ANALYSIS & PDE

Volume 10 No. 3 2017

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ANALYSIS AND PDE
Vol. 10, No. 3, 2017
dx.doi.org/10.2140/apde.2017.10.653

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For a family of systems of linear elasticity with rapidly oscillating periodic coefficients, we establish sharp boundary estimates with either Dirichlet or Neumann conditions, uniform down to the microscopic scale, without smoothness assumptions on the coefficients. Under additional smoothness conditions, these estimates, combined with the corresponding local estimates, lead to the full Rellich-type estimates in Lipschitz domains and Lipschitz estimates in $C^{1,\alpha}$ domains. The C^α, $W^{1,p}$, and L^p estimates in C^1 domains for systems with VMO coefficients are also studied. The approach is based on certain estimates on convergence rates. As a biproduct, we obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q = 2d/(d-1)$ and a Lipschitz domain Ω, with no smoothness assumption on the coefficients.

1. Introduction

The purpose of this paper is to establish sharp boundary estimates with either Dirichlet or Neumann conditions, uniform down to the microscopic scale, for a family of second-order elliptic systems in divergence form with rapidly oscillating coefficients, without any smoothness assumption on the coefficients. Under additional smoothness conditions, these estimates, combined with the corresponding local estimates, lead to the full Rellich-type estimates in Lipschitz domains and Lipschitz estimates in $C^{1,\alpha}$ domains. The C^α, $W^{1,p}$, and L^p estimates in C^1 domains for systems with VMO coefficients are also investigated. To fix the idea we shall consider the systems of linear elasticity with periodic coefficients in this paper. However, the same results, without the complications introduced by rigid displacements, hold for general second-order elliptic systems with periodic coefficients satisfying the stronger ellipticity

This work was supported in part by NSF grant DMS-1161154 .
MSC2010: primary 35B27, 35J55; secondary 74B05.
Keywords: homogenization, systems of elasticity, convergence rates, Rellich estimates, Lipschitz estimates.

condition (1-11) (the symmetry condition is also needed for Rellich estimates in Lipschitz domains). We further point out that although we restrict ourselves to the periodic case, our approach, which is based on certain estimates on convergence rates in H^1 and L^2, extends to nonperiodic settings, provided that the interior correctors or approximate correctors satisfy certain L^2 conditions. The compactness methods, which were introduced to the study of homogenization in [Avellaneda and Lin 1987] and have played an important role in establishing regularity results in the periodic setting (see, e.g., [Avellaneda and Lin 1987; 1989; Kenig et al. 2013; Kenig and Prange 2015]), are not used in this paper. As a biproduct of our new approach, we also obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q = 2d/(d-1)$ and a Lipschitz domain Ω, with no smoothness assumption on the coefficients.

More precisely, consider the systems of linear elasticity,

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_j}\right], \quad \varepsilon > 0. \tag{1-1}$$

We will assume that $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \le i, j, \alpha, \beta \le d$ is real, bounded measurable, and satisfies the elasticity condition

$$\begin{gathered} a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y) = a_{\alpha j}^{i\beta}(y), \\ \kappa_1|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha \xi_j^\beta \le \kappa_2|\xi|^2 \end{gathered} \tag{1-2}$$

for a.e. $y \in \mathbb{R}^d$ and for any symmetric matrix $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}$, where $\kappa_1, \kappa_2 > 0$ (the summation convention is used throughout the paper). We will also assume that $A(y)$ is 1-periodic; i.e.,

$$A(y+z) = A(y) \quad \text{for a.e. } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1-3}$$

Theorem 1.1. *Suppose that A satisfies conditions (1-2)–(1-3). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be the weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon = f \quad \text{on } \partial\Omega, \tag{1-4}$$

where $F \in L^p(\Omega; \mathbb{R}^d)$ for $p = 2d/(d+1)$ and $f \in H^1(\partial\Omega; \mathbb{R}^d)$. Then, for $\varepsilon \le r < \operatorname{diam}(\Omega)$,

$$\left\{ \frac{1}{r}\int_{\Omega_r} |\nabla u_\varepsilon|^2 \right\}^{1/2} \le C\left\{ \|F\|_{L^p(\Omega)} + \|f\|_{H^1(\partial\Omega)} \right\}, \tag{1-5}$$

where $\Omega_r = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < r\}$. The constant C depends only on d, κ_1, κ_2, and the Lipschitz character of Ω.

Let $\mathcal{R}$ denote the space of rigid displacements,

$$\mathcal{R} = \left\{ Mx + q : M^T = -M \in \mathbb{R}^{d\times d} \text{ and } q \in \mathbb{R}^d \right\}, \tag{1-6}$$

where $(Mx)^\alpha = M_i^\alpha x_i$ and M^T denotes the transpose of matrix M. By $u \perp \mathcal{R}$ we mean $u \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$, i.e., $\int_\Omega u \cdot \phi = 0$ for any $\phi \in \mathcal{R}$. We will use $\partial u_\varepsilon/\partial\nu_\varepsilon$ to denote the conormal derivative of u_ε associated with $\mathcal{L}_\varepsilon$.

Theorem 1.2. *Suppose that A and Ω satisfy the same conditions as in Theorem 1.1. Let $u_\varepsilon \in H^1(\Omega;\mathbb{R}^d)$ be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega, \tag{1-7}$$

where $F \in L^p(\Omega;\mathbb{R}^d)$ for $p = 2d/(d+1)$, $g \in L^2(\partial\Omega;\mathbb{R}^d)$ and $\int_\Omega F\cdot\phi + \int_{\partial\Omega} g\cdot\phi = 0$ for any $\phi \in \mathcal{R}$. Also assume that $u_\varepsilon \perp \mathcal{R}$. Then, for $\varepsilon \le r < \operatorname{diam}(\Omega)$,

$$\left\{\frac{1}{r}\int_{\Omega_r} |\nabla u_\varepsilon|^2\right\}^{1/2} \le C\big\{\|F\|_{L^p(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big\}, \tag{1-8}$$

where C depends only on d, κ_1, κ_2, and the Lipschitz character of Ω.

Estimates (1-5) and (1-8), which are scaling-invariant, may be regarded as the Rellich estimates, uniform down to the scale ε, in Lipschitz domains for the elasticity operators $\mathcal{L}_\varepsilon$. Indeed, if the coefficient matrix A is constant, then (1-5) and (1-8) hold for any $0 < r < \operatorname{diam}(\Omega)$. Suppose that $F = 0$ and $u_\varepsilon \in C^1(\overline{\Omega};\mathbb{R}^d)$. By letting $r \to 0$, one recovers the full Rellich estimates in Lipschitz domains,

$$\|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \le C\|u_\varepsilon\|_{H^1(\partial\Omega)} \quad \text{and} \quad \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \le C\left\|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right\|_{L^2(\partial\Omega)}, \tag{1-9}$$

which were proved in [Fabes et al. 1988; Dahlberg et al. 1988] for second-order elliptic systems with constant coefficients, using integration by parts (see [Kenig 1994] for references on related work on boundary value problems in Lipschitz domains). We should note that our proof of Theorems 1.1 and 1.2 uses the nontangential maximal function estimates in [Dahlberg et al. 1988]. On the other hand, under certain smoothness conditions on A, the Rellich estimates hold for the operator $\mathcal{L}_1$ on Lipschitz domains with $\operatorname{diam}(\Omega) \le 1$. By a blow-up argument as well as some localization procedures, this implies

$$\begin{aligned}
\|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} &\le C\big\{\|\nabla_{\tan} u_\varepsilon\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\big\},\\
\|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} &\le C\left\{\left\|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right\},
\end{aligned} \tag{1-10}$$

where $\nabla_{\tan} u_\varepsilon$ denotes the tangential derivative of u_ε on $\partial\Omega$. We emphasize that the estimates (1-10) are local and structure conditions such as periodicity are not needed. However, with the additional periodicity condition, one may combine the local estimates (1-10) with the estimates in Theorems 1.1 and 1.2 to obtain the full Rellich estimate (1-9), uniform in ε, for operators $\mathcal{L}_\varepsilon$ (see Remark 3.1). Thus we have been able to completely separate the large-scale regularity due to homogenization from the small-scale regularity due to smoothness of the coefficients.

Under the periodicity condition and the Hölder continuity condition on A, the uniform Rellich estimates (1-9) were proved in [Kenig and Shen 2011a; 2011b] for a family of elliptic operators $\{\mathcal{L}_\varepsilon\}$, where $\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla)$ and $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \le i, j \le d$ and $1 \le \alpha, \beta \le m$ satisfies the ellipticity condition

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \le \frac{1}{\mu}|\xi|^2 \tag{1-11}$$

for $y \in \mathbb{R}^d$ and $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times m}$ as well as the symmetry condition $A^* = A$, i.e., $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$. The results were used to establish the uniform solvability of the L^2 Dirichlet, regularity, and Neumann problems for the system $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Lipschitz domains. It is worth pointing out that the Rellich estimates (1-9) are not accessible by compactness methods. One of the key steps in [Kenig and Shen 2011a; 2011b] uses integration by parts and relies on the observation that $\mathcal{L}_1(Q) = Q(\mathcal{L}_1)$, where

$$Q(u)(x', x_d) = u(x', x_d + 1) - u(x', x_d).$$

As a result, the approach does not seem to apply if the coefficients are not periodic. We mention that even with periodic coefficients, the direct extension of the methods used in [Kenig and Shen 2011a; 2011b] is problematic for the system of elasticity, due to the weaker ellipticity condition and the lack of (uniform) Korn inequalities on boundary layers.

In this paper we develop a new approach to uniform boundary regularity in quantitative homogenization of elliptic equations and systems. Let u_0 denote the solution of the boundary value problem for the homogenized system with the same data. The basic idea is to consider the function

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_j^\beta(x/\varepsilon) K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right) \tag{1-12}$$

in Ω, where $\chi = (\chi_j^\beta)$ denotes the matrix of correctors, $K_\varepsilon^2 = K_\varepsilon \circ K_\varepsilon$ with K_ε being a smoothing operator at scale ε, and $\eta_\varepsilon \in C_0^\infty(\Omega)$ is a cut-off function with support in $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 3\varepsilon\}$. Using energy estimates for the operator $\mathcal{L}_\varepsilon$ as well as sharp boundary regularity estimates for u_0, we are able to bound

$$\varepsilon^{-1/2}\|w_\varepsilon\|_{H^1(\Omega)}$$

by the right-hand sides of estimates (1-5) and (1-8), respectively. This, together with sharp estimates for u_0, yields the desired estimates for

$$r^{-1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega_r)}$$

for $\varepsilon \leq r < \operatorname{diam}(\Omega)$. We mention that since $\mathcal{L}_0$ has constant coefficients, the sharp boundary estimates in Lipschitz domains in terms of nontangential maximal functions are known [Fabes et al. 1988; Dahlberg et al. 1988]. Also, because of the use of the smoothing operator K_ε, which is motivated by [Pastukhova 2006; Suslina 2013a] (also see [Griso 2004; Onofrei and Vernescu 2007; Kenig et al. 2012; Suslina 2013b]), we only need to assume that

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} \big(|\chi(y)|^2 + |\nabla\chi(y)|^2\big)\, dy < \infty,$$

and that a similar estimate holds for a dual corrector $\phi = (\phi_{kij}^{\alpha\beta})$ (see (2-5) for its definition). As such, it is possible to extend the approach to the almost-periodic or other nonperiodic settings. We plan to carry out this study in a separate work.

As we mentioned before, the estimates in Theorems 1.1 and 1.2 may be used to establish uniform solvability of L^2 boundary value problems for $\mathcal{L}_\varepsilon$ in Lipschitz domains [Kenig and Shen 2011a; 2011b]. They can also be used to obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q = 2d/(d-1)$ and a Lipschitz domain Ω, with no smoothness assumption on the coefficients.

Theorem 1.3. *Suppose that A and Ω satisfy the same conditions as in Theorem 1.1. Let u_ε be a weak solution to (1-4) or (1-7), and u_0 the weak solution of the homogenized system with the same data. Suppose that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. In the case of the Neumann problem (1-7) we further assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1-13}$$

where $q = p' = 2d/(d-1)$ and C depends only on d, κ_1, κ_2, and Ω.

We remark that if Ω is C^2 and $u_\varepsilon = 0$ or $\partial u_\varepsilon/\partial \nu_\varepsilon = 0$ on $\partial\Omega$, the $O(\varepsilon)$ estimate

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \|F\|_{L^2(\Omega)} \tag{1-14}$$

was proved in [Suslina 2013a; 2013b] for a broader class of elliptic operators with measurable periodic coefficients, which contains the systems of elasticity considered here (also see [Griso 2004; Onofrei and Vernescu 2007; Kenig et al. 2012; 2014] and their references for related work on convergence rates). Note that $q = 2d/(d-1) > 2$ and $\|u_0\|_{H^2(\Omega)} \le C\|F\|_{L^2(\Omega)}$ if Ω is C^2 and $\mathcal{L}_0(u_0) = F$ in Ω with $u_0 = 0$ or $\partial u_0/\partial \nu_0 = 0$ on $\partial\Omega$. Thus our estimate (1-13) is stronger than (1-14). In the case of scalar elliptic equations with Dirichlet condition $u_\varepsilon = 0$ on $\partial\Omega$, it is known that $\|u_\varepsilon - u_0\|_{L^q(\Omega)} \le C\varepsilon\|F\|_{L^p(\Omega)}$, where $1 < p < d$ and $1/q = 1/p - 1/d$ (see [Kenig et al. 2014, p. 1234]). Although the exponent $q = 2d/(d-1)$ may not be sharp, Theorem 1.3 seems to be the first result on the sharp $O(\varepsilon)$ estimate of $u_\varepsilon - u_0$ in $L^q(\Omega)$ with $q > 2$ for elliptic systems with bounded measurable periodic coefficients.

As we indicated above, the proof of Theorems 1.1 and 1.2 only uses the energy estimates in L^2 for $\mathcal{L}_\varepsilon$ and thus requires no smoothness assumptions on the coefficients. In the second part of this paper we apply the similar ideas in the L^p setting for $1 < p < \infty$. To do this we first establish the $W^{1,p}$ estimates for the systems

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(h) \quad \text{in } \Omega, \tag{1-15}$$

where $h = (h_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d\times d})$, with either the Dirichlet or Neumann boundary conditions, under the additional assumptions that Ω is C^1 and $A = A(y)$ belongs to $\mathrm{VMO}(\mathbb{R}^d)$. As a result, the L^p analogues of estimates (1-5) and (1-8) are proved under these additional conditions, which are more or less sharp. Consequently, by combining the L^p estimates on the boundary layer Ω_ε with local estimates for $\mathcal{L}_1$, which hold for Hölder continuous coefficients, we may obtain the uniform Rellich estimates in L^p for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in C^1 domains under the assumptions that A is Hölder continuous and satisfies (1-2)–(1-3). By the method of layer potentials, this will lead to the uniform solvability of the L^p Dirichlet, regularity, and Neumann problems in C^1 domains (details will be provided in a separate work). Previously, these results in L^p are known only in $C^{1,\alpha}$ domains for operators $\mathcal{L}_\varepsilon$ with Hölder continuous coefficients satisfying (1-11) and $A^* = A$ [Kenig et al. 2013]. We remark that the $W^{1,p}$ estimates (local or global) for operators with nonsmooth coefficients in nonsmooth domains are of interest in their own rights and have been studied extensively in recent years (see [Caffarelli and Peral 1998; Auscher and Qafsaoui 2002; Wang 2003; Byun and Wang 2004; 2005; Shen 2005; 2008; Krylov 2007; Dong and Kim 2010; Kenig et al. 2013; Geng 2012; Geng et al. 2012] and their references). Our approach to the $W^{1,p}$ estimates is based on a real-variable argument, which originated in [Caffarelli and Peral 1998] and further developed in [Wang 2003; Shen 2005; 2007]. The required (weak) reverse Hölder estimates at the boundary are proved by combining the interior Lipschitz estimates down to the scale ε with boundary C^α estimates.

Theorems 1.1 and 1.2 as well as their L^p analogues, given in Section 7, are the main contributions of this paper. For a comprehensive study in the boundary regularity for $\mathcal{L}_\varepsilon$, in Sections 8 and 9, we investigate the boundary Lipschitz estimates, uniform down to the scale ε, for solutions in $C^{1,\alpha}$ domains with the Dirichlet or Neumann conditions. Let

$$\begin{aligned} D_r &= \big\{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \psi(x') < x_d < \psi(x') + r\big\}, \\ \Delta_r &= \big\{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } x_d = \psi(x')\big\}, \end{aligned} \tag{1-16}$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a $C^{1,\alpha}$ function for some $\alpha > 0$ with $\psi(0) = 0$ and $\|\nabla\psi\|_{C^\alpha(\mathbb{R}^{d-1})} \le M$.

Theorem 1.4. *Suppose that A satisfies conditions (1-2)–(1-3). Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution to*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } D_1 \qquad \text{and} \qquad u_\varepsilon = f \quad \text{on } \Delta_1. \tag{1-17}$$

Then, for $\varepsilon \le r < 1$,

$$\left(⨍_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\left(⨍_{D_1} |\nabla u_\varepsilon|^2\right)^{1/2} + \|f\|_{C^{1,\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)}\right\}, \tag{1-18}$$

where $p > d$ and $\sigma \in (0, \alpha)$. The constant C depends only on d, κ_1, κ_2, p, σ, and (α, M).

