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KINETIC FORMULATION OF VORTEX VECTOR FIELDS

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This article focuses on gradient vector fields of unit Euclidean norm in $\mathbb{R}^N$. The stream functions associated to such vector fields solve the eikonal equation and the prototype is given by the distance function to a closed set. We introduce a kinetic formulation that characterizes stream functions whose level sets are either spheres or hyperplanes in dimension $N \geq 3$. Our main result proves that the kinetic formulation is a selection principle for the vortex vector field whose stream function is the distance function to a point.

1. Introduction

In this article, we analyze the following type of vortex vector field:

$$u^\star : \mathbb{R}^N \to \mathbb{R}^N, \quad u^\star(x) = \frac{x}{|x|} \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\}$$

in dimension $N \geq 2$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. This structure arises in many physical models such as micromagnetics, liquid crystals, superconductivity, elasticity. Clearly, $u^\star$ is smooth away from the origin: in fact, 0 is a topological singularity of degree 1 since the jacobian is $\det \nabla u^\star = V_N \delta_0$, where δ_0 is the Dirac measure at the origin and V_N is the volume of the unit ball in $\mathbb{R}^N$. Also, $u^\star$ is a curl-free unit-length vector field; i.e.,

$$|u^\star| = 1 \quad \text{and} \quad \nabla \times u^\star = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{1}$$

Moreover, there is a stream function $\psi^\star : \mathbb{R}^N \to \mathbb{R}$ associated to $u^\star$ by the equation

$$u^\star = \nabla \psi^\star;$$

indeed, one may consider $\psi^\star$ as the distance function at the origin, i.e., $\psi^\star(x) = |x|$ for $x \in \mathbb{R}^N$, and $\psi^\star$ represents the viscosity solution of the eikonal equation

$$|\nabla \psi^\star| = 1$$

under an appropriate boundary condition at infinity (e.g., $\lim_{|x|\to\infty}(\psi^\star(x) - |x|) = 0$).

Note that conversely, these properties characterize the vortex vector field: if $u : \mathbb{R}^N \to \mathbb{R}^N$ is a nonconstant vector field that is smooth away from the origin and satisfies (1) then $u = \pm u^\star$ in $\mathbb{R}^N$. Indeed, this classically follows by the method of characteristics: the flow associated to u by

$$\partial_t X(t, x) = u(X(t, x)) \tag{2}$$

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with the initial condition $X(0,x)=x$ for $x\neq 0$ yields straight lines $\{X(t,x)\}_t$ given by $X(t,x)=x+tu(x)$ along which u is constant, i.e., $u(X(t,x))=u(x)$. Since u is nonconstant and two characteristics can intersect only at the origin (which is the prescribed point-singularity of u), every characteristic passes through the origin[1] and therefore, u coincides with $u^\star$ or $-u^\star$. Caffarelli and Crandall [2010] proved this result under a weaker regularity hypothesis for the vector field $u=\nabla\psi$: if ψ is assumed only pointwise differentiable away from a set S of vanishing Hausdorff $\mathcal{H}^1$-measure (i.e., $\mathcal{H}^1(S)=0$) and $|\nabla\psi|=1$ in $\mathbb{R}^N\setminus S$, then $\psi=\pm\psi^\star$ (up to a translation and an additive constant). We also refer to [DiPerna and Lions 1989] for weaker regularity assumptions on u in the framework of Sobolev spaces.

Our aim is to prove a kinetic characterization of the vortex vector field that does not assume any initial regularity on u. This kinetic formulation will characterize stream functions whose level sets are totally umbilical hypersurfaces in dimension $N\geq 3$, i.e., either pieces of spheres or hyperplanes. In order to introduce the kinetic formulation of the vortex vector field, we start by presenting the case of dimension $N=2$ and then we extend it to dimensions $N\geq 3$.

1.1. ***Kinetic formulation in dimension $N = 2$.*** Let $\Omega\subset\mathbb{R}^2$ be an open set and $u:\Omega\to\mathbb{R}^2$ be a Lebesgue-measurable vector field that satisfies

$$|u|=1 \text{ a.e. in } \Omega \quad\text{and}\quad \nabla\times u=0 \text{ distributionally in } \Omega. \tag{3}$$

The main feature of the kinetic formulation relies on the concept of weak characteristic for a nonsmooth vector field u. We start by noting that (2) has a proper meaning only if some notion of trace of u can be defined on curves $\{X(t,x)\}_t$, which in general is a consequence of the regularity assumption on u (see [DiPerna and Lions 1989]). To overcome this difficulty, the following notion of "weak characteristic" is introduced for measurable vector fields u (see, e.g., [Lions, Perthame, and Tadmor 1994; Jabin and Perthame 2001]): for every direction $\xi\in\mathbb{S}^1$, one defines the function $\chi(\cdot,\xi):\Omega\to\{0,1\}$ by

$$\chi(x,\xi)=\begin{cases}1 & \text{for } u(x)\cdot\xi>0,\\ 0 & \text{for } u(x)\cdot\xi\leq 0.\end{cases} \tag{4}$$

In the case of a smooth vector field u in a neighborhood of a point $x_0\in\Omega$, then $\chi(\cdot,\xi)$ mimics the characteristic of u of normal direction $\xi=(\xi_1,\xi_2)$ (see Figure 1); formally, if $\xi^\perp=(-\xi_2,\xi_1)=\pm u(x_0)$, then either $\nabla\chi(\cdot,\xi)$ locally vanishes (if u is constant in a neighborhood of x_0), or $\nabla\chi(\cdot,\xi)$ is a measure concentrated on the characteristic $\{X(t,x_0)\}_t$ given by (2) with constant measure density $\pm\xi$. In other words, we have the following "kinetic formulation" of the problem (see, e.g., [DeSimone, Müller, Kohn and Otto 2001; Jabin and Perthame 2001]):

Proposition 1 (kinetic formulation in dimension $N=2$). *Let $\Omega\subset\mathbb{R}^2$ be an open set and $u:\Omega\to\mathbb{R}^2$ be a smooth vector field. If u satisfies* (3) *then*

$$\xi^\perp\cdot\nabla_x\chi(\cdot,\xi)=0 \quad \textit{distributionally in } \Omega \textit{ for every } \xi\in\mathbb{S}^1. \tag{5}$$

[1] This argument is clear in dimension $N=2$; for dimensions $N\geq 3$, one needs an additional argument showing that two characteristics are coplanar, as we will see later in the proof of Theorem 8.

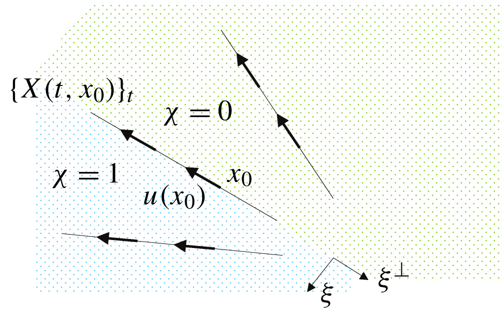

Figure 1. Characteristics of u.

We mention that the kinetic formulation (5) holds under the weaker Sobolev regularity $W^{1/p,p}$ for $p \in [1, 3]$ (see [Ignat 2011; 2012a; 2012b; De Lellis and Ignat 2015]). Note that the knowledge of $\chi(\cdot, \xi)$ in every direction $\xi \in \mathbb{S}^1$ determines completely a vector field u with $|u| = 1$ due to the averaging formula

$$u(x) = \frac{1}{2} \int_{\mathbb{S}^1} \xi \chi(x, \xi) \, d\mathcal{H}^1(\xi) \qquad \text{for a.e. } x \in \Omega. \tag{6}$$

Thanks to (6), we deduce that the kinetic formulation (5) incorporates the fact that $\nabla \times u = 0$ (see Proposition 5 below). Therefore, the curl-free condition will be no longer mentioned in the following statements whenever (5) is assumed to hold true for unit-length vector fields u.

The main question is whether the kinetic formulation (5) characterizes the vortex vector field in $\mathbb{R}^2$. First of all, (5) induces a regularizing effect for Lebesgue-measurable unit-length vector fields u. Indeed, the classical "kinetic averaging lemma" (see, e.g., [Golse, Lions, Perthame, and Sentis 1988]) shows that a measurable vector field $u : \Omega \to \mathbb{S}^1$ satisfying (5) belongs to $H^{1/2}_{\mathrm{loc}}(\Omega)$ due to the averaging formula (6).[2] Moreover, Jabin, Otto, and Perthame [2002] improved the regularizing effect by showing that u is locally Lipschitz away from vortex point-singularities[3] and u coincides with the vortex vector field around these singularities:

Theorem 2 [Jabin, Otto, and Perthame 2002]. *Let $\Omega \subset \mathbb{R}^2$ be an open set and $u : \Omega \to \mathbb{R}^2$ be a Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. in Ω together with the kinetic formulation* (5). *Then u is locally Lipschitz continuous inside Ω except at a locally finite number of singular points. Moreover, every singular point P of u corresponds to a vortex singularity of topological degree* 1 *of u; i.e., there exists a sign $\gamma = \pm 1$ such that*

$$u(x) = \gamma u^\star(x - P) \quad \textit{for every } x \neq P \textit{ in any convex neighborhood of } P \textit{ in } \Omega.$$

In particular, if $\Omega = \mathbb{R}^2$ and u is nonconstant, then u coincides with $u^\star$ or $-u^\star$ (up to a translation).

