi\|_{L^2} \lesssim \|S\psi\|_{L^2}. \quad (42)$$

Putting the excited-state and ground-state estimates together gives

$$\|\omega^{\frac{1}{2}}\psi\|_{L^2} + \|\omega|x|\psi\|_{L^2} + \|\nabla_r\psi\|_{L^2} \lesssim \omega^{\frac{1}{2}} \|S\psi\|_{L^2}. \quad (43)$$

Proof. For the excited-state estimates, we note

$$0 \leq \langle P_{\geq 1\omega}\psi, (-\Delta_x + \omega^2|x|^2 - 4\omega)P_{\geq 1\omega}\psi \rangle.$$

Adding $\frac{3}{2}\|\partial_z P_{\geq 1\omega}\psi\|_{L^2}^2 + \frac{1}{2}\|\nabla_x P_{\geq 1\omega}\psi\|_{L^2}^2 + \frac{1}{2}\|\omega|x|P_{\geq 1\omega}\psi\|_{L^2}^2 + \|\omega^{\frac{1}{2}}P_{\geq 1\omega}\psi\|_{L^2}^2$ to both sides, we get

$$\begin{aligned} \frac{3}{2}\|\partial_z P_{\geq 1\omega}\psi\|_{L^2}^2 + \frac{1}{2}\|\nabla_x P_{\geq 1\omega}\psi\|_{L^2}^2 + \frac{1}{2}\|\omega|x|P_{\geq 1\omega}\psi\|_{L^2}^2 + \|\omega^{\frac{1}{2}}P_{\geq 1\omega}\psi\|_{L^2}^2 \\ \leq \frac{3}{2}\langle P_{\geq 1\omega}\psi, (-\Delta_r + \omega^2|x|^2 - 2\omega)P_{\geq 1\omega}\psi \rangle. \end{aligned}$$

This proves (40).

For the ground-state estimate (41), it suffices to prove

$$\|\omega|x|P_{0\omega}\psi\|_{L^2} + \|\nabla_x P_{0\omega}\psi\|_{L^2} \lesssim C\omega^{\frac{1}{2}}\|\psi\|_{L^2},$$

because

$$\|\omega^{\frac{1}{2}}P_{0\omega}\psi\|_{L^2} = \omega^{\frac{1}{2}}\|P_{0\omega}\psi\|_{L^2} \leq \omega^{\frac{1}{2}}\|\psi\|_{L^2}.$$

We notice that

$$\|\omega|x|f\|_{L^2} + \|\nabla_x f\|_{L^2} \sim \|(-\Delta_x + \omega^2|x|^2)^{\frac{1}{2}}f\|_{L^2}.$$

This estimate has been proved by many authors (see, for example, [Thangavelu 1993]), but is usually known as a scattering space Σ estimate for PDE analysts. Then, since the eigenvalue for the ground-state Gaussian is exactly 2ω in 2D, we have

$$\|(-\Delta_x + \omega^2|x|^2)^{\frac{1}{2}}P_{0\omega}\psi\|_{L^2} = \sqrt{2}\omega^{\frac{1}{2}}\|P_{0\omega}\psi\|_{L^2} \leq \sqrt{2}\omega^{\frac{1}{2}}\|\psi\|_{L^2}.$$

So we have proved (41).

For (42), we notice that

$$\|\partial_z P_{0\omega}\psi\|_{L^2} = \|P_{0\omega}(\partial_z\psi)\|_{L^2} \leq \|\partial_z\psi\|_{L^2} \lesssim \|S\psi\|_{L^2}. \quad \square$$

Lemma 3.6. *We have the estimates*

$$\| |V_{N\omega 12}|^{\frac{1}{2}} S_1 P_{0\omega}^1 \psi_2 \|_{L_{r_1}^2} \lesssim \omega^{\frac{1}{2}} N^{\frac{1}{4}} \|S_1 \psi_2\|_{L^2}^{\frac{1}{2}} (N^{-\frac{1}{4}} \|S_1^2 \psi_2\|_{L^2}^{\frac{1}{2}}), \quad (44)$$

$$\| |V_{N\omega 12}|^{\frac{1}{2}} S_1 P_{\geq 1\omega}^1 \psi_2 \|_{L_{r_1}^2} \lesssim N^{\frac{1}{2}\beta + \frac{1}{2}} \omega^{\frac{1}{2}\beta} (N^{-\frac{1}{2}} \|S_1^2 \psi_2\|_{L_{r_1}^2}). \quad (45)$$

In particular, if $\omega \geq C_1 N^{\frac{\beta}{1-\beta}}$ then

$$\begin{aligned} & \int_{r_1} |V_{N\omega 12}| |\psi_1| |S_1 \psi_2| dr_1 \\ & \lesssim \omega N^{\frac{1}{4}} \|S_1 \psi_1\|_{L^2} \|S_1 \psi_2\|_{L^2}^{\frac{1}{2}} N^{-\frac{1}{4}} \|S_1^2 \psi_2\|_{L^2}^{\frac{1}{2}} + (N\omega)^{\frac{1}{2}\beta + \frac{1}{2}} \|S_1 \psi_1\|_{L^2} N^{-\frac{1}{2}} \|S_1^2 \psi_2\|_{L^2}. \end{aligned} \quad (46)$$

Proof. To prove (46), substituting $\psi_2 = P_{0\omega}^1 \psi_2 + P_{\geq 1\omega}^1 \psi_2$, we obtain

$$\int_{r_1} |V_{N\omega 12}| |\psi_1| |S_1 \psi_2| dr_1 \lesssim F_1 + F_2,$$

where

$$\begin{aligned} F_1 &= \int_{r_1} |V_{N\omega 12}| |\psi_1| |P_{0\omega}^1 S_1 \psi_2| dr_1 \\ &\leq \| |V_{N\omega 12}|^{\frac{1}{2}} \psi_1 \|_{L_{r_1}^2} \| |V_{N\omega 12}|^{\frac{1}{2}} P_{0\omega}^1 S_1 \psi_2 \|_{L_{r_1}^2} \\ &\leq \omega^{\frac{1}{2}} \|S_1 \psi_1\|_{L_{r_1}^2} \| |V_{N\omega 12}|^{\frac{1}{2}} P_{0\omega}^1 S_1 \psi_2 \|_{L_{r_1}^2}, \end{aligned}$$

$$\begin{aligned} F_2 &= \int_{r_1} |V_{N\omega 12}| |\psi_1| |P_{\geq 1\omega}^1 S_1 \psi_2| dr_1 \\ &\leq \omega^{\frac{1}{2}} \|S_1 \psi_1\|_{L_{r_1}^2} \| |V_{N\omega 12}|^{\frac{1}{2}} P_{\geq 1\omega}^1 S_1 \psi_2 \|_{L_{r_1}^2} \end{aligned}$$

by Cauchy–Schwarz and estimate (35). Hence we only need to prove (44) and (45).

On the one hand, using the fact that $P_{0\omega}^1 S_1 = (1 - \partial_{z_1}^2)^{\frac{1}{2}} P_{0\omega}^1$,

$$\begin{aligned} \||V_{N\omega 12}|^{\frac{1}{2}} S_1 P_{0\omega}^1 \psi_2 \|_{L_{r_1}^2} &= \||V_{N\omega 12}|^{\frac{1}{2}} (1 - \partial_{z_1}^2)^{\frac{1}{2}} P_{0\omega}^1 \psi_2 \|_{L_{r_1}^2} \\ &\leq \|V_{N\omega 12}\|_{L_{r_1}^1}^{\frac{1}{2}} \|(1 - \partial_{z_1}^2)^{\frac{1}{2}} P_{0\omega}^1 \psi_2\|_{L_{r_1}^\infty}. \end{aligned}$$

By Sobolev in z_1 and the estimate (33) in x_1 ,

$$\||V_{N\omega 12}|^{\frac{1}{2}} S_1 P_{0\omega}^1 \psi_2 \|_{L_{r_1}^2} \lesssim \omega^{\frac{1}{2}} \|(1 - \partial_{z_1}^2)^{\frac{1}{2}} \psi_2\|_{L_{r_1}^2}^{\frac{1}{2}} \|(1 - \partial_{z_1}^2) \psi_2\|_{L_{r_1}^2}^{\frac{1}{2}}.$$

That is, we get (44):

$$\||V_{N\omega 12}|^{\frac{1}{2}} S_1 P_{0\omega}^1 \psi_2 \|_{L_{r_1}^2} \lesssim \omega^{\frac{1}{2}} N^{\frac{1}{4}} \|S_1 \psi_2\|_{L^2}^{\frac{1}{2}} (N^{-\frac{1}{4}} \|S_1^2 \psi_2\|_{L^2}^{\frac{1}{2}}).$$

On the other hand,

$$\begin{aligned} \||V_{N\omega 12}|^{\frac{1}{2}} S_1 P_{\geq 1\omega}^1 \psi_2 \|_{L_{r_1}^2} &\lesssim \||V_{N\omega 12}|^{\frac{1}{2}} \|_{L^3} \|P_{\geq 1\omega}^1 S_1 \psi_2\|_{L_{r_1}^6} \\ &\lesssim (N\omega)^{\frac{1}{2}\beta} \|S_1^2 \psi_2\|_{L_{r_1}^2} \\ &= N^{\frac{1}{2}\beta + \frac{1}{2}} \omega^{\frac{1}{2}\beta} (N^{-\frac{1}{2}} \|S_1^2 \psi_2\|_{L_{r_1}^2}), \end{aligned}$$

which is (45). □

The $k = 1$ case. Recalling (32),

$$\langle \psi, (\alpha + N^{-1} H_{N,\omega} - 2\omega) \psi \rangle = \frac{1}{2} N^{-2} \sum_{1 \leq i \neq j \leq N} \langle H_{ij} \psi, \psi \rangle = \frac{1}{2} \langle H_{12} \psi, \psi \rangle,$$

where the second equality follows by symmetry. Hence we need to prove

$$\langle H_{12} \psi, \psi \rangle \geq \|S_1 \psi\|_{L^2}^2. \quad (47)$$

We prove (47) with the following lemma.

Lemma 3.7. *Recall $\alpha = C_3 \|V\|_{L^2}^2 + 1$. If $\omega \geq C_1 N^{\frac{\beta}{1-\beta}}$ and $\psi_j(r_1, r_2) = \psi_j(r_2, r_1)$ for $j = 1, 2$, then*

$$|\langle H_{12} \psi_1, \psi_2 \rangle_{r_1 r_2}| \lesssim \|S_1 \psi_1\|_{L_{r_1 r_2}^2} \|S_1 \psi_2\|_{L_{r_1 r_2}^2}. \quad (48)$$

Moreover,

$$\|S_1 \psi\|_{L^2}^2 \leq \langle H_{12} \psi, \psi \rangle \leq C \|S_1 \psi\|_{L^2}^2. \quad (49)$$

Proof. By Cauchy–Schwarz and (34),

$$\begin{aligned} |\langle \psi_1, H_{I12} \psi_2 \rangle_{r_1 r_2}| &= \omega^{-1} |\langle V_{N\omega 12} \psi_1, \psi_2 \rangle| \\ &\lesssim \left(\omega^{-1} \int |V_{N\omega 12}| |\psi_1|^2 \right)^{\frac{1}{2}} \left(\omega^{-1} \int |V_{N\omega 12}| |\psi_2|^2 \right)^{\frac{1}{2}} \\ &\lesssim \|S_1 \psi_1\|_{L^2} \|S_1 \psi_2\|_{L^2}. \end{aligned}$$

Thus

$$\begin{aligned} |\langle H_{12}\psi_1, \psi_2 \rangle_{r_1 r_2}| &\leq |\langle H_{K12}\psi_1, \psi_2 \rangle_{r_1 r_2}| + |\langle H_{I12}\psi_1, \psi_2 \rangle_{r_1 r_2}| \\ &\lesssim \|S_1\psi_1\|_{L^2_{r_1 r_2}} \|S_1\psi_2\|_{L^2_{r_1 r_2}}, \end{aligned}$$

which is (48). It remains to prove the first inequality in (49).

On the one hand, by (34), we have the lower bound for the potential term,

$$-\frac{1}{100}\langle \psi, (-\Delta_{r_1} + \omega^2|x_1|^2 - 2\omega)\psi \rangle_{r_1 r_2} - C_3\|V\|_{L^1}^2\|\psi\|_{L^2_{r_1 r_2}}^2 \leq \omega^{-1}\langle V_{N\omega 12}\psi, \psi \rangle_{r_1 r_2}.$$

Adding $\langle \psi, (\alpha - \Delta_{r_1} + \omega^2|x_1|^2 - 2\omega)\psi \rangle_{r_1 r_2}$ to both sides and noticing the trivial inequalities $\alpha - C_3\|V\|_{L^2}^2 = 1 \geq \frac{1}{2}$ and $\frac{99}{100} \geq \frac{1}{2}$, we have

$$\frac{1}{2}\langle \psi, (1 - \Delta_{r_1} + \omega^2|x_1|^2 - 2\omega)\psi \rangle_{r_1 r_2} \leq \langle \psi, (\alpha - \Delta_{r_1} + \omega^2|x_1|^2 - 2\omega + \omega^{-1}V_{N\omega 12})\psi \rangle_{r_1 r_2}. \quad (50)$$

On the other hand, we trivially have

$$\frac{1}{2}\langle \psi, (1 - \Delta_{r_2} + \omega^2|x_2|^2 - 2\omega)\psi \rangle_{r_1 r_2} \leq \langle \psi, (\alpha - \Delta_{r_2} + \omega^2|x_2|^2 - 2\omega)\psi \rangle_{r_1 r_2} \quad (51)$$

because $\alpha > \frac{1}{2}$.

Adding estimates (50) and (51) together, we have

$$\frac{1}{2}\langle \psi, S_1^2\psi \rangle + \frac{1}{2}\langle \psi, S_2^2\psi \rangle \leq \langle H_{12}\psi, \psi \rangle.$$

By symmetry in r_1 and r_2 , this is precisely (49). \square

The $k = 2$ case. The $k = 2$ energy estimate is the lower bound

$$\frac{1}{4}(\langle S_1^2 S_2^2 \psi, \psi \rangle + N^{-1}\langle S_1^4 \psi, \psi \rangle) \leq \langle (\alpha + N^{-1}H - 2\omega)^2 \psi, \psi \rangle.$$

We will prove it under the hypothesis

$$N^{\frac{\beta}{1-\beta}} \leq \omega \leq N^{\min(\frac{1-\beta}{\beta}, 2)}.$$

We substitute into (32) to obtain

$$\langle (\alpha + N^{-1}H - 2\omega)^2 \psi, \psi \rangle = \frac{1}{4}N^{-4} \sum_{\substack{1 \leq i_1 \neq j_1 \leq N \\ 1 \leq i_2 \neq j_2 \leq N}} \langle H_{i_1 j_1} H_{i_2 j_2} \psi, \psi \rangle = A_1 + A_2 + A_3,$$

where

- A_1 consists of those terms with $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$,
- A_2 consists of those terms with $|\{i_1, j_1\} \cap \{i_2, j_2\}| = 1$,
- A_3 consists of those terms with $|\{i_1, j_1\} \cap \{i_2, j_2\}| = 2$.

By symmetry, we have

$$\begin{aligned} A_1 &= \frac{1}{4}\langle H_{12}H_{34}\psi, \psi \rangle, \\ A_2 &= \frac{1}{2}N^{-1}\langle H_{12}H_{23}\psi, \psi \rangle, \\ A_3 &= \frac{1}{2}N^{-2}\langle H_{12}H_{12}\psi, \psi \rangle. \end{aligned}$$

We discard A_3 since $A_3 \geq 0$. By the analysis used in the $k = 1$ case,

$$A_1 \geq \frac{1}{4} \|S_1 S_3 \psi\|_{L^2}^2.$$

The main piece of work in the $k = 2$ case is to estimate A_2 . Substituting $H_{12} = H_{K12} + H_{I12}$ and $H_{23} = H_{K23} + H_{I23}$, we obtain the expansion

$$A_2 = B_0 + B_1 + B_2,$$

where

$$\begin{aligned} B_0 &= \frac{1}{2} N^{-1} \langle H_{K12} H_{K23} \psi, \psi \rangle, \\ B_1 &= \frac{1}{2} N^{-1} \langle H_{K12} H_{I23} \psi, \psi \rangle + \frac{1}{2} N^{-1} \langle H_{I12} H_{K23} \psi, \psi \rangle, \\ B_2 &= \frac{1}{2} N^{-1} \langle H_{I12} H_{I23} \psi, \psi \rangle. \end{aligned}$$

Let $\sigma = \alpha - 1 \geq 0$. First note that

$$B_0 = \frac{1}{2} N^{-1} \langle (S_1^2 + S_2^2 + 2\sigma)(S_2^2 + S_3^2 + 2\sigma) \psi, \psi \rangle.$$

Since S_1^2, S_2^2, S_3^2 all commute,

$$B_0 \geq \frac{1}{2} N^{-1} \langle S_2^4 \psi, \psi \rangle,$$

which is a component of the claimed lower bound.

Next, we consider B_1 . By symmetry

$$B_1 = N^{-1} \operatorname{Re} \langle H_{K12} H_{I23} \psi, \psi \rangle.$$

Since every term in B_1 is estimated, we do not drop the imaginary part. Decompose $I = P_{0\omega}^2 + P_{\geq 1\omega}^2$ in the right factor of ψ as

$$B_1 = B_{10} + B_{11} + B_{12},$$

where

$$\begin{aligned} B_{10} &= (N\omega)^{-1} \langle [(2\alpha - 1) + S_1^2] V_{N\omega 23} \psi, \psi \rangle, \\ B_{11} &= (N\omega)^{-1} \langle (-\Delta_{r_2} + \omega^2 |x_2|^2 - 2\omega) V_{N\omega 23} \psi, P_{0\omega}^2 \psi \rangle, \\ B_{12} &= (N\omega)^{-1} \langle (-\Delta_{r_2} + \omega^2 |x_2|^2 - 2\omega) V_{N\omega 23} \psi, P_{\geq 1\omega}^2 \psi \rangle. \end{aligned}$$

The term B_{10} is the simplest. In fact, by estimate (35) at the r_2 -coordinate, we have

$$\begin{aligned} |B_{10}| &= |(N\omega)^{-1} \langle [(2\alpha - 1) + S_1^2] V_{N\omega 23} \psi, \psi \rangle| \\ &\lesssim N^{-1} (\|S_2 \psi\|_{L^2}^2 + \|S_1 S_2 \psi\|_{L^2}^2). \end{aligned}$$

For B_{12} , we consider the four terms separately:

$$B_{12} = B_{121} + B_{122} + B_{123} + B_{124},$$

where

$$\begin{aligned} B_{121} &= (N\omega)^{\beta-1} \langle (\nabla V)_{N\omega 23} \psi, \nabla_{r_2} P_{\geq 1\omega}^2 \psi \rangle, \\ B_{122} &= (N\omega)^{-1} \langle V_{N\omega 23} \nabla_{r_2} \psi, \nabla_{r_2} P_{\geq 1\omega}^2 \psi \rangle, \\ B_{123} &= (N\omega)^{-1} \langle V_{N\omega 23} \omega |x_2| \psi, \omega |x_2| P_{\geq 1\omega}^2 \psi \rangle, \\ B_{124} &= -2(N\omega)^{-1} \langle V_{N\omega 23} \omega^{\frac{1}{2}} \psi, \omega^{\frac{1}{2}} P_{\geq 1\omega}^2 \psi \rangle. \end{aligned}$$

By (35) applied with r_1 replaced by r_3 , we obtain

$$|B_{121}| \lesssim (N\omega)^{\beta-1} \omega \|S_3 \psi\|_{L^2} \|\nabla_{r_2} P_{\geq 1\omega}^2 S_3 \psi\|_{L^2}.$$

By (40),

$$|B_{121}| \lesssim (N\omega)^{\beta-1} \omega \|S_3 \psi\|_{L^2} \|S_2 S_3 \psi\|_{L^2},$$

which yields the requirement $\omega \leq N^{\frac{1-\beta}{\beta}}$. By (35) applied with r_1 replaced by r_3 , we obtain

$$|B_{122}| \lesssim (N\omega)^{-1} \omega \|\nabla_{r_2} S_3 \psi\|_{L^2} \|\nabla_{r_2} P_{\geq 1\omega} S_3 \psi\|_{L^2}.$$

Utilizing (43) for the $\|\nabla_{r_2} S_3 \psi\|_{L^2}$ term and (40) for the $\|\nabla_{r_2} P_{\geq 1\omega} S_3 \psi\|_{L^2}$ term,

$$|B_{122}| \lesssim (N\omega)^{-1} \omega^{\frac{3}{2}} \|S_2 S_3\|_{L^2}^2.$$

This requires $\omega \leq N^2$. The terms B_{123} and B_{124} are estimated in the same way as B_{122} , yielding the requirement $\omega \leq N^2$. This completes the treatment of B_{12} .

For B_{11} , we move the operator $(-\Delta_{r_2} + \omega^2 |x_2|^2 - 2\omega)$ over to the right, and use the fact that $(-\Delta_{r_2} + \omega^2 |x_2|^2 - 2\omega) P_{0\omega}^2 \psi = -\partial_{z_2}^2 P_{0\omega}^2 \psi$ to obtain

$$B_{11} = B_{111} + B_{112},$$

where

$$\begin{aligned} B_{111} &= (N\omega)^{\beta-1} \langle (\partial_z V)_{N\omega 23} \psi, \partial_{z_2} P_{0\omega}^2 \psi \rangle, \\ B_{112} &= (N\omega)^{-1} \langle V_{N\omega 23} \partial_{z_2} \psi, \partial_{z_2} P_{0\omega}^2 \psi \rangle. \end{aligned}$$

By (35) applied with r_1 replaced by r_3 , we obtain

$$|B_{111}| \lesssim (N\omega)^{\beta-1} \omega \|S_3 \psi\|_{L^2} \|\partial_{z_2} P_{0\omega}^2 S_3 \psi\|_{L^2}.$$

Using (42) for the $\|\partial_{z_2} P_{0\omega}^2 S_3 \psi\|_{L^2}$ term (which saves us from the $\omega^{\frac{1}{2}}$ loss),

$$|B_{111}| \lesssim (N\omega)^{\beta-1} \omega \|S_3 \psi\|_{L^2} \|S_2 S_3 \psi\|_{L^2},$$

which again requires that $\omega \leq N^{\frac{1-\beta}{\beta}}$. By (35) applied with r_1 replaced by r_3 , we obtain

$$|B_{112}| \lesssim (N\omega)^{-1} \omega \|\partial_{z_2} S_3 \psi\|_{L^2} \|\partial_{z_2} P_{0\omega}^2 S_3 \psi\|_{L^2}.$$

Using (42),

$$|B_{112}| \lesssim (N\omega)^{-1} \omega \|S_2 S_3 \psi\|_{L^2}^2,$$

which has no requirement on ω . This completes the treatment of B_{11} , and hence also B_1 . Now let us consider B_2 :

$$B_2 = N^{-1}\omega^{-2}\langle V_{N\omega 12}V_{N\omega 23}\psi, \psi \rangle,$$

$$|B_2| \leq N^{-1}\omega^{-2} \int |V_{N\omega 23}| \left(\int_{r_1} |V_{N\omega 12}| |\psi(r_1, \dots, r_N)|^2 dr_1 \right) dr_2 \cdots dr_N.$$

In the parentheses, apply estimate (35) in the r_1 -coordinate to obtain

$$|B_2| \lesssim N^{-1}\omega^{-2}\omega \int_{r_2, \dots, r_N} |V_{N\omega 23}| \|S_1\psi\|_{L^2_{r_1}}^2 dr_2 \cdots dr_N.$$

By Fubini, the right-hand side is equal to

$$N^{-1}\omega^{-2}\omega \int_{r_1} \left(\int_{r_2, \dots, r_N} |V_{N\omega 23}| |S_1\psi(r_1, \dots, r_N)|^2 dr_2 \cdots dr_N \right) dr_1.$$

In the parentheses, apply estimate (35) in the r_2 -coordinate to obtain

$$|B_2| \lesssim N^{-1}\omega^{-2}\omega^2 \|S_1S_2\psi\|_{L^2}^2.$$

Hence B_2 is bounded without additional restriction on ω . Therefore we end the proof for the $k = 2$ case.

The k case implies the $k + 2$ case. We assume that (28) holds for k . Applying it with ψ replaced by $(\alpha + N^{-1}H_{N,\omega} - 2\omega)\psi$,

$$\frac{1}{2^k} \|S^{(k)}(\alpha + N^{-1}H_{N,\omega} - 2\omega)\psi\|_{L^2} \leq \langle (\alpha + N^{-1}H_{N,\omega} - 2\omega)^{k+2}\psi, \psi \rangle.$$

Hence, to prove (28) in the case $k + 2$, it suffices to prove

$$\frac{1}{4} (\|S^{(k+2)}\psi\|_{L^2}^2 + N^{-1}\|S_1S^{(k+1)}\psi\|_{L^2}^2) \leq \|S^{(k)}(\alpha + N^{-1}H_{N,\omega} - 2\omega)\psi\|_{L^2}^2. \quad (52)$$

To prove (52), we substitute (32) into

$$\langle S^{(k)}(\alpha + N^{-1}H_{N,\omega} - 2\omega)\psi, S^{(k)}(\alpha + N^{-1}H_{N,\omega} - 2\omega)\psi \rangle,$$

which gives

$$N^{-4} \sum_{\substack{1 \leq i_1 < j_1 \leq N \\ 1 \leq i_2 < j_2 \leq N}} \langle S^{(k)}H_{i_1j_1}\psi, S^{(k)}H_{i_2j_2}\psi \rangle.$$

We decompose into three terms

$$E_1 + E_2 + E_3$$

according to the location of i_1 and i_2 relative to k . We place no restriction on j_1, j_2 (other than $i_1 < j_1, i_2 < j_2$):

- E_1 consists of those terms for which $i_1 \leq k$ and $i_2 \leq k$.
- E_2 consists of those terms for which both $i_1 > k$ and $i_2 > k$.
- E_3 consists of those terms for which either ($i_1 \leq k$ and $i_2 > k$) or ($i_1 > k$ and $i_2 < k$).

We have $E_1 \geq 0$, and we discard this term. We extract the key lower bound from E_2 exactly as in the $k = 2$ case. In fact, inside E_2 , we know $H_{i_1 j_1}$ and $H_{i_2 j_2}$ commute with $S^{(k)}$ because $j_1 > i_1 > k$ and $j_2 > i_2 > k$; hence we indeed face the $k = 2$ case again. This leaves us with E_3 :

$$E_3 = 2N^{-4} \sum_{\substack{1 \leq i_1 < j_1 \leq N \\ 1 \leq i_2 < j_2 \leq N \\ i_1 \leq k, i_2 > k}} \operatorname{Re} \langle S^{(k)} H_{i_1 j_1} \psi, S^{(k)} H_{i_2 j_2} \psi \rangle.$$

We decompose E_3 as

$$E_3 = D_1 + D_2 + D_3,$$

where, in each case we require $i_1 \leq k$ and $i_2 > k$, but make the additional distinctions as follows:

- D_1 consists of those terms where $j_1 \leq k$.
- D_2 consists of those terms where $j_1 > k$ and $j_1 \in \{i_2, j_2\}$.
- D_3 consists of those terms where $j_1 > k$ and $j_1 \notin \{i_2, j_2\}$.

By symmetry,

$$\begin{aligned} D_1 &= k^2 N^{-2} \langle S_1 \cdots S_k H_{12} \psi, S_1 \cdots S_k H_{(k+1)(k+2)} \psi \rangle, \\ D_2 &= k N^{-2} \langle S_1 \cdots S_k H_{1(k+1)} \psi, S_1 \cdots S_k H_{(k+1)(k+2)} \psi \rangle, \\ D_3 &= N^{-1} \langle S_1 \cdots S_k H_{1(k+1)} \psi, S_1 \cdots S_k H_{(k+2)(k+3)} \psi \rangle. \end{aligned}$$

We begin with estimates for the term D_1 . We decompose it as

$$D_1 = D_{11} + D_{12},$$

where

$$\begin{aligned} D_{11} &= N^{-2} \langle H_{(k+1)(k+2)} [S_1 S_2, H_{12}] S_3 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{12} &= N^{-2} \langle H_{(k+1)(k+2)} H_{12} S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle. \end{aligned}$$

By Lemmas 3.7 and A.3, D_{12} is positive because $H_{(k+1)(k+2)}$ and H_{12} commutes. Therefore we discard D_{12} . For D_{11} , we take $[V_{N\omega 12}, S_1 S_2] \sim (N\omega)^{2\beta} (\Delta V)_{N\omega 12}$. This gives

$$|D_{11}| \lesssim N^{2\beta-2} \omega^{2\beta-1} \langle H_{(k+1)(k+2)} (\Delta V)_{N\omega 12} S_3 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle.$$

By using Lemma 3.7 in the r_{k+1} -coordinate to handle $H_{(k+1)(k+2)}$, we have

$$|D_{11}| \lesssim N^{2\beta-2} \omega^{2\beta-1} \| |(\Delta V)_{N\omega 12}|^{\frac{1}{2}} S_3 \cdots S_{k+1} \psi \|_{L^2} \| |(\Delta V)_{N\omega 12}|^{\frac{1}{2}} S_1 \cdots S_{k+1} \psi \|_{L^2}.$$

Using (35) in the first factor,

$$|D_{11}| \lesssim N^{2\beta-2} \omega^{2\beta-\frac{1}{2}} \| S_1 S_3 \cdots S_{k+1} \psi \|_{L^2} \| |(\Delta V)_{N\omega 12}|^{\frac{1}{2}} S_1 \cdots S_{k+1} \psi \|_{L^2}.$$

Decomposing ψ in the second factor into $P_{0\omega}^1 \psi + P_{\geq 1\omega}^1 \psi$ gives

$$\begin{aligned} |D_{11}| &\lesssim N^{2\beta-2} \omega^{2\beta-\frac{1}{2}} \| S_1 S_3 \cdots S_{k+1} \psi \|_{L^2} \\ &\quad \times \left(\| |(\Delta V)_{N\omega 12}|^{\frac{1}{2}} S_1 \cdots S_{k+1} P_{0\omega}^1 \psi \|_{L^2} + \| |(\Delta V)_{N\omega 12}|^{\frac{1}{2}} S_1 \cdots S_{k+1} P_{\geq 1\omega}^1 \psi \|_{L^2} \right). \end{aligned}$$

Applying Lemma 3.6,

$$|D_{11}| \lesssim N^{2\beta-2} \omega^{2\beta-\frac{1}{2}} \|S_1 S_3 \cdots S_{k+1} \psi\|_{L^2} \omega^{\frac{1}{2}} N^{\frac{1}{4}} \|S_1 \cdots S_{k+1} \psi\|_{L^2}^{\frac{1}{2}} (N^{-\frac{1}{4}} \|S_1^2 \cdots S_{k+1} \psi\|_{L^2}^{\frac{1}{2}}) \\ + N^{2\beta-2} \omega^{2\beta-\frac{1}{2}} \|S_1 S_3 \cdots S_{k+1} \psi\|_{L^2} N^{\frac{\beta}{2}+\frac{1}{2}} \omega^{\frac{\beta}{2}} (N^{-\frac{1}{2}} \|S_1^2 \cdots S_{k+1} \psi\|_{L^2}).$$

The coefficients simplify to $N^{2\beta-\frac{7}{4}} \omega^{2\beta}$ and $N^{\frac{5}{2}\beta-\frac{3}{2}} \omega^{\frac{5}{2}\beta-\frac{1}{2}}$. This gives the constraints

$$\omega \leq N^{\frac{7/4-2\beta}{2\beta}} \quad \text{and} \quad \omega \leq N^{\frac{3/5-\beta}{\beta-1/5}}.$$

The second one is the worst one. When combined with the lower bound $N^{\frac{\beta}{1-\beta}} \leq \omega$, it restricts us to $\beta \leq \frac{3}{7}$. Moreover, at $\beta = \frac{2}{5}$, the relation $\omega = N$ is within the allowable range.

We now find estimates for the term D_2 . We write

$$D_2 = D_{21} + D_{22},$$

where

$$D_{21} = N^{-2} \langle H_{(k+1)(k+2)} [S_1, H_{1(k+1)}] S_2 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{22} = N^{-2} \langle H_{(k+1)(k+2)} H_{1(k+1)} S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle.$$

Let us begin with D_{21} . We use

$$[S_1, H_{1(k+1)}] \sim (N\omega)^\beta \omega^{-1} (\nabla V)_{N\omega 1(k+1)}$$

and

$$H_{(k+1)(k+2)} = 2\sigma + S_{k+1}^2 + S_{k+2}^2 + \omega^{-1} V_{N\omega(k+1)(k+2)}$$

to get

$$D_{21} = D_{210} + D_{211} + D_{212} + D_{213},$$

where

$$D_{210} = 2\sigma N^{-1} (N\omega)^{\beta-1} \langle (\nabla V)_{N\omega 1(k+1)} S_2 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{211} = N^{-1} (N\omega)^{\beta-1} \langle S_{k+1}^2 (\nabla V)_{N\omega 1(k+1)} S_2 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{212} = N^{-1} (N\omega)^{\beta-1} \langle S_{k+2}^2 (\nabla V)_{N\omega 1(k+1)} S_2 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{213} = N^{-2} (N\omega)^\beta \omega^{-2} \langle V_{N\omega(k+1)(k+2)} (\nabla V)_{N\omega 1(k+1)} S_2 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle.$$

For D_{211} ,

$$D_{211} = N^{-1} (N\omega)^{\beta-1} \langle [S_{k+1}, (\nabla V)_{N\omega 1(k+1)}] S_2 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle \\ + N^{-1} (N\omega)^{\beta-1} \langle (\nabla V)_{N\omega 1(k+1)} S_2 \cdots S_k S_{k+1} \psi, S_1 \cdots S_k \psi \rangle.$$

The first piece is estimated the same way as D_{11} . For the second term, using Lemma 3.6 in the r_1 -coordinate,

$$|\cdot| \lesssim N^{-1} (N\omega)^{\beta-1} \omega N^{\frac{1}{4}} \|S_1 \cdots S_{k+1} \psi\|_{L^2} \|S_1 \cdots S_k \psi\|_{L^2}^{\frac{1}{2}} (N^{-\frac{1}{4}} \|S_1 S_1 \cdots S_k \psi\|_{L^2}) \\ + N^{-1} (N\omega)^{\beta-1} (N\omega)^{\frac{1}{2}\beta+\frac{1}{2}} \|S_1 \cdots S_{k+1} \psi\|_{L^2} (N^{-\frac{1}{2}} \|S_1 S_1 \cdots S_k \psi\|_{L^2}),$$

which gives the conditions $\omega \leq N^{\frac{7/4-\beta}{\beta}}$ and $\omega \leq N^{\frac{3-3\beta}{3\beta-1}}$. Since this results in conditions better than those produced for D_{11} , we neglect them.

For D_{213} , we apply estimate (35) in the r_{k+2} -coordinate and again in the r_{k+1} -coordinate to obtain

$$|D_{213}| \lesssim N^{-2}(N\omega)^\beta \omega^{-2} \omega^2 \|S_2 \cdots S_{k+2} \psi\|_{L^2} \|S_1 \cdots S_{k+2} \psi\|_{L^2}.$$

This gives the requirement $\omega \leq N^{\frac{2-\beta}{\beta}}$, which is clearly weaker than $\omega \leq N^{\frac{1-\beta}{\beta}}$, so we drop it. The terms D_{210} and D_{212} are estimated in the same way. In fact, utilizing estimate (35) in the r_{k+1} -coordinate yields

$$|D_{210}| \lesssim N^{-1}(N\omega)^{\beta-1} \omega \|S_2 \cdots S_k \psi\|_{L^2} \|S_1 \cdots S_k \psi\|_{L^2},$$

$$|D_{212}| \lesssim N^{-1}(N\omega)^{\beta-1} \omega \|S_2 \cdots S_{k+2} \psi\|_{L^2} \|S_1 \cdots S_{k+2} \psi\|_{L^2}.$$

They give the same weaker condition $\omega \leq N^{\frac{2-\beta}{\beta}}$.

We now turn to D_{22} . Since $H_{(k+1)(k+2)}$ and $H_{1(k+1)}$ do not commute, we cannot directly quote Lemma 3.7 and conclude it is positive. We estimate it. By the definition of H_{ij} , we only need to look at the terms

$$\begin{aligned} D_{220} &= N^{-2} \omega^{-1} \langle \sigma V_{N\omega 1(k+1)} S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{221} &= N^{-2} \omega^{-1} \langle S_{k+1}^2 V_{N\omega 1(k+1)} S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{222} &= N^{-2} \omega^{-1} \langle S_{k+2}^2 V_{N\omega 1(k+1)} S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{223} &= N^{-2} \omega^{-2} \langle V_{N\omega(k+1)(k+2)} V_{N\omega 1(k+1)} S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{224} &= N^{-2} \omega^{-1} \langle \sigma V_{N\omega(k+1)(k+2)} S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{225} &= N^{-2} \omega^{-1} \langle V_{N\omega(k+1)(k+2)} S_1^2 S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle, \\ D_{226} &= N^{-2} \omega^{-1} \langle V_{N\omega(k+1)(k+2)} S_{k+1}^2 S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle \end{aligned}$$

because all the other terms inside the expansion of D_{22} are positive. It is easy to tell the following: the terms D_{220} and D_{224} can be estimated in the same way as D_{210} , the terms D_{221} and D_{226} can be estimated in the same way as D_{211} , the terms D_{222} and D_{225} can be estimated in the same way as D_{212} , and the term D_{223} can be estimated in the same way as D_{213} . Moreover, all the D_{22} terms are better than the corresponding D_{21} terms since they do not have a $(N\omega)^\beta$ in front of them. Hence, we get no new restrictions from D_{22} and we conclude the estimate for D_{22} .

We now find estimates for the term D_3 . Commuting terms as usual,

$$D_3 = D_{31} + D_{32},$$

where

$$D_{31} = N^{-1} \langle H_{(k+2)(k+3)} [S_1, H_{1(k+1)}] S_2 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle,$$

$$D_{32} = N^{-1} \langle H_{(k+2)(k+3)} H_{1(k+1)} S_1 \cdots S_k \psi, S_1 \cdots S_k \psi \rangle.$$

Since $H_{(k+2)(k+3)}$ and $H_{1(k+1)}$ commute, D_{32} is positive due to Lemmas 3.7 and A.3. Thus we discard D_{32} . For D_{31} , we use that

$$[S_1, H_{1(k+1)}] \sim (N\omega)^\beta \omega^{-1} (\nabla V)_{N\omega 1(k+1)}$$

together with estimate (35) in the r_{k+1} -coordinate (to handle $[S_1, H_{1(k+1)}]$) and Lemma 3.7 in the r_{k+2} -coordinate (to handle $H_{(k+2)(k+3)}$):

$$|D_{31}| \lesssim N^{-1} (N\omega)^\beta \|S_2 \cdots S_{k+2} \psi\|_{L^2} \|S_1 \cdots S_{k+2} \psi\|_{L^2}.$$

This term again yields to the restriction

$$\omega \leq N^{\frac{1-\beta}{\beta}}.$$

So far, we have proved that all the terms in E_3 can be absorbed into the key lower bound exacted from E_2 for all N large enough as long as $C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_E(\beta)}$. Hence we have finished the two-step induction argument and established Theorem 3.1.

4. Compactness of the BBGKY sequence

Theorem 4.1. *Assume $C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}$. Then the sequence*

$$\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\} \subset \bigoplus_{k \geq 1} C([0, T], \mathcal{L}_k^1),$$

which satisfies the focusing “ $\infty - \infty$ ” BBGKY hierarchy (18), is compact with respect to the product topology τ_{prod} . For any limit point $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^N$, we have $\tilde{\gamma}^{(k)}$ is a symmetric nonnegative trace class operator with trace bounded by 1.

Proof. By the standard diagonalization argument, it suffices to show the compactness of $\tilde{\gamma}_{N,\omega}^{(k)}$ for fixed k with respect to the metric \hat{d}_k . By the Arzelà–Ascoli theorem, this is equivalent to the equicontinuity of $\tilde{\gamma}_{N,\omega}^{(k)}$. By [Erdős et al. 2010, Lemma 6.2], it suffices to prove that for every test function $J^{(k)}$ from a dense subset of $\mathcal{K}(L^2(\mathbb{R}^{3k}))$ and for every $\varepsilon > 0$, there exists $\delta(J^{(k)}, \varepsilon)$ such that for all $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| \leq \delta$, we can write

$$\sup_{N,\omega} |\text{Tr } J^{(k)} \tilde{\gamma}_{N,\omega}^{(k)}(t_1) - \text{Tr } J^{(k)} \tilde{\gamma}_{N,\omega}^{(k)}(t_2)| \leq \varepsilon. \tag{53}$$

Here, we assume that our compact operators $J^{(k)}$ have been cut off in frequency as in Lemma A.6. Assume $t_1 \leq t_2$. Inserting the decomposition (22) on the left and right sides of $\gamma_{N,\omega}^{(k)}$, we obtain

$$\tilde{\gamma}_{N,\omega}^{(k)} = \sum_{\alpha,\beta} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta,$$

where the sum is taken over all k -tuples α and β of the type described in (22).

To establish (53) it suffices to prove that, for each α and β , we have

$$\sup_{N,\omega} |\text{Tr } J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_1) - \text{Tr } J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_2)| \leq \varepsilon. \tag{54}$$

To this end, we establish the estimate

$$\begin{aligned} & |\text{Tr } J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_1) - \text{Tr } J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_2)| \\ & \lesssim C |t_2 - t_1| (\mathbf{1}_{\alpha=0 \text{ and } \beta=0} + \max(1, \omega^{1-\frac{1}{2}|\alpha|-\frac{1}{2}|\beta|}) \mathbf{1}_{\alpha \neq 0 \text{ or } \beta \neq 0}). \end{aligned} \tag{55}$$

At a glance, (55) seems not quite enough in the $|\alpha| = 0$ and $|\beta| = 1$ case (or vice versa) because it grows in ω . However, we can also prove the (comparatively simpler) bound

$$|\mathrm{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_2) - \mathrm{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_1)| \lesssim \omega^{-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|}, \quad (56)$$

which provides a better power of ω but no gain as $t_2 \rightarrow t_1$. Interpolating between (55) and (56) in the $|\alpha| = 0$ and $|\beta| = 1$ case (or vice versa), we acquire

$$|\mathrm{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_2) - \mathrm{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_1)| \lesssim |t_2 - t_1|^{\frac{1}{2}},$$

which suffices to establish (54).

Below, we prove (55) and (56). We first prove (55). The BBGKY hierarchy (18) yields

$$\partial_t \mathrm{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \quad (57)$$

where

$$\begin{aligned} \mathrm{I} &= -i\omega \sum_{j=1}^k \mathrm{Tr} J^{(k)} [-\Delta_{x_j} + |x_j|^2, \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta], \\ \mathrm{II} &= -i \sum_{j=1}^k \mathrm{Tr} J^{(k)} [-\partial_{z_j}^2, \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta], \\ \mathrm{III} &= \frac{-i}{N} \sum_{1 \leq i < j \leq k} \mathrm{Tr} J^{(k)} \mathcal{P}_\alpha [V_{N,\omega}(r_i - r_j), \tilde{\gamma}_{N,\omega}^{(k)}] \mathcal{P}_\beta, \\ \mathrm{IV} &= -i \frac{N-k}{N} \sum_{j=1}^k \mathrm{Tr} J^{(k)} \mathcal{P}_\alpha [V_{N,\omega}(r_j - r_{k+1}), \tilde{\gamma}_{N,\omega}^{(k+1)}] \mathcal{P}_\beta. \end{aligned}$$

We first consider I. When $\alpha = \beta = 0$,

$$\begin{aligned} \mathrm{I} &= -i\omega \sum_{j=1}^k \mathrm{Tr} J^{(k)} [-\Delta_{x_j} + |x_j|^2, \mathcal{P}_0 \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_0] \\ &= -i\omega \sum_{j=1}^k \mathrm{Tr} J^{(k)} [-2 - \Delta_{x_j} + |x_j|^2, \mathcal{P}_0 \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_0] = 0, \end{aligned}$$

since constants commute with everything. When $\alpha \neq 0$ or $\beta \neq 0$, we apply Lemma A.5 and integrate by parts to obtain

$$\begin{aligned} |\mathrm{I}| &\leq \omega \sum_{j=1}^k |\langle J^{(k)} H_j \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle - \langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, H_j \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle| \\ &\leq \omega \sum_{j=1}^k (|\langle J^{(k)} H_j \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle| + |\langle H_j J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle|), \end{aligned}$$

where $H_j = -\Delta_{x_j} + |x_j|^2$. Hence

$$|\text{I}| \lesssim \omega \sum_{j=1}^k (\|J^{(k)} H_j\|_{\text{op}} + \|H_j J^{(k)}\|_{\text{op}}) \|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\beta \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})}.$$

By the energy estimate (31),

$$\begin{aligned} |\text{I}| &= 0, & \text{if } \alpha = 0 \text{ and } \beta = 0, \\ |\text{I}| &\lesssim C_{k,J^{(k)}} \omega^{1-\frac{1}{2}|\alpha|-\frac{1}{2}|\beta|}, & \text{otherwise.} \end{aligned} \quad (58)$$

Next, consider II. Proceeding as in I, we have

$$|\text{II}| \leq \sum_{j=1}^k (|\langle J^{(k)} \partial_{z_j}^2 \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle| + |\langle \partial_{z_j}^2 J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle|).$$

That is,

$$|\text{II}| \leq \sum_{j=1}^k (\|J^{(k)} \partial_{z_j}^2\|_{\text{op}} + \|\partial_{z_j}^2 J^{(k)}\|_{\text{op}}) \|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\beta \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \leq C_{k,J^{(k)}}. \quad (59)$$

Now, consider III:

$$\begin{aligned} |\text{III}| &\leq N^{-1} \sum_{1 \leq i < j \leq k} |\langle J^{(k)} \mathcal{P}_\alpha V_{N,\omega}(r_i - r_j) \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle| \\ &\quad + N^{-1} \sum_{1 \leq i < j \leq k} |\langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta V_{N,\omega}(r_i - r_j) \tilde{\psi}_{N,\omega} \rangle|. \end{aligned}$$

That is,

$$\begin{aligned} |\text{III}| &\leq N^{-1} \sum_{1 \leq i < j \leq k} |\langle J^{(k)} \mathcal{P}_\alpha L_i L_j W_{ij} L_i L_j \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle| \\ &\quad + N^{-1} \sum_{1 \leq i < j \leq k} |\langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta L_i L_j W_{ij} L_i L_j \tilde{\psi}_{N,\omega} \rangle| \end{aligned}$$

if we write $L_i = (1 - \Delta_{r_i})^{\frac{1}{2}}$ and

$$W_{ij} = L_i^{-1} L_j^{-1} V_{N,\omega}(r_i - r_j) L_i^{-1} L_j^{-1}.$$

Hence

$$\begin{aligned} |\text{III}| &\leq N^{-1} \sum_{1 \leq i < j \leq k} \|J^{(k)} L_i L_j\|_{\text{op}} \|W_{ij}\|_{\text{op}} \|L_i L_j \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\beta \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \\ &\quad + N^{-1} \sum_{1 \leq i < j \leq k} \|L_i L_j J^{(k)}\|_{\text{op}} \|W_{ij}\|_{\text{op}} \|L_i L_j \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})}. \end{aligned}$$

Since $\|W_{ij}\|_{\text{op}} \lesssim \|V_{N,\omega}\|_{L^1} = \|V\|_{L^1}$ (independent of N, ω) by Lemma A.1, the energy estimates (Corollary 3.2) imply that

$$|\text{III}| \lesssim \frac{C_{k,J^{(k)}}}{N}. \quad (60)$$

Apply the same ideas to IV:

$$\begin{aligned}
 |\text{IV}| \leq \sum_{j=1}^k & \left| \langle J^{(k)} \mathcal{P}_\alpha L_j L_{k+1} W_{j(k+1)} L_j L_{k+1} \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle \right| \\
 & + \sum_{j=1}^k \left| \langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta L_j L_{k+1} W_{j(k+1)} L_j L_{k+1} \tilde{\psi}_{N,\omega} \rangle \right|.
 \end{aligned}$$

Then, since $J^{(k)} L_{k+1} = L_{k+1} J^{(k)}$,

$$\begin{aligned}
 |\text{IV}| & \leq \sum_{j=1}^k (\|J^{(k)} L_j\|_{\text{op}} + \|L_j J^{(k)}\|_{\text{op}}) \|W_{j(k+1)}\|_{\text{op}} \|L_j L_{k+1} \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|L_j \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \\
 & \lesssim C_{k,J^{(k)}}.
 \end{aligned} \tag{61}$$

Integrating (57) from t_1 to t_2 and applying the bounds obtained in (58)–(61), we obtain (55).

Finally, we prove (56). By Lemma A.5,

$$\begin{aligned}
 \left| \text{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_2) - \text{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_1) \right| & \leq 2 \sup_t \left| \langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}(t), \mathcal{P}_\beta \tilde{\psi}_{N,\omega}(t) \rangle \right| \\
 & \lesssim \|J^{(k)}\|_{\text{op}} \|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\beta \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})};
 \end{aligned}$$

that is,

$$\left| \text{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_2) - \text{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_1) \right| \lesssim \omega^{-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|}$$

once we apply (31). \square

With Theorem 4.1, we can start talking about the limit points of $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\}$. With the proofs of [X. Chen and Holmer 2013, Theorem 5 and Corollary 2], we arrive at the following corollary and theorem.

Corollary 4.2. *Let $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ be a limit point of $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\}$, with respect to the product topology τ_{prod} . Then $\tilde{\gamma}^{(k)}$ satisfies the a priori bound*

$$\text{Tr} L^{(k)} \tilde{\gamma}^{(k)} L^{(k)} \leq C^k \tag{62}$$

and takes the structure

$$\tilde{\gamma}^{(k)}(t, (\mathbf{x}_k, \mathbf{z}_k); (\mathbf{x}'_k, \mathbf{z}'_k)) = \left(\prod_{j=1}^k h_1(x_j) h_1(x'_j) \right) \tilde{\gamma}_z^{(k)}(t, \mathbf{z}_k; \mathbf{z}'_k), \tag{63}$$

where $\tilde{\gamma}_z^{(k)} = \text{Tr}_x \tilde{\gamma}^{(k)}$.

Theorem 4.3. *Assume $C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}$. Then the sequence*

$$\left\{ \Gamma_{z,N,\omega}(t) = \{\tilde{\gamma}_{z,N,\omega}^{(k)} = \text{Tr}_x \tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N \right\} \subset \bigoplus_{k \geq 1} C([0, T], \mathcal{L}_k^1(\mathbb{R}^k))$$

is compact with respect to the one-dimensional version of the product topology τ_{prod} used in Theorem 4.1.

5. Limit points satisfy GP hierarchy

Theorem 5.1. *Let $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ be a $C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}$ limit point of $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\}$ with respect to the product topology τ_{prod} . Then $\{\tilde{\gamma}_z^{(k)} = \text{Tr}_x \tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a solution to the coupled focusing Gross–Pitaevskii hierarchy (23) subject to initial data $\tilde{\gamma}_z^{(k)}(0) = |\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0 = |\int V(r) dr|$, which, rewritten in integral form, is*

$$\tilde{\gamma}_z^{(k)} = U^{(k)}(t)\tilde{\gamma}_z^{(k)}(0) + ib_0 \sum_{j=1}^k \int_0^t U^{(k)}(t-s) \text{Tr}_{z_{k+1}} \text{Tr}_x [\delta(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}(s)] ds, \quad (64)$$

where $U^{(k)}(t) = \prod_{j=1}^k e^{it\partial_{z_j}^2} e^{-it\partial_{z_j'}^2}$.

Remark. The proof of Theorem 5.1 is a bit special for the focusing case and is dimension- and scaling-dependent. So it does not follow from the 3D to 2D defocusing case [X. Chen and Holmer 2013, Theorem 4].

Proof. Passing to subsequences if necessary, we have

$$\begin{aligned} \lim_{\substack{N,\omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \sup_t \text{Tr} J^{(k)}(\tilde{\gamma}_{N,\omega}^{(k)}(t) - \tilde{\gamma}^{(k)}(t)) &= 0 \quad \forall J^{(k)} \in \mathcal{K}(L^2(\mathbb{R}^{3k})), \\ \lim_{\substack{N,\omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \sup_t \text{Tr} J_z^{(k)}(\tilde{\gamma}_{z,N,\omega}^{(k)}(t) - \tilde{\gamma}_z^{(k)}(t)) &= 0 \quad \forall J_z^{(k)} \in \mathcal{K}(L^2(\mathbb{R}^k)) \end{aligned} \quad (65)$$

via Theorems 4.1 and 4.3.

To establish (64), it suffices to test the limit point against the test functions $J_z^{(k)} \in \mathcal{K}(L^2(\mathbb{R}^k))$, as in the proof of Theorem 4.3. We will prove that the limit point satisfies

$$\text{Tr} J_z^{(k)} \tilde{\gamma}_z^{(k)}(0) = \text{Tr} J_z^{(k)} |\phi_0\rangle\langle\phi_0|^{\otimes k} \quad (66)$$

and

$$\text{Tr} J_z^{(k)} \tilde{\gamma}_z^{(k)}(t) = \text{Tr} J_z^{(k)} U^{(k)}(t) \tilde{\gamma}_z^{(k)}(0) + ib_0 \sum_{j=1}^k \int_0^t \text{Tr} J_z^{(k)} U^{(k)}(t-s) [\delta(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}(s)] ds. \quad (67)$$

To this end, we use the coupled focusing BBGKY hierarchy satisfied by $\tilde{\gamma}_{z,N,\omega}^{(k)}$, which, written in the form needed here, is

$$\text{Tr} J_z^{(k)} \tilde{\gamma}_{z,N,\omega}^{(k)}(t) = A + \frac{i}{N} \sum_{i < j}^k B + i \left(1 - \frac{k}{N}\right) \sum_{j=1}^k D,$$

where

$$\begin{aligned} A &= \text{Tr} J_z^{(k)} U^{(k)}(t) \tilde{\gamma}_{z,N,\omega}^{(k)}(0), \\ B &= \int_0^t \text{Tr} J_z^{(k)} U^{(k)}(t-s) [-V_{N,\omega}(r_i - r_j), \tilde{\gamma}_{N,\omega}^{(k)}(s)] ds, \\ D &= \int_0^t \text{Tr} J_z^{(k)} U^{(k)}(t-s) [-V_{N,\omega}(r_j - r_{k+1}), \tilde{\gamma}_{N,\omega}^{(k+1)}(s)] ds. \end{aligned}$$

By (65), we know

$$\begin{aligned} \lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \operatorname{Tr} J_z^{(k)} \tilde{\gamma}_{z, N, \omega}^{(k)}(t) &= \operatorname{Tr} J_z^{(k)} \tilde{\gamma}_z^{(k)}(t), \\ \lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \operatorname{Tr} J_z^{(k)} U^{(k)}(t) \tilde{\gamma}_{z, N, \omega}^{(k)}(0) &= \operatorname{Tr} J_z^{(k)} U^{(k)}(t) \tilde{\gamma}_z^{(k)}(0). \end{aligned}$$

With the argument in [Lieb et al. 2005, p. 64], we infer, from assumption (b) of Theorem 1.1,

$$\tilde{\gamma}_{N, \omega}^{(1)}(0) \rightarrow |h_1 \otimes \phi_0\rangle \langle h_1 \otimes \phi_0| \quad \text{strongly in trace norm;}$$

that is,

$$\tilde{\gamma}_{N, \omega}^{(k)}(0) \rightarrow |h_1 \otimes \phi_0\rangle \langle h_1 \otimes \phi_0|^{\otimes k} \quad \text{strongly in trace norm.}$$

Thus we have checked (66), the left-hand side of (67), and the first term on the right-hand side of (67) for the limit point. We are left to prove that

$$\begin{aligned} \lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \frac{B}{N} &= 0, \\ \lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \left(1 - \frac{k}{N}\right) D &= b_0 \int_0^t J_x^{(k)} U^{(k)}(t-s) [\delta(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}(s)] ds. \end{aligned}$$

We first use an argument similar to the estimates of II and III in the proof of Theorem 4.3 to prove that $|B|$ and $|D|$ are bounded for every finite time t . In fact, since $U^{(k)}$ is a unitary operator which commutes with Fourier multipliers, we have

$$\begin{aligned} |B| &\leq \int_0^t |\operatorname{Tr} J_z^{(k)} U^{(k)}(t-s) [V_{N, \omega}(r_i - r_j), \tilde{\gamma}_{N, \omega}^{(k)}(s)]| ds \\ &= \int_0^t ds |\operatorname{Tr} L_i^{-1} L_j^{-1} J_z^{(k)} L_i L_j U^{(k)}(t-s) W_{ij} L_i L_j \tilde{\gamma}_{N, \omega}^{(k)}(s) L_i L_j \\ &\quad - \operatorname{Tr} L_i L_j J_z^{(k)} L_i^{-1} L_j^{-1} U^{(k)}(t-s) L_i L_j \tilde{\gamma}_{N, \omega}^{(k)}(s) L_i L_j W_{ij}| \\ &\leq \int_0^t ds \|L_i^{-1} L_j^{-1} J_z^{(k)} L_i L_j\|_{\text{op}} \|U^{(k)}\|_{\text{op}} \|W_{ij}\| \operatorname{Tr} L_i L_j \tilde{\gamma}_{N, \omega}^{(k)}(s) L_i L_j \\ &\quad + \int_0^t ds \|L_i L_j J_z^{(k)} L_i^{-1} L_j^{-1}\|_{\text{op}} \|U^{(k)}\|_{\text{op}} \|W_{ij}\| \operatorname{Tr} L_i L_j \tilde{\gamma}_{N, \omega}^{(k)}(s) L_i L_j \\ &\leq C_J t. \end{aligned}$$

That is,

$$\lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \frac{B}{N} = \lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \frac{kD}{N} = 0.$$

We now use Lemma A.2 (stated and proved in Appendix A), which compares the δ -function and its approximation, to prove

$$\lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} D = b_0 \int_0^t \text{Tr} J_z^{(k)} U^{(k)}(t-s) [\delta(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}(s)] ds. \quad (68)$$

Pick a probability measure $\rho \in L^1(\mathbb{R}^3)$ and define $\rho_\alpha(r) = \alpha^{-3} \rho(r/\alpha)$. Letting $M_{s-t}^{(k)} = J_z^{(k)} U^{(k)}(t-s)$, we have

$$|\text{Tr} J_z^{(k)} U^{(k)}(t-s) (-V_{N,\omega}(r_j - r_{k+1}) \tilde{\gamma}_{N,\omega}^{(k+1)}(s) - b_0 \delta(r_j - r_{k+1}) \tilde{\gamma}^{(k+1)}(s))| = \text{I} + \text{II} + \text{III} + \text{IV},$$

where

$$\begin{aligned} \text{I} &= |\text{Tr} M_{s-t}^{(k)} (-V_{N,\omega}(r_j - r_{k+1}) - b_0 \delta(r_j - r_{k+1})) \tilde{\gamma}_{N,\omega}^{(k+1)}(s)|, \\ \text{II} &= b_0 |\text{Tr} M_{s-t}^{(k)} (\delta(r_j - r_{k+1}) - \rho_\alpha(r_j - r_{k+1})) \tilde{\gamma}_{N,\omega}^{(k+1)}(s)|, \\ \text{III} &= b_0 |\text{Tr} M_{s-t}^{(k)} \rho_\alpha(r_j - r_{k+1}) (\tilde{\gamma}_{N,\omega}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s))|, \\ \text{IV} &= b_0 |\text{Tr} M_{s-t}^{(k)} (\rho_\alpha(r_j - r_{k+1}) - \delta(r_j - r_{k+1})) \tilde{\gamma}^{(k+1)}(s)|. \end{aligned}$$

Consider I. Writing $V_\omega(r) = (1/\omega)V(x/\sqrt{\omega}, z)$, we have $V_{N,\omega} = (N\omega)^{3\beta} V_\omega((N\omega)^\beta r)$. Lemma A.2 then yields

$$\begin{aligned} \text{I} &\leq \frac{C b_0}{(N\omega)^{\beta\kappa}} \left(\int |V_\omega(r)| |r|^\kappa dr \right) (\|L_j J_z^{(k)} L_j^{-1}\|_{\text{op}} + \|L_j^{-1} J_z^{(k)} L_j\|_{\text{op}}) L_j L_{k+1} \tilde{\gamma}_{N,\omega}^{(k+1)}(s) L_j L_{k+1} \\ &= C_J \frac{\left(\int |V_\omega(r)| |r|^\kappa dr \right)}{(N\omega)^{\beta\kappa}}. \end{aligned}$$

Notice that $(\int |V_\omega(r)| |r|^\kappa dr)$ grows like $(\sqrt{\omega})^\kappa$, so

$$I \leq C_J \left(\frac{\sqrt{\omega}}{(N\omega)^\beta} \right)^\kappa,$$

which converges to zero as $N, \omega \rightarrow \infty$ in the way in which $N \geq \omega^{\frac{1}{2\beta}-1+}$. So we have proved

$$\lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} I = 0.$$

Similarly, for II and IV, via Lemma A.2, we have

$$\text{II} \leq C b_0 \alpha^\kappa (\|L_j J_z^{(k)} L_j^{-1}\|_{\text{op}} + \|L_j^{-1} J_z^{(k)} L_j\|_{\text{op}}) \text{Tr} L_j L_{k+1} \tilde{\gamma}_{N,\omega}^{(k+1)}(s) L_j L_{k+1} \leq C b_0 \alpha^\kappa C_{J_z^{(k)}} C^2,$$

where the second inequality follows from Corollary 3.2, and

$$\text{IV} \leq C b_0 \alpha^\kappa (\|L_j J_z^{(k)} L_j^{-1}\|_{\text{op}} + \|L_j^{-1} J_z^{(k)} L_j\|_{\text{op}}) \text{Tr} L_j L_{k+1} \tilde{\gamma}^{(k+1)}(s) L_j L_{k+1} \leq C b_0 \alpha^\kappa C_{J_z^{(k)}} C^2,$$

where the second inequality follows from Corollary 4.2; that is,

$$\text{II} \leq C_J \alpha^\kappa \quad \text{and} \quad \text{IV} \leq C_J \alpha^\kappa,$$

due to the energy estimate (Corollary 4.2). Hence II and IV converge to 0 as $\alpha \rightarrow 0$, uniformly in N, ω .

For III,

$$\begin{aligned} \text{III} \leq b_0 & \left| \text{Tr} J_{s-t}^{(k)} \rho_\alpha(r_j - r_{k+1}) \frac{1}{1 + \varepsilon L_{k+1}} (\tilde{\gamma}_{N,\omega}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s)) \right| \\ & + b_0 \left| \text{Tr} J_{s-t}^{(k)} \rho_\alpha(r_j - r_{k+1}) \frac{\varepsilon L_{k+1}}{1 + \varepsilon L_{k+1}} (\tilde{\gamma}_{N,\omega}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s)) \right|. \end{aligned}$$

The first term in the above estimate goes to zero as $N, \omega \rightarrow \infty$ for every $\varepsilon > 0$, since we have assumed condition (65) and $J_{s-t}^{(k)} \rho_\alpha(r_j - r_{k+1})(1 + \varepsilon L_{k+1})^{-1}$ is a compact operator. Due to the energy bounds on $\tilde{\gamma}_{N,\omega}^{(k+1)}$ and $\tilde{\gamma}^{(k+1)}$, the second term tends to zero as $\varepsilon \rightarrow 0$, uniformly in N and ω .