Theorem 1.5. *Suppose that A satisfies (1-2)–(1-3). Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution to*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } D_1 \qquad \text{and} \qquad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \Delta_1. \tag{1-19}$$

Then, for $\varepsilon \le r < 1$,

$$\left(⨍_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\left(⨍_{D_1} |\nabla u_\varepsilon|^2\right)^{1/2} + \|g\|_{C^{\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)}\right\}, \tag{1-20}$$

where $p > d$ and $\sigma \in (0, \alpha)$. The constant C depends only on d, κ_1, κ_2, p, σ, and (α, M).

As in the case of Rellich estimates, under additional smoothness conditions on A, using local Lipschitz estimates for $\mathcal{L}_1$ and a blow-up argument, one may derive from Theorems 1.4 and 1.5 the full boundary Lipschitz estimates

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \le C\left\{\left(⨍_{D_1} |u_\varepsilon|^2\right)^{1/2} + \|f\|_{C^{1,\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)}\right\} \tag{1-21}$$

for solutions of (1-17), and

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \le C\left\{\left(⨍_{D_1} |u_\varepsilon|^2\right)^{1/2} + \|g\|_{C^{\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)}\right\} \tag{1-22}$$

for solutions of (1-19). We remark that for elliptic systems satisfying the ellipticity condition (1-11), the periodicity condition (1-3) and the Hölder continuity condition, the estimate (1-21) was proved in [Avellaneda and Lin 1987], while (1-22) was established in [Kenig et al. 2013] under the additional symmetry condition $A^* = A$. This symmetry condition was removed recently in [Armstrong and Shen 2016]. However, our estimates in Theorems 1.4 and 1.5 are new for the system of elasticity.

Our proof of Theorems 1.4 and 1.5 also uses the function w_ε, given by (1-12). As a consequence of its estimates in L^2, for each $r \in (\varepsilon, \frac{1}{4})$, we are able to construct a function v such that $\mathcal{L}_0(v) = F$ in D_r with the same (Dirichlet or Neumann) data on Δ_r as u_ε, and

$$\left(\fint_{D_r} |u_\varepsilon - v|^2 \right)^{1/2} \le C(\varepsilon/r)^{1/2} \left\{ \left(\fint_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + \text{terms involving given data} \right\}.$$

This allows us to use a general scheme for establishing Lipschitz estimates down to the scale ε, which was formulated recently in [Armstrong and Smart 2016] and used for interior estimates in stochastic homogenization with random coefficients (also see [Armstrong and Mourrat 2016] as well as related work in [Gloria and Otto 2011; 2012; Gloria et al. 2014; 2015]). Our argument is similar to (and somewhat simpler and more transparent than) that in [Armstrong and Shen 2016], where the scheme was adapted to prove the full boundary Lipschitz estimates for second-order elliptic systems with almost-periodic and Hölder continuous coefficients. As indicated earlier, we have been able to completely avoid the use of compactness methods (even in the case of C^α estimates). Although it is possible to prove the interior Lipschitz estimates as well as the boundary C^α estimates, down to the scale ε without smoothness, by the compactness methods, as demonstrated in [Avellaneda and Lin 1987; Gu and Shen 2015], the compactness methods for boundary Lipschitz estimates require the same estimates for boundary correctors, which are not easy to establish [Avellaneda and Lin 1987; Kenig et al. 2013].

The paper is organized as follows. In Section 2 we establish some key convergence results in H^1. These results are used in Section 3 to prove Theorems 1.1 and 1.2. In Section 4 we study the convergence rates in L^q for $q = 2d/(d-1)$ and give the proof of Theorem 1.3, which uses the estimates in Theorems 1.1 and 1.2 as well as a duality argument. In Sections 5 and 6 we obtain the boundary C^α and $W^{1,p}$ estimates, respectively, in C^1 domains for operators with VMO coefficients. These estimates are used in Section 7 to establish the L^p analogues of (1-5) and (1-8) in C^1 domains. Finally, Theorem 1.4 is proved in Section 8, and Section 9 contains the proof of Theorem 1.5.

Throughout the paper we use $\fint_E u = (1/|E|) \int_E u$ to denote the average of u over the set E. We will use C and c to denote constants that may depend on d, κ_1, κ_2, A and Ω, but never on ε.

2. Convergence rates in H^1

In this section we establish certain results on convergence rates in H^1, which will play a crucial role in the proof of our main results. Throughout the section we assume that $A = A(y)$ satisfies (1-2)–(1-3) and Ω is a bounded Lipschitz domain in $\mathbb{R}^d$.

Let $\chi = (\chi_j^\beta(y)) = (\chi_j^{\alpha\beta}(y))$ denote the matrix of correctors for $\mathcal{L}_\varepsilon$, where $1 \le j, \alpha, \beta \le d$. This means that $\chi_j^\beta \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ is 1-periodic, $\int_Y \chi_j^\beta = 0$, and

$$\mathcal{L}_1(\chi_j^\beta) = -\mathcal{L}_1(P_j^\beta) \quad \text{in } \mathbb{R}^d, \tag{2-1}$$

where $Y = [0, 1)^d$ and $P_j^\beta = y_j(0, \dots, 1, \dots, 0)$ with 1 in the β-th position. The homogenized operator is given by $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$, where $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ is the matrix of effective coefficients with

$$\hat{a}_{ij}^{\alpha\beta} = \fint_Y \left\{ a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} (\chi_j^{\gamma\beta}) \right\}. \tag{2-2}$$

It is known that the constant matrix $\hat{A}$ satisfies the elasticity condition (1-2) [Oleĭnik et al. 1992; Jikov et al. 1994]. Define

$$b_{ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma}\frac{\partial}{\partial y_k}(\chi_j^{\gamma\beta}) - \hat{a}_{ij}^{\alpha\beta}. \tag{2-3}$$

By the definition of $\hat{A}$ and (2-1),

$$\int_Y b_{ij}^{\alpha\beta} = 0 \quad \text{and} \quad \frac{\partial}{\partial y_i}(b_{ij}^{\alpha\beta}) = 0. \tag{2-4}$$

It follows that there exist $\phi_{kij}^{\alpha\beta} \in H^1_{\text{loc}}(\mathbb{R}^d)$ such that $\phi_{kij}^{\alpha\beta}$ is 1-periodic,

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}(\phi_{kij}^{\alpha\beta}) \quad \text{and} \quad \phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta} \tag{2-5}$$

(see, e.g., [Jikov et al. 1994; Kenig et al. 2012]).

Fix $\varphi \in C_0^\infty\big(B(0, \frac{1}{4})\big)$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Define

$$K_\varepsilon(f)(x) = f * \varphi_\varepsilon(x) = \int_{\mathbb{R}^d} f(x-y)\varphi_\varepsilon(y)\,dy, \tag{2-6}$$

where $\varphi_\varepsilon(y) = \varepsilon^{-d}\varphi(y/\varepsilon)$.

Lemma 2.1. *Let $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Then for any $g \in L^p_{\text{loc}}(\mathbb{R}^d)$,*

$$\|g(x/\varepsilon)K_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq C \sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,1)} |g|^p\right)^{1/p} \|f\|_{L^p(\mathbb{R}^d)}, \tag{2-7}$$

where C depends only on d.

Proof. By Hölder's inequality,

$$|K_\varepsilon(f)(x)|^p \leq \frac{C}{|B(0,\varepsilon)|}\int_{\mathbb{R}^d} |f(y)|^p \chi_{B(x,\varepsilon)}(y)\,dy,$$

from which the estimate (2-7) follows readily by Fubini's theorem. □

It follows from (2-7) that if $g \in L^p_{\text{loc}}(\mathbb{R}^d)$ and is 1-periodic, then

$$\|g(x/\varepsilon)K_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq C\|g\|_{L^p(Y)}\|f\|_{L^p(\mathbb{R}^d)}. \tag{2-8}$$

Lemma 2.2. *Let $f \in W^{1,q}(\mathbb{R}^d)$ for some $1 < q < \infty$. Then*

$$\|K_\varepsilon(f) - f\|_{L^q(\mathbb{R}^d)} \leq C\varepsilon\|\nabla f\|_{L^q(\mathbb{R}^d)}. \tag{2-9}$$

Moreover, if $p = 2d/(d+1)$,

$$\begin{aligned} \|K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\leq C\varepsilon^{-1/2}\|f\|_{L^p(\mathbb{R}^d)}, \\ \|f - K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\leq C\varepsilon^{1/2}\|\nabla f\|_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{2-10}$$

The constant C depends only on d.

Proof. To see (2-9), we note that

$$\|f(\cdot - y) - f(\cdot)\|_{L^q(\mathbb{R}^d)} \le |y|\,\|\nabla f\|_{L^q(\mathbb{R}^d)}$$

for any $y \in \mathbb{R}^d$. Thus, by Minkowski's inequality,

$$\begin{aligned}\|K_\varepsilon(f) - f\|_{L^q(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d} \varphi_\varepsilon(y)\|f(\cdot - y) - f(\cdot)\|_{L^q(\mathbb{R}^d)}\,dy\\ &\le \int_{\mathbb{R}^d} \varphi_\varepsilon(y)|y|\,dy\,\|\nabla f\|_{L^q(\mathbb{R}^d)}\\ &= C\varepsilon\|\nabla f\|_{L^q(\mathbb{R}^d)}.\end{aligned}$$

Next, by Parseval's theorem and Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^d} |K_\varepsilon(f)|^2\,dx &= \int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi)|^2 |\hat{f}(\xi)|^2\,d\xi\\ &\le \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi)|^{2d}\,d\xi\right)^{1/d} \|\hat{f}\|^2_{L^{p'}(\mathbb{R}^d)}\\ &\le C\varepsilon^{-1}\|f\|^2_{L^p(\mathbb{R}^d)},\end{aligned}$$

where $\hat{f}$ denotes the Fourier transform of f, and we have used the Hausdorff–Young inequality $\|\hat{f}\|_{L^{p'}(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)}$. This gives the first inequality in (2-10). To see the second inequality, we note that $\hat{\varphi}(0) = \int_{\mathbb{R}^d} \varphi = 1$. It follows that

$$\begin{aligned}\|f - K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\le C\left\{\int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi) - \hat{\varphi}(0)|^{2d}|\xi|^{-2d}\,d\xi\right\}^{1/(2d)} \|\widehat{\nabla f}\|_{L^{p'}(\mathbb{R}^d)}\\ &\le C\varepsilon^{1/2}\|\nabla f\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

where we have used $|\hat{\varphi}(\xi) - \hat{\varphi}(0)| \le C|\xi|$ for the last step. □

Lemma 2.3. *Let $u_\varepsilon, u_0 \in H^1(\Omega; \mathbb{R}^d)$. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ in Ω and either $u_\varepsilon = u_0$ or $\partial u_\varepsilon/\partial\nu_\varepsilon = \partial u_0/\partial\nu_0$ on $\partial\Omega$. Let*

$$w_\varepsilon^\alpha = u_\varepsilon^\alpha - u_0^\alpha - \varepsilon\chi_j^{\alpha\beta}(x/\varepsilon)K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right),$$

where $K_\varepsilon^2 = K_\varepsilon \circ K_\varepsilon$, $\eta_\varepsilon \in C_0^\infty(\Omega)$ and $\operatorname{supp}(\eta_\varepsilon) \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge 3\varepsilon\}$. Then

$$\begin{aligned}\int_\Omega A(x/\varepsilon)\nabla w_\varepsilon \cdot \nabla w_\varepsilon\,dx &= \int_\Omega [\hat{A} - A(x/\varepsilon)][\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)] \cdot \nabla w_\varepsilon\,dx\\ &\quad - \int_\Omega B(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon\,dx\\ &\quad - \varepsilon\int_\Omega A(x/\varepsilon)\chi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon\,dx, \qquad (2\text{-}11)\end{aligned}$$

where $B(y) = (b_{ij}^{\alpha\beta}(y))$ is defined in (2-3).

Proof. We first note that if $u_\varepsilon = u_0$ on $\partial\Omega$, then $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$, as $K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \in C_0^\infty(\Omega)$. Since $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ in Ω, it follows that

$$\int_\Omega A(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla w_\varepsilon \, dx = \int_\Omega \hat{A}\nabla u_0 \cdot \nabla w_\varepsilon \, dx. \tag{2-12}$$

In the case of the Neumann condition $\partial u_\varepsilon/\partial\varepsilon = \partial u_0/\partial\nu_0$ on $\partial\Omega$, equation (2-12) continues to hold. This is because $w_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ and both sides of (2-12) are equal to

$$\langle \mathcal{L}_0(u_0), w_\varepsilon\rangle_{(H^1(\Omega))'\times H^1(\Omega)} + \left\langle \frac{\partial u_0}{\partial \nu_0}, w_\varepsilon \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}.$$

Using (2-12), we obtain

$$\begin{aligned}\int_\Omega A(x/\varepsilon)\nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx = &\int_\Omega [\hat{A} - A(x/\varepsilon)]\nabla u_0 \cdot \nabla w_\varepsilon \, dx \\ &- \int_\Omega A(x/\varepsilon)\nabla\chi(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\cdot\nabla w_\varepsilon \, dx \\ &- \varepsilon\int_\Omega A(x/\varepsilon)\chi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\cdot\nabla w_\varepsilon \, dx,\end{aligned}$$

from which the formal (2-11) follows by the definition of $B(y)$. □

Lemma 2.4. *Let $\phi(y) = (\phi_{kij}^{\alpha\beta}(y))$ be defined by (2-5). Then*

$$\int_\Omega B(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\cdot\nabla w_\varepsilon \, dx = -\varepsilon\int_\Omega \phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial w_\varepsilon^\alpha}{\partial x_i}\cdot\frac{\partial}{\partial x_k}K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)dx. \tag{2-13}$$

Proof. Using (2-5), we see that

$$\begin{aligned}B(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\cdot\nabla w_\varepsilon &= b_{ij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\cdot\frac{\partial w_\varepsilon^\alpha}{\partial x_i} \\ &= \varepsilon\frac{\partial}{\partial x_k}\left(\phi_{kij}^{\alpha\beta}(x/\varepsilon)\right)K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\cdot\frac{\partial w_\varepsilon^\alpha}{\partial x_i} \\ &= \varepsilon\frac{\partial}{\partial x_k}\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial w_\varepsilon^\alpha}{\partial x_i}\right\}K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right),\end{aligned}$$

from which equation (2-13) follows readily. □

Lemma 2.5. *Let u_ε ($\varepsilon \geq 0$) be a solution to the Dirichlet problem (1-4) or the Neumann problem (1-7). Let w_ε be defined as in Lemma 2.3 with η_ε satisfying*

$$\begin{cases}\eta_\varepsilon \in C_0^\infty(\Omega), \quad 0 \leq \eta \leq 1, \\ \operatorname{supp}(\eta_\varepsilon) \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 3\varepsilon\}, \\ \eta_\varepsilon = 1 \quad \textit{on } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 4\varepsilon\}, \\ |\nabla\eta_\varepsilon| \leq C\varepsilon^{-1}.\end{cases} \tag{2-14}$$

Then

$$\left|\int_\Omega A(x/\varepsilon)\nabla w_\varepsilon\cdot\nabla w_\varepsilon\,dx\right| \\ \le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\big\{\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}+\|(\nabla u_0)\eta_\varepsilon-K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}+\varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\}. \tag{2-15}$$

Proof. It follows from Lemmas 2.3 and 2.4 by the Cauchy inequality that

$$\begin{aligned}\left|\int_\Omega A(x/\varepsilon)\nabla w_\varepsilon\cdot\nabla w_\varepsilon\,dx\right| &\le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\big\{\|\nabla u_0-K^2_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}+\varepsilon\|\chi(x/\varepsilon)\nabla K^2_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\\ &\qquad+\varepsilon\|\phi(x/\varepsilon)\nabla K^2_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\}\\ &\le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\big\{\|\nabla u_0-K^2_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}+\varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\},\end{aligned}$$

where we have used Lemma 2.1 as well as the fact that $\chi,\phi\in L^2_{\text{loc}}(\mathbb{R}^d)$ and are 1-periodic for the last inequality. Observe that

$$\begin{aligned}\|\nabla u_0-K^2_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} &\le \|(\nabla u_0)(1-\eta_\varepsilon)\|_{L^2(\Omega)}+\|(\nabla u_0)\eta_\varepsilon-K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\\ &\qquad+\big\|K_\varepsilon\big((u_0)\eta_\varepsilon-K_\varepsilon((\nabla u_0)\eta_\varepsilon)\big)\big\|_{L^2(\Omega)}\\ &\le \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}+C\|(\nabla u_0)\eta_\varepsilon-K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}.\end{aligned}$$

Also,

$$\begin{aligned}\varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} &\le \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)}+\varepsilon\|K_\varepsilon((\nabla u_0)(\nabla\eta_\varepsilon))\|_{L^2(\Omega)}\\ &\le \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)}+C\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}.\end{aligned}$$

□

Finally, we are in a position to state and prove the main result of this section.