This result leads to the following interpretation of the kinetic formulation in dimension $N = 2$: equation (5) is a selection principle for the viscosity solutions of the eikonal equation $|\nabla \psi| = 1$ in the sense that the solutions ψ are smooth (more precisely, they belong to the Sobolev space $W^{2,\infty}_{\mathrm{loc}}$) away from

[2] For the improved regularizing effect for scalar conservation laws, see [Otto 2009; Golse and Perthame 2013].

[3] This regularity is optimal; see, e.g., Proposition 1 in [Ignat 2012b].

point-singularities. Clearly, these solutions are induced by the viscosity solutions of the eikonal equation under some appropriate boundary condition. Conversely, in the spirit of [Caffarelli and Crandall 2010], it was shown by Ignat [2012b] and De Lellis and Ignat [2015] that for any vector field u satisfying (3) together with an initial Sobolev regularity $W^{1/p,p}$, $p \in [1,3]$ (i.e., excluding jump line-singularities), the kinetic formulation (5) holds true and therefore, one obtains the regularizing effect in Theorem 2.

Remark 3. The result of Jabin, Otto, and Perthame [2002] was motivated by the study of zero-energy states in a line-energy Ginzburg–Landau model in dimension 2. More precisely, one considers the energy functional $E_\varepsilon : H^1(\Omega, \mathbb{R}^2) \to \mathbb{R}_+$ defined for $\varepsilon > 0$ as

$$E_\varepsilon(u_\varepsilon) = \varepsilon \int_\Omega |\nabla u_\varepsilon|^2\,dx + \frac{1}{\varepsilon}\int_\Omega (1-|u_\varepsilon|^2)^2\,dx + \frac{1}{\varepsilon}\|\nabla \times u_\varepsilon\|^2_{H^{-1}(\Omega)}, \quad u_\varepsilon \in H^1(\Omega, \mathbb{R}^2), \tag{7}$$

where Ω is a domain in $\mathbb{R}^2$ and $H^{-1}(\Omega)$ is the dual of the Sobolev space $H^1_0(\Omega)$. (We refer to [Ambrosio, De Lellis, and Mantegazza 1999; Aviles and Giga 1999; DeSimone, Müller, Kohn and Otto 2001; Jabin, Otto, and Perthame 2002; Jabin and Perthame 2001; Jin and Kohn 2000; Rivière and Serfaty 2001] for the analysis of this model.) A vector field $u : \Omega \to \mathbb{R}^2$ is called zero-energy state if there exists a family $\{u_\varepsilon \in H^1(\Omega, \mathbb{R}^2)\}_{\varepsilon\to 0}$ satisfying

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad E_\varepsilon(u_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

Obviously, a zero-energy state u satisfies (3). The result of Jabin, Otto, and Perthame [2002] shows that every zero-energy state u satisfies (5) and therefore, u shares the structure stated in Theorem 2.

1.2. ***Kinetic formulation in dimension $N \geq 3$.*** Our main interest consists in defining a kinetic formulation for the vortex vector field in dimension $N \geq 3$. Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a Lebesgue-measurable vector field. For every direction $\xi \in \mathbb{S}^{N-1}$, we consider the characteristic function $\chi(\,\cdot\,, \xi)$ defined at (4) and we denote the orthogonal hyperplane to ξ by

$$\xi^\perp := \{v \in \mathbb{R}^N : v \cdot \xi = 0\}.$$

Definition 4 (kinetic formulation). We say that a measurable vector field u satisfies the kinetic formulation if the following equation holds true:

$$v \cdot \nabla_x \chi(\,\cdot\,, \xi) = 0 \quad \text{distributionally in } \Omega \text{ for every } \xi \in \mathbb{S}^{N-1} \text{ and } v \in \xi^\perp. \tag{8}$$

Roughly speaking, (8) means that $\nabla_x \chi(\,\cdot\,, \xi)$ is a distribution pointing in direction $\pm\xi$. Note that the kinetic formulation (8) only carries out the information of the direction of the vector field u (i.e., it gives no information about the Euclidean norm of u). Imposing the unit-length constraint, u will satisfy a similar averaging formula (6) which justifies that the curl-free constraint $\nabla \times u = 0$ is incorporated in the kinetic formulation (8).

Proposition 5. *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}^N$ be Lebesgue measurable with $|u| = 1$ a.e. in Ω. Then*

$$u(x) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi\,\chi(x, \xi)\,d\mathcal{H}^{N-1}(\xi) \quad \textit{for a.e. } x \in \Omega, \tag{9}$$

where V_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$. Moreover, if u satisfies the kinetic formulation (8) *then $\nabla \times u = 0$ distributionally in Ω.*

Remark 6. We highlight that Proposition 1 is *false* in dimension $N \geq 3$; i.e., there are smooth curl-free vector fields with values into the unit sphere $\mathbb{S}^{N-1}$ that do not satisfy the kinetic formulation (8). For example, in dimension $N = 3$, considering the vortex-line vector field

$$u_0(x) = \frac{(x_1, x_2, 0)}{\sqrt{x_1^2 + x_2^2}} \quad \text{in } \Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 1\},$$

then u_0 is smooth in Ω and satisfies (3). However, (8) fails. Indeed, let $\xi = \frac{1}{\sqrt{2}}(1, 0, 1)$. Then $u_0(x) \cdot \xi = 0$ for $x \in \Omega$ is equivalent to $x_1 = 0$ and therefore,

$$\nabla_x \chi(\cdot, \xi) = e_1 \mathcal{H}^2 \llcorner \{x \in \Omega : x_1 = 0\},$$

where $e_1 = (1, 0, 0)$. Now, taking $v = \frac{1}{\sqrt{2}}(-1, 0, 1)$, we have $v \cdot \xi = 0$ (i.e., $v \in \xi^\perp$) and $v \cdot \nabla_x \chi(\cdot, \xi) \neq 0$ in $\mathcal{D}'(\Omega)$.

As Remark 6 has already revealed, the kinetic equation (8) in dimension $N \geq 3$ plays a different role than in dimension $N = 2$ because the gradient $\nabla \chi(\cdot, \xi)$ is expected to concentrate on hypersurfaces (not on the line characteristics of u). In fact, the geometric interpretation of (8) can be regarded in terms of the stream function ψ of a nonconstant vector field $u = \nabla \psi$: the level sets of ψ are expected to be pieces of spheres of codimension 1 where the characteristics of u represent the normal directions to these spheres.

Theorem 7. *Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be an open set and $\psi : \Omega \to \mathbb{R}$ be a smooth stream function such that $u = \nabla \psi$ satisfies the kinetic formulation* (8)*. Assume $|u|$ never vanishes on a level set $\{x \in \Omega : \psi(x) = \alpha\}$ for some $\alpha \in \mathbb{R}$ and let $\mathcal{S}$ be a connected component of $\{\psi = \alpha\}$. Then $\mathcal{S}$ is locally a totally umbilical hypersurface, that is, either a piece of an $(N-1)$-sphere or a piece of a hyperplane.*

Note that Theorem 7 fails in dimension $N = 2$: a level set of a smooth stream function ψ of $u = \nabla \psi$ satisfying (3) (and therefore, u satisfies the kinetic formulation (5) by Proposition 1) does not have, in general, constant curvature.[4]

2. Main results

Our main result shows that the kinetic formulation (8) is a characterization of the vortex vector field $u^\star$ in dimension $N \geq 3$.

Theorem 8. *Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a connected open set and $u : \Omega \to \mathbb{R}^N$ be a nonconstant Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. in Ω together with the kinetic equation* (8)*. Then u coincides with the vortex vector field $u^\star$ or $-u^\star$ up to a translation.*

Note that in dimension $N = 2$, this result is true for the domain $\Omega = \mathbb{R}^2$, but it is in general false for other domains Ω where there exist nonconstant smooth vector fields u in Ω different than vortex

[4]If $\Gamma \subset \mathbb{R}^2$ is a smooth curve of nonconstant curvature, then one takes ψ to be the distance function to Γ in a small neighborhood Ω of Γ (with the convention that Γ is withdrawn from that neighborhood, i.e., $\Gamma \cap \Omega = \varnothing$, so that ψ is smooth in Ω).

vector fields that satisfy (3) and thus, (5) (by Proposition 1). The main difference in dimension $N \geq 3$ is the following: if u is a smooth vector field with (3) that is neither constant nor a vortex vector field, then the kinetic formulation (8) doesn't hold for u (see Remark 6). Hence, in dimension $N \geq 3$, the zero-energy states of E_ε defined in (7) do not satisfy in general the kinetic equation (8). Therefore, the kinetic formulation (8) is more rigid in dimension $N \geq 3$ since it selects only the vortex vector fields, as they correspond to smooth solutions of the eikonal equation with level sets of constant sectional curvature (by Theorem 7).