Putting together the estimates for I–IV, we have justified limit (68). Hence, we have obtained Theorem 5.1. \square

Combining Corollary 4.2 and Theorem 5.1, we see that $\tilde{\gamma}_z^{(k)}$ in fact solves the 1D focusing Gross–Pitaevskii hierarchy with the desired coupling constant $b_0(\int |h_1(x)|^4 dx)$.

Corollary 5.2. *Let $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ be a $N \geq \omega^{v(\beta)+\varepsilon}$ limit point of $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\}$ with respect to the product topology τ_{prod} . Then $\{\tilde{\gamma}_z^{(k)} = \text{Tr}_x \tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a solution to the 1D Gross–Pitaevskii hierarchy (24) subject to initial data $\tilde{\gamma}_z^{(k)}(0) = |\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0(\int |h_1(x)|^4 dx)$, which, rewritten in integral form, is*

$$\begin{aligned} \tilde{\gamma}_z^{(k)} &= U^{(k)}(t) \tilde{\gamma}_z^{(k)}(0) \\ &+ i b_0 \left(\int |h_1(x)|^4 dx \right) \sum_{j=1}^k \int_0^t U^{(k)}(t-s) \text{Tr}_{z_{k+1}} [\delta(z_j - z_{k+1}), \tilde{\gamma}_z^{(k+1)}(s)] ds. \end{aligned} \quad (69)$$

Proof. This is a direct computation by plugging (63) into (64). \square

Appendix A: Basic operator facts and Sobolev-type lemmas

Lemma A.1 [Erdős et al. 2007, Lemma A.3]. *Let $L_j = (1 - \Delta_{r_j})^{\frac{1}{2}}$. Then we have*

$$\|L_i^{-1} L_j^{-1} V(r_i - r_j) L_i^{-1} L_j^{-1}\|_{\text{op}} \leq C \|V\|_{L^1}.$$

Lemma A.2. *Let $f \in L^1(\mathbb{R}^3)$ be such that $\int_{\mathbb{R}^3} \langle r \rangle^{\frac{1}{2}} |f(r)| dr < \infty$ and $\int_{\mathbb{R}^3} f(r) dr = 1$ but we allow that f not be nonnegative everywhere. Define $f_\alpha(r) = \alpha^{-3} f(r/\alpha)$. Then, for every $\kappa \in (0, \frac{1}{2})$, there exists $C_\kappa > 0$ such that*

$$\begin{aligned} & \left| \text{Tr} J^{(k)} (f_\alpha(r_j - r_{k+1}) - \delta(r_j - r_{k+1})) \gamma^{(k+1)} \right| \\ & \leq C_\kappa \left(\int |f(r)| |r|^\kappa dr \right) \alpha^\kappa (\|L_j J^{(k)} L_j^{-1}\|_{\text{op}} + \|L_j^{-1} J^{(k)} L_j\|_{\text{op}}) \text{Tr} L_j L_{k+1} \gamma^{(k+1)} L_j L_{k+1} \end{aligned}$$

for all nonnegative $\gamma^{(k+1)} \in \mathcal{L}^1(L^2(\mathbb{R}^{3k+3}))$.

Proof. This is the same as [X. Chen and Holmer 2016b, Lemma A.3; 2013, Lemma 2]. See [Kirkpatrick et al. 2011; T. Chen and Pavlović 2011; Erdős et al. 2007] for similar lemmas. \square

Lemma A.3 (some standard operator inequalities).

- (1) Suppose that $A \geq 0$, $P_j = P_j^*$, and $I = P_0 + P_1$. Then $A \leq 2P_0AP_0 + 2P_1AP_1$.
- (2) If $A \geq B \geq 0$, and $AB = BA$, then $A^\alpha \geq B^\alpha$ for any $\alpha \geq 0$.
- (3) If $A_1 \geq A_2 \geq 0$, $B_1 \geq B_2 \geq 0$ and $A_iB_j = B_jA_i$ for all $1 \leq i, j \leq 2$, then $A_1B_1 \geq A_2B_2$.
- (4) If $A \geq 0$ and $AB = BA$, then $A^{\frac{1}{2}}B = BA^{\frac{1}{2}}$.

Proof. For (1), $\|A^{\frac{1}{2}}f\|^2 = \|A^{\frac{1}{2}}(P_0 + P_1)f\|^2 \leq 2\|A^{\frac{1}{2}}P_0f\|^2 + 2\|A^{\frac{1}{2}}P_1f\|^2$. For (3), $A_1B_1 \geq A_2B_1 = B_1A_2 \geq B_2A_2 = A_2B_2$. The rest, (2) and (4), are standard facts in operator theory. See, for example, [Reed and Simon 1978; Stein and Shakarchi 2005, Proposition 6.3]. \square

Lemma A.4. Recall

$$\tilde{S} = (1 - \partial_z^2 + \omega(-2 - \Delta_x + |x|^2))^{\frac{1}{2}}.$$

We have

$$\tilde{S}^2 \gtrsim 1 - \Delta_r, \quad (70)$$

$$\tilde{S}^2 P_{\geq 1} \gtrsim P_{\geq 1}(1 - \partial_z^2 - \omega\Delta_x + \omega|x|^2)P_{\geq 1}, \quad (71)$$

$$\tilde{S}^2 P_{\geq 1} \gtrsim \omega P_{\geq 1}. \quad (72)$$

Proof. Directly from the definition of \tilde{S} , we have

$$\underbrace{P_{\geq 1}(1 - \partial_z^2 - \omega\Delta_x + \omega|x|^2)P_{\geq 1}}_{\text{all terms positive}} = 2\omega P_{\geq 1} + \tilde{S}^2 P_{\geq 1}. \quad (73)$$

The eigenvalues of the 2D Hermite operator $-\Delta_x + |x|^2$ are $\{2k + 2\}_{k=0}^\infty$. So

$$2\omega P_{\geq 1} \leq \omega(-2 - \Delta_x + |x|^2)P_{\geq 1} \leq \tilde{S}^2 P_{\geq 1}. \quad (74)$$

Inequalities (71) and (72) immediately follow from (73) and (74).

We now establish (70) using (71). On the one hand, we have

$$\tilde{S}^2 \geq (1 - \partial_z^2). \quad (75)$$

On the other hand,

$$P_0(-\Delta_x)P_0 \lesssim 1 \leq \tilde{S}^2 \quad (76)$$

since P_0 is merely the projection onto the smooth function $Ce^{-\frac{1}{2}|x|^2}$. Moreover, by (71),

$$P_{\geq 1}(-\Delta_x)P_{\geq 1} \leq \tilde{S}^2 P_{\geq 1} \leq \tilde{S}^2. \quad (77)$$

Thus Lemma A.3(1), (76) and (77) together imply,

$$-\Delta_x \lesssim \tilde{S}^2. \quad (78)$$

The claimed inequality (70) then follows from (75) and (78). \square

Lemma A.5. Suppose $\sigma : L^2(\mathbb{R}^{3k}) \rightarrow L^2(\mathbb{R}^{3k})$ has kernel

$$\sigma(\mathbf{r}_k, \mathbf{r}'_k) = \int \psi(\mathbf{r}_k, \mathbf{r}_{N-k}) \overline{\psi}(\mathbf{r}'_k, \mathbf{r}_{N-k}) d\mathbf{r}_{N-k}$$

for some $\psi \in L^2(\mathbb{R}^{3N})$, and let $A, B : L^2(\mathbb{R}^{3k}) \rightarrow L^2(\mathbb{R}^{3k})$. Then the composition $A\sigma B$ has kernel

$$(A\sigma B)(\mathbf{r}_k, \mathbf{r}'_k) = \int (A\psi)(\mathbf{r}_k, \mathbf{r}_{N-k}) (\overline{B^*\psi})(\mathbf{r}'_k, \mathbf{r}_{N-k}) d\mathbf{r}_{N-k}.$$

It follows that

$$\text{Tr } A\sigma B = \langle A\psi, B^*\psi \rangle.$$

Let \mathcal{K}_k denote the class of compact operators on $L^2(\mathbb{R}^{3k})$, let \mathcal{L}_k^1 denote the trace class operators on $L^2(\mathbb{R}^{3k})$, and let \mathcal{L}_k^2 denote the Hilbert–Schmidt operators on $L^2(\mathbb{R}^{3k})$. We have

$$\mathcal{L}_k^1 \subset \mathcal{L}_k^2 \subset \mathcal{K}_k.$$

For an operator J on $L^2(\mathbb{R}^{3k})$, let $|J| = (J^*J)^{\frac{1}{2}}$ and denote by $J(\mathbf{r}_k, \mathbf{r}'_k)$ the kernel of J and by $|J|(\mathbf{r}_k, \mathbf{r}'_k)$ the kernel of $|J|$, which satisfies $|J|(\mathbf{r}_k, \mathbf{r}'_k) \geq 0$. Let

$$\mu_1 \geq \mu_2 \geq \dots \geq 0$$

be the eigenvalues of $|J|$ repeated according to multiplicity (the *singular values* of J). Then

$$\begin{aligned} \|J\|_{\mathcal{K}_k} &= \|\mu_n\|_{\ell_n^\infty} = \mu_1 = \| |J| \|_{\text{op}} = \|J\|_{\text{op}}, \\ \|J\|_{\mathcal{L}_k^2} &= \|\mu_n\|_{\ell_n^2} = \|J(\mathbf{r}_k, \mathbf{r}'_k)\|_{L^2(\mathbf{r}_k, \mathbf{r}'_k)} = (\text{Tr } J^*J)^{\frac{1}{2}}, \\ \|J\|_{\mathcal{L}_k^1} &= \|\mu_n\|_{\ell_n^1} = \| |J|(\mathbf{r}_k, \mathbf{r}_k)\|_{L^1(\mathbf{r}_k)} = \text{Tr } |J|. \end{aligned}$$

The topology on \mathcal{K}_k coincides with the operator topology, and \mathcal{K}_k is a closed subspace of the space of bounded operators on $L^2(\mathbb{R}^{3k})$.

Lemma A.6. On the one hand, let χ be a smooth function on \mathbb{R}^3 such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. Let

$$(Q_M f)(\mathbf{r}_k) = \int e^{i\mathbf{r}_k \cdot \boldsymbol{\xi}_k} \prod_{j=1}^k \chi(M^{-1}\xi_j) \hat{f}(\boldsymbol{\xi}_k) d\boldsymbol{\xi}_k.$$

On the other hand, with respect to the spectral decomposition of $L^2(\mathbb{R}^2)$ corresponding to the operator $H_j = -\Delta_{x_j}^2 + |x_j|^2$, let X_M^j be the orthogonal projection onto the sum of the first M eigenspaces (in the x_j -variable only) and let

$$R_M = \prod_{j=1}^k X_M^j.$$

We then have the following:

- (1) Suppose that J is a compact operator. Then $J_M := R_M Q_M J Q_M R_M \rightarrow J$ in the operator norm.
- (2) $H_j J_M$, $J_M H_j$, $\Delta_{r_j} J_M$ and $J_M \Delta_{r_j}$ are all bounded.

- (3) *There exists a countable dense subset $\{T_i\}$ of the closed unit ball in the space of bounded operators on $L^2(\mathbb{R}^{3k})$ such that each T_i is compact and in fact for each i there exists M (depending on i) and $Y_i \in \mathcal{K}_k$ with $\|Y_i\|_{\text{op}} \leq 1$ such that $T_i = R_M Q_M Y_i Q_M R_M$.*

Proof. (1) If $S_n \rightarrow S$ strongly and $J \in \mathcal{K}_k$, then $S_n J \rightarrow S J$ in the operator norm and $J S_n \rightarrow J S$ in the operator norm.

(2) This is straightforward.

(3) Start with a subset $\{Y_n\}$ of the closed unit ball in the space of bounded operators on $L^2(\mathbb{R}^{3k})$ such that each Y_n is compact. Then let $\{T_i\}$ be an enumeration of the set $R_M Q_M Y_n Q_M R_M$, where M ranges over the dyadic integers. By (1) this collection will still be dense. The $\{Y_i\}$ in the statement of (3) is just a reindexing of $\{Y_n\}$. \square

Appendix B: Deducing Theorem 1.1 from Theorem 1.2

We first give the following lemma.

Lemma B.1. *Assume $\tilde{\psi}_{N,\omega}(0)$ satisfies (a), (b) and (c) in Theorem 1.1. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off such that $0 \leq \chi \leq 1$, $\chi(s) = 1$ for $0 \leq s \leq 1$ and $\chi(s) = 0$ for $s \geq 2$. For $\kappa > 0$, we define an approximation of $\tilde{\psi}_{N,\omega}(0)$ by*

$$\tilde{\psi}_{N,\omega}^\kappa(0) = \frac{\chi(\kappa(\tilde{H}_{N,\omega} - 2N\omega)/N)\tilde{\psi}_{N,\omega}(0)}{\|\chi(\kappa(\tilde{H}_{N,\omega} - 2N\omega)/N)\tilde{\psi}_{N,\omega}(0)\|}.$$

This approximation has the following properties:

- (i) $\tilde{\psi}_{N,\omega}^\kappa(0)$ verifies the energy condition

$$\langle \tilde{\psi}_{N,\omega}^\kappa(0), (\tilde{H}_{N,\omega} - 2N\omega)^k \tilde{\psi}_{N,\omega}^\kappa(0) \rangle \leq \frac{2^k N^k}{\kappa^k}.$$

- (ii) $\sup_{N,\omega} \|\tilde{\psi}_{N,\omega}(0) - \tilde{\psi}_{N,\omega}^\kappa(0)\|_{L^2} \leq C \kappa^{\frac{1}{2}}$.

- (iii) *For small enough $\kappa > 0$, we have $\tilde{\psi}_{N,\omega}^\kappa(0)$ is asymptotically factorized as well:*

$$\lim_{N,\omega \rightarrow \infty} \text{Tr} |\tilde{\gamma}_{N,\omega}^{\kappa,(1)}(0, x_1, z_1; x'_1, z'_1) - h(x_1)h(x'_1)\phi_0(z_1)\bar{\phi}_0(z'_1)| = 0,$$

where $\tilde{\gamma}_{N,\omega}^{\kappa,(1)}(0)$ is the one-particle marginal density associated with $\tilde{\psi}_{N,\omega}^\kappa(0)$, and ϕ_0 is the same as in assumption (b) in Theorem 1.1.

Proof. Let us write $\chi(\kappa(\tilde{H}_{N,\omega} - 2N\omega))$ as χ and $\tilde{\psi}_{N,\omega}(0)$ as $\tilde{\psi}_{N,\omega}$. This proof closely follows [Erdős et al. 2010, Proposition 8.1(i)–(ii); 2007, Proposition 5.1(iii)].

Property (i) follows by definition. In fact, denote the characteristic function of $[0, \lambda]$ by $\mathbf{1}(s \leq \lambda)$. We see that

$$\chi(\kappa(\tilde{H}_{N,\omega} - 2N\omega)/N) = \mathbf{1}(\tilde{H}_{N,\omega} - 2N\omega \leq 2N/\kappa) \chi(\kappa(\tilde{H}_{N,\omega} - 2N\omega)/N).$$

Thus

$$\begin{aligned} \langle \tilde{\psi}_{N,\omega}^\kappa(0), (\tilde{H}_{N,\omega} - 2N\omega)^k \tilde{\psi}_{N,\omega}^\kappa(0) \rangle &= \left\langle \frac{\chi \tilde{\psi}_{N,\omega}}{\|\chi \tilde{\psi}_{N,\omega}\|}, \mathbf{1}(\tilde{H}_{N,\omega} - 2N\omega \leq 2N/\kappa) (\tilde{H}_{N,\omega} - 2N\omega)^k \frac{\chi \tilde{\psi}_{N,\omega}}{\|\chi \tilde{\psi}_{N,\omega}\|} \right\rangle \\ &\leq \|\mathbf{1}(\tilde{H}_{N,\omega} - 2N\omega \leq 2N/\kappa) (\tilde{H}_{N,\omega} - 2N\omega)^k\|_{\text{op}} \\ &\leq \frac{2^k N^k}{\kappa^k}. \end{aligned}$$

We prove (ii) with a slightly modified proof of [Erdős et al. 2010, Proposition 8.1(ii)]. We still have

$$\begin{aligned} \|\tilde{\psi}_{N,\omega}^\kappa - \tilde{\psi}_{N,\omega}\|_{L^2} &\leq \|\chi \tilde{\psi}_{N,\omega} - \tilde{\psi}_{N,\omega}\|_{L^2} + \left\| \frac{\chi \tilde{\psi}_{N,\omega}}{\|\chi \tilde{\psi}_{N,\omega}\|} - \chi \tilde{\psi}_{N,\omega} \right\|_{L^2} \\ &\leq \|\chi \tilde{\psi}_{N,\omega} - \tilde{\psi}_{N,\omega}\|_{L^2} + |1 - \|\chi \tilde{\psi}_{N,\omega}\|| \\ &\leq 2\|\chi \tilde{\psi}_{N,\omega} - \tilde{\psi}_{N,\omega}\|_{L^2}, \end{aligned}$$

where

$$\begin{aligned} \|\chi \tilde{\psi}_{N,\omega} - \tilde{\psi}_{N,\omega}\|_{L^2}^2 &= \left\langle \psi_N, \left(1 - \chi\left(\frac{\kappa(\tilde{H}_{N,\omega} - 2N\omega)}{N}\right)\right)^2 \psi_N \right\rangle \\ &\leq \left\langle \psi_N, \mathbf{1}\left(\frac{\kappa(\tilde{H}_{N,\omega} - 2N\omega)}{N} \geq 1\right) \psi_N \right\rangle. \end{aligned}$$

To continue estimating, we notice that if $C \geq 0$, then $\mathbf{1}(s \geq 1) \leq \mathbf{1}(s + C \geq 1)$ for all s . So

$$\begin{aligned} \|\chi \tilde{\psi}_{N,\omega} - \tilde{\psi}_{N,\omega}\|_{L^2}^2 &\leq \left\langle \tilde{\psi}_{N,\omega}, \mathbf{1}\left(\frac{\kappa(\tilde{H}_{N,\omega} - 2N\omega)}{N} \geq 1\right) \tilde{\psi}_{N,\omega} \right\rangle \\ &\leq \left\langle \tilde{\psi}_{N,\omega}, \mathbf{1}\left(\frac{\kappa(\tilde{H}_{N,\omega} - 2N\omega + N\alpha)}{N} \geq 1\right) \tilde{\psi}_{N,\omega} \right\rangle. \end{aligned}$$

With the inequality $\mathbf{1}(s \geq 1) \leq s$ for all $s \geq 0$ and the fact that

$$\tilde{H}_{N,\omega} - 2N\omega + N\alpha \geq 0,$$

proved in Theorem 3.1, we arrive at

$$\begin{aligned} \|\chi \tilde{\psi}_{N,\omega} - \tilde{\psi}_{N,\omega}\|_{L^2}^2 &\leq \frac{\kappa}{N} \langle \tilde{\psi}_{N,\omega}, (\tilde{H}_{N,\omega} - 2N\omega + N\alpha) \tilde{\psi}_{N,\omega} \rangle \\ &\leq \frac{\kappa}{N} \langle \tilde{\psi}_{N,\omega}, (\tilde{H}_{N,\omega} - 2N\omega) \tilde{\psi}_{N,\omega} \rangle + \alpha \kappa \langle \tilde{\psi}_{N,\omega}, \tilde{\psi}_{N,\omega} \rangle. \end{aligned}$$

Using (a) and (c) in the assumptions of Theorem 1.1, we deduce that

$$\|\chi \tilde{\psi}_{N,\omega} - \tilde{\psi}_{N,\omega}\|_{L^2}^2 \leq C\kappa,$$

which implies

$$\|\tilde{\psi}_{N,\omega}^\kappa - \tilde{\psi}_{N,\omega}\|_{L^2} \leq C\kappa^{\frac{1}{2}}.$$

Property (iii) does not follow from the proof of [Erdős et al. 2010, Proposition 8.1(iii)] in which the positivity of V is used. Instead (iii) follows from the proof of [Erdős et al. 2007, Proposition 5.1(iii)],

which does not require V to hold a definite sign. Lemma B.1 follows the same proof as [Erdős et al. 2007, Proposition 5.1(iii)] if one replaces H_N by $(\tilde{H}_{N,\omega} - 2N\omega)$ and \hat{H}_N by

$$\sum_{j \geq k+1}^N (-\partial_{z_j} + \omega(-2 - \Delta_{x_j} + |x_j|^2)) + \frac{1}{N} \sum_{k+1 < i < j \leq N} V_{N,\omega}(r_i - r_j).$$

Notice that we are working with $V_{N,\omega} = (N\omega)^{3\beta} V_\omega((N\omega)^\beta r)$, where $V_\omega(r) = (1/\omega)V(x/\sqrt{\omega}, z)$; thus we get

$$(N\omega)^{\frac{3}{2}\beta} \|V_\omega\|_{L^2}^2 \sim \frac{(N\omega)^{\frac{3}{2}\beta}}{\omega}$$

instead of $N^{\frac{3}{2}\beta}$ in [Erdős et al. 2007, (5.20)] and hence we get $(N\omega)^{\frac{3}{2}\beta-1}$ in the estimate (5.18) of the same work, which tends to zero as $N, \omega \rightarrow \infty$ for $\beta \in (0, \frac{2}{3})$. \square

Via (i) and (iii) of Lemma B.1, $\tilde{\psi}_{N,\omega}^\kappa(0)$ verifies the hypothesis of Theorem 1.2 for small enough $\kappa > 0$. Therefore, for $\tilde{\gamma}_{N,\omega}^{\kappa,(1)}(t)$, the marginal density associated with $e^{it\tilde{H}_{N,\omega}}\tilde{\psi}_{N,\omega}^\kappa(0)$, Theorem 1.2 gives the convergence

$$\lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \text{Tr} \left| \tilde{\gamma}_{N,\omega}^{\kappa,(k)}(t, \mathbf{x}_k, \mathbf{z}_k; \mathbf{x}'_k, \mathbf{z}'_k) - \prod_{j=1}^k h_1(x_j)h_1(x'_j)\phi(t, z_j)\bar{\phi}(t, z'_j) \right| = 0 \quad (79)$$

for all small enough $\kappa > 0$, all $k \geq 1$, and all $t \in \mathbb{R}$.

For $\tilde{\gamma}_{N,\omega}^{(k)}(t)$ in Theorem 1.1, we notice that, $\forall J^{(k)} \in \mathcal{K}_k, \forall t \in \mathbb{R}$, we have

$$\begin{aligned} & \left| \text{Tr} J^{(k)}(\tilde{\gamma}_{N,\omega}^{(k)}(t) - |h_1 \otimes \phi(t)\rangle\langle h_1 \otimes \phi(t)|^{\otimes k}) \right| \\ & \leq \left| \text{Tr} J^{(k)}(\tilde{\gamma}_{N,\omega}^{(k)}(t) - \tilde{\gamma}_{N,\omega}^{\kappa,(k)}(t)) \right| + \left| \text{Tr} J^{(k)}(\tilde{\gamma}_{N,\omega}^{\kappa,(k)}(t) - |h_1 \otimes \phi(t)\rangle\langle h_1 \otimes \phi(t)|^{\otimes k}) \right| \\ & = \text{I} + \text{II}. \end{aligned}$$

Convergence (79) then takes care of II. To handle I, part (ii) of Lemma B.1 yields

$$\|e^{it\tilde{H}_{N,\omega}}\tilde{\psi}_{N,\omega}(0) - e^{it\tilde{H}_{N,\omega}}\tilde{\psi}_{N,\omega}^\kappa(0)\|_{L^2} = \|\tilde{\psi}_{N,\omega}(0) - \tilde{\psi}_{N,\omega}^\kappa(0)\|_{L^2} \leq C\kappa^{\frac{1}{2}},$$

which implies

$$\text{I} = \left| \text{Tr} J^{(k)}(\tilde{\gamma}_{N,\omega}^{(k)}(t) - \tilde{\gamma}_{N,\omega}^{\kappa,(k)}(t)) \right| \leq C\|J^{(k)}\|_{\text{op}}\kappa^{\frac{1}{2}}.$$

Since $\kappa > 0$ is arbitrary, we deduce that

$$\lim_{\substack{N, \omega \rightarrow \infty \\ C_1 N^{v_1(\beta)} \leq \omega \leq C_2 N^{v_2(\beta)}}} \left| \text{Tr} J^{(k)}(\tilde{\gamma}_{N,\omega}^{(k)}(t) - |h_1 \otimes \phi(t)\rangle\langle h_1 \otimes \phi(t)|^{\otimes k}) \right| = 0;$$

i.e., as trace class operators

$$\tilde{\gamma}_{N,\omega}^{(k)}(t) \rightarrow |h_1 \otimes \phi(t)\rangle\langle h_1 \otimes \phi(t)|^{\otimes k} \quad \text{weak*}.$$

Then again, Grümm’s convergence theorem upgrades the above weak* convergence to strong. Hence, we have concluded Theorem 1.1 via Theorem 1.2 and Lemma B.1.

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CONFORMALLY EUCLIDEAN METRICS ON \mathbb{R}^n WITH ARBITRARY TOTAL Q -CURVATURE

ALI HYDER

We study the existence of solution to the problem

$$(-\Delta)^{n/2}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \quad \kappa := \int_{\mathbb{R}^n} Qe^{nu} dx < \infty,$$

where $Q \geq 0$, $\kappa \in (0, \infty)$ and $n \geq 3$. Using ODE techniques, Martinazzi (for $n = 6$) and Huang and Ye (for $n = 4m + 2$) proved the existence of a solution to the above problem with $Q \equiv \text{constant} > 0$ and for every $\kappa \in (0, \infty)$. We extend these results in every dimension $n \geq 5$, thus completely answering the problem opened by Martinazzi. Our approach also extends to the case in which Q is nonconstant, and under some decay assumptions on Q we can also treat the cases $n = 3$ and $n = 4$.

1. Introduction

For a function $Q \in C^0(\mathbb{R}^n)$ we consider the problem

$$(-\Delta)^{n/2}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \quad \kappa := \int_{\mathbb{R}^n} Qe^{nu} dx < \infty, \quad (1)$$

where for n odd the nonlocal operator $(-\Delta)^{n/2}$ is defined on page 639.

Geometrically if u is a smooth solution of (1) then the conformal metric $g_u := e^{2u}|dx|^2$ (here $|dx|^2$ is the Euclidean metric on \mathbb{R}^n) has the Q -curvature Q , at least when $n \geq 2$. Moreover, the total Q -curvature of the metric g_u is κ .

Solutions to (1) have been classified in terms of their asymptotic behavior at infinity. More precisely we have the following:

Theorem A [Chen and Li 1991; Da Lio et al. 2015; Lin 1998; Martinazzi 2009a; Jin et al. 2015; Hyder 2015; Xu 2005]. *Let $n \geq 1$. Let u be a solution of*

$$(-\Delta)^{n/2}u = (n-1)!e^{nu} \quad \text{in } \mathbb{R}^n, \quad \kappa := (n-1)! \int_{\mathbb{R}^n} e^{nu} dx < \infty. \quad (2)$$

Then

$$u(x) = \frac{(n-1)!}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right) e^{nu(y)} dy + P(x) = -\frac{2\kappa}{\Lambda_1} \log|x| + P(x) + o(\log|x|) \quad \text{as } |x| \rightarrow \infty, \quad (3)$$

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where $\gamma_n := \frac{1}{2}(n-1)!|S^n|$, $\Lambda_1 := 2\gamma_n$, $o(\log|x|)/\log|x| \rightarrow 0$ as $|x| \rightarrow \infty$, P is a polynomial of degree at most $n-1$ and P is bounded from above. If $n \in \{3, 4\}$ then $\kappa \in (0, \Lambda_1]$ and $\kappa = \Lambda_1$ if and only if u is a spherical solution, that is,

$$u(x) = u_{\lambda, x_0}(x) := \log \frac{2\lambda}{1 + \lambda^2|x - x_0|^2} \quad (4)$$

for some $x_0 \in \mathbb{R}^n$ and $\lambda > 0$. Moreover u is spherical if and only if P is constant (which is always the case when $n \in \{1, 2\}$).

Chang and Chen [2001] showed the existence of nonspherical solutions to (2) in even dimension $n \geq 4$ for every $\kappa \in (0, \Lambda_1)$.

A partial converse to Theorem A has been proven in dimension 4 by Wei and Ye [2008] and extended by Hyder and Martinazzi [2015] for $n \geq 4$ even and Hyder [2016] for $n \geq 3$.

Theorem B [Wei and Ye 2008; Hyder and Martinazzi 2015; Hyder 2016]. *Let $n \geq 3$. Then for every $\kappa \in (0, \Lambda_1)$ and for every polynomial P with*

$$\deg(P) \leq n-1 \quad \text{and} \quad P(x) \xrightarrow{|x| \rightarrow \infty} -\infty,$$

there exists a solution u to (2) having the asymptotic behavior given by (3).

Although the assumption $\kappa \in (0, \Lambda_1]$ is a necessary condition for the existence of a solution to (2) for $n = 3, 4$, it is possible to have a solution for $\kappa > \Lambda_1$ arbitrarily large in higher dimension, as shown by Martinazzi [2013] for $n = 6$. Huang and Ye [2015] extended Martinazzi's result in arbitrary even dimension n of the form $n = 4m + 2$ for some $m \geq 1$, proving that for every $\kappa \in (0, \infty)$ there exists a solution to (2). The case $n = 4m$ remained open.

The ideas in [Martinazzi 2013; Huang and Ye 2015] are based upon ODE theory. One considers only radial solutions so that the equation in (2) becomes an ODE, and the result is obtained by choosing suitable initial conditions and letting one of the parameters go to $+\infty$ (or $-\infty$). However, this technique does not work if the dimension n is a multiple of 4, and things get even worse in odd dimension since $(-\Delta)^{n/2}$ is nonlocal and ODE techniques cannot be used.

In this paper we extend the works of [Martinazzi 2013; Huang and Ye 2015] and completely solve the cases left open; namely we prove that when $n \geq 5$, problem (2) has a solution for every $\kappa \in (0, \infty)$. In fact we do not need to assume that Q is constant, but only that it is radially symmetric with growth at infinity suitably controlled, or not even radially symmetric. Moreover, we are able to prescribe the asymptotic behavior of the solution u , as in (3), up to a polynomial of degree 4 which cannot be prescribed and in particular cannot be required to vanish when $\kappa \geq \Lambda_1$. This in turn, together with Theorem A, is consistent with the requirement $n \geq 5$, because only when $n \geq 5$ does the asymptotic expansion of u at infinity admit polynomials of degree 4.

We prove the following two theorems.

Theorem 1.1. *Let $n \geq 5$ be an integer. Let P be a polynomial on \mathbb{R}^n with degree at most $n-1$. Let $Q \in C^0(\mathbb{R}^n)$ be such that $Q(0) > 0$, $Q \geq 0$, Qe^{nP} is radially symmetric and*

$$\sup_{x \in \mathbb{R}^n} Q(x)e^{nP(x)} < \infty.$$

Then for every $\kappa > 0$ there exists a solution u to (1) such that

$$u(x) = -\frac{2\kappa}{\Lambda_1} \log |x| + P(x) + c_1|x|^2 - c_2|x|^4 + C + o(1) \quad \text{as } |x| \rightarrow \infty$$

for some $c_1, c_2 > 0$ and $C \in \mathbb{R}$. In fact, there exists a radially symmetric function v on \mathbb{R}^n and a constant c_v such that

$$v(x) = -\frac{2\kappa}{\Lambda_1} \log |x| + \frac{1}{2n} \Delta v(0)(|x|^4 - |x|^2) + o(1) \quad \text{as } |x| \rightarrow \infty,$$

and

$$u = P + v + c_v - |x|^4, \quad x \in \mathbb{R}^n.$$

Taking $Q = (n - 1)!$ and $P = 0$ in Theorem 1.1 one has the following corollary.

Corollary 1.2. *Let $n \geq 5$ and $\kappa \in (0, \infty)$. Then there exists a radially symmetric solution u to (2) such that*

$$u(x) = -\frac{2\kappa}{\Lambda_1} \log |x| + c_1|x|^2 - c_2|x|^4 + C + o(1) \quad \text{as } |x| \rightarrow \infty$$

for some $c_1, c_2 > 0$ and $C \in \mathbb{R}$.

Notice the polynomial part of the solution u in Theorem 1.1 is not exactly the prescribed polynomial P (compare to [Wei and Ye 2008; Hyder and Martinazzi 2015; Hyder 2016]). In general, without perturbing the polynomial part, it is not possible to find a solution for $\kappa \geq \Lambda_1$. For example, if P is nonincreasing and nonconstant then there is no solution u to (2) with $\kappa \geq \Lambda_1$ such that u has the asymptotic behavior (3) (see Lemma 3.6 below). This justifies the term $c_1|x|^2$ in Theorem 1.1. Then the additional term $-c_2|x|^4$ is also necessary to avoid that $u(x) \geq \frac{1}{2}c_1|x|^2$ for x large, which would contrast with the condition $\kappa < \infty$, at least if Q does not decay fast enough at infinity. In the latter case, the term $-c_2|x|^4$ can be avoided, and one obtains an existence result also in dimensions 3 and 4.

Theorem 1.3. *Let $n \geq 3$. Let $Q \in C_{\text{rad}}^0(\mathbb{R}^n)$ be such that $Q \geq 0$, $Q(0) > 0$ and*

$$\int_{\mathbb{R}^n} Q(x)e^{\lambda|x|^2} dx < \infty \quad \text{for every } \lambda > 0, \quad \int_{B_1(x)} \frac{Q(y)}{|x-y|^{n-1}} dy \xrightarrow{|x| \rightarrow \infty} 0.$$

Then for every $\kappa > 0$ there exists a radially symmetric solution u to (1).

The decay assumption on Q in Theorem 1.3 is sharp in the sense that if $Qe^{\lambda|x|^2} \notin L^1(\mathbb{R}^n)$ for some $\lambda > 0$, then problem (1) might not have a solution for every $\kappa > 0$. For instance, if $Q = e^{-\lambda|x|^2}$ for some $\lambda > 0$, then (1) with $n = 3, 4$ and $\kappa > \Lambda_1$ has no solution (see Lemma 3.5 below).

The proof of Theorem 1.1 is based on the Schauder fixed point theorem, and the main difficulty is to show that the ‘‘approximate solutions’’ are precompact (see in particular Lemma 2.2). We will do that using blow up analysis (see for instance [Adimurthi et al. 2006; Martinazzi 2009b; Robert 2006]). In general, if $\kappa \geq \Lambda_1$ one can expect blow up, but we will construct our approximate solutions carefully in a way that this does not happen. For instance in [Wei and Ye 2008; Hyder and Martinazzi 2015] one looks for solutions of the form $u = P + v + c_v$, where v satisfies the integral equation

$$v(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{nP(y)} e^{n(v(y)+c_v)} dy,$$

and c_v is a constant such that

$$\int_{\mathbb{R}^n} Q e^{n(P+v+c_v)} dx = \kappa.$$

With such a choice we would not be able to rule out blow up. Instead, by looking for solutions of the form

$$u = P + v + P_v + c_v,$$

where a posteriori $P_v = -|x|^4$, v satisfies

$$v(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y) e^{n(P(y)+P_v(y)+v(y)+c_v)} dy + \frac{1}{2n} (|x|^2 - |x|^4) |\Delta v(0)|, \quad (5)$$

and c_v is again a normalization constant, one can prove that the integral equation (5) enjoys sufficient compactness, essentially due to the term $\frac{1}{2n}|x|^2|\Delta v(0)|$ on the right-hand side. Indeed a sequence of (approximate) solutions v_k blowing up (for simplicity) at the origin, up to rescaling, leads to a sequence (η_k) of functions satisfying, for every $R > 0$,

$$\int_{B_R} |\Delta \eta_k - c_k| dx \leq C R^{n-2} + o(1) R^{n+2}, \quad o(1) \xrightarrow{k \rightarrow \infty} 0, \quad c_k > 0,$$

and converging to η_∞ , solving (for simplicity here we ignore some cases)

$$(-\Delta)^{n/2} \eta_\infty = e^{n\eta_\infty} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^{n\eta_\infty} dx < \infty,$$

and

$$\int_{B_R} |\Delta \eta_\infty - c_\infty| dx \leq C R^{n-2}, \quad c_\infty \geq 0, \quad (6)$$

where $c_\infty = 0$ corresponds to $\Delta \eta_\infty(0) = 0$ (see Subcase 1.1 in Lemma 2.2 with $x_k = 0$).

The estimate on $\|\Delta \eta_\infty\|_{L^1(B_R)}$ in (6) shows that the polynomial part P_∞ of η_∞ , as in (3), has degree at most 2, and hence $\Delta P_\infty \leq 0$ as P_∞ is bounded from above. Therefore, $c_\infty = 0 = \Delta P_\infty$, P_∞ is constant, and in particular η_∞ is a spherical solution by Theorem A, that is, $\eta_\infty = u_{\lambda, x_0}$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, where u_{λ, x_0} is given by (4). This leads to a contradiction as $\Delta \eta_\infty(0) = 0$ and $\Delta u_{\lambda, x_0} < 0$ in \mathbb{R}^n .

In this work we focus only on the case $Q \geq 0$ because the negative case is relatively well understood. For instance by a simple application of maximum principle, one can show that problem (1) has no solution with $Q \equiv \text{constant} < 0$, $n = 2$ and $\kappa > -\infty$, but when Q is nonconstant, solutions do exist, as shown by Chanillo and Kiessling [2000] under suitable assumptions. Martinazzi [2008] proved that in higher even dimension $n = 2m \geq 4$, problem (1) with $Q \equiv \text{constant} < 0$ has solutions for some κ , and it has been shown in [Hyder and Martinazzi 2015] that actually for every $\kappa \in (-\infty, 0)$ and Q a negative constant, (1) has a solution. The same result has been recently extended to odd dimension $n \geq 3$ in [Hyder 2016].

2. Proof of Theorem 1.1

We consider the space

$$X := \{v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric, } \|v\|_X < \infty\},$$

where

$$\|v\|_X := \sup_{x \in \mathbb{R}^n} \left(\sum_{|\alpha| \leq 3} (1 + |x|)^{|\alpha| - 4} |D^\alpha v(x)| + \sum_{3 < |\alpha| \leq n-1} |D^\alpha v(x)| \right).$$

For $v \in X$ we set

$$A_v := \max \left\{ 0, \sup_{|x| \geq 10} \frac{v(x) - v(0)}{|x|^4} \right\}, \quad P_v(x) := -|x|^4 - A_v|x|^4.$$

Then

$$v(x) + P_v(x) \leq v(0) - |x|^4 \quad \text{for } |x| \geq 10.$$

Let c_v be the constant determined by

$$\int_{\mathbb{R}^n} K e^{n(v+c_v)} dx = \kappa, \quad K := Q e^{nP} e^{nP_v},$$

where the functions Q and P satisfy the hypotheses in Theorem 1.1. Since $Q > 0$ in a neighborhood of the origin, by a dilation argument we can assume that $Q > 0$ on B_3 . More precisely, if u is a solution to (1) then for any $\lambda > 0$, we know $u_\lambda(x) := u(\lambda x) + \log \lambda$ is also a solution to (1) with Q replaced by Q_λ , where $Q_\lambda(x) := Q(\lambda x)$. Now for a suitable choice of $\lambda > 0$, one has $Q_\lambda > 0$ on B_3 .

The function $u = P + P_v + v + c_v$ satisfies

$$(-\Delta)^{n/2} u = Q e^{nu}, \quad \kappa = \int_{\mathbb{R}^n} Q e^{nu} dx$$

if and only if v satisfies

$$(-\Delta)^{n/2} v = K e^{n(v+c_v)}.$$

For odd integer n , the operator $(-\Delta)^{n/2}$ is defined as follows:

Definition. Let n be an odd integer. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We say that u is a solution of

$$(-\Delta)^{n/2} u = f \quad \text{in } \mathbb{R}^n$$

if $u \in W_{\text{loc}}^{n-1,1}(\mathbb{R}^n)$ and $\Delta^{(n-1)/2} u \in L_{1/2}(\mathbb{R}^n)$ and for every test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (-\Delta)^{(n-1)/2} u (-\Delta)^{1/2} \varphi dx = \langle f, \varphi \rangle.$$

Here $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space and the space $L_s(\mathbb{R}^n)$ is defined by

$$L_s(\mathbb{R}^n) := \left\{ u \in L_{\text{loc}}^1(\mathbb{R}^n) : \|u\|_{L_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < \infty \right\}, \quad s > 0.$$

For more details on the fractional Laplacian we refer the reader to [Di Nezza et al. 2012].

We define an operator $T : X \rightarrow X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1}{|x-y|} \right) K(y) e^{n(v(y)+c_v)} dy + \frac{1}{2n} (|x|^2 - |x|^4) |\Delta v(0)|.$$

Lemma 2.1. *Let v solve $tT(v) = v$ for some $0 < t \leq 1$. Then*

$$v(x) = \frac{t}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) K(y) e^{n(v(y)+c_v)} dy + \frac{t}{2n} (|x|^2 - |x|^4) |\Delta v(0)|, \tag{7}$$

$\Delta v(0) < 0$, and $v(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$. Moreover,

$$\sup_{x \in B_1^c} v(x) = v(1) = \inf_{x \in B_1} v(x),$$

and in particular $A_v = 0$.

Proof. Since v satisfies $tT(v) = v$, equation (7) follows from the definition of T . Differentiating under the integral sign and observing that $\Delta \log(1/|\cdot - y|) < 0$, from (7) one gets

$$\Delta v(x) < \frac{t}{2n} |\Delta v(0)| \Delta(|x|^2 - |x|^4), \quad x \in \mathbb{R}^n. \tag{8}$$

Taking $x = 0$ in (8) we obtain $\Delta v(0) < t|\Delta v(0)|$, which implies that $\Delta v(0) < 0$. Notice that the function

$$w(x) := v(x) + \frac{t}{2n} |\Delta v(0)| (|x|^4 - |x|^2)$$

is monotone decreasing as $\Delta w < 0$. This follows from (8) and the integral representation of radially symmetric functions given by

$$f(\xi) - f(\bar{\xi}) = \int_{\bar{\xi}}^{\xi} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r} \Delta f(x) dx dr, \quad 0 \leq \bar{\xi} < \xi, \quad \omega_{n-1} := |S^{n-1}|. \tag{9}$$

The monotonicity of w implies that $\sup_{x \in B_1^c} v(x) = v(1) = \inf_{x \in B_1} v(x)$, and hence $A_v = 0$. Finally, together with $|\Delta v(0)| > 0$, we conclude that $\lim_{|x| \rightarrow \infty} v(x) = -\infty$ as $\lim_{|x| \rightarrow \infty} w(x) \leq w(1)$. \square

Lemma 2.2. *Let $(v, t) \in X \times (0, 1]$ satisfy $v = tT(v)$. Then there exists $C > 0$ (independent of v and t) such that*

$$\sup_{B_{1/8}} w \leq C, \quad w := v + c_v + \frac{1}{n} \log t.$$

Proof. Let us assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence $w_k = v_k + c_{v_k} + \frac{1}{n} \log t_k$ such that $\max_{\bar{B}_{1/8}} w_k =: w_k(\theta_k) \rightarrow \infty$.

If θ_k is a point of local maxima of w_k , we set $x_k = \theta_k$. Otherwise, we can choose $x_k \in B_{1/4} \setminus B_{1/8}$ such that x_k is a point of local maxima of w_k and $w_k(x_k) \geq w_k(x)$ for every $x \in B_{|x_k|}$. This follows from the fact that

$$\inf_{B_{1/4} \setminus B_{1/8}} w_k \not\rightarrow \infty,$$

which is a consequence of

$$\int_{\mathbb{R}^n} K e^{nw_k} dx = t_k \kappa \leq \kappa, \quad K > 0 \text{ on } B_3.$$

We set $\mu_k := e^{-w_k(x_k)}$. We distinguish the following cases.

Case 1: Up to a subsequence, $t_k \mu_k^2 |\Delta v_k(0)| \rightarrow c_0 \in [0, \infty)$.

We set

$$\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k) = w_k(x_k + \mu_k x) - w_k(x_k).$$

Notice that by (7) we have, for some dimensional constant C_1 ,

$$\Delta\eta_k(x) = \mu_k^2 \Delta v_k(x_k + \mu_k x) = C_1 \frac{\mu_k^2}{\gamma_n} \int_{\mathbb{R}^n} \frac{K(y)e^{n\omega_k(y)}}{|x_k + \mu_k x - y|^2} dy + t_k \mu_k^2 \left(1 - \frac{4(n+2)}{2n} |x_k + \mu_k x|^2\right) |\Delta v_k(0)|,$$

so that

$$\begin{aligned} & \int_{B_R} \left| \Delta\eta_k(x) - t_k \mu_k^2 |\Delta v_k(0)| \left(1 - \frac{2(n+2)}{n} |x_k|^2\right) \right| dx \\ & \leq \frac{C_1}{\gamma_n} \int_{\mathbb{R}^n} K(y)e^{n\omega_k(y)} \int_{B_R} \frac{\mu_k^2 dx}{|x_k + \mu_k x - y|^2} dy + C t_k \mu_k^2 |\Delta v_k(0)| \int_{B_R} (\mu_k |x_k \cdot x| + \mu_k^2 |x|^2) dx \\ & \leq \frac{C_1}{\gamma_n} t_k \kappa \int_{B_R} \frac{1}{|x|^2} dx + C t_k \mu_k^2 |\Delta v_k(0)| \int_{B_R} (\mu_k |x| + \mu_k^2 |x|^2) dx \\ & \leq C \kappa t_k R^{n-2} + C t_k \mu_k^2 |\Delta v_k(0)| (\mu_k R^{n+1} + \mu_k^2 R^{n+2}). \end{aligned} \tag{10}$$

The function η_k satisfies

$$(-\Delta)^{n/2} \eta_k(x) = K(x_k + \mu_k x) e^{n\eta_k(x)} \quad \text{in } \mathbb{R}^n, \quad \eta_k(0) = 0.$$

Moreover, $\eta_k \leq C(R)$ on B_R . This follows easily if $|x_k| \leq \frac{1}{9}$, as in that case $\eta_k \leq 0$ on B_R for $k \geq k_0(R)$. On the other hand, for $\frac{1}{9} < |x_k| \leq \frac{1}{4}$ one can use Lemma 2.4 (below). Therefore, by Lemma A.3 (and Lemmas 2.6, 2.7 if n is odd), up to a subsequence, $\eta_k \rightarrow \eta$ in $C_{\text{loc}}^{n-1}(\mathbb{R}^n)$, where η satisfies

$$(-\Delta)^{n/2} \eta = K(x_\infty) e^{n\eta} \quad \text{in } \mathbb{R}^n, \quad K(x_\infty) \int_{\mathbb{R}^n} e^{n\eta} dx \leq t_\infty \kappa < \infty, \quad K(x_\infty) > 0,$$

where (up to a subsequence) $t_k \rightarrow t_\infty$ and $x_k \rightarrow x_\infty$. Notice that $t_\infty \in (0, 1]$, $x_\infty \in \bar{B}_{1/4}$ and for every $R > 0$, by (10)

$$\int_{B_R} |\Delta\eta - c_0 c_1| dx \leq C R^{n-2}, \quad c_1 =: 1 - \frac{2(n+2)}{n} |x_\infty|^2 > 0. \tag{11}$$

Hence by Theorem A we have

$$\eta(x) = P_0(x) - \alpha \log |x| + o(\log |x|) \quad \text{as } |x| \rightarrow \infty,$$

where P_0 is a polynomial of degree at most $n-1$, P_0 is bounded from above and α is a positive constant. In fact, by (11)

$$\int_{B_R} |\Delta P_0(x) - c_0 c_1| dx \leq C R^{n-2} \quad \text{for every } R > 0.$$

Since $c_0, c_1 \geq 0$, it follows that P_0 is a constant. This implies that η is a spherical solution and in particular $\Delta\eta < 0$ on \mathbb{R}^n , and therefore, again by (11), we have $c_0 = 0$.

We consider the following subcases.

Subcase 1.1: There exists $M > 0$ such that $|x_k|/\mu_k \leq M$.

We set $y_k := -x_k/\mu_k$. Then (up to a subsequence) $y_k \rightarrow y_\infty \in B_{M+1}$. Therefore,

$$\Delta\eta(y_\infty) = \lim_{k \rightarrow \infty} \Delta\eta_k(y_k) = \lim_{k \rightarrow \infty} \mu_k^2 \Delta v_k(0) = \frac{c_0}{t_\infty} = 0,$$

a contradiction as $\Delta\eta < 0$ on \mathbb{R}^n .

Subcase 1.2: Up to a subsequence, $|x_k|/\mu_k \rightarrow \infty$.

For any $N \in \mathbb{N}$ we can choose $\xi_{1,k}, \dots, \xi_{N,k} \in \mathbb{R}^n$ such that $|\xi_{i,k}| = |x_k|$ for all $i = 1, \dots, N$ and the balls $B_{2\mu_k}(\xi_{i,k})$ are disjoint for k large enough. Since the v_k are radially symmetric, the functions $\eta_{i,k} := v_k(\xi_{i,k} + \mu_k x) - v_k(\xi_{i,k}) \rightarrow \eta_i = \eta$ in $C_{\text{loc}}^{n-1}(\mathbb{R}^n)$. Therefore,

$$\lim_{k \rightarrow \infty} \int_{B_1} e^{n(v_k + c_{v_k})} dx \geq N \lim_{k \rightarrow \infty} \int_{B_{\mu_k}(\xi_{1,k})} e^{n(v_k + c_{v_k})} dx = N \frac{1}{t_\infty} \int_{B_1} e^{n\eta} dx.$$

This contradicts the fact that

$$\int_{B_1} K e^{n(v_k + c_{v_k})} dx \leq \kappa, \quad K > 0 \text{ on } B_3.$$

Case 2: Up to a subsequence, $t_k \mu_k^2 |\Delta v_k(0)| \rightarrow \infty$.

We choose $\rho_k > 0$ such that $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| = 1$. We set

$$\psi_k(x) = v_k(x_k + \rho_k \mu_k x) - v_k(x_k).$$

Then one can get (similar to (10))

$$\begin{aligned} & \int_{B_R} \left| \Delta \psi_k(x) - \left(1 - \frac{2(n+2)}{n} |x_k|^2 \right) \right| dx \\ & \leq C_1 \int_{\mathbb{R}^n} K(y) e^{nw_k(y)} \int_{B_R} \frac{\rho_k^2 \mu_k^2}{|x_k + \mu_k \rho_k x - y|^2} dx dy + C_2 \mu_k \rho_k \int_{B_R} (|x| + \mu_k \rho_k |x|^2) dx \xrightarrow{k \rightarrow \infty} 0, \end{aligned}$$

thanks to Lemma 2.5 (below). Moreover, together with Lemma 2.4, ψ_k satisfies

$$(-\Delta)^{n/2} \psi_k = o(1) \quad \text{in } B_R, \quad \psi_k(0) = 0, \quad \psi_k \leq C(R) \text{ on } B_R.$$

Hence, by Lemma A.3 (and Lemma 2.6 if n is odd), up to a subsequence, $\psi_k \rightarrow \psi$ in $C_{\text{loc}}^{n-1}(\mathbb{R}^n)$. Then ψ must satisfy

$$\int_{B_1} |\Delta \psi - c_0| dx = 0, \quad c_0 := 1 - \frac{2(n+2)}{n} |x_\infty|^2 > 0,$$

where (up to a subsequence) $x_k \rightarrow x_\infty$. This shows that $\Delta \psi(0) = c_0 > 0$, which is a contradiction as

$$\Delta \psi(0) = \lim_{k \rightarrow \infty} \Delta \psi_k(0) = \lim_{k \rightarrow \infty} \rho_k^2 \mu_k^2 \Delta v_k(x_k) \leq 0.$$

Here, $\Delta v_k(x_k) \leq 0$ follows from the fact that x_k is a point of local maxima of v_k . □

A consequence of the local uniform upper bounds of w are the following global uniform upper bounds:

Lemma 2.3. *There exists a constant $C > 0$ such that for all $(v, t) \in X \times (0, 1]$ with $v = tT(v)$ we have $|\Delta v(0)| \leq C$ and*

$$v(x) + c_v + \frac{1}{n} \log t \leq C \quad \text{on } \mathbb{R}^n.$$

Proof. By Lemma 2.2 we have

$$\sup_{B_{1/8}} w := \sup_{B_{1/8}} \left(v + c_v + \frac{1}{n} \log t \right) \leq C.$$

Differentiating under the integral sign from (7), and recalling that $\Delta v(0) < 0$, we obtain

$$\begin{aligned} |\Delta v(0)| &\leq C \int_{B_{1/8}} \frac{1}{|y|^2} K(y) e^{nw(y)} dy + C \int_{B_{1/8}^c} \frac{1}{|y|^2} K(y) e^{nw(y)} dy \\ &\leq C \sup_{B_{1/8}} K \int_{B_{1/8}} \frac{1}{|y|^2} dy + C \int_{B_{1/8}^c} K e^{nw} dy \leq C(\kappa, K). \end{aligned}$$

By (8) we get

$$\Delta v(x) \leq t |\Delta v(0)| \leq C, \quad x \in \mathbb{R}^n,$$

and hence, together with (9)

$$v(x) = v(0) + \int_0^{|x|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r} \Delta v(y) dy dr \leq v(0) + C|x|^2 \leq C + v(0), \quad x \in B_2.$$

The lemma follows from Lemmas 2.1 and 2.2. □

Proof of Theorem 1.1. Let $v \in X$ be a solution of $v = tT(v)$ for some $0 < t \leq 1$. Then $A_v = 0$ and $|\Delta v(0)| \leq C$, thanks to Lemmas 2.1 and 2.3. Hence, for $0 \leq |\beta| \leq n - 1$,

$$\begin{aligned} |D^\beta v(x)| &\leq C \int_{\mathbb{R}^n} \left| D^\beta \log \left(\frac{1}{|x-y|} \right) \right| K(y) e^{n(v(y)+c_v+(1/n)\log t)} dy + C |D^\beta (|x|^2 - |x|^4)| \\ &\leq C \int_{\mathbb{R}^n} \left| D^\beta \log \left(\frac{1}{|x-y|} \right) \right| e^{-|y|^4} dy + C |D^\beta (|x|^2 - |x|^4)|, \end{aligned}$$

where in the second inequality we have used that

$$v(x) + c_v + \frac{1}{n} \log t \leq C, \quad C \text{ is independent of } v \text{ and } t,$$

which follows from Lemma 2.3. Now as in Lemma 2.8 one can show that

$$\|v\|_X \leq M,$$

and therefore, by Lemma A.1, the operator T has a fixed point (say) v . Then

$$u = P + v + c_v - |x|^4$$

is a solution to the problem (1) and u has the asymptotic behavior given by

$$u(x) = P(x) - \frac{2\kappa}{\Lambda_1} \log |x| + \frac{1}{2n} \Delta v(0) (|x|^4 - |x|^2) - |x|^4 + c_v + o(1) \quad \text{as } |x| \rightarrow \infty. \quad \square$$

Now we give a proof of the technical lemmas used in the proof of Lemma 2.2.

Lemma 2.4. *Let $\varepsilon > 0$. Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (7) or (14) for all $k \in \mathbb{N}$. Let $x_k \in B_1 \setminus B_\varepsilon$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v'_k(x_k) = 0$. Then*

$$v_k(x_k + x) - v_k(x_k) \leq C(n, \varepsilon) |x|^2 t_k |\Delta v_k(0)|, \quad x \in B_1.$$

Proof. If $|x_k + x| \leq |x_k|$ then $v_k(x_k + x) - v_k(x_k) \leq 0$ as $v_k(x_k) \geq v_k(y)$ for every $y \in B_{|x_k|}$. For $|x_k| < |x_k + x|$, setting $a = a(k, x) := x_k + x$, and together with (9) we obtain

$$\begin{aligned} v_k(x_k + x) - v_k(x_k) &= \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1}r^{n-1}} \int_{B_r \setminus B_{|x_k|}} \Delta v_k(x) \, dx \, dr \\ &\leq \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1}r^{n-1}} \int_{B_{|a|} \setminus B_{|x_k|}} t_k |\Delta v_k(0)| \, dx \, dp \\ &\leq C(n)t_k |\Delta v_k(0)| (|B_{|a|}| - |B_{|x_k|}|) \left(\frac{1}{|x_k|^{n-2}} - \frac{1}{|a|^{n-2}} \right) \\ &\leq C(n, \varepsilon)t_k |x|^2 |\Delta v_k(0)|, \end{aligned}$$

where in the first equality we have used that

$$0 = v'_k(x_k) = \frac{1}{\omega_{n-1}|x_k|^{n-1}} \int_{B_{|x_k|}} \Delta v_k \, dx.$$

Hence we have the lemma. □

Lemma 2.5. *Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (7) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v'_k(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n} \log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| \leq C$ and $\rho_k \mu_k \rightarrow 0$. Then for any $R_0 > 0$,*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} K(y) e^{nw_k(y)} \int_{B_{R_0}} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2} \, dx \, dy =: \lim_{k \rightarrow \infty} I_k = 0.$$

Proof. In order to prove the lemma we fix $R > 0$ (large). We split B_{R_0} into

$$A_1(R, y) := \{x \in B_{R_0} : |x_k + \rho_k \mu_k x - y| > R \rho_k \mu_k\}, \quad A_2(R, y) := B_{R_0} \setminus A_1(R, y).$$

Then we can write $I_k = I_{1,k} + I_{2,k}$, where

$$I_{i,k} := \int_{\mathbb{R}^n} K(y) e^{nw_k(y)} \int_{A_i(R,y)} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2} \, dx \, dy, \quad i = 1, 2.$$

Changing the variable $y \mapsto x_k + \rho_k \mu_k y$ and by Fubini's theorem, one gets

$$\begin{aligned} I_{2,k} &= \rho_k^n \int_{B_{R_0}} \int_{\mathbb{R}^n} K(x_k + \rho_k \mu_k y) e^{n\eta_k(y)} \frac{1}{|x - y|^2} \chi_{|x-y| \leq R} \, dy \, dx \\ &\leq \rho_k^n \int_{B_{R_0}} \int_{B_{R+R_0}} K(x_k + \rho_k \mu_k y) e^{n\eta_k(y)} \frac{1}{|x - y|^2} \, dy \, dx \\ &\leq C(n, \varepsilon) (\sup_{B_{R+R_0+1}} K e^{n\eta_k}) (R + R_0)^n R_0^{n-2} \rho_k^n, \end{aligned}$$

where $\eta_k(y) := w_k(x_k + \rho_k \mu_k y) - w_k(x_k)$. If $x_k \rightarrow 0$ then $\eta_k \leq 0$ on B_{R+R_0+1} for k large. Otherwise, for k large, $\rho_k \mu_k y \in B_1$ for every $y \in B_{R+R_0+1}$ and hence, by Lemma 2.4

$$\eta_k(y) = v_k(x_k + \rho_k \mu_k y) - v_k(x_k) \leq C |\rho_k \mu_k y|^2 t_k |\Delta v_k(0)| \leq C(R, R_0).$$

Therefore,

$$\lim_{k \rightarrow \infty} I_{2,k} = 0.$$

Using the definition of c_v we bound

$$I_{1,k} \leq \frac{|B_{R_0}|}{R^2} \int_{\mathbb{R}^n} K(y)e^{nw_k(y)} dy \leq C(n, \kappa, R_0) \frac{1}{R^2}.$$

Since $R > 0$ is arbitrary, we conclude the lemma. □

We need the following two lemmas only for n odd.

Lemma 2.6. *Let $n \geq 5$. Let v be given by (7). For any $r > 0$ and $\xi \in \mathbb{R}^n$ we set*

$$w(x) = v(rx + \xi), \quad x \in \mathbb{R}^n.$$

Then there exists $C > 0$ (independent of v, t, r, ξ) such that for every multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = n - 1$ we have $\|D^\alpha w\|_{L^{1/2}(\mathbb{R}^n)} \leq Ct(1 + r^4|\Delta v(0)|)$. Moreover, for any $\varepsilon > 0$ there exists $R > 0$ (independent of r, ξ and t) such that

$$\int_{B_R^c} \frac{|D^\alpha w(x)|}{1 + |x|^{n+1}} dx < \varepsilon t(1 + r^4|\Delta v(0)|), \quad |\alpha| = n - 1.$$

Proof. Differentiating under the integral sign we obtain

$$|D^\alpha w(x)| \leq Ct \int_{\mathbb{R}^n} \frac{r^{n-1}}{|rx + \xi - y|^{n-1}} f(y) dy + Ctr^4|\Delta v(0)|, \quad f(y) := K(y)e^{n(v(y)+c_v)}.$$

If $n > 5$ then the above inequality is true without the term $Ctr^4|\Delta v(0)|$. Using a change of variable $y \mapsto \xi + ry$, we get

$$\int_{\Omega} \frac{|D^\alpha w(x)|}{1 + |x|^{n+1}} dx \leq Ctr^n \int_{\mathbb{R}^n} f(\xi + ry) \int_{\Omega} \frac{1}{|x - y|^{n-1}} \frac{1}{1 + |x|^{n+1}} dx dy + Ctr^4|\Delta v(0)| \int_{\Omega} \frac{dx}{1 + |x|^{n+1}}.$$

The lemma follows by taking $\Omega = \mathbb{R}^n$ or B_R^c . □

Lemma 2.7. *Let $\eta_k \rightarrow \eta$ in $C_{loc}^{n-1}(\mathbb{R}^n)$. We assume that for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{B_R^c} \frac{|\Delta^{(n-1)/2} \eta_k(x)|}{1 + |x|^{n+1}} dx < \varepsilon \quad \text{for } k = 1, 2, \dots \tag{12}$$

We further assume that

$$(-\Delta)^{n/2} \eta_k = K(x_k + \mu_k x) e^{n\eta_k} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} |K(x_k + \mu_k x)| e^{n\eta_k(x)} dx \leq C,$$

where $x_k \rightarrow x_\infty, \mu_k \rightarrow 0, K$ is a continuous function and $K(x_\infty) > 0$. Then $e^{n\eta} \in L^1(\mathbb{R}^n)$ and η satisfies

$$(-\Delta)^{n/2} \eta = K(x_\infty) e^{n\eta} \quad \text{in } \mathbb{R}^n.$$

Proof. First notice that $\Delta^{(n-1)/2}\eta_k \rightarrow \Delta^{(n-1)/2}\eta$ in $L_{1/2}(\mathbb{R}^n)$, thanks to (12) and the convergence $\eta_k \rightarrow \eta$ in $C_{\text{loc}}^{n-1}(\mathbb{R}^n)$.

We claim that η satisfies $(-\Delta)^{n/2}\eta = K(x_\infty)e^{n\eta}$ in \mathbb{R}^n in the sense of distribution.

In order to prove the claim we let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} K(x_k + \mu_k x) e^{n\eta_k(x)} \varphi(x) dx = \int_{\mathbb{R}^n} K(x_\infty) e^{n\eta(x)} \varphi(x) dx,$$

and

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} (-\Delta)^{(n-1)/2} \eta_k (-\Delta)^{1/2} \varphi dx = \int_{\mathbb{R}^n} (-\Delta)^{(n-1)/2} \eta (-\Delta)^{1/2} \varphi dx.$$

We conclude the claim.