Theorem 2.6. *Suppose that $A(y)$ satisfies (1-2)–(1-3). Let Ω be a bounded Lipschitz domain. Let u_ε ($\varepsilon\ge 0$) be the solutions to the Dirichlet problem (1-4) in Ω with $f\in H^1(\partial\Omega;\mathbb{R}^d)$ and $F\in L^p(\Omega;\mathbb{R}^d)$, where $p=2d/(d+1)$. Then*

$$\left\|u_\varepsilon-u_0-\varepsilon\chi^\beta_j(x/\varepsilon)K^2_\varepsilon\Big(\frac{\partial u^\beta_0}{\partial x_j}\eta_\varepsilon\Big)\right\|_{H^1_0(\Omega)}\le C\varepsilon^{1/2}\big\{\|f\|_{H^1(\partial\Omega)}+\|F\|_{L^p(\Omega)}\big\}, \tag{2-16}$$

where $\eta_\varepsilon\in C^\infty_0(\Omega)$ satisfies (2-14). The constant C depends only on d, κ_1, κ_2, and the Lipschitz character of Ω.

Proof. Let w_ε denote the function on the left-hand side of (2-16). Since $w_\varepsilon\in H^1_0(\Omega;\mathbb{R}^d)$, it follows from (2-15) by the first Korn inequality [Oleĭnik et al. 1992] that

$$\|w_\varepsilon\|_{H^1_0(\Omega)}\le C\big\{\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}+\|(\nabla u_0)\eta_\varepsilon-K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}+\varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\}. \tag{2-17}$$

To bound the right-hand side of (2-17), we write $u_0=v+h$, where

$$v(x)=\int_\Omega\Gamma_0(x-y)F(y)\,dy$$

and $\Gamma_0(x)$ denotes the matrix of fundamental solutions for the homogenized operator $\mathcal{L}_0$ in $\mathbb{R}^d$, with pole at the origin. Note that $\mathcal{L}_0(v)=F$ in Ω, and by the well known singular integral and fractional integral estimates,

$$\|\nabla^2 v\|_{L^p(\mathbb{R}^d)} + \|\nabla v\|_{L^{p'}(\mathbb{R}^d)} \le C_p \|F\|_{L^p(\Omega)}, \tag{2-18}$$

where we have used the observation $1/p' = 1/p - 1/d$. Let $\boldsymbol{e} = (e_1, \ldots, e_d) \in C_0^1(\mathbb{R}^d; \mathbb{R}^d)$ be a vector field such that $\langle \boldsymbol{e}, n\rangle \ge c_0 > 0$ on $\partial\Omega$ and $|\nabla \boldsymbol{e}| \le C r_0^{-1}$, where $r_0 = \operatorname{diam}(\Omega)$ and n denotes the outward unit normal to $\partial\Omega$. It follows from the divergence theorem that

$$\begin{aligned} c_0 \int_{\partial\Omega} |\nabla v|^2 \, d\sigma &\le \int_{\partial\Omega} |\nabla v|^2 \langle \boldsymbol{e}, n\rangle \, d\sigma \\ &= \int_\Omega |\nabla v|^2 \operatorname{div}(\boldsymbol{e}) \, dx + \int_\Omega e_i \frac{\partial}{\partial x_i} \nabla v \cdot \nabla v \, dx \\ &\le C \left\{ r_0^{-1} \int_\Omega |\nabla v|^2 \, dx + \int_\Omega |\nabla v| |\nabla^2 v| \, dx \right\} \\ &\le C \big\{ r_0^{-1} \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla v\|_{L^{p'}(\Omega)} \|\nabla^2 v\|_{L^p(\Omega)} \big\} \le C \|F\|^2_{L^p(\Omega)}, \end{aligned} \tag{2-19}$$

where we have used (2-18) for the last step. Note that the same argument also gives $\|\nabla v\|_{L^2(S_t)} \le C\|F\|_{L^p(\Omega)}$, where $S_t = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) = t\}$ for $0 < t < cr_0$. Consequently, by the coarea formula, we obtain

$$\left\{ \frac{1}{r} \int_{\widetilde{\Omega}_r} |\nabla v|^2 \, dx \right\}^{1/2} \le C \|F\|_{L^p(\Omega)}, \tag{2-20}$$

where $0 < r < \operatorname{diam}(\Omega)$ and $\widetilde{\Omega}_r = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) < r\}$.

Next, we observe that $\mathcal{L}_0(h) = 0$ in Ω and

$$\begin{aligned} \|h\|_{H^1(\partial\Omega)} &\le \|f\|_{H^1(\partial\Omega)} + \|v\|_{H^1(\partial\Omega)} \\ &\le \|f\|_{H^1(\partial\Omega)} + C\|F\|_{L^p(\Omega)}, \end{aligned}$$

where we have used (2-19) for the last inequality. It follows from the estimates for solutions of the L^2 regularity problem in Lipschitz domains for the operator $\mathcal{L}_0$ in [Dahlberg et al. 1988; Verchota 1986] that

$$\|(\nabla h)^*\|_{L^2(\partial\Omega)} \le C \big\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \big\}, \tag{2-21}$$

where $(\nabla h)^*$ denotes the nontangential maximal function of ∇h. This, together with (2-20), gives

$$\|\nabla u_0\|_{L^2(\Omega_r)} \le C r^{1/2} \big\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \big\} \tag{2-22}$$

for any $0 < r < \operatorname{diam}(\Omega)$. As a result, the first term on the right-hand side of (2-17) is bounded by $C\varepsilon^{1/2}\{\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)}\}$.

To handle the third term on the right-hand side of (2-17), we use Lemma 2.2 to obtain

$$\begin{aligned} \varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} &\le \varepsilon \|K_\varepsilon((\nabla^2 v)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon \|K_\varepsilon((\nabla^2 h)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\le C\varepsilon^{1/2} \|(\nabla^2 v)\eta_\varepsilon\|_{L^p(\Omega)} + C\varepsilon \|(\nabla^2 h)\eta_\varepsilon\|_{L^2(\Omega)} \\ &\le C\varepsilon^{1/2} \|F\|_{L^p(\Omega)} + C\varepsilon \|\nabla^2 h\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})}. \end{aligned} \tag{2-23}$$

Since $\mathcal{L}_0(\nabla h) = 0$ in Ω, we may use the interior estimate for $\mathcal{L}_0$,

$$|\nabla^2 h(x)| \le \frac{C}{\delta(x)} \left(\fint_{B(x,\delta(x)/8)} |\nabla h|^2 \right)^{1/2},$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, to show that

$$\begin{aligned} \|\nabla^2 h\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})} &\le C \|(\nabla h)[\delta(x)]^{-1}\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \\ &\le C\varepsilon^{-1/2} \big\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \big\}, \end{aligned} \tag{2-24}$$

where the last inequality follows from (2-21). This, together with (2-23), gives

$$\varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon^{1/2} \big\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \big\}. \tag{2-25}$$

Finally, to bound the second term on the right-hand side of (2-17), we again write $u_0 = v + h$ as before. Note that by Lemma 2.2,

$$\begin{aligned} \|(\nabla v)\eta_\varepsilon - K_\varepsilon((\nabla v)\eta_\varepsilon)\|_{L^2(\Omega)} &\le \|\nabla v - K_\varepsilon(\nabla v)\|_{L^2(\mathbb{R}^d)} + \|(\nabla v)(1-\eta_\varepsilon)\|_{L^2(\Omega)} + \|K_\varepsilon((\nabla v)(1-\eta_\varepsilon))\|_{L^2(\Omega)} \\ &\le C\varepsilon^{1/2} \|\nabla^2 v\|_{L^p(\mathbb{R}^d)} + C\|\nabla v\|_{L^2(\widetilde{\Omega}_{8\varepsilon})} \\ &\le C\varepsilon^{1/2} \|F\|_{L^p(\Omega)}, \end{aligned}$$

where we have used (2-18) and (2-20) for the last inequality. Also, by Lemma 2.2,

$$\begin{aligned} \|(\nabla h)\eta_\varepsilon - K_\varepsilon((\nabla h)\eta_\varepsilon)\|_{L^2(\Omega)} &\le C\varepsilon \|\nabla((\nabla h)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\le C \big\{ \varepsilon \|\nabla^2 h\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})} + \|\nabla h\|_{L^2(\Omega_{4\varepsilon})} \big\} \\ &\le C\varepsilon^{1/2} \big\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \big\}. \end{aligned}$$

Consequently, the second term on the right-hand side of (2-17) is dominated by the right-hand side of (2-16). This completes the proof of Theorem 2.6. □

The next theorem is an analogue of Theorem 2.6 for the Neumann boundary conditions.

Theorem 2.7. *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Let Ω be a bounded Lipschitz domain. Let u_ε ($\varepsilon \ge 0$) be the solutions to the Neumann problem (1-7) in Ω with $g \in L^2(\partial\Omega; \mathbb{R}^d)$ and $F \in L^p(\Omega; \mathbb{R}^d)$, where $p = 2d/(d+1)$. Also assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Then*

$$\left\| u_\varepsilon - u_0 - \varepsilon \chi_j^\beta(x/\varepsilon) K_\varepsilon^2 \left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \right\|_{H^1(\Omega)} \le C\varepsilon^{1/2} \big\{ \|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)} \big\}, \tag{2-26}$$

where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14). The constant C depends only on d, κ_1, κ_2, and the Lipschitz character of Ω.

Proof. The proof, which uses the estimate in Lemma 2.5, is similar to that of Theorem 2.6. We will only point out the differences and leave the details to the reader.

Let w_ε denote the function on the left-hand side of (2-26). Let

$$\big\{ \varphi_j : j = 1, \ldots, J = \tfrac{1}{2} d(d+1) \big\}$$

be an orthonormal basis of $\mathcal{R}$, as a subspace of $L^2(\Omega;\mathbb{R}^d)$. By the second Korn inequality [Oleĭnik et al. 1992],

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C\left|\int_\Omega A(x/\varepsilon)\nabla w_\varepsilon\cdot\nabla w_\varepsilon\,dx\right| + C\sum_{j=1}^J\left|\int_\Omega w_\varepsilon\cdot\varphi_j\,dx\right|. \tag{2-27}$$

Since $u_\varepsilon, u_0 \perp \mathcal{R}$, it follows that

$$\begin{aligned}\left|\int_\Omega w_\varepsilon\cdot\varphi_j\,dx\right| &\le C\varepsilon\|\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\\ &\le C\varepsilon\|\nabla u_0\|_{L^2(\Omega)}.\end{aligned}$$

This, together with (2-27) and Lemma 2.5, shows that

$$\begin{aligned}&\|w_\varepsilon\|_{H^1(\Omega)}\\ &\le C\big\{\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}+\varepsilon\|\nabla u_0\|_{L^2(\Omega)}+\|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}+\varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\}.\end{aligned} \tag{2-28}$$

To bound the right-hand side of (2-28), we write $u_0 = v + h$, where v is the same as in the proof of Theorem 2.6. Since $\mathcal{L}_0(h) = 0$ in Ω and

$$\begin{aligned}\left\|\frac{\partial h}{\partial \nu_0}\right\|_{L^2(\partial\Omega)} &\le \left\|\frac{\partial u_0}{\partial \nu_0}\right\|_{L^2(\partial\Omega)} + \left\|\frac{\partial v}{\partial \nu_0}\right\|_{L^2(\partial\Omega)}\\ &\le C\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\},\end{aligned}$$

we may use the estimates in [Dahlberg et al. 1988; Verchota 1986] for solutions of the L^2 Neumann problem for $\mathcal{L}_0$ in Lipschitz domains to obtain

$$\begin{aligned}\|(\nabla h)^*\|_{L^2(\partial\Omega)} &\le C\left\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)} + \sum_{j=1}^J\left|\int_\Omega h\cdot\varphi_j\right|\right\}\\ &\le C\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\},\end{aligned} \tag{2-29}$$

where we have used the assumption $u_0 \perp \mathcal{R}$. With the nontangential maximal function estimate (2-29) at our disposal, the rest of the proof is exactly the same as that of Theorem 2.6. □

Remark 2.8. Since

$$\|\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \le C\|\nabla u_0\|_{L^2(\Omega)},$$

it follows from the estimate (2-16) that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\big\{\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^2(\Omega)}\big\}, \tag{2-30}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0) = F$ in Ω and $u_\varepsilon = u_0 = f$ on $\partial\Omega$. Similarly, the estimate (2-26) implies

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)}\big\}, \tag{2-31}$$

where u_ε, u_0 are given in Theorem 2.7. These $O(\varepsilon^{1/2})$ estimates in L^2 are not sharp (see Section 4), but they will be sufficient for us to establish the boundary C^α and Lipschitz estimates.

3. Proof of Theorems 1.1 and 1.2

Theorems 1.1 and 1.2 are consequences of Theorems 2.6 and 2.7, respectively. We give the proof of Theorem 1.1. Theorem 1.2 follows from Theorem 2.7 in the same manner.

Without loss of generality we may assume that

$$\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} = 1.$$

Let w_ε denote the function on the left-hand side of (2-16). By Theorem 2.6, for $\varepsilon \le r < \operatorname{diam}(\Omega)$,

$$\begin{aligned}
\|\nabla u_\varepsilon\|_{L^2(\Omega_r)} &\le \|\nabla u_0\|_{L^2(\Omega_r)} + \|\nabla w_\varepsilon\|_{L^2(\Omega)} + \varepsilon \big\|\nabla\{\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\big\|_{L^2(\Omega_r)}\\
&\le Cr^{1/2} + \big\|\nabla\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\big\|_{L^2(\Omega_r)} + \varepsilon\big\|\chi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\big\|_{L^2(\Omega_r)}\\
&\le Cr^{1/2} + C\|K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + C\varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})},
\end{aligned}$$

where we have used (2-22) and Lemma 2.1 as well as the fact that the operator K_ε is a convolution with a kernel supported in $B(0, \varepsilon/4)$. Note that by (2-22) and (2-25),

$$\|K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} \le C\|\nabla u_0\|_{L^2(\Omega_{3r})} \le Cr^{1/2},$$

and

$$\begin{aligned}
\varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} &\le \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + \varepsilon\|K_\varepsilon((\nabla u_0)(\nabla\eta_\varepsilon))\|_{L^2(\Omega_{2r})}\\
&\le \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + C\|\nabla u_0\|_{L^2(\Omega_{3r})}\\
&\le Cr^{1/2}.
\end{aligned}$$

The proof of Theorem 1.1 is complete.

Remark 3.1. Under certain smoothness conditions on A, it is possible to extend the Rellich estimates in [Dahlberg et al. 1988] for the Lamé systems with constant coefficients to the operator $\mathcal{L}_1$ with variable coefficients satisfying the condition (1-2). We refer the reader to [Kenig and Shen 2011b], where this is done in the case that the coefficients satisfy the ellipticity condition (1-11). It follows that if $\mathcal{L}_1(u) = 0$ in D_2, where D_r is defined by (1-16) with $\psi(0) = 0$ and $\|\nabla\psi\|_\infty \le M$, then

$$\begin{cases}
\displaystyle\int_{\partial D_r} |\nabla u|^2\, d\sigma \le C\int_{\partial D_r}\left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma + C\int_{D_r}|\nabla u|^2\, dx,\\
\displaystyle\int_{\partial D_r} |\nabla u|^2\, d\sigma \le C\int_{\partial D_r}|\nabla_{\tan} u|^2\, d\sigma + C\int_{D_r}|\nabla u|^2\, dx
\end{cases}\tag{3-1}$$

for any $r \in \left(1, \frac{3}{2}\right)$, where C depends only on d, A, and M. By integrating both sides of the inequalities in (3-1) with respect to r over $\left(1, \frac{3}{2}\right)$, we obtain

$$\begin{cases}
\displaystyle\int_{\Delta_1} |\nabla u|^2\, d\sigma \le C\int_{\Delta_2}\left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma + C\int_{D_2}|\nabla u|^2\, dx,\\
\displaystyle\int_{\Delta_1} |\nabla u|^2\, d\sigma \le C\int_{\Delta_2}|\nabla_{\tan} u|^2\, d\sigma + C\int_{D_2}|\nabla u|^2\, dx,
\end{cases}\tag{3-2}$$

where $\Delta_r = \{(x', \psi(x')) \in \mathbb{R}^d : |x'|<r \text{ and } x_d=\psi(x')\}$. We now take advantage of the fact that the dependence of C on ψ is only through M. Since $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ implies $\mathcal{L}_1\{u_\varepsilon(\varepsilon x)\} = 0$, one may deduce from (3-2) that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $D_{2\varepsilon}$, then

$$\begin{cases} \displaystyle\int_{\Delta_\varepsilon} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\Delta_{2\varepsilon}} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2 d\sigma + \frac{C}{\varepsilon} \int_{D_{2\varepsilon}} |\nabla u_\varepsilon|^2 \, dx, \\ \displaystyle\int_{\Delta_\varepsilon} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\Delta_{2\varepsilon}} |\nabla_{\tan} u_\varepsilon|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{D_{2\varepsilon}} |\nabla u_\varepsilon|^2 \, dx. \end{cases} \tag{3-3}$$

Now, suppose that $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ and $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω, where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$. By covering $\partial\Omega$ with a finite number of suitable balls of size $c\varepsilon$, it follows from (3-3) that

$$\begin{cases} \displaystyle\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2 d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^2 \, dx, \\ \displaystyle\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^2 \, dx. \end{cases} \tag{3-4}$$

Notice that up to this point, we have only used the smoothness condition of A, not the periodicity of A. With the additional periodicity condition we may invoke the estimates in Theorems 1.1 and 1.2 to bound the volume integrals of $|\nabla u_\varepsilon|^2$ over the boundary layer $\Omega_{c\varepsilon}$. This yields the full Rellich estimates,

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2 d\sigma \tag{3-5}$$

if $u_\varepsilon \perp \mathcal{R}$, and

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^2 \, d\sigma + C r_0^{-2} \int_{\partial\Omega} |u_\varepsilon|^2 \, d\sigma. \tag{3-6}$$

It is well known that estimates (3-5)–(3-6) may be used to solve the L^2 boundary value problems in Lipschitz domains by the method of layer potentials. We refer the reader to [Kenig and Shen 2011b] for the case where $A(y)$ satisfies (1-11). The details for the system of linear elasticity have been carried out in a separate work [Geng et al. 2017].