Let us explain the strategy of the proof of Theorem 8. The key point relies on a relation of order of the level sets of the stream function associated to u: for every two Lebesgue points $x, y \in \Omega$ of u such that the segment $[x, y]$ lies in Ω and for every direction $\xi \in \mathbb{S}^{N-1}$ orthogonal to $x - y$, one has

$$u(x) \cdot \xi > 0 \quad \Longrightarrow \quad u(y) \cdot \xi \geq 0.$$

The next step consists in defining the trace of u on each segment $\Sigma \subset \Omega$; more precisely, similar to the procedure of [Jabin, Otto, and Perthame 2002], there exists a trace $\tilde{u} \in L^\infty(\Sigma, \mathbb{S}^{N-1})$ of u such that $u(P) = \tilde{u}(P)$ for each Lebesgue point $P \in \Sigma$ of u. Moreover, if the trace $\tilde{u}$ of u is collinear with the segment Σ at some Lebesgue point, then $\tilde{u}$ is $\mathcal{H}^1$-almost everywhere collinear with Σ (which coincides with the classical principle of characteristics for smooth vector fields u). The final step consists in proving that every two characteristics are coplanar. Then one concludes by the following geometrical fact specific to dimension $N \geq 3$:

Proposition 9. *Let $N \geq 3$ and $\mathcal{D}$ be a set of lines in $\mathbb{R}^N$ such that every two lines of $\mathcal{D}$ are coplanar*, *but $\mathcal{D}$ is not planar* (*i.e.*, *there is no* 2*-dimensional plane containing $\mathcal{D}$*). *Then either all lines of $\mathcal{D}$ are collinear*, *or all lines of $\mathcal{D}$ pass through a same point* (*that is a vortex point*).

In view of Theorem 8, it is natural to ask if one can characterize other types of unit-length curl-free vector fields u by weakening the kinetic formulation (8), in particular, vector fields having a vortex-line singularity. In dimension $N \geq 3$, the prototype of a vortex-line vector field is given by

$$u_0(x', x_N) = \nabla |x'|,$$

where $x = (x', x_N)$ and $x' = (x_1, \ldots, x_{N-1})$; clearly, u_0 is smooth away from the vortex-line $\{x \in \mathbb{R}^N : x' = 0\}$ where (3) holds true. Defining

$$\mathcal{E} := \{\xi \in \mathbb{S}^{N-1} : \xi_N = 0\} = \mathbb{S}^{N-2} \times \{0\},$$

using the notation (4), we have that u_0 satisfies the following kinetic formulation in $\Omega = \mathbb{R}^N$:

$$\forall \xi \in \mathcal{E},\ \forall v \in \xi^\perp, \quad v \cdot \nabla_x \chi(\cdot, \xi) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{10}$$

Note that (10) is a weakened form of (8): the quantity $v \cdot \nabla_x \chi(\cdot, \xi)$ vanishes for directions $\xi \in \mathcal{E}$ (and $v \in \xi^\perp$) and fails to vanish for $\mathcal{H}^{N-1}$-a.e. direction $\xi \in \mathbb{S}^{N-1}$. As opposed to (8) (in view of (9)), the kinetic formulation (10) does not force a unit-length vector field u to be curl-free; it only implies that

$$\nabla' \times \frac{u'}{|u'|} = 0 \quad \text{in } \{|u'| \neq 0\} = \{u \neq \pm e_N\},$$

where $e_N = (0, \dots, 0, 1)$, $u' = (u_1, \dots, u_{N-1})$ and $\nabla' = (\partial_1, \dots, \partial_{N-1})$. Since we are looking for a characterization of vortex-line vector fields (that are in particular curl-free), we will impose that

$$\partial_k u_N = \partial_N u_k \quad \text{in } \Omega \ \text{ for } k = 1, \dots, N-1. \tag{11}$$

We will prove the following result:

Theorem 10. *Let $N \geq 4$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. on Ω together with* (10) *and* (11). *Then in every ball included in $\{x \in \Omega : u(x) \neq \pm e_N\}$, there exists a stream function $\psi = \psi(\alpha, \beta)$ solving the eikonal equation in dimension* 2 *such that*

$$u(x) = \nabla_x[\psi(\alpha, \beta)],$$

where

(1) *either $\alpha = |x' - P'|$ and $\beta = x_N$ for some point $P' \in \mathbb{R}^{N-1}$;*

(2) *or $\alpha = w' \cdot x'$ and $\beta = x_N$ for some vector $w' \in \mathbb{S}^{N-2}$.*

Therefore, the weakened kinetic formulation (10), together with (11), is not enough to select vortex-line vector fields which correspond to the stream function $\psi(\alpha, \beta) = \pm\alpha$ in case (1) of Theorem 10. Similar results to Theorem 10 hold for similar kinetic formulations corresponding to vector fields having vortex-sheet singularities of dimension k in $\mathbb{R}^N$ with $N \geq k + 3$.

The outline of this paper is as follows: in Section 3, we characterize the level sets of smooth stream functions associated to vector fields that satisfy the kinetic formulation (8). In particular, we prove Proposition 1 and Theorem 7. Section 4 is devoted to proving fine properties of Lebesgue points of u needed in Section 5, where the notion of the trace on lines for a vector field u satisfying (8) is defined. Section 6 is the core of this paper: using this notion of trace and the geometric arguments of Proposition 9, we prove our main result in Theorem 8. Section 7 deals with the study of the weakened kinetic formulation (10).

3. Level sets of the stream function

This section is devoted to the study of the level sets of smooth stream functions ψ associated to vector fields $u = \nabla\psi$ satisfying (8). We start by proving that $|\nabla\psi|$ is locally constant on each level set of ψ.

Lemma 11. *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $\psi : \Omega \to \mathbb{R}$ be a smooth stream function such that $u = \nabla\psi$ satisfies the kinetic formulation* (8). *Assume $|u|$ never vanishes on a level set $\{x \in \Omega : \psi(x) = \alpha\}$ for some $\alpha \in \mathbb{R}$ and let $\mathcal{S}$ be a connected component of $\{\psi = \alpha\}$. Then $|u|$ is constant on $\mathcal{S}$. Moreover, there exists a neighborhood ω of $\mathcal{S}$, a smooth solution $\tilde{\psi} : \omega \to \mathbb{R}$ of the eikonal equation and a diffeomorphism $t \mapsto F(t)$ such that $\psi = F(\tilde{\psi})$ in ω (in particular, $\nabla\tilde{\psi}$ satisfies* (8)).

Proof. Since $|u| \neq 0$ on $\mathcal{S}$ and u is smooth in Ω, we can define

$$v = \frac{u}{|u|} \quad \text{in a neighborhood of } \mathcal{S}.$$

For simplicity of notation, we suppose that Ω is this neighborhood, i.e., $|u| \neq 0$ in Ω. Then v satisfies (8) because u satisfies it, too; since v is smooth in Ω, Proposition 5 implies $\nabla \times v = 0$ in Ω. (The proof of Proposition 5 is independent of Lemma 11; we will admit it here and prove it later in Section 4.) As a consequence, in any simply connected domain $\omega \subset \Omega$, the Poincaré lemma yields the existence of a smooth function $\tilde{\psi}$ such that $v = u/|u| = \nabla\tilde{\psi}$ in ω, i.e.,

$$\nabla\psi = u = |u|v = |u|\nabla\tilde{\psi} \quad \text{in } \omega.$$

Therefore, ψ and $\tilde{\psi}$ have the same level sets in ω. Without loss of generality, we may assume that $\tilde{\psi} = 0$ on $\omega \cap \mathcal{S}$. Now, for every $P' \in \omega \cap \mathcal{S}$, we consider the flow associated to v,

$$\begin{cases} \dot{X}(P', t) = \nabla\tilde{\psi}(X(P', t)), \\ X(P', 0) = P'. \end{cases} \tag{12}$$

Call $I_{P'}$ the maximal interval where the solution $X(P', \cdot)$ exists. Obviously, the flow is unique and smooth, satisfying

$$\ddot{X}(P', t) = \nabla^2\tilde{\psi}(X) \cdot \dot{X} = \nabla^2\tilde{\psi}(X) \cdot \nabla\tilde{\psi}(X) = 0 \quad \text{in } I_{P'}$$

because $\nabla^2\tilde{\psi}$ is a symmetric matrix and $|\nabla\tilde{\psi}| = 1$ in ω. Consequently, $\dot{X}(P', \cdot)$ is constant in $I_{P'}$ so that

$$\nabla\tilde{\psi}(X(P', t)) = \nabla\tilde{\psi}(P'), \quad \frac{d}{dt}[\tilde{\psi}(X(P', t))] = 1, \quad X(P', t) = P' + t\nabla\tilde{\psi}(P').$$