To complete the lemma first notice that $e^{n\eta} \in L^1(\mathbb{R}^n)$, which follows from the fact that for any $R > 0$

$$\int_{B_R} e^{n\eta} dx = \lim_{k \rightarrow \infty} \int_{B_R} e^{n\eta_k} dx = \lim_{k \rightarrow \infty} \int_{B_R} \frac{K(x_k + \mu_k x)}{K(x_\infty)} e^{n\eta_k(x)} dx \leq \frac{C}{K(x_\infty)}.$$

We fix a function $\psi \in C_c^\infty(B_2)$ such that $\psi = 1$ on B_1 . For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we set $\varphi_k(x) = \varphi(x)\psi(x/k)$. The lemma follows by taking $k \rightarrow \infty$, thanks to the previous claim. \square

Lemma 2.8. *The operator $T : X \rightarrow X$ is compact.*

Proof. Let v_k be a bounded sequence in X . Then (up to a subsequence) $\{v_k(0)\}$, $\{\Delta v_k(0)\}$, $\{A_{v_k}\}$ and $\{c_{v_k}\}$ are convergent sequences. Therefore, $|\Delta v_k(0)|(|x|^2 - |x|^4)$ converges to some function in X . To conclude the lemma, it is sufficient to show that up to a subsequence $\{f_k\}$ converges in X , where f_k is defined by

$$f_k(x) = \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y) e^{nP(y)} e^{nP_{v_k}(y)} e^{n(v_k(y)+c_{v_k})} dy.$$

Differentiating under the integral sign, for $0 < |\beta| \leq n-1$, one gets

$$|D^\beta f_k(x)| \leq C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{|\beta|}} Q(y) e^{nP(y)} e^{nP_{v_k}(y)} e^{n(v_k(y)+c_{v_k})} dy \leq C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{|\beta|}} e^{-|y|^4} dy \leq C,$$

where the second inequality follows from the uniform bounds

$$|v_k(0)| \leq C, \quad |c_{v_k}| \leq C, \quad Qe^{nP} \leq C, \quad \text{and} \quad v_k(x) + P_{v_k}(x) \leq v_k(0) - |x|^4. \quad (13)$$

Indeed, for $0 < |\beta| \leq n-1$

$$\lim_{R \rightarrow \infty} \sup_k \sup_{x \in B_R^c} |D^\beta f_k(x)| = 0,$$

and for every $0 < s < 1$ we have $\|D^{n-1} f_k\|_{C^{0,s}(B_R)} \leq C(R, s)$. Finally, using (13) we have the bound

$$|f_k(x)| \leq C \int_{\mathbb{R}^n} |\log|x-y|| e^{-|y|^4} dy \leq C \log(2+|x|).$$

Thus, by Ascoli's theorem, up to a subsequence, $f_k \rightarrow f$ in $C_{\text{loc}}^{n-1}(\mathbb{R}^n)$ for some $f \in C^{n-1}(\mathbb{R}^n)$, and the global uniform estimates of f_k and $D^\beta f_k$ would imply that $f_k \rightarrow f$ in X . \square

3. Proof of Theorem 1.3

We consider the space

$$X := \{v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric, } \|v\|_X < \infty\},$$

where

$$\|v\|_X := \sup_{x \in \mathbb{R}^n} \left(\sum_{|\alpha| \leq 1} (1 + |x|)^{|\alpha|-2} |D^\alpha v(x)| + \sum_{1 < |\alpha| \leq n-1} |D^\alpha v(x)| \right).$$

For $v \in X$, let c_v be the constant determined by

$$\int_{\mathbb{R}^n} Q e^{n(v+c_v)} dy = \kappa,$$

where Q satisfies the hypothesis in Theorem 1.3. Again by a dilation argument we can assume $Q > 0$ on B_3 .

We define an operator $T : X \rightarrow X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y) e^{n(v(y)+c_v)} dy + \frac{1}{2n} |\Delta v(0)| |x|^2.$$

As in Lemma 2.8 one can show that the operator T is compact.

The proofs of the following two lemmas are similar to those of Lemmas 2.1 and 2.5 respectively.

Lemma 3.1. *Let v solve $tT(v) = v$ for some $0 < t \leq 1$. Then $\Delta v(0) < 0$, and*

$$v(x) = \frac{t}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y) e^{n(v(y)+c_v)} dy + \frac{t}{2n} |\Delta v(0)| |x|^2. \tag{14}$$

Lemma 3.2. *Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (14) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v'_k(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n} \log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $\rho_k^2 t_k \mu_k^2 |\Delta v_k(0)| \leq C$ and $\rho_k \mu_k \rightarrow 0$. Then for any $R_0 > 0$*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} Q(y) e^{n w_k(y)} \int_{B_{R_0}} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2} dx dy = 0.$$

Now we prove similar local uniform upper bounds to those in Lemma 2.2.

Lemma 3.3. *Let $(v, t) \in X \times (0, 1]$ satisfy (14). Then there exists $C > 0$ (independent of v and t) such that*

$$\sup_{B_{1/8}} w \leq C, \quad w := v + c_v + \frac{1}{n} \log t.$$

Proof. The proof is very similar to that of Lemma 2.2. Here we briefly sketch it.

We assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence of (v_k, t_k) and a sequence of points x_k in $B_{1/4}$ such that

$$w_k(x_k) \rightarrow \infty, \quad w_k \leq w_k(x_k) \text{ on } B_{|x_k|}, \quad x_k \text{ is a point of local maxima of } v_k.$$

We set $\mu_k := e^{-w_k(x_k)}$ and we distinguish following cases.

Case 1: Up to a subsequence, $t_k \mu_k^2 |\Delta v_k(0)| \rightarrow c_0 \in [0, \infty)$.

We set $\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k)$. Then we have

$$\int_{B_R} |\Delta \eta_k - t_k \mu_k^2 |\Delta v_k(0)| | dx \leq C t_k R^{n-2}.$$

Now one can proceed exactly as in Case 1 in Lemma 2.2.

Case 2: Up to a subsequence, $t_k \mu_k^2 |\Delta v_k(0)| \rightarrow \infty$.

We set $\psi_k(x) = v_k(x_k + \rho_k \mu_k x) - v_k(x_k)$, where ρ_k is determined by $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| = 1$. Then by Lemma 3.2

$$\int_{B_R} |\Delta \psi_k - 1| dx = o(1) \quad \text{as } k \rightarrow \infty.$$

Similar to Case 2 in Lemma 2.2 one can get a contradiction. \square

With the help of Lemma 3.3 we prove:

Lemma 3.4. *There exists a constant $M > 0$ such that for all $(v, t) \in X \times (0, 1]$ satisfying (14) we have $\|v\| \leq M$.*

Proof. Let $(v, t) \in X \times (0, 1]$ satisfy (14). We set $w := v + c_v + \frac{1}{n} \log t$.

First we show that $|\Delta v(0)| \leq C$ for some $C > 0$ independent of v and t . Indeed, differentiating under the integral sign, from (14), and together with Lemma 3.3, we get

$$\begin{aligned} |\Delta v(0)|(1+t) &\leq C \int_{\mathbb{R}^n} \frac{1}{|y|^2} Q(y) e^{nw(y)} dy \\ &= C \int_{B_{1/8}} \frac{1}{|y|^2} Q(y) e^{nw(y)} dy + C \int_{B_{1/8}^c} \frac{1}{|y|^2} Q(y) e^{nw(y)} dy \\ &\leq C \int_{B_{1/8}} \frac{1}{|y|^2} Q(y) dy + C\kappa \leq C. \end{aligned}$$

Hence $|\Delta v(0)| \leq C$.

We define a function $\xi(x) := v(x) - (t/2n)|\Delta v(0)||x|^2$. Then ξ is monotone decreasing on $(0, \infty)$, which follows from the fact that $\Delta \xi \leq 0$. Therefore,

$$\begin{aligned} w(x) &= \xi(x) + c_v + \frac{1}{n} \log t + \frac{t}{2n} |\Delta v(0)| |x|^2 \\ &\leq \xi\left(\frac{1}{8}\right) + c_v + \frac{1}{n} \log t + \frac{t}{2n} |\Delta v(0)| |x|^2 \\ &\leq w\left(\frac{1}{8}\right) + \frac{t}{2n} |\Delta v(0)| |x|^2. \end{aligned}$$

Hence, $w(x) \leq \lambda(1 + |x|^2)$ on \mathbb{R}^n for some $\lambda > 0$ independent of v and t . Using this in (14) one can show

$$|v(x)| \leq C \log(2 + |x|) + C|x|^2,$$

and differentiating under the integral sign, from (14)

$$|D^\beta v(x)| \leq C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{|\beta|}} Q(y) e^{\lambda(1+|y|^2)} dy + C |D^\beta |x|^2|, \quad 0 < |\beta| \leq n-1.$$

The lemma follows easily. \square

Proof of Theorem 1.3. By the Schauder fixed point theorem (see Lemma A.1), the operator T has a fixed point, thanks to Lemma 3.4. Let v be a fixed point of T . Then $u = v + c_v$ is a solution of (1). \square

Now we prove the nonexistence results stated in the Introduction.

Lemma 3.5. *Let $n \in \{3, 4\}$. If $Q(x) = e^{-\lambda|x|^2}$ for some $\lambda > 0$ then there is no solution to (1) with $\kappa > \Lambda_1$. If $Q \in C^1_{\text{rad}}(\mathbb{R}^n)$ is of the form $Q = e^\xi$ and it satisfies*

$$Q' \leq 0, \quad |x \cdot \nabla Q(x)| \leq C, \quad \frac{\xi(x)}{|x|^2} \xrightarrow{|x| \rightarrow \infty} 0,$$

then there is no radially symmetric solution to (1) with $\kappa > \Lambda_1$.

Proof. First we consider the case when $Q = e^{-\lambda|x|^2}$. Let u be a solution to (1) with $Q = e^{-\lambda|x|^2}$. Then the function $w(x) := u - (\lambda/n)|x|^2$ satisfies

$$(-\Delta)^{n/2} w = e^{nw}, \quad \kappa = \int_{\mathbb{R}^n} Q e^{nu} dx = \int_{\mathbb{R}^n} e^{nw} dx < \infty.$$

It follows from [Lin 1998; Jin et al. 2015] that $\kappa \leq \Lambda_1$.

In order to prove the lemma for $Q = e^\xi$, we assume by contradiction that there is a solution u to (1) with $\kappa > \Lambda_1$. We set

$$v(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right) Q(y) e^{nu(y)} dy, \quad h := u - v.$$

Then $v(x) = -(2\kappa/\Lambda_1) \log|x| + o(\log|x|)$ as $|x| \rightarrow \infty$. Notice that h is radially symmetric and $(-\Delta)^{n/2} h = 0$ on \mathbb{R}^n . Therefore, $h(x) = c_1 + c_2|x|^2$ for some $c_1, c_2 \in \mathbb{R}$. This follows easily if $n = 4$. For $n = 3$, first notice that $\Delta h \in L_{1/2}(\mathbb{R}^3)$. Hence, by [Jin et al. 2015, Lemma 15] $\Delta h \equiv \text{constant}$. Now radial symmetry of h implies that $h(x) = c_1 + c_2|x|^2$.

From a Pohozaev-type identity in [Xu 2005, Theorem 2.1], we get

$$\frac{\kappa}{\gamma_n} \left(\frac{\kappa}{\gamma_n} - 2 \right) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} (x \cdot \nabla K(x)) e^{nv(x)} dx, \quad K := Q e^{nh}. \tag{15}$$

Since $\kappa > \Lambda_1 = 2\gamma_n$, from (15) we deduce that $x \cdot \nabla K(x) > 0$ for some $x \in \mathbb{R}^n$. Using that $Q e^{nu} \in L^1(\mathbb{R}^n)$ and that $\xi(x) = o(|x|^2)$ at infinity, one has $c_2 \leq 0$. Therefore, $x \cdot \nabla K(x) \leq 0$ in \mathbb{R}^n , a contradiction. \square

The proof of the following lemma is similar to that of Lemma 3.5.

Lemma 3.6. *Let $\kappa \geq \Lambda_1$. Let P be a nonconstant and nonincreasing radially symmetric polynomial of degree at most $n - 1$. Then there is no solution u to (2) (with $n \geq 3$) such that u has the asymptotic behavior given by*

$$u(x) = -\frac{2\kappa}{\Lambda_1} \log|x| + P(x) + o(\log|x|) \quad \text{as } |x| \rightarrow \infty.$$

Appendix

Lemma A.1 [Gilbarg and Trudinger 1998, Theorem 11.3]. *Let T be a compact mapping of a Banach space X into itself, and suppose that there exists a constant M such that*

$$\|x\|_X < M$$

for all $x \in X$ and $t \in (0, 1]$ satisfying $tTx = x$. Then T has a fixed point.

The following identity (16) is due to Pizzetti [1909]. Simple proofs of (16) and (17) can be found in Lemma 3 and Proposition 4, respectively, of [Martinazzi 2009a].

Lemma A.2 [Pizzetti 1909; Martinazzi 2009a]. *Let $\Delta^m h = 0$ in $B_{4R} \subset \mathbb{R}^n$. For any $x \in B_R$ and $0 < r < R - |x|$ we have*

$$\frac{1}{|B_r|} \int_{B_r(x)} h(z) dz = \sum_{i=0}^{m-1} c_i r^{2i} \Delta^i h(x), \tag{16}$$

where

$$c_0 = 1, \quad c_i = c(i, n) > 0 \quad \text{for } i \geq 1.$$

Moreover, for every $k \geq 0$ there exists $C = C(k, R) > 0$ such that

$$\|h\|_{C^k(B_R)} \leq C \|h\|_{L^1(B_{4R})}. \tag{17}$$

Lemma A.3. *Let $R > 0$ and $B_R \subset \mathbb{R}^n$. Let $u_k \in C^{n-1,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (\frac{1}{2}, 1)$ be such that*

$$u_k(0) = 0, \quad \|u_k^+\|_{L^\infty(B_R)} \leq C, \quad \|(-\Delta)^{n/2} u_k\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R} |\Delta u_k| dx \leq C.$$

If n is an odd integer, we also assume that $\|\Delta^{(n-1)/2} u_k\|_{L^{1/2}(\mathbb{R}^n)} \leq C$. Then (up to a subsequence) $u_k \rightarrow u$ in $C^{n-1}(B_{R/8})$.

Proof. First we prove the lemma for n even.

We write $u_k = w_k + h_k$, where

$$\begin{cases} (-\Delta)^{n/2} w_k = (-\Delta)^{n/2} u_k & \text{in } B_R, \\ \Delta^j w_k = 0 & \text{on } \partial B_R, \quad j = 0, 1, \dots, \frac{1}{2}(n-2). \end{cases}$$

Then by standard elliptic estimates, the w_k are uniformly bounded in $C^{n-1,\beta}(B_R)$. Therefore,

$$|h_k(0)| \leq C, \quad \|h_k^+\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R} |\Delta h_k| dx \leq C.$$

Since the h_k are $\frac{n}{2}$ -harmonic, the Δh_k are $(\frac{n}{2}-1)$ -harmonic in B_R , and by (17) we obtain

$$\|\Delta h_k\|_{C^n(B_{R/4})} \leq C \|\Delta h_k\|_{L^1(B_R)} \leq C.$$

Using the identity (16) we have the bound

$$\begin{aligned} \frac{1}{|B_R|} \int_{B_R(0)} h_k^-(z) dz &= \frac{1}{|B_R|} \int_{B_R(0)} h_k^+(z) dz - \frac{1}{|B_R|} \int_{B_R(0)} h_k(z) dz \\ &= \frac{1}{|B_R|} \int_{B_R(0)} h_k^+(z) dz - h_k(0) - \sum_{i=1}^{n/2-1} c_i R^{2i} \Delta^i h_k(0) \leq C, \end{aligned}$$

and hence

$$\int_{B_R} |h_k(z)| dz = \int_{B_R} h_k^+(z) dz + \int_{B_R} h_k^-(z) dz \leq C.$$

Again by (17) we obtain

$$\|h_k\|_{C^n(B_{R/4})} \leq C \|h_k\|_{L^1(B_R)} \leq C.$$

Thus, the u_k are uniformly bounded in $C^{n-1,\beta}(B_{R/4})$ and (up to a subsequence) $u_k \rightarrow u$ in $C^{n-1}(B_{R/4})$ for some $u \in C^{n-1}(B_{R/4})$.

It remains to prove the lemma for n odd.

If n is odd then $\frac{1}{2}(n-1)$ is an integer. We split $\Delta^{(n-1)/2} u_k = w_k + h_k$, where

$$\begin{cases} (-\Delta)^{1/2} w_k = (-\Delta)^{1/2} \Delta^{(n-1)/2} u_k & \text{in } B_R, \\ w_k = 0 & \text{in } B_R^c. \end{cases}$$

Then by Lemmas A.4 and A.5 one has $\|\Delta^{(n-1)/2} u_k\|_{C^{1/2}(B_{R/2})} \leq C$. Now one can proceed as in the case of even integer. □

Lemma A.4 [Jin et al. 2015, Proposition 22]. *Let $u \in L_\sigma(\mathbb{R}^n)$ for some $\sigma \in (0, 1)$ and $(-\Delta)^\sigma u = 0$ in B_{2R} . Then for every $k \in \mathbb{N}$,*

$$\|\nabla^k u\|_{C^0(B_R)} \leq C(n, \sigma, k) \frac{1}{R^k} \left(R^{2\sigma} \int_{\mathbb{R}^n \setminus B_{2R}} \frac{|u(x)|}{|x|^{n+2\sigma}} dx + \frac{\|u\|_{L^1(B_{2R})}}{R^n} \right),$$

where $\alpha \in (0, 1)$ and k is a nonnegative integer.

Lemma A.5 [Ros-Oton and Serra 2014, Proposition 1.1]. *Let $\sigma \in (0, 1)$. Let u be a solution of*

$$\begin{cases} (-\Delta)^\sigma u = f & \text{in } B_R, \\ u = 0 & \text{in } B_R^c. \end{cases}$$

Then

$$\|u\|_{C^\sigma(\mathbb{R}^n)} \leq C(R, \sigma) \|f\|_{L^\infty(B_R)}.$$

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BOUNDARY ESTIMATES IN ELLIPTIC HOMOGENIZATION

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For a family of systems of linear elasticity with rapidly oscillating periodic coefficients, we establish sharp boundary estimates with either Dirichlet or Neumann conditions, uniform down to the microscopic scale, without smoothness assumptions on the coefficients. Under additional smoothness conditions, these estimates, combined with the corresponding local estimates, lead to the full Rellich-type estimates in Lipschitz domains and Lipschitz estimates in $C^{1,\alpha}$ domains. The C^α , $W^{1,p}$, and L^p estimates in C^1 domains for systems with VMO coefficients are also studied. The approach is based on certain estimates on convergence rates. As a byproduct, we obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q = 2d/(d - 1)$ and a Lipschitz domain Ω , with no smoothness assumption on the coefficients.

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1. Introduction

The purpose of this paper is to establish sharp boundary estimates with either Dirichlet or Neumann conditions, uniform down to the microscopic scale, for a family of second-order elliptic systems in divergence form with rapidly oscillating coefficients, without any smoothness assumption on the coefficients. Under additional smoothness conditions, these estimates, combined with the corresponding local estimates, lead to the full Rellich-type estimates in Lipschitz domains and Lipschitz estimates in $C^{1,\alpha}$ domains. The C^α , $W^{1,p}$, and L^p estimates in C^1 domains for systems with VMO coefficients are also investigated. To fix the idea we shall consider the systems of linear elasticity with periodic coefficients in this paper. However, the same results, without the complications introduced by rigid displacements, hold for general second-order elliptic systems with periodic coefficients satisfying the stronger ellipticity

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condition (1-11) (the symmetry condition is also needed for Rellich estimates in Lipschitz domains). We further point out that although we restrict ourselves to the periodic case, our approach, which is based on certain estimates on convergence rates in H^1 and L^2 , extends to nonperiodic settings, provided that the interior correctors or approximate correctors satisfy certain L^2 conditions. The compactness methods, which were introduced to the study of homogenization in [Avellaneda and Lin 1987] and have played an important role in establishing regularity results in the periodic setting (see, e.g., [Avellaneda and Lin 1987; 1989; Kenig et al. 2013; Kenig and Prange 2015]), are not used in this paper. As a byproduct of our new approach, we also obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q = 2d/(d-1)$ and a Lipschitz domain Ω , with no smoothness assumption on the coefficients.

More precisely, consider the systems of linear elasticity,

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i} \left[a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_j} \right], \quad \varepsilon > 0. \quad (1-1)$$

We will assume that $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \leq i, j, \alpha, \beta \leq d$ is real, bounded measurable, and satisfies the elasticity condition

$$\begin{aligned} a_{ij}^{\alpha\beta}(y) &= a_{ji}^{\beta\alpha}(y) = a_{\alpha j}^{i\beta}(y), \\ \kappa_1 |\xi|^2 &\leq a_{ij}^{\alpha\beta}(y) \xi_i^\alpha \xi_j^\beta \leq \kappa_2 |\xi|^2 \end{aligned} \quad (1-2)$$

for a.e. $y \in \mathbb{R}^d$ and for any symmetric matrix $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d \times d}$, where $\kappa_1, \kappa_2 > 0$ (the summation convention is used throughout the paper). We will also assume that $A(y)$ is 1-periodic; i.e.,

$$A(y+z) = A(y) \quad \text{for a.e. } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \quad (1-3)$$

Theorem 1.1. *Suppose that A satisfies conditions (1-2)–(1-3). Let Ω be a bounded Lipschitz domain in \mathbb{R}^d . Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be the weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega, \quad (1-4)$$

where $F \in L^p(\Omega; \mathbb{R}^d)$ for $p = 2d/(d+1)$ and $f \in H^1(\partial\Omega; \mathbb{R}^d)$. Then, for $\varepsilon \leq r < \operatorname{diam}(\Omega)$,

$$\left\{ \frac{1}{r} \int_{\Omega_r} |\nabla u_\varepsilon|^2 \right\}^{1/2} \leq C \{ \|F\|_{L^p(\Omega)} + \|f\|_{H^1(\partial\Omega)} \}, \quad (1-5)$$

where $\Omega_r = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < r\}$. The constant C depends only on d, κ_1, κ_2 , and the Lipschitz character of Ω .

Let \mathcal{R} denote the space of rigid displacements,

$$\mathcal{R} = \{Mx + q : M^T = -M \in \mathbb{R}^{d \times d} \text{ and } q \in \mathbb{R}^d\}, \quad (1-6)$$

where $(Mx)^\alpha = M_i^\alpha x_i$ and M^T denotes the transpose of matrix M . By $u \perp \mathcal{R}$ we mean $u \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$, i.e., $\int_\Omega u \cdot \phi = 0$ for any $\phi \in \mathcal{R}$. We will use $\partial u_\varepsilon / \partial \nu_\varepsilon$ to denote the conormal derivative of u_ε associated with \mathcal{L}_ε .

Theorem 1.2. *Suppose that A and Ω satisfy the same conditions as in Theorem 1.1. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega, \tag{1-7}$$

where $F \in L^p(\Omega; \mathbb{R}^d)$ for $p = 2d/(d + 1)$, $g \in L^2(\partial\Omega; \mathbb{R}^d)$ and $\int_\Omega F \cdot \phi + \int_{\partial\Omega} g \cdot \phi = 0$ for any $\phi \in \mathcal{R}$. Also assume that $u_\varepsilon \perp \mathcal{R}$. Then, for $\varepsilon \leq r < \text{diam}(\Omega)$,

$$\left\{ \frac{1}{r} \int_{\Omega_r} |\nabla u_\varepsilon|^2 \right\}^{1/2} \leq C \{ \|F\|_{L^p(\Omega)} + \|g\|_{L^2(\partial\Omega)} \}, \tag{1-8}$$

where C depends only on d, κ_1, κ_2 , and the Lipschitz character of Ω .

Estimates (1-5) and (1-8), which are scaling-invariant, may be regarded as the Rellich estimates, uniform down to the scale ε , in Lipschitz domains for the elasticity operators \mathcal{L}_ε . Indeed, if the coefficient matrix A is constant, then (1-5) and (1-8) hold for any $0 < r < \text{diam}(\Omega)$. Suppose that $F = 0$ and $u_\varepsilon \in C^1(\bar{\Omega}; \mathbb{R}^d)$. By letting $r \rightarrow 0$, one recovers the full Rellich estimates in Lipschitz domains,

$$\|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \leq C \|u_\varepsilon\|_{H^1(\partial\Omega)} \quad \text{and} \quad \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \leq C \left\| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right\|_{L^2(\partial\Omega)}, \tag{1-9}$$

which were proved in [Fabes et al. 1988; Dahlberg et al. 1988] for second-order elliptic systems with constant coefficients, using integration by parts (see [Kenig 1994] for references on related work on boundary value problems in Lipschitz domains). We should note that our proof of Theorems 1.1 and 1.2 uses the nontangential maximal function estimates in [Dahlberg et al. 1988]. On the other hand, under certain smoothness conditions on A , the Rellich estimates hold for the operator \mathcal{L}_1 on Lipschitz domains with $\text{diam}(\Omega) \leq 1$. By a blow-up argument as well as some localization procedures, this implies

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} &\leq C \{ \|\nabla_{\text{tan}} u_\varepsilon\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \}, \\ \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} &\leq C \left\{ \left\| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \right\}, \end{aligned} \tag{1-10}$$

where $\nabla_{\text{tan}} u_\varepsilon$ denotes the tangential derivative of u_ε on $\partial\Omega$. We emphasize that the estimates (1-10) are local and structure conditions such as periodicity are not needed. However, with the additional periodicity condition, one may combine the local estimates (1-10) with the estimates in Theorems 1.1 and 1.2 to obtain the full Rellich estimate (1-9), uniform in ε , for operators \mathcal{L}_ε (see Remark 3.1). Thus we have been able to completely separate the large-scale regularity due to homogenization from the small-scale regularity due to smoothness of the coefficients.

Under the periodicity condition and the Hölder continuity condition on A , the uniform Rellich estimates (1-9) were proved in [Kenig and Shen 2011a; 2011b] for a family of elliptic operators $\{\mathcal{L}_\varepsilon\}$, where $\mathcal{L}_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla)$ and $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$ satisfies the ellipticity condition

$$\mu |\xi|^2 \leq a_{ij}^{\alpha\beta}(y) \xi_i^\alpha \xi_j^\beta \leq \frac{1}{\mu} |\xi|^2 \tag{1-11}$$

for $y \in \mathbb{R}^d$ and $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d \times m}$ as well as the symmetry condition $A^* = A$, i.e., $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$. The results were used to establish the uniform solvability of the L^2 Dirichlet, regularity, and Neumann problems for

the system $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Lipschitz domains. It is worth pointing out that the Rellich estimates (1-9) are not accessible by compactness methods. One of the key steps in [Kenig and Shen 2011a; 2011b] uses integration by parts and relies on the observation that $\mathcal{L}_1(Q) = Q(\mathcal{L}_1)$, where

$$Q(u)(x', x_d) = u(x', x_d + 1) - u(x', x_d).$$

As a result, the approach does not seem to apply if the coefficients are not periodic. We mention that even with periodic coefficients, the direct extension of the methods used in [Kenig and Shen 2011a; 2011b] is problematic for the system of elasticity, due to the weaker ellipticity condition and the lack of (uniform) Korn inequalities on boundary layers.

In this paper we develop a new approach to uniform boundary regularity in quantitative homogenization of elliptic equations and systems. Let u_0 denote the solution of the boundary value problem for the homogenized system with the same data. The basic idea is to consider the function

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_j^\beta(x/\varepsilon) K_\varepsilon^2 \left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \quad (1-12)$$

in Ω , where $\chi = (\chi_j^\beta)$ denotes the matrix of correctors, $K_\varepsilon^2 = K_\varepsilon \circ K_\varepsilon$ with K_ε being a smoothing operator at scale ε , and $\eta_\varepsilon \in C_0^\infty(\Omega)$ is a cut-off function with support in $\{x \in \Omega : \text{dist}(x, \partial\Omega) \geq 3\varepsilon\}$. Using energy estimates for the operator \mathcal{L}_ε as well as sharp boundary regularity estimates for u_0 , we are able to bound

$$\varepsilon^{-1/2} \|w_\varepsilon\|_{H^1(\Omega)}$$

by the right-hand sides of estimates (1-5) and (1-8), respectively. This, together with sharp estimates for u_0 , yields the desired estimates for

$$r^{-1/2} \|\nabla u_\varepsilon\|_{L^2(\Omega_r)}$$

for $\varepsilon \leq r < \text{diam}(\Omega)$. We mention that since \mathcal{L}_0 has constant coefficients, the sharp boundary estimates in Lipschitz domains in terms of nontangential maximal functions are known [Fabes et al. 1988; Dahlberg et al. 1988]. Also, because of the use of the smoothing operator K_ε , which is motivated by [Pastukhova 2006; Suslina 2013a] (also see [Griso 2004; Onofrei and Vernescu 2007; Kenig et al. 2012; Suslina 2013b]), we only need to assume that

$$\sup_{x \in \mathbb{R}^d} \int_{B(x, 1)} (|\chi(y)|^2 + |\nabla \chi(y)|^2) dy < \infty,$$

and that a similar estimate holds for a dual corrector $\phi = (\phi_{kij}^{\alpha\beta})$ (see (2-5) for its definition). As such, it is possible to extend the approach to the almost-periodic or other nonperiodic settings. We plan to carry out this study in a separate work.

As we mentioned before, the estimates in Theorems 1.1 and 1.2 may be used to establish uniform solvability of L^2 boundary value problems for \mathcal{L}_ε in Lipschitz domains [Kenig and Shen 2011a; 2011b]. They can also be used to obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q = 2d/(d-1)$ and a Lipschitz domain Ω , with no smoothness assumption on the coefficients.

Theorem 1.3. *Suppose that A and Ω satisfy the same conditions as in Theorem 1.1. Let u_ε be a weak solution to (1-4) or (1-7), and u_0 the weak solution of the homogenized system with the same data. Suppose that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. In the case of the Neumann problem (1-7) we further assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1-13}$$

where $q = p' = 2d/(d - 1)$ and C depends only on d, κ_1, κ_2 , and Ω .

We remark that if Ω is C^2 and $u_\varepsilon = 0$ or $\partial u_\varepsilon/\partial \nu_\varepsilon = 0$ on $\partial\Omega$, the $O(\varepsilon)$ estimate

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|F\|_{L^2(\Omega)} \tag{1-14}$$

was proved in [Suslina 2013a; 2013b] for a broader class of elliptic operators with measurable periodic coefficients, which contains the systems of elasticity considered here (also see [Griso 2004; Onofrei and Vernescu 2007; Kenig et al. 2012; 2014] and their references for related work on convergence rates). Note that $q = 2d/(d - 1) > 2$ and $\|u_0\|_{H^2(\Omega)} \leq C\|F\|_{L^2(\Omega)}$ if Ω is C^2 and $\mathcal{L}_0(u_0) = F$ in Ω with $u_0 = 0$ or $\partial u_0/\partial \nu_0 = 0$ on $\partial\Omega$. Thus our estimate (1-13) is stronger than (1-14). In the case of scalar elliptic equations with Dirichlet condition $u_\varepsilon = 0$ on $\partial\Omega$, it is known that $\|u_\varepsilon - u_0\|_{L^q(\Omega)} \leq C\varepsilon \|F\|_{L^p(\Omega)}$, where $1 < p < d$ and $1/q = 1/p - 1/d$ (see [Kenig et al. 2014, p. 1234]). Although the exponent $q = 2d/(d - 1)$ may not be sharp, Theorem 1.3 seems to be the first result on the sharp $O(\varepsilon)$ estimate of $u_\varepsilon - u_0$ in $L^q(\Omega)$ with $q > 2$ for elliptic systems with bounded measurable periodic coefficients.

As we indicated above, the proof of Theorems 1.1 and 1.2 only uses the energy estimates in L^2 for \mathcal{L}_ε and thus requires no smoothness assumptions on the coefficients. In the second part of this paper we apply the similar ideas in the L^p setting for $1 < p < \infty$. To do this we first establish the $W^{1,p}$ estimates for the systems

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(h) \quad \text{in } \Omega, \tag{1-15}$$

where $h = (h_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d \times d})$, with either the Dirichlet or Neumann boundary conditions, under the additional assumptions that Ω is C^1 and $A = A(y)$ belongs to $\operatorname{VMO}(\mathbb{R}^d)$. As a result, the L^p analogues of estimates (1-5) and (1-8) are proved under these additional conditions, which are more or less sharp. Consequently, by combining the L^p estimates on the boundary layer Ω_ε with local estimates for \mathcal{L}_1 , which hold for Hölder continuous coefficients, we may obtain the uniform Rellich estimates in L^p for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in C^1 domains under the assumptions that A is Hölder continuous and satisfies (1-2)–(1-3). By the method of layer potentials, this will lead to the uniform solvability of the L^p Dirichlet, regularity, and Neumann problems in C^1 domains (details will be provided in a separate work). Previously, these results in L^p are known only in $C^{1,\alpha}$ domains for operators \mathcal{L}_ε with Hölder continuous coefficients satisfying (1-11) and $A^* = A$ [Kenig et al. 2013]. We remark that the $W^{1,p}$ estimates (local or global) for operators with nonsmooth coefficients in nonsmooth domains are of interest in their own rights and have been studied extensively in recent years (see [Caffarelli and Peral 1998; Auscher and Qafsaoui 2002; Wang 2003; Byun and Wang 2004; 2005; Shen 2005; 2008; Krylov 2007; Dong and Kim 2010; Kenig et al. 2013; Geng 2012; Geng et al. 2012] and their references). Our approach to the $W^{1,p}$ estimates is based on a real-variable argument, which originated in [Caffarelli and Peral 1998] and further developed

in [Wang 2003; Shen 2005; 2007]. The required (weak) reverse Hölder estimates at the boundary are proved by combining the interior Lipschitz estimates down to the scale ε with boundary C^α estimates.

Theorems 1.1 and 1.2 as well as their L^p analogues, given in Section 7, are the main contributions of this paper. For a comprehensive study in the boundary regularity for \mathcal{L}_ε , in Sections 8 and 9, we investigate the boundary Lipschitz estimates, uniform down to the scale ε , for solutions in $C^{1,\alpha}$ domains with the Dirichlet or Neumann conditions. Let

$$\begin{aligned} D_r &= \{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \psi(x') < x_d < \psi(x') + r\}, \\ \Delta_r &= \{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } x_d = \psi(x')\}, \end{aligned} \quad (1-16)$$

where $\psi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a $C^{1,\alpha}$ function for some $\alpha > 0$ with $\psi(0) = 0$ and $\|\nabla\psi\|_{C^\alpha(\mathbb{R}^{d-1})} \leq M$.

Theorem 1.4. *Suppose that A satisfies conditions (1-2)–(1-3). Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution to*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } D_1 \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \Delta_1. \quad (1-17)$$

Then, for $\varepsilon \leq r < 1$,

$$\left(\int_{D_r} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left\{ \left(\int_{D_1} |\nabla u_\varepsilon|^2 \right)^{1/2} + \|f\|_{C^{1,\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)} \right\}, \quad (1-18)$$

where $p > d$ and $\sigma \in (0, \alpha)$. The constant C depends only on $d, \kappa_1, \kappa_2, p, \sigma$, and (α, M) .

Theorem 1.5. *Suppose that A satisfies (1-2)–(1-3). Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution to*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } D_1 \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \Delta_1. \quad (1-19)$$

Then, for $\varepsilon \leq r < 1$,

$$\left(\int_{D_r} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left\{ \left(\int_{D_1} |\nabla u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^\sigma(\Delta_1)} + \|F\|_{L^p(D_1)} \right\}, \quad (1-20)$$

where $p > d$ and $\sigma \in (0, \alpha)$. The constant C depends only on $d, \kappa_1, \kappa_2, p, \sigma$, and (α, M) .

As in the case of Rellich estimates, under additional smoothness conditions on A , using local Lipschitz estimates for \mathcal{L}_1 and a blow-up argument, one may derive from Theorems 1.4 and 1.5 the full boundary Lipschitz estimates

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \leq C \left\{ \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|f\|_{C^{1,\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)} \right\} \quad (1-21)$$

for solutions of (1-17), and

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \leq C \left\{ \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^\sigma(\Delta_1)} + \|F\|_{L^p(D_1)} \right\} \quad (1-22)$$

for solutions of (1-19). We remark that for elliptic systems satisfying the ellipticity condition (1-11), the periodicity condition (1-3) and the Hölder continuity condition, the estimate (1-21) was proved in [Avellaneda and Lin 1987], while (1-22) was established in [Kenig et al. 2013] under the additional symmetry condition $A^* = A$. This symmetry condition was removed recently in [Armstrong and Shen 2016]. However, our estimates in Theorems 1.4 and 1.5 are new for the system of elasticity.

Our proof of Theorems 1.4 and 1.5 also uses the function w_ε , given by (1-12). As a consequence of its estimates in L^2 , for each $r \in (\varepsilon, \frac{1}{4})$, we are able to construct a function v such that $\mathcal{L}_0(v) = F$ in D_r with the same (Dirichlet or Neumann) data on Δ_r as u_ε , and

$$\left(\int_{D_r} |u_\varepsilon - v|^2 \right)^{1/2} \leq C(\varepsilon/r)^{1/2} \left\{ \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + \text{terms involving given data} \right\}.$$

This allows us to use a general scheme for establishing Lipschitz estimates down to the scale ε , which was formulated recently in [Armstrong and Smart 2016] and used for interior estimates in stochastic homogenization with random coefficients (also see [Armstrong and Mourrat 2016] as well as related work in [Gloria and Otto 2011; 2012; Gloria et al. 2014; 2015]). Our argument is similar to (and somewhat simpler and more transparent than) that in [Armstrong and Shen 2016], where the scheme was adapted to prove the full boundary Lipschitz estimates for second-order elliptic systems with almost-periodic and Hölder continuous coefficients. As indicated earlier, we have been able to completely avoid the use of compactness methods (even in the case of C^α estimates). Although it is possible to prove the interior Lipschitz estimates as well as the boundary C^α estimates, down to the scale ε without smoothness, by the compactness methods, as demonstrated in [Avellaneda and Lin 1987; Gu and Shen 2015], the compactness methods for boundary Lipschitz estimates require the same estimates for boundary correctors, which are not easy to establish [Avellaneda and Lin 1987; Kenig et al. 2013].

The paper is organized as follows. In Section 2 we establish some key convergence results in H^1 . These results are used in Section 3 to prove Theorems 1.1 and 1.2. In Section 4 we study the convergence rates in L^q for $q = 2d/(d - 1)$ and give the proof of Theorem 1.3, which uses the estimates in Theorems 1.1 and 1.2 as well as a duality argument. In Sections 5 and 6 we obtain the boundary C^α and $W^{1,p}$ estimates, respectively, in C^1 domains for operators with VMO coefficients. These estimates are used in Section 7 to establish the L^p analogues of (1-5) and (1-8) in C^1 domains. Finally, Theorem 1.4 is proved in Section 8, and Section 9 contains the proof of Theorem 1.5.

Throughout the paper we use $\int_E u = (1/|E|) \int_E u$ to denote the average of u over the set E . We will use C and c to denote constants that may depend on d, κ_1, κ_2, A and Ω , but never on ε .

2. Convergence rates in H^1

In this section we establish certain results on convergence rates in H^1 , which will play a crucial role in the proof of our main results. Throughout the section we assume that $A = A(y)$ satisfies (1-2)–(1-3) and Ω is a bounded Lipschitz domain in \mathbb{R}^d .

Let $\chi = (\chi_j^\beta(y)) = (\chi_j^{\alpha\beta}(y))$ denote the matrix of correctors for \mathcal{L}_ε , where $1 \leq j, \alpha, \beta \leq d$. This means that $\chi_j^\beta \in H^1_{loc}(\mathbb{R}^d; \mathbb{R}^d)$ is 1-periodic, $\int_Y \chi_j^\beta = 0$, and

$$\mathcal{L}_1(\chi_j^\beta) = -\mathcal{L}_1(P_j^\beta) \quad \text{in } \mathbb{R}^d, \tag{2-1}$$

where $Y = [0, 1)^d$ and $P_j^\beta = y_j(0, \dots, 1, \dots, 0)$ with 1 in the β -th position. The homogenized operator is given by $\mathcal{L}_0 = -\text{div}(\hat{A}\nabla)$, where $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ is the matrix of effective coefficients with

$$\hat{a}_{ij}^{\alpha\beta} = \int_Y \left\{ a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} (\chi_j^{\gamma\beta}) \right\}. \tag{2-2}$$

It is known that the constant matrix \hat{A} satisfies the elasticity condition (1-2) [Oleřnik et al. 1992; Jikov et al. 1994]. Define

$$b_{ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} (\chi_j^{\gamma\beta}) - \hat{a}_{ij}^{\alpha\beta}. \quad (2-3)$$

By the definition of \hat{A} and (2-1),

$$\int_Y b_{ij}^{\alpha\beta} = 0 \quad \text{and} \quad \frac{\partial}{\partial y_i} (b_{ij}^{\alpha\beta}) = 0. \quad (2-4)$$

It follows that there exist $\phi_{kij}^{\alpha\beta} \in H_{\text{loc}}^1(\mathbb{R}^d)$ such that $\phi_{kij}^{\alpha\beta}$ is 1-periodic,

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k} (\phi_{kij}^{\alpha\beta}) \quad \text{and} \quad \phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta} \quad (2-5)$$

(see, e.g., [Jikov et al. 1994; Kenig et al. 2012]).

Fix $\varphi \in C_0^\infty(B(0, \frac{1}{4}))$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Define

$$K_\varepsilon(f)(x) = f * \varphi_\varepsilon(x) = \int_{\mathbb{R}^d} f(x-y) \varphi_\varepsilon(y) dy, \quad (2-6)$$

where $\varphi_\varepsilon(y) = \varepsilon^{-d} \varphi(y/\varepsilon)$.

Lemma 2.1. *Let $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Then for any $g \in L_{\text{loc}}^p(\mathbb{R}^d)$,*

$$\|g(x/\varepsilon) K_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq C \sup_{x \in \mathbb{R}^d} \left(\int_{B(x,1)} |g|^p \right)^{1/p} \|f\|_{L^p(\mathbb{R}^d)}, \quad (2-7)$$

where C depends only on d .

Proof. By Hölder's inequality,

$$|K_\varepsilon(f)(x)|^p \leq \frac{C}{|B(0, \varepsilon)|} \int_{\mathbb{R}^d} |f(y)|^p \chi_{B(x, \varepsilon)}(y) dy,$$

from which the estimate (2-7) follows readily by Fubini's theorem. \square

It follows from (2-7) that if $g \in L_{\text{loc}}^p(\mathbb{R}^d)$ and is 1-periodic, then

$$\|g(x/\varepsilon) K_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq C \|g\|_{L^p(Y)} \|f\|_{L^p(\mathbb{R}^d)}. \quad (2-8)$$

Lemma 2.2. *Let $f \in W^{1,q}(\mathbb{R}^d)$ for some $1 < q < \infty$. Then*

$$\|K_\varepsilon(f) - f\|_{L^q(\mathbb{R}^d)} \leq C \varepsilon \|\nabla f\|_{L^q(\mathbb{R}^d)}. \quad (2-9)$$

Moreover, if $p = 2d/(d+1)$,

$$\begin{aligned} \|K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\leq C \varepsilon^{-1/2} \|f\|_{L^p(\mathbb{R}^d)}, \\ \|f - K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\leq C \varepsilon^{1/2} \|\nabla f\|_{L^p(\mathbb{R}^d)}. \end{aligned} \quad (2-10)$$

The constant C depends only on d .

Proof. To see (2-9), we note that

$$\|f(\cdot - y) - f(\cdot)\|_{L^q(\mathbb{R}^d)} \leq |y| \|\nabla f\|_{L^q(\mathbb{R}^d)}$$

for any $y \in \mathbb{R}^d$. Thus, by Minkowski's inequality,

$$\begin{aligned} \|K_\varepsilon(f) - f\|_{L^q(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} \varphi_\varepsilon(y) \|f(\cdot - y) - f(\cdot)\|_{L^q(\mathbb{R}^d)} dy \\ &\leq \int_{\mathbb{R}^d} \varphi_\varepsilon(y) |y| dy \|\nabla f\|_{L^q(\mathbb{R}^d)} \\ &= C\varepsilon \|\nabla f\|_{L^q(\mathbb{R}^d)}. \end{aligned}$$

Next, by Parseval's theorem and Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}^d} |K_\varepsilon(f)|^2 dx &= \int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\ &\leq \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi)|^{2d} d\xi \right)^{1/d} \|\hat{f}\|_{L^{p'}(\mathbb{R}^d)}^2 \\ &\leq C\varepsilon^{-1} \|f\|_{L^p(\mathbb{R}^d)}^2, \end{aligned}$$

where \hat{f} denotes the Fourier transform of f , and we have used the Hausdorff–Young inequality $\|\hat{f}\|_{L^{p'}(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}$. This gives the first inequality in (2-10). To see the second inequality, we note that $\hat{\varphi}(0) = \int_{\mathbb{R}^d} \varphi = 1$. It follows that

$$\begin{aligned} \|f - K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\leq C \left\{ \int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi) - \hat{\varphi}(0)|^{2d} |\xi|^{-2d} d\xi \right\}^{1/(2d)} \|\widehat{\nabla f}\|_{L^{p'}(\mathbb{R}^d)} \\ &\leq C\varepsilon^{1/2} \|\nabla f\|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

where we have used $|\hat{\varphi}(\xi) - \hat{\varphi}(0)| \leq C|\xi|$ for the last step. □

Lemma 2.3. *Let $u_\varepsilon, u_0 \in H^1(\Omega; \mathbb{R}^d)$. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ in Ω and either $u_\varepsilon = u_0$ or $\partial u_\varepsilon / \partial \nu_\varepsilon = \partial u_0 / \partial \nu_0$ on $\partial\Omega$. Let*

$$w_\varepsilon^\alpha = u_\varepsilon^\alpha - u_0^\alpha - \varepsilon \chi_j^{\alpha\beta}(x/\varepsilon) K_\varepsilon^2 \left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right),$$

where $K_\varepsilon^2 = K_\varepsilon \circ K_\varepsilon$, $\eta_\varepsilon \in C_0^\infty(\Omega)$ and $\text{supp}(\eta_\varepsilon) \subset \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq 3\varepsilon\}$. Then

$$\begin{aligned} \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla w_\varepsilon dx &= \int_{\Omega} [\hat{A} - A(x/\varepsilon)] [\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)] \cdot \nabla w_\varepsilon dx \\ &\quad - \int_{\Omega} B(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon dx \\ &\quad - \varepsilon \int_{\Omega} A(x/\varepsilon) \chi(x/\varepsilon) \nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon dx, \end{aligned} \tag{2-11}$$

where $B(y) = (b_{ij}^{\alpha\beta}(y))$ is defined in (2-3).

Proof. We first note that if $u_\varepsilon = u_0$ on $\partial\Omega$, then $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$, as $K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \in C_0^\infty(\Omega)$. Since $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ in Ω , it follows that

$$\int_{\Omega} A(x/\varepsilon) \nabla u_\varepsilon \cdot \nabla w_\varepsilon \, dx = \int_{\Omega} \hat{A} \nabla u_0 \cdot \nabla w_\varepsilon \, dx. \quad (2-12)$$

In the case of the Neumann condition $\partial u_\varepsilon / \partial \varepsilon = \partial u_0 / \partial \nu_0$ on $\partial\Omega$, equation (2-12) continues to hold. This is because $w_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ and both sides of (2-12) are equal to

$$\langle \mathcal{L}_0(u_0), w_\varepsilon \rangle_{(H^1(\Omega))' \times H^1(\Omega)} + \left\langle \frac{\partial u_0}{\partial \nu_0}, w_\varepsilon \right\rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}.$$

Using (2-12), we obtain

$$\begin{aligned} \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx &= \int_{\Omega} [\hat{A} - A(x/\varepsilon)] \nabla u_0 \cdot \nabla w_\varepsilon \, dx \\ &\quad - \int_{\Omega} A(x/\varepsilon) \nabla \chi(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon \, dx \\ &\quad - \varepsilon \int_{\Omega} A(x/\varepsilon) \chi(x/\varepsilon) \nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon \, dx, \end{aligned}$$

from which the formal (2-11) follows by the definition of $B(y)$. \square

Lemma 2.4. *Let $\phi(y) = (\phi_{kij}^{\alpha\beta}(y))$ be defined by (2-5). Then*

$$\int_{\Omega} B(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon \, dx = -\varepsilon \int_{\Omega} \phi_{kij}^{\alpha\beta}(x/\varepsilon) \frac{\partial w_\varepsilon^\alpha}{\partial x_i} \cdot \frac{\partial}{\partial x_k} K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon\right) \, dx. \quad (2-13)$$

Proof. Using (2-5), we see that

$$\begin{aligned} B(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon &= b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon\right) \cdot \frac{\partial w_\varepsilon^\alpha}{\partial x_i} \\ &= \varepsilon \frac{\partial}{\partial x_k} \left(\phi_{kij}^{\alpha\beta}(x/\varepsilon) \right) K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon\right) \cdot \frac{\partial w_\varepsilon^\alpha}{\partial x_i} \\ &= \varepsilon \frac{\partial}{\partial x_k} \left\{ \phi_{kij}^{\alpha\beta}(x/\varepsilon) \frac{\partial w_\varepsilon^\alpha}{\partial x_i} \right\} K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon\right), \end{aligned}$$

from which equation (2-13) follows readily. \square

Lemma 2.5. *Let u_ε ($\varepsilon \geq 0$) be a solution to the Dirichlet problem (1-4) or the Neumann problem (1-7). Let w_ε be defined as in Lemma 2.3 with η_ε satisfying*

$$\begin{cases} \eta_\varepsilon \in C_0^\infty(\Omega), & 0 \leq \eta \leq 1, \\ \text{supp}(\eta_\varepsilon) \subset \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq 3\varepsilon\}, \\ \eta_\varepsilon = 1 & \text{on } \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq 4\varepsilon\}, \\ |\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}. \end{cases} \quad (2-14)$$

Then

$$\begin{aligned} & \left| \int_{\Omega} A(x/\varepsilon) \nabla w_{\varepsilon} \cdot \nabla w_{\varepsilon} \, dx \right| \\ & \leq C \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \left\{ \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + \|(\nabla u_0)\eta_{\varepsilon} - K_{\varepsilon}((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} + \varepsilon \|K_{\varepsilon}((\nabla^2 u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} \right\}. \end{aligned} \quad (2-15)$$

Proof. It follows from Lemmas 2.3 and 2.4 by the Cauchy inequality that

$$\begin{aligned} \left| \int_{\Omega} A(x/\varepsilon) \nabla w_{\varepsilon} \cdot \nabla w_{\varepsilon} \, dx \right| & \leq C \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \left\{ \|\nabla u_0 - K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} + \varepsilon \|\chi(x/\varepsilon) \nabla K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} \right. \\ & \qquad \qquad \qquad \left. + \varepsilon \|\phi(x/\varepsilon) \nabla K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} \right\} \\ & \leq C \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \left\{ \|\nabla u_0 - K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} + \varepsilon \|\nabla K_{\varepsilon}((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} \right\}, \end{aligned}$$

where we have used Lemma 2.1 as well as the fact that $\chi, \phi \in L^2_{\text{loc}}(\mathbb{R}^d)$ and are 1-periodic for the last inequality. Observe that

$$\begin{aligned} \|\nabla u_0 - K_{\varepsilon}^2((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} & \leq \|(\nabla u_0)(1 - \eta_{\varepsilon})\|_{L^2(\Omega)} + \|(\nabla u_0)\eta_{\varepsilon} - K_{\varepsilon}((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} \\ & \qquad \qquad \qquad + \|K_{\varepsilon}((u_0)\eta_{\varepsilon} - K_{\varepsilon}((\nabla u_0)\eta_{\varepsilon}))\|_{L^2(\Omega)} \\ & \leq \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + C \|(\nabla u_0)\eta_{\varepsilon} - K_{\varepsilon}((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)}. \end{aligned}$$

Also,

$$\begin{aligned} \varepsilon \|\nabla K_{\varepsilon}((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} & \leq \varepsilon \|K_{\varepsilon}((\nabla^2 u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} + \varepsilon \|K_{\varepsilon}((\nabla u_0)(\nabla \eta_{\varepsilon}))\|_{L^2(\Omega)} \\ & \leq \varepsilon \|K_{\varepsilon}((\nabla^2 u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} + C \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}. \end{aligned} \quad \square$$

Finally, we are in a position to state and prove the main result of this section.

Theorem 2.6. *Suppose that $A(y)$ satisfies (1-2)–(1-3). Let Ω be a bounded Lipschitz domain. Let u_{ε} ($\varepsilon \geq 0$) be the solutions to the Dirichlet problem (1-4) in Ω with $f \in H^1(\partial\Omega; \mathbb{R}^d)$ and $F \in L^p(\Omega; \mathbb{R}^d)$, where $p = 2d/(d + 1)$. Then*

$$\left\| u_{\varepsilon} - u_0 - \varepsilon \chi_j^{\beta}(x/\varepsilon) K_{\varepsilon}^2 \left(\frac{\partial u_0^{\beta}}{\partial x_j} \eta_{\varepsilon} \right) \right\|_{H_0^1(\Omega)} \leq C \varepsilon^{1/2} \left\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \right\}, \quad (2-16)$$

where $\eta_{\varepsilon} \in C_0^{\infty}(\Omega)$ satisfies (2-14). The constant C depends only on d, κ_1, κ_2 , and the Lipschitz character of Ω .

Proof. Let w_{ε} denote the function on the left-hand side of (2-16). Since $w_{\varepsilon} \in H_0^1(\Omega; \mathbb{R}^d)$, it follows from (2-15) by the first Korn inequality [Oleřnik et al. 1992] that

$$\|w_{\varepsilon}\|_{H_0^1(\Omega)} \leq C \left\{ \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + \|(\nabla u_0)\eta_{\varepsilon} - K_{\varepsilon}((\nabla u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} + \varepsilon \|K_{\varepsilon}((\nabla^2 u_0)\eta_{\varepsilon})\|_{L^2(\Omega)} \right\}. \quad (2-17)$$

To bound the right-hand side of (2-17), we write $u_0 = v + h$, where

$$v(x) = \int_{\Omega} \Gamma_0(x - y) F(y) \, dy$$

and $\Gamma_0(x)$ denotes the matrix of fundamental solutions for the homogenized operator \mathcal{L}_0 in \mathbb{R}^d , with pole at the origin. Note that $\mathcal{L}_0(v) = F$ in Ω , and by the well known singular integral and fractional integral

estimates,

$$\|\nabla^2 v\|_{L^p(\mathbb{R}^d)} + \|\nabla v\|_{L^{p'}(\mathbb{R}^d)} \leq C_p \|F\|_{L^p(\Omega)}, \quad (2-18)$$

where we have used the observation $1/p' = 1/p - 1/d$. Let $\mathbf{e} = (e_1, \dots, e_d) \in C_0^1(\mathbb{R}^d; \mathbb{R}^d)$ be a vector field such that $\langle \mathbf{e}, n \rangle \geq c_0 > 0$ on $\partial\Omega$ and $|\nabla \mathbf{e}| \leq Cr_0^{-1}$, where $r_0 = \text{diam}(\Omega)$ and n denotes the outward unit normal to $\partial\Omega$. It follows from the divergence theorem that

$$\begin{aligned} c_0 \int_{\partial\Omega} |\nabla v|^2 d\sigma &\leq \int_{\partial\Omega} |\nabla v|^2 \langle \mathbf{e}, n \rangle d\sigma \\ &= \int_{\Omega} |\nabla v|^2 \text{div}(\mathbf{e}) dx + \int_{\Omega} e_i \frac{\partial}{\partial x_i} \nabla v \cdot \nabla v dx \\ &\leq C \left\{ r_0^{-1} \int_{\Omega} |\nabla v|^2 dx + \int_{\Omega} |\nabla v| |\nabla^2 v| dx \right\} \\ &\leq C \{ r_0^{-1} \|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^{p'}(\Omega)} \|\nabla^2 v\|_{L^p(\Omega)} \} \leq C \|F\|_{L^p(\Omega)}^2, \end{aligned} \quad (2-19)$$

where we have used (2-18) for the last step. Note that the same argument also gives $\|\nabla v\|_{L^2(S_t)} \leq C \|F\|_{L^p(\Omega)}$, where $S_t = \{x \in \mathbb{R}^d : \text{dist}(x, \partial\Omega) = t\}$ for $0 < t < cr_0$. Consequently, by the coarea formula, we obtain

$$\left\{ \frac{1}{r} \int_{\tilde{\Omega}_r} |\nabla v|^2 dx \right\}^{1/2} \leq C \|F\|_{L^p(\Omega)}, \quad (2-20)$$

where $0 < r < \text{diam}(\Omega)$ and $\tilde{\Omega}_r = \{x \in \mathbb{R}^d : \text{dist}(x, \partial\Omega) < r\}$.

Next, we observe that $\mathcal{L}_0(h) = 0$ in Ω and

$$\begin{aligned} \|h\|_{H^1(\partial\Omega)} &\leq \|f\|_{H^1(\partial\Omega)} + \|v\|_{H^1(\partial\Omega)} \\ &\leq \|f\|_{H^1(\partial\Omega)} + C \|F\|_{L^p(\Omega)}, \end{aligned}$$

where we have used (2-19) for the last inequality. It follows from the estimates for solutions of the L^2 regularity problem in Lipschitz domains for the operator \mathcal{L}_0 in [Dahlberg et al. 1988; Verchota 1986] that

$$\|(\nabla h)^*\|_{L^2(\partial\Omega)} \leq C \{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \}, \quad (2-21)$$

where $(\nabla h)^*$ denotes the nontangential maximal function of ∇h . This, together with (2-20), gives

$$\|\nabla u_0\|_{L^2(\tilde{\Omega}_r)} \leq Cr^{1/2} \{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \} \quad (2-22)$$

for any $0 < r < \text{diam}(\Omega)$. As a result, the first term on the right-hand side of (2-17) is bounded by $C\varepsilon^{1/2} \{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \}$.

To handle the third term on the right-hand side of (2-17), we use Lemma 2.2 to obtain

$$\begin{aligned} \varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} &\leq \varepsilon \|K_\varepsilon((\nabla^2 v)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon \|K_\varepsilon((\nabla^2 h)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\leq C\varepsilon^{1/2} \|(\nabla^2 v)\eta_\varepsilon\|_{L^p(\Omega)} + C\varepsilon \|(\nabla^2 h)\eta_\varepsilon\|_{L^2(\Omega)} \\ &\leq C\varepsilon^{1/2} \|F\|_{L^p(\Omega)} + C\varepsilon \|\nabla^2 h\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})}. \end{aligned} \quad (2-23)$$

Since $\mathcal{L}_0(\nabla h) = 0$ in Ω , we may use the interior estimate for \mathcal{L}_0 ,

$$|\nabla^2 h(x)| \leq \frac{C}{\delta(x)} \left(\int_{B(x, \delta(x)/8)} |\nabla h|^2 \right)^{1/2},$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$, to show that

$$\begin{aligned} \|\nabla^2 h\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} &\leq C \|(\nabla h)[\delta(x)]^{-1}\|_{L^2(\Omega \setminus \Omega_\varepsilon)} \\ &\leq C\varepsilon^{-1/2} \{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \}, \end{aligned} \tag{2-24}$$

where the last inequality follows from (2-21). This, together with (2-23), gives

$$\varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \}. \tag{2-25}$$

Finally, to bound the second term on the right-hand side of (2-17), we again write $u_0 = v + h$ as before. Note that by Lemma 2.2,

$$\begin{aligned} \|(\nabla v)\eta_\varepsilon - K_\varepsilon((\nabla v)\eta_\varepsilon)\|_{L^2(\Omega)} &\leq \|\nabla v - K_\varepsilon(\nabla v)\|_{L^2(\mathbb{R}^d)} + \|(\nabla v)(1 - \eta_\varepsilon)\|_{L^2(\Omega)} + \|K_\varepsilon((\nabla v)(1 - \eta_\varepsilon))\|_{L^2(\Omega)} \\ &\leq C\varepsilon^{1/2} \|\nabla^2 v\|_{L^p(\mathbb{R}^d)} + C\|\nabla v\|_{L^2(\tilde{\Omega}_{8\varepsilon})} \\ &\leq C\varepsilon^{1/2} \|F\|_{L^p(\Omega)}, \end{aligned}$$

where we have used (2-18) and (2-20) for the last inequality. Also, by Lemma 2.2,

$$\begin{aligned} \|(\nabla h)\eta_\varepsilon - K_\varepsilon((\nabla h)\eta_\varepsilon)\|_{L^2(\Omega)} &\leq C\varepsilon \|\nabla((\nabla h)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\leq C \{ \varepsilon \|\nabla^2 h\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \|\nabla h\|_{L^2(\Omega_{4\varepsilon})} \} \\ &\leq C\varepsilon^{1/2} \{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \}. \end{aligned}$$

Consequently, the second term on the right-hand side of (2-17) is dominated by the right-hand side of (2-16). This completes the proof of Theorem 2.6. \square

The next theorem is an analogue of Theorem 2.6 for the Neumann boundary conditions.

Theorem 2.7. *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Let Ω be a bounded Lipschitz domain. Let u_ε ($\varepsilon \geq 0$) be the solutions to the Neumann problem (1-7) in Ω with $g \in L^2(\partial\Omega; \mathbb{R}^d)$ and $F \in L^p(\Omega; \mathbb{R}^d)$, where $p = 2d/(d + 1)$. Also assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Then*

$$\left\| u_\varepsilon - u_0 - \varepsilon \chi_j^\beta(x/\varepsilon) K_\varepsilon^2 \left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \right\|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)} \}, \tag{2-26}$$

where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14). The constant C depends only on d, κ_1, κ_2 , and the Lipschitz character of Ω .

Proof. The proof, which uses the estimate in Lemma 2.5, is similar to that of Theorem 2.6. We will only point out the differences and leave the details to the reader.

Let w_ε denote the function on the left-hand side of (2-26). Let

$$\{ \varphi_j : j = 1, \dots, J = \frac{1}{2}d(d + 1) \}$$

be an orthonormal basis of \mathcal{R} , as a subspace of $L^2(\Omega; \mathbb{R}^d)$. By the second Korn inequality [Oleřnik et al. 1992],

$$\|w_\varepsilon\|_{H^1(\Omega)} \leq C \left| \int_\Omega A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla w_\varepsilon dx \right| + C \sum_{j=1}^J \left| \int_\Omega w_\varepsilon \cdot \varphi_j dx \right|. \quad (2-27)$$

Since $u_\varepsilon, u_0 \perp \mathcal{R}$, it follows that

$$\begin{aligned} \left| \int_\Omega w_\varepsilon \cdot \varphi_j dx \right| &\leq C\varepsilon \|\chi(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\leq C\varepsilon \|\nabla u_0\|_{L^2(\Omega)}. \end{aligned}$$

This, together with (2-27) and Lemma 2.5, shows that

$$\begin{aligned} \|w_\varepsilon\|_{H^1(\Omega)} &\leq C \left\{ \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + \varepsilon \|\nabla u_0\|_{L^2(\Omega)} + \|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \right\}. \end{aligned} \quad (2-28)$$

To bound the right-hand side of (2-28), we write $u_0 = v + h$, where v is the same as in the proof of Theorem 2.6. Since $\mathcal{L}_0(h) = 0$ in Ω and

$$\begin{aligned} \left\| \frac{\partial h}{\partial \nu_0} \right\|_{L^2(\partial\Omega)} &\leq \left\| \frac{\partial u_0}{\partial \nu_0} \right\|_{L^2(\partial\Omega)} + \left\| \frac{\partial v}{\partial \nu_0} \right\|_{L^2(\partial\Omega)} \\ &\leq C \left\{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)} \right\}, \end{aligned}$$

we may use the estimates in [Dahlberg et al. 1988; Verchota 1986] for solutions of the L^2 Neumann problem for \mathcal{L}_0 in Lipschitz domains to obtain

$$\begin{aligned} \|(\nabla h)^*\|_{L^2(\partial\Omega)} &\leq C \left\{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)} + \sum_{j=1}^J \left| \int_\Omega h \cdot \varphi_j \right| \right\} \\ &\leq C \left\{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)} \right\}, \end{aligned} \quad (2-29)$$

where we have used the assumption $u_0 \perp \mathcal{R}$. With the nontangential maximal function estimate (2-29) at our disposal, the rest of the proof is exactly the same as that of Theorem 2.6. \square

Remark 2.8. Since

$$\|\chi(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \leq C \|\nabla u_0\|_{L^2(\Omega)},$$

it follows from the estimate (2-16) that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \left\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^2(\Omega)} \right\}, \quad (2-30)$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0) = F$ in Ω and $u_\varepsilon = u_0 = f$ on $\partial\Omega$. Similarly, the estimate (2-26) implies

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \left\{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)} \right\}, \quad (2-31)$$

where u_ε, u_0 are given in Theorem 2.7. These $O(\varepsilon^{1/2})$ estimates in L^2 are not sharp (see Section 4), but they will be sufficient for us to establish the boundary C^α and Lipschitz estimates.