4. Convergence rates in L^q for $q = 2d/(d-1)$

We now establish sharp $O(\varepsilon)$ estimates for $\|u_\varepsilon - u_0\|_{L^q(\Omega)}$ with $q = 2d/(d-1)$, using Theorems 1.1 and 1.2 and a duality argument. Throughout this section we will assume that Ω is a bounded Lipschitz domain and $A = A(y)$ satisfies (1-2)–(1-3).

We start with the Dirichlet boundary condition.

Lemma 4.1. *Let u_ε ($\varepsilon \ge 0$) be the solution of (1-4). Suppose that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then*

$$\left\| u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) - v_\varepsilon \right\|_{H_0^1(\Omega)} \le C\varepsilon \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)}, \tag{4-1}$$

where $\tilde{u}_0 \in H^2(\mathbb{R}^d; \mathbb{R}^d)$ *is an extension of* u_0 *and* $v_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ *is the weak solution to*

$$\mathcal{L}_\varepsilon(v_\varepsilon) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad v_\varepsilon = -\varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) \quad \text{on } \partial\Omega. \tag{4-2}$$

Proof. Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) - v_\varepsilon.$$

Using $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ and $\mathcal{L}_\varepsilon(v_\varepsilon) = 0$ in Ω, a direct computation shows that

$$\begin{aligned}\mathcal{L}_\varepsilon(w_\varepsilon) &= -\frac{\partial}{\partial x_i}\left\{[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\frac{\partial u_0^\beta}{\partial x_j}\right\} - \mathcal{L}_\varepsilon\left\{\varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right)\right\} \\ &= -\frac{\partial}{\partial x_i}\left\{[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)\right]\right\} + \frac{\partial}{\partial x_i}\left\{b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)\right\} \\ &\qquad + \varepsilon \frac{\partial}{\partial x_i}\left\{a_{ij}^{\alpha\beta}(x/\varepsilon)\chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k}\right)\right\},\end{aligned} \tag{4-3}$$

where $b_{ij}^{\alpha\beta}$ is defined by (2-3). Using (2-5), we see that

$$\frac{\partial}{\partial x_i}\left\{b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)\right\} = \varepsilon \frac{\partial}{\partial x_i}\left\{\phi_{ikj}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j}\right)\right\}. \tag{4-4}$$

Indeed, the left-hand side of (4-4) equals

$$b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_i \partial x_j}\right),$$

while the right-hand side equals

$$b_{kj}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j}\right) + \phi_{ikj}^{\alpha\beta}(x/\varepsilon)\frac{\partial^2}{\partial x_i \partial x_k} K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)$$

and the second term is zero due to the skew-symmetry $\phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}$.

It follows from (4-3) and (4-4) by Lemmas 2.1 and 2.2 that

$$\|\mathcal{L}_\varepsilon(w_\varepsilon)\|_{H^{-1}(\Omega)} \le C\varepsilon \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)},$$

where C depends only on d, κ_1, κ_2, and Ω. Since $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$, this gives the estimate (4-1) by the energy estimate. □

The following theorem establishes the sharp $O(\varepsilon)$ estimate in L^q with $q = 2d/(d-1)$ for the Dirichlet boundary condition.

Theorem 4.2. *Suppose that A satisfies (1-2)–(1-3). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let u_ε $(\varepsilon \ge 0)$ be the weak solution to Dirichlet problem (1-4). Assume that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{4-5}$$

where $q = 2d/(d-1)$ and C depends only on d, κ_1, κ_2, and Ω.

Proof. We begin by choosing $\tilde{u}_0 \in H^2(\mathbb{R}^d;\mathbb{R}^d)$ such that $\tilde{u}_0 = u_0$ in Ω and $\|\tilde{u}_0\|_{H^2(\mathbb{R}^d)} \le C\|u_0\|_{H^2(\Omega)}$, where C depends only on Ω. Since Ω is Lipschitz, this is possible by an extension theorem due to A. Calderón [Stein 1970, Theorem 5, p. 181]. Next, since $H_0^1(\Omega) \subset L^q(\Omega)$ and

$$\left\| \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) \right\|_{L^q(\Omega)} \le C\|\nabla \tilde{u}_0\|_{L^q(\mathbb{R}^d)} \le C\|u_0\|_{H^2(\Omega)},$$

in view of Lemma 4.1, it suffices to show that

$$\|v_\varepsilon\|_{L^q(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{4-6}$$

where v_ε is given by (4-2).

To this end we fix $G \in L^p(\Omega;\mathbb{R}^d)$, where $p = q' = 2d/(d+1)$, and let $h_\varepsilon \in H_0^1(\Omega;\mathbb{R}^d)$ be the weak solution to

$$\mathcal{L}_\varepsilon(h_\varepsilon) = G \quad \text{in } \Omega \qquad \text{and} \qquad h_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{4-7}$$

It follows from (4-2), (4-7), and the divergence theorem that

$$\begin{aligned}
\int_\Omega v_\varepsilon \cdot G\, dx &= -\int_{\partial\Omega} v_\varepsilon \cdot \frac{\partial h_\varepsilon}{\partial \nu_\varepsilon}\, d\sigma \\
&= \varepsilon \int_{\partial\Omega} \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) \cdot \frac{\partial h_\varepsilon}{\partial \nu_\varepsilon} (\eta_\varepsilon - 1)\, d\sigma \\
&= \int_\Omega \frac{\partial \chi_k^{\alpha\gamma}}{\partial x_i}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k}\right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j} (\eta_\varepsilon - 1)\, dx \\
&\quad + \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_i \partial x_k}\right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j} (\eta_\varepsilon - 1)\, dx \\
&\quad - \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k}\right) G^\alpha (\eta_\varepsilon - 1)\, dx \\
&\quad + \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k}\right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j} \frac{\partial \eta_\varepsilon}{\partial x_i}\, dx,
\end{aligned}$$

where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14). This implies

$$\begin{aligned}
\left| \int_\Omega v_\varepsilon \cdot G\, dx \right| &\le C \int_\Omega |\nabla \chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |\nabla h_\varepsilon| |\eta_\varepsilon - 1|\, dx \\
&\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla^2 \tilde{u}_0)| |\nabla h_\varepsilon| |\eta_\varepsilon - 1|\, dx \\
&\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |G| |\eta_\varepsilon - 1|\, dx \\
&\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |\nabla h_\varepsilon| |\nabla \eta_\varepsilon|\, dx.
\end{aligned} \tag{4-8}$$

Note that by Cauchy's inequality and (2-14), the first and fourth terms on the right-hand side of (4-8) are bounded by

$$C\Big(\int_{\Omega_{4\varepsilon}} \big|\big(|\nabla\chi(x/\varepsilon)|+|\chi(x/\varepsilon)|\big)K_\varepsilon(\nabla\tilde{u}_0)\big|^2\,dx\Big)^{1/2}\Big(\int_{\Omega_{4\varepsilon}} |\nabla h_\varepsilon|^2\,dx\Big)^{1/2} \le C\Big(\int_{\widetilde{\Omega}_{5\varepsilon}} |\nabla\tilde{u}_0|^2\,dx\Big)^{1/2}\Big(\int_{\Omega_{4\varepsilon}} |\nabla h_\varepsilon|^2\,dx\Big)^{1/2},$$

where $\Omega_r=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)<r\}$, $\widetilde{\Omega}_r=\{x\in\mathbb{R}^d:\operatorname{dist}(x,\partial\Omega)<r\}$, and we have used Lemma 2.1 for the last inequality. Using the divergence theorem, as in (2-19), one may prove that

$$\|\nabla\tilde{u}_0\|_{L^2(S_r)}\le C\|\tilde{u}_0\|^{1/2}_{H^1(\mathbb{R}^d)}\|\tilde{u}_0\|^{1/2}_{H^2(\mathbb{R}^d)},$$

where $S_r=\{x\in\mathbb{R}^d:\operatorname{dist}(x,\partial\Omega)=r\}$. It follows by the coarea formula that

$$\|\nabla\tilde{u}_0\|_{L^2(\widetilde{\Omega}_r)}\le Cr^{1/2}\|\tilde{u}_0\|^{1/2}_{H^1(\mathbb{R}^d)}\|\tilde{u}_0\|^{1/2}_{H^2(\mathbb{R}^d)}. \tag{4-9}$$

This, together with the estimate in Theorem 1.1 for h_ε, shows that the first and fourth terms on the right-hand side of (4-8) are bounded by

$$C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)},$$

where $p=q'=2d/(d+1)$. Finally, we note that the second and third terms on the right-hand side of (4-8) are bounded by

$$C\varepsilon\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)}\|\nabla h_\varepsilon\|_{L^2(\Omega)}+C\varepsilon\|\nabla\tilde{u}_0\|_{L^q(\mathbb{R}^d)}\|G\|_{L^p(\Omega)}\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)}.$$

As a result, we have proved that

$$\Big|\int_\Omega v_\varepsilon\cdot G\,dx\Big|\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)},$$

which, by duality, gives the estimate (4-6) and completes the proof. □

Next we consider the solutions with the Neumann boundary conditions.

Lemma 4.3. *Let u_ε ($\varepsilon\ge 0$) be the solutions of (1-7) such that $u_\varepsilon\perp\mathcal{R}$. Suppose that $u_0\in H^2(\Omega;\mathbb{R}^d)$. Then*

$$\Big\|u_\varepsilon-u_0-\varepsilon\chi_k(x/\varepsilon)K_\varepsilon\Big(\frac{\partial\tilde{u}_0}{\partial x_k}\Big)-v_\varepsilon\Big\|_{H^1(\Omega)}\le C\varepsilon\big\{\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)}+\|\nabla\tilde{u}_0\|_{L^2(\mathbb{R}^d)}\big\}, \tag{4-10}$$

where $\tilde{u}_0$ is an extension of u_0 and $v_\varepsilon\in H^1(\Omega;\mathbb{R}^d)$ is the weak solution to

$$\begin{cases}\mathcal{L}_\varepsilon(v_\varepsilon)=0 \quad \text{in } \Omega,\\[2pt] \dfrac{\partial v_\varepsilon}{\partial\nu_\varepsilon}=\dfrac{\varepsilon}{2}\Big(n_k\dfrac{\partial}{\partial x_i}-n_i\dfrac{\partial}{\partial x_k}\Big)\Big\{\phi_{kij}(x/\varepsilon)K_\varepsilon\Big(\dfrac{\partial\tilde{u}_0}{\partial x_j}\Big)\Big\} \quad \text{on } \partial\Omega,\\[2pt] v_\varepsilon\perp\mathcal{R}.\end{cases} \tag{4-11}$$

Proof. Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) - v_\varepsilon.$$

Using $\partial u_\varepsilon/\partial \nu_\varepsilon = \partial u_0/\partial \nu_0$ on $\partial\Omega$, a direct computation shows that

$$\begin{aligned}
\frac{\partial w_\varepsilon}{\partial \nu_\varepsilon} &= \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} - \frac{\partial}{\partial \nu_\varepsilon}\left\{\varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right)\right\} - \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} \\
&= n_i[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\right)\right] - n_i b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\right) \\
&\qquad - n_i a_{ij}^{\alpha\beta}(x/\varepsilon)\cdot \varepsilon \chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k}\right) - \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}.
\end{aligned} \tag{4-12}$$

Using (2-5), we also see that

$$\begin{aligned}
n_i b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right) + \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} &= \varepsilon n_i \frac{\partial}{\partial x_k}[\phi_{kij}^{\alpha\beta}(x/\varepsilon)] K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right) + \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} \\
&= \frac{\varepsilon}{2}\left(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i}\right)[\phi_{kij}^{\alpha\beta}(x/\varepsilon)] K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right) + \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} \\
&= -\varepsilon n_i \phi_{kij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j}\right).
\end{aligned} \tag{4-13}$$

As a result, we obtain

$$\begin{aligned}
\frac{\partial w_\varepsilon}{\partial \nu_\varepsilon} &= n_i[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\right)\right] \\
&\qquad + \varepsilon n_i \phi_{kij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j}\right) - n_i a_{ij}^{\alpha\beta}(x/\varepsilon)\cdot \varepsilon \chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k}\right).
\end{aligned} \tag{4-14}$$

Next, we note that as in the proof of Lemma 4.1,

$$\begin{aligned}
\mathcal{L}_\varepsilon(w_\varepsilon) &= -\frac{\partial}{\partial x_i}\left\{[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)\right]\right\} - \varepsilon \frac{\partial}{\partial x_i}\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j}\right)\right\} \\
&\qquad + \varepsilon \frac{\partial}{\partial x_i}\left\{a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k}\right)\right\}.
\end{aligned} \tag{4-15}$$

Thus, by (1-2) and the energy estimate,

$$\begin{aligned}
&\|\nabla w_\varepsilon + (\nabla w_\varepsilon)^T\|_{L^2(\Omega)} \\
&\qquad \le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\left\{\|\nabla u_0 - K_\varepsilon(\nabla \tilde{u}_0)\|_{L^2(\Omega)} + \varepsilon\|\phi(x/\varepsilon) K_\varepsilon(\nabla^2 \tilde{u}_0)\|_{L^2(\Omega)} + \varepsilon\|\chi(x/\varepsilon) K_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega)}\right\} \\
&\qquad \le C\varepsilon\|\nabla w_\varepsilon\|_{L^2(\Omega)}\|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where we have used Lemmas 2.1 and 2.2 for the last step. By the second Korn inequality, this implies

$$\begin{aligned}\|w_\varepsilon\|_{H^1(\Omega)} &\le C\varepsilon\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)} + C\sum_{j=1}^J\left|\int_\Omega w_\varepsilon\cdot\varphi_j\,dx\right|\\ &\le C\varepsilon\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)} + C\varepsilon\|\chi(x/\varepsilon)K_\varepsilon(\nabla\tilde{u}_0)\|_{L^2(\Omega)} \le C\varepsilon\big\{\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)} + \|\nabla\tilde{u}_0\|_{L^2(\mathbb{R}^d)}\big\},\end{aligned}$$

where $\{\varphi_j : j=1,\dots,J\}$ forms an orthonormal basis of $\mathcal{R}$, as a subspace of $L^2(\Omega;\mathbb{R}^d)$. □

The next theorem is an analogue of Theorem 4.2 for the Neumann boundary conditions.

Theorem 4.4. *Suppose that A satisfies (1-2)–(1-3). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let u_ε $(\varepsilon\ge 0)$ be the weak solutions to the Neumann problem (1-7) with the property $u_\varepsilon\perp\mathcal{R}$. Assume that $u_0\in H^2(\Omega;\mathbb{R}^d)$. Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{4-16}$$

where $q=2d/(d-1)$ and C depends only on d, κ_1, κ_2, and Ω.

Proof. As in the proof of Theorem 4.2, it suffices to show that

$$\|v_\varepsilon\|_{L^q(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{4-17}$$

where v_ε is given by (4-11). To this end we fix $G\in L^p(\Omega;\mathbb{R}^d)$ with $G\perp\mathcal{R}$ and let $h_\varepsilon\in H^1(\Omega;\mathbb{R}^d)$ be the weak solution to

$$\mathcal{L}_\varepsilon(h_\varepsilon) = G \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial h_\varepsilon}{\partial\nu_\varepsilon} = 0 \quad \text{on } \partial\Omega, \tag{4-18}$$

with the property $h_\varepsilon\perp\mathcal{R}$. It follows from (4-18), (4-11), and Green's formula that

$$\begin{aligned}\int_\Omega v_\varepsilon\cdot G\,dx &= \int_\Omega A(x/\varepsilon)\nabla v_\varepsilon\cdot\nabla h_\varepsilon\,dx = \int_{\partial\Omega}\frac{\partial v_\varepsilon}{\partial\nu_\varepsilon}\cdot h_\varepsilon\,d\sigma\\ &= \frac{\varepsilon}{2}\int_{\partial\Omega}\left(n_k\frac{\partial}{\partial x_i} - n_i\frac{\partial}{\partial x_k}\right)\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde{u}_0^\beta}{\partial x_j}\right)\right\}\cdot h_\varepsilon^\alpha\,d\sigma\\ &= -\frac{\varepsilon}{2}\int_{\partial\Omega}\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde{u}_0^\beta}{\partial x_j}\right)\cdot\left(n_k\frac{\partial}{\partial x_i} - n_i\frac{\partial}{\partial x_k}\right)h_\varepsilon^\alpha\cdot(1-\eta_\varepsilon)\,d\sigma\\ &= -\varepsilon\int_\Omega\frac{\partial}{\partial x_k}\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde{u}_0^\beta}{\partial x_j}\right)(1-\eta_\varepsilon)\right\}\cdot\frac{\partial h_\varepsilon^\alpha}{\partial x_i}\,dx,\end{aligned}$$

where $\eta_\varepsilon\in C_0^\infty(\Omega)$ satisfies (2-14) and we have used the divergence theorem as well as (2-5) for the last inequality. This leads to

$$\begin{aligned}\left|\int_\Omega v_\varepsilon\cdot G\,dx\right| &\le C\int_{\Omega_{4\varepsilon}}|\nabla\phi(x/\varepsilon)||K_\varepsilon(\nabla\tilde{u}_0)||\nabla h_\varepsilon|\,dx\\ &\quad + C\varepsilon\int_{\Omega_{4\varepsilon}}|\phi(x/\varepsilon)||K_\varepsilon(\nabla^2\tilde{u}_0)||\nabla h_\varepsilon|\,dx\\ &\quad + C\varepsilon\int_{\Omega_{4\varepsilon}}|\phi(x/\varepsilon)||K_\varepsilon(\nabla\tilde{u}_0)||\nabla\eta_\varepsilon||\nabla h_\varepsilon|\,dx.\end{aligned} \tag{4-19}$$

Note that by the Cauchy inequality, the first and third term on the right-hand side of (4-19) are bounded by

$$
\begin{aligned}
C\big\|\big(|\nabla\phi(x/\varepsilon)|+|\phi(x/\varepsilon)|\big)K_\varepsilon(\nabla\tilde u_0)\big\|_{L^2(\Omega_{4\varepsilon})}\|\nabla h_\varepsilon\|_{L^2(\Omega_{4\varepsilon})} &\le C\|\nabla\tilde u_0\|_{L^2(\widetilde\Omega_{5\varepsilon})}\|\nabla h_\varepsilon\|_{L^2(\Omega_{4\varepsilon})}\\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)},
\end{aligned}
$$

where we have used Lemma 2.2 for the first inequality and Theorem 1.2 as well as estimate (4-9) for the second. Also, the second term on the right-hand side of (4-19) is bounded by

$$
C\varepsilon\|\phi(x/\varepsilon)K_\varepsilon(\nabla^2\tilde u_0)\|_{L^2(\Omega)}\|\nabla h_\varepsilon\|_{L^2(\Omega)}\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)}.
$$

Hence we have proved that for any $G\in L^p(\Omega;\mathbb{R}^d)$ with the property $G\perp\mathcal{A}$,

$$
\left|\int_\Omega v_\varepsilon\cdot G\,dx\right|\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)}.
$$

Since $v_\varepsilon\perp\mathcal{A}$, this gives the estimate (4-17) by duality and completes the proof. □

Note that by combining Theorems 4.2 and 4.4, one obtains Theorem 1.3.