Therefore, since $\tilde{\psi} = 0$ on $\omega \cap \mathcal{S}$, we have

$$\tilde{\psi}(X(P', t)) = t \quad \text{for all } P' \in \omega \cap \mathcal{S} \text{ and } t \in I_{P'}.$$

Identifying the level sets of $\tilde{\psi}$ (and of ψ, too) using the flow, i.e., $\{\tilde{\psi} = t\} = \{X(P', t) : P' \in \omega \cap \mathcal{S}\}$, we can define

$$F(t) := \psi(X(P', t)) \quad \text{for} \quad P' \in \omega \cap \mathcal{S},\ t \in I_{P'}.$$

The function F is a diffeomorphism: F is smooth (because ψ and X are smooth) and we have

$$\frac{d}{dt}F(t) = \nabla\psi(X(P', t)) \cdot \dot{X}(P', t) \overset{(12)}{=} \nabla\psi(X(P', t)) \cdot \frac{\nabla\psi}{|\nabla\psi|}(X(P', t)) = |u|(X(P', t)) \neq 0.$$

In particular, $|u|$ is constant on $\{\tilde{\psi} = 0\} = \{\psi = F(0)\} = \omega \cap \mathcal{S}$. Since ω was arbitrarily chosen, we deduce that $|u|$ is locally constant on $\mathcal{S}$; because $\mathcal{S}$ is connected, it follows that $|u|$ is constant on $\mathcal{S}$. Since the flow $\{X(P', t) : P' \in \mathcal{S}, t \in I_{P'}\}$ covers a neighborhood of $\mathcal{S}$, the last statement of the lemma follows. □

3.1. *The case of dimension $N = 2$.* In the special case of dimension $N = 2$, we start by proving that every smooth curl-free vector field of unit length satisfies the kinetic formulation (5). This result can be found already in [DeSimone, Müller, Kohn and Otto 2001; Jabin and Perthame 2001]. For completeness, we will present two easy and self-contained proofs. The first one is based on the geometry of the flow (2) (as heuristically described in Section 1), while the second proof is based on the concept of entropy introduced in [DeSimone, Müller, Kohn and Otto 2001].

Proof of Proposition 1: first method. We can assume that $\xi = e_1$ and $\xi^\perp = e_2$ (otherwise, one considers a rotation $R \in SO(2)$ such that $e_1 = R\xi$ and $\tilde{u}(x) := Ru(R^{-1}x)$ in a neighborhood of a point $x \in \Omega$).

Naturally, Ω can be written as a countable union of squares whose edges are parallel with e_1 and e_2. Therefore, using a partition of unity, it is enough to prove the statement for $\Omega = (-1, 1)^2$:

$$\forall \varphi \in C_c^\infty(\Omega), \quad 0 = \int_\Omega \varphi\, \xi^\perp \cdot \nabla_x \chi(x, \xi)\, dx \overset{\xi = e_1}{=} \int_\Omega \varphi\, \partial_2 \chi(x, e_1)\, dx = -\int_{\Omega \cap \{u_1 > 0\}} \partial_2 \varphi\, dx.$$

For that, we consider the flow (2) and by the proof of Lemma 11, we have, for every $x \in \Omega$, that $\{X(t, x)\}_t$ is a straight line given by $X(t, x) = x + tu(x)$ and $u(X(t, x)) = u(x)$ for all t. Since u is smooth, there is no crossing between two characteristics in Ω. We claim that

$$\Omega \cap \{u_1 > 0\} = \bigsqcup_{k \in K} A_k,$$

where $\{A_k\}_{k \in K}$ is a (at most) countable set of pairwise disjoint rectangles of type $(a_k, b_k) \times (-1, 1) \subset \Omega = (-1, 1)^2$. Note first that $\Omega \cap \{u_1 = 0\}$ is the intersection of Ω by vertical lines. Indeed, if $u_1(x) = 0$, then $u(x) \parallel e_2$. By the characteristic method, for all t, we have $u_1(x + tu(x)) = 0$ and u_1 vanishes on the vertical line passing through x. Now $\{x_1 \in (-1, 1) : u_1(x_1, 0) = 0\}^c$ is an open set in $(-1, 1)$ and therefore, we can write

$$\left\{x_1 \in (-1, 1) : u_1(x_1, 0) = 0\right\}^c = \bigsqcup_{k \in \widetilde{K}} (a_k, b_k),$$

where $\widetilde{K}$ is at most countable. For $k \in \widetilde{K}$, we define $A_k := (a_k, b_k) \times (-1, 1)$. By continuity, u_1 is either positive or negative on A_k. Defining $K := \{k \in \widetilde{K} : u_1 > 0 \text{ on } A_k\}$, the claim is proved. Now, for $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega \cap \{u_1 > 0\}} \partial_2 \varphi = \sum_k \int_{A_k} \partial_2 \varphi = \sum_k \int_{a_k}^{b_k} \int_{-1}^{1} \partial_2 \varphi = 0,$$

because $\partial_2 \varphi$ can be seen as a signed Radon measure for $\varphi \in C_c^\infty(\Omega)$ and the proposition is proved. $\square$

Proof of Proposition 1: second method. The following proof links the kinetic formulation (5) with the theory of entropy solutions for scalar conservation laws (see, e.g., [DeSimone, Müller, Kohn and Otto 2001]). Indeed, if u is a smooth vector field satisfying (3), then formally, $u_1 = -h(u_2) := \pm\sqrt{1 - u_2^2}$ so that $\nabla \times u = 0$ can be rewritten as

$$\partial_1 u_2 + \partial_2 [h(u_2)] = 0; \tag{13}$$

thus, u_2 can be formally interpreted as a solution of the above scalar conservation law in the variables (time, space) $= (x_1, x_2)$. Based on the concept of entropy solution of (13) introduced via the pairs (entropy, entropy-flux), the following applications (called "elementary entropies") were used in [DeSimone, Müller, Kohn and Otto 2001]. More precisely, for every $\xi \in \mathbb{S}^1$, the map $\Phi^\xi : \mathbb{S}^1 \to \mathbb{R}^2$ is defined as

$$\text{for } z \in \mathbb{S}^1, \quad \Phi^\xi(z) = \begin{cases} \xi^\perp & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \le 0. \end{cases}$$

Then the kinetic formulation (5) can be written as

$$\nabla \cdot \left[\Phi^\xi(u)\right] = 0 \quad \text{distributionally in } \Omega. \tag{14}$$

In order to prove (14), we will approximate Φ^ξ by a sequence of smooth maps $\{\Phi_k : \mathbb{S}^1 \to \mathbb{R}^2\}$ such that $\{\Phi_k\}$ is uniformly bounded, $\lim_k \Phi_k(z) = \Phi^\xi(z)$ for every $z \in \mathbb{S}^1$ and Φ_k satisfies (14) for every k. Following the ideas in [DeSimone, Müller, Kohn and Otto 2001] (see also [Ignat and Merlet 2012]), this smoothing result comes from the following observation: there exists a (unique) 2π-periodic piecewise C^1 function $\varphi : \mathbb{R} \to \mathbb{R}$ associated to Φ^ξ via the equation

$$\Phi^\xi(z) = -\varphi'(\theta)z + \varphi(\theta)z^\perp \quad \text{for every } z = e^{i\theta} \in \mathbb{S}^1. \tag{15}$$

In fact, φ is given by

$$\varphi(\theta) = \Phi^\xi(z)\cdot z^\perp = \xi\cdot z\,\mathbb{1}_{\{z\cdot\xi>0\}} = \cos(\theta-\theta_0)\mathbb{1}_{\{\theta-\theta_0\in(-\pi/2,\pi/2)\}} \quad \text{for } z = e^{i\theta},\ \theta \in (-\pi+\theta_0, \pi+\theta_0),$$

where $\xi = e^{i\theta_0} \in \mathbb{S}^1$ with $\theta_0 \in (-\pi, \pi]$. In (15), the distributional derivative φ' is given by

$$\varphi'(\theta) = -\sin(\theta-\theta_0)\mathbb{1}_{\{\theta-\theta_0\in(-\pi/2,\pi/2)\}} \quad \text{for } \theta \in (-\pi+\theta_0, \pi+\theta_0).$$

Now, one regularizes φ by 2π-periodic functions $\varphi_k \in C^\infty(\mathbb{R})$ that are uniformly bounded in $W^{1,\infty}(\mathbb{R})$ with $\lim_k \varphi_k(\theta) = \varphi(\theta)$ and $\lim_k \varphi_k'(\theta) = \varphi'(\theta)$ for every $\theta \in \mathbb{R}$. Then we define Φ_k as in (15) for the functions φ_k:

$$\Phi_k(z) = -\varphi_k'(\theta)z + \varphi_k(\theta)z^\perp \quad \text{for } z = e^{i\theta} \in \mathbb{S}^1.$$