3. Proof of Theorems 1.1 and 1.2

Theorems 1.1 and 1.2 are consequences of Theorems 2.6 and 2.7, respectively. We give the proof of Theorem 1.1. Theorem 1.2 follows from Theorem 2.7 in the same manner.

Without loss of generality we may assume that

$$\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} = 1.$$

Let w_ε denote the function on the left-hand side of (2-16). By Theorem 2.6, for $\varepsilon \leq r < \text{diam}(\Omega)$,

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^2(\Omega_r)} &\leq \|\nabla u_0\|_{L^2(\Omega_r)} + \|\nabla w_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla\{\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\}\|_{L^2(\Omega_r)} \\ &\leq Cr^{1/2} + \|\nabla\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_r)} + \varepsilon \|\chi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_r)} \\ &\leq Cr^{1/2} + C\|K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + C\varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})}, \end{aligned}$$

where we have used (2-22) and Lemma 2.1 as well as the fact that the operator K_ε is a convolution with a kernel supported in $B(0, \varepsilon/4)$. Note that by (2-22) and (2-25),

$$\|K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} \leq C\|\nabla u_0\|_{L^2(\Omega_{3r})} \leq Cr^{1/2},$$

and

$$\begin{aligned} \varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} &\leq \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + \varepsilon\|K_\varepsilon((\nabla u_0)(\nabla\eta_\varepsilon))\|_{L^2(\Omega_{2r})} \\ &\leq \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + C\|\nabla u_0\|_{L^2(\Omega_{3r})} \\ &\leq Cr^{1/2}. \end{aligned}$$

The proof of Theorem 1.1 is complete.

Remark 3.1. Under certain smoothness conditions on A , it is possible to extend the Rellich estimates in [Dahlberg et al. 1988] for the Lamé systems with constant coefficients to the operator \mathcal{L}_1 with variable coefficients satisfying the condition (1-2). We refer the reader to [Kenig and Shen 2011b], where this is done in the case that the coefficients satisfy the ellipticity condition (1-11). It follows that if $\mathcal{L}_1(u) = 0$ in D_2 , where D_r is defined by (1-16) with $\psi(0) = 0$ and $\|\nabla\psi\|_\infty \leq M$, then

$$\begin{cases} \int_{\partial D_r} |\nabla u|^2 d\sigma \leq C \int_{\partial D_r} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma + C \int_{D_r} |\nabla u|^2 dx, \\ \int_{\partial D_r} |\nabla u|^2 d\sigma \leq C \int_{\partial D_r} |\nabla_{\tan} u|^2 d\sigma + C \int_{D_r} |\nabla u|^2 dx \end{cases} \quad (3-1)$$

for any $r \in (1, \frac{3}{2})$, where C depends only on d , A , and M . By integrating both sides of the inequalities in (3-1) with respect to r over $(1, \frac{3}{2})$, we obtain

$$\begin{cases} \int_{\Delta_1} |\nabla u|^2 d\sigma \leq C \int_{\Delta_2} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma + C \int_{D_2} |\nabla u|^2 dx, \\ \int_{\Delta_1} |\nabla u|^2 d\sigma \leq C \int_{\Delta_2} |\nabla_{\tan} u|^2 d\sigma + C \int_{D_2} |\nabla u|^2 dx, \end{cases} \quad (3-2)$$

where $\Delta_r = \{(x', \psi(x')) \in \mathbb{R}^d : |x'| < r \text{ and } x_d = \psi(x')\}$. We now take advantage of the fact that the dependence of C on ψ is only through M . Since $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ implies $\mathcal{L}_1\{u_\varepsilon(\varepsilon x)\} = 0$, one may deduce from (3-2) that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $D_{2\varepsilon}$, then

$$\begin{cases} \int_{\Delta_\varepsilon} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\Delta_{2\varepsilon}} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 d\sigma + \frac{C}{\varepsilon} \int_{D_{2\varepsilon}} |\nabla u_\varepsilon|^2 dx, \\ \int_{\Delta_\varepsilon} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\Delta_{2\varepsilon}} |\nabla_{\tan} u_\varepsilon|^2 d\sigma + \frac{C}{\varepsilon} \int_{D_{2\varepsilon}} |\nabla u_\varepsilon|^2 dx. \end{cases} \quad (3-3)$$

Now, suppose that $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ and $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω , where Ω is a bounded Lipschitz domain in \mathbb{R}^d . By covering $\partial\Omega$ with a finite number of suitable balls of size $c\varepsilon$, it follows from (3-3) that

$$\begin{cases} \int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^2 dx, \\ \int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^2 d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^2 dx. \end{cases} \quad (3-4)$$

Notice that up to this point, we have only used the smoothness condition of A , not the periodicity of A . With the additional periodicity condition we may invoke the estimates in Theorems 1.1 and 1.2 to bound the volume integrals of $|\nabla u_\varepsilon|^2$ over the boundary layer $\Omega_{c\varepsilon}$. This yields the full Rellich estimates,

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 d\sigma \quad (3-5)$$

if $u_\varepsilon \perp \mathcal{R}$, and

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^2 d\sigma + Cr_0^{-2} \int_{\partial\Omega} |u_\varepsilon|^2 d\sigma. \quad (3-6)$$

It is well known that estimates (3-5)–(3-6) may be used to solve the L^2 boundary value problems in Lipschitz domains by the method of layer potentials. We refer the reader to [Kenig and Shen 2011b] for the case where $A(y)$ satisfies (1-11). The details for the system of linear elasticity have been carried out in a separate work [Geng et al. 2017].

4. Convergence rates in L^q for $q = 2d/(d-1)$

We now establish sharp $O(\varepsilon)$ estimates for $\|u_\varepsilon - u_0\|_{L^q(\Omega)}$ with $q = 2d/(d-1)$, using Theorems 1.1 and 1.2 and a duality argument. Throughout this section we will assume that Ω is a bounded Lipschitz domain and $A = A(y)$ satisfies (1-2)–(1-3).

We start with the Dirichlet boundary condition.

Lemma 4.1. *Let u_ε ($\varepsilon \geq 0$) be the solution of (1-4). Suppose that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then*

$$\left\| u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) - v_\varepsilon \right\|_{H_0^1(\Omega)} \leq C\varepsilon \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)}, \quad (4-1)$$

where $\tilde{u}_0 \in H^2(\mathbb{R}^d; \mathbb{R}^d)$ is an extension of u_0 and $v_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ is the weak solution to

$$\mathcal{L}_\varepsilon(v_\varepsilon) = 0 \quad \text{in } \Omega \quad \text{and} \quad v_\varepsilon = -\varepsilon \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) \quad \text{on } \partial\Omega. \tag{4-2}$$

Proof. Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) - v_\varepsilon.$$

Using $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ and $\mathcal{L}_\varepsilon(v_\varepsilon) = 0$ in Ω , a direct computation shows that

$$\begin{aligned} \mathcal{L}_\varepsilon(w_\varepsilon) &= -\frac{\partial}{\partial x_i} \left\{ [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \frac{\partial u_0^\beta}{\partial x_j} \right\} - \mathcal{L}_\varepsilon \left\{ \varepsilon \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) \right\} \\ &= -\frac{\partial}{\partial x_i} \left\{ [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon \left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j} \right) \right] \right\} + \frac{\partial}{\partial x_i} \left\{ b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j} \right) \right\} \\ &\quad + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k} \right) \right\}, \end{aligned} \tag{4-3}$$

where $b_{ij}^{\alpha\beta}$ is defined by (2-3). Using (2-5), we see that

$$\frac{\partial}{\partial x_i} \left\{ b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j} \right) \right\} = \varepsilon \frac{\partial}{\partial x_i} \left\{ \phi_{ikj}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j} \right) \right\}. \tag{4-4}$$

Indeed, the left-hand side of (4-4) equals

$$b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_i \partial x_j} \right),$$

while the right-hand side equals

$$b_{kj}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j} \right) + \phi_{ikj}^{\alpha\beta}(x/\varepsilon) \frac{\partial^2}{\partial x_i \partial x_k} K_\varepsilon \left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j} \right)$$

and the second term is zero due to the skew-symmetry $\phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}$.

It follows from (4-3) and (4-4) by Lemmas 2.1 and 2.2 that

$$\|\mathcal{L}_\varepsilon(w_\varepsilon)\|_{H^{-1}(\Omega)} \leq C\varepsilon \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)},$$

where C depends only on d, κ_1, κ_2 , and Ω . Since $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$, this gives the estimate (4-1) by the energy estimate. □

The following theorem establishes the sharp $O(\varepsilon)$ estimate in L^q with $q = 2d/(d - 1)$ for the Dirichlet boundary condition.

Theorem 4.2. *Suppose that A satisfies (1-2)–(1-3). Let Ω be a bounded Lipschitz domain in \mathbb{R}^d . Let u_ε ($\varepsilon \geq 0$) be the weak solution to Dirichlet problem (1-4). Assume that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{4-5}$$

where $q = 2d/(d - 1)$ and C depends only on d, κ_1, κ_2 , and Ω .

Proof. We begin by choosing $\tilde{u}_0 \in H^2(\mathbb{R}^d; \mathbb{R}^d)$ such that $\tilde{u}_0 = u_0$ in Ω and $\|\tilde{u}_0\|_{H^2(\mathbb{R}^d)} \leq C\|u_0\|_{H^2(\Omega)}$, where C depends only on Ω . Since Ω is Lipschitz, this is possible by an extension theorem due to A. Calderón [Stein 1970, Theorem 5, p. 181]. Next, since $H_0^1(\Omega) \subset L^q(\Omega)$ and

$$\left\| \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) \right\|_{L^q(\Omega)} \leq C \|\nabla \tilde{u}_0\|_{L^q(\mathbb{R}^d)} \leq C \|u_0\|_{H^2(\Omega)},$$

in view of Lemma 4.1, it suffices to show that

$$\|v_\varepsilon\|_{L^q(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \quad (4-6)$$

where v_ε is given by (4-2).

To this end we fix $G \in L^p(\Omega; \mathbb{R}^d)$, where $p = q' = 2d/(d+1)$, and let $h_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$ be the weak solution to

$$\mathcal{L}_\varepsilon(h_\varepsilon) = G \quad \text{in } \Omega \quad \text{and} \quad h_\varepsilon = 0 \quad \text{on } \partial\Omega. \quad (4-7)$$

It follows from (4-2), (4-7), and the divergence theorem that

$$\begin{aligned} \int_\Omega v_\varepsilon \cdot G \, dx &= - \int_{\partial\Omega} v_\varepsilon \cdot \frac{\partial h_\varepsilon}{\partial \nu_\varepsilon} \, d\sigma \\ &= \varepsilon \int_{\partial\Omega} \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) \cdot \frac{\partial h_\varepsilon}{\partial \nu_\varepsilon} (\eta_\varepsilon - 1) \, d\sigma \\ &= \int_\Omega \frac{\partial \chi_k^{\alpha\gamma}}{\partial x_i}(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k} \right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j} (\eta_\varepsilon - 1) \, dx \\ &\quad + \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_i \partial x_k} \right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j} (\eta_\varepsilon - 1) \, dx \\ &\quad - \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k} \right) G^\alpha (\eta_\varepsilon - 1) \, dx \\ &\quad + \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k} \right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j} \frac{\partial \eta_\varepsilon}{\partial x_i} \, dx, \end{aligned}$$

where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14). This implies

$$\begin{aligned} \left| \int_\Omega v_\varepsilon \cdot G \, dx \right| &\leq C \int_\Omega |\nabla \chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |\nabla h_\varepsilon| |\eta_\varepsilon - 1| \, dx \\ &\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla^2 \tilde{u}_0)| |\nabla h_\varepsilon| |\eta_\varepsilon - 1| \, dx \\ &\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |G| |\eta_\varepsilon - 1| \, dx \\ &\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |\nabla h_\varepsilon| |\nabla \eta_\varepsilon| \, dx. \end{aligned} \quad (4-8)$$

Note that by Cauchy’s inequality and (2-14), the first and fourth terms on the right-hand side of (4-8) are bounded by

$$C \left(\int_{\Omega_{4\varepsilon}} (|\nabla \chi(x/\varepsilon)| + |\chi(x/\varepsilon)|) K_\varepsilon |\nabla \tilde{u}_0|^2 dx \right)^{1/2} \left(\int_{\Omega_{4\varepsilon}} |\nabla h_\varepsilon|^2 dx \right)^{1/2} \\ \leq C \left(\int_{\tilde{\Omega}_{5\varepsilon}} |\nabla \tilde{u}_0|^2 dx \right)^{1/2} \left(\int_{\Omega_{4\varepsilon}} |\nabla h_\varepsilon|^2 dx \right)^{1/2},$$

where $\Omega_r = \{x \in \Omega : \text{dist}(x, \partial\Omega) < r\}$, $\tilde{\Omega}_r = \{x \in \mathbb{R}^d : \text{dist}(x, \partial\Omega) < r\}$, and we have used Lemma 2.1 for the last inequality. Using the divergence theorem, as in (2-19), one may prove that

$$\|\nabla \tilde{u}_0\|_{L^2(S_r)} \leq C \|\tilde{u}_0\|_{H^1(\mathbb{R}^d)}^{1/2} \|\tilde{u}_0\|_{H^2(\mathbb{R}^d)}^{1/2},$$

where $S_r = \{x \in \mathbb{R}^d : \text{dist}(x, \partial\Omega) = r\}$. It follows by the coarea formula that

$$\|\nabla \tilde{u}_0\|_{L^2(\tilde{\Omega}_r)} \leq Cr^{1/2} \|\tilde{u}_0\|_{H^1(\mathbb{R}^d)}^{1/2} \|\tilde{u}_0\|_{H^2(\mathbb{R}^d)}^{1/2}. \tag{4-9}$$

This, together with the estimate in Theorem 1.1 for h_ε , shows that the first and fourth terms on the right-hand side of (4-8) are bounded by

$$C\varepsilon \|u_0\|_{H^2(\Omega)} \|G\|_{L^p(\Omega)},$$

where $p = q' = 2d/(d + 1)$. Finally, we note that the second and third terms on the right-hand side of (4-8) are bounded by

$$C\varepsilon \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)} \|\nabla h_\varepsilon\|_{L^2(\Omega)} + C\varepsilon \|\nabla \tilde{u}_0\|_{L^q(\mathbb{R}^d)} \|G\|_{L^p(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|G\|_{L^p(\Omega)}.$$

As a result, we have proved that

$$\left| \int_{\Omega} v_\varepsilon \cdot G dx \right| \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|G\|_{L^p(\Omega)},$$

which, by duality, gives the estimate (4-6) and completes the proof. □

Next we consider the solutions with the Neumann boundary conditions.

Lemma 4.3. *Let u_ε ($\varepsilon \geq 0$) be the solutions of (1-7) such that $u_\varepsilon \perp \mathcal{R}$. Suppose that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then*

$$\left\| u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) - v_\varepsilon \right\|_{H^1(\Omega)} \leq C\varepsilon \left\{ \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)} + \|\nabla \tilde{u}_0\|_{L^2(\mathbb{R}^d)} \right\}, \tag{4-10}$$

where \tilde{u}_0 is an extension of u_0 and $v_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ is the weak solution to

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon) = 0 & \text{in } \Omega, \\ \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} = \frac{\varepsilon}{2} \left(n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k} \right) \left\{ \phi_{kij}(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_j} \right) \right\} & \text{on } \partial\Omega, \\ v_\varepsilon \perp \mathcal{R}. \end{cases} \tag{4-11}$$

Proof. Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) - v_\varepsilon.$$

Using $\partial u_\varepsilon / \partial v_\varepsilon = \partial u_0 / \partial v_0$ on $\partial\Omega$, a direct computation shows that

$$\begin{aligned} \frac{\partial w_\varepsilon}{\partial v_\varepsilon} &= \frac{\partial u_0}{\partial v_0} - \frac{\partial u_0}{\partial v_\varepsilon} - \frac{\partial}{\partial v_\varepsilon} \left\{ \varepsilon \chi_k(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0}{\partial x_k} \right) \right\} - \frac{\partial v_\varepsilon}{\partial v_\varepsilon} \\ &= n_i [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \right] - n_i b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \\ &\quad - n_i a_{ij}^{\alpha\beta}(x/\varepsilon) \cdot \varepsilon \chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k} \right) - \frac{\partial v_\varepsilon}{\partial v_\varepsilon}. \end{aligned} \quad (4-12)$$

Using (2-5), we also see that

$$\begin{aligned} n_i b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j} \right) + \frac{\partial v_\varepsilon}{\partial v_\varepsilon} &= \varepsilon n_i \frac{\partial}{\partial x_k} [\phi_{kij}^{\alpha\beta}(x/\varepsilon)] K_\varepsilon \left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j} \right) + \frac{\partial v_\varepsilon}{\partial v_\varepsilon} \\ &= \frac{\varepsilon}{2} \left(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i} \right) [\phi_{kij}^{\alpha\beta}(x/\varepsilon)] K_\varepsilon \left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j} \right) + \frac{\partial v_\varepsilon}{\partial v_\varepsilon} \\ &= -\varepsilon n_i \phi_{kij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j} \right). \end{aligned} \quad (4-13)$$

As a result, we obtain

$$\begin{aligned} \frac{\partial w_\varepsilon}{\partial v_\varepsilon} &= n_i [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \right] \\ &\quad + \varepsilon n_i \phi_{kij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j} \right) - n_i a_{ij}^{\alpha\beta}(x/\varepsilon) \cdot \varepsilon \chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k} \right). \end{aligned} \quad (4-14)$$

Next, we note that as in the proof of Lemma 4.1,

$$\begin{aligned} \mathcal{L}_\varepsilon(w_\varepsilon) &= -\frac{\partial}{\partial x_i} \left\{ [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon \left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j} \right) \right] \right\} - \varepsilon \frac{\partial}{\partial x_i} \left\{ \phi_{kij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j} \right) \right\} \\ &\quad + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon \left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k} \right) \right\}. \end{aligned} \quad (4-15)$$

Thus, by (1-2) and the energy estimate,

$$\begin{aligned} &\|\nabla w_\varepsilon + (\nabla w_\varepsilon)^T\|_{L^2(\Omega)} \\ &\leq C \|\nabla w_\varepsilon\|_{L^2(\Omega)} \left\{ \|\nabla u_0 - K_\varepsilon(\nabla \tilde{u}_0)\|_{L^2(\Omega)} + \varepsilon \|\phi(x/\varepsilon) K_\varepsilon(\nabla^2 \tilde{u}_0)\|_{L^2(\Omega)} + \varepsilon \|\chi(x/\varepsilon) K_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega)} \right\} \\ &\leq C \varepsilon \|\nabla w_\varepsilon\|_{L^2(\Omega)} \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

where we have used Lemmas 2.1 and 2.2 for the last step. By the second Korn inequality, this implies

$$\begin{aligned} \|w_\varepsilon\|_{H^1(\Omega)} &\leq C\varepsilon\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)} + C\sum_{j=1}^J\left|\int_{\Omega}w_\varepsilon\cdot\varphi_j\,dx\right| \\ &\leq C\varepsilon\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)} + C\varepsilon\|\chi(x/\varepsilon)K_\varepsilon(\nabla\tilde{u}_0)\|_{L^2(\Omega)} \leq C\varepsilon\{\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)} + \|\nabla\tilde{u}_0\|_{L^2(\mathbb{R}^d)}\}, \end{aligned}$$

where $\{\varphi_j : j = 1, \dots, J\}$ forms an orthonormal basis of \mathcal{R} , as a subspace of $L^2(\Omega; \mathbb{R}^d)$. □

The next theorem is an analogue of Theorem 4.2 for the Neumann boundary conditions.

Theorem 4.4. *Suppose that A satisfies (1-2)–(1-3). Let Ω be a bounded Lipschitz domain in \mathbb{R}^d . Let u_ε ($\varepsilon \geq 0$) be the weak solutions to the Neumann problem (1-7) with the property $u_\varepsilon \perp \mathcal{R}$. Assume that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{4-16}$$

where $q = 2d/(d - 1)$ and C depends only on d, κ_1, κ_2 , and Ω .

Proof. As in the proof of Theorem 4.2, it suffices to show that

$$\|v_\varepsilon\|_{L^q(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{4-17}$$

where v_ε is given by (4-11). To this end we fix $G \in L^p(\Omega; \mathbb{R}^d)$ with $G \perp \mathcal{R}$ and let $h_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be the weak solution to

$$\mathcal{L}_\varepsilon(h_\varepsilon) = G \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial h_\varepsilon}{\partial \nu_\varepsilon} = 0 \quad \text{on } \partial\Omega, \tag{4-18}$$

with the property $h_\varepsilon \perp \mathcal{R}$. It follows from (4-18), (4-11), and Green’s formula that

$$\begin{aligned} \int_{\Omega}v_\varepsilon\cdot G\,dx &= \int_{\Omega}A(x/\varepsilon)\nabla v_\varepsilon\cdot\nabla h_\varepsilon\,dx = \int_{\partial\Omega}\frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}\cdot h_\varepsilon\,d\sigma \\ &= \frac{\varepsilon}{2}\int_{\partial\Omega}\left(n_k\frac{\partial}{\partial x_i} - n_i\frac{\partial}{\partial x_k}\right)\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde{u}_0^\beta}{\partial x_j}\right)\right\}\cdot h_\varepsilon^\alpha\,d\sigma \\ &= -\frac{\varepsilon}{2}\int_{\partial\Omega}\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde{u}_0^\beta}{\partial x_j}\right)\cdot\left(n_k\frac{\partial}{\partial x_i} - n_i\frac{\partial}{\partial x_k}\right)h_\varepsilon^\alpha\cdot(1-\eta_\varepsilon)\,d\sigma \\ &= -\varepsilon\int_{\Omega}\frac{\partial}{\partial x_k}\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde{u}_0^\beta}{\partial x_j}\right)(1-\eta_\varepsilon)\right\}\cdot\frac{\partial h_\varepsilon^\alpha}{\partial x_i}\,dx, \end{aligned}$$

where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14) and we have used the divergence theorem as well as (2-5) for the last inequality. This leads to

$$\begin{aligned} \left|\int_{\Omega}v_\varepsilon\cdot G\,dx\right| &\leq C\int_{\Omega_{4\varepsilon}}|\nabla\phi(x/\varepsilon)||K_\varepsilon(\nabla\tilde{u}_0)||\nabla h_\varepsilon|\,dx \\ &\quad + C\varepsilon\int_{\Omega_{4\varepsilon}}|\phi(x/\varepsilon)||K_\varepsilon(\nabla^2\tilde{u}_0)||\nabla h_\varepsilon|\,dx \\ &\quad + C\varepsilon\int_{\Omega_{4\varepsilon}}|\phi(x/\varepsilon)||K_\varepsilon(\nabla\tilde{u}_0)||\nabla\eta_\varepsilon||\nabla h_\varepsilon|\,dx. \end{aligned} \tag{4-19}$$

Note that by the Cauchy inequality, the first and third term on the right-hand side of (4-19) are bounded by

$$\begin{aligned} C \left\| (|\nabla\phi(x/\varepsilon)| + |\phi(x/\varepsilon)|) K_\varepsilon(\nabla\tilde{u}_0) \right\|_{L^2(\Omega_{4\varepsilon})} \|\nabla h_\varepsilon\|_{L^2(\Omega_{4\varepsilon})} &\leq C \|\nabla\tilde{u}_0\|_{L^2(\tilde{\Omega}_{5\varepsilon})} \|\nabla h_\varepsilon\|_{L^2(\Omega_{4\varepsilon})} \\ &\leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|G\|_{L^p(\Omega)}, \end{aligned}$$

where we have used Lemma 2.2 for the first inequality and Theorem 1.2 as well as estimate (4-9) for the second. Also, the second term on the right-hand side of (4-19) is bounded by

$$C\varepsilon \|\phi(x/\varepsilon) K_\varepsilon(\nabla^2\tilde{u}_0)\|_{L^2(\Omega)} \|\nabla h_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|G\|_{L^p(\Omega)}.$$

Hence we have proved that for any $G \in L^p(\Omega; \mathbb{R}^d)$ with the property $G \perp \mathcal{A}$,

$$\left| \int_{\Omega} v_\varepsilon \cdot G \, dx \right| \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|G\|_{L^p(\Omega)}.$$

Since $v_\varepsilon \perp \mathcal{A}$, this gives the estimate (4-17) by duality and completes the proof. \square

Note that by combining Theorems 4.2 and 4.4, one obtains Theorem 1.3.

5. C^α estimates in C^1 domains

In this section we investigate uniform boundary C^α estimates in C^1 domains. The results will be used in the next section to establish uniform boundary $W^{1,p}$ estimates in C^1 domains. Throughout the section we will assume that the defining function ψ in D_r and Δ_r is C^1 and $\psi(0) = 0$. To quantify the C^1 condition we further assume that

$$\sup\{|\nabla\psi(x') - \nabla\psi(y')| : x', y' \in \mathbb{R}^{d-1} \text{ and } |x' - y'| \leq t\} \leq \tau(t), \quad (5-1)$$

where $\tau(t) \rightarrow 0$ as $t \rightarrow 0^+$.

The rescaling argument is used frequently in this paper. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_{2r} and $u_\varepsilon = f$ on Δ_{2r} . Let $w(x) = u_\varepsilon(rx)$. Then

$$\mathcal{L}_{\varepsilon/r}(w) = G \quad \text{in } \tilde{D}_2 \quad \text{and} \quad w = g \quad \text{on } \tilde{\Delta}_2,$$

where $G(x) = r^2 F(rx)$, $g(x) = f(rx)$, and

$$\begin{aligned} \tilde{D}_2 &= \{(x', x_d) \in \mathbb{R}^d : |x'| < 2 \text{ and } \psi_r(x') < x_d < \psi_r(x') + 2\}, \\ \tilde{\Delta}_2 &= \{(x', x_d) \in \mathbb{R}^d : |x'| < 2 \text{ and } x_d = \psi_r(x')\}, \end{aligned}$$

with $\psi_r(x') = r^{-1}\psi(rx')$. Note that $\psi_r(0) = 0$ and $\|\nabla\psi_r\|_\infty = \|\nabla\psi\|_\infty$. Moreover, if ψ is C^1 and satisfies (5-1), then ψ_r satisfies (5-1) uniformly in r for $0 < r \leq 1$.

Lemma 5.1. *Let $0 < \varepsilon \leq r \leq 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} with $u_\varepsilon = 0$ on Δ_{2r} . Then there exists $v \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(v) = 0$ in D_r , $v = 0$ on Δ_r , and*

$$\left(\int_{D_r} |u_\varepsilon - v|^2 \right)^{1/2} \leq C(\varepsilon/r)^{1/2} \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2}, \quad (5-2)$$

where $\|\nabla\psi\|_\infty \leq M$, and C depends only on d, κ_1, κ_2 , and M .

Proof. By rescaling we may assume $r = 1$. By Caccioppoli's inequality,

$$\left(\int_{D_{3/2}} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left(\int_{D_2} |u_\varepsilon|^2 \right)^{1/2}. \tag{5-3}$$

It follows from (5-3) and the coarea formula that there exists $t \in [\frac{5}{4}, \frac{3}{2}]$ such that

$$\|\nabla u_\varepsilon\|_{L^2(\partial D_t \setminus \Delta_2)} + \|u_\varepsilon\|_{L^2(\partial D_t \setminus D_2)} \leq C \|u_\varepsilon\|_{L^2(D_2)}. \tag{5-4}$$

Let v be the solution to the Dirichlet problem: $\mathcal{L}_0(v) = 0$ in D_t and $v = u_\varepsilon$ on ∂D_t . Note that $v = 0$ on Δ_1 , and by Remark 2.8,

$$\|u_\varepsilon - v\|_{L^2(D_t)} \leq C\varepsilon^{1/2} \|u_\varepsilon\|_{H^1(\partial D_t)}. \tag{5-5}$$

This, together with (5-4), gives

$$\|u_\varepsilon - v\|_{L^2(D_1)} \leq \|u_\varepsilon - v\|_{L^2(D_t)} \leq C\varepsilon^{1/2} \|u_\varepsilon\|_{L^2(D_2)}. \quad \square$$

Theorem 5.2. *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_1 with $u_\varepsilon = 0$ on Δ_1 , where the defining function ψ in D_1 and Δ_1 is C^1 . Then, for any $\alpha \in (0, 1)$ and $\varepsilon \leq r \leq \frac{1}{2}$,*

$$\left(\int_{D_r} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C_\alpha r^{\alpha-1} \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2}, \tag{5-6}$$

where C_α depends only on $d, \alpha, \kappa_1, \kappa_2$, and the function $\tau(t)$ in (5-1).

Proof. Fix $\beta \in (\alpha, 1)$. For each $r \in [\varepsilon, \frac{1}{2}]$, let $v = v_r$ be the function given by Lemma 5.1. By the boundary C^β estimates in C^1 domains for the operator \mathcal{L}_0 (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]),

$$\left(\int_{D_{\theta r}} |v|^2 \right)^{1/2} \leq C_0 \theta^\beta \left(\int_{D_r} |v|^2 \right)^{1/2}$$

for any $\theta \in (0, 1)$, where C_0 depends only on $d, \kappa_1, \kappa_2, \beta$ and $\tau(t)$. It follows that

$$\begin{aligned} \left(\int_{D_{\theta r}} |u_\varepsilon|^2 \right)^{1/2} &\leq \left(\int_{D_{\theta r}} |v|^2 \right)^{1/2} + C \left(\int_{D_{\theta r}} |u_\varepsilon - v|^2 \right)^{1/2} \\ &\leq C\theta^\beta \left(\int_{D_r} |v|^2 \right)^{1/2} + C\theta^{-d/2} \left(\int_{D_r} |u_\varepsilon - v|^2 \right)^{1/2} \\ &\leq C_1\theta^\beta \left(\int_{D_r} |u_\varepsilon|^2 \right)^{1/2} + C_1\theta^{-d/2} (\varepsilon/r)^{1/2} \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} \end{aligned}$$

for any $\varepsilon \leq r \leq \frac{1}{2}$. We now choose $\theta \in (0, \frac{1}{4})$ so small that $C_1\theta^{\beta-\alpha} < \frac{1}{4}$. With θ fixed, choose $N > 1$ large so that

$$C_1 2^\alpha \theta^{-d/2-\alpha} N^{-1/2} \leq \frac{1}{4}.$$

It follows that if $r \geq N\varepsilon$,

$$\phi(\theta r) \leq \frac{1}{4} \{\phi(r) + \phi(2r)\}, \tag{5-7}$$

where

$$\phi(r) = r^{-\alpha} \left(\int_{D_r} |u_\varepsilon|^2 \right)^{1/2}.$$

By integration we may deduce from (5-7) that

$$\int_{\theta a}^{\theta/2} \phi(r) \frac{dr}{r} \leq \frac{1}{4} \int_a^{1/2} \phi(r) \frac{dr}{r} + \frac{1}{4} \int_{2a}^1 \phi(r) \frac{dr}{r},$$

where $N\varepsilon \leq a < \frac{1}{2}$. This implies

$$\int_{\theta a}^1 \phi(r) \frac{dr}{r} \leq C \int_{\theta/2}^1 \phi(r) \frac{dr}{r} \leq C\phi(1).$$

Hence, $\phi(r) \leq C\phi(1)$ for any $r \in [\varepsilon, 1]$, and the estimate (5-6) now follows by Caccioppoli's inequality. \square

Remark 5.3. Under the stronger assumption that the defining function ϕ for D_1 is $C^{1,\sigma}$ for some $\sigma > 0$, we will show in Section 8 that the estimate (5-6) holds for $\alpha = 1$. In particular, it follows from the argument in Section 7 that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(0, 1)$, then

$$\left(\int_{B(0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left(\int_{B(0,1)} |\nabla u_\varepsilon|^2 \right)^{1/2} \quad (5-8)$$

for any $\varepsilon \leq r < 1$. This is the interior Lipschitz estimate down to the scale ε .

A function A is said to belong to $\text{VMO}(\mathbb{R}^d)$ if the left-hand side of (5-9) goes to zero as $t \rightarrow 0^+$. To quantify this assumption we assume that

$$\sup_{\substack{x \in \mathbb{R}^d \\ 0 < r < t}} \int_{B(x,r)} \left| A(y) - \int_{B(x,r)} A \right| dy \leq \rho(t), \quad (5-9)$$

where $\rho(t) \rightarrow 0$ as $t \rightarrow 0^+$.

The following corollary was essentially proved in [Avellaneda and Lin 1987] by a compactness method.

Corollary 5.4. *Suppose that A satisfies (1-2)–(1-3). Also assume that $A \in \text{VMO}(\mathbb{R}^d)$. Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_1 with $u_\varepsilon = 0$ on Δ_1 . Then, for any $\alpha \in (0, 1)$,*

$$\|u_\varepsilon\|_{C^\alpha(D_{1/2})} \leq C_\alpha \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2}, \quad (5-10)$$

where C_α depends only on $d, \kappa_1, \kappa_2, \alpha$, and the functions $\tau(t), \rho(t)$.

Proof. We may assume $0 < \varepsilon < \frac{1}{2}$, as the case of $\varepsilon \geq \frac{1}{2}$ is local. Since $\mathcal{L}_1(u_\varepsilon(\varepsilon x)) = 0$, it follows from the boundary C^α estimates in C^1 domains (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]) for the operator \mathcal{L}_1 by rescaling that if $\alpha \in (0, 1)$ and $0 < r < \varepsilon$,

$$\left(\int_{D_r} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C(r/\varepsilon)^{\alpha-1} \left(\int_{D_\varepsilon} |\nabla u_\varepsilon|^2 \right)^{1/2},$$

where C depends only on $d, \kappa_1, \kappa_2, \alpha, \tau(t)$ and $\rho(t)$. This, together with Theorem 5.2, shows that the estimate (5-6) holds for any $0 < r < \frac{1}{2}$. By combining (5-6) with a similar interior estimate, we obtain

$$r^{\alpha-1} \left(\int_{B(x,r) \cap D_{1/2}} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \|u_\varepsilon\|_{L^2(D_1)} \tag{5-11}$$

for any $0 < r < c$ and $x \in D_{1/2}$. The estimate (5-10) follows from (5-11) by Campanato’s characterization of Hölder spaces. \square

The rest of this section is devoted to the boundary C^α estimates for solutions with the Neumann boundary conditions.

Lemma 5.5. *Let $0 < \varepsilon \leq r \leq 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} with $\partial u_\varepsilon / \partial \nu_\varepsilon = 0$ on Δ_{2r} . Then there exists a function $w \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = 0$, $\partial w / \partial \nu_0 = 0$ in Δ_r , and*

$$\left(\int_{D_r} |u_\varepsilon - w|^2 \right)^{1/2} \leq C(\varepsilon/r)^{1/2} \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2}, \tag{5-12}$$

where $\|\nabla \psi\|_\infty \leq M$, and C depends only on d, κ_1, κ_2 , and M .

Proof. By rescaling we may assume $r = 1$. As in the proof of Lemma 5.1, there exists $t \in [\frac{5}{4}, \frac{3}{2}]$ such that

$$\|u_\varepsilon\|_{L^2(\partial D_t \setminus \Delta_2)} + \|\nabla u_\varepsilon\|_{L^2(\partial D_t \setminus \Delta_2)} \leq C \|u_\varepsilon\|_{L^2(D_2)}.$$

Let ϕ_ε be a function in \mathcal{R} such that $u_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(D_t; \mathbb{R}^d)$. Let v be the solution to the Neumann problem: $\mathcal{L}_0(v) = 0$ in D_t and $\partial v / \partial \nu_0 = \partial u_\varepsilon / \partial \nu_\varepsilon$ on ∂D_t , with $v \perp \mathcal{R}$. It follows from Remark 2.8 that

$$\begin{aligned} \|u_\varepsilon - \phi_\varepsilon - v\|_{L^2(D_t)} &\leq \|u_\varepsilon - \phi_\varepsilon - v\|_{L^2(D_t)} \\ &\leq C\varepsilon^{1/2} \left\| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right\|_{L^2(\partial D_t)} \\ &\leq C\varepsilon^{1/2} \|u_\varepsilon\|_{L^2(D_2)}. \end{aligned}$$

It is easy to see that the function $w = v + \phi_\varepsilon$ satisfies the desired conditions. \square

Theorem 5.6. *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_1 with $\partial u_\varepsilon / \partial \nu_\varepsilon = 0$ on Δ_1 , where the defining function ψ in D_1 and Δ_1 is C^1 . Then, for any $\alpha \in (0, 1)$ and $\varepsilon \leq r \leq 1$,*

$$\left(\int_{D_r} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C_\alpha r^{\alpha-1} \left(\int_{D_1} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{5-13}$$

where C depends only on $d, \alpha, \kappa_1, \kappa_2$, and the function $\tau(t)$.

Proof. Fix $\beta \in (\alpha, 1)$. For each $r \in [\varepsilon, \frac{1}{2}]$, let $w = w_r$ be the function given by Lemma 5.5. By the boundary C^β estimates in C^1 domains for the operator \mathcal{L}_0 ,

$$\inf_{q \in \mathbb{R}^d} \left(\int_{D_{\theta r}} |w - q|^2 \right)^{1/2} \leq C_0 \theta^\beta \inf_{q \in \mathbb{R}^d} \left(\int_{D_r} |w - q|^2 \right)^{1/2},$$

where C_0 depends only on $d, \beta, \kappa_1, \kappa_2$, and $\tau(t)$. This, together with Lemma 5.5, gives

$$\begin{aligned} \inf_{q \in \mathbb{R}^d} \left(\int_{D_{\theta r}} |u_\varepsilon - q|^2 \right)^{1/2} &\leq C \inf_{q \in \mathbb{R}^d} \left(\int_{D_{\theta r}} |w - q|^2 \right)^{1/2} + \left(\int_{D_{\theta r}} |u_\varepsilon - w|^2 \right)^{1/2} \\ &\leq C\theta^\beta \inf_{q \in \mathbb{R}^d} \left(\int_{D_r} |w - q|^2 \right)^{1/2} + C_0\theta^{-d/2} \left(\int_{D_r} |u_\varepsilon - w|^2 \right)^{1/2} \\ &\leq C\theta^\beta \inf_{q \in \mathbb{R}^d} \left(\int_{D_r} |u_\varepsilon - q|^2 \right)^{1/2} + C\theta^{-d/2}(\varepsilon/r)^{1/2} \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2}. \end{aligned}$$

By replacing u_ε with $u_\varepsilon - q$, we obtain

$$\phi(\theta r) \leq C\theta^{\beta-\alpha}\phi(r) + C\theta^{-\alpha-d/2}(\varepsilon/r)^{1/2}\phi(2r)$$

for any $r \in [\varepsilon, \frac{1}{2}]$, where

$$\phi(r) = r^{-\alpha} \inf_{q \in \mathbb{R}^d} \left(\int_{D_r} |u_\varepsilon - q|^2 \right)^{1/2}.$$

By the integration argument used in the proof of Theorem 5.2, we may conclude that $\phi(r) \leq C\phi(1)$ for $r \in [\varepsilon, \frac{1}{2}]$, which yields (5-13) by Caccioppoli's inequality. \square

Remark 5.7. Under the stronger condition that the defining function for D_1 and Δ_1 is $C^{1,\sigma}$ for some $\sigma > 0$, we will show in Section 9 that the estimate (5-13) holds for $\alpha = 1$.

The following corollary was essentially proved in [Kenig et al. 2013] by a compactness method.

Corollary 5.8. *Suppose that A satisfies (1-2)–(1-3). Also assume that $A \in \text{VMO}(\mathbb{R}^d)$. Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_1 with $\partial u_\varepsilon / \partial \nu_\varepsilon = 0$ on Δ_1 . Then, for any $\alpha \in (0, 1)$,*

$$\|u_\varepsilon\|_{C^\alpha(D_{1/2})} \leq C_\alpha \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2}, \quad (5-14)$$

where C_α depends only on $d, \kappa_1, \kappa_2, \alpha$, and the functions $\tau(t), \rho(t)$.

Proof. As in the case of the Dirichlet boundary condition, the additional smoothness assumption $A \in \text{VMO}(\mathbb{R}^d)$ ensures that the estimate (5-13) holds for any $r \in (0, \frac{1}{2})$ (see, e.g., [Byun and Wang 2005] for estimates for \mathcal{L}_1). This, together with the interior estimates, gives the estimate (5-14) by the use of Campanato's characterization of Hölder spaces. \square

6. $W^{1,p}$ estimates in C^1 domains

In this section we study the uniform $W^{1,p}$ estimates in C^1 domains. Throughout the section we will assume that $A = A(y)$ satisfies (1-2)–(1-3), $A \in \text{VMO}(\mathbb{R}^d)$, and Ω is C^1 . Our goal is to prove the following two theorems.

Theorem 6.1. *Suppose that A satisfies (1-2)–(1-3). Also assume that $A \in \text{VMO}(\mathbb{R}^d)$. Let $1 < p < \infty$ and Ω be a bounded C^1 domain in \mathbb{R}^d . Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ be a weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f) \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega, \quad (6-1)$$

where $f = (f_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d \times d})$. Then

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq C_p \|f\|_{L^p(\Omega)}, \tag{6-2}$$

where C_p depends only on d, p, A , and Ω .

Theorem 6.2. *Suppose that A satisfies the same conditions as in Theorem 6.1. Let $1 < p < \infty$ and Ω be a bounded C^1 domain in \mathbb{R}^d . Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f) \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -n \cdot f \quad \text{on } \partial\Omega, \tag{6-3}$$

where $f = (f_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d \times d})$. Assume that $u_\varepsilon \perp \mathcal{R}$. Then

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq C_p \|f\|_{L^p(\Omega)}, \tag{6-4}$$

where C_p depends only on d, p, A , and Ω .

Recall that a function u_ε is called a weak solution of (6-1) if $u_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}^d)$ and

$$\int_\Omega a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial u_\varepsilon^\beta}{\partial x_j} \cdot \frac{\partial \varphi^\alpha}{\partial x_i} dx = - \int_\Omega f_i^\alpha \cdot \frac{\partial \varphi^\alpha}{\partial x_i} dx \tag{6-5}$$

for any $\varphi = (\varphi^\alpha) \in C_0^\infty(\Omega; \mathbb{R}^d)$. Similarly, u_ε is called a weak solution of (6-3) if $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ and (6-5) holds for any $\varphi = (\varphi^\alpha) \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$. Under the assumptions that $A \in \text{VMO}(\mathbb{R}^d)$ and Ω is C^1 , the existence and uniqueness of solutions of (6-1) and (6-3) are more or less well known (see [Auscher and Qafsaoui 2002; Byun and Wang 2004; 2005] for references). The main interest here is that the constants C in the $W^{1,p}$ estimates (6-2) and (6-4) are independent of ε . We mention that for \mathcal{L}_ε with coefficients satisfying (1-3), (1-11) and the Hölder continuity condition, estimates (6-2) and (6-4) were established in [Avellaneda and Lin 1987; 1991; Shen 2008; Kenig et al. 2013]. The results were extended to the case of almost-periodic coefficients in [Armstrong and Shen 2016]. Also, for \mathcal{L}_ε with coefficients satisfying (1-2)–(1-3) in Lipschitz domains, some partial results may be found in [Geng et al. 2012].

Theorems 6.1 and 6.2 are proved by a real-variable argument. The required weak reverse Hölder inequalities (6-6) and (6-2) for $p > 2$ are established by combining local estimates for \mathcal{L}_1 and boundary Hölder estimates in Section 4 with the interior Lipschitz estimates, up to the scale ε .

Lemma 6.3. *Let $u_\varepsilon \in H^1(B(x_0, 2r); \mathbb{R}^d)$ be a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(x_0, 2r)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$. Then, for any $2 < p < \infty$,*

$$\left(\int_{B(x_0,r)} |\nabla u_\varepsilon|^p \right)^{1/p} \leq C_p \left(\int_{B(x_0,2r)} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{6-6}$$

where C_p depends only on d, p, κ_1, κ_2 , and the function $\rho(t)$ in (5-9).

Proof. By translation and dilation we may assume that $x_0 = 0$ and $r = 1$. We may also assume that $0 < \varepsilon < \frac{1}{4}$. The case $\varepsilon \geq \frac{1}{4}$ for $B(0, 1)$ is local, since $A(x/\varepsilon)$ satisfies the smoothness condition (5-9)

uniformly in ε . For each $y \in B(0, 1)$, we use the local $W^{1,p}$ estimates for the operator \mathcal{L}_1 (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]) and a simple blow-up argument to show that

$$\left(\int_{B(y, \varepsilon/2)} |\nabla u_\varepsilon|^p \right)^{1/p} \leq C \left(\int_{B(y, \varepsilon)} |\nabla u_\varepsilon|^2 \right)^{1/2}. \tag{6-7}$$

By the interior Lipschitz estimate, up to the scale ε , we have

$$\left(\int_{B(y, \varepsilon)} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left(\int_{B(y, 1)} |\nabla u_\varepsilon|^2 \right)^{1/2}. \tag{6-8}$$

We point out that the estimate (6-8) will be proved in Section 8 with no smoothness assumption on A (see Theorem 8.6). Hence, for any $y \in B(0, 1)$,

$$\begin{aligned} \left(\int_{B(y, \varepsilon/2)} |\nabla u_\varepsilon|^p \right)^{1/p} &\leq C \left(\int_{B(y, 1)} |\nabla u_\varepsilon|^2 \right)^{1/2} \\ &\leq C \|\nabla u_\varepsilon\|_{L^2(B(0, 2))}. \end{aligned} \tag{6-9}$$

By covering $B(0, 1)$ with balls of radius $\varepsilon/2$, we may deduce (6-6) readily from (6-9). □

Lemma 6.4. *Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} with either $u_\varepsilon = 0$ or $\partial u_\varepsilon / \partial \nu_\varepsilon = 0$ in Δ_{2r} , where $0 < r \leq 1$. Then, for any $2 < p < \infty$,*

$$\left(\int_{D_r} |\nabla u_\varepsilon|^p \right)^{1/p} \leq C_p \left(\int_{D_{2r}} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{6-10}$$

where C depends only on $d, p, \kappa_1, \kappa_2, \tau(t)$ in (5-1), and $\rho(t)$ in (5-9).

Proof. Note that the function $r^{-1}\psi(rx')$ satisfies the condition (5-1) uniformly for $0 < r \leq 1$. Thus, by rescaling, it suffices to prove the lemma for $r = 1$. Using Lemma 6.3, Theorem 5.2 and Theorem 5.6, we obtain

$$\begin{aligned} \left(\int_{B(y, \delta(y)/8)} |\nabla u_\varepsilon|^p \right)^{1/p} &\leq C \left(\int_{B(y, \delta(y)/4)} |\nabla u_\varepsilon|^2 \right)^{1/2} \\ &\leq C_\alpha [\delta(y)]^{\alpha-1} \|\nabla u_\varepsilon\|_{L^2(D_2)} \end{aligned} \tag{6-11}$$

for any $\alpha \in (0, 1)$, where $y \in D_1$ and $\delta(y) = \text{dist}(y, \partial D_2)$. We now fix $\alpha \in (1 - \frac{1}{p}, 1)$. It follows from (6-11) that

$$\int_{D_1} \left(\int_{B(y, \delta(y)/8)} |\nabla u_\varepsilon|^p dx \right) dy \leq C \|\nabla u_\varepsilon\|_{L^2(D_2)}^p. \tag{6-12}$$

Using the fact that $\delta(x) \approx \delta(y)$ if $y \in D_1$ and $|y - x| < \frac{1}{8}\delta(y)$, it is not hard to verify that (6-12) implies (6-10). □

Proof of Theorems 6.1 and 6.2. By duality and a density argument it suffices to consider the case where $p > 2$ and $f = (f_i^\alpha) \in C_0^1(\Omega; \mathbb{R}^{d \times d})$. Furthermore, by a real-variable argument, which originated in [Caffarelli and Peral 1998] and further developed in [Shen 2005; 2007], one only needs to establish weak reverse Hölder inequalities for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(x_0, r) \cap \Omega$ with either $u_\varepsilon = 0$ or $\partial u_\varepsilon / \partial \nu_\varepsilon = 0$

on $B(x_0, r) \cap \partial\Omega$, where $x_0 \in \bar{\Omega}$ and $0 < r < c_0 \text{diam}(\Omega)$. These inequalities are exactly those given by Lemmas 6.3 and 6.4. We omit the details and refer the reader to [Shen 2005; 2008; Geng 2012] for details in the case of scalar elliptic equations. \square

Remark 6.5. Suppose that A and Ω satisfy the same conditions as in Theorem 6.1. By some fairly standard extension and duality arguments (see, e.g., [Kenig et al. 2013]), one may deduce from Theorem 6.1 that the solution of the Dirichlet problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(h) + F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega$$

satisfies

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq C_p \{ \|h\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} + \|f\|_{W^{1/p,p}(\partial\Omega)} \}$$

for any $1 < p < \infty$, where $W^{\alpha,p}(\partial\Omega)$ denotes the Sobolev space on $\partial\Omega$ of order α with exponent p . Similarly, the solutions of the Neumann problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(h) + F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -n \cdot h + g \quad \text{on } \partial\Omega$$

with $u_\varepsilon \perp \mathcal{R}$ satisfies

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq C_p \{ \|h\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\partial\Omega)} \},$$

where $W^{-1/p,p}(\partial\Omega)$ is the dual of $W^{1/p,p'}(\partial\Omega)$.

7. L^p estimates in C^1 domains

The $W^{1,p}$ estimates in the last section allow us to establish the Rellich-type estimates in L^p , down to the scale ε , in C^1 domains under the additional assumption that A belongs to $\text{VMO}(\mathbb{R}^d)$.

Theorem 7.1. *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Also assume that $A \in \text{VMO}(\mathbb{R}^d)$. Let $1 < p < \infty$ and Ω be a bounded C^1 domain in \mathbb{R}^d . Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ be a weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{in } \partial\Omega, \tag{7-1}$$

where $F \in L^p(\Omega; \mathbb{R}^d)$ and $f \in W^{1,p}(\partial\Omega; \mathbb{R}^d)$. Then, for any $\varepsilon \leq r < \text{diam}(\Omega)$,

$$\left\{ \frac{1}{r} \int_{\Omega_r} |\nabla u_\varepsilon|^p \right\}^{1/p} \leq C_p \{ \|F\|_{L^p(\Omega)} + \|f\|_{W^{1,p}(\partial\Omega)} \}, \tag{7-2}$$

where $\Omega_r = \{x \in \mathbb{R}^d : \text{dist}(x, \partial\Omega) < r\}$. The constant C_p depends only on d, p, A and Ω .

Theorem 7.2. *Suppose that A and Ω satisfy the same conditions as in Theorem 7.1. Let $1 < p < \infty$. Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{in } \partial\Omega, \tag{7-3}$$

where $F \in L^p(\Omega; \mathbb{R}^d)$, $g \in L^p(\partial\Omega; \mathbb{R}^d)$ and $\int_{\Omega} F + \int_{\partial\Omega} g = 0$. Also assume that $u_{\varepsilon} \perp \mathcal{R}$. Then, for any $\varepsilon \leq r < \text{diam}(\Omega)$,

$$\left\{ \frac{1}{r} \int_{\Omega_r} |\nabla u_{\varepsilon}|^p \right\}^{1/p} \leq C_p \{ \|F\|_{L^p(\Omega)} + \|g\|_{L^p(\partial\Omega)} \}, \quad (7-4)$$

where C_p depends only on d, p, A and Ω .

The proof of Theorems 7.1 and 7.2 follows a similar line of argument as for Theorems 1.1 and 1.2 by considering

$$w_{\varepsilon} = u_{\varepsilon} - u_0 - \varepsilon \chi_j^{\beta}(x/\varepsilon) K_{\varepsilon} \left(\frac{\partial u_0^{\beta}}{\partial x_j} \eta_{\varepsilon} \right), \quad (7-5)$$

where u_0 is the solution of the homogenized problem, K_{ε} is a smoothing operator defined by (2-6), and $\eta_{\varepsilon} \in C_0^{\infty}(\Omega)$ is a cut-off function satisfying (2-14).

Throughout this section we will assume that Ω is C^1 and A satisfies (1-2)–(1-3) and (5-9).

Lemma 7.3. *Let u_{ε} ($\varepsilon \geq 0$) be the solutions of the Dirichlet problems (7-1). Let w_{ε} be defined by (7-5). Then*

$$\|w_{\varepsilon}\|_{W^{1,p}(\Omega)} \leq C_p \varepsilon^{1/p} \{ \|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} \}, \quad (7-6)$$

where C_p depends only on d, p, A and Ω .

Proof. A direct computation shows that

$$\begin{aligned} \mathcal{L}_{\varepsilon}(w_{\varepsilon}) &= -\frac{\partial}{\partial x_i} \left\{ [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \left[\frac{\partial u_0^{\beta}}{\partial x_j} - K_{\varepsilon} \left(\frac{\partial u_0^{\beta}}{\partial x_j} \eta_{\varepsilon} \right) \right] \right\} \\ &\quad + \frac{\partial}{\partial x_i} \left\{ b_{ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon} \left(\frac{\partial u_0^{\beta}}{\partial x_j} \eta_{\varepsilon} \right) \right\} \\ &\quad + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) \frac{\partial}{\partial x_j} \left(K_{\varepsilon} \left(\frac{\partial u_0^{\gamma}}{\partial x_k} \eta_{\varepsilon} \right) \right) \right\}, \end{aligned}$$

where $b_{ij}^{\alpha\beta}(y)$ is defined by (2-3). Using (2-5), we obtain

$$\frac{\partial}{\partial x_i} \left\{ b_{ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon} \left(\frac{\partial u_0^{\beta}}{\partial x_j} \eta_{\varepsilon} \right) \right\} = -\varepsilon \frac{\partial}{\partial x_i} \left\{ \phi_{kij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_k} \left(K_{\varepsilon} \left(\frac{\partial u_0^{\beta}}{\partial x_j} \eta_{\varepsilon} \right) \right) \right\}.$$

It follows that

$$\begin{aligned} \mathcal{L}_{\varepsilon}(w_{\varepsilon}) &= -\frac{\partial}{\partial x_i} \left\{ [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \left[\frac{\partial u_0^{\beta}}{\partial x_j} - K_{\varepsilon} \left(\frac{\partial u_0^{\beta}}{\partial x_j} \eta_{\varepsilon} \right) \right] \right\} \\ &\quad - \varepsilon \frac{\partial}{\partial x_i} \left\{ \phi_{kij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_k} \left(K_{\varepsilon} \left(\frac{\partial u_0^{\beta}}{\partial x_j} \eta_{\varepsilon} \right) \right) \right\} \\ &\quad + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) \frac{\partial}{\partial x_j} \left(K_{\varepsilon} \left(\frac{\partial u_0^{\gamma}}{\partial x_k} \eta_{\varepsilon} \right) \right) \right\}. \quad (7-7) \end{aligned}$$

Since $w_\varepsilon = 0$ on $\partial\Omega$, we may apply the $W^{1,p}$ estimate in Theorem 6.1 to obtain

$$\begin{aligned} \|w_\varepsilon\|_{W^{1,p}(\Omega)} &\leq C \left\{ \|\nabla u_0 - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + \varepsilon \|\phi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} \right. \\ &\quad \left. + \varepsilon \|\chi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} \right\} \\ &\leq C \left\{ \|\nabla u_0 - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + \varepsilon \|\nabla((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} \right\} \\ &\leq C \left\{ \|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon \|(\nabla^2 u_0)\eta_\varepsilon\|_{L^2(\Omega)} \right\}, \end{aligned} \tag{7-8}$$

where we have used Lemmas 2.1 and 2.2 for the second and third inequalities.

We now write $u_0 = v + w$, where

$$v(x) = \int_{\Omega} \Gamma_0(x - y)F(y) dy \tag{7-9}$$

and $\Gamma_0(x - y)$ denotes the matrix of fundamental solutions for the operator \mathcal{L}_0 in \mathbb{R}^d , with pole at the origin. Note that $\|v\|_{W^{2,p}(\mathbb{R}^d)} \leq C_p \|F\|_{L^p(\Omega)}$ and

$$\|\nabla v\|_{L^p(S_t)} \leq C_p \|F\|_{L^p(\Omega)},$$

where $S_t = \{x \in \mathbb{R}^d : \text{dist}(x, \partial\Omega) = t\}$ for t small (see the proof of Theorem 2.6). It follows that

$$\|\nabla v\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon \|\nabla^2 v\|_{L^p(\Omega)} \leq C\varepsilon^{1/p} \|F\|_{L^p(\Omega)}. \tag{7-10}$$

Finally, we observe that $\mathcal{L}_0(w) = 0$ in Ω and

$$\|w\|_{W^{1,p}(\partial\Omega)} \leq \|f\|_{W^{1,p}(\partial\Omega)} + \|v\|_{W^{1,p}(\partial\Omega)} \leq C \left\{ \|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} \right\}.$$

It follows from the solvability of the L^p regularity problem for the operator \mathcal{L}_0 in C^1 domain Ω , which follows from [Fabes et al. 1978; Lewis et al. 1993; Hofmann et al. 2015], that

$$\|(\nabla w)^*\|_{L^p(\partial\Omega)} \leq C \left\{ \|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} \right\}.$$

Also, using the interior estimate

$$|\nabla^2 w(x)| \leq \frac{C}{\delta(x)} \left(\int_{B(x, \delta(x)/8)} |\nabla w|^p \right)^{1/p},$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$, we may show that

$$\begin{aligned} \int_{\Omega \setminus \Omega_{3\varepsilon}} |\nabla^2 w|^p dx &\leq C \int_{\Omega \setminus \Omega_{2\varepsilon}} |\nabla w(x)|^p [\delta(x)]^{-p} dx \\ &\leq C\varepsilon^{1-p} \|(\nabla w)^*\|_{L^p(\partial\Omega)}^p \leq C\varepsilon^{1-p} \left\{ \|f\|_{W^{1,p}(\partial\Omega)}^p + \|F\|_{L^p(\Omega)}^p \right\}. \end{aligned}$$

As a result, we obtain

$$\|\nabla w\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon \|(\nabla^2 w)\eta_\varepsilon\|_{L^p(\Omega)} \leq C\varepsilon^{1/p} \left\{ \|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} \right\}.$$

This, together with the estimate (7-10) for v , gives

$$\|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon \|(\nabla^2 u_0)\eta_\varepsilon\|_{L^p(\Omega)} \leq C\varepsilon^{1/p} \left\{ \|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} \right\}, \tag{7-11}$$

which, in view of (7-8), completes the proof. □

Proof of Theorem 7.1. Without loss of generality we may assume that

$$\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} = 1.$$

Let $\varepsilon \leq r < \text{diam}(\Omega)$. It follows from Lemma 7.3 that

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^p(\Omega_r)} &\leq \|\nabla u_0\|_{L^p(\Omega_r)} + C\|\nabla\chi(x/\varepsilon)K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega_r)} + C\varepsilon\|\chi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega_r)} + C\varepsilon^{1/p} \\ &\leq C\|\nabla u_0\|_{L^p(\Omega_{2r})} + C\varepsilon\|\nabla((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + C\varepsilon^{1/p} \\ &\leq C\|\nabla u_0\|_{L^p(\Omega_{2r})} + C\varepsilon^{1/p}, \end{aligned} \quad (7-12)$$

where we have used Lemma 2.1 for the second inequality and (7-11) for the third. An inspection of the proof of Lemma 7.3 shows that

$$\|\nabla u_0\|_{L^p(\Omega_{2r})} \leq Cr^{1/p},$$

which, in view of (7-12), gives

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_r)} \leq Cr^{1/p}. \quad \square$$

To prove Theorem 7.2, we need the following lemma.

Lemma 7.4. *Let u_ε ($\varepsilon \geq 0$) be solutions of the Neumann problem (7-3). Also assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Let w_ε be defined by (7-5). Then*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega)} \leq C_p\varepsilon^{1/p}\{\|g\|_{L^p(\partial\Omega)} + \|F\|_{L^p(\Omega)}\}, \quad (7-13)$$

where C_p depends only on d, p, A and Ω .

Proof. The proof is similar to that of Lemma 7.3. Let ϕ_ε be a function in \mathcal{R} such that $w_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. It follows from the formula (7-7) and the $W^{1,p}$ estimates in Theorem 6.2 that

$$\|w_\varepsilon - \phi_\varepsilon\|_{W^{1,p}(\Omega)} \leq C\{\|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 u_0)\eta_\varepsilon\|_{L^2(\Omega)}\}. \quad (7-14)$$

To estimate the right-hand side of (7-14), we proceed as in the proof of Lemma 7.3, but use the nontangential maximal function estimate [Fabes et al. 1978; Lewis et al. 1993; Hofmann et al. 2015]

$$\|(\nabla w)^*\|_{L^p(\partial\Omega)} \leq C\left\|\frac{\partial w}{\partial\nu_0}\right\|_{L^p(\partial\Omega)},$$

where $\mathcal{L}_0(w) = 0$ in Ω and $w \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. As a result, we obtain

$$\|w_\varepsilon - \phi_\varepsilon\|_{W^{1,p}(\Omega)} \leq C\varepsilon^{1/p}\{\|g\|_{L^p(\partial\Omega)} + \|F\|_{L^p(\Omega)}\}. \quad (7-15)$$

Finally, note that since $u_\varepsilon - u_0 \perp \mathcal{R}$,

$$\begin{aligned} \|\phi_\varepsilon\|_{W^{1,p}(\Omega)} &\leq C\varepsilon\|\chi(x/\varepsilon)K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} \\ &\leq C\varepsilon\|\nabla u_0\|_{L^p(\Omega)}. \end{aligned}$$

This, together with (7-15), yields the estimate (7-13). \square

Proof of Theorem 7.2. The estimate (7-4) follows from (7-13), as in the case of the Dirichlet conditions. We omit the details. \square

Remark 7.5. Under certain smoothness condition on A , such as Hölder continuity, it is possible to solve the L^p Dirichlet, regularity, and Neumann problems for $\mathcal{L}_1(u) = 0$ in C^1 domains for any $1 < p < \infty$. By the same localization procedure and blow-up argument as in Remark 3.1, this implies

$$\begin{cases} \int_{\partial\Omega} |\nabla u_\varepsilon|^p d\sigma \leq C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^p d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^p dx, \\ \int_{\partial\Omega} |\nabla u_\varepsilon|^p d\sigma \leq C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^p d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^p dx, \end{cases} \tag{7-16}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω . It then follows from Theorems 7.1 and 7.2 that

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^p d\sigma \leq C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^p d\sigma \tag{7-17}$$

if $u_\varepsilon \perp \mathcal{R}$, and

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^p d\sigma \leq C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^p d\sigma + C \int_{\partial\Omega} |u_\varepsilon|^p d\sigma. \tag{7-18}$$

As in the case $p = 2$, by the method of layer potentials, estimates (7-17)–(7-18) lead to the uniform solvability of the L^p Dirichlet, regularity, and Neumann problems in C^1 domains. The details will be given elsewhere.