5. C^α estimates in C^1 domains

In this section we investigate uniform boundary C^α estimates in C^1 domains. The results will be used in the next section to establish uniform boundary $W^{1,p}$ estimates in C^1 domains. Throughout the section we will assume that the defining function ψ in D_r and Δ_r is C^1 and $\psi(0)=0$. To quantify the C^1 condition we further assume that

$$
\sup\Big\{|\nabla\psi(x')-\nabla\psi(y')|:x',y'\in\mathbb{R}^{d-1}\text{ and }|x'-y'|\le t\Big\}\le\tau(t),\tag{5-1}
$$

where $\tau(t)\to0$ as $t\to0^+$.

The rescaling argument is used frequently in this paper. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon)=F$ in D_{2r} and $u_\varepsilon=f$ on Δ_{2r}. Let $w(x)=u_\varepsilon(rx)$. Then

$$
\mathcal{L}_{\varepsilon/r}(w)=G\quad\text{in }\widetilde D_2\qquad\text{and}\qquad w=g\quad\text{on }\tilde\Delta_2,
$$

where $G(x)=r^2F(rx)$, $g(x)=f(rx)$, and

$$
\begin{aligned}
\widetilde D_2&=\Big\{(x',x_d)\in\mathbb{R}^d:|x'|<2\text{ and }\psi_r(x')<x_d<\psi_r(x')+2\Big\},\\
\tilde\Delta_2&=\Big\{(x',x_d)\in\mathbb{R}^d:|x'|<2\text{ and }x_d=\psi_r(x')\Big\},
\end{aligned}
$$

with $\psi_r(x')=r^{-1}\psi(rx')$. Note that $\psi_r(0)=0$ and $\|\nabla\psi_r\|_\infty=\|\nabla\psi\|_\infty$. Moreover, if ψ is C^1 and satisfies (5-1), then ψ_r satisfies (5-1) uniformly in r for $0<r\le1$.

Lemma 5.1. *Let $0<\varepsilon\le r\le1$. Let $u_\varepsilon\in H^1(D_{2r};\mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon)=0$ in D_{2r} with $u_\varepsilon=0$ on Δ_{2r}. Then there exists $v\in H^1(D_r;\mathbb{R}^d)$ such that $\mathcal{L}_0(v)=0$ in D_r, $v=0$ on Δ_r, and*

$$
\left(\fint_{D_r}|u_\varepsilon-v|^2\right)^{1/2}\le C(\varepsilon/r)^{1/2}\left(\fint_{D_{2r}}|u_\varepsilon|^2\right)^{1/2},\tag{5-2}
$$

where $\|\nabla\psi\|_\infty\le M$, and C depends only on d, κ_1, κ_2, and M.

Proof. By rescaling we may assume $r=1$. By Caccioppoli's inequality,

$$\left(\fint_{D_{3/2}} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left(\fint_{D_2} |u_\varepsilon|^2\right)^{1/2}. \tag{5-3}$$

It follows from (5-3) and the coarea formula that there exists $t\in\left[\frac{5}{4},\frac{3}{2}\right]$ such that

$$\|\nabla u_\varepsilon\|_{L^2(\partial D_t\setminus\Delta_2)} + \|u_\varepsilon\|_{L^2(\partial D_t\setminus D_2)} \le C\|u_\varepsilon\|_{L^2(D_2)}. \tag{5-4}$$

Let v be the solution to the Dirichlet problem: $\mathcal{L}_0(v)=0$ in D_t and $v=u_\varepsilon$ on ∂D_t. Note that $v=0$ on Δ_1, and by Remark 2.8,

$$\|u_\varepsilon - v\|_{L^2(D_t)} \le C\varepsilon^{1/2}\|u_\varepsilon\|_{H^1(\partial D_t)}. \tag{5-5}$$

This, together with (5-4), gives

$$\|u_\varepsilon - v\|_{L^2(D_1)} \le \|u_\varepsilon - v\|_{L^2(D_t)} \le C\varepsilon^{1/2}\|u_\varepsilon\|_{L^2(D_2)}.$$ □

Theorem 5.2. *Suppose that $A=A(y)$ satisfies (1-2)–(1-3). Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon)=0$ in D_1 with $u_\varepsilon=0$ on Δ_1, where the defining function ψ in D_1 and Δ_1 is C^1. Then, for any $\alpha\in(0,1)$ and $\varepsilon\le r\le\frac{1}{2}$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C_\alpha r^{\alpha-1}\left(\fint_{D_1} |u_\varepsilon|^2\right)^{1/2}, \tag{5-6}$$

where C_α depends only on d, α, κ_1, κ_2, and the function $\tau(t)$ in (5-1).

Proof. Fix $\beta\in(\alpha,1)$. For each $r\in\left[\varepsilon,\frac{1}{2}\right]$, let $v=v_r$ be the function given by Lemma 5.1. By the boundary C^β estimates in C^1 domains for the operator $\mathcal{L}_0$ (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]),

$$\left(\fint_{D_{\theta r}} |v|^2\right)^{1/2} \le C_0\theta^\beta\left(\fint_{D_r} |v|^2\right)^{1/2}$$

for any $\theta\in(0,1)$, where C_0 depends only on d, κ_1, κ_2, β and $\tau(t)$. It follows that

$$\begin{aligned}
\left(\fint_{D_{\theta r}} |u_\varepsilon|^2\right)^{1/2} &\le \left(\fint_{D_{\theta r}} |v|^2\right)^{1/2} + C\left(\fint_{D_{\theta r}} |u_\varepsilon - v|^2\right)^{1/2}\\
&\le C\theta^\beta\left(\fint_{D_r} |v|^2\right)^{1/2} + C\theta^{-d/2}\left(\fint_{D_r} |u_\varepsilon - v|^2\right)^{1/2}\\
&\le C_1\theta^\beta\left(\fint_{D_r} |u_\varepsilon|^2\right)^{1/2} + C_1\theta^{-d/2}(\varepsilon/r)^{1/2}\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2}
\end{aligned}$$

for any $\varepsilon\le r\le\frac{1}{2}$. We now choose $\theta\in\left(0,\frac{1}{4}\right)$ so small that $C_1\theta^{\beta-\alpha}<\frac{1}{4}$. With θ fixed, choose $N>1$ large so that

$$C_1 2^\alpha\theta^{-d/2-\alpha}N^{-1/2}\le\tfrac{1}{4}.$$

It follows that if $r\ge N\varepsilon$,

$$\phi(\theta r)\le\tfrac{1}{4}\{\phi(r)+\phi(2r)\}, \tag{5-7}$$

where

$$\phi(r) = r^{-\alpha}\left(\fint_{D_r} |u_\varepsilon|^2\right)^{1/2}.$$

By integration we may deduce from (5-7) that

$$\int_{\theta a}^{\theta/2} \phi(r)\frac{dr}{r} \le \frac{1}{4}\int_a^{1/2} \phi(r)\frac{dr}{r} + \frac{1}{4}\int_{2a}^1 \phi(r)\frac{dr}{r},$$

where $N\varepsilon \le a < \frac{1}{2}$. This implies

$$\int_{\theta a}^1 \phi(r)\frac{dr}{r} \le C\int_{\theta/2}^1 \phi(r)\frac{dr}{r} \le C\phi(1).$$

Hence, $\phi(r) \le C\phi(1)$ for any $r \in [\varepsilon, 1]$, and the estimate (5-6) now follows by Caccioppoli's inequality. □

Remark 5.3. Under the stronger assumption that the defining function ϕ for D_1 is $C^{1,\sigma}$ for some $\sigma > 0$, we will show in Section 8 that the estimate (5-6) holds for $\alpha = 1$. In particular, it follows from the argument in Section 7 that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(0, 1)$, then

$$\left(\fint_{B(0,r)} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left(\fint_{B(0,1)} |\nabla u_\varepsilon|^2\right)^{1/2} \tag{5-8}$$

for any $\varepsilon \le r < 1$. This is the interior Lipschitz estimate down to the scale ε.

A function A is said to belong to $\mathrm{VMO}(\mathbb{R}^d)$ if the left-hand side of (5-9) goes to zero as $t \to 0^+$. To quantify this assumption we assume that

$$\sup_{\substack{x\in\mathbb{R}^d\\ 0<r<t}} \fint_{B(x,r)} \left| A(y) - \fint_{B(x,r)} A \right| dy \le \rho(t), \tag{5-9}$$

where $\rho(t) \to 0$ as $t \to 0^+$.

The following corollary was essentially proved in [Avellaneda and Lin 1987] by a compactness method.

Corollary 5.4. *Suppose that A satisfies (1-2)–(1-3). Also assume that $A \in \mathrm{VMO}(\mathbb{R}^d)$. Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_1 with $u_\varepsilon = 0$ on Δ_1. Then, for any $\alpha \in (0, 1)$,*

$$\|u_\varepsilon\|_{C^\alpha(D_{1/2})} \le C_\alpha\left(\fint_{D_1} |u_\varepsilon|^2\right)^{1/2}, \tag{5-10}$$

where C_α depends only on d, κ_1, κ_2, α, and the functions $\tau(t)$, $\rho(t)$.

Proof. We may assume $0 < \varepsilon < \frac{1}{2}$, as the case of $\varepsilon \ge \frac{1}{2}$ is local. Since $\mathcal{L}_1(u_\varepsilon(\varepsilon x)) = 0$, it follows from the boundary C^α estimates in C^1 domains (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]) for the operator $\mathcal{L}_1$ by rescaling that if $\alpha \in (0, 1)$ and $0 < r < \varepsilon$,

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C(r/\varepsilon)^{\alpha-1}\left(\fint_{D_\varepsilon} |\nabla u_\varepsilon|^2\right)^{1/2},$$

where C depends only on d, κ_1, κ_2, α, $\tau(t)$ and $\rho(t)$. This, together with Theorem 5.2, shows that the estimate (5-6) holds for any $0 < r < \frac{1}{2}$. By combining (5-6) with a similar interior estimate, we obtain

$$r^{\alpha-1}\left(\fint_{B(x,r)\cap D_{1/2}} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\|u_\varepsilon\|_{L^2(D_1)} \tag{5-11}$$

for any $0 < r < c$ and $x \in D_{1/2}$. The estimate (5-10) follows from (5-11) by Campanato's characterization of Hölder spaces. □

The rest of this section is devoted to the boundary C^α estimates for solutions with the Neumann boundary conditions.

Lemma 5.5. *Let $0 < \varepsilon \le r \le 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} with $\partial u_\varepsilon/\partial \nu_\varepsilon = 0$ on Δ_{2r}. Then there exists a function $w \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = 0$, $\partial w/\partial \nu_0 = 0$ in Δ_r, and*

$$\left(\fint_{D_r} |u_\varepsilon - w|^2\right)^{1/2} \le C(\varepsilon/r)^{1/2}\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2}, \tag{5-12}$$

where $\|\nabla\psi\|_\infty \le M$, and C depends only on d, κ_1, κ_2, and M.

Proof. By rescaling we may assume $r = 1$. As in the proof of Lemma 5.1, there exists $t \in \left[\frac{5}{4}, \frac{3}{2}\right]$ such that

$$\|u_\varepsilon\|_{L^2(\partial D_t \setminus \Delta_2)} + \|\nabla u_\varepsilon\|_{L^2(\partial D_t \setminus \Delta_2)} \le C\|u_\varepsilon\|_{L^2(D_2)}.$$

Let ϕ_ε be a function in $\mathcal{R}$ such that $u_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(D_t; \mathbb{R}^d)$. Let v be the solution to the Neumann problem: $\mathcal{L}_0(v) = 0$ in D_t and $\partial v/\partial \nu_0 = \partial u_\varepsilon/\partial \nu_\varepsilon$ on ∂D_t, with $v \perp \mathcal{R}$. It follows from Remark 2.8 that

$$\begin{aligned}\|u_\varepsilon - \phi_\varepsilon - v\|_{L^2(D_1)} &\le \|u_\varepsilon - \phi_\varepsilon - v\|_{L^2(D_t)}\\ &\le C\varepsilon^{1/2}\left\|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right\|_{L^2(\partial D_t)}\\ &\le C\varepsilon^{1/2}\|u_\varepsilon\|_{L^2(D_2)}.\end{aligned}$$

It is easy to see that the function $w = v + \phi_\varepsilon$ satisfies the desired conditions. □

Theorem 5.6. *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Let u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_1 with $\partial u_\varepsilon/\partial \nu_\varepsilon = 0$ on Δ_1, where the defining function ψ in D_1 and Δ_1 is C^1. Then, for any $\alpha \in (0, 1)$ and $\varepsilon \le r \le 1$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C_\alpha r^{\alpha-1}\left(\fint_{D_1} |\nabla u_\varepsilon|^2\right)^{1/2}, \tag{5-13}$$

where C depends only on d, α, κ_1, κ_2, and the function $\tau(t)$.

Proof. Fix $\beta \in (\alpha, 1)$. For each $r \in \left[\varepsilon, \frac{1}{2}\right]$, let $w = w_r$ be the function given by Lemma 5.5. By the boundary C^β estimates in C^1 domains for the operator $\mathcal{L}_0$,

$$\inf_{q\in\mathbb{R}^d}\left(\fint_{D_{\theta r}} |w - q|^2\right)^{1/2} \le C_0\theta^\beta \inf_{q\in\mathbb{R}^d}\left(\fint_{D_r} |w - q|^2\right)^{1/2},$$

where C_0 depends only on d, β, κ_1, κ_2, and $\tau(t)$. This, together with Lemma 5.5, gives

$$\begin{aligned}\inf_{q\in\mathbb{R}^d}\left(\fint_{D_{\theta r}} |u_\varepsilon - q|^2\right)^{1/2} &\le C \inf_{q\in\mathbb{R}^d}\left(\fint_{D_{\theta r}} |w - q|^2\right)^{1/2} + \left(\fint_{D_{\theta r}} |u_\varepsilon - w|^2\right)^{1/2}\\ &\le C\theta^\beta \inf_{q\in\mathbb{R}^d}\left(\fint_{D_r} |w - q|^2\right)^{1/2} + C_0\theta^{-d/2}\left(\fint_{D_r} |u_\varepsilon - w|^2\right)^{1/2}\\ &\le C\theta^\beta \inf_{q\in\mathbb{R}^d}\left(\fint_{D_r} |u_\varepsilon - q|^2\right)^{1/2} + C\theta^{-d/2}(\varepsilon/r)^{1/2}\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2}.\end{aligned}$$

By replacing u_ε with $u_\varepsilon - q$, we obtain

$$\phi(\theta r) \le C\theta^{\beta-\alpha}\phi(r) + C\theta^{-\alpha-d/2}(\varepsilon/r)^{1/2}\phi(2r)$$

for any $r \in \left[\varepsilon, \frac{1}{2}\right]$, where

$$\phi(r) = r^{-\alpha} \inf_{q\in\mathbb{R}^d}\left(\fint_{D_r} |u_\varepsilon - q|^2\right)^{1/2}.$$

By the integration argument used in the proof of Theorem 5.2, we may conclude that $\phi(r) \le C\phi(1)$ for $r \in \left[\varepsilon, \frac{1}{2}\right]$, which yields (5-13) by Caccioppoli's inequality. □

Remark 5.7. Under the stronger condition that the defining function for D_1 and Δ_1 is $C^{1,\sigma}$ for some $\sigma > 0$, we will show in Section 9 that the estimate (5-13) holds for $\alpha = 1$.