Let us now check that $\{\Phi_k\}_k$ are indeed the desired (smooth) approximating maps of Φ^ξ. For that, first, note that differentiating the above equation defining Φ_k, one obtains

$$\frac{\partial \Phi_k}{\partial \theta}(z)\cdot z^\perp = 0 \qquad \text{for every } z = e^{i\theta} \in \mathbb{S}^1. \tag{16}$$

Next, we prove that Φ_k satisfies (14). Indeed, we can locally write $u = e^{i\Theta}$ in every ball $B \subset \Omega$ for some smooth lifting $\Theta : B \to \mathbb{R}$ that satisfies

$$\nabla\Theta\cdot u = \nabla\times u = 0 \quad \text{in } B.$$

This means that $\nabla\Theta = \lambda u^\perp$ in B for some smooth function $\lambda : B \to \mathbb{R}$. Therefore, it follows that

$$\nabla\cdot[\Phi_k(u)] = \frac{\partial\Phi_k}{\partial\theta}(e^{i\Theta})\cdot\nabla\Theta = \lambda\frac{\partial\Phi_k}{\partial\theta}(u)\cdot u^\perp \overset{(16)}{=} 0 \quad \text{in } B.$$

Passing to limit $k \to \infty$, the dominated convergence theorem yields

$$\int_B \Phi^\xi(u)\cdot\nabla\zeta\,dx = 0 \quad \text{for every } \zeta \in C_c^\infty(B).$$

The conclusion is now straightforward. □

Note that another interest of this second method is that it can be adapted to vector fields $u \in W^{1/p,p}$ for $p \in [1, 3]$. For such vector fields, there is a priori no trace of u on a segment, so the flow (2) does not have a proper meaning anymore; see [Ignat 2012b; De Lellis and Ignat 2015] for more details.

3.2. ***The case of dimension $N \geq 3$.*** The aim of this subsection is to prove Theorem 7. We divide the proof in several steps, each being stated as a lemma.

Lemma 12. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a smooth vector field satisfying* (8). *We define*

$$\widetilde{\Omega} := \left\{ x \in \Omega : u(x) \neq 0,\ \nabla\left(\frac{u}{|u|}\right)(x) \neq 0 \right\}$$

and for every $x \in \widetilde{\Omega}$,

$$\mathbb{S}_x := u(x)^\perp \cap \mathbb{S}^{N-1} = \left\{ \xi \in \mathbb{S}^{N-1} : u(x) \cdot \xi = 0 \right\} \approx \mathbb{S}^{N-2}.$$

Then we have for all $x \in \widetilde{\Omega}$ and for $\mathcal{H}^{N-2}$-a.e. $\xi \in \mathbb{S}_x$ that the set

$$\{ y \in \widetilde{\Omega} : u(y) \cdot \xi = 0 \} = \widetilde{\Omega} \cap \partial \{ u \cdot \xi > 0 \}$$

is a hyperplane around x that is oriented by the normal vector ξ. Moreover,

$$\nabla_x \chi(\,\cdot\,, \xi) = \pm \xi \mathcal{H}^{N-1} \llcorner \partial\{u \cdot \xi > 0\} \qquad \text{locally around } x. \tag{17}$$

Proof. As in the proof of Lemma 11, we set $v = u/|u|$ on $\widetilde{\Omega}$. Then v is a smooth unit-length vector field in $\widetilde{\Omega}$ that satisfies (8) (because u satisfies it, too) and by Proposition 5, we have that v is curl-free in $\widetilde{\Omega}$. Let $x \in \widetilde{\Omega}$; in particular, $\nabla v(x) \neq 0$. First, we show that $\{ y \in \widetilde{\Omega} : u(y) \cdot \xi = 0 \}$ is a smooth $(N-1)$-manifold around x. Since v is curl-free, we know that $\nabla v(x) = (\partial_j v_i(x))_{i,j}$ is symmetric. By differentiating the relation $|v(x)| = 1$, it follows that

$$\nabla v(x)^T v(x) = \nabla v(x)\, v(x) = 0,$$

which means $v(x) \in \operatorname{Ker} \nabla v(x)$. We will prove that

$$\mathcal{H}^{N-2}(\mathbb{S}_x \cap \operatorname{Ker} \nabla v(x)) = 0.$$

Assume by contradiction that $\mathbb{S}_x \cap \operatorname{Ker} \nabla v(x)$ has positive $\mathcal{H}^{N-2}$-measure. Since $\operatorname{Ker} \nabla v(x)$ is a linear space, we have $\mathbb{S}_x \subset \operatorname{Ker} \nabla v(x)$, that is, $\nabla v(x)\xi = 0$ for all $\xi \in \mathbb{S}_x$. Moreover, since $v(x) \in \operatorname{Ker} \nabla v(x)$ and $\mathbb{S}_x \subset v(x)^\perp$, it follows that $\nabla v(x) = 0$, which is a contradiction with the assumption $\nabla v(x) \neq 0$. Therefore, $\nabla v(x)\xi \neq 0$ for $\mathcal{H}^{N-2}$-a.e. $\xi \in \mathbb{S}_x$ and $\{ y \in \widetilde{\Omega} : v(y) \cdot \xi = 0 \} = \{ y \in \widetilde{\Omega} : u(y) \cdot \xi = 0 \}$ is a smooth $(N-1)$-manifold around x.

It remains to prove that this manifold is a piece of hyperplane oriented by ξ where (17) holds true. For that, let $\varphi \in C_c^\infty(\widetilde{\Omega}, \mathbb{R}^N)$ be supported in a ball $B \subset \widetilde{\Omega}$ centered at x. By the Gauss theorem, we have

$$-\langle \nabla_x \chi(\,\cdot\,, \xi), \varphi \rangle = \int_B \nabla \cdot \varphi(y) \chi(y, \xi)\, dy = \int_{\{y \in B : u(y) \cdot \xi > 0\}} \nabla \cdot \varphi\, dy = \int_{B \cap \partial\{u \cdot \xi > 0\}} \varphi \cdot \nu\, d\mathcal{H}^{N-1}(y),$$

where ν is the unit outer normal vector to the $(N-1)$-manifold $\partial\{u(y) \cdot \xi > 0\}$. This proves that locally around x, we have

$$\nabla_x \chi(x, \xi) = -\nu \mathcal{H}^{N-1} \llcorner \big(B \cap \partial\{u \cdot \xi > 0\}\big).$$

Because of (8), we know that $\nabla_x \chi(x, \xi)$ and ξ are collinear. Since ν is smooth on $B \cap \partial\{u \cdot \xi > 0\}$, this implies $\nu = \xi$ or $\nu = -\xi$ on $B \cap \partial\{u \cdot \xi > 0\}$. The conclusion is now straightforward. $\square$

We now state the following result, which is the key point in proving Theorem 7.

Lemma 13. *Under the hypotheses of Theorem 7, every point $x \in \mathcal{S}$ is an umbilical point; i.e., there exists $\lambda(x) \in \mathbb{R}$ such that*

$$Du(x) = \lambda(x)\mathrm{Id} : T_x\mathcal{S} \to \mathbb{R}^{N-1},$$

where u is proportional to the Gauss map on $\mathcal{S}$, $T_x\mathcal{S}$ is the tangent plane to the hypersurface $\mathcal{S}$ at x and Id *is the identity matrix.*

Proof. Recall that $|u|$ is constant on $\mathcal{S}$ by Lemma 11 so that $u/|u|$ is the normal vector (i.e., the Gauss map) at the hypersurface $\mathcal{S}$. Therefore,

$$D\left(\frac{u}{|u|}\bigg|_{\mathcal{S}}\right) = \frac{1}{|u|} D(u|_{\mathcal{S}}) \quad \text{in } \mathcal{S},$$

where $D(u|_{\mathcal{S}})$ is the differential of u restricted to $\mathcal{S}$ as a map with values into the sphere $\mathbb{S}^{N-1}$ (up to the multiplicative constant $|u|$). As in the proofs of Lemmas 11 and 12, we may assume that u never vanishes in Ω and set $v = u/|u|$ in Ω. Then v is a smooth unit-length vector field in Ω that satisfies (8) and by Proposition 5, v is curl-free so that locally $v = \nabla\tilde{\psi}$ for a smooth stream function $\tilde{\psi}$. Since $\nabla\psi = u = |u|\nabla\tilde{\psi}$, we know that ψ and $\tilde{\psi}$ have the same level sets; in particular, $\mathcal{S}$ is a level set of $\tilde{\psi}$. Therefore, replacing u by v, we may assume in the following that

$$|u| = 1 \quad \text{in } \Omega.$$

Let $x \in \mathcal{S}$. We want to show that x is an umbilical point of $\mathcal{S}$. This is clear if $\nabla u(x) = 0$. Therefore, we assume in the following that $x \in \widetilde{\Omega} \cap \mathcal{S}$, as defined in Lemma 12; i.e.,