8. Lipschitz estimates in $C^{1,\alpha}$ domains, part I

In this section we investigate the Lipschitz estimates, down to the scale ε , in $C^{1,\alpha}$ domains with Dirichlet boundary conditions and give the proof of Theorem 1.4. The Neumann boundary conditions will be treated in the next section. The proof of Theorems 1.4 and 1.5 is based on a general scheme for establishing Lipschitz estimates at large scales in homogenization, recently formulated in [Armstrong and Smart 2016] for interior estimates. Our approach to the boundary Lipschitz estimates in $C^{1,\alpha}$ domains is similar to that used in [Armstrong and Shen 2016] for elliptic systems with almost-periodic coefficients. We remark that Lemma 8.5, which is a continuous version of Lemma 3.1 in [Armstrong and Shen 2016] and whose proof is simpler, makes the argument more transparent.

Let D_r and Δ_r be defined by (1-16) with $\psi(0) = 0$ and $\|\nabla\psi\|_\infty \leq M$.

Lemma 8.1. *Let $u_\varepsilon \in H^1(D_2; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_2 with $u_\varepsilon = f$ on Δ_2 . Then there exists $v \in H^1(D_1; \mathbb{R}^d)$ such that $\mathcal{L}_0(v) = F$ in D_1 , $v = f$ on Δ_1 , and*

$$\|u_\varepsilon - v\|_{L^2(D_1)} \leq C\varepsilon^{1/2} \left\{ \|u_\varepsilon\|_{L^2(D_2)} + \|F\|_{L^2(D_2)} + \|f\|_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)} \right\}, \tag{8-1}$$

where C depends only on d, κ_1, κ_2 , and M .

Proof. By Caccioppoli’s inequality,

$$\int_{D_{3/2}} |\nabla u_\varepsilon|^2 \leq C \left\{ \int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|f\|_{L^\infty(\Delta_2)}^2 + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)}^2 \right\}.$$

By the coarea formula this implies that there exists some $t \in [\frac{5}{4}, \frac{3}{2}]$ such that

$$\int_{\partial D_t \setminus \Delta_2} (|\nabla u_\varepsilon|^2 + |u_\varepsilon|^2) \leq C \left\{ \int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|f\|_{L^\infty(\Delta_2)}^2 + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)}^2 \right\}.$$

Let v be the weak solution to the Dirichlet problem,

$$\mathcal{L}_0(v) = F \quad \text{in } D_t \quad \text{and} \quad v = u_\varepsilon \quad \text{on } \partial D_t.$$

It follows from Remark 2.8 that

$$\begin{aligned} \|u_\varepsilon - v\|_{L^2(D_t)} &\leq \|u_\varepsilon - v\|_{L^2(D_t)} \\ &\leq C\varepsilon^{1/2} \{ \|u_\varepsilon\|_{H^1(\partial D_t)} + \|F\|_{L^2(D_t)} \} \\ &\leq C\varepsilon^{1/2} \{ \|u_\varepsilon\|_{L^2(D_2)} + \|F\|_{L^2(D_2)} + \|f\|_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)} \}, \end{aligned}$$

where C depends only on d, κ_1, κ_2 , and M . □

Lemma 8.2. *Let $\varepsilon \leq r < 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_{2r} with $u_\varepsilon = f$ on Δ_{2r} . Then there exists $v \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(v) = F$ in D_r , $v = f$ on Δ_r , and*

$$\begin{aligned} &\left(\int_{D_r} |u_\varepsilon - v|^2 \right)^{1/2} \\ &\leq C(\varepsilon/r)^{1/2} \left\{ \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left(\int_{D_{2r}} |F|^2 \right)^{1/2} + \|f\|_{L^\infty(\Delta_{2r})} + r \|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} \right\}, \end{aligned} \tag{8-2}$$

where C depends only on d, κ_1, κ_2 , and M .

Proof. This follows from Lemma 8.1 by rescaling. □

In the rest of this section we will assume that the defining function ψ in the definition of D_r and Δ_r is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$ with $\psi(0) = 0$ and $\|\nabla \psi\|_{C^\alpha(\mathbb{R}^{d-1})} \leq M$.

Lemma 8.3. *Let v be a solution of $\mathcal{L}_0(v) = F$ in D_r with $v = f$ on Δ_r . For $0 < t \leq r$, define*

$$\begin{aligned} G(t; v) = \frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ q \in \mathbb{R}^d}} &\left\{ \left(\int_{D_t} |v - Mx - q|^2 \right)^{1/2} + t^2 \left(\int_{D_t} |F|^p \right)^{1/p} + \|f - Mx - q\|_{L^\infty(\Delta_t)} \right. \\ &\left. + t \|\nabla_{\tan}(f - Mx - q)\|_{L^\infty(\Delta_t)} + t^{1+\sigma} \|\nabla_{\tan}(f - Mx - q)\|_{C^{0,\sigma}(\Delta_t)} \right\}, \end{aligned} \tag{8-3}$$

where $p > d$ and $\sigma \in (0, \alpha)$. Then there exists $\theta \in (0, \frac{1}{4})$, depending only on $d, p, \kappa_1, \kappa_2, \sigma, \alpha$ and M , such that

$$G(\theta r; v) \leq \frac{1}{2} G(r; v). \tag{8-4}$$

Proof. The lemma follows from the boundary $C^{1,\alpha}$ estimates for elasticity systems with constant coefficients. We refer the reader to [Armstrong and Shen 2016, Lemma 7.1] for the case $\mathcal{L}_0(v) = 0$. The argument for the general case $F \in L^p$ with $p > d$ is the same. □

Lemma 8.4. Let $0 < \varepsilon < \frac{1}{2}$. Let u_ε be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_1 with $u_\varepsilon = f$ on Δ_1 . Define

$$H(r) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ q \in \mathbb{R}^d}} \left\{ \left(\int_{D_r} |u_\varepsilon - Mx - q|^2 \right)^{1/2} + r^2 \left(\int_{D_r} |F|^p \right)^{1/p} + \|f - Mx - q\|_{L^\infty(\Delta_r)} + r \|\nabla_{\tan}(f - Mx - q)\|_{L^\infty(\Delta_r)} + r^{1+\sigma} \|\nabla_{\tan}(f - Mx - q)\|_{C^{0,\sigma}(\Delta_r)} \right\} \quad (8-5)$$

and

$$\Phi(r) = \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left\{ \left(\int_{D_{2r}} |u_\varepsilon - q|^2 \right)^{1/2} + r^2 \left(\int_{D_{2r}} |F|^p \right)^{1/p} + \|f - q\|_{L^\infty(\Delta_{2r})} + r \|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} \right\}, \quad (8-6)$$

where $p > d$ and $\sigma \in (0, \alpha)$. Then

$$H(\theta r) \leq \frac{1}{2} H(r) + C \left(\frac{\varepsilon}{r} \right)^{1/2} \Phi(2r) \quad (8-7)$$

for any $r \in [\varepsilon, \frac{1}{2}]$, where $\theta \in (0, \frac{1}{4})$ is given by Lemma 8.3.

Proof. Fix $r \in [\varepsilon, \frac{1}{2}]$. Let v be a solution of $\mathcal{L}_0(v) = F$ in D_r with $v = f$ on Δ_r . Observe that

$$\begin{aligned} H(\theta r) &\leq \frac{1}{\theta r} \left(\int_{D_{\theta r}} |u_\varepsilon - v|^2 \right)^{1/2} + G(\theta r; v) \\ &\leq \frac{1}{\theta r} \left(\int_{D_{\theta r}} |u_\varepsilon - v|^2 \right)^{1/2} + \frac{1}{2} G(r; v) \\ &\leq \frac{C}{r} \left(\int_{D_r} |u_\varepsilon - v|^2 \right)^{1/2} + \frac{1}{2} H(r), \end{aligned}$$

where we have used Lemma 8.3 for the second inequality. This, together with Lemma 8.2, gives

$$H(\theta r) \leq \frac{1}{2} H(r) + C \left(\frac{\varepsilon}{r} \right)^{1/2} \frac{1}{r} \left\{ \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left(\int_{D_{2r}} |F|^2 \right)^{1/2} + \|f\|_{L^\infty(\Delta_{2r})} + r \|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} \right\}.$$

Since $H(r)$ remains invariant if we subtract a constant from u_ε , the inequality (8-7) follows. \square

Lemma 8.5. Let $H(r)$ and $h(r)$ be two nonnegative continuous functions on the interval $(0, 1]$. Let $0 < \varepsilon < \frac{1}{4}$. Suppose that there exists a constant C_0 such that

$$\begin{cases} \max_{r \leq t \leq 2r} H(t) \leq C_0 H(2r), \\ \max_{r \leq t, s \leq 2r} |h(t) - h(s)| \leq C_0 H(2r) \end{cases} \quad (8-8)$$

for any $r \in [\varepsilon, \frac{1}{2}]$. We further assume that

$$H(\theta r) \leq \frac{1}{2} H(r) + C_0 \omega(\varepsilon/r) \{H(2r) + h(2r)\} \quad (8-9)$$

for any $r \in [\varepsilon, \frac{1}{2}]$, where $\theta \in (0, \frac{1}{4})$ and ω is a nonnegative increasing function $[0, 1]$ such that $\omega(0) = 0$ and

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty. \quad (8-10)$$

Then

$$\max_{\varepsilon \leq r \leq 1} \{H(r) + h(r)\} \leq C\{H(1) + h(1)\}, \quad (8-11)$$

where C depends only on C_0, θ , and ω .

Proof. It follows from (8-8) that

$$h(r) \leq h(2r) + C_0 H(2r)$$

for any $\varepsilon \leq r \leq \frac{1}{2}$. Hence,

$$\begin{aligned} \int_a^{1/2} \frac{h(r)}{r} dr &\leq \int_a^{1/2} \frac{h(2r)}{r} dr + C_0 \int_a^{1/2} \frac{H(2r)}{r} dr \\ &= \int_{2a}^1 \frac{h(r)}{r} dr + C_0 \int_{2a}^1 \frac{H(r)}{r} dr, \end{aligned}$$

where $\varepsilon \leq a \leq \frac{1}{4}$. This implies

$$\begin{aligned} \int_a^{2a} \frac{h(r)}{r} dr &\leq \int_{1/2}^1 \frac{h(r)}{r} dr + C \int_{2a}^1 \frac{H(r)}{r} dr \\ &\leq C\{h(1) + H(1)\} + C \int_{2a}^1 \frac{H(r)}{r} dr, \end{aligned}$$

which, by (8-8), gives

$$\begin{aligned} h(a) &\leq C \left\{ H(2a) + h(1) + H(1) + \int_{2a}^1 \frac{H(r)}{r} dr \right\} \\ &\leq C \left\{ h(1) + H(1) + \int_a^1 \frac{H(r)}{r} dr \right\} \end{aligned} \quad (8-12)$$

for any $a \in [\varepsilon, \frac{1}{4}]$.

Next, we use (8-9) and (8-12) to obtain

$$H(\theta r) \leq \frac{1}{2} H(r) + C\omega(\varepsilon/r)\{h(1) + H(1)\} + C\omega(\varepsilon/r) \int_r^1 \frac{H(r)}{r} dr.$$

It follows that

$$\int_{\alpha\theta\varepsilon}^{\theta} \frac{H(r)}{r} dr \leq \frac{1}{2} \int_{\alpha\varepsilon}^1 \frac{H(r)}{r} dr + C_\alpha \{h(1) + H(1)\} + C \int_{\alpha\varepsilon}^1 \omega(\varepsilon/r) \left\{ \int_r^1 \frac{H(t)}{t} dt \right\} \frac{dr}{r},$$

where $\alpha > 1$ and we have used the condition (8-10). Using (8-10) and the observation that

$$\int_{\alpha\varepsilon}^1 \omega(\varepsilon/r) \left\{ \int_r^1 \frac{H(t)}{t} dt \right\} \frac{dr}{r} = \int_{\alpha\varepsilon}^1 H(t) \left\{ \int_{\varepsilon/t}^{1/\alpha} \frac{\omega(s)}{s} ds \right\} \frac{dt}{t} \leq (4C)^{-1} \int_{\alpha\varepsilon}^1 H(t) \frac{dt}{t}$$

if $\alpha > \alpha_0(\omega)$, we see that

$$\int_{\alpha\theta\varepsilon}^{\theta} \frac{H(r)}{r} dr \leq \frac{1}{2} \int_{\alpha\varepsilon}^1 \frac{H(r)}{r} dr + C_\alpha \{h(1) + H(1)\} + \frac{1}{4} \int_{\alpha\varepsilon}^1 \frac{H(r)}{r} dr.$$

It follows that

$$\int_{\varepsilon}^1 \frac{H(r)}{r} dr \leq C\{h(1) + H(1)\}, \tag{8-13}$$

which, together with (8-8) and (8-12), yields the estimate (8-11). □

Proof of Theorem 1.4. We may assume that $0 < \varepsilon < \frac{1}{4}$. Let u_ε be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_1 with $u_\varepsilon = f$ on Δ_1 , where $F \in L^p(D_1)$ for some $p > d$ and $f \in C^{1,\sigma}(\Delta_1)$ for some $\sigma \in (0, \alpha)$. For $r \in (0, 1)$, we define the function $H(r)$ by (8-5). It is easy to see that $H(t) \leq CH(2r)$ if $t \in (r, 2r)$.

Next, we let $h(r) = |M_r|$, where M_r is the $d \times d$ matrix such that

$$H(r) = \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left\{ \left(\int_{D_r} |u_\varepsilon - M_r x - q|^2 \right)^{1/2} + r^2 \left(\int_{D_r} |F|^p \right)^{1/p} + \|f - M_r x - q\|_{L^\infty(\Delta_r)} + r \|\nabla_{\tan}(f - M_r x - q)\|_{L^\infty(\Delta_r)} + r^{1+\sigma} \|\nabla_{\tan}(f - M_r x - q)\|_{C^{0,\sigma}(\Delta_r)} \right\}.$$

Let $t, s \in [r, 2r]$. Using

$$\begin{aligned} |M_t - M_s| &\leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left(\int_{D_r} |(M_t - M_s)x - q|^2 \right)^{1/2} \\ &\leq \frac{C}{t} \inf_{q \in \mathbb{R}^d} \left(\int_{D_t} |u_\varepsilon - M_t x - q|^2 \right)^{1/2} + \frac{C}{s} \inf_{q \in \mathbb{R}^d} \left(\int_{D_s} |u_\varepsilon - M_s x - q|^2 \right)^{1/2} \\ &\leq C\{H(t) + H(s)\} \\ &\leq CH(2r), \end{aligned}$$

we obtain

$$\max_{r \leq t, s \leq 2r} |h(t) - h(s)| \leq CH(2r).$$

Furthermore, if Φ is defined by (8-6), then

$$\Phi(r) \leq H(2r) + h(2r).$$

In view of Lemma 8.4 this gives

$$H(\theta r) \leq \frac{1}{2}H(r) + C\omega(\varepsilon/r)\{H(2r) + h(2r)\}$$

for $r \in [\varepsilon, \frac{1}{2}]$, where $\omega(t) = t^{1/2}$. Thus the functions $H(r)$ and $h(r)$ satisfy the conditions (8-8), (8-9) and (8-10) in Lemma 8.5. Consequently, we obtain that for $r \in [\varepsilon, \frac{1}{2}]$,

$$\begin{aligned} \inf_{q \in \mathbb{R}^d} \frac{1}{r} \left(\int_{D_r} |u_\varepsilon - q|^2 \right)^{1/2} &\leq C\{H(r) + h(r)\} \\ &\leq C\{H(1) + h(1)\} \\ &\leq C \left\{ \left(\int_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|F\|_{L^p(D_1)} + \|f\|_{C^{1,\sigma}(\Delta_1)} \right\}, \end{aligned}$$

which, together with Caccioppoli's inequality, gives the estimate (1-18). □

The argument used in this section may be used to prove the interior Lipschitz estimates, down to the scale ε .

Theorem 8.6. *Suppose that A satisfies (1-2)–(1-3). Let $u_\varepsilon \in H^1(B(x_0, R); \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$, where $F \in L^p(B(x_0, R); \mathbb{R}^d)$ for some $p > d$. Then, for $\varepsilon \leq r < R$,*

$$\left(\int_{B(x_0, r)} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left\{ \left(\int_{B(x_0, R)} |\nabla u_\varepsilon|^2 \right)^{1/2} + R \left(\int_{B(x_0, R)} |F|^p \right)^{1/p} \right\}, \quad (8-14)$$

where C depends only on d, κ_1, κ_2 , and p .

9. Lipschitz estimates in $C^{1,\alpha}$ domains, part II

In this section we study the Lipschitz estimate, down to the scale ε , with Neumann boundary conditions, and give the proof of Theorem 1.5. Throughout this section we will assume that the defining function ψ in D_r and Δ_r is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$ and $\|\nabla \psi\|_{C^\alpha(\mathbb{R}^{d-1})} \leq M$.

Lemma 9.1. *Let Ω be a bounded Lipschitz domain. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $\partial u_\varepsilon / \partial \nu_\varepsilon = g$ on $\partial\Omega$. Then there exists $w \in H^1(\Omega; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = F$ in Ω , $\partial w / \partial \nu_0 = g$ on $\partial\Omega$, and*

$$\|u_\varepsilon - w\|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)} \}. \quad (9-1)$$

Proof. Choose $\phi_\varepsilon \in \mathcal{R}$ such that $u_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. Let u_0 be the weak solution to the Neumann problem: $\mathcal{L}_0(u_0) = F$ in Ω and $\partial u_0 / \partial \nu_0 = g$ on $\partial\Omega$ with the property $u_0 \perp \mathcal{R}$. It follows from Remark 2.8 that

$$\|u_\varepsilon - \phi_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)} \}.$$

By letting $w = u_0 + \phi_\varepsilon$ this gives (9-1). □

Lemma 9.2. *Let $\varepsilon \leq r < 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_{2r} with $\partial u_\varepsilon / \partial \nu_\varepsilon = g$ on Δ_{2r} . Then there exists $w \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = F$ in D_r , $\partial w / \partial \nu_0 = g$ on Δ_r , and*

$$\left(\int_{D_r} |u_\varepsilon - w|^2 \right)^{1/2} \leq C(\varepsilon/r)^{1/2} \left\{ \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left(\int_{D_{2r}} |F|^2 \right)^{1/2} + r \|g\|_{L^\infty(\Delta_{2r})} \right\}, \quad (9-2)$$

where C depends only on d, κ_1, κ_2 , and M .

Proof. By rescaling we may assume $r = 1$. As in the case of Dirichlet conditions in Lemma 8.2, the desired estimate follows from Lemma 9.1 by using the coarea formula and the Caccioppoli inequality

$$\int_{D_{3/2}} |\nabla u_\varepsilon|^2 \leq C \left\{ \int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|g\|_{L^\infty(\Delta_2)}^2 \right\}, \quad (9-3)$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_2 and $\partial u_\varepsilon / \partial \nu_\varepsilon = g$ on Δ_2 . □

Lemma 9.3. *Let w be a solution of $\mathcal{L}_0(w) = F$ in D_r with $\partial w / \partial \nu_0 = g$ on Δ_r . For $0 < t \leq r$, define*

$$I(t; w) = \frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ q \in \mathbb{R}^d}} \left\{ \left(\int_{D_t} |w - Mx - q|^2 \right)^{1/2} + t^2 \left(\int_{D_t} |F|^p \right)^{1/p} + t \left\| \frac{\partial}{\partial \nu_0} (w - Mx) \right\|_{L^\infty(\Delta_t)} + t^{1+\sigma} \left\| \frac{\partial}{\partial \nu_0} (w - Mx) \right\|_{C^{0,\sigma}(\Delta_t)} \right\}, \quad (9-4)$$

where $p > d$ and $\sigma \in (0, \alpha)$. Then there exists $\theta \in (0, \frac{1}{4})$, depending only on $d, p, \kappa_1, \kappa_2, \sigma, \alpha$ and M , such that

$$I(\theta r; w) \leq \frac{1}{2} I(r; w). \quad (9-5)$$

Proof. By rescaling we may assume $r = 1$. The lemma then follows from the boundary $C^{1,\sigma}$ estimates with Neumann boundary conditions in $C^{1,\alpha}$ domains for elasticity systems with constant coefficients. \square

Lemma 9.4. *Let $0 < \varepsilon \leq \frac{1}{2}$. Let u_ε be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_1 with $\partial u_\varepsilon / \partial \nu_\varepsilon = g$ on Δ_1 , where $F \in L^p(D_1; \mathbb{R}^d)$ for some $p > d$ and $g \in C^\sigma(\Delta_1; \mathbb{R}^d)$ for some $\sigma \in (0, \alpha)$. Define*

$$J(r) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ q \in \mathbb{R}^d}} \left\{ \left(\int_{D_r} |u_\varepsilon - Mx - q|^2 \right)^{1/2} + r^2 \left(\int_{D_r} |F|^p \right)^{1/p} + r \left\| g - \frac{\partial}{\partial \nu_0} (Mx) \right\|_{L^\infty(\Delta_r)} + r^{1+\sigma} \left\| g - \frac{\partial}{\partial \nu_0} (Mx) \right\|_{C^{0,\sigma}(\Delta_r)} \right\} \quad (9-6)$$

and

$$\Psi(r) = \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left\{ \left(\int_{D_{2r}} |u_\varepsilon - q|^2 \right)^{1/2} + r^2 \left(\int_{D_{2r}} |F|^p \right)^{1/p} + r \|g\|_{L^\infty(\Delta_{2r})} \right\}. \quad (9-7)$$

Then

$$J(\theta r) \leq \frac{1}{2} J(r) + C(\varepsilon/r)^{1/2} \Psi(2r) \quad (9-8)$$

for any $r \in [\varepsilon, \frac{1}{2}]$, where $\theta \in (0, \frac{1}{4})$ is given by Lemma 9.3.

Proof. Fix $r \in [\varepsilon, \frac{1}{2}]$. Let w be the function in $H^1(D_r; \mathbb{R}^d)$ given by Lemma 9.2. Then

$$\begin{aligned} J(\theta r) &\leq I(\theta r; w) + \frac{1}{\theta r} \left(\int_{D_{\theta r}} |u_\varepsilon - w|^2 \right)^{1/2} \\ &\leq \frac{1}{2} I(r; w) + \frac{1}{\theta r} \left(\int_{D_{\theta r}} |u_\varepsilon - w|^2 \right)^{1/2} \\ &\leq \frac{1}{2} J(r) + \frac{C}{r} \left(\int_{D_r} |u_\varepsilon - w|^2 \right)^{1/2}, \end{aligned}$$

where we have used Lemma 9.3 for the second inequality. In view of Lemma 9.2, this gives

$$J(\theta r) \leq \frac{1}{2} J(r) + \frac{C}{r} \left\{ \left(\int_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left(\int_{D_{2r}} |F|^p \right)^{1/p} + r \|g\|_{L^\infty(\Delta_{2r})} \right\},$$

from which the estimate (9-8) follows, as the function $J(r)$ is invariant if we replace u_ε by $u_\varepsilon - q$ for any $q \in \mathbb{R}^d$. \square

Proof of Theorem 1.5. With Lemma 9.4 at our disposal, Theorem 1.5 follows from Lemma 8.5, as in the case of Dirichlet boundary conditions. We omit the details. \square

As we indicate in the Introduction, under additional smoothness conditions, the full Lipschitz estimates, uniform in ε , follow from Theorem 1.4, Theorem 1.5, and local Lipschitz estimates by a blow-up argument.

Corollary 9.5. *Suppose that A satisfies (1-2)–(1-3). Also assume that A is Hölder continuous. Let $u_\varepsilon \in H^1(B(0, 1); \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(0, 1)$, where $F \in L^p(B(0, 1); \mathbb{R}^d)$ for some $p > d$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(B(0, 1/2))} \leq C_p \left\{ \|u_\varepsilon\|_{L^2(B(0, 1))} + \|F\|_{L^p(B(0, 1))} \right\}, \quad (9-9)$$

where C_p depends only on d , p and A .

Corollary 9.6. *Suppose that A satisfies (1-2)–(1-3). Also assume that A is Hölder continuous. Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}(u_\varepsilon) = F$ in D_1 with $u_\varepsilon = f$ on Δ_1 , where the defining function ψ in D_1 and Δ_1 is $C^{1, \alpha}$ with $\|\nabla \psi\|_{C^\alpha(\mathbb{R}^{d-1})} \leq M$ for some $\alpha > 0$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \leq C \left\{ \|u_\varepsilon\|_{L^2(D_1)} + \|F\|_{L^p(D_1)} + \|f\|_{C^{1, \sigma}(\Delta_1)} \right\}, \quad (9-10)$$

where $p > d$, $\sigma \in (0, \alpha)$, and C depends only on d , p , σ , A , α and M .

Corollary 9.7. *Suppose that A , D_1 and Δ_1 satisfy the same conditions as in Corollary 9.6. Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}(u_\varepsilon) = F$ in D_1 with $\partial u_\varepsilon / \partial \nu_\varepsilon = g$ on Δ_1 . Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \leq C \left\{ \|u_\varepsilon\|_{L^2(D_1)} + \|F\|_{L^p(D_1)} + \|g\|_{C^\sigma(\Delta_1)} \right\}, \quad (9-11)$$

where $p > d$, $\sigma \in (0, \alpha)$, and C depends only on d , p , σ , A , α and M .

As we mentioned in Introduction, for \mathcal{L}_ε with coefficients satisfying (1-11), (1-3) and the Hölder continuity condition, estimates (9-9) and (9-10) were proved in [Avellaneda and Lin 1987], while (9-11) was established in [Kenig et al. 2013; Armstrong and Shen 2016].

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CONVEX INTEGRATION FOR THE MONGE–AMPÈRE EQUATION IN TWO DIMENSIONS

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This paper concerns the questions of flexibility and rigidity of solutions to the Monge–Ampère equation, which arises as a natural geometrical constraint in prestrained nonlinear elasticity. In particular, we focus on degenerate, i.e., “flexible”, weak solutions that can be constructed through methods of convex integration à la Nash and Kuiper and establish the related h -principle for the Monge–Ampère equation in two dimensions.

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1. Introduction

In this paper we study the $C^{1,\alpha}$ solutions to the Monge–Ampère equation in two dimensions,

$$\mathit{Det} \nabla^2 v := -\frac{1}{2} \operatorname{curl} \operatorname{curl} (\nabla v \otimes \nabla v) = f \quad \text{in } \Omega \subset \mathbb{R}^2. \quad (1-1)$$

Our results concern the dichotomy of “rigidity vs. flexibility”, in the spirit of the analogous results and techniques appearing in the contexts of the low codimension isometric immersion problem [Nash 1954; Kuiper 1955a; 1955b; Borisov 1959; 2004; Conti et al. 2012] and Onsager’s conjecture for Euler equations [Székelyhidi 2013; De Lellis and Székelyhidi 2009; 2013; Constantin et al. 1994; Eyink 1994].

In the first, main part of the paper we show that below the regularity threshold $\alpha < \frac{1}{7}$, the very weak $C^{1,\alpha}(\bar{\Omega})$ solutions to (1-1), as defined below, are dense in the set of all continuous functions (see Theorems 1.1 and 1.2). These flexibility statements are a consequence of the convex integration h -principle, which is a method proposed in [Gromov 1986] for solving certain partial differential relations

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and which turns out to be applicable to our setting of the Monge–Ampère equation as well. Here, we directly adapt the iteration method of Nash [1954] and Kuiper [1955a; 1955b] in order to construct the oscillatory solutions to (1-1).¹

In the second part of the paper we prove that the same class of very weak solutions fails the above flexibility in the regularity regime $\alpha > \frac{2}{3}$. Our results are parallel with those concerning isometric immersions [Borisov 1959; Conti et al. 2012; Pakzad 2004], Euler equations [Constantin et al. 1994; Eyink 1994], the Perona–Malik equation [Kim and Yan 2015a; 2015b], the active scalar equation [Isett and Vicol 2015], and should also be compared with results on the regularity of Sobolev solutions to the Monge–Ampère equation [Pakzad 2004; Šverák 1991; Lewicka et al. 2017; Jerrard and Pakzad 2017], whose study is important in the context of nonlinear elasticity, and with the rigidity results for the Monge–Ampère functions [Jerrard 2008; 2010].

The weak determinant Hessian. Let $\Omega \subset \mathbb{R}^2$ be an open set. Given a function $v \in W_{\text{loc}}^{1,2}(\Omega)$, we define its very weak Hessian (denoted by \mathcal{H}_2^* in [Iwaniec 2001; Fonseca and Malý 2005]) as

$$\text{Det } \nabla^2 v = -\frac{1}{2} \text{curl curl}(\nabla v \otimes \nabla v),$$

understood in the sense of distributions. A straightforward approximation argument shows that if $v \in W_{\text{loc}}^{2,2}$ then $L_{\text{loc}}^1(\Omega) \ni \text{Det } \nabla^2 v = \det \nabla^2 v$ a.e. in Ω , where $\nabla^2 v$ stands for the Hessian matrix field of v . We also remark that this notion of the very weak Hessian is distinct from the distributional Hessian $\text{Det } \nabla^2 v = \text{Det } \nabla(\nabla v)$ (denoted by $\mathcal{H}u$ in [Iwaniec 2001; Fonseca and Malý 2005]), which is defined through the distributional determinant Det ,

$$\text{Det } \nabla \psi = -\text{div}(\psi_2 \nabla^\perp \psi_1) = \partial_2(\psi_2 \partial_1 \psi_1) - \partial_1(\psi_2 \partial_2 \psi_1) \quad \text{for } \psi = (\psi_1, \psi_2) \in W^{1,4/3}(\Omega, \mathbb{R}^2).$$

Contrary to the distributional Hessian, the very weak Hessian is not continuous with respect to the weak topology. Indeed, an example of a sequence $v_n \in W^{1,2}(\Omega)$ is constructed in [Iwaniec 2001], where $\text{Det } \nabla^2 v = -1$ while v_n converges weakly to 0. One consequence of the proof of our Theorem 1.1 below is that $\text{Det } \nabla^2$ is actually weakly discontinuous everywhere in $W^{1,2}(\Omega)$ (see Corollary 6.2).

Here is our first main result:

Theorem 1.1. *Let $f \in L^{7/6}(\Omega)$ on an open, bounded, simply connected $\Omega \subset \mathbb{R}^2$. Fix an exponent*

$$\alpha < \frac{1}{7}.$$

Then the set of $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ solutions to (1-1) is dense in the space $\mathcal{C}^0(\bar{\Omega})$. More precisely, for every $v_0 \in \mathcal{C}^0(\bar{\Omega})$ there exists a sequence $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$, converging uniformly to v_0 and satisfying

$$\text{Det } \nabla^2 v_n = f \quad \text{in } \Omega. \tag{1-2}$$

When $f \in L^p(\Omega)$ and $p \in (1, \frac{7}{6})$, the same result is true for any $\alpha < 1 - \frac{1}{p}$.

¹We remark that the recent work of De Lellis, Inauen and Székelyhidi [De Lellis et al. 2015] showed that the flexibility exponent $\frac{1}{7}$ can be improved to $\frac{1}{5}$ in the case of the isometric immersion problem in two dimensions. We expect similar improvement to be possible also in the present case of equation (1-1); this will be investigated in our future work.

In order to better understand Theorem 1.1, we point out a connection between the solutions to (1-1) and the isometric immersions of Riemannian metrics, motivated by a study of nonlinear elastic plates. Since on a simply connected domain Ω , the kernel of the differential operator curl curl consists of the fields of the form $\text{sym } \nabla w$, a solution to (1-1) with the vanishing right-hand side $f \equiv 0$ can be characterized by the criterion

$$\exists w : \Omega \rightarrow \mathbb{R}^2 \quad \text{such that} \quad \frac{1}{2} \nabla v \otimes \nabla v + \text{sym } \nabla w = 0 \quad \text{in } \Omega. \tag{1-3}$$

The equation in (1-3) can be seen as an equivalent condition for the one-parameter family of deformations

$$\phi_\varepsilon = \text{id} + \varepsilon v e_3 + \varepsilon^2 w : \Omega \rightarrow \mathbb{R}^3,$$

given through the out-of-plane displacement v and the in-plane displacement w (albeit with different orders of magnitude ε and ε^2), to form a second-order infinitesimal isometry (bending), i.e., to induce the change of metric on the plate Ω whose second-order terms in ε disappear:

$$(\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon - \text{Id}_2 = o(\varepsilon^2).$$

In this context, we take the cue about Theorem 1.1 from the celebrated work of Nash [1954] and Kuiper [1955a; 1955b], where they show the density of codimension-one C^1 isometric immersions of Riemannian manifolds in the set of short mappings. Since we are now dealing with the second-order infinitesimal isometries rather than the exact isometries, the classical metric pull-back equation

$$y^* g_e = h$$

for a mapping y from (Ω, h) into \mathbb{R}^3 equipped with the standard Euclidean metric g_e is replaced by the compatibility equation of the tensor $T(v, w) = \frac{1}{2} \nabla v \otimes \nabla v + \text{sym } \nabla w$ with a matrix field A_0 that satisfies $-\text{curl curl } A_0 = f$:

$$T(v, w) = A_0. \tag{1-4}$$

Note that there are many potential choices for A_0 ; for example, one may take $A_0(x) = \lambda(x) \text{Id}_2$ with $\Delta \lambda = -f$ in Ω . Again, equation (1-4) states precisely that the metric $(\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon$ agrees with the given metric $h = \text{Id}_2 + 2\varepsilon^2 A_0$ on Ω , up to terms of order ε^2 . The Gauss curvature κ of the metric h satisfies

$$\kappa(h) = \kappa(\text{Id}_2 + 2\varepsilon^2 A_0) = -\varepsilon^2 \text{curl curl } A_0 + o(\varepsilon^2),$$

while $\kappa((\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon) = -\varepsilon^2 \text{curl curl}(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym } w) + o(\varepsilon^2)$, so the problem (1-1) can also be interpreted as seeking all appropriately regular out-of-plane displacements v that can be matched, by a higher order in-plane displacement perturbation w , to achieve the prescribed Gauss curvature f of Ω , at its highest-order term.

In this paper, similar to the isometric immersion case, we show that solutions to (1-4) are ample. We design a scheme inspired by the work of Nash and Kuiper, which pushes a “short infinitesimal isometry”, i.e., a couple (v_0, w_0) such that $T(v_0, w_0) < A_0$, towards an exact solution to (1-4) in successive small steps. Note that both $y^* g_e = (\nabla y)^T \nabla y$ and the term $\nabla v \otimes \nabla v$ in $T(v, w)$ have a quadratic structure, which is crucial in the analysis of [Nash 1954; Kuiper 1955a; 1955b] and also of this paper. Here, not only does the presence of the linear term $\text{sym } \nabla w$ in $T(u, w)$ not destroy the adaptation of the Nash–Kuiper scheme, but it actually allows for this construction to work.

Convex integration for the Monge–Ampère equation in two dimensions. As we will see in Section 4, Theorem 1.1 follows easily from the statement of our next main result:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $v_0 \in \mathcal{C}^1(\bar{\Omega})$, $w_0 \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ and $A_0 \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}_{\text{sym}}^{2 \times 2})$, for some $\beta \in (0, 1)$, be such that*

$$\exists c_0 > 0 \quad \text{such that} \quad A_0 - \left(\frac{1}{2} \nabla v_0 \otimes \nabla v_0 + \text{sym } \nabla w_0\right) > c_0 \text{Id}_2 \quad \text{in } \bar{\Omega}. \quad (1-5)$$

Then, for every exponent α in the range

$$0 < \alpha < \min\left\{\frac{1}{7}, \frac{1}{2}\beta\right\},$$

there exist sequences $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $w_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ which converge uniformly to v_0 and w_0 , respectively, and which satisfy

$$A_0 = \frac{1}{2} \nabla v_n \otimes \nabla v_n + \text{sym } \nabla w_n \quad \text{in } \bar{\Omega}. \quad (1-6)$$

The above result is the Monge–Ampère analogue of [Conti et al. 2012, Theorem 1], where the authors improved on the Nash–Kuiper method to obtain higher regularity within the flexibility regime. In our paper, we adapt similar methods to the system (1-6).

The term *convex integration* usually refers to a collection of approaches that allow for constructing anomalous solutions to nonlinear PDEs; in particular, flexibility-type results for the isometric immersion problem were obtained via the above-mentioned iteration scheme of Nash and Kuiper. From a geometric perspective, they are special cases of *h-principle*, a notion which was developed by Gromov [1986] for studying partial differential relations; see also [Eliashberg and Mishachev 2002]. From another perspective, one seeks weak solutions of a differential inclusion $Lu(x) \in K$ in Ω by investigating certain classes of subsolutions, e.g., functions u that satisfy $Lu(x) \in \text{conv } K$, where the original constraint set K is replaced by its convex hull $\text{conv } K$ [Tartar 1979; Dacorogna and Marcellini 1997; Müller and Šverák 2003]. This approach leads to the density of very weak solutions, satisfying $Lu \in L^\infty(\Omega)$, in the set of subsolutions. When K is a continuum, the regularity may be improved to $Lu \in \mathcal{C}^0(\Omega)$ by applying the correcting iterations.

Recently, similar techniques were advanced in the context of fluid dynamics and yielded many interesting results for the Euler equations. De Lellis and Székelyhidi [2009] proved the existence of weak solutions with bounded velocity and pressure, their nonuniqueness and the existence of energy-decreasing solutions. In [De Lellis and Székelyhidi 2013], using iteration methods à la Nash and Kuiper, they proved the existence of continuous periodic solutions of the three-dimensional incompressible Euler equations, which dissipate the total kinetic energy. These results are to be contrasted with [Constantin et al. 1994; Eyink 1994], where it was shown that $\mathcal{C}^{0,\alpha}$ solutions of the Euler equations are energy conservative if $\alpha > \frac{1}{3}$. There have been several improvements of [De Lellis and Székelyhidi 2009; 2013] recently, towards a proof of Onsager’s conjecture, which puts the Hölder regularity threshold for the energy conservation of the weak solutions to the Euler equations at $\mathcal{C}^{0,1/3}$ [Isett 2012; 2013; 2016; Buckmaster et al. 2013; 2015; 2016; Choffrut and Székelyhidi 2014]. The stationary incompressible Euler equation has been studied in [Choffrut and Székelyhidi 2014], where the existence of bounded anomalous solutions

has been proved. The authors indicate that in two dimensions, the relaxation set corresponding to the appropriate subsolutions is smaller than in the case of the evolutionary equations. In this context, we noticed a connection between our reformulation of the Monge–Ampère equation and the steady-state Euler equation, which lead to our modest Corollary 4.1.

In this paper we use a direct iteration method to construct exact solutions of (1-1). The recasting of the statement and the proof in the language of convex integration might shed more light on the structure of the Monge–Ampère equation, but it would not improve the results and therefore we do not address this task. We note, however, that constructing Lipschitz continuous piecewise affine approximating solutions to (1-6) for $A_0 \equiv 0$ is quite straightforward and could be used to prove a convex integration density result via the Baire category method, as was done in [De Lellis and Székelyhidi 2009] for the Euler equations (see also Figure 1 and the corresponding explanation).

Rigidity versus flexibility. The flexibility results obtained in view of the h -principle are usually coupled with the rigidity results for more regular solutions. Rigidity of isometric immersions of elliptic metrics for $C^{1,\alpha}$ isometries [Borisov 1959; De Lellis and Székelyhidi 2009] with $\alpha > \frac{2}{3}$, or the energy conservation of weak solutions of the Euler equations for $C^{0,\alpha}$ solutions with $\alpha > \frac{1}{3}$, are results of this type. For the Monge–Ampère equations, we recall two recent statements regarding solutions with Sobolev regularity: Following the well-known unpublished work by Šverák [1991], we proved in [Lewicka et al. 2017] that if $v \in W^{2,2}(\Omega)$ is a solution to (1-1) with $f \in L^1(\Omega)$ and $f \geq c > 0$ in Ω , then in fact v must be C^1 and globally convex (or concave). On the other hand, if $f = 0$ then likewise $v \in C^1(\Omega)$ and v must be developable [Pakzad 2004] (see also [Jerrard 2008; 2010; Jerrard and Pakzad 2017]). A clear statement of rigidity is still lacking for the general f , as is the case for isometric immersions, where rigidity results are usually formulated only for elliptic [Conti et al. 2012] or Euclidean metrics [Pakzad 2004; Liu and Pakzad 2015; Jerrard and Pakzad 2017].

In this paper, we prove the rigidity properties of solutions to (1-1) in the Hölder regularity context when $f \equiv 0$. Namely, we prove:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and let*

$$\frac{2}{3} < \alpha < 1.$$

If $v \in C^{1,\alpha}(\bar{\Omega})$ is a solution to $\text{Det } \nabla^2 v = 0$ in $\bar{\Omega}$, then v must be developable. More precisely, for all $x \in \Omega$ either v is affine in a neighbourhood of x , or there exists a segment l_x joining $\partial\Omega$ on its both ends such that ∇v is constant on l_x .

We also announce the following parallel rigidity result for $f \geq c > 0$, which will be the subject of the forthcoming paper [Lewicka and Pakzad \geq 2017]:

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and let*

$$\frac{2}{3} < \alpha < 1.$$

If $v \in C^{1,\alpha}(\bar{\Omega})$ is a solution to $\text{Det } \nabla^2 v = f$ in $\bar{\Omega}$, where f is a positive Dini continuous function, then v is convex. In fact, it is also an Alexandrov solution to $\det \nabla^2 v = f$ in Ω .

In proving Theorem 1.3, we use a commutator estimate for deriving a degree formula in Proposition 7.1. Similar commutator estimates are used in [Constantin et al. 1994] for the Euler equations and in [Conti et al. 2012] for the isometric immersion problem; this is not surprising, since the presence of a quadratic term plays a major role in all three cases, allowing for the efficiency of the convex integration and iteration methods. Let us also mention that it is still unknown which value of α is the critical value for the rigidity-flexibility dichotomy, but it is conjectured to be $\frac{1}{3}$, $\frac{1}{2}$ or $\frac{2}{3}$.

Notation. By $\mathbb{R}_{\text{sym}}^{2 \times 2}$ we denote the space of symmetric 2×2 matrices, and by $\mathbb{R}_{\text{sym}, >}^{2 \times 2}$ we denote the cone of symmetric, positive definite 2×2 matrices. The space of Hölder continuous functions $\mathcal{C}^{k, \alpha}(\bar{\Omega})$ consists of restrictions of functions $f \in \mathcal{C}^{k, \alpha}(\mathbb{R}^2)$ to $\Omega \subset \mathbb{R}^2$. Then, the $\mathcal{C}^k(\bar{\Omega})$ norm of such a restriction is denoted by $\|f\|_k$, while its Hölder norm $\mathcal{C}^{k, \alpha}(\bar{\Omega})$ is $\|f\|_{k, \alpha}$. By $C > 0$ we denote a universal constant which is independent of all parameters, unless indicated otherwise.

2. The \mathcal{C}^1 approximations: preliminary results

In this and the next section we prove a weaker version of the result in Theorem 1.2. Namely:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $v_0 \in \mathcal{C}^\infty(\bar{\Omega})$, $w_0 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ and $A_0 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}_{\text{sym}}^{2 \times 2})$ be such that*

$$\exists c_0 > 0 \text{ such that } A_0 - \left(\frac{1}{2} \nabla v_0 \otimes \nabla v_0 + \text{sym } \nabla w_0\right) > c_0 \text{Id}_2 \text{ in } \bar{\Omega}. \tag{2-1}$$

Then there exist sequences $v_n \in \mathcal{C}^1(\bar{\Omega})$ and $w_n \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ which converge uniformly to v_0 and w_0 respectively, and which satisfy

$$A_0 = \frac{1}{2} \nabla v_n \otimes \nabla v_n + \text{sym } \nabla w_n \text{ in } \bar{\Omega}. \tag{2-2}$$

We start with a series of preliminary lemmas whose details we provide for the sake of completeness. The first is an observation in convex integration, pertaining to solving an appropriate differential inclusion to be used for constructing the one-dimensional oscillatory perturbations in v_n and w_n . As always, $C > 0$ is a universal constant, independent of all parameters, in particular independent of the function a below.

Lemma 2.2. *Let $a \in \mathcal{C}^\infty(\bar{\Omega})$ be a nonnegative function on an open and bounded set $\Omega \subset \mathbb{R}^2$. There exists a smooth 1-periodic field $\Gamma = (\Gamma_1, \Gamma_2) \in \mathcal{C}^\infty(\bar{\Omega} \times \mathbb{R}, \mathbb{R}^2)$ such that the following holds for all $(x, t) \in \bar{\Omega} \times \mathbb{R}$:*

$$\begin{aligned} \Gamma(x, t + 1) &= \Gamma(x, t), \\ \frac{1}{2} |\partial_t \Gamma_1(x, t)|^2 + \partial_t \Gamma_2(x, t) &= a(x)^2, \end{aligned} \tag{2-3}$$

together with the uniform bounds

$$\begin{aligned} |\Gamma_1(x, t)| + |\partial_t \Gamma_1(x, t)| &\leq Ca(x), & |\nabla_x \Gamma_1(x, t)| &\leq C|\nabla a(x)|, \\ |\Gamma_2(x, t)| + |\partial_t \Gamma_2(x, t)| &\leq Ca(x)^2, & |\nabla_x \Gamma_2(x, t)| &\leq C|a(x)||\nabla a(x)|. \end{aligned} \tag{2-4}$$

Proof. Firstly, note that there exists a smooth 1-periodic function $\gamma \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2)$ such that for all $t \in \mathbb{R}$,

$$\gamma(t + 1) = \gamma(t), \quad \int_0^1 \gamma(t) dt = (0, 0), \quad \gamma(t) \in P := \left\{ (s_1, s_2) \in \mathbb{R}^2 : \frac{1}{2} s_1^2 + s_2 = 1, |s_1| \leq 2 \right\}.$$

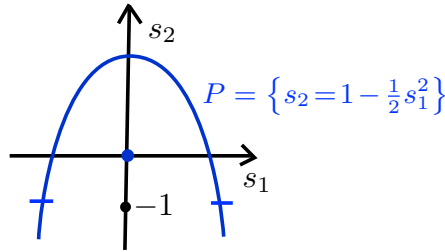


Figure 1. The parabola P in the one-dimensional convex integration problem of Lemma 2.2.

The existence of γ is a consequence of the fundamental lemma of convex integration, since the intended average $(0, 0)$ lies in the convex hull of the parabola P (see Figure 1). Indeed, one can take

$$\gamma(t) = (2 \cos(2\pi t), -\cos(4\pi t)) \in P.$$

It is now enough to ensure that $\partial_t \Gamma_1 = a(x)\gamma_1(x)$ and $\partial_t \Gamma_2 = a(x)^2\gamma_2(x)$ to obtain (2-3). Namely

$$\Gamma_1(x, t) = \frac{a(x)}{\pi} \sin(2\pi t), \quad \Gamma_2(x, t) = -\frac{a(x)^2}{4\pi} \sin(4\pi t).$$

We see directly that the bounds in (2-4) hold. □

To compare with the problem of isometric immersions, note that in that context, a one-dimensional convex integration lemma is similarly proved in [Székelyhidi 2013, Figure 2, p. 11], where instead of a parabola, the constraint set consists of a full circle.

We will also need a special case of [Conti et al. 2012, Lemma 3] about decomposition of positive definite symmetric matrices into rank-one matrices.

Lemma 2.3. *There exists a sufficiently small constant $r_0 > 0$ such that the following holds. For every positive definite symmetric matrix $G_0 \in \mathbb{R}_{\text{sym}, >}^{2 \times 2}$, there are three unit vectors $\{\xi_k \in \mathbb{R}^2\}_{k=1}^3$ and three linear functions $\{\Phi_k : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}\}_{k=1}^3$ such that for any $G \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ we have*

$$\forall G \in \mathbb{R}_{\text{sym}}^{2 \times 2}, \quad G = \sum_{k=1}^3 \Phi_k(G) \xi_k \otimes \xi_k, \tag{2-5}$$

and each Φ_k is strictly positive on the ball $B(G_0, r(G_0)) \subset \mathbb{R}_{\text{sym}}^{2 \times 2}$ with radius $r(G_0) = r_0/|G_0^{-1/2}|^2$.

Proof. (1) First, assume that $G_0 = \text{Id}_2$. Set

$$\zeta_1 = \frac{1}{\sqrt{12}}(2 + \sqrt{2}, -2 + \sqrt{2}), \quad \zeta_2 = \frac{1}{\sqrt{12}}(-2 + \sqrt{2}, 2 + \sqrt{2}), \quad \zeta_3 = \frac{1}{\sqrt{2}}(1, 1).$$

In order to check that the matrices

$$\zeta_1 \otimes \zeta_1 = \frac{1}{12} \begin{bmatrix} 6+4\sqrt{2} & -2 \\ -2 & 6-4\sqrt{2} \end{bmatrix}, \quad \zeta_2 \otimes \zeta_2 = \frac{1}{12} \begin{bmatrix} 6-4\sqrt{2} & -2 \\ -2 & 6+4\sqrt{2} \end{bmatrix}, \quad \zeta_3 \otimes \zeta_3 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

form a basis of the three-dimensional space $\mathbb{R}_{\text{sym}}^{2 \times 2}$, we validate that

$$\det \left(\frac{1}{12} \begin{bmatrix} 6+4\sqrt{2} & 6-4\sqrt{2} & 6 \\ -2 & -2 & 6 \\ 6-4\sqrt{2} & 6+4\sqrt{2} & 6 \end{bmatrix} \right) \neq 0.$$

Consequently, there exist linear mappings $\{\Psi_k : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}\}_{k=1}^3$ yielding the unique decomposition

$$\forall G \in \mathbb{R}_{\text{sym}}^{2 \times 2}, \quad G = \sum_{k=1}^3 \Psi_k(G) \zeta_k \otimes \zeta_k. \quad (2-6)$$

Now, since $\text{Id}_2 = \frac{3}{4}\zeta_1 \otimes \zeta_1 + \frac{3}{4}\zeta_2 \otimes \zeta_2 + \frac{1}{2}\zeta_3 \otimes \zeta_3$, the continuity of each function Ψ_k implies its positivity in a neighbourhood of Id_2 of some appropriate radius r_0 .

(2) For an arbitrary $G_0 \in \mathbb{R}_{\text{sym}, >}^{2 \times 2}$ we set

$$\forall k = 1, \dots, 3, \quad \xi_k = \frac{1}{|G_0^{1/2} \zeta_k|} G_0^{1/2} \zeta_k \quad \text{and} \quad \Phi_k(G) = |G_0^{1/2} \zeta_k|^2 \Psi_k(G_0^{-1/2} G G_0^{-1/2}).$$

Then, in view of (2-6) we obtain (2-5):

$$\forall G \in \mathbb{R}_{\text{sym}}^{2 \times 2}, \quad G = G_0^{-1/2} \left(\sum_{k=1}^3 \Psi_k(G_0^{-1/2} G G_0^{-1/2}) \zeta_k \otimes \zeta_k \right) G_0^{1/2} = \sum_{k=1}^3 \Phi_k(G) \xi_k \otimes \xi_k.$$

Finally, if $|G - G_0| < r(G_0)$ then $|G_0^{-1/2} G G_0^{-1/2} - \text{Id}_2| \leq |G_0^{-1/2}|^2 |G - G_0| < r_0$, and so indeed $\Phi_k(G) > 0$, since $\Psi_k(G_0^{-1/2} G G_0^{-1/2}) > 0$. \square

The above result can be localized in the following manner, similar to [Székelyhidi 2013, Lemma 3.3]:

Lemma 2.4. *There exist sequences of unit vectors $\{\eta_k \in \mathbb{R}^2\}_{k=1}^\infty$ and nonnegative smooth functions $\{\phi_k \in C_c^\infty(\mathbb{R}_{\text{sym}, >}^{2 \times 2})\}_{k=1}^\infty$ such that*

$$\forall G \in \mathbb{R}_{\text{sym}, >}^{2 \times 2}, \quad G = \sum_{k=1}^\infty \phi_k(G)^2 \eta_k \otimes \eta_k \quad (2-7)$$

and such that:

- (i) For all $G \in \mathbb{R}_{\text{sym}, >}^{2 \times 2}$, at most N_0 terms of the sum in (2-7) are nonzero. The constant N_0 is independent of G .
- (ii) For every compact $K \subset \mathbb{R}_{\text{sym}, >}^{2 \times 2}$, there exists a finite set of indices $J(K) \subset \mathbb{N}$ such that $\phi_k(G) = 0$ for all $k \notin J(K)$ and $G \in K$.

Proof. (1) Let r_0 be as in Lemma 2.3 and additionally ensure that

$$r_0 < \frac{1}{8}. \quad (2-8)$$

Recall that for each $G \in \mathbb{R}_{\text{sym}, >}^{2 \times 2}$ we have defined $r(G) = r_0/|G^{-1/2}|^2$ and that $B(G, r(G)) \subset \mathbb{R}_{\text{sym}, >}^{2 \times 2}$. We first construct a locally finite covering of $\mathbb{R}_{\text{sym}, >}^{2 \times 2}$ with properties corresponding to (i) and (ii).

Since the set $\mathbb{R}_{\text{sym}, >}^{2 \times 2}$ is a cone, we have

$$\mathbb{R}_{\text{sym}, >}^{2 \times 2} = \bigcup_{k \in \mathbb{Z}} 2^k \mathcal{C}_0, \quad \text{where } \mathcal{C}_0 = \{G \in \mathbb{R}_{\text{sym}, >}^{2 \times 2} : \frac{1}{2} \leq |G| \leq 1\}. \quad (2-9)$$

The collection $\{B(G, r(G))\}_{G \in \mathcal{C}_0}$ covers the sector \mathcal{C}_0 by balls that have uniformly bounded radii $r(G) \leq r_0|G|/\sqrt{2} \leq r_0$. Hence, by the Besicovitch covering theorem, it has a countable subcovering $\mathcal{G}_0 = \bigcup_{\sigma=1}^{\sigma_0} \mathcal{G}_0^\sigma$, consisting of $\sigma_0 \in \mathbb{N}$ countable families $\{\mathcal{G}_0^\sigma\}_{\sigma=1}^{\sigma_0}$ of pairwise disjoint balls.

Note that for all $c > 0$ one has $r(cG) = cr(G)$ and so $B(cG, r(cG)) = cB(G, r(G))$. Consequently, the collections $\mathcal{G}_k^\sigma = \{2^k B : B \in \mathcal{G}_0^\sigma\}$ each consist of countably many pairwise disjoint balls, and $\mathcal{G}_k = \bigcup_{\sigma=1}^{\sigma_0} \mathcal{G}_k^\sigma$ is a covering of the dilated sector $2^k \mathcal{C}_0$ for every $k \in \mathbb{Z}$. Define

$$\forall \sigma = 1, \dots, \sigma_0, \quad \mathcal{G}_{\text{even}}^\sigma = \bigcup_{2|k} \mathcal{G}_k^\sigma \quad \text{and} \quad \mathcal{G}_{\text{odd}}^\sigma = \bigcup_{2 \nmid (k+1)} \mathcal{G}_k^\sigma. \tag{2-10}$$

Clearly, in view of (2-9), the $2\sigma_0$ families in (2-10) form a covering of $\mathbb{R}_{\text{sym}, >}^{2 \times 2}$, namely

$$\mathcal{G} = \bigcup_{\sigma=1}^{\sigma_0} \mathcal{G}_{\text{even}}^\sigma \cup \bigcup_{\sigma=1}^{\sigma_0} \mathcal{G}_{\text{odd}}^\sigma.$$

We now prove that each of the families in \mathcal{G} consists of pairwise disjoint balls. We argue by contradiction. Assume that

$$\exists G \in B(G_1, r(G_1)) \cap B(G_2, r(G_2)) \quad \text{for some } B(G_1, r(G_1)) \in \mathcal{G}_{2k_1}^\sigma, \quad B(G_2, r(G_2)) \in \mathcal{G}_{2k_2}^\sigma.$$

Without loss of generality we may take $k_1 = 0$ and $k_2 = k \geq 1$, so that

$$\frac{1}{2} \leq |G_1| \leq 1 \quad \text{and} \quad 2^{2k-1} \leq |G_2| \leq 2^{2k}.$$

This yields a contradiction with (2-8), in view of

$$\begin{aligned} 2^{2k-1} - 1 &\leq |G_2| - |G_1| \leq |G_2 - G_1| \leq |G_2 - G| + |G - G_1| \\ &\leq r(G_2) + r(G_1) = r_0 \left(\frac{1}{|G_2^{-1/2}|^2} + \frac{1}{|G_1^{-1/2}|^2} \right) \leq \frac{r_0}{\sqrt{2}} (|G_2| + |G_1|) \leq r_0(2^{2k} + 1). \end{aligned}$$

(2) Note that \mathcal{G} can be assumed locally finite, by paracompactness. We write $\mathcal{G} = \{B_i = B(G_i, r(G_i))\}_{i=1}^\infty$ and let $\{\theta_i \in C_c^\infty(B_i)\}_{i=1}^\infty$ be a partition of unity subordinated to \mathcal{G} . For each $i \in \mathbb{N}$, let $\{\xi_{k,G_i}\}_{k=1}^3$ and $\{\Phi_{k,G_i}\}_{k=1}^3$ be the unit vectors and the linear functions as in Lemma 2.3. Then

$$\forall G \in \mathbb{R}_{\text{sym}, >}^{2 \times 2}, \quad G = \sum_{i \in \mathbb{N}} \theta_i(G) G = \sum_{i \in \mathbb{N}} \sum_{k=1}^3 \theta_i(G) \Phi_{k,G_i}(G) \xi_{k,G_i} \otimes \xi_{k,G_i},$$

and we see that (2-7) holds by taking

$$\eta_{i,k} = \xi_{k,G_i} \quad \text{and} \quad \phi_{i,k} = (\theta_i \Phi_{k,G_i}).$$

Since $\text{supp } \phi_{i,k} \subset B_i$ and since each G belongs to at most $2\sigma_0$ balls B_i , we see that (i) holds with $N_0 = 6\sigma_0$. On the other hand, condition (ii) follows by the local finiteness of \mathcal{G} . □

3. The C^1 approximations: a proof of Theorem 2.1

The first result in the approximating sequence construction is what corresponds to a ‘‘step’’ in the terminology of Nash and Kuiper.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set. Given are functions $v \in C^\infty(\bar{\Omega})$ and $w \in C^\infty(\bar{\Omega}, \mathbb{R}^2)$, a nonnegative function $a \in C^\infty(\bar{\Omega})$, and a unit vector $\eta \in \mathbb{R}^2$. Then, for every $\lambda > 1$ there exist approximations $\tilde{v}_\lambda \in C^\infty(\bar{\Omega})$ and $\tilde{w}_\lambda \in C^\infty(\bar{\Omega}, \mathbb{R}^2)$ satisfying the bounds*

$$\begin{aligned} \left\| \left(\frac{1}{2} \nabla \tilde{v}_\lambda \otimes \nabla \tilde{v}_\lambda + \text{sym} \nabla \tilde{w}_\lambda \right) - \left(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym} \nabla w + a^2 \eta \otimes \eta \right) \right\|_0 \\ \leq \frac{C}{\lambda} \|a\|_0 (\|\nabla a\|_0 + \|\nabla^2 v\|_0) + \frac{C}{\lambda^2} \|\nabla a\|_0^2, \end{aligned} \quad (3-1)$$

$$\|\tilde{v}_\lambda - v\|_0 \leq \frac{C}{\lambda} \|a\|_0 \quad \text{and} \quad \|\tilde{w}_\lambda - w\|_0 \leq \frac{C}{\lambda} \|a\|_0 (\|a\|_0 + \|\nabla v\|_0), \quad (3-2)$$

and for all $x \in \bar{\Omega}$,

$$|\nabla \tilde{v}_\lambda(x) - \nabla v(x)| \leq C a(x) + \frac{C}{\lambda} \|\nabla a\|_0, \quad (3-3)$$

$$|\nabla \tilde{w}_\lambda(x) - \nabla w(x)| \leq C a(x) (\|a\|_0 + \|\nabla v\|_0) + \frac{C}{\lambda} (\|a\|_0 (\|\nabla a\|_0 + \|\nabla^2 v\|_0) + \|\nabla a\|_0 \|\nabla v\|_0).$$

Proof. Using the 1-periodic functions Γ_i from Lemma 2.2, we define \tilde{v}_λ and \tilde{w}_λ as λ -periodic perturbations of v , w in the direction η :

$$\begin{aligned} \tilde{v}_\lambda(x) &= v(x) + \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta), \\ \tilde{w}_\lambda(x) &= w(x) - \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta) \nabla v(x) + \frac{1}{\lambda} \Gamma_2(x, \lambda x \cdot \eta) \eta. \end{aligned} \quad (3-4)$$

The error estimates in (3-2) follow immediately from (2-4). The pointwise error estimates (3-3) follow from (2-4) in view of

$$\begin{aligned} \nabla \tilde{v}_\lambda(x) &= \nabla v(x) + \frac{1}{\lambda} \nabla_x \Gamma_1(x, \lambda x \cdot \eta) + \partial_t \Gamma_1(x, \lambda x \cdot \eta) \eta, \\ \nabla \tilde{w}_\lambda(x) &= \nabla w(x) - \frac{1}{\lambda} \nabla v(x) \otimes \nabla_x \Gamma_1(x, \lambda x \cdot \eta) - \partial_t \Gamma_1(x, \lambda x \cdot \eta) \eta \otimes \nabla v(x) - \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta) \nabla^2 v(x) \\ &\quad + \frac{1}{\lambda} \eta \otimes \nabla_x \Gamma_2(x, \lambda x \cdot \eta) + \partial_t \Gamma_2(x, \lambda x \cdot \eta) \eta \otimes \eta. \end{aligned}$$

Finally, we compute

$$\begin{aligned} &\frac{1}{2} \nabla \tilde{v}_\lambda(x) \otimes \nabla \tilde{v}_\lambda(x) - \frac{1}{2} \nabla v(x) \otimes \nabla v(x) \\ &= \boxed{\frac{1}{\lambda} \text{sym}(\nabla v(x) \otimes \nabla_x \Gamma_1(x, \lambda x \cdot \eta)) + \partial_t \Gamma_1(x, \lambda x \cdot \eta) \text{sym}(\nabla v(x) \otimes \eta)} + \boxed{\frac{1}{2} |\partial_t \Gamma_1(x, \lambda x \cdot \eta)|^2 \eta \otimes \eta} \\ &\quad + \frac{1}{\lambda} \partial_t \Gamma_1(x, \lambda x \cdot \eta) \text{sym}(\eta \otimes \nabla_x \Gamma_1(x, \lambda x \cdot \eta)) + \frac{1}{2\lambda^2} \nabla_x \Gamma_1(x, \lambda x \cdot \eta) \otimes \nabla_x \Gamma_1(x, \lambda x \cdot \eta), \end{aligned}$$

and

$$\begin{aligned} \text{sym} \nabla \tilde{w}_\lambda(x) - \text{sym} \nabla w(x) &= \boxed{-\frac{1}{\lambda} \text{sym}(\nabla v(x) \otimes \nabla_x \Gamma_1(x, \lambda x \cdot \eta)) - \partial_t \Gamma_1(x, \lambda x \cdot \eta) \text{sym}(\nabla v(x) \otimes \eta)} \\ &\quad - \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta) \nabla^2 v(x) + \frac{1}{\lambda} \text{sym}(\eta \otimes \nabla_x \Gamma_2(x, \lambda x \cdot \eta)) + \boxed{\partial_t \Gamma_2(x, \lambda x \cdot \eta) \eta \otimes \eta}. \end{aligned}$$

We see that the terms in boxes cancel out, while the terms in double boxes add up to $a(x)^2\eta \otimes \eta$ by (2-3). Consequently,

$$\begin{aligned} & \left(\frac{1}{2}\nabla\tilde{v}_\lambda(x)\otimes\nabla\tilde{v}_\lambda(x)+\text{sym}\nabla\tilde{w}_\lambda(x)\right)-\left(\frac{1}{2}\nabla v(x)\otimes\nabla v(x)+\text{sym}\nabla w(x)+a(x)^2\eta\otimes\eta\right) \\ &= \frac{1}{\lambda}\left(\partial_t\Gamma_1(x,\lambda x\cdot\eta)\text{sym}(\eta\otimes\nabla_x\Gamma_1(x,\lambda x\cdot\eta))-\Gamma_1(x,\lambda x\cdot\eta)\nabla^2v(x)+\text{sym}(\eta\otimes\nabla_x\Gamma_2(x,\lambda x\cdot\eta))\right) \\ & \quad +\frac{1}{2\lambda^2}\nabla_x\Gamma_1(x,\lambda x\cdot\eta)\otimes\nabla_x\Gamma_1(x,\lambda x\cdot\eta), \end{aligned}$$

which implies (3-1) in view of the bounds in (2-4). □

We now complete the “stage” in the approximating sequence construction.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $v \in C^\infty(\bar{\Omega})$, $w \in C^\infty(\bar{\Omega}, \mathbb{R}^2)$ and $A \in C^\infty(\bar{\Omega}, \mathbb{R}_{\text{sym}}^{2 \times 2})$ be such that the deficit function \mathcal{D} defined below is positive definite in $\bar{\Omega}$:*

$$\exists c > 0 \quad \text{such that} \quad \mathcal{D} = A - \left(\frac{1}{2}\nabla v \otimes \nabla v + \text{sym} \nabla w\right) > c \text{Id}_2 \quad \text{in } \bar{\Omega}. \tag{3-5}$$

Fix $\varepsilon > 0$. Then there exist $\tilde{v} \in C^\infty(\bar{\Omega})$ and $\tilde{w} \in C^\infty(\bar{\Omega}, \mathbb{R}^2)$ such that the new deficit $\tilde{\mathcal{D}}$ is still positive definite, and bounded by ε together with the error in the approximations \tilde{v}, \tilde{w} ; namely,

$$\exists \tilde{c} > 0 \quad \text{such that} \quad \tilde{\mathcal{D}} = A - \left(\frac{1}{2}\nabla\tilde{v} \otimes \nabla\tilde{v} + \text{sym} \nabla\tilde{w}\right) > \tilde{c} \text{Id}_2 \quad \text{in } \bar{\Omega}, \tag{3-6}$$

$$\|\tilde{\mathcal{D}}\|_0 < \varepsilon \quad \text{and} \quad \|\tilde{v} - v\|_0 + \|\tilde{w} - w\|_0 < \varepsilon. \tag{3-7}$$

Moreover, we have the uniform gradient error bounds

$$\begin{aligned} \|\nabla\tilde{v} - \nabla v\|_0 &\leq CN_0^{1/2}\|\mathcal{D}\|_0^{1/2} \\ \|\nabla\tilde{w} - \nabla w\|_0 &\leq CN_0(\|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2})\|\mathcal{D}\|_0^{1/2}, \end{aligned} \tag{3-8}$$

where the constant $N_0 \in \mathbb{N}$ is as in Lemma 2.4.