The following corollary was essentially proved in [Kenig et al. 2013] by a compactness method.

Corollary 5.8. *Suppose that A satisfies (1-2)–(1-3). Also assume that $A \in \mathrm{VMO}(\mathbb{R}^d)$. Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_1 with $\partial u_\varepsilon/\partial\nu_\varepsilon = 0$ on Δ_1. Then, for any $\alpha \in (0, 1)$,*

$$\|u_\varepsilon\|_{C^\alpha(D_{1/2})} \le C_\alpha\left(\fint_{D_1} |u_\varepsilon|^2\right)^{1/2}, \tag{5-14}$$

where C_α depends only on d, κ_1, κ_2, α, and the functions $\tau(t)$, $\rho(t)$.

Proof. As in the case of the Dirichlet boundary condition, the additional smoothness assumption $A \in \mathrm{VMO}(\mathbb{R}^d)$ ensures that the estimate (5-13) holds for any $r \in \left(0, \frac{1}{2}\right)$ (see, e.g., [Byun and Wang 2005] for estimates for $\mathcal{L}_1$). This, together with the interior estimates, gives the estimate (5-14) by the use of Campanato's characterization of Hölder spaces. □

6. $W^{1,p}$ estimates in C^1 domains

In this section we study the uniform $W^{1,p}$ estimates in C^1 domains. Throughout the section we will assume that $A = A(y)$ satisfies (1-2)–(1-3), $A \in \mathrm{VMO}(\mathbb{R}^d)$, and Ω is C^1. Our goal is to prove the following two theorems.

Theorem 6.1. *Suppose that A satisfies (1-2)–(1-3). Also assume that $A \in \mathrm{VMO}(\mathbb{R}^d)$. Let $1 < p < \infty$ and Ω be a bounded C^1 domain in $\mathbb{R}^d$. Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ be a weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f) \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{6-1}$$

where $f = (f_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d\times d})$. *Then*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \|f\|_{L^p(\Omega)}, \tag{6-2}$$

where C_p *depends only on* d, p, A, *and* Ω.

Theorem 6.2. *Suppose that* A *satisfies the same conditions as in Theorem 6.1. Let* $1 < p < \infty$ *and* Ω *be a bounded* C^1 *domain in* $\mathbb{R}^d$. *Let* $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ *be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f) \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -n\cdot f \quad \text{on } \partial\Omega, \tag{6-3}$$

where $f = (f_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d\times d})$. *Assume that* $u_\varepsilon \perp \mathcal{R}$. *Then*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \|f\|_{L^p(\Omega)}, \tag{6-4}$$

where C_p *depends only on* d, p, A, *and* Ω.

Recall that a function u_ε is called a weak solution of (6-1) if $u_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}^d)$ and

$$\int_\Omega a_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_\varepsilon^\beta}{\partial x_j}\cdot\frac{\partial \varphi^\alpha}{\partial x_i}\,dx = -\int_\Omega f_i^\alpha \cdot \frac{\partial\varphi^\alpha}{\partial x_i}\,dx \tag{6-5}$$

for any $\varphi = (\varphi^\alpha) \in C_0^\infty(\Omega; \mathbb{R}^d)$. Similarly, u_ε is called a weak solution of (6-3) if $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ and (6-5) holds for any $\varphi = (\varphi^\alpha) \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$. Under the assumptions that $A \in \mathrm{VMO}(\mathbb{R}^d)$ and Ω is C^1, the existence and uniqueness of solutions of (6-1) and (6-3) are more or less well known (see [Auscher and Qafsaoui 2002; Byun and Wang 2004; 2005] for references). The main interest here is that the constants C in the $W^{1,p}$ estimates (6-2) and (6-4) are independent of ε. We mention that for $\mathcal{L}_\varepsilon$ with coefficients satisfying (1-3), (1-11) and the Hölder continuity condition, estimates (6-2) and (6-4) were established in [Avellaneda and Lin 1987; 1991; Shen 2008; Kenig et al. 2013]. The results were extended to the case of almost-periodic coefficients in [Armstrong and Shen 2016]. Also, for $\mathcal{L}_\varepsilon$ with coefficients satisfying (1-2)–(1-3) in Lipschitz domains, some partial results may be found in [Geng et al. 2012].

Theorems 6.1 and 6.2 are proved by a real-variable argument. The required weak reverse Hölder inequalities (6-6) and (6-2) for $p > 2$ are established by combining local estimates for $\mathcal{L}_1$ and boundary Hölder estimates in Section 4 with the interior Lipschitz estimates, up to the scale ε.

Lemma 6.3. *Let* $u_\varepsilon \in H^1(B(x_0, 2r); \mathbb{R}^d)$ *be a weak solution to* $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ *in* $B(x_0, 2r)$ *for some* $x_0 \in \mathbb{R}^d$ *and* $r > 0$. *Then, for any* $2 < p < \infty$,

$$\left(\fint_{B(x_0,r)} |\nabla u_\varepsilon|^p\right)^{1/p} \le C_p\left(\fint_{B(x_0,2r)} |\nabla u_\varepsilon|^2\right)^{1/2}, \tag{6-6}$$

where C_p *depends only on* d, p, κ_1, κ_2, *and the function* $\rho(t)$ *in* (5-9).

Proof. By translation and dilation we may assume that $x_0 = 0$ and $r = 1$. We may also assume that $0 < \varepsilon < \frac{1}{4}$. The case $\varepsilon \ge \frac{1}{4}$ for $B(0, 1)$ is local, since $A(x/\varepsilon)$ satisfies the smoothness condition (5-9) uniformly in ε. For each $y \in B(0,1)$, we use the local $W^{1,p}$ estimates for the operator $\mathcal{L}_1$ (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]) and a simple blow-up argument to show that

$$\left(\fint_{B(y,\varepsilon/2)} |\nabla u_\varepsilon|^p\right)^{1/p} \le C\left(\fint_{B(y,\varepsilon)} |\nabla u_\varepsilon|^2\right)^{1/2}. \tag{6-7}$$

By the interior Lipschitz estimate, up to the scale ε, we have

$$\left(\fint_{B(y,\varepsilon)} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left(\fint_{B(y,1)} |\nabla u_\varepsilon|^2\right)^{1/2}. \tag{6-8}$$

We point out that the estimate (6-8) will be proved in Section 8 with no smoothness assumption on A (see Theorem 8.6). Hence, for any $y \in B(0,1)$,

$$\begin{aligned}\left(\fint_{B(y,\varepsilon/2)} |\nabla u_\varepsilon|^p\right)^{1/p} &\le C\left(\fint_{B(y,1)} |\nabla u_\varepsilon|^2\right)^{1/2}\\ &\le C\|\nabla u_\varepsilon\|_{L^2(B(0,2))}.\end{aligned} \tag{6-9}$$

By covering $B(0,1)$ with balls of radius $\varepsilon/2$, we may deduce (6-6) readily from (6-9). □

Lemma 6.4. *Let $u_\varepsilon \in H^1(D_{2r};\mathbb{R}^d)$ be a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{2r} with either $u_\varepsilon = 0$ or $\partial u_\varepsilon/\partial\nu_\varepsilon = 0$ in Δ_{2r}, where $0 < r \le 1$. Then, for any $2 < p < \infty$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^p\right)^{1/p} \le C_p\left(\fint_{D_{2r}} |\nabla u_\varepsilon|^2\right)^{1/2}, \tag{6-10}$$

where C depends only on d, p, κ_1, κ_2, $\tau(t)$ in (5-1), and $\rho(t)$ in (5-9).

Proof. Note that the function $r^{-1}\psi(rx')$ satisfies the condition (5-1) uniformly for $0 < r \le 1$. Thus, by rescaling, it suffices to prove the lemma for $r = 1$. Using Lemma 6.3, Theorem 5.2 and Theorem 5.6, we obtain

$$\begin{aligned}\left(\fint_{B(y,\delta(y)/8)} |\nabla u_\varepsilon|^p\right)^{1/p} &\le C\left(\fint_{B(y,\delta(y)/4)} |\nabla u_\varepsilon|^2\right)^{1/2}\\ &\le C_\alpha[\delta(y)]^{\alpha-1}\|\nabla u_\varepsilon\|_{L^2(D_2)}\end{aligned} \tag{6-11}$$

for any $\alpha \in (0,1)$, where $y \in D_1$ and $\delta(y) = \operatorname{dist}(y, \partial D_2)$. We now fix $\alpha \in \left(1 - \frac{1}{p}, 1\right)$. It follows from (6-11) that

$$\int_{D_1}\left(\fint_{B(y,\delta(y)/8)} |\nabla u_\varepsilon|^p\,dx\right)dy \le C\|\nabla u_\varepsilon\|^p_{L^2(D_2)}. \tag{6-12}$$

Using the fact that $\delta(x) \approx \delta(y)$ if $y \in D_1$ and $|y - x| < \frac{1}{8}\delta(y)$, it is not hard to verify that (6-12) implies (6-10). □

Proof of Theorems 6.1 and 6.2. By duality and a density argument it suffices to consider the case where $p > 2$ and $f = (f_i^\alpha) \in C_0^1(\Omega;\mathbb{R}^{d\times d})$. Furthermore, by a real-variable argument, which originated in [Caffarelli and Peral 1998] and further developed in [Shen 2005; 2007], one only needs to establish weak reverse Hölder inequalities for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(x_0, r) \cap \Omega$ with either $u_\varepsilon = 0$ or $\partial u_\varepsilon/\partial\nu_\varepsilon = 0$

on $B(x_0,r)\cap\partial\Omega$, where $x_0\in\overline{\Omega}$ and $0<r<c_0\operatorname{diam}(\Omega)$. These inequalities are exactly those given by Lemmas 6.3 and 6.4. We omit the details and refer the reader to [Shen 2005; 2008; Geng 2012] for details in the case of scalar elliptic equations. □

Remark 6.5. Suppose that A and Ω satisfy the same conditions as in Theorem 6.1. By some fairly standard extension and duality arguments (see, e.g., [Kenig et al. 2013]), one may deduce from Theorem 6.1 that the solution of the Dirichlet problem

$$\mathcal{L}_\varepsilon(u_\varepsilon)=\operatorname{div}(h)+F \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon=f \quad \text{on } \partial\Omega$$

satisfies

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)}\le C_p\big\{\|h\|_{L^p(\Omega)}+\|F\|_{L^p(\Omega)}+\|f\|_{W^{1/p,p}(\partial\Omega)}\big\}$$

for any $1<p<\infty$, where $W^{\alpha,p}(\partial\Omega)$ denotes the Sobolev space on $\partial\Omega$ of order α with exponent p. Similarly, the solutions of the Neumann problem

$$\mathcal{L}_\varepsilon(u_\varepsilon)=\operatorname{div}(h)+F \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}=-n\cdot h+g \quad \text{on } \partial\Omega$$

with $u_\varepsilon\perp\mathcal{R}$ satisfies

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)}\le C_p\big\{\|h\|_{L^p(\Omega)}+\|F\|_{L^p(\Omega)}+\|g\|_{W^{-1/p,p}(\partial\Omega)}\big\},$$

where $W^{-1/p,p}(\partial\Omega)$ is the dual of $W^{1/p,p'}(\partial\Omega)$.

7. L^p estimates in C^1 domains

The $W^{1,p}$ estimates in the last section allow us to establish the Rellich-type estimates in L^p, down to the scale ε, in C^1 domains under the additional assumption that A belongs to $\mathrm{VMO}(\mathbb{R}^d)$.

Theorem 7.1. *Suppose that $A=A(y)$ satisfies (1-2)–(1-3). Also assume that $A\in\mathrm{VMO}(\mathbb{R}^d)$. Let $1<p<\infty$ and Ω be a bounded C^1 domain in $\mathbb{R}^d$. Let $u_\varepsilon\in W^{1,p}(\Omega;\mathbb{R}^d)$ be a weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon)=F \quad \textit{in } \Omega \qquad \textit{and} \qquad u_\varepsilon=f \quad \textit{in } \partial\Omega, \tag{7-1}$$

where $F\in L^p(\Omega;\mathbb{R}^d)$ and $f\in W^{1,p}(\partial\Omega;\mathbb{R}^d)$. Then, for any $\varepsilon\le r<\operatorname{diam}(\Omega)$,

$$\left\{\frac{1}{r}\int_{\Omega_r}|\nabla u_\varepsilon|^p\right\}^{1/p}\le C_p\big\{\|F\|_{L^p(\Omega)}+\|f\|_{W^{1,p}(\partial\Omega)}\big\}, \tag{7-2}$$

where $\Omega_r=\{x\in\mathbb{R}^d:\operatorname{dist}(x,\partial\Omega)<r\}$. The constant C_p depends only on d, p, A and Ω.

Theorem 7.2. *Suppose that A and Ω satisfy the same conditions as in Theorem 7.1. Let $1<p<\infty$. Let $u_\varepsilon\in W^{1,p}(\Omega;\mathbb{R}^d)$ be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon)=F \quad \textit{in } \Omega \quad \textit{and} \quad \frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}=g \quad \textit{in } \partial\Omega, \tag{7-3}$$

where $F \in L^p(\Omega; \mathbb{R}^d)$, $g \in L^p(\partial\Omega; \mathbb{R}^d)$ and $\int_\Omega F + \int_{\partial\Omega} g = 0$. Also assume that $u_\varepsilon \perp \mathcal{R}$. Then, for any $\varepsilon \le r < \operatorname{diam}(\Omega)$,

$$\left\{ \frac{1}{r} \int_{\Omega_r} |\nabla u_\varepsilon|^p \right\}^{1/p} \le C_p \left\{ \|F\|_{L^p(\Omega)} + \|g\|_{L^p(\partial\Omega)} \right\}, \tag{7-4}$$

where C_p depends only on d, p, A and Ω.

The proof of Theorems 7.1 and 7.2 follows a similar line of argument as for Theorems 1.1 and 1.2 by considering

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_j^\beta(x/\varepsilon) K_\varepsilon \left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right), \tag{7-5}$$

where u_0 is the solution of the homogenized problem, K_ε is a smoothing operator defined by (2-6), and $\eta_\varepsilon \in C_0^\infty(\Omega)$ is a cut-off function satisfying (2-14).

Throughout this section we will assume that Ω is C^1 and A satisfies (1-2)–(1-3) and (5-9).

Lemma 7.3. *Let u_ε ($\varepsilon \ge 0$) be the solutions of the Dirichlet problems (7-1). Let w_ε be defined by (7-5). Then*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \varepsilon^{1/p} \left\{ \|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} \right\}, \tag{7-6}$$

where C_p depends only on d, p, A and Ω.

Proof. A direct computation shows that

$$\begin{aligned}
\mathcal{L}_\varepsilon(w_\varepsilon) = &-\frac{\partial}{\partial x_i}\left\{ [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \right] \right\} \\
&+ \frac{\partial}{\partial x_i}\left\{ b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \right\} \\
&+ \varepsilon \frac{\partial}{\partial x_i}\left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) \frac{\partial}{\partial x_j}\left(K_\varepsilon\left(\frac{\partial u_0^\gamma}{\partial x_k} \eta_\varepsilon \right) \right) \right\},
\end{aligned}$$

where $b_{ij}^{\alpha\beta}(y)$ is defined by (2-3). Using (2-5), we obtain

$$\frac{\partial}{\partial x_i}\left\{ b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \right\} = -\varepsilon \frac{\partial}{\partial x_i}\left\{ \phi_{kij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_k}\left(K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \right) \right\}.$$

It follows that

$$\begin{aligned}
\mathcal{L}_\varepsilon(w_\varepsilon) = &-\frac{\partial}{\partial x_i}\left\{ [\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)] \left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \right] \right\} \\
&- \varepsilon \frac{\partial}{\partial x_i}\left\{ \phi_{kij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_k}\left(K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j} \eta_\varepsilon \right) \right) \right\} \\
&+ \varepsilon \frac{\partial}{\partial x_i}\left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_k^{\beta\gamma}(x/\varepsilon) \frac{\partial}{\partial x_j}\left(K_\varepsilon\left(\frac{\partial u_0^\gamma}{\partial x_k} \eta_\varepsilon \right) \right) \right\}.
\end{aligned} \tag{7-7}$$

Since $w_\varepsilon = 0$ on $\partial\Omega$, we may apply the $W^{1,p}$ estimate in Theorem 6.1 to obtain

$$\begin{aligned}\|w_\varepsilon\|_{W^{1,p}(\Omega)} &\le C\big\{\|\nabla u_0 - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + \varepsilon\|\phi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} \\ &\qquad\qquad + \varepsilon\|\chi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)}\big\} \\ &\le C\big\{\|\nabla u_0 - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + \varepsilon\|\nabla((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)}\big\} \\ &\le C\big\{\|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 u_0)\eta_\varepsilon\|_{L^2(\Omega)}\big\},\end{aligned} \tag{7-8}$$

where we have used Lemmas 2.1 and 2.2 for the second and third inequalities.