$$\nabla u(x) \neq 0.$$

Since (9) holds for the unit-length vector field u, by differentiating (9), we obtain

$$\nabla u(x) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi \otimes \nabla_x \chi(x, \xi),$$

where V_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$. The above integrand is to be understood as an absolutely continuous measure with respect to the Hausdorff $\mathcal{H}^{N-2}$ measure concentrated on the set $\mathbb{S}_x$ (defined at Lemma 12). For that, we check first that the support of the integrand lies on $\mathbb{S}_x$. Indeed, if $\xi \in \mathbb{S}^{N-1}$ with $u(x) \cdot \xi \neq 0$, then $\nabla_x \chi(\,\cdot\,, \xi) = 0$ in the open set $\{u \cdot \xi \neq 0\}$ around x. Therefore, the integrand has support on the set $\xi \in \mathbb{S}_x$, where (17) holds true for $\mathcal{H}^{N-2}$-a.e. $\xi \in \mathbb{S}_x$ and the density of the measure is equal to $\pm\xi \otimes \xi \mathcal{H}^{N-2} \llcorner \mathbb{S}_x$. Since $\mathbb{S}_x \subset u(x)^\perp = T_x\mathcal{S}$, the density $\xi \otimes \xi$ with $\xi \in \mathbb{S}_x$ already identifies $\nabla u(x) \equiv Du(x)$. Next we compute this quantity by exploring the sign of the density $\pm\xi \otimes \xi$:

Case $N = 3$. We show that there are at most two nonzero vectors $\pm\xi_0 \in \mathbb{S}_x \approx \mathbb{S}^1$ such that $\nabla u(x)\xi_0 = 0$. Assume by contradiction that there are more than two vectors as above; i.e., there exists another nonzero vector $\tilde{\xi}_0 \neq \pm\xi_0$ in $\mathbb{S}_x$ such that $\nabla u(x)\xi_0 = \nabla u(x)\tilde{\xi}_0 = 0$. Because of $|u| = 1$, we know that $\nabla u(x)u(x) = 0$. Since the set $\{u(x), \xi_0, \tilde{\xi}_0\}$ spans $\mathbb{R}^3$, we have $\nabla u(x) = 0$, which contradicts the hypothesis $x \in \widetilde{\Omega}$. Therefore, $\nabla u(x)\xi \neq 0$ for every $\xi \in \mathbb{S}_x \backslash \{\pm\xi_0\}$ (or for every $\xi \in \mathbb{S}_x$ if ξ_0 does not exist) and by Lemma 12,

$\partial\{u(y)\cdot\xi > 0\}$ is a smooth surface around x oriented by ξ. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two connected components of $\mathbb{S}_x \setminus \{\pm\xi_0\}$ (with the convention that $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$ in the case $\nabla u(x)\xi \neq 0$ for every $\xi \in \mathbb{S}_x$). For $j = 1, 2$, we associate to a point $\xi \in \mathcal{C}_j$ the unit outer normal vector field $\nu(\xi) \in \{\pm\xi\}$ to the plane $\partial\{u\cdot\xi > 0\}$ around x. Since the map $\xi \in \mathcal{C}_j \to \nu(\xi)$ is smooth (by the implicit function theorem) and $\mathcal{C}_j$ is connected, we deduce that ν is constant on $\mathcal{C}_j$. Thus it follows that

$$\pi\nabla u(x) = \gamma_1\int_{\mathcal{C}_1}\xi\otimes\xi\,d\xi + \gamma_2\int_{\mathcal{C}_2}\xi\otimes\xi\,d\xi,$$

with $V_2 = \pi$ and $\gamma_{1,2} \in \{\pm 1\}$ (with the convention that $\gamma_1 = \gamma_2 = \pm 1/2$ if $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$). It remains to show that $\int_{\mathcal{C}_j}\xi\otimes\xi\,d\xi$ is proportional to the identity matrix Id, $j = 1, 2$. Up to a rotation, we can suppose that $u(x) = e_3$ and $\mathcal{C}_1 = \{\xi \in \mathbb{S}^1\times\{0\} : \xi_2 > 0\} \approx \{(\cos\theta, \sin\theta) : \theta \in (0,\pi)\}$. We have

$$\int_{\mathcal{C}_1}\xi\otimes\xi\,d\xi \approx \int_0^\pi \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix} d\theta = \frac{\pi}{2}\mathrm{Id}$$

(the conclusion follows similarly if $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$).

Case $N > 3$. Let $\mathcal{C} = \operatorname{Ker}\nabla u(x) \cap \mathbb{S}_x$. We know that $u(x) \in \operatorname{Ker}\nabla u(x)$ and $u(x)$ is orthogonal to $\mathbb{S}_x$, which is isomorphic to $\mathbb{S}^{N-2}$. Since $\nabla u(x) \neq 0$ (i.e., the dimension of $\operatorname{Ker}\nabla u(x)$ is at most $N-1$), we have two situations (as in the case $N = 3$):

• Situation 1: $\dim\operatorname{Ker}\nabla u(x) = N-1$, which leads to $\mathcal{C}$ isomorphic to $\mathbb{S}^{N-3}$. In this situation, $\mathbb{S}_x\setminus\mathcal{C}$ is the partition of two connected sets $\mathcal{C}_1$ and $\mathcal{C}_2$ that are isomorphic to the half-sphere

$$\mathbb{S}^{N-2}_+ = \{\xi = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{S}^{N-2} : \xi_1 > 0\}.$$

The same argument as in the case $N = 3$ shows that the sign of the unit outer normal field $\nu(\xi) \in \{\pm\xi\}$ to the hyperplane $\partial\{u\cdot\xi > 0\}$ is constant when ξ covers $\mathcal{C}_j$, $j = 1, 2$, so that

$$V_{N-1}\nabla u(x) = \gamma_1\int_{\mathcal{C}_1}\xi\otimes\xi\,d\xi + \gamma_2\int_{\mathcal{C}_2}\xi\otimes\xi\,d\xi,$$

with $\gamma_1, \gamma_2 \in \{\pm 1\}$.

• Situation 2: $\dim\operatorname{Ker}\nabla u(x) \le N-2$, which leads to the manifold $\mathcal{C}$ of dimension $\le N-4$. In other words, $\mathbb{S}_x\setminus\mathcal{C}$ is connected and covers a.e. point of $\mathbb{S}_x$. The above formula holds for $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$ and $\gamma_1 = \gamma_2 = \pm 1/2$.

We now compute $\nabla u(x)$. For that, we may assume (up to a rotation) that $u(x) = e_N$ and $\mathcal{C}_1 = \mathbb{S}^{N-2}_+$. Since $\mathbb{S}^{N-2}_+$ is invariant under the change of coordinate $\xi_d \mapsto -\xi_d$ for some $2 \le d \le N-1$, we have for every $1 \le j \le N-1$ with $j \neq d$,

$$\int_{\mathbb{S}^{N-2}_+}\xi_j\xi_d\,d\xi = -\int_{\mathbb{S}^{N-2}_+}\xi_j\xi_d\,d\xi = 0,$$

leading to

$$\int_{\mathbb{S}^{N-2}_+}\xi\otimes\xi\,d\xi = \int_{\mathbb{S}^{N-2}_+}\begin{pmatrix}\xi_1^2 & & 0\\ & \ddots & \\ 0 & & \xi_{N-1}^2\end{pmatrix} d\xi = \frac{\mathcal{H}^{N-2}(\mathbb{S}^{N-2})}{2(N-1)}\mathrm{Id}. \qquad \square$$

Proof of Theorem 7. By Lemma 13, every point in $\mathcal{S}$ is an umbilical point. Such a hypersurface is called *totally umbilical*. A classical result in differential geometry states that a totally umbilical hypersurface is either a piece of an $(N-1)$-sphere or a piece of a hyperplane (see, e.g., [Hicks 1965, Chapter 2, page 36]). □

We have the following consequence of Lemma 11 and Theorem 8 (whose proof is independent of this section):

Corollary 14. *Under the hypotheses of Theorem 7, there exists a neighborhood ω of $\mathcal{S}$ and a diffeomorphism $t \to F(t)$ such that either $\psi = F(|x-P|)$ for every $x \in \omega$ for a point $P \in \mathbb{R}^N$, or $\psi = F(x \cdot \xi)$ for every $x \in \omega$ for a vector $\xi \in \mathbb{S}^{N-1}$.*

4. Several properties on the set of Lebesgue points

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^N)$. Recall that $x_0 \in \Omega$ is a *Lebesgue point* of u if there exists a vector $u_0 \in \mathbb{R}^N$ such that

$$\lim_{r\to 0} \fint_{B_r(x_0)} |u(x) - u_0|\, dx = 0. \tag{18}$$

In this case, we write $u(x_0) = u_0$, which is the limit of the average $\fint$ of u on the ball $B_r(x_0)$ as $r \to 0$. We denote by Leb $\subset \Omega$ the set of Lebesgue points of u. It is well known that $\mathcal{H}^N(\Omega \setminus \text{Leb}) = 0$ and one can replace the ball $B_r(x_0)$ by the cube $x_0 + (-r, r)^N$ in the definition (18) to recover the same set of Lebesgue points.