Proof. (1) Note that the image $\mathcal{D}(\bar{\Omega})$ is a compact subset of $\mathbb{R}_{\text{sym},>}^{2 \times 2}$. By Lemma 2.4 and rearranging the indices, if needed, so that $J(\mathcal{D}(\bar{\Omega})) = \{1, \dots, N\}$ in (ii), we get

$$\forall x \in \bar{\Omega}, \quad \mathcal{D}(x) = \sum_{k=1}^N b_k(x)^2 \eta_k \otimes \eta_k, \quad \text{where } b_k = \phi_k \circ \mathcal{D} \in C^\infty(\bar{\Omega}). \tag{3-9}$$

Let now $a_k = (1 - \delta)^{1/2}b_k$, with $\delta > 0$ so small that

$$\mathcal{D} - \sum_{k=1}^N a_k^2 \eta_k \otimes \eta_k = \delta \mathcal{D} \quad \text{and} \quad \delta \|\mathcal{D}\|_0 < \frac{1}{2}\varepsilon. \tag{3-10}$$

We set $v_1 = v$, $w_1 = w$. For $k = 1, \dots, N$ we inductively define $v_{k+1} \in C^\infty(\bar{\Omega})$ and $w_{k+1} \in C^\infty(\bar{\Omega}, \mathbb{R}^2)$, by means of Proposition 3.1 applied to v_k, w_k, a_k, η_k and with $\lambda_k > 1$ sufficiently large, as indicated below. We then finally set $\tilde{v} = v_{N+1}$ and $\tilde{w} = w_{N+1}$.

(2) To prove the estimates (3-6)–(3-8), we start by observing that since by Lemma 2.4(i) at most N_0 terms in the expansion (3-9) are nonzero, we have

$$\begin{aligned} \sum_{k=1}^N a_k(x) &\leq \sum_{k=1}^N b_k(x) \leq N_0^{1/2} \left(\sum_{k=1}^N b_k(x)^2 \right)^{1/2} = N_0^{1/2} (\text{Trace } \mathcal{D}(x))^{1/2} \\ &\leq N_0^{1/2} (\sqrt{2} |\mathcal{D}(x)|)^{1/2} \leq CN_0^{1/2} \|\mathcal{D}\|_0^{1/2}. \end{aligned} \quad (3-11)$$

Further, by (3-1) and (3-10),

$$\begin{aligned} \tilde{\mathcal{D}} &= \mathcal{D} - \left(\left(\frac{1}{2} \nabla \tilde{v} \otimes \nabla \tilde{v} + \text{sym } \nabla \tilde{w} \right) - \left(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym } \nabla w \right) \right) \\ &= \mathcal{D} - \sum_{k=1}^N \left(\left(\frac{1}{2} \nabla v_{k+1} \otimes \nabla v_{k+1} + \text{sym } \nabla w_{k+1} \right) - \left(\frac{1}{2} \nabla v_k \otimes \nabla v_k + \text{sym } \nabla w_k \right) \right) \\ &= \left(\mathcal{D} - \sum_{k=1}^N a_k^2 \eta_k \otimes \eta_k \right) - \sum_{k=1}^N \left(\left(\frac{1}{2} \nabla v_{k+1} \otimes \nabla v_{k+1} + \text{sym } \nabla w_{k+1} \right) - \left(\frac{1}{2} \nabla v_k \otimes \nabla v_k + \text{sym } \nabla w_k + a_k^2 \eta_k \otimes \eta_k \right) \right) \\ &= \delta \mathcal{D} + \sum_{k=1}^N \mathcal{O} \left(\frac{1}{\lambda_k} (\|a_k\|_0 \|\nabla a_k\|_0 + \|\nabla a_k\|_0^2 + \|a_k\|_0 \|\nabla^2 v_k\|_0) \right). \end{aligned}$$

Choosing at each step λ_k sufficiently large with respect to the given a_k and the already generated v_k , we may ensure the smallness of the error term in the right-hand side above and hence the positive definiteness of $\tilde{\mathcal{D}}$ in (3-6), because of the uniform positive definiteness of $\delta \mathcal{D} > c\delta \text{Id}_2$ in $\bar{\Omega}$. Likewise, the first inequality in (3-7) follows already when the error is smaller than $\frac{1}{2}\varepsilon$.

The same reasoning proves the error bounds on $\tilde{v} - v$ and $\tilde{w} - w$ in (3-7), in view of (3-2):

$$\begin{aligned} \tilde{v}(x) - v(x) &= \sum_{k=1}^N (v_{k+1}(x) - v_k(x)) = \sum_{k=1}^N \mathcal{O} \left(\frac{1}{\lambda_k} \|a_k\|_0 \right), \\ \tilde{w}(x) - w(x) &= \sum_{k=1}^N (w_{k+1}(x) - w_k(x)) = \sum_{k=1}^N \mathcal{O} \left(\frac{1}{\lambda_k} (\|a_k\|_0^2 + \|\nabla a_k\|_0 \|\nabla v_k\|_0) \right). \end{aligned}$$

(3) To obtain the first error bound in (3-8), use (3-3) and (3-11):

$$|\nabla \tilde{v}(x) - \nabla v(x)| \leq \sum_{k=1}^N |\nabla v_{k+1}(x) - \nabla v_k(x)| \leq C \sum_{k=1}^N a_k(x) + \sum_{k=1}^N \mathcal{O} \left(\frac{1}{\lambda_k} \|a_k\|_0^2 \right) \leq CN_0^{1/2} \|\mathcal{D}\|_0^{1/2},$$

where again, by adjusting λ_k at each step, we ensure the controllability of the error term with respect to the nonnegative quantity $N_0^{1/2} \|\mathcal{D}\|_0^{1/2}$. Likewise,

$$\forall k = 1, \dots, N, \quad |\nabla v_k(x)| \leq |\nabla v(x)| + \sum_{i=1}^{k-1} |\nabla v_{i+1}(x) - \nabla v_i(x)| \leq \|\nabla v\|_0 + CN_0^{1/2} \|\mathcal{D}\|_0^{1/2},$$

and obviously by (3-11),

$$a_k(x) \leq \sum_{i=1}^{k-1} a_i(x) \leq CN_0^{1/2} \|\mathcal{D}\|_0^{1/2},$$

which by (3-11) yield

$$\sum_{k=1}^N a_k(x)(\|a_k\|_0 + \|\nabla v_k\|_0) \leq C(\|\nabla v\|_0 + N_0^{1/2}\|\mathcal{D}\|_0^{1/2}) \sum_{k=1}^N a_k(x) \leq CN_0(\|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2})\|\mathcal{D}\|_0^{1/2}.$$

Consequently and by (3-3), we get the last gradient error bound in (3-8):

$$\begin{aligned} & |\nabla \tilde{w}(x) - \nabla w(x)| \\ & \leq \sum_{k=1}^N |\nabla w_{k+1}(x) - \nabla w_k(x)| \\ & \leq C \sum_{k=1}^N a_k(x)(\|a_k\|_0 + \|\nabla v_k\|_0) + \sum_{k=1}^N \mathcal{O}\left(\frac{1}{\lambda_k}(\|a_k\|_0 \|\nabla a_k\|_0 + \|a_k\|_0 \|\nabla^2 v_k\|_0 + \|\nabla a_k\|_0 \|\nabla v_k\|_0)\right) \\ & \leq CN_0(\|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2})\|\mathcal{D}\|_0^{1/2}. \end{aligned}$$

This concludes the proof of the stage approximation construction. □

We now finally give:

Proof of Theorem 2.1. (1) Fix $\varepsilon > 0$. It suffices to construct $v \in C^1(\bar{\Omega})$ and $w \in C^1(\bar{\Omega}, \mathbb{R}^2)$ such that

$$A_0 = \frac{1}{2} \nabla v \otimes \nabla v + \text{sym } \nabla w \quad \text{in } \bar{\Omega} \tag{3-12}$$

and

$$\|v - v_0\|_0 + \|w - w_0\|_0 < \varepsilon. \tag{3-13}$$

The exact solution (v, w) of (3-12) will be obtained as the C^1 limit of sequences of successive approximations $\{v_k \in C^\infty(\bar{\Omega}), w_k \in C^\infty(\bar{\Omega}, \mathbb{R}^2)\}_{k=0}^\infty$, where v_0 and w_0 are given in the statement of the theorem and satisfy (2-1), while v_{k+1} and w_{k+1} are defined inductively by means of Proposition 3.2 applied to v_k, w_k and $\varepsilon_k > 0$, under the requirement

$$\sum_{k=1}^\infty \varepsilon_k < \varepsilon \quad \text{and} \quad \sum_{k=1}^\infty \varepsilon_k^{1/2} < 1. \tag{3-14}$$

In agreement with our notation convention, we introduce the k -th deficit \mathcal{D}_k , which is positive definite by (3-6):

$$\forall k \geq 0, \quad \mathcal{D}_k := A_0 - \left(\frac{1}{2} \nabla v_k \otimes \nabla v_k + \text{sym } \nabla w_k\right) \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}_{\text{sym}, >}^{2 \times 2}).$$

By (3-7) it follows that

$$\|v_k - v\|_0 + \|w_k - w\|_0 \leq \sum_{i=0}^{k-1} \|v_{i+1} - v_i\|_0 + \sum_{i=0}^{k-1} \|w_{i+1} - w_i\|_0 < \sum_{i=1}^{k-1} \varepsilon_i < \sum_{i=1}^\infty \varepsilon_i.$$

Thus, $\{v_k\}_{k=0}^\infty$ and $\{w_k\}_{k=0}^\infty$ converge uniformly in $\bar{\Omega}$, respectively, to v and w which satisfy (3-13) in view of (3-14).

(2) We now show that this convergence is in C^1 . Indeed, by (3-7) $\|D_k\|_0 < \varepsilon_k$, so by (3-8)

$$\|\nabla v_{k+m} - \nabla v_k\|_0 \leq \sum_{i=k}^{m-1} \|\nabla v_{i+1} - \nabla v_i\|_0 \leq CN_0^{1/2} \sum_{i=k}^{m-1} \|D_i\|_0^{1/2} \leq CN_0^{1/2} \sum_{i=k}^{m-1} \varepsilon_i^{1/2}. \tag{3-15}$$

In particular, in view of (3-14) the sequence $\{\|\nabla v_k\|_0\}_{k=0}^\infty$ is bounded, so we further have

$$\|\nabla w_{k+m} - \nabla w_k\|_0 \leq \sum_{i=k}^{m-1} \|\nabla w_{i+1} - \nabla w_i\|_0 \leq CN_0 \sum_{i=k}^{m-1} (\|\nabla v_i\|_0 + \|D_i\|_0^{1/2}) \|D_i\|_0^{1/2} \leq \tilde{C} N_0 \sum_{i=k}^{m-1} \varepsilon_i^{1/2}, \tag{3-16}$$

where the constant \tilde{C} is independent of k and m . Through the above assertions (3-15) and (3-16), in view of the second condition in (3-14), we conclude that $\{v_k\}_{k=1}^\infty$ and $\{w_k\}_{k=0}^\infty$ are Cauchy sequences that converge in $C^1(\bar{\Omega})$ to $v \in C^1(\bar{\Omega})$ and $w \in C^1(\bar{\Omega}, \mathbb{R}^2)$, respectively. Finally,

$$\|A_0 - (\frac{1}{2}\nabla v \otimes \nabla v + \text{sym } \nabla w)\|_0 = \lim_{k \rightarrow \infty} \|D_k\|_0 \leq \lim_{k \rightarrow \infty} \varepsilon_k = 0$$

implies (3-12) and completes the proof of Theorem 2.1. □

Remark 3.3. In addition to the uniform convergence postulated in Theorem 2.1, one also has

$$\forall n, \quad \|\nabla v_n\|_0 \leq \|\nabla v_0\|_0 + CN_0^{1/2}.$$

Using notation as in the proof above and recalling (3-15) and (3-14), this bound follows by

$$\|\nabla v - \nabla v_0\|_0 = \lim_{k \rightarrow \infty} \|\nabla v_k - \nabla v_0\|_0 \leq \lim_{k \rightarrow \infty} \left(CN_0^{1/2} \sum_{i=0}^{k-1} \varepsilon_i^{1/2} \right) \leq CN_0^{1/2}.$$

4. The $C^{1,\alpha}$ approximations: a proof of Theorem 1.1, preliminary results and some heuristics towards the proof of Theorem 1.2

Theorem 1.1 follows easily from Theorem 1.2, which will be proved in the next section.

Proof of Theorem 1.1. Since $C^1(\bar{\Omega})$ is dense in $C^0(\bar{\Omega})$, we may without loss of generality assume that $v_0 \in C^1(\bar{\Omega})$. Set $w_0 = 0$ and $A_0 = (\lambda + c)\text{Id} \in C^{0,\beta}(\bar{\Omega}, \mathbb{R}_{\text{sym}}^{2 \times 2})$, where c is a constant and λ is constructed as follows.

Extend the function f to $f \in L^p(\Omega_\varepsilon)$ defined on an open smooth set $\Omega_\varepsilon \supset \bar{\Omega}$ and solve

$$-\Delta \lambda = f \quad \text{in } \Omega_\varepsilon, \quad \lambda = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Since $\lambda \in W^{2,p}(\Omega_\varepsilon)$, Morrey’s theorem implies that $\lambda \in C^{0,\beta}(\bar{\Omega})$ for every $\beta \in (0, 1)$ when $p \geq 2$, and for $\beta = 2 - \frac{2}{p}$ when $p \in (1, 2)$. Also, for c large enough, condition (1-5) on the positive definiteness of the defect is satisfied. On the other hand,

$$-\text{curl curl } A_0 = -\Delta(\lambda + c) = f,$$

so the result follows directly from Theorem 1.2, since $\frac{1}{2}(2 - \frac{2}{p}) \geq \frac{1}{7}$ is equivalent to $p \geq \frac{7}{6}$. □

Our next simple corollary concerns the steady-state Euler equations with the exchanged roles of the given pressure q and the unknown forcing term $\nabla^\perp g$.

Corollary 4.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $q \in C^{0,\beta}(\bar{\Omega})$ for some $\beta \in (0, 1)$ and fix $\varepsilon > 0$. Then for every exponent α in the range $0 < \alpha < \min\{\frac{1}{7}, \frac{1}{2}\beta\}$, there exist sequences $\{u_n \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^2)\}_{n=1}^\infty$ and $\{g_n \in C^{0,\alpha}(\bar{\Omega})\}_{n=1}^\infty$ solving in Ω the system*

$$\operatorname{div}(u_n \otimes u_n) - \nabla q = \nabla^\perp g_n, \quad \operatorname{div} u_n = 0, \tag{4-1}$$

and such that $u_n = \nabla^\perp v_n$ and $g_n = \operatorname{curl} w_n$, where each $v_n \in C^{1,\alpha}(\bar{\Omega})$ and $w_n \in C^1(\bar{\Omega}, \mathbb{R}^2)$, while the sequence $\{v_n\}_{n=1}^\infty$ is dense in $C^0(\bar{\Omega})$ and $\|w_n\|_0 < \varepsilon$ for every $n \geq 1$.

Proof. As before, since $C^1(\bar{\Omega})$ is dense in $C^0(\bar{\Omega})$, it is enough to take $v_0 \in C^1(\bar{\Omega})$ and approximate it by a sequence $\{v_n \in C^{1,\alpha}(\bar{\Omega})\}_{n=1}^\infty$ with the properties as in the statement of the corollary. Let $w_0 = 0$ and let $c > 0$ be a sufficiently large constant, so that $(q + c)\operatorname{Id}_2 - \nabla v_0 \otimes \nabla v_0$ is strictly positive definite in $\bar{\Omega}$. By Theorem 1.2, there exist sequences $v_n \in C^{1,\alpha}(\bar{\Omega})$ and $w_n \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ which converge uniformly to v_0 and w_0 and which satisfy

$$(q + c)\operatorname{Id}_2 = \nabla v_n \otimes \nabla v_n + 2 \operatorname{sym} \nabla w_n \quad \text{in } \bar{\Omega}.$$

Taking the cofactor of both sides in the above matrix identity, we get

$$(q + c)\operatorname{Id}_2 = \nabla^\perp v_n \otimes \nabla^\perp v_n + 2 \operatorname{cof}(\operatorname{sym} \nabla w_n).$$

Taking the row-wise divergence, we obtain (4-1) with $u_n = \nabla^\perp v_n$ and $g_n = \operatorname{curl} w_n$, since $\operatorname{div} \operatorname{cof} \nabla w_n = 0$, while $(\operatorname{div} \operatorname{cof}(\nabla w_n)^T)^\perp = -\nabla(\operatorname{curl} w_n)$. □

Towards a proof of Theorem 1.2 we will derive a sequence of approximation results, and then combine them with Theorem 2.1 in Section 6. For completeness, we first prove a simple, useful result:

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Given are functions $f \in C^N(\bar{\Omega}, \mathbb{R}^n)$ and $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$. Then*

$$\forall k = 0, \dots, N, \quad \|\psi \circ f\|_k \leq M \|f\|_k,$$

where the constant $M > 0$ depends on the dimensions n, m , the differentiability order N , the domain Ω , the norm $\|\psi\|_N$ on the compact set $f(\bar{\Omega})$ and the norm $\|f\|_0$, but it does not depend on the higher norms of f .

Proof. The statement is obvious for $k = 0$. Fix $k \in \{1, \dots, N\}$ and let $m = (m_1, \dots, m_k)$ be any k -tuple of nonnegative integers such that $\sum_{i=1}^k m_i = k$. Defining $|m| = \sum_{i=1}^k m_i$ and using the interpolation inequality [Adams and Fournier 2003]

$$\forall i = 1, \dots, k, \quad \|f\|_i \leq M_0 \|f\|_0^{1-i/k} \|f\|_k^{i/k},$$

valid with a constant $M_0 > 0$ depending on n, N and Ω , we get

$$\prod_{i=1}^k \|\nabla^i f\|_0^{m_i} \leq M_0^{|m|} \prod_{i=1}^k \|f\|_0^{m_i - m_i/k} \|f\|_k^{m_i/k} = M_0^{|m|} \|f\|_0^{|m|-1} \|f\|_k,$$

with $|m| := m_1 + \dots + m_j$. Calculating the partial derivatives in $\nabla^k(\psi \circ f)$ by the Faà di Bruno formula gives hence the desired estimate

$$\|\nabla^k(\psi \circ f)\|_0 \leq M \sum_m \prod_{i=1}^k \|\nabla^i f\|_0^{m_i} \leq M \|f\|_k.$$

Above, the summation extends over all multiindices $m = (m_1, \dots, m_k)$ with the properties listed at the beginning of the proof. \square

We recall the following estimates which have been proved in [Conti et al. 2012]:

Lemma 4.3. *Let $\varphi \in C_c^\infty(B(0, 1), \mathbb{R})$ be a standard mollifier supported on the ball $B(0, 1) \subset \mathbb{R}^n$, that is, a nonnegative, smooth and radially symmetric function such that $\int_{\mathbb{R}^n} \varphi = 1$. Denote*

$$\forall l \in (0, 1), \quad \varphi_l(x) = \frac{1}{l^n} \varphi\left(\frac{x}{l}\right).$$

Then, for every $f, g \in C^0(\mathbb{R}^n)$ we have

$$\forall k, j \geq 0, \quad \|f * \varphi_l\|_{k+j} \leq \frac{C}{l^k} \|f\|_j, \quad (4-2)$$

$$\forall k \geq 0, \quad \|f * \varphi_l - f\|_k \leq \frac{C}{l^{k-2}} \|f\|_2, \quad (4-3)$$

$$\forall \alpha \in (0, 1], \quad \|f * \varphi_l - f\|_0 \leq C l^\alpha \|f\|_{0,\alpha}, \quad (4-4)$$

$$\forall \alpha \in (0, 1], \quad \|f * \varphi_l\|_1 \leq \frac{C}{l^{1-\alpha}} \|f\|_{0,\alpha}, \quad (4-5)$$

$$\forall k \geq 0, \forall \alpha \in (0, 1], \quad \|(fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)\|_k \leq \frac{C}{l^{k-2\alpha}} \|f\|_{0,\alpha} \|g\|_{0,\alpha}, \quad (4-6)$$

with the uniform constants $C > 0$ depending only on the smoothness exponents k, j, α .

Proof. The estimate (4-2) follows directly from the definition of convolution. To prove (4-3), note that for every $x \in \mathbb{R}^n$,

$$\begin{aligned} |\nabla^k(f * \varphi_l - f)(x)| &= \left| \int_{\mathbb{R}^n} \varphi_l(y) (\nabla^k f(x-y) - \nabla^k f(x)) dy \right| \\ &= \left| \int_{\mathbb{R}^n} \nabla^k \varphi_l(y) (f(x-y) - f(x)) dy \right| = \frac{1}{l^k} \left| \int_{\mathbb{R}^n} \frac{1}{l^n} \nabla^k \varphi\left(\frac{y}{l}\right) (\nabla f(x) \cdot y + r_x(y)) dy \right| \\ &= \frac{1}{l^k} \left| \int_{\mathbb{R}^n} \frac{1}{l^n} \nabla^k \varphi\left(\frac{y}{l}\right) r_x(y) dy \right| \leq \frac{C}{l^k} \sup_{x \in \mathbb{R}^n, |y| < l} |r_x(y)| \leq \frac{C}{l^{k-2}} \|f\|_2, \end{aligned}$$

where we integrated by parts, discarded the contribution with the symmetric term $\nabla f(x) \cdot y$, which integrates to 0, and estimated the Taylor's formula remainder term

$$r_x(y) = f(x-y) - f(x) - \nabla f(x) \cdot y = \|f\|_2 \mathcal{O}(|y|^2).$$

The proof of (4-4) follows similarly by

$$|\nabla^k(f * \varphi_l - f)(x)| = \left| \int_{\mathbb{R}^n} \varphi_l(y) |y|^\alpha \frac{f(x-y) - f(x)}{|y|^\alpha} dy \right| \leq Cl^\alpha \|f\|_{0,\alpha} \int_{\mathbb{R}^n} \varphi_l(y) dy \leq Cl^\alpha \|f\|_{0,\alpha},$$

while for (4-5) we write

$$\begin{aligned} |\nabla(f * \varphi_l)(x)| &= \left| \int_{\mathbb{R}^n} f(x-y) \frac{1}{l^{n+1}} \nabla \varphi_l\left(\frac{y}{l}\right) dy \right| = \frac{1}{l} \left| \int_{\mathbb{R}^n} \frac{f(x-y) - f(x)}{|y|^\alpha} \frac{|y|^\alpha}{l} \frac{1}{l^n} \nabla \varphi_l\left(\frac{y}{l}\right) dy \right| \\ &\leq Cl^{\alpha-1} \|f\|_{0,\alpha} \int_{\mathbb{R}^n} \frac{1}{l^n} \left| \nabla \varphi_l\left(\frac{y}{l}\right) \right| dy \leq \frac{C}{l^{1-\alpha}} \|f\|_{0,\alpha}. \end{aligned}$$

Finally, for the crucial commutator estimate (4-6) we refer to [Conti et al. 2012, Lemma 1]. \square

A heuristic overview of the next two sections.

Let us attempt to follow the construction in Sections 2 and 3, but with the goal of controlling the higher Hölder norms of the iterations, and hence also quantifying the growth of the \mathcal{C}^2 norms of v , w . Let $A \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}_{\text{sym}}^{2 \times 2})$ be the target matrix field and let $v_1 \in \mathcal{C}^\infty(\bar{\Omega})$, $w_1 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)$ be given at an input of a “stage”. As in Proposition 3.2, we decompose the defect $\mathcal{D} = A - (\frac{1}{2} \nabla v_1 \otimes \nabla v_1 + \text{sym} \nabla w_1)$ into a linear combination $\sum_{k=1}^N a_k^2 \eta_k \otimes \eta_k$ of rank-one symmetric matrices with smooth coefficients given by Lemma 2.4. We define

$$v_{k+1}(x) = v_k(x) + \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta_k), \quad w_{k+1}(x) = w_k(x) - \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta_k) \nabla v_k(x) + \frac{1}{\lambda} \Gamma_2(x, \lambda x \cdot \eta_k) \eta_k.$$

This yields, by applying Lemma 4.2 to $\psi(x) = x^2$ and $f = a_k$,

$$\forall m = 0, \dots, 3, \quad \|\nabla^m v_{k+1} - \nabla^m v_k\|_0 \leq C \sum_{\substack{i+j=m \\ 0 \leq i, j \leq m}} \|a_k\|_i \lambda^{j-1},$$

$$\forall m = 0, \dots, 2, \quad \|\nabla^m w_{k+1} - \nabla^m w_k\|_0 \leq C \sum_{\substack{i+j=m \\ 0 \leq i, j \leq m}} \|a_k\|_i \lambda^{j-1} + C \sum_{\substack{i+j+s=m \\ 0 \leq i, j, s \leq m}} \|a_k\|_i \lambda^{j-1} \|\nabla^{s+1} v_k\|_0,$$

On the other hand, applying Lemma 4.2 to $\psi = \phi_k$ defined in Lemma 2.4 and to $f = \mathcal{D}$, we get

$$\forall k = 1, \dots, N, \quad \|a_k\|_2 \leq C(\|v_1\|_3^2 + \|w_1\|_3 + \|A\|_2).$$

Now, in order to control the $\mathcal{C}^{1,\alpha}$ norm of v_{N+1} through interpolation, we need to control the norm $\|v_{N+1}\|_2$, which in turn depends on $\|a_k\|_2$. The above estimate shows that at the end of each stage, the \mathcal{C}^2 norm of a_k is determined by the \mathcal{C}^3 norms of the given v_1 and w_1 of the previous stage. Further, the \mathcal{C}^2 norm of w_{N+1} is only controlled by the \mathcal{C}^3 norm of v_0 and also of all the a_k . One might hope to control $\|a_k\|_3$ if the deficit \mathcal{D} is small enough, but the dependence of $\|w_{N+1}\|_2$ on $\|v_0\|_3$ cannot be easily bypassed. Recalling that we need infinitely many stages in the construction, this implies that a direct estimate cannot be obtained in this manner, unless we deal with analytic data similarly to [Borisov 2004]. We thus need to modify the previous simplistic approach.

The appropriate modification is achieved by introducing a mollification before each stage. This technique was first introduced in [Conti et al. 2012] for the isometric immersion problem, in order to control the loss of regularity through the stages and to improve on results in [Borisov 2004]. Indeed, we

note that the loss of derivatives in the above estimates is accompanied by a similar gain in the powers of λ , in a manner such that the total order of derivatives, plus the order of powers needed to control $\|v_{N+1}\|_2$ and $\|w_{N+1}\|_2$ is constant. If we replace v_1 and w_1 by their mollifications on the scale $l \sim \lambda^{-1}$, each derivative loss can be estimated by one power of λ , and $\|v_0\|_2$ and $\|w_0\|_2$ will control $\|v_{N+1}\|_2$ and $\|w_{N+1}\|_2$. One problem still remains to be taken care of: does the deficit \mathcal{D} decrease at the end of each stage? As the calculation below will show, a mollification of order λ^{-1} does not suffice to this end, and we need to mollify at a larger scale of $l > \lambda^{-1}$.

This is indeed how we want proceed. In practice, we let the mollification scale be $l = \delta/M$ and we treat ∇v “like a ”, controlling its j -th norm by δl^{-j} . We then “sacrifice” one l in order to gain one δ ; instead of $\|\nabla(v * \varphi_l)\|_j \leq C\|v\|_1 l^{-j}$, we use $\|\nabla(v * \varphi_l)\|_j \leq C(\|v\|_2 l) l^{-j}$, choosing l such that $l\|v\|_2 < \delta$ and obtaining the desired bound (5-2).

Finally, note that the loss of N powers of $\lambda l > 1$ in the control of the \mathcal{C}^2 norms at the end of each stage is the main reason why the described scheme does not deliver better than $\mathcal{C}^{1,1/7}$ estimates, even for the optimal $N = 3$ from the decomposition in Lemma 2.3.

5. The $\mathcal{C}^{1,\alpha}$ approximations: a “step” and a “stage” in a proof of Theorem 1.2

In this section, we develop the approximation technique that will be used for a proof of Theorem 1.2 in the next section. The first result is a variant of Proposition 3.1 in which we accomplish the “step” of the Nash–Kuiper construction with extra estimates on the higher derivatives.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set. Given are functions $v \in \mathcal{C}^3(\bar{\Omega})$, $w \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$, a nonnegative function $a \in \mathcal{C}^3(\bar{\Omega})$ and a unit vector $\eta \in \mathbb{R}^2$. Let $\delta, l \in (0, 1)$ be two parameter constants such that*

$$\|a\|_m \leq \frac{\delta}{l^m} \quad \forall m = 0, \dots, 3, \quad \text{and} \quad \|\nabla v\|_m \leq \frac{\delta}{l^m} \quad \forall m = 1, 2. \quad (5-1)$$

Then for every $\lambda > 1/l$ there exist approximating functions $\tilde{v}_\lambda \in \mathcal{C}^3(\bar{\Omega})$ and $\tilde{w}_\lambda \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ satisfying the following bounds, with a universal constant $C > 0$ independent of all parameters:

$$\left\| \left(\frac{1}{2} \nabla \tilde{v}_\lambda \otimes \nabla \tilde{v}_\lambda + \text{sym } \nabla \tilde{w}_\lambda \right) - \left(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym } \nabla w + a^2 \eta \otimes \eta \right) \right\|_0 \leq C \frac{\delta^2}{\lambda l}, \quad (5-2)$$

$$\|\tilde{v}_\lambda - v\|_m \leq C \delta \lambda^{m-1} \quad \forall m = 0, \dots, 3, \quad (5-3)$$

$$\|\tilde{w}_\lambda - w\|_m \leq C \delta \lambda^{m-1} (1 + \|\nabla v\|_0) \quad \forall m = 0, \dots, 2. \quad (5-4)$$

Proof. We define $\tilde{v}_\lambda, \tilde{w}_\lambda$ as in the proof of Proposition 3.1:

$$\tilde{v}_\lambda(x) = v(x) + \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta), \quad \tilde{w}_\lambda(x) = w(x) - \frac{1}{\lambda} \Gamma_1(x, \lambda x \cdot \eta) \nabla v(x) + \frac{1}{\lambda} \Gamma_2(x, \lambda x \cdot \eta).$$

Firstly, (5-2) follows immediately from (3-1) in view of (5-1), because $\lambda l > 1$:

$$\frac{1}{\lambda} \|a\|_0 (\|\nabla a\|_0 + \|\nabla^2 v\|_0) + \frac{1}{\lambda^2} \|\nabla a\|_0^2 \leq 2 \frac{\delta}{\lambda} \frac{\delta}{l} + \frac{1}{\lambda^2} \frac{\delta^2}{l^2} \leq 3 \frac{\delta^2}{\lambda l}.$$

To check (5-3), we compute directly as in Lemma 2.2:

$$\|\nabla^m(\tilde{v}_\lambda - v)\|_0 \leq \frac{C}{\lambda} \|\nabla^m \Gamma_1(x, \lambda x \cdot \eta)\|_0 \leq \frac{C}{\lambda} \sum_{\substack{i+j=m \\ 0 \leq i, j \leq m}} \|a\|_j \lambda^j \leq \frac{C}{\lambda} \sum_{i=0}^m \frac{\delta}{l^i} \lambda^{m-i} \leq C\delta\lambda^{m-1}$$

by (5-1) and noting again $\lambda l > 1$. Similarly,

$$\begin{aligned} \|\nabla^m(\tilde{w}_\lambda - w)\|_0 &\leq \frac{C}{\lambda} (\|\nabla^m \Gamma_2(x, \lambda x \cdot \eta)\|_0 + \|\nabla^m \Gamma_1(x, \lambda x \cdot \eta) \nabla v\|_0) \\ &\leq \frac{C}{\lambda} \left(\sum_{\substack{i+j=m \\ 0 \leq i, j \leq m}} \|a^2\|_i \lambda^j + \sum_{\substack{i+j+s=m \\ 0 \leq i, j, s \leq m}} \|a\|_i \lambda^j \|\nabla v\|_s \right) \\ &\leq \frac{C}{\lambda} \left(\sum_{i=1}^m \frac{\delta}{l^i} \lambda^{m-i} + \sum_{\substack{0 \leq i+s \leq m \\ 0 \leq i, s \leq m}} \frac{\delta}{l^i} \lambda^{m-(i+s)} \frac{\delta}{l^s} + \sum_{\substack{i+j=m \\ 0 \leq i, j \leq m}} \frac{\delta}{l^i} \lambda^j \|\nabla v\|_0 \right) \\ &\leq \frac{C}{\lambda} \left(\sum_{i=1}^m \frac{\delta}{l^i} \lambda^{m-i} \right) (1 + 1 + \|\nabla v\|_0) \leq C\delta\lambda^{m-1} (1 + \|\nabla v\|_0), \end{aligned}$$

where we applied Lemma 4.2 to $\psi(x) = x^2$ and $f = a$ in view of (5-1) yielding $\|a\|_0 \leq 1$, so that $\|a^2\|_i \leq C\|a\|_i \leq C\delta/l^i$. This achieves (5-4) and completes the proof of the proposition. \square

We now accomplish the ‘‘stage’’ in the Hölder regular approximation construction.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain. Let $v \in C^2(\bar{\Omega})$, $w \in C^2(\bar{\Omega}, \mathbb{R}^2)$ and $A \in C^{0,\beta}(\bar{\Omega}, \mathbb{R}_{\text{sym}}^{2 \times 2})$ for some $\beta \in (0, 1)$ be such that the deficit \mathcal{D} is appropriately small:*

$$\mathcal{D} = A - \left(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym} \nabla w\right), \quad 0 < \|\mathcal{D}\|_0 < \delta_0 \ll 1. \tag{5-5}$$

Then, for every two parameter constants M, σ satisfying

$$M > \max\{\|v\|_2, \|w\|_2, 1\} \quad \text{and} \quad \sigma > 1, \tag{5-6}$$

there exist $\tilde{v} \in C^2(\bar{\Omega})$ and $\tilde{w} \in C^2(\bar{\Omega}, \mathbb{R}^2)$ such that the following error bounds hold for \tilde{v}, \tilde{w} and the new deficit $\tilde{\mathcal{D}} = A - (\frac{1}{2} \nabla \tilde{v} \otimes \nabla \tilde{v} + \text{sym} \nabla \tilde{w})$:

$$\|\tilde{\mathcal{D}}\|_0 \leq C \left(\frac{\|A\|_{0,\beta}}{M^\beta} \|\mathcal{D}\|_0^{\beta/2} + \frac{1}{\sigma} \|\mathcal{D}\|_0 \right), \tag{5-7}$$

$$\|\tilde{v} - v\|_1 \leq C \|\mathcal{D}\|_0^{1/2} \quad \text{and} \quad \|\tilde{w} - w\|_1 \leq C(1 + \|\nabla v\|_0) \|\mathcal{D}\|_0^{1/2}, \tag{5-8}$$

$$\|\tilde{v}\|_2 \leq CM\sigma^3 \quad \text{and} \quad \|\tilde{w}\|_2 \leq C(1 + \|\nabla v\|_0) M\sigma^3. \tag{5-9}$$

The constant $C > 0$ is universal and independent of all parameters.

Proof. Analogously to [Conti et al. 2012, Proposition 4], the proof is split into three parts.

Part 1: mollification. Let $\varphi \in C_c^\infty(B(0, 1))$ be the standard mollifier in two dimensions, as in Lemma 4.3. Since v , w and A can be extended on the whole \mathbb{R}^2 , with all their relevant norms increased at most C times (C depends here on the curvature of the boundary $\partial\Omega$), we may define

$$\mathbf{v} = v * \varphi_l, \quad \mathbf{w} := w * \varphi_l, \quad \mathfrak{A} := A * \varphi_l \quad \text{with } l = \frac{\|\mathcal{D}\|_0^{1/2}}{M} < 1.$$

Applying Lemma 4.3 and noting (5-6), we immediately get the following uniform error bounds for \mathbf{v} , \mathbf{w} , \mathfrak{A} and for the induced deficit $\mathfrak{D} = \mathfrak{A} - (\frac{1}{2}\nabla\mathbf{v} \otimes \nabla\mathbf{v} + \text{sym } \nabla\mathbf{w})$:

$$\begin{aligned} \|\mathbf{v} - v\|_1 + \|\mathbf{w} - w\|_1 &\leq Cl(\|v\|_2 + \|w\|_2) \leq C\|\mathcal{D}\|_0^{1/2}, \\ \|\mathfrak{A} - A\|_0 &\leq Cl^\beta \|A\|_{0,\beta}, \\ \|\mathfrak{D}\|_m &\leq \|\mathcal{D} * \varphi_l\|_m + \|(\nabla v * \varphi_l) \otimes (\nabla v * \varphi_l) - (\nabla v \otimes \nabla v) * \varphi_l\|_m \\ &\leq \frac{C}{l^m} \|\mathcal{D}\| + \frac{C}{l^{m-2}} \|v\|_2^2 \leq \frac{C}{l^m} \|\mathcal{D}\|_0 \quad \forall m = 0, \dots, 3. \end{aligned} \tag{5-10}$$

In the proof of the last inequality above, we used (4-6) with the Hölder exponent $\alpha = 1$.

We note that so far we have simply exchanged the lower regularity fields v , w , A with their smooth approximations, at the expense of the error that, as we shall see below, is compatible with the that postulated in (5-7)–(5-9). The following estimate, however, reflects the advantage of averaging through mollification that results in the control of the C^3 norm of \mathbf{v} by the C^2 norm:

$$\forall m = 1, 2, \quad \|\nabla\mathbf{v}\|_m \leq \|\mathbf{v}\|_{m+1} \leq \frac{C}{l^{m-1}} \|v\|_2 \leq \frac{C}{l^m} \|\mathcal{D}\|_0^{1/2}, \tag{5-11}$$

where again we used Lemma 4.3 and (5-6). Note that the scaling bound (5-11) is consistent with the second requirement in (5-1) of Proposition 5.1. We also record the simple bound

$$\|\mathbf{w}\|_2 \leq C\|w\|_2 \leq CM. \tag{5-12}$$

Part 2: modification and positive definiteness. Contrary to the “stage” construction in the proof of Proposition 3.2, we do not know whether the original defect \mathcal{D} (and hence the induced defect \mathfrak{D}) is positive definite, so that Lemma 2.4 could be used. In any case, we need to keep the number of terms in the decomposition (3-9) into rank-one matrices as small as possible.

We now further modify \mathbf{w} in order to use the optimal decomposition in (2-5). Let r_0 be as in Lemma 2.3 and define

$$\mathbf{w}' = \mathbf{w} - 2 \frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0} \text{id}_2, \quad \mathfrak{D}' = \mathfrak{A} - \left(\frac{1}{2}\nabla\mathbf{v} \otimes \nabla\mathbf{v} + \text{sym } \nabla\mathbf{w}'\right).$$

Clearly, by (5-10) we get

$$\|\mathbf{w}' - \mathbf{w}\|_2 \leq C(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0) \leq C\|\mathcal{D}\|_0. \tag{5-13}$$

Note now that

$$\mathfrak{D}'(x) = 2 \frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0} \text{Id}_2 + \mathfrak{D}(x) = 2 \frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0} \left(\text{Id}_2 + \frac{r_0}{2(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)} \mathfrak{D} \right) \quad \forall x \in \bar{\Omega}.$$

By Lemma 2.3 we may apply (2-5) to the scaled defect

$$G = \text{Id}_2 + \frac{r_0}{2(\|\mathcal{D}\|_0 + \|\mathcal{D}\|_0)} \mathcal{D}$$

and arrive at

$$\mathcal{D}'(x) = \sum_{k=1}^3 2 \frac{(\|\mathcal{D}\|_0 + \|\mathcal{D}\|_0)}{r_0} \Phi_k(G(x)) \xi_k \otimes \xi_k = \sum_{k=1}^3 a_k^2(x) \xi_k \otimes \xi_k \quad \forall x \in \bar{\Omega}, \quad (5-14)$$

where

$$\left\{ a_k = \left(2 \frac{(\|\mathcal{D}\|_0 + \|\mathcal{D}\|_0)}{r_0} \Phi_k \circ G \right)^{1/2} \right\}_{k=1}^3$$

are positive smooth functions on $\bar{\Omega}$. We claim that

$$\forall k = 1, \dots, 3, \forall m = 0, \dots, 3, \quad \|a_k\|_m \leq \frac{C}{l^m} \|\mathcal{D}\|_0^{1/2}. \quad (5-15)$$

Indeed, for $m = 0$ this inequality follows directly by $\|\mathcal{D}\|_0 \leq C \|\mathcal{D}\|_0$. For $m = 1, \dots, 3$ we use Lemma 4.2 on each $\psi = \Phi_k^{1/2}$ and $f = G$, where noting that $\|G\|_0 \leq C$ and recalling (5-10) yields

$$\begin{aligned} \|a_k\|_m &\leq \left(2 \frac{(\|\mathcal{D}\|_0 + \|\mathcal{D}\|_0)}{r_0} \right)^{1/2} C \|G\|_m \\ &\leq C (\|\mathcal{D}\|_0 + \|\mathcal{D}\|_0)^{1/2} \left(C + \frac{r_0}{2(\|\mathcal{D}\|_0 + \|\mathcal{D}\|_0)} \|\mathcal{D}\|_m \right) \\ &\leq C \left((\|\mathcal{D}\|_0 + \|\mathcal{D}\|_0)^{1/2} + \frac{1}{(\|\mathcal{D}\|_0 + \|\mathcal{D}\|_0)^{1/2}} \frac{1}{l^m} \|\mathcal{D}\|_0 \right) \leq C \left(\|\mathcal{D}\|_0^{1/2} + \frac{1}{l^m} \|\mathcal{D}\|_0^{1/2} \right) \end{aligned} \quad (5-16)$$

and hence achieves (5-15). Note that the scaling bound (5-15) is consistent with the first requirement in (5-1) of Proposition 5.1.

Part 3: iterating the one-dimensional oscillations. We set $v_1 = \mathbf{v}$, $w_1 = \mathbf{w}$ and inductively define $v_{k+1} \in \mathcal{C}^3(\bar{\Omega})$ and $w_{k+1} \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ for $k = 1, 2, 3$ by means of Proposition 5.1 applied to v_k, w_k , the function a_k and the unit vector ξ_k appearing in (5-14), with the parameters

$$l_k = \frac{l}{\sigma^{k-1}} < 1, \quad \lambda_k = \frac{\sigma}{l_k} = \frac{1}{l_{k+1}} > \frac{1}{l_k},$$

and with the remaining three parameters

$$\delta_3 \geq \delta_2 \geq \delta_1 = \max_{m=1,2} \{l^m \|\nabla \mathbf{v}\|_m\} + \max_{\substack{m=0,\dots,3 \\ k=1,\dots,3}} \{l^m \|a_k\|_m\} \quad (5-17)$$

as indicated below. We then finally set $\tilde{v} = v_4$ and $\tilde{w} = w_4$.

We start by checking that the assumptions of Proposition 5.1 are satisfied. Namely, we claim that $\delta_k, l_k \in (0, 1)$, together with

$$\|a_k\|_m \leq \frac{\delta_k}{l_k^m} \quad \forall m = 0, \dots, 3 \quad \text{and} \quad \|\nabla v_k\|_m \leq \frac{\delta_k}{l_k^m} \quad \forall m = 1, 2, \quad (5-18)$$

at each iteration step $k = 1, 2, 3$, if only the constant δ_0 in (5-5) is appropriately small.

Indeed, $\delta_1 \leq C\|\mathcal{D}\|_0^{1/2}$ in view of (5-11) and (5-15), so $\delta_1 < 1$ if only $\delta_0 \ll 1$. Further, by the definition (5-17) it follows that

$$\|a_k\|_m = \frac{1}{l^m} l^m \|a_k\|_m \leq \frac{\delta_1}{l^m} \leq \frac{\delta_k}{l_k^m},$$

so the first assertion in (5-18) holds. For the second assertion, we see directly that it holds when $k = 1$, as

$$\|\nabla v_1\|_m = \frac{1}{l^m} l^m \|\nabla \mathbf{v}\|_m \leq \frac{\delta_1}{l^m}.$$

On the other hand, using induction on k and exploiting (5-3), we get

$$\begin{aligned} \|\nabla v_{k+1}\|_m &\leq \|\nabla v_k\|_m + \|\nabla v_{k+1} - \nabla v_k\|_m \leq \frac{\delta_k}{l_k^m} + C\delta_k \lambda_k^m \\ &\leq \delta_k \left(\frac{1}{l_{k+1}^m} + \frac{C}{l_{k+1}^m} \right) = C \frac{\delta_k}{l_{k+1}^m} \leq \frac{\delta_{k+1}}{l_{k+1}^m} \quad \forall m = 1, 2, \forall k = 1, 2. \end{aligned}$$

The proof of (5-18) is now complete for the choice $\delta_{k+1} = C\delta_k$, where $C > 1$ is, as always, an appropriately large universal constant. Consequently, $\delta_2, \delta_3 \leq C\|\mathcal{D}\|_0^{1/2} < 1$ if only $\delta_0 \ll 1$.

(4) We now directly verify the concluding estimates of Proposition 5.2. We have, in view of the definition of \mathfrak{D}' and (5-14),

$$\begin{aligned} \tilde{\mathcal{D}} &= A - \mathfrak{A} + \mathfrak{D}' + \left(\frac{1}{2} \nabla v_1 \otimes \nabla v_1 + \text{sym } \nabla w_1 \right) - \left(\frac{1}{2} \nabla v_4 \otimes \nabla v_4 + \text{sym } \nabla w_4 \right) \\ &= A - \mathfrak{A} - \sum_{k=1}^3 \left(\left(\frac{1}{2} \nabla v_{k+1} \otimes \nabla v_{k+1} + \text{sym } \nabla w_{k+1} \right) - \left(\frac{1}{2} \nabla v_k \otimes \nabla v_k + \text{sym } \nabla w_k + a_k \xi_k \otimes \xi_k \right) \right), \end{aligned}$$

and thus by (5-10), (5-2) and the definition of l , (5-7) follows:

$$\begin{aligned} \|\tilde{\mathcal{D}}\|_0 &\leq \|A - \mathfrak{A}\|_0 + C \sum_{k=1}^3 \frac{\delta_k^2}{\lambda_k l_k} \leq C \left(l^\beta \|A\|_{0,\beta} + \delta_3^2 \sum_{k=1}^3 \frac{1}{\lambda_k l_k} \right) \\ &\leq C \left(\frac{\|\mathcal{D}\|_0^{\beta/2}}{M^\beta} \|A\|_{0,\beta} + 3 \frac{\delta_3^2}{\sigma} \right) \leq C \left(\frac{\|\mathcal{D}\|_0^{\beta/2}}{M^\beta} \|A\|_{0,\beta} + \frac{1}{\sigma} \|\mathcal{D}\|_0 \right). \end{aligned}$$

We now check (5-8), using (5-10), (5-13) and (5-4):

$$\begin{aligned} \|\tilde{v} - v\|_1 &\leq \|\mathbf{v} - v\|_1 + \sum_{k=1}^3 \|v_{k+1} - v_k\|_1 \leq C\|\mathcal{D}\|_0^{1/2} + C \sum_{k=1}^3 \delta_k \leq C\|\mathcal{D}\|_0^{1/2}, \\ \|\tilde{w} - w\|_1 &\leq \|\mathbf{w} - w\|_1 + \|\mathbf{w}' - \mathbf{w}\|_1 + \sum_{k=1}^3 \|w_{k+1} - w_k\|_1 \\ &\leq C \left(\|\mathcal{D}\|_0^{1/2} + \|\mathcal{D}\|_0 + \sum_{k=1}^3 \delta_k (1 + \|\nabla v_k\|_0) \right) \leq C\|\mathcal{D}\|_0^{1/2} \left(1 + \sum_{k=1}^3 \|\nabla v_k\|_0 \right) \\ &\leq C\|\mathcal{D}\|_0^{1/2} \left(1 + \|\nabla v\|_0 + \|\mathbf{v} - v\|_1 + \sum_{k=1}^2 \|v_{k+1} - v_k\|_1 \right) \\ &\leq C\|\mathcal{D}\|_0^{1/2} (1 + \|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2}) \leq C\|\mathcal{D}\|_0^{1/2} (1 + \|\nabla v\|_0). \end{aligned} \tag{5-19}$$

Finally, the first bound in (5-9) follows by (5-11) and (5-3),

$$\begin{aligned} \|\tilde{v}\|_2 &\leq \|v\|_2 + \sum_{k=1}^3 \|v_{k+1} - v_k\|_2 \leq \frac{C}{l} \|\mathcal{D}\|_0^{1/2} + C \sum_{k=1}^3 \delta_k \lambda_k \\ &\leq \frac{C}{l} \|\mathcal{D}\|_0^{1/2} + C \delta_3 \sum_{k=1}^3 \frac{\sigma^k}{l} \leq \frac{C}{l} \|\mathcal{D}\|_0^{1/2} (1 + \sigma^3) \leq CM\sigma^3, \end{aligned}$$

while the second bound is obtained by

$$\begin{aligned} \|\tilde{w}\|_2 &\leq \|w\|_2 + \|w' - w\|_2 + \sum_{k=1}^3 \|w_{k+1} - w_k\|_2 \leq C \left(M + \|\mathcal{D}\|_0 + \sum_{k=1}^3 \delta_k \lambda_k (1 + \|\nabla v_k\|_0) \right) \\ &\leq C \left(M + \delta_3 \sum_{k=1}^3 \frac{\sigma^3}{l} (1 + \|\nabla v_k\|_0) \right) \leq CM \left(1 + \sigma^3 + \sigma^3 \sum_{k=1}^3 \|\nabla v_k\|_0 \right) \\ &\leq CM\sigma^3 \left(1 + \sum_{k=1}^3 \|\nabla v_k\|_0 \right) \leq CM\sigma^3 (1 + \|\nabla v\|_0) \end{aligned}$$

in view of (5-12), (5-13) and reasoning as in (5-19). \square

6. The $C^{1,\alpha}$ approximations: a proof of Theorem 1.2

We are now in a position to state the final intermediary approximation result, parallel to [Conti et al. 2012, Theorem 1].

Theorem 6.1. *Assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded domain. Given are functions $v \in C^2(\bar{\Omega})$, $w \in C^2(\bar{\Omega}, \mathbb{R}^2)$ and $A \in C^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2 \times 2}_{\text{sym}})$ for some $\beta \in (0, 1)$, such that the deficit \mathcal{D} below is appropriately small:*

$$\mathcal{D} = A - \left(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym} \nabla w \right), \quad 0 < \|\mathcal{D}\|_0 < \delta_0 \ll 1. \quad (6-1)$$

Fix the exponent

$$0 < \alpha < \min\left\{ \frac{1}{7}, \frac{1}{2}\beta \right\}. \quad (6-2)$$

Then, there exist $\bar{v} \in C^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ such that

$$\frac{1}{2} \nabla \bar{v} \otimes \nabla \bar{v} + \text{sym} \nabla \bar{w} = A, \quad (6-3)$$

$$\|\bar{v} - v\|_1 \leq C \|\mathcal{D}\|_0^{1/2} \quad \text{and} \quad \|\bar{w} - w\|_1 \leq C (1 + \|\nabla \tilde{v}\|_0) \|\mathcal{D}\|_0^{1/2}, \quad (6-4)$$

where $C > 0$ is a constant depending on α but independent of all other parameters.

Proof. The exact solution to (6-3) will be obtained as the $C^{1,\alpha}$ limit of sequences of successive approximations $\{v_k \in C^2(\bar{\Omega}), w_k \in C^2(\bar{\Omega}, \mathbb{R}^2)\}_{k=1}^\infty$.

Part 1: induction on stages. We set $v_0 = v$ and $w_0 = w$. Given v_k and w_k , define v_{k+1} and w_{k+1} by applying Proposition 5.2 with parameters σ and M_k that will be appropriately chosen below and that

satisfy

$$M_k > \max\{\|v_k\|_2, \|w_k\|_2, 1\} \quad \text{and} \quad \sigma > 1. \quad (6-5)$$

Following our notational convention, we define the k -th deficit $\mathcal{D}_k = A - (\frac{1}{2}\nabla v_k \otimes \nabla v_k + \text{sym} \nabla w_k)$. In view of Proposition 5.2, we get

$$\|\mathcal{D}_{k+1}\|_0 \leq C \left(\frac{\|A\|_{0,\beta}}{M_k^\beta} \|\mathcal{D}_k\|_0^{\beta/2} + \frac{1}{\sigma} \|\mathcal{D}_k\|_0 \right), \quad (6-6)$$

$$\|v_{k+1} - v_k\|_1 \leq C \|\mathcal{D}_k\|_0^{1/2} \quad \text{and} \quad \|w_{k+1} - w_k\|_1 \leq C(1 + \|\nabla v_n\|_0) \|\mathcal{D}_k\|_0^{1/2}, \quad (6-7)$$

$$\|v_{k+1}\|_2 \leq C M_k \sigma^3 \quad \text{and} \quad \|w_{k+1}\|_2 \leq C(1 + \|\nabla v_k\|_0) M_k \sigma^3, \quad (6-8)$$

provided that (5-5) holds for each \mathcal{D}_k . We shall now validate this requirement, with the parameters

$$M_k = (\mathfrak{C}(1 + \|\nabla v_0\|_0) \sigma^3)^k M_0. \quad (6-9)$$

In fact, we will inductively prove that one can have

$$\|\mathcal{D}_k\|_0 \leq \frac{1}{\sigma^{sk}} \|\mathcal{D}\|_0 \quad \text{with any } 0 < s < \min\left\{1, \frac{6\beta}{2-\beta}\right\}. \quad (6-10)$$

Fix s as indicated in (6-10). Clearly, (6-10) and (6-5) hold for $k = 0$. By (6-6) and the induction assumption we obtain the bound

$$\sigma^{s(k+1)} \frac{\|\mathcal{D}_{k+1}\|_0}{\|\mathcal{D}\|_0} \leq \frac{C \|A\|_{0,\beta} \|\mathcal{D}\|_0^{\beta/2-1} \sigma^s}{M_0^\beta} \frac{1}{\mathfrak{C}^{k\beta}} \left(\frac{\sigma^{(1-\beta/2)(s-6\beta/(2-\beta))}}{(1 + \|\nabla v_0\|_0)^\beta} \right)^k + C \sigma^{s-1}. \quad (6-11)$$

We see that in view of the condition on s in (6-10), both σ^{s-1} and $\sigma^{(1-\beta/2)(s-6\beta/(2-\beta))}$ are smaller than 1. Further, it is possible to choose $\sigma > 1$ so that the second term in (6-11) is smaller than $\frac{1}{2}$ and so that the quotient term in parentheses above is also smaller than 1. Then, choose M_0 so that (6-5) holds for $k = 0$ together with

$$\frac{C \|A\|_{0,\beta} \|\mathcal{D}\|_0^{\beta/2-1} \sigma^s}{M_0^\beta} < \frac{1}{2}.$$

This results in the first term in (6-11) being smaller than $\frac{1}{2}$ if $\mathfrak{C} \geq 1$. Consequently, we get that $\sigma^{s(k+1)} \|\mathcal{D}_{k+1}\|_0 / \|\mathcal{D}\|_0 \leq 1$ as needed in (6-10).

Observe now that by (6-7) and by the established (6-10),

$$\begin{aligned} \forall k \geq 0, \quad \|\nabla v_k\|_0 &\leq \|\nabla v_0\|_0 + \sum_{i=0}^{k-1} \|v_{i+1} - v_i\|_1 \leq \|\nabla v_0\|_0 + C \sum_{i=0}^{k-1} \|\mathcal{D}_i\|_0^{1/2} \\ &\leq \|\nabla v_0\|_0 + C \left(\sum_{i=0}^{\infty} \frac{1}{\sigma^{si/2}} \right) \|\mathcal{D}\|_0^{1/2} = \|\nabla v_0\|_0 + \frac{C}{1 - \sigma^{-s/2}} \|\mathcal{D}\|_0^{1/2} \\ &\leq \|\nabla v_0\|_0 + C \|\mathcal{D}\|_0^{1/2} \end{aligned} \quad (6-12)$$

if only, say, $\sigma^s > 4$, which can be easily achieved through the choice of σ . Now, by (6-8) and (6-12),

$$\begin{aligned} \frac{\|v_{k+1}\|_2}{M_{k+1}} &\leq \frac{1}{\mathfrak{C}} \frac{C}{(1 + \|\nabla v_0\|_0)}, \\ \frac{\|w_{k+1}\|_2}{M_{k+1}} &\leq \frac{1}{\mathfrak{C}} \frac{C(1 + \|\nabla v_k\|_0)}{(1 + \|\nabla v_0\|_0)} \leq \frac{1}{\mathfrak{C}} \frac{C(1 + \|\nabla v_0\|_0 + \|\mathcal{D}\|_0^{1/2})}{(1 + \|\nabla v_0\|_0)}. \end{aligned}$$

Hence, taking the constant $\mathfrak{C} \gg 1$ large enough, we see that both quantities above can be made smaller than 1, proving therefore the required (6-5).

Part 2: $C^{1,\alpha}$ control of the approximating sequences v_n and w_n . Let now α be an exponent as in (6-2). Choose s satisfying (6-10) and

$$\alpha(6 + s) - s < 0. \tag{6-13}$$

It is an easy calculation that s satisfying (6-10) and (6-13) exists if and only if the exponent α is in the range (6-2). Indeed, (6-13) is equivalent to $\alpha < s/(6 + s)$, while (6-10) is equivalent to

$$0 < \frac{s}{6 + s} < \min\left\{\frac{1}{7}, \frac{1}{2}\beta\right\}.$$

We will prove that the sequences $\{v_k, w_k\}_{k=0}^\infty$ are Cauchy in $C^{1,\alpha}(\bar{\Omega})$. Firstly, by (6-7), (6-12), (6-10),

$$\begin{aligned} \|v_{k+1} - v_k\|_1 &\leq C \|\mathcal{D}_k\|_0^{1/2} \leq \frac{C}{\sigma^{sk/2}} \|\mathcal{D}\|_0^{1/2}, \\ \|w_{k+1} - w_k\|_1 &\leq C(1 + \|\nabla v_k\|_0) \|\mathcal{D}_k\|_0^{1/2} \leq \frac{C}{\sigma^{sk/2}} (1 + \|\nabla v_0\|_0 + \|\mathcal{D}\|_0^{1/2}) \|\mathcal{D}\|_0^{1/2}, \end{aligned} \tag{6-14}$$

so we see right away that they are Cauchy in $C^1(\bar{\Omega})$. On the other hand, by (6-8), (6-12), (6-10),

$$\|v_{k+1} - v_k\|_2 + \|w_{k+1} - w_k\|_2 \leq C(1 + \|\nabla v_k\|_0) M_k \sigma^3 \leq C(1 + \|\nabla v_0\|_0 + \|\mathcal{D}\|_0^{1/2}) (\mathfrak{C}(1 + \|\nabla v_0\|_0) \sigma^3)^k M_0,$$

so the sequences have the tendency to diverge in $C^2(\bar{\Omega})$. Interpolating now the $C^{1,\alpha}$ norm by [Adams and Fournier 2003],

$$\|f\|_{0,\alpha} \leq \|f\|_1^\alpha \|f\|_0^{1-\alpha},$$

we obtain

$$\begin{aligned} \|\nabla(v_{k+1} - v_k)\|_{0,\alpha} + \|\nabla(w_{k+1} - w_k)\|_{0,\alpha} &\leq C_0^\alpha (C_0 \sigma^3)^{k\alpha} M_0^\alpha \cdot C_0^{1-\alpha} \frac{1}{\sigma^{sk(1-\alpha)/2}} \\ &= C_0 M_0^\alpha (C_0^\alpha)^h (\sigma^{\frac{1}{2}(\alpha(6+s)-s)})^k, \end{aligned} \tag{6-15}$$

where by C_0 we denoted an upper bound of all quantities involving C, v_0, \mathcal{D} . It is clear that choosing σ sufficiently large (so that $C_0 \sigma^{3-s/2} < 1$), the resulting bound (6-15) implies that $\{\nabla v_k, \nabla w_k\}_{k=0}^\infty$ are Cauchy in $C^{0,\alpha}(\bar{\Omega})$, provided that (6-13) holds. We see that the choice of exponent range in (6-2) so that the above construction technique works, is optimal.