We now write $u_0 = v + w$, where

$$v(x) = \int_\Omega \Gamma_0(x-y)F(y)\,dy \tag{7-9}$$

and $\Gamma_0(x-y)$ denotes the matrix of fundamental solutions for the operator $\mathcal{L}_0$ in $\mathbb{R}^d$, with pole at the origin. Note that $\|v\|_{W^{2,p}(\mathbb{R}^d)} \le C_p\|F\|_{L^p(\Omega)}$ and

$$\|\nabla v\|_{L^p(S_t)} \le C_p\|F\|_{L^p(\Omega)},$$

where $S_t = \{x \in \mathbb{R}^d : \operatorname{dist}(x,\partial\Omega)=t\}$ for t small (see the proof of Theorem 2.6). It follows that

$$\|\nabla v\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|\nabla^2 v\|_{L^p(\Omega)} \le C\varepsilon^{1/p}\|F\|_{L^p(\Omega)}. \tag{7-10}$$

Finally, we observe that $\mathcal{L}_0(w) = 0$ in Ω and

$$\|w\|_{W^{1,p}(\partial\Omega)} \le \|f\|_{W^{1,p}(\partial\Omega)} + \|v\|_{W^{1,p}(\partial\Omega)} \le C\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}.$$

It follows from the solvability of the L^p regularity problem for the operator $\mathcal{L}_0$ in C^1 domain Ω, which follows from [Fabes et al. 1978; Lewis et al. 1993; Hofmann et al. 2015], that

$$\|(\nabla w)^*\|_{L^p(\partial\Omega)} \le C\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}.$$

Also, using the interior estimate

$$|\nabla^2 w(x)| \le \frac{C}{\delta(x)}\bigg(\fint_{B(x,\delta(x)/8)} |\nabla w|^p\bigg)^{1/p},$$

where $\delta(x) = \operatorname{dist}(x,\partial\Omega)$, we may show that

$$\begin{aligned}\int_{\Omega\setminus\Omega_{3\varepsilon}} |\nabla^2 w|^p\,dx &\le C\int_{\Omega\setminus\Omega_{2\varepsilon}} |\nabla w(x)|^p[\delta(x)]^{-p}\,dx \\ &\le C\varepsilon^{1-p}\|(\nabla w)^*\|^p_{L^p(\partial\Omega)} \le C\varepsilon^{1-p}\big\{\|f\|^p_{W^{1,p}(\partial\Omega)} + \|F\|^p_{L^p(\Omega)}\big\}.\end{aligned}$$

As a result, we obtain

$$\|\nabla w\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 w)\eta_\varepsilon\|_{L^p(\Omega)} \le C\varepsilon^{1/p}\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}.$$

This, together with the estimate (7-10) for v, gives

$$\|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 u_0)\eta_\varepsilon\|_{L^p(\Omega)} \le C\varepsilon^{1/p}\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}, \tag{7-11}$$

which, in view of (7-8), completes the proof. □

Proof of Theorem 7.1. Without loss of generality we may assume that

$$\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} = 1.$$

Let $\varepsilon \le r < \operatorname{diam}(\Omega)$. It follows from Lemma 7.3 that

$$\begin{aligned}
&\|\nabla u_\varepsilon\|_{L^p(\Omega_r)} \\
&\quad\le \|\nabla u_0\|_{L^p(\Omega_r)} + C\|\nabla\chi(x/\varepsilon)K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega_r)} + C\varepsilon\|\chi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega_r)} + C\varepsilon^{1/p} \\
&\quad\le C\|\nabla u_0\|_{L^p(\Omega_{2r})} + C\varepsilon\|\nabla((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + C\varepsilon^{1/p} \\
&\quad\le C\|\nabla u_0\|_{L^p(\Omega_{2r})} + C\varepsilon^{1/p},
\end{aligned} \tag{7-12}$$

where we have used Lemma 2.1 for the second inequality and (7-11) for the third. An inspection of the proof of Lemma 7.3 shows that

$$\|\nabla u_0\|_{L^p(\Omega_{2r})} \le Cr^{1/p},$$

which, in view of (7-12), gives

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_r)} \le Cr^{1/p}. \qquad \square$$

To prove Theorem 7.2, we need the following lemma.

Lemma 7.4. *Let u_ε $(\varepsilon \ge 0)$ be solutions of the Neumann problem (7-3). Also assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Let w_ε be defined by (7-5). Then*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p\varepsilon^{1/p}\big\{\|g\|_{L^p(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}, \tag{7-13}$$

where C_p depends only on d, p, A and Ω.

Proof. The proof is similar to that of Lemma 7.3. Let ϕ_ε be a function in $\mathcal{R}$ such that $w_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(\Omega;\mathbb{R}^d)$. It follows from the formula (7-7) and the $W^{1,p}$ estimates in Theorem 6.2 that

$$\|w_\varepsilon - \phi_\varepsilon\|_{W^{1,p}(\Omega)} \le C\big\{\|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 u_0)\eta_\varepsilon\|_{L^2(\Omega)}\big\}. \tag{7-14}$$

To estimate the right-hand side of (7-14), we proceed as in the proof of Lemma 7.3, but use the nontangential maximal function estimate [Fabes et al. 1978; Lewis et al. 1993; Hofmann et al. 2015]

$$\|(\nabla w)^*\|_{L^p(\partial\Omega)} \le C\left\|\frac{\partial w}{\partial\nu_0}\right\|_{L^p(\partial\Omega)},$$

where $\mathcal{L}_0(w) = 0$ in Ω and $w \perp \mathcal{R}$ in $L^2(\Omega;\mathbb{R}^d)$. As a result, we obtain

$$\|w_\varepsilon - \phi_\varepsilon\|_{W^{1,p}(\Omega)} \le C\varepsilon^{1/p}\big\{\|g\|_{L^p(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}. \tag{7-15}$$

Finally, note that since $u_\varepsilon - u_0 \perp \mathcal{R}$,

$$\begin{aligned}
\|\phi_\varepsilon\|_{W^{1,p}(\Omega)} &\le C\varepsilon\|\chi(x/\varepsilon)K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} \\
&\le C\varepsilon\|\nabla u_0\|_{L^p(\Omega)}.
\end{aligned}$$

This, together with (7-15), yields the estimate (7-13). $\square$

Proof of Theorem 7.2. The estimate (7-4) follows from (7-13), as in the case of the Dirichlet conditions. We omit the details. □

Remark 7.5. Under certain smoothness condition on A, such as Hölder continuity, it is possible to solve the L^p Dirichlet, regularity, and Neumann problems for $\mathcal{L}_1(u) = 0$ in C^1 domains for any $1 < p < \infty$. By the same localization procedure and blow-up argument as in Remark 3.1, this implies

$$\begin{cases} \displaystyle\int_{\partial\Omega} |\nabla u_\varepsilon|^p \, d\sigma \le C \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^p d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^p \, dx, \\ \displaystyle\int_{\partial\Omega} |\nabla u_\varepsilon|^p \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^p \, d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^p \, dx, \end{cases} \tag{7-16}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω. It then follows from Theorems 7.1 and 7.2 that

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^p \, d\sigma \le C \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^p d\sigma \tag{7-17}$$

if $u_\varepsilon \perp \mathcal{R}$, and

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^p \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^p \, d\sigma + C \int_{\partial\Omega} |u_\varepsilon|^p \, d\sigma. \tag{7-18}$$

As in the case $p = 2$, by the method of layer potentials, estimates (7-17)–(7-18) lead to the uniform solvability of the L^p Dirichlet, regularity, and Neumann problems in C^1 domains. The details will be given elsewhere.

8. Lipschitz estimates in $C^{1,\alpha}$ domains, part I

In this section we investigate the Lipschitz estimates, down to the scale ε, in $C^{1,\alpha}$ domains with Dirichlet boundary conditions and give the proof of Theorem 1.4. The Neumann boundary conditions will be treated in the next section. The proof of Theorems 1.4 and 1.5 is based on a general scheme for establishing Lipschitz estimates at large scales in homogenization, recently formulated in [Armstrong and Smart 2016] for interior estimates. Our approach to the boundary Lipschitz estimates in $C^{1,\alpha}$ domains is similar to that used in [Armstrong and Shen 2016] for elliptic systems with almost-periodic coefficients. We remark that Lemma 8.5, which is a continuous version of Lemma 3.1 in [Armstrong and Shen 2016] and whose proof is simpler, makes the argument more transparent.

Let D_r and Δ_r be defined by (1-16) with $\psi(0) = 0$ and $\|\nabla\psi\|_\infty \le M$.

Lemma 8.1. *Let $u_\varepsilon \in H^1(D_2; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_2 with $u_\varepsilon = f$ on Δ_2. Then there exists $v \in H^1(D_1; \mathbb{R}^d)$ such that $\mathcal{L}_0(v) = F$ in D_1, $v = f$ on Δ_1, and*

$$\|u_\varepsilon - v\|_{L^2(D_1)} \le C\varepsilon^{1/2} \big\{ \|u_\varepsilon\|_{L^2(D_2)} + \|F\|_{L^2(D_2)} + \|f\|_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)} \big\}, \tag{8-1}$$

where C depends only on d, κ_1, κ_2, and M.

Proof. By Caccioppoli's inequality,

$$\int_{D_{3/2}} |\nabla u_\varepsilon|^2 \le C \left\{ \int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|f\|^2_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|^2_{L^\infty(\Delta_2)} \right\}.$$

By the coarea formula this implies that there exists some $t \in \left[\frac{5}{4}, \frac{3}{2}\right]$ such that

$$\int_{\partial D_t \setminus \Delta_2} (|\nabla u_\varepsilon|^2 + |u_\varepsilon|^2) \le C \left\{ \int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|f\|^2_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|^2_{L^\infty(\Delta_2)} \right\}.$$

Let v be the weak solution to the Dirichlet problem,

$$\mathcal{L}_0(v) = F \quad \text{in } D_t \qquad \text{and} \qquad v = u_\varepsilon \quad \text{on } \partial D_t.$$

It follows from Remark 2.8 that

$$\begin{aligned} \|u_\varepsilon - v\|_{L^2(D_1)} &\le \|u_\varepsilon - v\|_{L^2(D_t)} \\ &\le C\varepsilon^{1/2}\big\{\|u_\varepsilon\|_{H^1(\partial D_t)} + \|F\|_{L^2(D_t)}\big\} \\ &\le C\varepsilon^{1/2}\big\{\|u_\varepsilon\|_{L^2(D_2)} + \|F\|_{L^2(D_2)} + \|f\|_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)}\big\}, \end{aligned}$$

where C depends only on d, κ_1, κ_2, and M. $\square$

Lemma 8.2. *Let $\varepsilon \le r < 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_{2r} with $u_\varepsilon = f$ on Δ_{2r}. Then there exists $v \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(v) = F$ in D_r, $v = f$ on Δ_r, and*

$$\begin{aligned} &\left(\fint_{D_r} |u_\varepsilon - v|^2 \right)^{1/2} \\ &\quad \le C(\varepsilon/r)^{1/2} \left\{ \left(\fint_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left(\fint_{D_{2r}} |F|^2 \right)^{1/2} + \|f\|_{L^\infty(\Delta_{2r})} + r\|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} \right\}, \end{aligned} \tag{8-2}$$

where C depends only on d, κ_1, κ_2, and M.

Proof. This follows from Lemma 8.1 by rescaling. $\square$

In the rest of this section we will assume that the defining function ψ in the definition of D_r and Δ_r is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$ with $\psi(0) = 0$ and $\|\nabla \psi\|_{C^\alpha(\mathbb{R}^{d-1})} \le M$.

Lemma 8.3. *Let v be a solution of $\mathcal{L}_0(v) = F$ in D_r with $v = f$ on Δ_r. For $0 < t \le r$, define*

$$\begin{aligned} G(t; v) = \frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{d\times d} \\ q \in \mathbb{R}^d}} \Bigg\{ &\left(\fint_{D_t} |v - Mx - q|^2 \right)^{1/2} + t^2 \left(\fint_{D_t} |F|^p \right)^{1/p} + \|f - Mx - q\|_{L^\infty(\Delta_t)} \\ &+ t\|\nabla_{\tan}(f - Mx - q)\|_{L^\infty(\Delta_t)} + t^{1+\sigma} \|\nabla_{\tan}(f - Mx - q)\|_{C^{0,\sigma}(\Delta_t)} \Bigg\}, \end{aligned} \tag{8-3}$$

where $p > d$ and $\sigma \in (0, \alpha)$. Then there exists $\theta \in \left(0, \frac{1}{4}\right)$, depending only on d, p, κ_1, κ_2, σ, α and M, such that

$$G(\theta r; v) \le \tfrac{1}{2} G(r; v). \tag{8-4}$$

Proof. The lemma follows from the boundary $C^{1,\alpha}$ estimates for elasticity systems with constant coefficients. We refer the reader to [Armstrong and Shen 2016, Lemma 7.1] for the case $\mathcal{L}_0(v) = 0$. The argument for the general case $F \in L^p$ with $p > d$ is the same. $\square$

Lemma 8.4. *Let $0<\varepsilon<\frac{1}{2}$. Let u_ε be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon)=F$ in D_1 with $u_\varepsilon=f$ on Δ_1. Define*

$$H(r)=\frac{1}{r}\inf_{\substack{M\in\mathbb{R}^{d\times d}\\ q\in\mathbb{R}^d}}\Bigg\{\bigg(\fint_{D_r}|u_\varepsilon-Mx-q|^2\bigg)^{1/2}+r^2\bigg(\fint_{D_r}|F|^p\bigg)^{1/p}+\|f-Mx-q\|_{L^\infty(\Delta_r)} \\ +r\|\nabla_{\tan}(f-Mx-q)\|_{L^\infty(\Delta_r)}+r^{1+\sigma}\|\nabla_{\tan}(f-Mx-q)\|_{C^{0,\sigma}(\Delta_r)}\Bigg\} \tag{8-5}$$

and

$$\Phi(r)=\frac{1}{r}\inf_{q\in\mathbb{R}^d}\Bigg\{\bigg(\fint_{D_{2r}}|u_\varepsilon-q|^2\bigg)^{1/2}+r^2\bigg(\fint_{D_{2r}}|F|^p\bigg)^{1/p}+\|f-q\|_{L^\infty(\Delta_{2r})}+r\|\nabla_{\tan}f\|_{L^\infty(\Delta_{2r})}\Bigg\}, \tag{8-6}$$

where $p>d$ and $\sigma\in(0,\alpha)$. Then

$$H(\theta r)\le\tfrac{1}{2}H(r)+C\Big(\frac{\varepsilon}{r}\Big)^{1/2}\Phi(2r) \tag{8-7}$$

for any $r\in\big[\varepsilon,\frac{1}{2}\big]$, where $\theta\in\big(0,\frac{1}{4}\big)$ is given by Lemma 8.3.

Proof. Fix $r\in\big[\varepsilon,\frac{1}{2}\big]$. Let v be a solution of $\mathcal{L}_0(v)=F$ in D_r with $v=f$ on Δ_r. Observe that

$$\begin{aligned}H(\theta r)&\le\frac{1}{\theta r}\bigg(\fint_{D_{\theta r}}|u_\varepsilon-v|^2\bigg)^{1/2}+G(\theta r;v)\\&\le\frac{1}{\theta r}\bigg(\fint_{D_{\theta r}}|u_\varepsilon-v|^2\bigg)^{1/2}+\tfrac{1}{2}G(r;v)\\&\le\frac{C}{r}\bigg(\fint_{D_r}|u_\varepsilon-v|^2\bigg)^{1/2}+\tfrac{1}{2}H(r),\end{aligned}$$

where we have used Lemma 8.3 for the second inequality. This, together with Lemma 8.2, gives

$$H(\theta r)\le\tfrac{1}{2}H(r)+C\Big(\frac{\varepsilon}{r}\Big)^{1/2}\frac{1}{r}\Bigg\{\bigg(\fint_{D_{2r}}|u_\varepsilon|^2\bigg)^{1/2}+r^2\bigg(\fint_{D_{2r}}|F|^2\bigg)^{1/2}+\|f\|_{L^\infty(\Delta_{2r})}+r\|\nabla_{\tan}f\|_{L^\infty(\Delta_{2r})}\Bigg\}.$$

Since $H(r)$ remains invariant if we subtract a constant from u_ε, the inequality (8-7) follows. □

Lemma 8.5. *Let $H(r)$ and $h(r)$ be two nonnegative continuous functions on the interval $(0,1]$. Let $0<\varepsilon<\frac{1}{4}$. Suppose that there exists a constant C_0 such that*

$$\begin{cases}\max\limits_{r\le t\le 2r}H(t)\le C_0H(2r),\\ \max\limits_{r\le t,s\le 2r}|h(t)-h(s)|\le C_0H(2r)\end{cases} \tag{8-8}$$

for any $r\in\big[\varepsilon,\frac{1}{2}\big]$. We further assume that

$$H(\theta r)\le\tfrac{1}{2}H(r)+C_0\omega(\varepsilon/r)\{H(2r)+h(2r)\} \tag{8-9}$$

for any $r\in\big[\varepsilon,\frac{1}{2}\big]$, where $\theta\in\big(0,\frac{1}{4}\big)$ and ω is a nonnegative increasing function $[0,1]$ such that $\omega(0)=0$ and

$$\int_0^1\frac{\omega(t)}{t}\,dt<\infty. \tag{8-10}$$

Then

$$\max_{\varepsilon \le r \le 1} \{H(r) + h(r)\} \le C\{H(1) + h(1)\}, \tag{8-11}$$

where C depends only on C_0, θ, and ω.