Proof of Proposition 5. We start by proving (9) for a fixed vector $u(x) \in \mathbb{S}^{N-1}$. By rotating the axes if necessary, we may assume that $u(x) = e_N$. Then we compute

$$\int_{\mathbb{S}^{N-1}} \xi \chi(x,\xi)\, d\mathcal{H}^{N-1}(\xi) = \int_{\mathbb{S}^{N-1}\cap\{\xi_N>0\}} \xi\, d\mathcal{H}^{N-1}(\xi) = \left(\int_{\mathbb{S}^{N-1}\cap\{\xi_N>0\}} \xi_N\, d\mathcal{H}^{N-1}(\xi)\right) e_N$$

because the integrand is odd in the variables ξ_j for $j = 1, \dots, N-1$. Defining $\xi' := (\xi_1, \dots, \xi_{N-1})$, the half-sphere $\mathbb{S}^{N-1} \cap \{\xi_N > 0\}$ is the graph of the map

$$\xi' \in B^{N-1} \mapsto \xi_N = \sqrt{1 - |\xi'|^2}$$

so that we have

$$\int_{\mathbb{S}^{N-1}\cap\{\xi_N>0\}} \xi_N\, d\mathcal{H}^{N-1}(\xi) = \int_{B^{N-1}} \sqrt{1-|\xi'|^2} \frac{d\xi'}{\sqrt{1-|\xi'|^2}} = \mathcal{H}^{N-1}(B^{N-1}) = V_{N-1}.$$

The second statement naturally reduces (by a slicing argument) to the case of dimension $N = 2$. In that case, for any $\varphi \in C^\infty_c(\Omega)$, we have $\nabla \times u = \partial_1 u_2 - \partial_2 u_1$ and

$$\begin{aligned}\int_\Omega \varphi \nabla \times u\, dx &= -\int_\Omega \nabla^\perp \varphi \cdot u\, dx \\ &\overset{(6)}{=} \frac{1}{2} \int_\Omega \int_{\mathbb{S}^1} \nabla\varphi \cdot \xi^\perp \chi(x,\xi)\, d\mathcal{H}^1(\xi)\, dx = \frac{1}{2} \int_{\mathbb{S}^1} d\mathcal{H}^1(\xi) \int_\Omega \nabla\varphi \cdot \xi^\perp \chi(x,\xi)\, dx \overset{(5)}{=} 0.\end{aligned}$$ □

The following lemma yields the relation between the Lebesgue points of u and Lebesgue points of the functions $\{\chi(\,\cdot\,,\xi)\}_{\xi\in\mathbb{S}^{N-1}}$ defined in (4).

Lemma 15. *Let $\Omega\subset\mathbb{R}^N$ be an open set and $u\in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$.*

(i) *If $|u|=1$ a.e. in Ω and x_0 is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ for almost every $\xi\in\mathbb{S}^{N-1}$, then x_0 is a Lebesgue point of u and* (9) *holds at x_0.*

(ii) *Let x_0 be a Lebesgue point of u and $\xi\in\mathbb{S}^{N-1}$. If $u(x_0)\cdot\xi\neq 0$, then x_0 is a Lebesgue point of $\chi(\,\cdot\,,\xi)$. Conversely, if x_0 is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(x_0,\xi)=1$ (resp. $=0$), then $u(x_0)\cdot\xi\geq 0$ (resp. ≤ 0).*

Proof. To prove (i), we apply Proposition 5. Indeed, if x_0 is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ for a.e. $\xi\in\mathbb{S}^{N-1}$, then Fubini's theorem implies

$$\begin{aligned}
&⨍_{B_r(x_0)}\left|u(x)-\frac{1}{V_{N-1}}\int_{\mathbb{S}^{N-1}}\xi\,\chi(x_0,\xi)\,d\mathcal{H}^{N-1}(\xi)\right|dx\\
&\qquad\overset{(9)}{\leq}\frac{1}{V_{N-1}}⨍_{B_r(x_0)}\int_{\mathbb{S}^{N-1}}\bigl|\xi\bigl(\chi(x,\xi)-\chi(x_0,\xi)\bigr)\bigr|\,d\mathcal{H}^{N-1}(\xi)\,dx\\
&\qquad\leq\frac{1}{V_{N-1}}\int_{\mathbb{S}^{N-1}}\left(⨍_{B_r(x_0)}|\chi(x,\xi)-\chi(x_0,\xi)|\,dx\right)d\mathcal{H}^{N-1}(\xi)\xrightarrow{r\to 0}0,
\end{aligned}$$

where we used the dominated convergence theorem.

Next we prove (ii). We treat the case $u(x_0)\cdot\xi>0$. For that, we have

$$\begin{aligned}
\int_{B_r(x_0)}\bigl|\chi(x,\xi)-1\bigr|\,dx&=\frac{1}{u(x_0)\cdot\xi}\int_{B_r(x_0)\cap\{u\cdot\xi\leq 0\}}u(x_0)\cdot\xi\,dx\\
&\leq\frac{1}{u(x_0)\cdot\xi}\int_{B_r(x_0)\cap\{u\cdot\xi\leq 0\}}\underbrace{(u(x_0)\cdot\xi-u(x)\cdot\xi)}_{\geq u(x_0)\cdot\xi>0}\,dx\leq\frac{1}{u(x_0)\cdot\xi}\int_{B_r(x_0)}|u(x)-u(x_0)|\,dx.
\end{aligned}$$

Since x_0 is a Lebesgue point of u, it follows that x_0 is a Lebesgue point for $\chi(\,\cdot\,,\xi)$ with $\chi(x_0,\xi)=1$. The case $u(x_0)\cdot\xi<0$ can be shown similarly and we obtain $\chi(x_0,\xi)=0$. The last statement is a direct consequence of the above lines (using a contradiction argument). $\square$

Remark 16. (a) Note that the condition $u(x_0)\cdot\xi\neq 0$ is essential in Lemma 15(ii). Indeed, if one considers the vortex vector field $u(x)=x/|x|$ for $x\in\mathbb{R}^N\setminus\{0\}$, then for every $\xi\in\mathbb{S}^{N-1}$, any point $x_0\in\xi^{\perp}\setminus\{0\}$ is a Lebesgue point of u (because u is smooth around x_0) and satisfies

$$u(x_0)\cdot\xi=0,$$

but x_0 is not a Lebesgue point of $\chi(\,\cdot\,,\xi)$ because

$$⨍_{B_r(x_0)}\left|\chi(x,\xi)-⨍_{B_r(x_0)}\chi(\,\cdot\,,\xi)\right|dx=⨍_{B_r(x_0)}\tfrac{1}{2}\,dx\not\to 0\quad\text{ as } r\to 0,$$

where we used that

$$⨍_{B_r(x_0)}\chi(x,\xi)\,dx=\frac{\mathcal{H}^N\bigl(\{x\in B_r(x_0):x\cdot\xi>0\}\bigr)}{\mathcal{H}^N(B_r(x_0))}\overset{x=y+x_0}{=}\frac{\mathcal{H}^N\bigl(\{y\in B_r(0):y\cdot\xi>0\}\bigr)}{\mathcal{H}^N(B_r(0))}=\frac{1}{2}.$$

(b) Note that in Lemma 15(ii), one cannot conclude in general that $u(x_0)\cdot\xi>0$ provided that $\chi(x_0,\xi)=1$. Indeed, consider for example $\xi=e_N$, $u(x)\cdot\xi=u_N(x):=|x|$ for $x\in\mathbb{R}^N$ and set $x_0=0$; then $\chi(\,\cdot\,,\xi)=1$ in $\mathbb{R}^N\setminus\{x_0\}$, x_0 is a Lebesgue point of u_N and $\chi(\,\cdot\,,\xi)$ with $\chi(x_0,\xi)=1$, but $u_N(x_0)=0$.