Part 3: Concluding, we see that $\{v_k, w_k\}_{k=0}^\infty$ converge to some $\bar{v} \in C^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$. Since the defects in the approximating sequence obeys $\lim_{k \rightarrow \infty} \|\mathcal{D}_k\|_0 = 0$ by (6-10), we immediately get (6-3).

Additionally, by (6-14),

$$\begin{aligned}\|\bar{v} - v\|_1 &\leq \sum_{k=0}^{\infty} \|v_{k+1} - v_k\|_1 \leq C \left(\sum_{k=0}^{\infty} \frac{1}{\sigma^{sk/2}} \right) \|\mathcal{D}\|_0^{1/2} = \frac{C}{1 - \sigma^{-s/2}} \|\mathcal{D}\|_0^{1/2} \leq C \|\mathcal{D}\|_0^{1/2}, \\ \|\bar{w} - w\|_1 &\leq \sum_{k=0}^{\infty} \|w_{k+1} - w_k\|_1 \leq C \left(\sum_{k=0}^{\infty} \frac{1}{\sigma^{sk/2}} \right) (1 + \|\nabla v\|_0) \|\mathcal{D}\|_0^{1/2} \leq C(1 + \|\nabla v\|_0) \|\mathcal{D}\|_0^{1/2},\end{aligned}$$

completing the proof of (6-4). \square

We are now ready to give:

Proof of Theorem 1.2. Fix a sufficiently small $\varepsilon > 0$. We will construct $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R}^2)$ such that

$$A_0 = \frac{1}{2} \nabla \bar{v} \otimes \nabla \bar{v} + \text{sym } \nabla \bar{w} \quad \text{in } \bar{\Omega} \quad (6-16)$$

and

$$\|\bar{v} - v_0\|_0 + \|\bar{w} - w_0\|_0 < \varepsilon. \quad (6-17)$$

In order to apply Theorem 6.1, we need to decrease the deficit $A_0 - (\frac{1}{2} \nabla v_0 \otimes \nabla v_0 + \text{sym } \nabla w_0)$ so that it obeys (6-1). This will be done in three steps.

First, let $\tilde{v}_0 \in \mathcal{C}^\infty(\bar{\Omega})$, $\tilde{w}_0 \in \mathcal{C}^\infty(\Omega, \mathbb{R}^2)$ and $\tilde{A}_0 \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{2 \times 2}_{\text{sym}})$ be such that

$$\begin{aligned}\|\tilde{v}_0 - v_0\|_1 + \|\tilde{w}_0 - w_0\|_1 + \|\tilde{A}_0 - A_0\|_0 &< \varepsilon^2, \\ \exists \tilde{c}_0 > 0 \quad \text{such that} \quad A_0 - \left(\frac{1}{2} \nabla \tilde{v}_0 \otimes \nabla \tilde{v}_0 + \text{sym } \nabla \tilde{w}_0 \right) &> \tilde{c}_0 \text{Id}_2 \quad \text{in } \bar{\Omega}.\end{aligned} \quad (6-18)$$

Second, by Theorem 2.1 and Remark 3.3, there exist $v \in \mathcal{C}^1(\bar{\Omega})$ and $w \in \mathcal{C}^1(\Omega, \mathbb{R}^2)$ such that

$$\begin{aligned}\tilde{A}_0 &= \frac{1}{2} \nabla v \otimes \nabla v + \text{sym } \nabla w \quad \text{in } \bar{\Omega}, \\ \|v - \tilde{v}_0\|_0 + \|w - \tilde{w}_0\|_0 &< \varepsilon^2 \quad \text{and} \quad \|\nabla v - \nabla \tilde{v}_0\|_0 \leq C.\end{aligned} \quad (6-19)$$

Third, let $\tilde{v} \in \mathcal{C}^2(\bar{\Omega})$ and $\tilde{w} \in \mathcal{C}^2(\Omega, \mathbb{R}^2)$ be such that

$$\|v - \tilde{v}\|_1 + \|w - \tilde{w}\|_1 < \varepsilon^2. \quad (6-20)$$

By (6-19), (6-20) and (6-18), we get

$$\begin{aligned}\|A_0 - \left(\frac{1}{2} \nabla \tilde{v} \otimes \nabla \tilde{v} + \text{sym } \nabla \tilde{w} \right)\|_0 &\leq \|A_0 - \tilde{A}_0\|_0 + \left\| \left(\frac{1}{2} \nabla \tilde{v} \otimes \nabla \tilde{v} + \text{sym } \nabla \tilde{w} \right) - \left(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym } \nabla w \right) \right\|_0 \\ &\leq \|A_0 - \tilde{A}_0\|_0 + (\|\nabla v\|_0 + \|\nabla \tilde{v}\|_0) \|\nabla v - \nabla \tilde{v}\|_0 + \|\nabla w - \nabla \tilde{w}\|_0 \\ &\leq \varepsilon^2 + (2\|\nabla v_0\|_0 + 2\varepsilon^2 + C)\varepsilon^2 + \varepsilon^2 < \delta_0,\end{aligned} \quad (6-21)$$

as required in Theorem 6.1, if only ε is small enough. We now apply Theorem 6.1 to \tilde{v} , \tilde{w} and the original field A_0 , and get $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R}^2)$ satisfying (6-16) and such that

$$\begin{aligned}\|\bar{v} - v_0\|_0 + \|\bar{w} - w_0\|_0 &\leq C(1 + \|\nabla \tilde{v}\|_0) \|A_0 - \left(\frac{1}{2} \nabla \tilde{v} \otimes \nabla \tilde{v} + \text{sym } \nabla \tilde{w} \right)\|_0 + 3\varepsilon^2 \\ &\leq C(1 + \varepsilon^2 + \|\nabla v_0\|_0)^2 \varepsilon^2 + 3\varepsilon^2\end{aligned}$$

by (6-4), (6-21), (6-20), (6-19) and (6-18). Clearly (6-17) follows, if ε is small enough. \square

The following corollary is of independent interest:

Corollary 6.2. *Let Ω, f, p, α be as in the statement of Theorem 1.1. Let $q \geq 2$. Then, for all $v_0 \in W^{1,q}(\Omega)$, there exists a sequence $v_n \in C^{1,\alpha}(\bar{\Omega})$ weakly converging to v_0 in $W^{1,q}(\Omega)$, and such that $\text{Det } \nabla^2 v_n = f$ in Ω .*

Proof. Let $\bar{v}_n \in C^1(\bar{\Omega})$ converge to v_0 in $W^{1,q}(\Omega)$. For every \bar{v}_n , consider the approximating sequence $\{v_{n,k} \in C^{1,\alpha}(\bar{\Omega})\}_{k=1}^\infty$ as in Theorem 1.1, converging uniformly to \bar{v}_n . Define now $\{v_n\}$ to be an appropriate diagonal sequence, so that it converges to v_0 in $L^q(\Omega)$. We will check that $\{v_n\}$ is bounded in $W^{1,q}$.

The boundedness of $\|v_n\|_{L^q}$ is clear from the convergence statement. On the other hand, the proof of Theorem 1.2 gives, by (6-4), (6-18), (6-19), (6-20) and (6-21),

$$|\nabla v_n(x)| \leq |\nabla \bar{v}_n(x)| + 2\varepsilon^2 + C + C\delta_0^{1/2} \leq |\nabla \bar{v}_n(x)| + C \quad \forall x \in \Omega.$$

Consequently, $\|\nabla v_n\|_{L^q} \leq \|\nabla \bar{v}_n\|_{L^q} + C \leq C$, which concludes the proof. □

7. Rigidity results for $\alpha > \frac{2}{3}$: a proof of Theorem 1.3

The crucial element in the proof of the rigidity Theorems 1.3 and 1.4 is the following result, which is the “small slope analogue” of [Conti et al. 2012, Proposition 6]:

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, simply connected domain. Assume that for some $\alpha \in (\frac{2}{3}, 1)$, the function $v \in C^{1,\alpha}(\bar{\Omega})$ is a solution to*

$$\text{Det } \nabla^2 v = f \quad \text{in } \bar{\Omega},$$

where $f \in L^p(\Omega)$ and $p > 1$. Then the following degree formula holds true for every open subset U compactly contained in Ω and every $g \in L^\infty(\mathbb{R}^2)$ with $\text{supp } g \subset \mathbb{R}^2 \setminus \nabla v(\partial U)$:

$$\int_U (g \circ \nabla v) f = \int_{\mathbb{R}^2} g(y) \text{deg}(\nabla v, U, y) \, dy. \tag{7-1}$$

Above, $\text{deg}(\psi, U, y)$ denotes the Brouwer degree of a continuous function $\psi : \bar{U} \rightarrow \mathbb{R}^2$ at a point $y \in \mathbb{R}^2 \setminus \psi(\partial U)$.

Proof. (1) Fix U and g as in the statement of the proposition. We refer to [Lloyd 1978] for the definition and properties of the Brouwer degree; recall first that $\text{deg}(\nabla v, U, \cdot)$ is well defined on the open set $\mathbb{R}^2 \setminus \nabla v(\partial U)$. In fact, this function is constant on each connected component $\{U_i\}_{i=0}^\infty$ of $\mathbb{R}^2 \setminus \nabla v(\partial U)$ and it equals 0 on the only unbounded component $U_0 \subset \mathbb{R}^2 \setminus \nabla v(\bar{U})$. Thus, without loss of generality, we may assume that g is compactly supported and that $\text{supp } g \subset \bigcup_{k=1}^\infty U_k$. By compactness, there must be $\text{supp } g \subset \bigcup_{k=1}^N U_k$ for some N , and consequently the integral in the right-hand side of (7-1) is well defined.

Let now $\{g_i \in C_c^\infty(\bigcup_{k=1}^N U_k)\}_{i=1}^\infty$ be a sequence pointwise converging to g and such that $\|g_i\|_0 \leq \|g\|_{L^\infty}$ for all i . It is sufficient to prove the formula (7-1) for each g_i and pass to the limit by the dominated convergence theorem. To simplify the notation, we drop the index i , and so in what follows we assume that $g \in C_c^\infty(\mathbb{R}^2 \setminus \nabla v(\partial U))$.

As in the proof of Theorem 1.1, let $A \in W^{2,p}(\Omega) \cap C^{0,\beta}(\bar{\Omega})$ be such that $\text{curl curl } A = -f$. Here, we take $\beta = \min\{2 - \frac{2}{p}, \alpha\} \in (0, 1)$. Consequently, in view of the simple connectedness of Ω , there exists $w \in C^{1,\beta}(\bar{\Omega}, \mathbb{R}^2)$ such that

$$A = \frac{1}{2} \nabla v \otimes \nabla v + \text{sym } \nabla w.$$

For a standard 2-dimensional mollifier $\varphi \in C_c^\infty(B(0, 1))$ as in Lemma 4.3, define

$$\forall l \in (0, 1), \quad v_l = v * \varphi_l, \quad w_l = w * \varphi_l, \quad A_l = A * \varphi_l,$$

and apply the degree formula (change of variable formula [Evans and Gariepy 1992; Ambrosio et al. 2000]) to the smooth functions g and ∇v_l , noting that for sufficiently small l , we have $g \in C_c^\infty(\mathbb{R}^2 \setminus \nabla v_l(\partial U))$:

$$\int_U (g \circ \nabla v_l) \det \nabla^2 v_l = \int_{\mathbb{R}^2} g(y) \deg(\nabla v_l, U, y) dy. \quad (7-2)$$

We see that ∇v_l converge uniformly to ∇v , so by [Kavian 1993, Proposition 2.1] we obtain that for l sufficiently small, and for all $y \in \text{supp } g$, we have $\deg(\nabla v, U, y) = \deg(\nabla v_l, U, y)$. Thus

$$\lim_{l \rightarrow 0} \int_{\mathbb{R}^2} g(y) \deg(\nabla v_l, U, y) dy = \int_{\mathbb{R}^2} g(y) \deg(\nabla v, U, y) dy.$$

Another proof of integrability of the Brouwer degree, in a more general context, can be found in [Olbermann 2015]. Now, to conclude the proof in view of (7-2), it suffices to show that

$$\lim_{l \rightarrow 0} \int_U (g \circ \nabla v_l) \det \nabla^2 v_l = \int_U (g \circ \nabla v) f. \quad (7-3)$$

(2) Following [Conti et al. 2012; Constantin et al. 1994] we use a commutator estimate to get (7-3). As $f = -\text{curl curl } A$, we have

$$\begin{aligned} \left| \int_U (g \circ \nabla v_l) \det \nabla^2 v_l - (g \circ \nabla v) f \right| &\leq \left| \int_U (g \circ \nabla v_l) (\det \nabla^2 v_l + \text{curl curl } A_l) \right| \\ &\quad + \left| \int_U (g \circ \nabla v_l) \text{curl curl } (A_l - A) \right| + \left| \int_U ((g \circ \nabla v_l) - (g \circ \nabla v)) f \right|. \end{aligned} \quad (7-4)$$

The second term above is bounded by $C \int_U |\nabla^2 A_l - \nabla^2 A| \leq C \|A_l - A\|_{W^{2,p}(\Omega)}$, hence it converges to 0. The third term also converges to 0 by the dominated convergence theorem, since $g \circ \nabla v_l$ converges to $g \circ \nabla v$. In order to deal with the first term in (7-4), observe that $\det \nabla^2 v_l = -\text{curl curl}(\frac{1}{2} \nabla v_l \otimes \nabla v_l + \text{sym } \nabla w_l)$ and integrate by parts, in view of $g \circ \nabla v_l = 0$ on ∂U :

$$\begin{aligned} \left| \int_U (g \circ \nabla v_l) (\det \nabla^2 v_l + \text{curl curl } A_l) \right| &= \left| \int_U \langle \nabla^\perp (g \circ \nabla v_l), \text{curl}(\frac{1}{2} \nabla v_l \otimes \nabla v_l + \text{sym } \nabla w_l - A_l) \rangle \right| \\ &\leq C \|\nabla g\|_0 \|\nabla^2 v_l\|_0 \|\nabla v_l \otimes \nabla v_l - (\nabla v \otimes \nabla v) * \varphi_l\|_1 \\ &\leq C \frac{1}{l^{1-\alpha}} \|\nabla v\|_{0,\alpha} \cdot \frac{1}{l^{1-2\alpha}} \|\nabla v\|_{0,\alpha}^2 = C \frac{1}{l^{2-3\alpha}} \|\nabla v\|_{0,\alpha}^3, \end{aligned} \quad (7-5)$$

where we used Lemma 4.3. Clearly, for $\alpha > \frac{2}{3}$ the right-hand side in (7-5) converges to 0 as $l \rightarrow 0$. By (7-4), this implies (7-3) and concludes the proof. \square

Below, we present all the details of the proof of Theorem 1.3. The proof of Theorem 1.4 will be postponed to [Lewicka and Pakzad ≥ 2017].

Proof of Theorem 1.3. (1) By Proposition 7.1 it follows that for all open sets $U \subset \bar{U} \subset \Omega$,

$$\deg(\nabla v, U, y) = 0 \quad \forall y \in \mathbb{R}^2 \setminus \nabla v(\partial U). \tag{7-6}$$

We would like to conclude [Pogorelov 1956; 1973] that the image set $\nabla v(U)$ is of measure 0. This will result in the developability of v , by the main statement of [Korobkov 2007]. However, we note that (Malý, personal communication, 2016) for each $\alpha \in (0, 1)$, there exists a map in $C^{0,\alpha}(\Omega, \mathbb{R}^2)$ whose local degree vanishes everywhere, but whose image is onto the unit square. This example can be constructed through a similar approach to that in [Malý and Martio 1995, Section 5]. Therefore, we will additionally exploit the gradient structure of ∇v , using ideas of [Kirchheim 2001, Chapter 2], in combination with the commutator estimate technique of the proof of Proposition 7.1.

Let $v_l = v * \varphi_l$ be as in the proof of Proposition 7.1 and for every $\delta > 0$ define

$$u_{l,\delta}(x_1, x_2) = \nabla v_l(x_1, x_2) + \delta(-x_2, x_1), \quad u_\delta(x_1, x_2) = \nabla v(x_1, x_2) + \delta(-x_2, x_1).$$

Fix an open set U with smooth boundary and compactly contained in Ω . Let $g \in C_c^\infty(\mathbb{R}^2 \setminus \nabla v(\partial U))$, and use the change of variable formula to g and $u_{l,\delta}$:

$$\int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) = \int_{\mathbb{R}^2} g(y) \deg(u_{l,\delta}, U, y) \, dy, \tag{7-7}$$

where we noted that $\det \nabla u_{l,\delta} = \det \nabla^2 v_l + \delta^2$. The integral in the right-hand side of (7-7) is well defined for sufficiently small l and δ , because then $y \in \text{supp } g$ implies $y \notin u_{l,\delta}(\partial U)$.

Passing to the limit, we immediately obtain

$$\lim_{l \rightarrow 0} \int_{\mathbb{R}^2} g(y) \deg(u_{l,\delta}, U, y) \, dy = \int_{\mathbb{R}^2} g(y) \deg(u_\delta, U, y) \, dy, \tag{7-8}$$

while to the left hand side of (7-7) we apply the estimate

$$\left| \int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) - (g \circ u_\delta)\delta^2 \right| \leq \left| \int_U (g \circ u_{l,\delta}) \det \nabla^2 v_l \right| + \left| \int_U (g \circ u_{l,\delta} - g \circ u_\delta)\delta^2 \right|.$$

The second term above clearly converges to 0 as $l \rightarrow 0$, because $u_{l,\delta}$ converge to u_δ . The first term also converges to 0 as $\alpha > \frac{2}{3}$, where we reason exactly as in (7-4) and (7-5), keeping in mind that $f = 0$. We hence conclude

$$\lim_{l \rightarrow 0} \int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) = \int_U (g \circ u_\delta)\delta^2.$$

In view of (7-8) and (7-7) this implies

$$\forall 0 < \delta \ll 1, \quad \int_U (g \circ u_\delta)\delta^2 = \int_{\mathbb{R}^2} g(y) \deg(u_\delta, U, y) \, dy.$$

Consequently,

$$\forall 0 < \delta \ll 1, \quad \forall y \in u_\delta(U) \setminus u_\delta(\partial U), \quad \deg(u_\delta, U, y) \geq 1. \tag{7-9}$$

(2) We now claim that

$$\nabla v(U) \subset \nabla v(\partial U). \quad (7-10)$$

To prove (7-10) we argue by contradiction, assuming that for some $x_0 \in U$ there is $y_0 = \nabla v(x_0) \in \nabla v(U) \setminus \nabla v(\partial U)$. Note that for δ small enough, we have $y_0 \notin u_\delta(\partial U)$, because u_δ converges uniformly to ∇v as $\delta \rightarrow 0$. We distinguish two cases:

- (i) There exist sequences $\{x_k \in U\}_{k=1}^\infty$ and $\delta_k \rightarrow 0^+$ as $k \rightarrow \infty$ such that $y_0 = u_{\delta_k}(x_k)$ for all k . In view of (7-9) we get $\deg(u_{\delta_k}, U, y_0) \geq 1$, contradicting (7-6).
- (ii) For all δ small enough, $y_0 \notin u_\delta(\bar{U})$. In this case, we must have $\deg(u_\delta, U, y_0) = 0$. But on the other hand, there exists a ball $B(y_0, 2r) \subset \mathbb{R}^2 \setminus \nabla v(\partial U)$, so also $B(y_0, r) \subset \mathbb{R}^2 \setminus u_\delta(\partial U)$ for all small δ . Consequently, continuity of the degree yields that $\deg(u_\delta, U, z) = 0$ for every $z \in B(y_0, r)$. In particular, $\deg(u_\delta, U, u_\delta(x_0)) = 0$, because $\lim_{\delta \rightarrow 0} u_\delta(x_0) = \nabla v(x_0) = y_0$. This finally contradicts (7-9), as $u_\delta(x_0) \in u_\delta(U) \setminus u_\delta(\partial U)$.

Our claim (7-10) is now established. Since the set $\nabla v(\partial U)$ is the image of a Hausdorff one-dimensional set ∂U under a $\mathcal{C}^{0,\alpha}$, $\alpha > \frac{1}{2}$, deformation ∇v , it has Lebesgue measure 0 (see [Conti et al. 2012, Lemma 4]). Thus $\nabla v(U)$ must have measure 0 for every smooth U compactly contained in Ω . The same then must be true for the entire set Ω , i.e., $|\nabla v(\Omega)| = 0$, and we consequently obtain

$$\text{Int}(\nabla v(\Omega)) = \emptyset. \quad (7-11)$$

(3) By [Korobkov 2009, Corollary 1.1.2], condition (7-11) implies that every point $y \in \Omega$ has a convex open neighbourhood Ω_y such that for every point $x \in \Omega_y$ there is a line L_x passing through x so that ∇v is constant on $L_x \cap \Omega_y$. The same result in the present dimensionality has been first established in [Korobkov 2007]; see also the footnote on p. 875 in [Korobkov 2009] for an explanation.

We now prove that v is developable. Fix $x_0 \in \Omega$ and let $[y, z] \subset \bar{\Omega}$ be the maximal segment passing through x_0 on which $\nabla v = \nabla v(x_0)$ is constant. Assume that $[y, z]$ does not extend to the boundary $\partial\Omega$, i.e., $y \in \Omega$. We will prove that then ∇v must be constant in an open neighbourhood of x_0 . In fact, we will show that

$$V = \text{Int}((\nabla v)^{-1}(\nabla v(x_0))) \supset (y, z). \quad (7-12)$$

Let $(p, q) = L_y \cap \Omega_y$. By the maximality of $[y, z]$, the segment (p, q) is not an extension of (is not parallel to) $[y, z]$. Also, $\nabla v = \nabla v(x_0)$ on (p, q) . Take any $y_1 \in (y, z) \cap \Omega_y$ and define the open triangle $T = \text{Int}(\text{span}\{p, q, y_1\})$. It is easy to notice that every line passing through any point $x \in T$ must intersect at least one of the segments (p, q) or (y, y_1) . Since $T \subset \Omega_y$, it follows that $\nabla v(x) = \nabla v(x_0)$. Hence

$$(y, y_1) \subset T \subset V$$

and, in particular, the set V in (7-12) is nonempty.

To prove (7-12) assume, by contradiction, that there exists $y_2 \in [y_1, z]$ so that

$$(y, y_2) \subset V \quad \text{but} \quad (y, y_3) \not\subset V \quad \forall y_3 \in (y_2, z). \quad (7-13)$$

Now, the intersection $\Omega_{y_2} \cap V$ contains an open arc C crossing the segment $(y, y_2) \cap \Omega_{y_2}$. As above, we argue that every point in a sufficiently small open neighbourhood of the segment $I = (y, z) \cap \Omega_{y_2}$ must have the property that every line passing through it intersects C or I , where $\nabla v = \nabla v(x_0)$. Consequently $I \subset V$, contradicting (7-13) and establishing (7-12). \square

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KINETIC FORMULATION OF VORTEX VECTOR FIELDS

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This article focuses on gradient vector fields of unit Euclidean norm in \mathbb{R}^N . The stream functions associated to such vector fields solve the eikonal equation and the prototype is given by the distance function to a closed set. We introduce a kinetic formulation that characterizes stream functions whose level sets are either spheres or hyperplanes in dimension $N \geq 3$. Our main result proves that the kinetic formulation is a selection principle for the vortex vector field whose stream function is the distance function to a point.

1. Introduction

In this article, we analyze the following type of vortex vector field:

$$u^* : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad u^*(x) = \frac{x}{|x|} \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\}$$

in dimension $N \geq 2$, where $|\cdot|$ is the Euclidean norm in \mathbb{R}^N . This structure arises in many physical models such as micromagnetics, liquid crystals, superconductivity, elasticity. Clearly, u^* is smooth away from the origin: in fact, 0 is a topological singularity of degree 1 since the jacobian is $\det \nabla u^* = V_N \delta_0$, where δ_0 is the Dirac measure at the origin and V_N is the volume of the unit ball in \mathbb{R}^N . Also, u^* is a curl-free unit-length vector field; i.e.,

$$|u^*| = 1 \quad \text{and} \quad \nabla \times u^* = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{1}$$

Moreover, there is a stream function $\psi^* : \mathbb{R}^N \rightarrow \mathbb{R}$ associated to u^* by the equation

$$u^* = \nabla \psi^*;$$

indeed, one may consider ψ^* as the distance function at the origin, i.e., $\psi^*(x) = |x|$ for $x \in \mathbb{R}^N$, and ψ^* represents the viscosity solution of the eikonal equation

$$|\nabla \psi^*| = 1$$

under an appropriate boundary condition at infinity (e.g., $\lim_{|x| \rightarrow \infty} (\psi^*(x) - |x|) = 0$).

Note that conversely, these properties characterize the vortex vector field: if $u : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a nonconstant vector field that is smooth away from the origin and satisfies (1) then $u = \pm u^*$ in \mathbb{R}^N . Indeed, this classically follows by the method of characteristics: the flow associated to u by

$$\partial_t X(t, x) = u(X(t, x)) \tag{2}$$

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with the initial condition $X(0, x) = x$ for $x \neq 0$ yields straight lines $\{X(t, x)\}_t$ given by $X(t, x) = x + tu(x)$ along which u is constant, i.e., $u(X(t, x)) = u(x)$. Since u is nonconstant and two characteristics can intersect only at the origin (which is the prescribed point-singularity of u), every characteristic passes through the origin¹ and therefore, u coincides with u^* or $-u^*$. Caffarelli and Crandall [2010] proved this result under a weaker regularity hypothesis for the vector field $u = \nabla\psi$: if ψ is assumed only pointwise differentiable away from a set S of vanishing Hausdorff \mathcal{H}^1 -measure (i.e., $\mathcal{H}^1(S) = 0$) and $|\nabla\psi| = 1$ in $\mathbb{R}^N \setminus S$, then $\psi = \pm\psi^*$ (up to a translation and an additive constant). We also refer to [DiPerna and Lions 1989] for weaker regularity assumptions on u in the framework of Sobolev spaces.

Our aim is to prove a kinetic characterization of the vortex vector field that does not assume any initial regularity on u . This kinetic formulation will characterize stream functions whose level sets are totally umbilical hypersurfaces in dimension $N \geq 3$, i.e., either pieces of spheres or hyperplanes. In order to introduce the kinetic formulation of the vortex vector field, we start by presenting the case of dimension $N = 2$ and then we extend it to dimensions $N \geq 3$.

1.1. Kinetic formulation in dimension $N = 2$. Let $\Omega \subset \mathbb{R}^2$ be an open set and $u : \Omega \rightarrow \mathbb{R}^2$ be a Lebesgue-measurable vector field that satisfies

$$|u| = 1 \text{ a.e. in } \Omega \quad \text{and} \quad \nabla \times u = 0 \text{ distributionally in } \Omega. \quad (3)$$

The main feature of the kinetic formulation relies on the concept of weak characteristic for a nonsmooth vector field u . We start by noting that (2) has a proper meaning only if some notion of trace of u can be defined on curves $\{X(t, x)\}_t$, which in general is a consequence of the regularity assumption on u (see [DiPerna and Lions 1989]). To overcome this difficulty, the following notion of “weak characteristic” is introduced for measurable vector fields u (see, e.g., [Lions, Perthame, and Tadmor 1994; Jabin and Perthame 2001]): for every direction $\xi \in \mathbb{S}^1$, one defines the function $\chi(\cdot, \xi) : \Omega \rightarrow \{0, 1\}$ by

$$\chi(x, \xi) = \begin{cases} 1 & \text{for } u(x) \cdot \xi > 0, \\ 0 & \text{for } u(x) \cdot \xi \leq 0. \end{cases} \quad (4)$$

In the case of a smooth vector field u in a neighborhood of a point $x_0 \in \Omega$, then $\chi(\cdot, \xi)$ mimics the characteristic of u of normal direction $\xi = (\xi_1, \xi_2)$ (see Figure 1); formally, if $\xi^\perp = (-\xi_2, \xi_1) = \pm u(x_0)$, then either $\nabla\chi(\cdot, \xi)$ locally vanishes (if u is constant in a neighborhood of x_0), or $\nabla\chi(\cdot, \xi)$ is a measure concentrated on the characteristic $\{X(t, x_0)\}_t$ given by (2) with constant measure density $\pm\xi$. In other words, we have the following “kinetic formulation” of the problem (see, e.g., [DeSimone, Müller, Kohn and Otto 2001; Jabin and Perthame 2001]):

Proposition 1 (kinetic formulation in dimension $N = 2$). *Let $\Omega \subset \mathbb{R}^2$ be an open set and $u : \Omega \rightarrow \mathbb{R}^2$ be a smooth vector field. If u satisfies (3) then*

$$\xi^\perp \cdot \nabla_x \chi(\cdot, \xi) = 0 \quad \text{distributionally in } \Omega \text{ for every } \xi \in \mathbb{S}^1. \quad (5)$$

¹This argument is clear in dimension $N = 2$; for dimensions $N \geq 3$, one needs an additional argument showing that two characteristics are coplanar, as we will see later in the proof of Theorem 8.

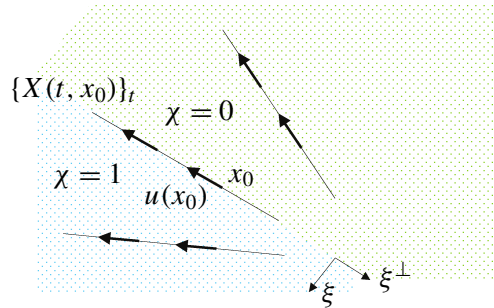


Figure 1. Characteristics of u .

We mention that the kinetic formulation (5) holds under the weaker Sobolev regularity $W^{1/p,p}$ for $p \in [1, 3]$ (see [Ignat 2011; 2012a; 2012b; De Lellis and Ignat 2015]). Note that the knowledge of $\chi(\cdot, \xi)$ in every direction $\xi \in \mathbb{S}^1$ determines completely a vector field u with $|u| = 1$ due to the averaging formula

$$u(x) = \frac{1}{2} \int_{\mathbb{S}^1} \xi \chi(x, \xi) d\mathcal{H}^1(\xi) \quad \text{for a.e. } x \in \Omega. \tag{6}$$

Thanks to (6), we deduce that the kinetic formulation (5) incorporates the fact that $\nabla \times u = 0$ (see Proposition 5 below). Therefore, the curl-free condition will be no longer mentioned in the following statements whenever (5) is assumed to hold true for unit-length vector fields u .

The main question is whether the kinetic formulation (5) characterizes the vortex vector field in \mathbb{R}^2 . First of all, (5) induces a regularizing effect for Lebesgue-measurable unit-length vector fields u . Indeed, the classical “kinetic averaging lemma” (see, e.g., [Golse, Lions, Perthame, and Sentis 1988]) shows that a measurable vector field $u : \Omega \rightarrow \mathbb{S}^1$ satisfying (5) belongs to $H_{\text{loc}}^{1/2}(\Omega)$ due to the averaging formula (6).² Moreover, Jabin, Otto, and Perthame [2002] improved the regularizing effect by showing that u is locally Lipschitz away from vortex point-singularities³ and u coincides with the vortex vector field around these singularities:

Theorem 2 [Jabin, Otto, and Perthame 2002]. *Let $\Omega \subset \mathbb{R}^2$ be an open set and $u : \Omega \rightarrow \mathbb{R}^2$ be a Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. in Ω together with the kinetic formulation (5). Then u is locally Lipschitz continuous inside Ω except at a locally finite number of singular points. Moreover, every singular point P of u corresponds to a vortex singularity of topological degree 1 of u ; i.e., there exists a sign $\gamma = \pm 1$ such that*

$$u(x) = \gamma u^*(x - P) \quad \text{for every } x \neq P \text{ in any convex neighborhood of } P \text{ in } \Omega.$$

In particular, if $\Omega = \mathbb{R}^2$ and u is nonconstant, then u coincides with u^ or $-u^*$ (up to a translation).*

This result leads to the following interpretation of the kinetic formulation in dimension $N = 2$: equation (5) is a selection principle for the viscosity solutions of the eikonal equation $|\nabla \psi| = 1$ in the sense that the solutions ψ are smooth (more precisely, they belong to the Sobolev space $W_{\text{loc}}^{2,\infty}$) away from

²For the improved regularizing effect for scalar conservation laws, see [Otto 2009; Golse and Perthame 2013].

³This regularity is optimal; see, e.g., Proposition 1 in [Ignat 2012b].

point-singularities. Clearly, these solutions are induced by the viscosity solutions of the eikonal equation under some appropriate boundary condition. Conversely, in the spirit of [Caffarelli and Crandall 2010], it was shown by Ignat [2012b] and De Lellis and Ignat [2015] that for any vector field u satisfying (3) together with an initial Sobolev regularity $W^{1/p,p}$, $p \in [1, 3]$ (i.e., excluding jump line-singularities), the kinetic formulation (5) holds true and therefore, one obtains the regularizing effect in Theorem 2.

Remark 3. The result of Jabin, Otto, and Perthame [2002] was motivated by the study of zero-energy states in a line-energy Ginzburg–Landau model in dimension 2. More precisely, one considers the energy functional $E_\varepsilon : H^1(\Omega, \mathbb{R}^2) \rightarrow \mathbb{R}_+$ defined for $\varepsilon > 0$ as

$$E_\varepsilon(u_\varepsilon) = \varepsilon \int_{\Omega} |\nabla u_\varepsilon|^2 dx + \frac{1}{\varepsilon} \int_{\Omega} (1 - |u_\varepsilon|^2)^2 dx + \frac{1}{\varepsilon} \|\nabla \times u_\varepsilon\|_{H^{-1}(\Omega)}^2, \quad u_\varepsilon \in H^1(\Omega, \mathbb{R}^2), \quad (7)$$

where Ω is a domain in \mathbb{R}^2 and $H^{-1}(\Omega)$ is the dual of the Sobolev space $H_0^1(\Omega)$. (We refer to [Ambrosio, De Lellis, and Mantegazza 1999; Aviles and Giga 1999; DeSimone, Müller, Kohn and Otto 2001; Jabin, Otto, and Perthame 2002; Jabin and Perthame 2001; Jin and Kohn 2000; Rivière and Serfaty 2001] for the analysis of this model.) A vector field $u : \Omega \rightarrow \mathbb{R}^2$ is called zero-energy state if there exists a family $\{u_\varepsilon \in H^1(\Omega, \mathbb{R}^2)\}_{\varepsilon \rightarrow 0}$ satisfying

$$u_\varepsilon \rightarrow u \quad \text{in } L^1(\Omega) \quad \text{and} \quad E_\varepsilon(u_\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Obviously, a zero-energy state u satisfies (3). The result of Jabin, Otto, and Perthame [2002] shows that every zero-energy state u satisfies (5) and therefore, u shares the structure stated in Theorem 2.

1.2. Kinetic formulation in dimension $N \geq 3$. Our main interest consists in defining a kinetic formulation for the vortex vector field in dimension $N \geq 3$. Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \rightarrow \mathbb{R}^N$ be a Lebesgue-measurable vector field. For every direction $\xi \in \mathbb{S}^{N-1}$, we consider the characteristic function $\chi(\cdot, \xi)$ defined at (4) and we denote the orthogonal hyperplane to ξ by

$$\xi^\perp := \{v \in \mathbb{R}^N : v \cdot \xi = 0\}.$$

Definition 4 (kinetic formulation). We say that a measurable vector field u satisfies the kinetic formulation if the following equation holds true:

$$v \cdot \nabla_x \chi(\cdot, \xi) = 0 \quad \text{distributionally in } \Omega \quad \text{for every } \xi \in \mathbb{S}^{N-1} \text{ and } v \in \xi^\perp. \quad (8)$$

Roughly speaking, (8) means that $\nabla_x \chi(\cdot, \xi)$ is a distribution pointing in direction $\pm \xi$. Note that the kinetic formulation (8) only carries out the information of the direction of the vector field u (i.e., it gives no information about the Euclidean norm of u). Imposing the unit-length constraint, u will satisfy a similar averaging formula (6) which justifies that the curl-free constraint $\nabla \times u = 0$ is incorporated in the kinetic formulation (8).

Proposition 5. Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \rightarrow \mathbb{R}^N$ be Lebesgue measurable with $|u| = 1$ a.e. in Ω . Then

$$u(x) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi \chi(x, \xi) d\mathcal{H}^{N-1}(\xi) \quad \text{for a.e. } x \in \Omega, \quad (9)$$

where V_{N-1} is the volume of the unit ball in \mathbb{R}^{N-1} . Moreover, if u satisfies the kinetic formulation (8) then $\nabla \times u = 0$ distributionally in Ω .

Remark 6. We highlight that Proposition 1 is *false* in dimension $N \geq 3$; i.e., there are smooth curl-free vector fields with values into the unit sphere \mathbb{S}^{N-1} that do not satisfy the kinetic formulation (8). For example, in dimension $N = 3$, considering the vortex-line vector field

$$u_0(x) = \frac{(x_1, x_2, 0)}{\sqrt{x_1^2 + x_2^2}} \quad \text{in } \Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 1\},$$

then u_0 is smooth in Ω and satisfies (3). However, (8) fails. Indeed, let $\xi = \frac{1}{\sqrt{2}}(1, 0, 1)$. Then $u_0(x) \cdot \xi = 0$ for $x \in \Omega$ is equivalent to $x_1 = 0$ and therefore,

$$\nabla_x \chi(\cdot, \xi) = e_1 \mathcal{H}^2 \llcorner \{x \in \Omega : x_1 = 0\},$$

where $e_1 = (1, 0, 0)$. Now, taking $v = \frac{1}{\sqrt{2}}(-1, 0, 1)$, we have $v \cdot \xi = 0$ (i.e., $v \in \xi^\perp$) and $v \cdot \nabla_x \chi(\cdot, \xi) \neq 0$ in $\mathcal{D}'(\Omega)$.

As Remark 6 has already revealed, the kinetic equation (8) in dimension $N \geq 3$ plays a different role than in dimension $N = 2$ because the gradient $\nabla \chi(\cdot, \xi)$ is expected to concentrate on hypersurfaces (not on the line characteristics of u). In fact, the geometric interpretation of (8) can be regarded in terms of the stream function ψ of a nonconstant vector field $u = \nabla \psi$: the level sets of ψ are expected to be pieces of spheres of codimension 1 where the characteristics of u represent the normal directions to these spheres.

Theorem 7. *Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be an open set and $\psi : \Omega \rightarrow \mathbb{R}$ be a smooth stream function such that $u = \nabla \psi$ satisfies the kinetic formulation (8). Assume $|u|$ never vanishes on a level set $\{x \in \Omega : \psi(x) = \alpha\}$ for some $\alpha \in \mathbb{R}$ and let \mathcal{S} be a connected component of $\{\psi = \alpha\}$. Then \mathcal{S} is locally a totally umbilical hypersurface, that is, either a piece of an $(N-1)$ -sphere or a piece of a hyperplane.*

Note that Theorem 7 fails in dimension $N = 2$: a level set of a smooth stream function ψ of $u = \nabla \psi$ satisfying (3) (and therefore, u satisfies the kinetic formulation (5) by Proposition 1) does not have, in general, constant curvature.⁴

2. Main results

Our main result shows that the kinetic formulation (8) is a characterization of the vortex vector field u^* in dimension $N \geq 3$.

Theorem 8. *Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a connected open set and $u : \Omega \rightarrow \mathbb{R}^N$ be a nonconstant Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. in Ω together with the kinetic equation (8). Then u coincides with the vortex vector field u^* or $-u^*$ up to a translation.*

Note that in dimension $N = 2$, this result is true for the domain $\Omega = \mathbb{R}^2$, but it is in general false for other domains Ω where there exist nonconstant smooth vector fields u in Ω different than vortex

⁴If $\Gamma \subset \mathbb{R}^2$ is a smooth curve of nonconstant curvature, then one takes ψ to be the distance function to Γ in a small neighborhood Ω of Γ (with the convention that Γ is withdrawn from that neighborhood, i.e., $\Gamma \cap \Omega = \emptyset$, so that ψ is smooth in Ω).

vector fields that satisfy (3) and thus, (5) (by Proposition 1). The main difference in dimension $N \geq 3$ is the following: if u is a smooth vector field with (3) that is neither constant nor a vortex vector field, then the kinetic formulation (8) doesn't hold for u (see Remark 6). Hence, in dimension $N \geq 3$, the zero-energy states of E_ε defined in (7) do not satisfy in general the kinetic equation (8). Therefore, the kinetic formulation (8) is more rigid in dimension $N \geq 3$ since it selects only the vortex vector fields, as they correspond to smooth solutions of the eikonal equation with level sets of constant sectional curvature (by Theorem 7).

Let us explain the strategy of the proof of Theorem 8. The key point relies on a relation of order of the level sets of the stream function associated to u : for every two Lebesgue points $x, y \in \Omega$ of u such that the segment $[x, y]$ lies in Ω and for every direction $\xi \in \mathbb{S}^{N-1}$ orthogonal to $x - y$, one has

$$u(x) \cdot \xi > 0 \implies u(y) \cdot \xi \geq 0.$$

The next step consists in defining the trace of u on each segment $\Sigma \subset \Omega$; more precisely, similar to the procedure of [Jabin, Otto, and Perthame 2002], there exists a trace $\tilde{u} \in L^\infty(\Sigma, \mathbb{S}^{N-1})$ of u such that $u(P) = \tilde{u}(P)$ for each Lebesgue point $P \in \Sigma$ of u . Moreover, if the trace \tilde{u} of u is collinear with the segment Σ at some Lebesgue point, then \tilde{u} is \mathcal{H}^1 -almost everywhere collinear with Σ (which coincides with the classical principle of characteristics for smooth vector fields u). The final step consists in proving that every two characteristics are coplanar. Then one concludes by the following geometrical fact specific to dimension $N \geq 3$:

Proposition 9. *Let $N \geq 3$ and \mathcal{D} be a set of lines in \mathbb{R}^N such that every two lines of \mathcal{D} are coplanar, but \mathcal{D} is not planar (i.e., there is no 2-dimensional plane containing \mathcal{D}). Then either all lines of \mathcal{D} are collinear, or all lines of \mathcal{D} pass through a same point (that is a vortex point).*

In view of Theorem 8, it is natural to ask if one can characterize other types of unit-length curl-free vector fields u by weakening the kinetic formulation (8), in particular, vector fields having a vortex-line singularity. In dimension $N \geq 3$, the prototype of a vortex-line vector field is given by

$$u_0(x', x_N) = \nabla |x'|,$$

where $x = (x', x_N)$ and $x' = (x_1, \dots, x_{N-1})$; clearly, u_0 is smooth away from the vortex-line $\{x \in \mathbb{R}^N : x' = 0\}$ where (3) holds true. Defining

$$\mathcal{E} := \{\xi \in \mathbb{S}^{N-1} : \xi_N = 0\} = \mathbb{S}^{N-2} \times \{0\},$$

using the notation (4), we have that u_0 satisfies the following kinetic formulation in $\Omega = \mathbb{R}^N$:

$$\forall \xi \in \mathcal{E}, \forall v \in \xi^\perp, \quad v \cdot \nabla_x \chi(\cdot, \xi) = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (10)$$

Note that (10) is a weakened form of (8): the quantity $v \cdot \nabla_x \chi(\cdot, \xi)$ vanishes for directions $\xi \in \mathcal{E}$ (and $v \in \xi^\perp$) and fails to vanish for \mathcal{H}^{N-1} -a.e. direction $\xi \in \mathbb{S}^{N-1}$. As opposed to (8) (in view of (9)), the kinetic formulation (10) does not force a unit-length vector field u to be curl-free; it only implies that

$$\nabla' \times \frac{u'}{|u'|} = 0 \quad \text{in } \{|u'| \neq 0\} = \{u \neq \pm e_N\},$$

where $e_N = (0, \dots, 0, 1)$, $u' = (u_1, \dots, u_{N-1})$ and $\nabla' = (\partial_1, \dots, \partial_{N-1})$. Since we are looking for a characterization of vortex-line vector fields (that are in particular curl-free), we will impose that

$$\partial_k u_N = \partial_N u_k \quad \text{in } \Omega \text{ for } k = 1, \dots, N - 1. \tag{11}$$

We will prove the following result:

Theorem 10. *Let $N \geq 4$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \rightarrow \mathbb{R}^N$ be a Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. on Ω together with (10) and (11). Then in every ball included in $\{x \in \Omega : u(x) \neq \pm e_N\}$, there exists a stream function $\psi = \psi(\alpha, \beta)$ solving the eikonal equation in dimension 2 such that*

$$u(x) = \nabla_x[\psi(\alpha, \beta)],$$

where

- (1) either $\alpha = |x' - P'|$ and $\beta = x_N$ for some point $P' \in \mathbb{R}^{N-1}$;
- (2) or $\alpha = w' \cdot x'$ and $\beta = x_N$ for some vector $w' \in \mathbb{S}^{N-2}$.

Therefore, the weakened kinetic formulation (10), together with (11), is not enough to select vortex-line vector fields which correspond to the stream function $\psi(\alpha, \beta) = \pm\alpha$ in case (1) of Theorem 10. Similar results to Theorem 10 hold for similar kinetic formulations corresponding to vector fields having vortex-sheet singularities of dimension k in \mathbb{R}^N with $N \geq k + 3$.

The outline of this paper is as follows: in Section 3, we characterize the level sets of smooth stream functions associated to vector fields that satisfy the kinetic formulation (8). In particular, we prove Proposition 1 and Theorem 7. Section 4 is devoted to proving fine properties of Lebesgue points of u needed in Section 5, where the notion of the trace on lines for a vector field u satisfying (8) is defined. Section 6 is the core of this paper: using this notion of trace and the geometric arguments of Proposition 9, we prove our main result in Theorem 8. Section 7 deals with the study of the weakened kinetic formulation (10).

3. Level sets of the stream function

This section is devoted to the study of the level sets of smooth stream functions ψ associated to vector fields $u = \nabla\psi$ satisfying (8). We start by proving that $|\nabla\psi|$ is locally constant on each level set of ψ .

Lemma 11. *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $\psi : \Omega \rightarrow \mathbb{R}$ be a smooth stream function such that $u = \nabla\psi$ satisfies the kinetic formulation (8). Assume $|u|$ never vanishes on a level set $\{x \in \Omega : \psi(x) = \alpha\}$ for some $\alpha \in \mathbb{R}$ and let S be a connected component of $\{\psi = \alpha\}$. Then $|u|$ is constant on S . Moreover, there exists a neighborhood ω of S , a smooth solution $\tilde{\psi} : \omega \rightarrow \mathbb{R}$ of the eikonal equation and a diffeomorphism $t \mapsto F(t)$ such that $\psi = F(\tilde{\psi})$ in ω (in particular, $\nabla\tilde{\psi}$ satisfies (8)).*

Proof. Since $|u| \neq 0$ on S and u is smooth in Ω , we can define

$$v = \frac{u}{|u|} \quad \text{in a neighborhood of } S.$$

For simplicity of notation, we suppose that Ω is this neighborhood, i.e., $|u| \neq 0$ in Ω . Then v satisfies (8) because u satisfies it, too; since v is smooth in Ω , Proposition 5 implies $\nabla \times v = 0$ in Ω . (The proof

of Proposition 5 is independent of Lemma 11; we will admit it here and prove it later in Section 4.) As a consequence, in any simply connected domain $\omega \subset \Omega$, the Poincaré lemma yields the existence of a smooth function $\tilde{\psi}$ such that $v = u/|u| = \nabla\tilde{\psi}$ in ω , i.e.,

$$\nabla\psi = u = |u|v = |u|\nabla\tilde{\psi} \quad \text{in } \omega.$$

Therefore, ψ and $\tilde{\psi}$ have the same level sets in ω . Without loss of generality, we may assume that $\tilde{\psi} = 0$ on $\omega \cap \mathcal{S}$. Now, for every $P' \in \omega \cap \mathcal{S}$, we consider the flow associated to v ,

$$\begin{cases} \dot{X}(P', t) = \nabla\tilde{\psi}(X(P', t)), \\ X(P', 0) = P'. \end{cases} \quad (12)$$

Call $I_{P'}$ the maximal interval where the solution $X(P', \cdot)$ exists. Obviously, the flow is unique and smooth, satisfying

$$\ddot{X}(P', t) = \nabla^2\tilde{\psi}(X) \cdot \dot{X} = \nabla^2\tilde{\psi}(X) \cdot \nabla\tilde{\psi}(X) = 0 \quad \text{in } I_{P'}$$

because $\nabla^2\tilde{\psi}$ is a symmetric matrix and $|\nabla\tilde{\psi}| = 1$ in ω . Consequently, $\dot{X}(P', \cdot)$ is constant in $I_{P'}$ so that

$$\nabla\tilde{\psi}(X(P', t)) = \nabla\tilde{\psi}(P'), \quad \frac{d}{dt}[\tilde{\psi}(X(P', t))] = 1, \quad X(P', t) = P' + t\nabla\tilde{\psi}(P').$$

Therefore, since $\tilde{\psi} = 0$ on $\omega \cap \mathcal{S}$, we have

$$\tilde{\psi}(X(P', t)) = t \quad \text{for all } P' \in \omega \cap \mathcal{S} \text{ and } t \in I_{P'}.$$

Identifying the level sets of $\tilde{\psi}$ (and of ψ , too) using the flow, i.e., $\{\tilde{\psi} = t\} = \{X(P', t) : P' \in \omega \cap \mathcal{S}\}$, we can define

$$F(t) := \psi(X(P', t)) \quad \text{for } P' \in \omega \cap \mathcal{S}, t \in I_{P'}.$$

The function F is a diffeomorphism: F is smooth (because ψ and X are smooth) and we have

$$\frac{d}{dt}F(t) = \nabla\psi(X(P', t)) \cdot \dot{X}(P', t) \stackrel{(12)}{=} \nabla\psi(X(P', t)) \cdot \frac{\nabla\psi}{|\nabla\psi|}(X(P', t)) = |u|(X(P', t)) \neq 0.$$

In particular, $|u|$ is constant on $\{\tilde{\psi} = 0\} = \{\psi = F(0)\} = \omega \cap \mathcal{S}$. Since ω was arbitrarily chosen, we deduce that $|u|$ is locally constant on \mathcal{S} ; because \mathcal{S} is connected, it follows that $|u|$ is constant on \mathcal{S} . Since the flow $\{X(P', t) : P' \in \mathcal{S}, t \in I_{P'}\}$ covers a neighborhood of \mathcal{S} , the last statement of the lemma follows. \square

3.1. The case of dimension $N = 2$. In the special case of dimension $N = 2$, we start by proving that every smooth curl-free vector field of unit length satisfies the kinetic formulation (5). This result can be found already in [DeSimone, Müller, Kohn and Otto 2001; Jabin and Perthame 2001]. For completeness, we will present two easy and self-contained proofs. The first one is based on the geometry of the flow (2) (as heuristically described in Section 1), while the second proof is based on the concept of entropy introduced in [DeSimone, Müller, Kohn and Otto 2001].

Proof of Proposition 1: first method. We can assume that $\xi = e_1$ and $\xi^\perp = e_2$ (otherwise, one considers a rotation $R \in \text{SO}(2)$ such that $e_1 = R\xi$ and $\tilde{u}(x) := Ru(R^{-1}x)$ in a neighborhood of a point $x \in \Omega$).

Naturally, Ω can be written as a countable union of squares whose edges are parallel with e_1 and e_2 . Therefore, using a partition of unity, it is enough to prove the statement for $\Omega = (-1, 1)^2$:

$$\forall \varphi \in C_c^\infty(\Omega), \quad 0 = \int_{\Omega} \varphi \xi^\perp \cdot \nabla_x \chi(x, \xi) dx \stackrel{\xi=e_1}{=} \int_{\Omega} \varphi \partial_2 \chi(x, e_1) dx = - \int_{\Omega \cap \{u_1 > 0\}} \partial_2 \varphi dx.$$

For that, we consider the flow (2) and by the proof of Lemma 11, we have, for every $x \in \Omega$, that $\{X(t, x)\}_t$ is a straight line given by $X(t, x) = x + tu(x)$ and $u(X(t, x)) = u(x)$ for all t . Since u is smooth, there is no crossing between two characteristics in Ω . We claim that

$$\Omega \cap \{u_1 > 0\} = \bigsqcup_{k \in K} A_k,$$

where $\{A_k\}_{k \in K}$ is a (at most) countable set of pairwise disjoint rectangles of type $(a_k, b_k) \times (-1, 1) \subset \Omega = (-1, 1)^2$. Note first that $\Omega \cap \{u_1 = 0\}$ is the intersection of Ω by vertical lines. Indeed, if $u_1(x) = 0$, then $u(x) \parallel e_2$. By the characteristic method, for all t , we have $u_1(x + tu(x)) = 0$ and u_1 vanishes on the vertical line passing through x . Now $\{x_1 \in (-1, 1) : u_1(x_1, 0) = 0\}^c$ is an open set in $(-1, 1)$ and therefore, we can write

$$\{x_1 \in (-1, 1) : u_1(x_1, 0) = 0\}^c = \bigsqcup_{k \in \tilde{K}} (a_k, b_k),$$

where \tilde{K} is at most countable. For $k \in \tilde{K}$, we define $A_k := (a_k, b_k) \times (-1, 1)$. By continuity, u_1 is either positive or negative on A_k . Defining $K := \{k \in \tilde{K} : u_1 > 0 \text{ on } A_k\}$, the claim is proved. Now, for $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega \cap \{u_1 > 0\}} \partial_2 \varphi = \sum_k \int_{A_k} \partial_2 \varphi = \sum_k \int_{a_k}^{b_k} \int_{-1}^1 \partial_2 \varphi = 0,$$

because $\partial_2 \varphi$ can be seen as a signed Radon measure for $\varphi \in C_c^\infty(\Omega)$ and the proposition is proved. \square

Proof of Proposition 1: second method. The following proof links the kinetic formulation (5) with the theory of entropy solutions for scalar conservation laws (see, e.g., [DeSimone, Müller, Kohn and Otto 2001]). Indeed, if u is a smooth vector field satisfying (3), then formally, $u_1 = -h(u_2) := \pm\sqrt{1 - u_2^2}$ so that $\nabla \times u = 0$ can be rewritten as

$$\partial_1 u_2 + \partial_2 [h(u_2)] = 0; \tag{13}$$

thus, u_2 can be formally interpreted as a solution of the above scalar conservation law in the variables (time, space) = (x_1, x_2) . Based on the concept of entropy solution of (13) introduced via the pairs (entropy, entropy-flux), the following applications (called “elementary entropies”) were used in [DeSimone, Müller, Kohn and Otto 2001]. More precisely, for every $\xi \in \mathbb{S}^1$, the map $\Phi^\xi : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ is defined as

$$\text{for } z \in \mathbb{S}^1, \quad \Phi^\xi(z) = \begin{cases} \xi^\perp & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \leq 0. \end{cases}$$

Then the kinetic formulation (5) can be written as

$$\nabla \cdot [\Phi^\xi(u)] = 0 \quad \text{distributionally in } \Omega. \tag{14}$$

In order to prove (14), we will approximate Φ^ξ by a sequence of smooth maps $\{\Phi_k : \mathbb{S}^1 \rightarrow \mathbb{R}^2\}$ such that $\{\Phi_k\}$ is uniformly bounded, $\lim_k \Phi_k(z) = \Phi^\xi(z)$ for every $z \in \mathbb{S}^1$ and Φ_k satisfies (14) for every k . Following the ideas in [DeSimone, Müller, Kohn and Otto 2001] (see also [Ignat and Merlet 2012]), this smoothing result comes from the following observation: there exists a (unique) 2π -periodic piecewise C^1 function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ associated to Φ^ξ via the equation

$$\Phi^\xi(z) = -\varphi'(\theta)z + \varphi(\theta)z^\perp \quad \text{for every } z = e^{i\theta} \in \mathbb{S}^1. \tag{15}$$

In fact, φ is given by

$$\varphi(\theta) = \Phi^\xi(z) \cdot z^\perp = \xi \cdot z \mathbb{1}_{\{z \cdot \xi > 0\}} = \cos(\theta - \theta_0) \mathbb{1}_{\{\theta - \theta_0 \in (-\pi/2, \pi/2)\}} \quad \text{for } z = e^{i\theta}, \theta \in (-\pi + \theta_0, \pi + \theta_0),$$

where $\xi = e^{i\theta_0} \in \mathbb{S}^1$ with $\theta_0 \in (-\pi, \pi]$. In (15), the distributional derivative φ' is given by

$$\varphi'(\theta) = -\sin(\theta - \theta_0) \mathbb{1}_{\{\theta - \theta_0 \in (-\pi/2, \pi/2)\}} \quad \text{for } \theta \in (-\pi + \theta_0, \pi + \theta_0).$$

Now, one regularizes φ by 2π -periodic functions $\varphi_k \in C^\infty(\mathbb{R})$ that are uniformly bounded in $W^{1,\infty}(\mathbb{R})$ with $\lim_k \varphi_k(\theta) = \varphi(\theta)$ and $\lim_k \varphi'_k(\theta) = \varphi'(\theta)$ for every $\theta \in \mathbb{R}$. Then we define Φ_k as in (15) for the functions φ_k :

$$\Phi_k(z) = -\varphi'_k(\theta)z + \varphi_k(\theta)z^\perp \quad \text{for } z = e^{i\theta} \in \mathbb{S}^1.$$

Let us now check that $\{\Phi_k\}_k$ are indeed the desired (smooth) approximating maps of Φ^ξ . For that, first, note that differentiating the above equation defining Φ_k , one obtains

$$\frac{\partial \Phi_k}{\partial \theta}(z) \cdot z^\perp = 0 \quad \text{for every } z = e^{i\theta} \in \mathbb{S}^1. \tag{16}$$

Next, we prove that Φ_k satisfies (14). Indeed, we can locally write $u = e^{i\Theta}$ in every ball $B \subset \Omega$ for some smooth lifting $\Theta : B \rightarrow \mathbb{R}$ that satisfies

$$\nabla \Theta \cdot u = \nabla \times u = 0 \quad \text{in } B.$$

This means that $\nabla \Theta = \lambda u^\perp$ in B for some smooth function $\lambda : B \rightarrow \mathbb{R}$. Therefore, it follows that

$$\nabla \cdot [\Phi_k(u)] = \frac{\partial \Phi_k}{\partial \theta}(e^{i\Theta}) \cdot \nabla \Theta = \lambda \frac{\partial \Phi_k}{\partial \theta}(u) \cdot u^\perp \stackrel{(16)}{=} 0 \quad \text{in } B.$$

Passing to limit $k \rightarrow \infty$, the dominated convergence theorem yields

$$\int_B \Phi^\xi(u) \cdot \nabla \zeta \, dx = 0 \quad \text{for every } \zeta \in C_c^\infty(B).$$

The conclusion is now straightforward. □

Note that another interest of this second method is that it can be adapted to vector fields $u \in W^{1/p,p}$ for $p \in [1, 3]$. For such vector fields, there is a priori no trace of u on a segment, so the flow (2) does not have a proper meaning anymore; see [Ignat 2012b; De Lellis and Ignat 2015] for more details.

3.2. The case of dimension $N \geq 3$. The aim of this subsection is to prove Theorem 7. We divide the proof in several steps, each being stated as a lemma.

Lemma 12. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \rightarrow \mathbb{R}^N$ be a smooth vector field satisfying (8). We define*

$$\tilde{\Omega} := \left\{ x \in \Omega : u(x) \neq 0, \nabla \left(\frac{u}{|u|} \right) (x) \neq 0 \right\}$$

and for every $x \in \tilde{\Omega}$,

$$\mathbb{S}_x := u(x)^\perp \cap \mathbb{S}^{N-1} = \{ \xi \in \mathbb{S}^{N-1} : u(x) \cdot \xi = 0 \} \approx \mathbb{S}^{N-2}.$$

Then we have for all $x \in \tilde{\Omega}$ and for \mathcal{H}^{N-2} -a.e. $\xi \in \mathbb{S}_x$ that the set

$$\{ y \in \tilde{\Omega} : u(y) \cdot \xi = 0 \} = \tilde{\Omega} \cap \partial \{ u \cdot \xi > 0 \}$$

is a hyperplane around x that is oriented by the normal vector ξ . Moreover,

$$\nabla_x \chi(\cdot, \xi) = \pm \xi \mathcal{H}^{N-1} \llcorner \partial \{ u \cdot \xi > 0 \} \quad \text{locally around } x. \tag{17}$$

Proof. As in the proof of Lemma 11, we set $v = u/|u|$ on $\tilde{\Omega}$. Then v is a smooth unit-length vector field in $\tilde{\Omega}$ that satisfies (8) (because u satisfies it, too) and by Proposition 5, we have that v is curl-free in $\tilde{\Omega}$. Let $x \in \tilde{\Omega}$; in particular, $\nabla v(x) \neq 0$. First, we show that $\{ y \in \tilde{\Omega} : u(y) \cdot \xi = 0 \}$ is a smooth $(N-1)$ -manifold around x . Since v is curl-free, we know that $\nabla v(x) = (\partial_j v_i(x))_{i,j}$ is symmetric. By differentiating the relation $|v(x)| = 1$, it follows that

$$\nabla v(x)^T v(x) = \nabla v(x) v(x) = 0,$$

which means $v(x) \in \text{Ker } \nabla v(x)$. We will prove that

$$\mathcal{H}^{N-2}(\mathbb{S}_x \cap \text{Ker } \nabla v(x)) = 0.$$

Assume by contradiction that $\mathbb{S}_x \cap \text{Ker } \nabla v(x)$ has positive \mathcal{H}^{N-2} -measure. Since $\text{Ker } \nabla v(x)$ is a linear space, we have $\mathbb{S}_x \subset \text{Ker } \nabla v(x)$, that is, $\nabla v(x)\xi = 0$ for all $\xi \in \mathbb{S}_x$. Moreover, since $v(x) \in \text{Ker } \nabla v(x)$ and $\mathbb{S}_x \subset v(x)^\perp$, it follows that $\nabla v(x) = 0$, which is a contradiction with the assumption $\nabla v(x) \neq 0$. Therefore, $\nabla v(x)\xi \neq 0$ for \mathcal{H}^{N-2} -a.e. $\xi \in \mathbb{S}_x$ and $\{ y \in \tilde{\Omega} : v(y) \cdot \xi = 0 \} = \{ y \in \tilde{\Omega} : u(y) \cdot \xi = 0 \}$ is a smooth $(N-1)$ -manifold around x .

It remains to prove that this manifold is a piece of hyperplane oriented by ξ where (17) holds true. For that, let $\varphi \in C_c^\infty(\tilde{\Omega}, \mathbb{R}^N)$ be supported in a ball $B \subset \tilde{\Omega}$ centered at x . By the Gauss theorem, we have

$$-\langle \nabla_x \chi(\cdot, \xi), \varphi \rangle = \int_B \nabla \cdot \varphi(y) \chi(y, \xi) dy = \int_{\{y \in B : u(y) \cdot \xi > 0\}} \nabla \cdot \varphi dy = \int_{B \cap \partial \{u \cdot \xi > 0\}} \varphi \cdot \nu d\mathcal{H}^{N-1}(y),$$

where ν is the unit outer normal vector to the $(N-1)$ -manifold $\partial \{ u(y) \cdot \xi > 0 \}$. This proves that locally around x , we have

$$\nabla_x \chi(x, \xi) = -\nu \mathcal{H}^{N-1} \llcorner (B \cap \partial \{ u \cdot \xi > 0 \}).$$

Because of (8), we know that $\nabla_x \chi(x, \xi)$ and ξ are collinear. Since ν is smooth on $B \cap \partial \{ u \cdot \xi > 0 \}$, this implies $\nu = \xi$ or $\nu = -\xi$ on $B \cap \partial \{ u \cdot \xi > 0 \}$. The conclusion is now straightforward. \square

We now state the following result, which is the key point in proving Theorem 7.