Proof. It follows from (8-8) that

$$h(r) \le h(2r) + C_0 H(2r)$$

for any $\varepsilon \le r \le \frac{1}{2}$. Hence,

$$\begin{aligned}\int_a^{1/2} \frac{h(r)}{r}\, dr &\le \int_a^{1/2} \frac{h(2r)}{r}\, dr + C_0 \int_a^{1/2} \frac{H(2r)}{r}\, dr \\ &= \int_{2a}^{1} \frac{h(r)}{r}\, dr + C_0 \int_{2a}^{1} \frac{H(r)}{r}\, dr,\end{aligned}$$

where $\varepsilon \le a \le \frac{1}{4}$. This implies

$$\begin{aligned}\int_a^{2a} \frac{h(r)}{r}\, dr &\le \int_{1/2}^{1} \frac{h(r)}{r}\, dr + C \int_{2a}^{1} \frac{H(r)}{r}\, dr \\ &\le C\{h(1) + H(1)\} + C \int_{2a}^{1} \frac{H(r)}{r}\, dr,\end{aligned}$$

which, by (8-8), gives

$$\begin{aligned}h(a) &\le C\left\{H(2a) + h(1) + H(1) + \int_{2a}^{1} \frac{H(r)}{r}\, dr\right\} \\ &\le C\left\{h(1) + H(1) + \int_{a}^{1} \frac{H(r)}{r}\, dr\right\}\end{aligned} \tag{8-12}$$

for any $a \in \left[\varepsilon, \frac{1}{4}\right]$.

Next, we use (8-9) and (8-12) to obtain

$$H(\theta r) \le \tfrac{1}{2} H(r) + C\omega(\varepsilon/r)\{h(1) + H(1)\} + C\omega(\varepsilon/r) \int_r^1 \frac{H(r)}{r}\, dr.$$

It follows that

$$\int_{\alpha\theta\varepsilon}^{\theta} \frac{H(r)}{r}\, dr \le \frac{1}{2} \int_{\alpha\varepsilon}^{1} \frac{H(r)}{r}\, dr + C_\alpha\{h(1) + H(1)\} + C \int_{\alpha\varepsilon}^{1} \omega(\varepsilon/r)\left\{\int_r^1 \frac{H(t)}{t}\, dt\right\} \frac{dr}{r},$$

where $\alpha > 1$ and we have used the condition (8-10). Using (8-10) and the observation that

$$\int_{\alpha\varepsilon}^{1} \omega(\varepsilon/r)\left\{\int_r^1 \frac{H(t)}{t}\, dt\right\} \frac{dr}{r} = \int_{\alpha\varepsilon}^{1} H(t)\left\{\int_{\varepsilon/t}^{1/\alpha} \frac{\omega(s)}{s}\, ds\right\} \frac{dt}{t} \le (4C)^{-1} \int_{\alpha\varepsilon}^{1} H(t) \frac{dt}{t}$$

if $\alpha > \alpha_0(\omega)$, we see that

$$\int_{\alpha\theta\varepsilon}^{\theta} \frac{H(r)}{r}\, dr \le \frac{1}{2} \int_{\alpha\varepsilon}^{1} \frac{H(r)}{r}\, dr + C_\alpha\{h(1) + H(1)\} + \frac{1}{4} \int_{\alpha\varepsilon}^{1} \frac{H(r)}{r}\, dr.$$

It follows that

$$\int_\varepsilon^1 \frac{H(r)}{r}\, dr \le C\{h(1)+H(1)\}, \tag{8-13}$$

which, together with (8-8) and (8-12), yields the estimate (8-11). □

Proof of Theorem 1.4. We may assume that $0<\varepsilon<\frac{1}{4}$. Let u_ε be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon)=F$ in D_1 with $u_\varepsilon=f$ on Δ_1, where $F\in L^p(D_1)$ for some $p>d$ and $f\in C^{1,\sigma}(\Delta_1)$ for some $\sigma\in(0,\alpha)$. For $r\in(0,1)$, we define the function $H(r)$ by (8-5). It is easy to see that $H(t)\le CH(2r)$ if $t\in(r,2r)$.

Next, we let $h(r)=|M_r|$, where M_r is the $d\times d$ matrix such that

$$\begin{aligned}H(r)=\frac{1}{r}\inf_{q\in\mathbb{R}^d}\Bigg\{\bigg(\fint_{D_r}|u_\varepsilon-M_rx-q|^2\bigg)^{1/2}+r^2\bigg(\fint_{D_r}|F|^p\bigg)^{1/p}+\|f-M_rx-q\|_{L^\infty(\Delta_r)}\\ +r\|\nabla_{\tan}(f-M_rx-q)\|_{L^\infty(\Delta_r)}+r^{1+\sigma}\|\nabla_{\tan}(f-M_rx-q)\|_{C^{0,\sigma}(\Delta_r)}\Bigg\}.\end{aligned}$$

Let $t,s\in[r,2r]$. Using

$$\begin{aligned}|M_t-M_s|&\le\frac{C}{r}\inf_{q\in\mathbb{R}^d}\bigg(\fint_{D_r}|(M_t-M_s)x-q|^2\bigg)^{1/2}\\&\le\frac{C}{t}\inf_{q\in\mathbb{R}^d}\bigg(\fint_{D_t}|u_\varepsilon-M_tx-q|^2\bigg)^{1/2}+\frac{C}{s}\inf_{q\in\mathbb{R}^d}\bigg(\fint_{D_s}|u_\varepsilon-M_sx-q|^2\bigg)^{1/2}\\&\le C\{H(t)+H(s)\}\\&\le CH(2r),\end{aligned}$$

we obtain

$$\max_{r\le t,s\le 2r}|h(t)-h(s)|\le CH(2r).$$

Furthermore, if Φ is defined by (8-6), then

$$\Phi(r)\le H(2r)+h(2r).$$

In view of Lemma 8.4 this gives

$$H(\theta r)\le\tfrac{1}{2}H(r)+C\omega(\varepsilon/r)\{H(2r)+h(2r)\}$$

for $r\in\left[\varepsilon,\frac{1}{2}\right]$, where $\omega(t)=t^{1/2}$. Thus the functions $H(r)$ and $h(r)$ satisfy the conditions (8-8), (8-9) and (8-10) in Lemma 8.5. Consequently, we obtain that for $r\in\left[\varepsilon,\frac{1}{2}\right]$,

$$\begin{aligned}\inf_{q\in\mathbb{R}^d}\frac{1}{r}\bigg(\fint_{D_r}|u_\varepsilon-q|^2\bigg)^{1/2}&\le C\{H(r)+h(r)\}\\&\le C\{H(1)+h(1)\}\\&\le C\Bigg\{\bigg(\fint_{D_1}|u_\varepsilon|^2\bigg)^{1/2}+\|F\|_{L^p(D_1)}+\|f\|_{C^{1,\sigma}(\Delta_1)}\Bigg\},\end{aligned}$$

which, together with Caccioppoli's inequality, gives the estimate (1-18). □

The argument used in this section may be used to prove the interior Lipschitz estimates, down to the scale ε.

Theorem 8.6. *Suppose that A satisfies (1-2)–(1-3). Let $u_\varepsilon \in H^1(B(x_0, R); \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$, where $F \in L^p(B(x_0, R); \mathbb{R}^d)$ for some $p > d$. Then, for $\varepsilon \le r < R$,*

$$\left(\fint_{B(x_0,r)} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\left(\fint_{B(x_0,R)} |\nabla u_\varepsilon|^2\right)^{1/2} + R\left(\fint_{B(x_0,R)} |F|^p\right)^{1/p}\right\}, \tag{8-14}$$

where C depends only on d, κ_1, κ_2, and p.

9. Lipschitz estimates in $C^{1,\alpha}$ domains, part II

In this section we study the Lipschitz estimate, down to the scale ε, with Neumann boundary conditions, and give the proof of Theorem 1.5. Throughout this section we will assume that the defining function ψ in D_r and Δ_r is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$ and $\|\nabla\psi\|_{C^\alpha(\mathbb{R}^{d-1})} \le M$.

Lemma 9.1. *Let Ω be a bounded Lipschitz domain. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $\partial u_\varepsilon/\partial\nu_\varepsilon = g$ on $\partial\Omega$. Then there exists $w \in H^1(\Omega; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = F$ in Ω, $\partial w/\partial\nu_0 = g$ on $\partial\Omega$, and*

$$\|u_\varepsilon - w\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)}\big\}. \tag{9-1}$$

Proof. Choose $\phi_\varepsilon \in \mathcal{R}$ such that $u_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. Let u_0 be the weak solution to the Neumann problem: $\mathcal{L}_0(u_0) = F$ in Ω and $\partial u_0/\partial\nu_0 = g$ on $\partial\Omega$ with the property $u_0 \perp \mathcal{R}$. It follows from Remark 2.8 that

$$\|u_\varepsilon - \phi_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)}\big\}.$$

By letting $w = u_0 + \phi_\varepsilon$ this gives (9-1). □

Lemma 9.2. *Let $\varepsilon \le r < 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_{2r} with $\partial u_\varepsilon/\partial\nu_\varepsilon = g$ on Δ_{2r}. Then there exists $w \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = F$ in D_r, $\partial w/\partial\nu_0 = g$ on Δ_r, and*

$$\left(\fint_{D_r} |u_\varepsilon - w|^2\right)^{1/2} \le C(\varepsilon/r)^{1/2}\left\{\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{D_{2r}} |F|^2\right)^{1/2} + r\|g\|_{L^\infty(\Delta_{2r})}\right\}, \tag{9-2}$$

where C depends only on d, κ_1, κ_2, and M.

Proof. By rescaling we may assume $r = 1$. As in the case of Dirichlet conditions in Lemma 8.2, the desired estimate follows from Lemma 9.1 by using the coarea formula and the Caccioppoli inequality

$$\int_{D_{3/2}} |\nabla u_\varepsilon|^2 \le C\left\{\int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|g\|^2_{L^\infty(\Delta_2)}\right\}, \tag{9-3}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_2 and $\partial u_\varepsilon/\partial\nu_\varepsilon = g$ on Δ_2. □

Lemma 9.3. *Let w be a solution of $\mathcal{L}_0(w) = F$ in D_r with $\partial w/\partial \nu_0 = g$ on Δ_r. For $0 < t \le r$, define*

$$I(t; w) = \frac{1}{t} \inf_{\substack{M\in\mathbb{R}^{d\times d}\\ q\in\mathbb{R}^d}} \Bigg\{ \bigg(\fint_{D_t} |w - Mx - q|^2 \bigg)^{1/2} + t^2 \bigg(\fint_{D_t} |F|^p \bigg)^{1/p} + t \bigg\| \frac{\partial}{\partial \nu_0}(w - Mx) \bigg\|_{L^\infty(\Delta_t)} + t^{1+\sigma} \bigg\| \frac{\partial}{\partial \nu_0}(w - Mx) \bigg\|_{C^{0,\sigma}(\Delta_t)} \Bigg\}, \tag{9-4}$$

where $p > d$ and $\sigma \in (0, \alpha)$. Then there exists $\theta \in \big(0, \frac{1}{4}\big)$, depending only on d, p, κ_1, κ_2, σ, α and M, such that

$$I(\theta r; w) \le \tfrac{1}{2} I(r; w). \tag{9-5}$$

Proof. By rescaling we may assume $r = 1$. The lemma then follows from the boundary $C^{1,\sigma}$ estimates with Neumann boundary conditions in $C^{1,\alpha}$ domains for elasticity systems with constant coefficients. □

Lemma 9.4. *Let $0 < \varepsilon < \frac{1}{2}$. Let u_ε be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_1 with $\partial u_\varepsilon/\partial \nu_\varepsilon = g$ on Δ_1, where $F \in L^p(D_1; \mathbb{R}^d)$ for some $p > d$ and $g \in C^\sigma(\Delta_1; \mathbb{R}^d)$ for some $\sigma \in (0, \alpha)$. Define*

$$J(r) = \frac{1}{r} \inf_{\substack{M\in\mathbb{R}^{d\times d}\\ q\in\mathbb{R}^d}} \Bigg\{ \bigg(\fint_{D_r} |u_\varepsilon - Mx - q|^2 \bigg)^{1/2} + r^2 \bigg(\fint_{D_r} |F|^p \bigg)^{1/p} + r \bigg\| g - \frac{\partial}{\partial \nu_0}(Mx) \bigg\|_{L^\infty(\Delta_r)} + r^{1+\sigma} \bigg\| g - \frac{\partial}{\partial \nu_0}(Mx) \bigg\|_{C^{0,\sigma}(\Delta_r)} \Bigg\} \tag{9-6}$$

and

$$\Psi(r) = \frac{1}{r} \inf_{q\in\mathbb{R}^d} \Bigg\{ \bigg(\fint_{D_{2r}} |u_\varepsilon - q|^2 \bigg)^{1/2} + r^2 \bigg(\fint_{D_{2r}} |F|^p \bigg)^{1/p} + r \|g\|_{L^\infty(\Delta_{2r})} \Bigg\}. \tag{9-7}$$

Then

$$J(\theta r) \le \tfrac{1}{2} J(r) + C(\varepsilon/r)^{1/2} \Psi(2r) \tag{9-8}$$

for any $r \in \big[\varepsilon, \frac{1}{2}\big]$, where $\theta \in \big(0, \frac{1}{4}\big)$ is given by Lemma 9.3.

Proof. Fix $r \in \big[\varepsilon, \frac{1}{2}\big]$. Let w be the function in $H^1(D_r; \mathbb{R}^d)$ given by Lemma 9.2. Then

$$\begin{aligned} J(\theta r) &\le I(\theta r; w) + \frac{1}{\theta r} \bigg(\fint_{D_{\theta r}} |u_\varepsilon - w|^2 \bigg)^{1/2} \\ &\le \tfrac{1}{2} I(r; w) + \frac{1}{\theta r} \bigg(\fint_{D_{\theta r}} |u_\varepsilon - w|^2 \bigg)^{1/2} \\ &\le \tfrac{1}{2} J(r) + \frac{C}{r} \bigg(\fint_{D_r} |u_\varepsilon - w|^2 \bigg)^{1/2}, \end{aligned}$$

where we have used Lemma 9.3 for the second inequality. In view of Lemma 9.2, this gives

$$J(\theta r) \le \tfrac{1}{2} J(r) + \frac{C}{r} \Bigg\{ \bigg(\fint_{D_{2r}} |u_\varepsilon|^2 \bigg)^{1/2} + r^2 \bigg(\fint_{D_{2r}} |F|^p \bigg)^{1/p} + r \|g\|_{L^\infty(\Delta_{2r})} \Bigg\},$$

from which the estimate (9-8) follows, as the function $J(r)$ is invariant if we replace u_ε by $u_\varepsilon - q$ for any $q \in \mathbb{R}^d$. □

Proof of Theorem 1.5. With Lemma 9.4 at our disposal, Theorem 1.5 follows from Lemma 8.5, as in the case of Dirichlet boundary conditions. We omit the details. □

As we indicate in the Introduction, under additional smoothness conditions, the full Lipschitz estimates, uniform in ε, follow from Theorem 1.4, Theorem 1.5, and local Lipschitz estimates by a blow-up argument.

Corollary 9.5. *Suppose that A satisfies (1-2)–(1-3). Also assume that A is Hölder continuous. Let $u_\varepsilon \in H^1(B(0,1);\mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(0,1)$, where $F \in L^p(B(0,1);\mathbb{R}^d)$ for some $p > d$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(B(0,1/2))} \le C_p\bigl\{\|u_\varepsilon\|_{L^2(B(0,1))} + \|F\|_{L^p(B(0,1))}\bigr\}, \tag{9-9}$$

where C_p depends only on d, p and A.

Corollary 9.6. *Suppose that A satisfies (1-2)–(1-3). Also assume that A is Hölder continuous. Let $u_\varepsilon \in H^1(D_1;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}(u_\varepsilon) = F$ in D_1 with $u_\varepsilon = f$ on Δ_1, where the defining function ψ in D_1 and Δ_1 is $C^{1,\alpha}$ with $\|\nabla\psi\|_{C^\alpha(\mathbb{R}^{d-1})} \le M$ for some $\alpha > 0$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \le C\bigl\{\|u_\varepsilon\|_{L^2(D_1)} + \|F\|_{L^p(D_1)} + \|f\|_{C^{1,\sigma}(\Delta_1)}\bigr\}, \tag{9-10}$$

where $p > d$, $\sigma \in (0,\alpha)$, and C depends only on d, p, σ, A, α and M.

Corollary 9.7. *Suppose that A, D_1 and Δ_1 satisfy the same conditions as in Corollary 9.6. Let $u_\varepsilon \in H^1(D_1;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}(u_\varepsilon) = F$ in D_1 with $\partial u_\varepsilon/\partial\nu_\varepsilon = g$ on Δ_1. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \le C\bigl\{\|u_\varepsilon\|_{L^2(D_1)} + \|F\|_{L^p(D_1)} + \|g\|_{C^\sigma(\Delta_1)}\bigr\}, \tag{9-11}$$

where $p > d$, $\sigma \in (0,\alpha)$, and C depends only on d, p, σ, A, α and M.

As we mentioned in Introduction, for $\mathcal{L}_\varepsilon$ with coefficients satisfying (1-11), (1-3) and the Hölder continuity condition, estimates (9-9) and (9-10) were proved in [Avellaneda and Lin 1987], while (9-11) was established in [Kenig et al. 2013; Armstrong and Shen 2016].

Acknowledgements

The author thanks Carlos E. Kenig for several very helpful discussions regarding this work. The author also would like to thank Scott N. Armstrong for insightful conversations and discussions regarding the work [Armstrong and Smart 2016].

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Received 9 Aug 2016. Revised 21 Nov 2016. Accepted 22 Jan 2017.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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ANALYSIS & PDE

Volume 10 No. 3 2017