Lemma 13. *Under the hypotheses of Theorem 7, every point $x \in \mathcal{S}$ is an umbilical point; i.e., there exists $\lambda(x) \in \mathbb{R}$ such that*

$$Du(x) = \lambda(x)\text{Id} : T_x\mathcal{S} \rightarrow \mathbb{R}^{N-1},$$

where u is proportional to the Gauss map on \mathcal{S} , $T_x\mathcal{S}$ is the tangent plane to the hypersurface \mathcal{S} at x and Id is the identity matrix.

Proof. Recall that $|u|$ is constant on \mathcal{S} by Lemma 11 so that $u/|u|$ is the normal vector (i.e., the Gauss map) at the hypersurface \mathcal{S} . Therefore,

$$D\left(\frac{u}{|u|}\Big|_{\mathcal{S}}\right) = \frac{1}{|u|}D(u|_{\mathcal{S}}) \quad \text{in } \mathcal{S},$$

where $D(u|_{\mathcal{S}})$ is the differential of u restricted to \mathcal{S} as a map with values into the sphere \mathbb{S}^{N-1} (up to the multiplicative constant $|u|$). As in the proofs of Lemmas 11 and 12, we may assume that u never vanishes in Ω and set $v = u/|u|$ in Ω . Then v is a smooth unit-length vector field in Ω that satisfies (8) and by Proposition 5, v is curl-free so that locally $v = \nabla\tilde{\psi}$ for a smooth stream function $\tilde{\psi}$. Since $\nabla\psi = u = |u|\nabla\tilde{\psi}$, we know that ψ and $\tilde{\psi}$ have the same level sets; in particular, \mathcal{S} is a level set of $\tilde{\psi}$. Therefore, replacing u by v , we may assume in the following that

$$|u| = 1 \quad \text{in } \Omega.$$

Let $x \in \mathcal{S}$. We want to show that x is an umbilical point of \mathcal{S} . This is clear if $\nabla u(x) = 0$. Therefore, we assume in the following that $x \in \tilde{\Omega} \cap \mathcal{S}$, as defined in Lemma 12; i.e.,

$$\nabla u(x) \neq 0.$$

Since (9) holds for the unit-length vector field u , by differentiating (9), we obtain

$$\nabla u(x) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi \otimes \nabla_x \chi(x, \xi),$$

where V_{N-1} is the volume of the unit ball in \mathbb{R}^{N-1} . The above integrand is to be understood as an absolutely continuous measure with respect to the Hausdorff \mathcal{H}^{N-2} measure concentrated on the set \mathbb{S}_x (defined at Lemma 12). For that, we check first that the support of the integrand lies on \mathbb{S}_x . Indeed, if $\xi \in \mathbb{S}^{N-1}$ with $u(x) \cdot \xi \neq 0$, then $\nabla_x \chi(\cdot, \xi) = 0$ in the open set $\{u \cdot \xi \neq 0\}$ around x . Therefore, the integrand has support on the set $\xi \in \mathbb{S}_x$, where (17) holds true for \mathcal{H}^{N-2} -a.e. $\xi \in \mathbb{S}_x$ and the density of the measure is equal to $\pm \xi \otimes \xi \mathcal{H}^{N-2} \llcorner \mathbb{S}_x$. Since $\mathbb{S}_x \subset u(x)^\perp = T_x\mathcal{S}$, the density $\xi \otimes \xi$ with $\xi \in \mathbb{S}_x$ already identifies $\nabla u(x) \equiv Du(x)$. Next we compute this quantity by exploring the sign of the density $\pm \xi \otimes \xi$:

Case $N = 3$. We show that there are at most two nonzero vectors $\pm \xi_0 \in \mathbb{S}_x \approx \mathbb{S}^1$ such that $\nabla u(x)\xi_0 = 0$. Assume by contradiction that there are more than two vectors as above; i.e., there exists another nonzero vector $\tilde{\xi}_0 \neq \pm \xi_0$ in \mathbb{S}_x such that $\nabla u(x)\xi_0 = \nabla u(x)\tilde{\xi}_0 = 0$. Because of $|u| = 1$, we know that $\nabla u(x)u(x) = 0$. Since the set $\{u(x), \xi_0, \tilde{\xi}_0\}$ spans \mathbb{R}^3 , we have $\nabla u(x) = 0$, which contradicts the hypothesis $x \in \tilde{\Omega}$. Therefore, $\nabla u(x)\xi \neq 0$ for every $\xi \in \mathbb{S}_x \setminus \{\pm \xi_0\}$ (or for every $\xi \in \mathbb{S}_x$ if ξ_0 does not exist) and by Lemma 12,

$\partial\{u(y) \cdot \xi > 0\}$ is a smooth surface around x oriented by ξ . Let \mathcal{C}_1 and \mathcal{C}_2 be the two connected components of $\mathbb{S}_x \setminus \{\pm \xi_0\}$ (with the convention that $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$ in the case $\nabla u(x)\xi \neq 0$ for every $\xi \in \mathbb{S}_x$). For $j = 1, 2$, we associate to a point $\xi \in \mathcal{C}_j$ the unit outer normal vector field $\nu(\xi) \in \{\pm \xi\}$ to the plane $\partial\{u \cdot \xi > 0\}$ around x . Since the map $\xi \in \mathcal{C}_j \rightarrow \nu(\xi)$ is smooth (by the implicit function theorem) and \mathcal{C}_j is connected, we deduce that ν is constant on \mathcal{C}_j . Thus it follows that

$$\pi \nabla u(x) = \gamma_1 \int_{\mathcal{C}_1} \xi \otimes \xi \, d\xi + \gamma_2 \int_{\mathcal{C}_2} \xi \otimes \xi \, d\xi,$$

with $V_2 = \pi$ and $\gamma_{1,2} \in \{\pm 1\}$ (with the convention that $\gamma_1 = \gamma_2 = \pm 1/2$ if $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$). It remains to show that $\int_{\mathcal{C}_j} \xi \otimes \xi \, d\xi$ is proportional to the identity matrix Id , $j = 1, 2$. Up to a rotation, we can suppose that $u(x) = e_3$ and $\mathcal{C}_1 = \{\xi \in \mathbb{S}^1 \times \{0\} : \xi_2 > 0\} \approx \{(\cos \theta, \sin \theta) : \theta \in (0, \pi)\}$. We have

$$\int_{\mathcal{C}_1} \xi \otimes \xi \, d\xi \approx \int_0^\pi \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix} d\theta = \frac{\pi}{2} \text{Id}$$

(the conclusion follows similarly if $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$).

Case $N > 3$. Let $\mathcal{C} = \text{Ker } \nabla u(x) \cap \mathbb{S}_x$. We know that $u(x) \in \text{Ker } \nabla u(x)$ and $u(x)$ is orthogonal to \mathbb{S}_x , which is isomorphic to \mathbb{S}^{N-2} . Since $\nabla u(x) \neq 0$ (i.e., the dimension of $\text{Ker } \nabla u(x)$ is at most $N - 1$), we have two situations (as in the case $N = 3$):

- Situation 1: $\dim \text{Ker } \nabla u(x) = N - 1$, which leads to \mathcal{C} isomorphic to \mathbb{S}^{N-3} . In this situation, $\mathbb{S}_x \setminus \mathcal{C}$ is the partition of two connected sets \mathcal{C}_1 and \mathcal{C}_2 that are isomorphic to the half-sphere

$$\mathbb{S}_+^{N-2} = \{\xi = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{S}^{N-2} : \xi_1 > 0\}.$$

The same argument as in the case $N = 3$ shows that the sign of the unit outer normal field $\nu(\xi) \in \{\pm \xi\}$ to the hyperplane $\partial\{u \cdot \xi > 0\}$ is constant when ξ covers \mathcal{C}_j , $j = 1, 2$, so that

$$V_{N-1} \nabla u(x) = \gamma_1 \int_{\mathcal{C}_1} \xi \otimes \xi \, d\xi + \gamma_2 \int_{\mathcal{C}_2} \xi \otimes \xi \, d\xi,$$

with $\gamma_1, \gamma_2 \in \{\pm 1\}$.

- Situation 2: $\dim \text{Ker } \nabla u(x) \leq N - 2$, which leads to the manifold \mathcal{C} of dimension $\leq N - 4$. In other words, $\mathbb{S}_x \setminus \mathcal{C}$ is connected and covers a.e. point of \mathbb{S}_x . The above formula holds for $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$ and $\gamma_1 = \gamma_2 = \pm 1/2$.

We now compute $\nabla u(x)$. For that, we may assume (up to a rotation) that $u(x) = e_N$ and $\mathcal{C}_1 = \mathbb{S}_+^{N-2}$. Since \mathbb{S}_+^{N-2} is invariant under the change of coordinate $\xi_d \mapsto -\xi_d$ for some $2 \leq d \leq N - 1$, we have for every $1 \leq j \leq N - 1$ with $j \neq d$,

$$\int_{\mathbb{S}_+^{N-2}} \xi_j \xi_d \, d\xi = - \int_{\mathbb{S}_+^{N-2}} \xi_j \xi_d \, d\xi = 0,$$

leading to

$$\int_{\mathbb{S}_+^{N-2}} \xi \otimes \xi \, d\xi = \int_{\mathbb{S}_+^{N-2}} \begin{pmatrix} \xi_1^2 & & 0 \\ & \ddots & \\ 0 & & \xi_{N-1}^2 \end{pmatrix} d\xi = \frac{\mathcal{H}^{N-2}(\mathbb{S}^{N-2})}{2(N-1)} \text{Id}. \quad \square$$

Proof of Theorem 7. By Lemma 13, every point in S is an umbilical point. Such a hypersurface is called *totally umbilical*. A classical result in differential geometry states that a totally umbilical hypersurface is either a piece of an $(N-1)$ -sphere or a piece of a hyperplane (see, e.g., [Hicks 1965, Chapter 2, page 36]). \square

We have the following consequence of Lemma 11 and Theorem 8 (whose proof is independent of this section):

Corollary 14. *Under the hypotheses of Theorem 7, there exists a neighborhood ω of S and a diffeomorphism $t \rightarrow F(t)$ such that either $\psi = F(|x - P|)$ for every $x \in \omega$ for a point $P \in \mathbb{R}^N$, or $\psi = F(x \cdot \xi)$ for every $x \in \omega$ for a vector $\xi \in \mathbb{S}^{N-1}$.*

4. Several properties on the set of Lebesgue points

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^N)$. Recall that $x_0 \in \Omega$ is a *Lebesgue point* of u if there exists a vector $u_0 \in \mathbb{R}^N$ such that

$$\lim_{r \rightarrow 0} \int_{B_r(x_0)} |u(x) - u_0| dx = 0. \tag{18}$$

In this case, we write $u(x_0) = u_0$, which is the limit of the average f of u on the ball $B_r(x_0)$ as $r \rightarrow 0$. We denote by $\text{Leb} \subset \Omega$ the set of Lebesgue points of u . It is well known that $\mathcal{H}^N(\Omega \setminus \text{Leb}) = 0$ and one can replace the ball $B_r(x_0)$ by the cube $x_0 + (-r, r)^N$ in the definition (18) to recover the same set of Lebesgue points.

Proof of Proposition 5. We start by proving (9) for a fixed vector $u(x) \in \mathbb{S}^{N-1}$. By rotating the axes if necessary, we may assume that $u(x) = e_N$. Then we compute

$$\int_{\mathbb{S}^{N-1}} \xi \chi(x, \xi) d\mathcal{H}^{N-1}(\xi) = \int_{\mathbb{S}^{N-1} \cap \{\xi_N > 0\}} \xi d\mathcal{H}^{N-1}(\xi) = \left(\int_{\mathbb{S}^{N-1} \cap \{\xi_N > 0\}} \xi_N d\mathcal{H}^{N-1}(\xi) \right) e_N$$

because the integrand is odd in the variables ξ_j for $j = 1, \dots, N-1$. Defining $\xi' := (\xi_1, \dots, \xi_{N-1})$, the half-sphere $\mathbb{S}^{N-1} \cap \{\xi_N > 0\}$ is the graph of the map

$$\xi' \in B^{N-1} \mapsto \xi_N = \sqrt{1 - |\xi'|^2}$$

so that we have

$$\int_{\mathbb{S}^{N-1} \cap \{\xi_N > 0\}} \xi_N d\mathcal{H}^{N-1}(\xi) = \int_{B^{N-1}} \sqrt{1 - |\xi'|^2} \frac{d\xi'}{\sqrt{1 - |\xi'|^2}} = \mathcal{H}^{N-1}(B^{N-1}) = V_{N-1}.$$

The second statement naturally reduces (by a slicing argument) to the case of dimension $N = 2$. In that case, for any $\varphi \in C_c^\infty(\Omega)$, we have $\nabla \times u = \partial_1 u_2 - \partial_2 u_1$ and

$$\begin{aligned} \int_{\Omega} \varphi \nabla \times u dx &= - \int_{\Omega} \nabla^\perp \varphi \cdot u dx \\ &\stackrel{(6)}{=} \frac{1}{2} \int_{\Omega} \int_{\mathbb{S}^1} \nabla \varphi \cdot \xi^\perp \chi(x, \xi) d\mathcal{H}^1(\xi) dx = \frac{1}{2} \int_{\mathbb{S}^1} d\mathcal{H}^1(\xi) \int_{\Omega} \nabla \varphi \cdot \xi^\perp \chi(x, \xi) dx \stackrel{(5)}{=} 0. \quad \square \end{aligned}$$

The following lemma yields the relation between the Lebesgue points of u and Lebesgue points of the functions $\{\chi(\cdot, \xi)\}_{\xi \in \mathbb{S}^{N-1}}$ defined in (4).

Lemma 15. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^N)$.*

- (i) *If $|u| = 1$ a.e. in Ω and x_0 is a Lebesgue point of $\chi(\cdot, \xi)$ for almost every $\xi \in \mathbb{S}^{N-1}$, then x_0 is a Lebesgue point of u and (9) holds at x_0 .*
- (ii) *Let x_0 be a Lebesgue point of u and $\xi \in \mathbb{S}^{N-1}$. If $u(x_0) \cdot \xi \neq 0$, then x_0 is a Lebesgue point of $\chi(\cdot, \xi)$. Conversely, if x_0 is a Lebesgue point of $\chi(\cdot, \xi)$ with $\chi(x_0, \xi) = 1$ (resp. $= 0$), then $u(x_0) \cdot \xi \geq 0$ (resp. ≤ 0).*

Proof. To prove (i), we apply Proposition 5. Indeed, if x_0 is a Lebesgue point of $\chi(\cdot, \xi)$ for a.e. $\xi \in \mathbb{S}^{N-1}$, then Fubini's theorem implies

$$\begin{aligned} \int_{B_r(x_0)} \left| u(x) - \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi \chi(x_0, \xi) d\mathcal{H}^{N-1}(\xi) \right| dx \\ \stackrel{(9)}{\leq} \frac{1}{V_{N-1}} \int_{B_r(x_0)} \int_{\mathbb{S}^{N-1}} |\xi (\chi(x, \xi) - \chi(x_0, \xi))| d\mathcal{H}^{N-1}(\xi) dx \\ \leq \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \left(\int_{B_r(x_0)} |\chi(x, \xi) - \chi(x_0, \xi)| dx \right) d\mathcal{H}^{N-1}(\xi) \xrightarrow{r \rightarrow 0} 0, \end{aligned}$$

where we used the dominated convergence theorem.

Next we prove (ii). We treat the case $u(x_0) \cdot \xi > 0$. For that, we have

$$\begin{aligned} \int_{B_r(x_0)} |\chi(x, \xi) - 1| dx &= \frac{1}{u(x_0) \cdot \xi} \int_{B_r(x_0) \cap \{u \cdot \xi \leq 0\}} u(x_0) \cdot \xi dx \\ &\leq \frac{1}{u(x_0) \cdot \xi} \int_{B_r(x_0) \cap \{u \cdot \xi \leq 0\}} \underbrace{(u(x_0) \cdot \xi - u(x) \cdot \xi)}_{\geq u(x_0) \cdot \xi > 0} dx \leq \frac{1}{u(x_0) \cdot \xi} \int_{B_r(x_0)} |u(x) - u(x_0)| dx. \end{aligned}$$

Since x_0 is a Lebesgue point of u , it follows that x_0 is a Lebesgue point for $\chi(\cdot, \xi)$ with $\chi(x_0, \xi) = 1$. The case $u(x_0) \cdot \xi < 0$ can be shown similarly and we obtain $\chi(x_0, \xi) = 0$. The last statement is a direct consequence of the above lines (using a contradiction argument). □

Remark 16. (a) Note that the condition $u(x_0) \cdot \xi \neq 0$ is essential in Lemma 15(ii). Indeed, if one considers the vortex vector field $u(x) = x/|x|$ for $x \in \mathbb{R}^N \setminus \{0\}$, then for every $\xi \in \mathbb{S}^{N-1}$, any point $x_0 \in \xi^\perp \setminus \{0\}$ is a Lebesgue point of u (because u is smooth around x_0) and satisfies

$$u(x_0) \cdot \xi = 0,$$

but x_0 is not a Lebesgue point of $\chi(\cdot, \xi)$ because

$$\int_{B_r(x_0)} \left| \chi(x, \xi) - \int_{B_r(x_0)} \chi(\cdot, \xi) \right| dx = \int_{B_r(x_0)} \frac{1}{2} dx \not\rightarrow 0 \quad \text{as } r \rightarrow 0,$$

where we used that

$$\int_{B_r(x_0)} \chi(x, \xi) dx = \frac{\mathcal{H}^N(\{x \in B_r(x_0) : x \cdot \xi > 0\})}{\mathcal{H}^N(B_r(x_0))} \stackrel{x=y+x_0}{=} \frac{\mathcal{H}^N(\{y \in B_r(0) : y \cdot \xi > 0\})}{\mathcal{H}^N(B_r(0))} = \frac{1}{2}.$$

(b) Note that in Lemma 15(ii), one cannot conclude in general that $u(x_0) \cdot \xi > 0$ provided that $\chi(x_0, \xi) = 1$. Indeed, consider for example $\xi = e_N$, $u(x) \cdot \xi = u_N(x) := |x|$ for $x \in \mathbb{R}^N$ and set $x_0 = 0$; then $\chi(\cdot, \xi) = 1$ in $\mathbb{R}^N \setminus \{x_0\}$, x_0 is a Lebesgue point of u_N and $\chi(\cdot, \xi)$ with $\chi(x_0, \xi) = 1$, but $u_N(x_0) = 0$.

We now prove one of the key tools in the proof of Theorem 8, which mimics the relation of the ordering of level sets of a stream function when (8) holds true. It is a generalization of Proposition 3.1 in [Jabin, Otto, and Perthame 2002] to the case of dimension N :

Proposition 17 (ordering). *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^N)$ satisfy the kinetic formulation (8). Assume that $y, z \in \text{Leb}$ are two different Lebesgue points of u such that the closed segment $[yz]$ is included in Ω . Then for every direction $\xi \in \mathbb{S}^{N-1}$ with $\xi \in (z - y)^\perp$, we have*

$$u(y) \cdot \xi > 0 \text{ (resp. } < 0) \implies u(z) \cdot \xi \geq 0 \text{ (resp. } \leq 0); \tag{19}$$

moreover, y and z are Lebesgue points of $\chi(\cdot, \xi)$ and $\chi(y, \xi) = \chi(z, \xi)$. As a consequence, if $u \neq 0$ a.e. in Ω , then for a.e. $y \in \Omega$, \mathcal{H}^{N-1} -a.e. $\xi \in \mathbb{S}^{N-1}$ and \mathcal{H}^{N-1} -a.e. $v \in \xi^\perp$ with the segment $[y, y + v]$ included in Ω , we have that y and $y + v$ are Lebesgue points of u and

$$\chi(y, \xi) = \chi(y + v, \xi). \tag{20}$$

Proof. First, we consider the case $u(y) \cdot \xi > 0$. By Lemma 15(ii), y is a Lebesgue point of $\chi(\cdot, \xi)$ and $\chi(y, \xi) = 1$. Let

$$\left\{ \rho_\varepsilon(\cdot) = \frac{1}{\varepsilon^N} \rho\left(\frac{\cdot}{\varepsilon}\right) \right\}_{\varepsilon > 0}$$

be a standard family of mollifiers, where ρ is a nonnegative radial smooth function having as support the unit ball $\text{supp } \rho = B_1 \subset \mathbb{R}^N$ and $\int_{B_1} \rho \, dx = 1$. Set the convoluted function

$$\chi_\varepsilon := \rho_\varepsilon * \chi(\cdot, \xi)$$

in a neighborhood $\omega \subset \Omega$ of the segment $[yz]$ for $\varepsilon > 0$ sufficiently small. Then χ_ε is smooth in ω and for every Lebesgue point $x \in \omega$ of $\chi(\cdot, \xi)$ we have $\chi_\varepsilon(x) \rightarrow \chi(x, \xi)$ as $\varepsilon \rightarrow 0$ because

$$\begin{aligned} |\chi_\varepsilon(x) - \chi(x, \xi)| &= \left| \int_{B_\varepsilon(0)} (\chi(x - \tilde{x}, \xi) - \chi(x, \xi)) \rho_\varepsilon(\tilde{x}) \, d\tilde{x} \right| \\ &\leq \frac{\sup \rho}{\varepsilon^N} \int_{B_\varepsilon(0)} |\chi(x - \tilde{x}, \xi) - \chi(x, \xi)| \, d\tilde{x} \\ &\leq C \int_{B_\varepsilon(x)} |\chi(\tilde{y}, \xi) - \chi(x, \xi)| \, d\tilde{y} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

In particular, $\lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(y) = \chi(y, \xi) = 1$. Let $v = z - y$. We will show that $\chi(y + v, \xi) = 1$. For that, we have $v \in \xi^\perp$ and

$$v \cdot \nabla_x \chi_\varepsilon = v \cdot \nabla_x \chi(\cdot, \xi) * \rho_\varepsilon \stackrel{(8)}{=} 0 \text{ in } \omega.$$

Then

$$\chi_\varepsilon(y + v) - \chi_\varepsilon(y) = \int_0^1 v \cdot \nabla_x \chi_\varepsilon(y + tv) \, dt = 0$$

so that

$$\lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(z) = \lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(y) = \chi(y, \xi) = 1.$$

This implies that $u(z) \cdot \xi \geq 0$. Assume by contradiction that $u(z) \cdot \xi < 0$. By Lemma 15(ii), z is a Lebesgue point of $\chi(\cdot, \xi)$ with $\chi(z, \xi) = 0$ so that

$$\lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(z) = \chi(z, \xi) = 0,$$

which contradicts the above statement. We prove now the following:

Claim. If $\chi_\varepsilon(z) \rightarrow 1$ as $\varepsilon \rightarrow 0$, then z is a Lebesgue point of $\chi(\cdot, \xi)$ with $\chi(z, \xi) = 1$.

Proof of Claim. Let $\{\varepsilon_k\}$ be a sequence converging to 0 as $k \rightarrow \infty$. For k large enough, we define $f_k : B_1 \rightarrow \{0, 1\}$ by $f_k(x) = \chi(z - \varepsilon_k x, \xi)$ for every $x \in B_1$. Then the sequence $\{f_k\}$ is bounded in $L^2(B_1)$ and up to a subsequence, $f_k \rightharpoonup f$ weakly in $L^2(B_1)$, where the limit $f : B_1 \rightarrow \mathbb{R}$ takes values in $[0, 1]$. Therefore, we have for our smooth mollifier $\rho \in L^2(B_1)$ that

$$\int_{B_1} \rho f_k dx \rightarrow \int_{B_1} \rho f dx \quad \text{as } k \rightarrow \infty.$$

Note now that by the change of variable $\tilde{x} = z - \varepsilon_k x$ we obtain by our assumption:

$$\int_{B_1} \rho(x) f_k(x) dx = \int_{B_{\varepsilon_k}(z)} \rho_{\varepsilon_k}(z - \tilde{x}) \chi(\tilde{x}, \xi) d\tilde{x} = \chi_{\varepsilon_k}(z) \rightarrow 1 \quad \text{as } k \rightarrow \infty;$$

therefore, $\int_{B_1} \rho f dx = 1$. Since 1 is the maximal value of f and ρ is nonnegative with the integral on B_1 equal to 1, we deduce that $f = 1$ in $\text{supp } \rho = B_1$. It follows by the change of variable $\tilde{x} = z - \varepsilon_k x$ that

$$\int_{B_{\varepsilon_k}(z)} |\chi(\tilde{x}, \xi) - 1| d\tilde{x} = 1 - \int_{B_1(0)} f_k(x) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

because $f_k \rightharpoonup 1$ weakly in $L^2(B_1)$. Since the limit is unique for every subsequence $\varepsilon_k \rightarrow 0$, we conclude that z is a Lebesgue point of $\chi(\cdot, \xi)$ with $\chi(z, \xi) = 1$, which proves the claim. \square

For the case $u(y) \cdot \xi < 0$, i.e., $\chi(y, \xi) = 0$ by Lemma 15(ii), one applies the above argument by replacing ξ with $-\xi$ and obtains that z is a Lebesgue point of $\chi(\cdot, -\xi)$ with $\chi(z, -\xi) = 1$. It follows that z is a Lebesgue point of $\chi(\cdot, \xi)$ with $\chi(z, \xi) = 0$ because

$$\int_{B_r(z)} |\chi(x, \xi)| dx \leq \frac{\mathcal{H}^N(\{x \in B_r(z) : u(x) \cdot \xi \geq 0\})}{\mathcal{H}^N(B_r(z))} = 1 - \int_{B_r(z)} \chi(x, -\xi) dx \rightarrow 0$$

as $r \rightarrow 0$. One also concludes that $u(z) \cdot \xi \leq 0$ by Lemma 15(ii).

For the last statement, we have for a.e. $y \in \Omega$ that y is a Lebesgue point of u with $u(y) \neq 0$. Then for \mathcal{H}^{N-1} -a.e. direction $\xi \in \mathbb{S}^{N-1}$, we have that $u(y) \cdot \xi \neq 0$ and $y + v$ is a Lebesgue point of u for \mathcal{H}^{N-1} -a.e. $v \in \xi^\perp$ with the segment $[y, y + v] \subset \Omega$. By the above argument, we get (20). \square

5. Notion of the trace on lines

The $H^{1/2}$ -regularity for N -dimensional unit-length vector fields u satisfying the kinetic formulation (8) (see [Golse, Lions, Perthame, and Sentis 1988]) is a priori not enough to define the notion of the trace of u on 1-dimensional lines. However, using the ideas in [Jabin, Otto, and Perthame 2002] for dimension 2, we will define a notion of the trace of u on segments (in the sense of Lebesgue points) in any dimension $N \geq 2$.

Proposition 18 (trace). *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \rightarrow \mathbb{S}^{N-1}$ be a Lebesgue-measurable vector field satisfying the kinetic formulation (8). Assume that the segment*

$$L := \{0\}^{N-1} \times [-1, 1] \text{ is included in } \Omega.$$

Then there exists a Lebesgue-measurable function $\tilde{u} : (-1, 1) \rightarrow \mathbb{R}^N$ such that

$$\lim_{r \rightarrow 0} \int_{(-r,r)^{N-1}} \int_{-1}^1 |u(x', x_N) - \tilde{u}(x_N)| dx_N dx' = 0, \tag{21}$$

where $x = (x', x_N)$, $x' = (x_1, \dots, x_{N-1})$. Moreover, for \mathcal{H}^1 -a.e. $x_N \in (-1, 1)$,

$$\tilde{u}(x_N) = \lim_{r \rightarrow 0} \int_{(-r,r)^{N-1}} u(x', x_N) dx' \quad \text{and} \quad |\tilde{u}(x_N)| = 1. \tag{22}$$

Finally, every Lebesgue point $x \in \text{Leb}$ of u lying inside L is a Lebesgue point of \tilde{u} and $u(x) = \tilde{u}(x_N)$. The vector field \tilde{u} is called the trace of u on the segment L .

Proof. To simplify the writing, we assume that $\Omega = \mathbb{R}^N$. We divide the proof into several steps:

Step 1: defining the 1-dimensional function $\tilde{\chi}(\cdot, \xi)$ for suitable directions $\xi \in \mathbb{S}^{N-1}$. Let \mathcal{D} be the set of directions $\xi \in \mathbb{S}^{N-1}$ such that $\xi_N \neq 0$ and (20) holds true for the triple $(y, y + v, \xi)$ for a.e. $y \in \Omega$ and \mathcal{H}^{N-1} -a.e. $v \in \xi^\perp$ (with the segment $[y, y + v] \subset \Omega$, where y and $y + v$ are Lebesgue points of u). By Proposition 17, we know that \mathcal{D} covers \mathbb{S}^{N-1} up to a set of zero \mathcal{H}^{N-1} -measure. For such a direction $\xi \in \mathcal{D}$, we can choose a point $y_\xi \in \Omega$ (in a neighborhood of L) such that the map $\xi \in \mathcal{D} \mapsto y_\xi \in \Omega$ is Lebesgue measurable, the point $y_\xi + t\xi \in \Omega$ is a Lebesgue point of $\chi(\cdot, \xi)$ for \mathcal{H}^1 -a.e. $t \in \mathbb{R}$, the function $t \mapsto \chi(y_\xi + t\xi, \xi)$ is \mathcal{H}^1 -measurable (by Fubini's theorem) and (20) holds true for the triple $(y_\xi + t\xi, y_\xi + t\xi + v, \xi)$ for \mathcal{H}^{N-1} -a.e. $v \in \xi^\perp$ and \mathcal{H}^1 -a.e. t . Define the 1-dimensional function

$$s \mapsto \tilde{\chi}(s, \xi) := \chi(y_\xi + (s - y_\xi \cdot \xi)\xi, \xi) \in \{0, 1\}.$$

Then we have that for a.e. $x \in \Omega$ in a neighborhood of L ,

$$\tilde{\chi}(x \cdot \xi, \xi) = \chi(y_\xi - y_\xi \cdot \xi \xi + x \cdot \xi \xi, \xi) \stackrel{(20)}{=} \chi(x, \xi), \tag{23}$$

because

$$v = y_\xi - y_\xi \cdot \xi \xi + x \cdot \xi \xi - x \in \xi^\perp.$$

Step 2: for $\xi \in \mathcal{D}$ and for every Lebesgue point $P = (0, \dots, 0, P_N) \in L$ of $\chi(\cdot, \xi)$ with $P_N \in (-1, 1)$, the point $P \cdot \xi$ is a Lebesgue point of $\tilde{\chi}(\cdot, \xi)$ and $\tilde{\chi}(P_N \xi_N, \xi) = \chi(P, \xi)$. Indeed, since $\xi_N \neq 0$, we have

$$\begin{aligned} & \int_{P_N \xi_N - r |\xi_N|}^{P_N \xi_N + r |\xi_N|} |\tilde{\chi}(t, \xi) - \chi(P, \xi)| dt \\ &= \int_{(-r, r)^{N-1}} dx' \int_{P_N - r}^{P_N + r} |\tilde{\chi}(\tilde{x}_N \xi_N, \xi) - \chi(P, \xi)| d\tilde{x}_N \quad (\text{since } t = \tilde{x}_N \xi_N) \\ &= \int_{(-r, r)^{N-1}} dx' \int_{P_N - x' \cdot \xi' / \xi_N - r}^{P_N - x' \cdot \xi' / \xi_N + r} \underbrace{|\tilde{\chi}(x' \cdot \xi' + x_N \xi_N, \xi) - \chi(P, \xi)|}_{\stackrel{(23)}{=} |\chi(x, \xi)|} dx_N \quad (\text{since } x' \cdot \xi' + x_N \xi_N = \tilde{x}_N \xi_N) \\ &\leq \int_{(-r, r)^{N-1}} dx' \frac{1}{2r} \int_{P_N - \tilde{r}}^{P_N + \tilde{r}} |\chi(x, \xi) - \chi(P, \xi)| dx_N \\ &\leq C \int_{P + (-\tilde{r}, \tilde{r})^N} |\chi(x, \xi) - \chi(P, \xi)| dx \rightarrow 0 \quad \text{as } r \rightarrow 0, \end{aligned}$$

where we used that $|x' \cdot \xi'| \leq r \sqrt{N-1}$ for $x' \in (-r, r)^{N-1}$ and $\tilde{r} = (\sqrt{N-1}/|\xi_N| + 1)r$. Thus, $P_N \xi_N$ is a Lebesgue point of $\tilde{\chi}(\cdot, \xi)$. In particular, we have by Fubini's theorem, for every $\alpha > 0$,

$$\begin{aligned} & \int_{-\alpha r}^{\alpha r} d\tilde{t} \int_{P_N \xi_N - r |\xi_N| + \tilde{t}}^{P_N \xi_N + r |\xi_N| + \tilde{t}} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dt \\ &= \frac{1}{4\alpha |\xi_N| r^2} \int_{-\alpha r}^{\alpha r} \int_{P_N \xi_N - r(|\xi_N| + \alpha)}^{P_N \xi_N + r(|\xi_N| + \alpha)} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| \mathbb{1}_{(P_N \xi_N - r|\xi_N| + \tilde{t}, P_N \xi_N + r|\xi_N| + \tilde{t})}(t) dt d\tilde{t} \\ &= \frac{1}{4\alpha |\xi_N| r^2} \int_{P_N \xi_N - r(|\xi_N| + \alpha)}^{P_N \xi_N + r(|\xi_N| + \alpha)} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dt \int_{-\alpha r}^{\alpha r} \mathbb{1}_{(-P_N \xi_N - r|\xi_N| + t, -P_N \xi_N + r|\xi_N| + t)}(\tilde{t}) d\tilde{t} \\ &\leq \frac{1}{2|\xi_N| r} \int_{P_N \xi_N - r(|\xi_N| + \alpha)}^{P_N \xi_N + r(|\xi_N| + \alpha)} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dt \rightarrow 0 \quad \text{as } r \rightarrow 0. \end{aligned} \tag{24}$$

Step 3: proof of (21). For $\xi \in \mathcal{D}$, we have, for small $r > 0$,

$$\begin{aligned} & \int_{(-r, r)^{N-1}} \int_{-1}^1 |\chi(x, \xi) - \tilde{\chi}(x_N \xi_N, \xi)| dx' dx_N \\ & \stackrel{(23)}{=} \int_{(-r, r)^{N-1}} \int_{-1}^1 |\tilde{\chi}(x' \cdot \xi' + x_N \xi_N, \xi) - \tilde{\chi}(x_N \xi_N, \xi)| dx' dx_N \\ & \leq \frac{1}{|\xi_N|} \sup_{|\tilde{t}| \leq r \sqrt{N-1}} \int_{-|\xi_N|}^{|\xi_N|} |\tilde{\chi}(t + \tilde{t}, \xi) - \tilde{\chi}(t, \xi)| dt \quad (\text{since } t = x_N \xi_N) \end{aligned}$$

because $|x' \cdot \xi'| \leq r \sqrt{N-1}$. Since the 1-dimensional function $t \mapsto \tilde{\chi}(t, \xi)$ belongs to L^∞ , its L^1 -modulus of continuity present in the above right-hand side tends to 0 as $r \rightarrow 0$, which leads to

$$\lim_{r \rightarrow 0} \int_{(-r, r)^{N-1}} \int_{-1}^1 |\chi(x, \xi) - \tilde{\chi}(x_N \xi_N, \xi)| dx' dx_N = 0.$$

This formula can be interpreted as the notion of the trace of $\chi(\cdot, \xi)$ on the segment L and yields (21). Indeed, due to (9), we define for a.e. $x_N \in (-1, 1)$,

$$\tilde{u}(x_N) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi \tilde{\chi}(x_N \xi_N, \xi) d\mathcal{H}^{N-1}(\xi)$$

and we obtain, by Fubini's theorem,

$$\begin{aligned} & \int_{(-r,r)^{N-1}} \int_{-1}^1 |u(x', x_N) - \tilde{u}(x_N)| dx' dx_N \\ & \leq \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \left(\int_{(-r,r)^{N-1}} \int_{-1}^1 |\chi(x, \xi) - \tilde{\chi}(x_N \xi_N, \xi)| dx' dx_N \right) d\mathcal{H}^{N-1}(\xi) \xrightarrow{r \rightarrow 0} 0, \end{aligned}$$

where we used the dominated convergence theorem.

Step 4: proof of (22). By Step 3, we deduce that

$$\int_{(-r,r)^{N-1}} u(x', \cdot) dx' \xrightarrow{r \rightarrow 0} \tilde{u} \quad \text{in } L^1((-1, 1));$$

therefore, the first statement in (22) follows immediately. Moreover,

$$\begin{aligned} \int_{-1}^1 |\tilde{u}(x_N) - 1| dx_N &= \int_{(-r,r)^{N-1}} \int_{-1}^1 |\tilde{u}(x_N) - u(x', x_N)| dx' dx_N \\ &\leq \int_{(-r,r)^{N-1}} \int_{-1}^1 |\tilde{u}(x_N) - u(x', x_N)| dx' dx_N \xrightarrow{(21)} 0 \quad \text{as } r \rightarrow 0; \end{aligned}$$

thus, $|\tilde{u}(x_N)| = 1$ for \mathcal{H}^1 -a.e. $x_N \in (-1, 1)$.

Step 5: conclusion. Let $P = (0, \dots, 0, P_N) \in \text{Leb}$ be a Lebesgue point of u with $P_N \in (-1, 1)$. We want to show that P_N is a Lebesgue point of \tilde{u} and $\tilde{u}(P_N) = u(P)$. For that, we know by Lemma 15 that P is a Lebesgue point of $\chi(\cdot, \xi)$ for every direction $\xi \in \mathbb{S}^{N-1}$ with $u(P) \cdot \xi \neq 0$. If in addition $\xi \in \mathcal{D}$, we know by Step 2 that $P \cdot \xi$ is also a Lebesgue point of $\tilde{\chi}(\cdot, \xi)$. By the same argument as in Step 3, we have

$$\begin{aligned} & \int_{P+(-r,r)^N} |u(x', x_N) - \tilde{u}(x_N)| dx' dx_N \\ & \leq \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \int_{P+(-r,r)^N} \left| \underbrace{\chi(x, \xi)}_{\stackrel{(23)}{=} \tilde{\chi}(x' \cdot \xi' + x_N \xi_N, \xi)} - \tilde{\chi}(x_N \xi_N, \xi) \right| dx' dx_N d\mathcal{H}^{N-1}(\xi) \\ & \leq \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) \left[\int_{P+(-r,r)^N} |\tilde{\chi}(x' \cdot \xi' + x_N \xi_N, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dx \right. \\ & \quad \left. + \int_{P_N-r}^{P_N+r} |\tilde{\chi}(x_N \xi_N, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dx_N \right] \\ & \leq \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) \int_{(-r,r)^{N-1}} dx' \int_{P_N \xi_N - r |\xi_N| + x' \cdot \xi'}^{P_N \xi_N + r |\xi_N| + x' \cdot \xi'} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dt \\ & \quad + \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) \int_{P_N \xi_N - r |\xi_N|}^{P_N \xi_N + r |\xi_N|} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dt. \end{aligned}$$

Using the dominated convergence theorem twice, we conclude that the above right-hand side vanishes as $r \rightarrow 0$. Indeed, the second integrand converges to 0 as $r \rightarrow 0$ by Step 2 for a.e. $\xi \in \mathbb{S}^{N-1}$. For the first integrand, we proceed as follows: for \mathcal{H}^{N-1} -a.e. direction ξ , we may assume that $|\xi'| > 0$ and $\xi_N \neq 0$ so that there exists a rotation $R' \in \text{SO}(N - 1)$ with $R'\xi' = |\xi'|e_1$ and we have by the change of variables $\tilde{x}' = R'x'$ and $\hat{r} = r\sqrt{N - 1}$,

$$\begin{aligned} & \int_{(-r,r)^{N-1}} dx' \int_{P_N \xi_N - r|\xi_N| + x' \cdot \xi'}^{P_N \xi_N + r|\xi_N| + x' \cdot \xi'} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dt \\ & \leq C \int_{\{|\tilde{x}'| < \hat{r}\}} d\tilde{x}' \int_{P_N \xi_N - r|\xi_N| + \tilde{x}_1 |\xi'|}^{P_N \xi_N + r|\xi_N| + \tilde{x}_1 |\xi'|} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dt \\ & \leq C \int_{-|\xi'| \hat{r}}^{|\xi'| \hat{r}} \int_{P_N \xi_N - r|\xi_N| + \tilde{t}}^{P_N \xi_N + r|\xi_N| + \tilde{t}} |\tilde{\chi}(t, \xi) - \tilde{\chi}(P_N \xi_N, \xi)| dt d\tilde{t} \stackrel{(24)}{\rightarrow} 0 \quad \text{as } r \rightarrow 0. \quad \square \end{aligned}$$

6. Proof of Theorem 8

We start by showing some preliminary results that reveal the geometric consequences of the kinetic formulation (8). The following lemma is the first step in proving that u is constant along the characteristics and is reminiscent of the ideas presented in [Jabin, Otto, and Perthame 2002]:

Lemma 19. *Let $\Omega \subset \mathbb{R}^N$ be an open set such that $L = \{0\}^{N-1} \times [-1, 1] \subset \Omega$ and $u : \Omega \rightarrow \mathbb{S}^{N-1}$ be a Lebesgue-measurable vector field satisfying the kinetic formulation (8). Assume that the origin $O \in \Omega \cap \text{Leb}$ is a Lebesgue point of u and $u(O) = e_N$. Then for every Lebesgue point $x_N \in (-1, 1)$ of \tilde{u} , we have*

$$\tilde{u}(x_N) = \pm e_N,$$

where \tilde{u} is the trace of u on L defined at Proposition 18.

Proof. Without loss of generality we assume that Ω is a convex open neighborhood of L . By Proposition 18, we know that O is also a Lebesgue point of \tilde{u} and $\tilde{u}(0) = e_N$; moreover, $|\tilde{u}| = 1$ a.e. in $(-1, 1)$. Let $x_N \in (-1, 1) \setminus \{0\}$ be a Lebesgue point of \tilde{u} such that \mathcal{H}^{N-1} -a.e. $z \in \Omega \cap (x_N e_N + e_N^\perp)$ is a Lebesgue point of u and such that the following holds true (see Proposition 18):

$$\lim_{r \rightarrow 0} \int_{(-r,r)^{N-1}} |u(x', x_N) - \tilde{u}(x_N)| dx' = 0. \tag{25}$$

Our goal is to prove that the component $\tilde{u}_i(x_N)$ of $\tilde{u}(x_N)$ in direction e_i vanishes for every $i = 1, \dots, N - 1$. For that, we follow the ideas in [Jabin, Otto, and Perthame 2002]. Let $\varepsilon > 0$ be small and define the following subsets E_i^- and E_i^+ (depending on ε) of the hyperplane $(x_N e_N + e_N^\perp)$ for $1 \leq i \leq N - 1$:

$$E_i^\pm = \{z \in \Omega \cap \text{Leb} : z_N = x_N, \varepsilon |x_N| \geq \pm z_i > 0\}.$$

By our assumption, the sets E_i^\pm contain many points (e.g., for $i = 1$, the set E_1^+ covers the $(N-1)$ -parallelepiped $(0, r) \times (-r, r)^{N-2} \times \{x_N\}$ up to a set of zero \mathcal{H}^{N-1} -measure for $r < \varepsilon$). For $z \in E_i^+$, we set $y = -z_i e_N + x_N e_i$ if $x_N > 0$ and $y = z_i e_N - x_N e_i$ if $x_N < 0$. Obviously, $z \cdot y = 0$; that is, $y \in z^\perp$.

By the convexity of Ω , the segment $[Oz]$ lies in Ω so that by Proposition 17 we have if $x_N > 0$ (resp. $x_N < 0$), then $u(O) \cdot y = -z_i < 0$ (resp. $u(O) \cdot y = z_i > 0$) so that $u(z) \cdot y \leq 0$ (resp. ≥ 0). It follows that

$$u_i(z) \leq \frac{z_i}{x_N} u_N(z) \leq \varepsilon \quad \left(\text{resp. } u_i(z) \geq \frac{-z_i}{|x_N|} u_N(z) \geq -\varepsilon \right),$$

because $|u_N(z)| \leq 1$. Similarly, for $z \in E_i^-$, one uses $y = z_i e_N - x_N e_i$ if $x_N > 0$ and $y = -z_i e_N + x_N e_i$ if $x_N < 0$ and deduces that $u_i(z) \geq -\varepsilon$ if $x_N > 0$ and $u_i(z) \leq \varepsilon$ if $x_N < 0$. We conclude that $\tilde{u}_i(x_N) \in [-\varepsilon, \varepsilon]$. Indeed, let us set $i = 1$ for simplicity of notation; by (25), we have

$$\tilde{u}_1(x_N) = \lim_{r \rightarrow 0} \int_{(0,r) \times (-r,r)^{N-2}} u_1(x', x_N) dx' \leq \varepsilon \quad \text{if } x_N > 0 \quad (\text{resp. } \geq -\varepsilon \text{ if } x_N < 0)$$

and also,

$$\tilde{u}_1(x_N) = \lim_{r \rightarrow 0} \int_{(-r,0) \times (-r,r)^{N-2}} u_1(x', x_N) dx' \geq -\varepsilon \quad \text{if } x_N > 0 \quad (\text{resp. } \leq \varepsilon \text{ if } x_N < 0).$$

Passing to the limit $\varepsilon \rightarrow 0$, we conclude that $\tilde{u}_i(x_N) = 0$ for $i = 1$ (similarly, for every $1 \leq i \leq N - 1$). Obviously, \mathcal{H}^1 -a.e. $x_N \in (-1, 1)$ satisfies this property. As a consequence, if $P_N \in (-1, 1)$ is a Lebesgue point of \tilde{u} then for every $1 \leq i \leq N - 1$,

$$\tilde{u}_i(P_N) = \lim_{r \rightarrow 0} \int_{P_N-r}^{P_N+r} \tilde{u}_i(x_N) dx_N = 0.$$

Since $|\tilde{u}(P_N)| = 1$, we deduce that $\tilde{u}_N(P_N) = \pm 1$, that is, $\tilde{u}(P_N) = \pm e_N$. □

We now prove the main result:

Proof of Theorem 8. We first treat the case where Ω is a ball and then the general case of a connected open set.

Case I: Ω is a ball. Since u is not a constant vector field, there exist two Lebesgue points $P_0, P_1 \in \Omega \cap \text{Leb}$ of u such that

$$u(P_0) \neq u(P_1).$$

Let D_0 (resp. D_1) be the line directed by $u(P_0)$ (resp. $u(P_1)$) that passes through P_0 (resp. P_1).

Step 1: we show that D_0 and D_1 are coplanar. Assume by contradiction that D_0 and D_1 are not coplanar; in particular $|u(P_0) \cdot u(P_1)| < 1$. Set $A \in D_0$ and $B \in D_1$ such that

$$0 < |A - B| = \min_{x \in D_0, y \in D_1} |x - y|.$$

Obviously, the segment $[AB]$ is orthogonal to D_0 and D_1 . Let O be the middle point of the segment $[AB]$ (see Figure 2). Let

$$w_1 = u(P_0), \quad w_2 = \frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} \quad \text{and} \quad w_3 = \alpha u(P_0) + \beta u(P_1),$$

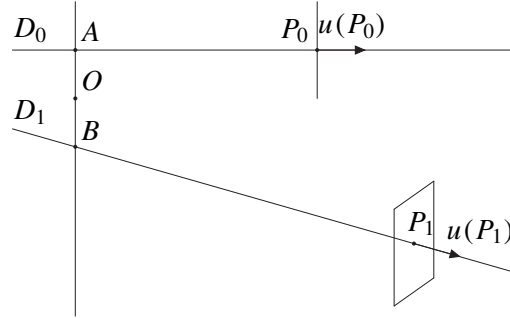


Figure 2. Two noncoplanar lines D_0 and D_1 .

where

$$\alpha = \frac{-u(P_0) \cdot u(P_1)}{\sqrt{1 - (u(P_0) \cdot u(P_1))^2}} \quad \text{and} \quad \beta = \frac{1}{\sqrt{1 - (u(P_0) \cdot u(P_1))^2}} > 0. \quad (26)$$

The choice of α and β is done in order to ensure that $w_1 \cdot w_3 = 0$ and $|w_3|^2 = 1$, which finally yields the orthonormal basis w_1, w_2 and w_3 . Note now that the vectors $u(P_0)$ and $u(P_1)$ have the following components in the basis (w_1, w_2, w_3) :

$$u(P_0) = (1, 0, 0) \quad \text{and} \quad u(P_1) = \left(-\frac{\alpha}{\beta}, 0, \frac{1}{\beta} \right).$$

We want to find the expression of $\overrightarrow{P_0 P_1}$ in that basis, too. For that, we have

$$\overrightarrow{P_0 P_1} = \overrightarrow{P_0 A} + \overrightarrow{AB} + \overrightarrow{B P_1},$$

which implies the existence of three real numbers $\lambda, \tilde{\lambda}, \hat{\lambda} \in \mathbb{R}$ with $\tilde{\lambda} \neq 0$ such that

$$\overrightarrow{P_0 P_1} = \lambda w_1 + \tilde{\lambda} w_2 + \hat{\lambda} u(P_1) = \lambda w_1 + \tilde{\lambda} w_2 + \hat{\lambda} \left(\frac{1}{\beta} w_3 - \frac{\alpha}{\beta} w_1 \right).$$

Thus, $\overrightarrow{P_0 P_1}$ has the following components in the basis (w_1, w_2, w_3) :

$$\overrightarrow{P_0 P_1} = \left(\lambda - \frac{\alpha}{\beta} \hat{\lambda}, \tilde{\lambda}, \frac{\hat{\lambda}}{\beta} \right).$$

Define the vector $\xi := (1, s, -\beta) \neq 0$, written in our basis where

$$s := \frac{\hat{\lambda}(\alpha + \beta)}{\beta \tilde{\lambda}} - \frac{\lambda}{\tilde{\lambda}}.$$

Then $[P_0 P_1] \subset \Omega$ (since Ω is a ball) and

$$\begin{aligned} \overrightarrow{P_0 P_1} \cdot \xi &= 0, \quad \text{i.e., } \xi \in P_0 P_1^\perp, \\ u(P_0) \cdot \xi &= 1 > 0, \quad u(P_1) \cdot \xi = u(P_0)u(P_1) - 1 < 0, \end{aligned}$$

which contradicts Proposition 17. Thus, D_0 and D_1 are coplanar.

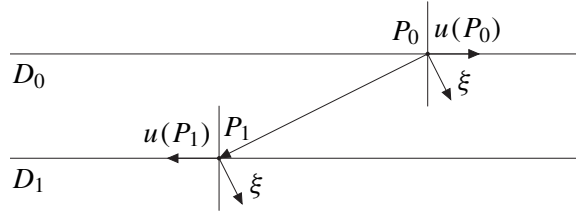


Figure 3. Two parallel lines D_0 and D_1 .

Step 2: we show that D_0 and D_1 must intersect (D_0 might coincide with D_1). Assume by contradiction that D_0 and D_1 are parallel and $D_0 \neq D_1$. This means that $u(P_0) = -u(P_1)$ (because of our choice $u(P_0) \neq u(P_1)$). Set (w_1, w_2) to be an orthonormal basis in the 2-dimensional plane Π determined by D_0 and D_1 with $w_1 = u(P_0)$. In the basis (w_1, w_2) , we write $\overrightarrow{P_0P_1} = (\lambda, \tilde{\lambda})$, where $\tilde{\lambda} \neq 0$ (since $D_0 \neq D_1$), and set $\xi = (-\tilde{\lambda}, \lambda)$ to be an orthogonal vector to $\overrightarrow{P_0P_1}$ in Π (see Figure 3). Then one checks that $u(P_0) \cdot \xi = -\tilde{\lambda}$ and $u(P_1) \cdot \xi = \tilde{\lambda}$ have different signs, which again contradicts Proposition 17.

Step 3: there exists a point $O \in D_0$ with $O \neq P_0, P_1$ and a sign $\gamma \in \{\pm 1\}$ such that

$$u(P_i) = \gamma \frac{\overrightarrow{OP_i}}{|\overrightarrow{OP_i}|}, \quad i = 0, 1.$$

If $D_0 = D_1$, then $u(P_0) = -u(P_1)$, so any point $O \in D_0$ located between P_0 and P_1 leads to the conclusion. Otherwise, $D_0 \neq D_1$ and we define $\{O\} = D_0 \cap D_1$. First, we prove that $O \neq P_0, P_1$. Assume by contradiction that $O = P_0 \in D_0 \cap D_1$. Then by Proposition 18 we know that P_0 and P_1 are Lebesgue points of the trace \tilde{u} of u on the segment $D_1 \cap \Omega$ (directed by $u(P_1)$) with $\tilde{u}(P_0) = u(P_0)$ and $\tilde{u}(P_1) = u(P_1)$ so that by Lemma 19, we should have $u(P_0)$ is parallel with $u(P_1)$, which is a contradiction with $D_0 \neq D_1$. So, $O \neq P_0, P_1$. Next, note that for any orthogonal vector ξ to $\overrightarrow{P_0P_1}$ in the plane determined by D_0 and D_1 , we have by Proposition 17 that $u(P_0) \cdot \xi$ and $u(P_1) \cdot \xi$ have the same sign, i.e.,

$$(u(P_0) \cdot \xi) \cdot (u(P_1) \cdot \xi) \geq 0. \quad (27)$$

Write now

$$\overrightarrow{OP_0} = \lambda u(P_0) \quad \text{and} \quad \overrightarrow{OP_1} = \tilde{\lambda} u(P_1)$$

with $\lambda, \tilde{\lambda}$ nonzero real numbers. The conclusion of Step 3 is equivalent to proving that λ and $\tilde{\lambda}$ have the same sign. For that, as in Step 1, we choose the orthonormal basis $w_1 = u(P_0)$ and $w_2 = \alpha u(P_0) + \beta u(P_1)$ with $\alpha \in \mathbb{R}$ and $\beta > 0$ given in (26) (recall that $|u(P_0) \cdot u(P_1)| < 1$ because of the assumption $D_0 \neq D_1$). Since $\overrightarrow{P_0P_1} = \overrightarrow{OP_1} - \overrightarrow{OP_0} = \tilde{\lambda} u(P_1) - \lambda u(P_0)$, we write, in the basis (w_1, w_2) ,

$$u(P_0) = (1, 0), \quad u(P_1) = \left(-\frac{\alpha}{\beta}, \frac{1}{\beta}\right), \quad \overrightarrow{P_0P_1} = \left(-\lambda - \frac{\alpha}{\beta} \tilde{\lambda}, \frac{\tilde{\lambda}}{\beta}\right).$$

Then for the orthogonal vector $\xi = (\tilde{\lambda}, \lambda\beta + \alpha\tilde{\lambda}) \neq 0$ to $\overrightarrow{P_0P_1}$, we have by (27) that

$$0 \leq (u(P_0) \cdot \xi) \cdot (u(P_1) \cdot \xi) = \tilde{\lambda} \cdot \lambda.$$

Step 4: conclusion. For every Lebesgue point $P \in \text{Leb} \cap \Omega$ of u , we consider the line D passing through P and directed by $u(P)$. Call \mathcal{D} the set of these lines. Obviously, \mathcal{D} covers \mathcal{H}^N -almost all of the ball Ω (since $\mathcal{H}^N(\Omega \setminus \text{Leb}) = 0$); in particular, \mathcal{D} is not planar. By Step 1, we know that every two lines in \mathcal{D} are coplanar. Then Proposition 9 (whose proof is presented below) implies that either all these lines are parallel, or they pass through the same point O . Since u is nonconstant, we deduce by Step 2 that only the last situation holds true. By Step 3, we conclude that $u = \gamma u^*(\cdot - O)$ a.e. in Ω .

Case II: Ω is a connected open set. By Case I, we know that in every open ball $B \subset \Omega$ around a Lebesgue point of u , the vector field u is either a vortex-type vector field in B , or u is constant in B . Since u is nonconstant in Ω , there exists a Lebesgue point P_0 of u and a ball $B_0 \subset \Omega$ around P_0 such that u is a vortex-type vector field in B_0 ; say for simplicity $u = u^*$. Let $P \neq P_0$ be any other Lebesgue point of u . Since Ω is path-connected, there exists a path $\Gamma \subset \Omega$ from P_0 to P . Then we can cover the path Γ by a finite number of open balls $\{B_j\}_{0 \leq j \leq n}$ such that $P \in B_n$, $B_j \cap B_{j+1} \neq \emptyset$ for $0 \leq j \leq n - 1$ and u is either constant or a vortex-type vector field in any B_j . Since $u = u^*$ in B_0 and $B_0 \cap B_1$ is a nonempty open set, the analysis in Case I yields $u = u^*$ in B_1 and by induction, $u = u^*$ in B_n , which is a neighborhood of P . \square

Let us now present the proof of the geometric result in Proposition 9, which is independent of the previous results:

Proof of Proposition 9. Assume that there are two lines $D_0, D_1 \in \mathcal{D}$ that are not collinear. Since D_0 and D_1 are coplanar, they intersect at a point P . Call Π the plane determined by D_0 and D_1 . We show that all the lines in \mathcal{D} pass through P . Let $D_2 \in \mathcal{D}$ be any line not included in Π (such a line exists because \mathcal{D} is not planar). We know that D_2 is coplanar with D_0 and D_1 , respectively. Then D_2 cannot be parallel with D_0 (otherwise, $D_2 \parallel D_0$ and $D_2 \cap D_1 \neq \emptyset$ imply that $D_2 \subset \Pi$, which is a contradiction with our assumption). Similarly, D_2 cannot be parallel with D_1 . Therefore, D_2 intersects both D_0 and D_1 . Since D_2 is not included in Π , the intersection points coincide with P . Let now $D_3 \in \mathcal{D}$ be any line included in Π (different than D_0 and D_1). Then D_3 is not included in the plane determined by D_1 and D_2 . The previous argument leads again to $P \in D_3$, which concludes our proof. \square

7. Vector fields of vortex-line type

We will prove the characterization of the weakened kinetic formulation (10) in Theorem 10. This result is in the spirit of Corollary 14 and leads to vector fields that have vortex-line singularities.

Proof of Theorem 10. For $x \in \mathbb{R}^N$, recall the notation $x = (x', x_N)$ with $x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$. As the result is local in the set $\{u_N \neq \pm 1\}$, we will assume that $\omega = B' \times (-1, 1)$ is included in that set, where B' is the unit ball in \mathbb{R}^{N-1} . Let $\xi' \in \mathbb{S}^{N-2}$ and $\xi = (\xi', 0) \in \mathcal{E}$. Since $e_N \in \xi^\perp$, we deduce by (10) that

$$e_N \cdot \nabla_x \chi(\cdot, \xi) = \partial_N \chi(\cdot, \xi) = 0 \quad \text{in } \mathcal{D}'(\omega). \tag{28}$$

We know that the point (x', t) is a Lebesgue point of $\chi(\cdot, \xi)$ for \mathcal{H}^{N-1} -a.e. $x' \in B'$ and \mathcal{H}^1 -a.e. $t \in (-1, 1)$ and the convolution argument in the proof of Proposition 17 yields

$$\chi(x, \xi) = \chi(x + te_N, \xi) \quad \text{for } \mathcal{H}^N\text{-a.e. } x \in \omega \text{ and } \mathcal{H}^1\text{-a.e. } t.$$

Then one can define the measurable function $\tilde{\chi}(\cdot, \xi') : B' \rightarrow \{0, 1\}$ by

$$\tilde{\chi}(x', \xi') := \chi(x, \xi) = \mathbb{1}_{\{x \in \omega : u'(x) \cdot \xi' > 0\}} \quad \text{for } \mathcal{H}^N\text{-a.e. } x = (x', t) \in \omega.$$

Set

$$\tilde{u}(x') = \frac{1}{V_{N-2}} \int_{\mathbb{S}^{N-2}} \xi' \tilde{\chi}(x', \xi') d\mathcal{H}^{N-2}(\xi'), \quad x' \in B'.$$

Thanks to (9),

$$\tilde{u}(x') = \frac{u'(x)}{|u'(x)|} \quad \text{for } \mathcal{H}^N\text{-a.e. } x = (x', t) \in \omega \subset \{|u'| > 0\}.$$

In particular, $\tilde{\chi}(x', \xi') = \mathbb{1}_{\{x' \in B' : \tilde{u}(x') \cdot \xi' > 0\}}$ in B' for every $\xi' \in \mathbb{S}^{N-2}$. Therefore, we deduce by (10) that $\tilde{u} : B' \rightarrow \mathbb{S}^{N-2}$ satisfies

$$\forall \xi' \in \mathbb{S}^{N-2}, \forall v' \in (\xi')^\perp, \quad v' \cdot \nabla_{x'} \tilde{\chi}(x', \xi') = 0 \text{ in } B',$$

where $\nabla_{x'} = (\partial_1, \dots, \partial_{N-1})$. As $N - 1 \geq 3$, Theorem 8 yields either $\tilde{u}(x') = w'$ for almost every $x' \in B'$, where $w' \in \mathbb{S}^{N-2}$ is a constant vector, or $\tilde{u}(x') = \gamma(x' - P')/|x' - P'|$ for almost every $x' \in B'$, where $\gamma \in \{\pm 1\}$ and $P' \in \mathbb{R}^{N-1}$ is some point. This means that for a.e. $x \in \omega$,

$$\text{either } u'(x) = |u'(x)|w' \quad \text{or} \quad u'(x) = \gamma|u'(x)| \frac{x' - P'}{|x' - P'|}.$$

Case 1. Let $u'(x) = |u'(x)|w'$ for a.e. $x \in \omega$. By (11), we have for $k \in \{1, \dots, N - 1\}$,

$$\partial_k u_N = \partial_N u_k = w_k \partial_N(|u'|) \quad \text{in } \omega, \tag{29}$$

which yields, for all $k, j \in \{1, \dots, N - 1\}$,

$$w_j \partial_k u_N = w_k \partial_j u_N \quad \text{in } \omega.$$

Therefore, $u_N(x) = g(\alpha, x_N)$ in ω for some 2-dimensional function g with the new variable $\alpha := \alpha(x) = x' \cdot w'$. Moreover, by (29), the function g satisfies the following: since $w_k \neq 0$ for some $k \in \{1, \dots, N - 1\}$ (because $w \in \mathbb{S}^{N-1}$), the equation $|u'|^2 + u_N^2 = 1$ a.e. in ω implies

$$w_k \partial_\alpha g = \partial_k u_N \stackrel{(29)}{=} w_k \partial_N(|u'|) = w_k \partial_N(\sqrt{1 - g^2}).$$

The Poincaré lemma yields the existence of a stream function $\psi(\alpha, x_N)$ such that $g = \partial_N \psi$ and $\sqrt{1 - g^2} = \partial_\alpha \psi$ so that $u(x) = \nabla_x[\psi(\alpha, x_N)]$ and therefore, ψ satisfies the 2-dimensional eikonal equation

$$(\partial_\alpha \psi)^2 + (\partial_N \psi)^2 = 1.$$

Case 2. Let $u'(x) = \gamma|u'(x)|(x' - P')/|x' - P'|$ for a.e. $x \in \omega$. As above, we have, for $k \in \{1, \dots, N - 1\}$,

$$\partial_k u_N = \partial_N u_k = \gamma \frac{x_k - P_k}{|x' - P'|} \partial_N(|u'|) \quad \text{in } \omega \tag{30}$$

and we deduce that, for all $k, j \in \{1, \dots, N - 1\}$,

$$(x_j - P_j) \partial_k u_N = (x_k - P_k) \partial_j u_N \quad \text{in } \omega.$$

Therefore, $u_N(x) = g(\alpha, x_N)$ in ω for some 2-dimensional function g with the new variable $\alpha := \alpha(x) = |x'|$. By (30), we conclude as above that there exists a stream function ψ solving the eikonal equation in the variables (α, x_N) such that

$$u(x) = \nabla_x[\psi(\alpha, x_N)]. \quad \square$$

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