We now prove one of the key tools in the proof of Theorem 8, which mimics the relation of the ordering of level sets of a stream function when (8) holds true. It is a generalization of Proposition 3.1 in [Jabin, Otto, and Perthame 2002] to the case of dimension N:

Proposition 17 (ordering). *Let $N\ge 2$, $\Omega\subset\mathbb{R}^N$ be an open set and $u\in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ satisfy the kinetic formulation* (8). *Assume that $y,z\in$ Leb are two different Lebesgue points of u such that the closed segment $[yz]$ is included in Ω. Then for every direction $\xi\in\mathbb{S}^{N-1}$ with $\xi\in(z-y)^\perp$, we have*

$$u(y)\cdot\xi>0\ (\textit{resp.}<0)\quad\Longrightarrow\quad u(z)\cdot\xi\ge 0\ (\textit{resp.}\le 0);\tag{19}$$

moreover, y and z are Lebesgue points of $\chi(\,\cdot\,,\xi)$ and $\chi(y,\xi)=\chi(z,\xi)$. As a consequence, if $u\neq 0$ a.e. in Ω, then for a.e. $y\in\Omega$, $\mathcal{H}^{N-1}$-a.e. $\xi\in\mathbb{S}^{N-1}$ and $\mathcal{H}^{N-1}$-a.e. $v\in\xi^\perp$ with the segment $[y,y+v]$ included in Ω, we have that y and $y+v$ are Lebesgue points of u and

$$\chi(y,\xi)=\chi(y+v,\xi).\tag{20}$$

Proof. First, we consider the case $u(y)\cdot\xi>0$. By Lemma 15(ii), y is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ and $\chi(y,\xi)=1$. Let

$$\left\{\rho_\varepsilon(\,\cdot\,)=\frac{1}{\varepsilon^N}\rho\left(\frac{\cdot}{\varepsilon}\right)\right\}_{\varepsilon>0}$$

be a standard family of mollifiers, where ρ is a nonnegative radial smooth function having as support the unit ball $\operatorname{supp}\rho=B_1\subset\mathbb{R}^N$ and $\int_{B_1}\rho\,dx=1$. Set the convoluted function

$$\chi_\varepsilon:=\rho_\varepsilon*\chi(\,\cdot\,,\xi)$$

in a neighborhood $\omega\subset\Omega$ of the segment $[yz]$ for $\varepsilon>0$ sufficiently small. Then χ_ε is smooth in ω and for every Lebesgue point $x\in\omega$ of $\chi(\,\cdot\,,\xi)$ we have $\chi_\varepsilon(x)\to\chi(x,\xi)$ as $\varepsilon\to 0$ because

$$\begin{aligned}|\chi_\varepsilon(x)-\chi(x,\xi)|&=\left|\int_{B_\varepsilon(0)}\big(\chi(x-\tilde{x},\xi)-\chi(x,\xi)\big)\rho_\varepsilon(\tilde{x})\,d\tilde{x}\right|\\&\le\frac{\sup\rho}{\varepsilon^N}\int_{B_\varepsilon(0)}\big|\chi(x-\tilde{x},\xi)-\chi(x,\xi)\big|\,d\tilde{x}\\&\le C\fint_{B_\varepsilon(x)}\big|\chi(\tilde{y},\xi)-\chi(x,\xi)\big|\,d\tilde{y}\xrightarrow{\varepsilon\to 0}0.\end{aligned}$$

In particular, $\lim_{\varepsilon\to 0}\chi_\varepsilon(y)=\chi(y,\xi)=1$. Let $v=z-y$. We will show that $\chi(y+v,\xi)=1$. For that, we have $v\in\xi^\perp$ and

$$v\cdot\nabla_x\chi_\varepsilon=v\cdot\nabla_x\chi(\,\cdot\,,\xi)*\rho_\varepsilon\overset{(8)}{=}0\quad\text{in }\omega.$$

Then

$$\chi_\varepsilon(y+v)-\chi_\varepsilon(y)=\int_0^1 v\cdot\nabla_x\chi_\varepsilon(y+tv)\,dt=0$$

so that

$$\lim_{\varepsilon\to 0}\chi_\varepsilon(z)=\lim_{\varepsilon\to 0}\chi_\varepsilon(y)=\chi(y,\xi)=1.$$

This implies that $u(z)\cdot\xi\geq 0$. Assume by contradiction that $u(z)\cdot\xi<0$. By Lemma 15(ii), z is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(z,\xi)=0$ so that

$$\lim_{\varepsilon\to 0}\chi_\varepsilon(z)=\chi(z,\xi)=0,$$

which contradicts the above statement. We prove now the following:

Claim. If $\chi_\varepsilon(z)\to 1$ as $\varepsilon\to 0$, then z is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(z,\xi)=1$.

Proof of Claim. Let $\{\varepsilon_k\}$ be a sequence converging to 0 as $k\to\infty$. For k large enough, we define $f_k:B_1\to\{0,1\}$ by $f_k(x)=\chi(z-\varepsilon_k x,\xi)$ for every $x\in B_1$. Then the sequence $\{f_k\}$ is bounded in $L^2(B_1)$ and up to a subsequence, $f_k\rightharpoonup f$ weakly in $L^2(B_1)$, where the limit $f:B_1\to\mathbb{R}$ takes values in [0, 1]. Therefore, we have for our smooth mollifier $\rho\in L^2(B_1)$ that

$$\int_{B_1}\rho f_k\,dx\to\int_{B_1}\rho f\,dx\quad\text{as } k\to\infty.$$

Note now that by the change of variable $\tilde{x}=z-\varepsilon_k x$ we obtain by our assumption:

$$\int_{B_1}\rho(x)f_k(x)\,dx=\int_{B_{\varepsilon_k}(z)}\rho_{\varepsilon_k}(z-\tilde{x})\chi(\tilde{x},\xi)\,d\tilde{x}=\chi_{\varepsilon_k}(z)\to 1\quad\text{as } k\to\infty;$$

therefore, $\int_{B_1}\rho f\,dx=1$. Since 1 is the maximal value of f and ρ is nonnegative with the integral on B_1 equal to 1, we deduce that $f=1$ in $\operatorname{supp}\rho=B_1$. It follows by the change of variable $\tilde{x}=z-\varepsilon_k x$ that

$$⨍_{B_{\varepsilon_k}(z)}|\chi(\tilde{x},\xi)-1|\,d\tilde{x}=1-⨍_{B_1(0)}f_k(x)\,dx\to 0\quad\text{as } k\to\infty,$$

because $f_k\rightharpoonup 1$ weakly in $L^2(B_1)$. Since the limit is unique for every subsequence $\varepsilon_k\to 0$, we conclude that z is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(z,\xi)=1$, which proves the claim. □

For the case $u(y)\cdot\xi<0$, i.e., $\chi(y,\xi)=0$ by Lemma 15(ii), one applies the above argument by replacing ξ with $-\xi$ and obtains that z is a Lebesgue point of $\chi(\,\cdot\,,-\xi)$ with $\chi(z,-\xi)=1$. It follows that z is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(z,\xi)=0$ because

$$⨍_{B_r(z)}|\chi(x,\xi)|\,dx\leq\frac{\mathcal{H}^N\big(\{x\in B_r(z):u(x)\cdot\xi\geq 0\}\big)}{\mathcal{H}^N(B_r(z))}=1-⨍_{B_r(z)}\chi(x,-\xi)\,dx\to 0$$

as $r\to 0$. One also concludes that $u(z)\cdot\xi\leq 0$ by Lemma 15(ii).

For the last statement, we have for a.e. $y\in\Omega$ that y is a Lebesgue point of u with $u(y)\neq 0$. Then for $\mathcal{H}^{N-1}$-a.e. direction $\xi\in\mathbb{S}^{N-1}$, we have that $u(y)\cdot\xi\neq 0$ and $y+v$ is a Lebesgue point of u for $\mathcal{H}^{N-1}$-a.e. $v\in\xi^\perp$ with the segment $[y,y+v]\subset\Omega$. By the above argument, we get (20). □

5. Notion of the trace on lines

The $H^{1/2}$-regularity for N-dimensional unit-length vector fields u satisfying the kinetic formulation (8) (see [Golse, Lions, Perthame, and Sentis 1988]) is a priori not enough to define the notion of the trace of u on 1-dimensional lines. However, using the ideas in [Jabin, Otto, and Perthame 2002] for dimension 2, we will define a notion of the trace of u on segments (in the sense of Lebesgue points) in any dimension $N \geq 2$.

Proposition 18 (trace). *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{S}^{N-1}$ be a Lebesgue-measurable vector field satisfying the kinetic formulation* (8). *Assume that the segment*

$$L := \{0\}^{N-1} \times [-1, 1] \textit{ is included in } \Omega.$$

Then there exists a Lebesgue-measurable function $\tilde{u} : (-1, 1) \to \mathbb{R}^N$ such that

$$\lim_{r \to 0} \fint_{(-r,r)^{N-1}} \int_{-1}^{1} |u(x', x_N) - \tilde{u}(x_N)| \, dx_N \, dx' = 0, \tag{21}$$

where $x = (x', x_N)$, $x' = (x_1, \dots, x_{N-1})$. Moreover, for $\mathcal{H}^1$-a.e. $x_N \in (-1, 1)$,

$$\tilde{u}(x_N) = \lim_{r \to 0} \fint_{(-r,r)^{N-1}} u(x', x_N) \, dx' \quad \textit{and} \quad |\tilde{u}(x_N)| = 1. \tag{22}$$

Finally, every Lebesgue point $x \in \mathrm{Leb}$ of u lying inside L is a Lebesgue point of $\tilde{u}$ and $u(x) = \tilde{u}(x_N)$. The vector field $\tilde{u}$ is called the trace of u on the segment L.

Proof. To simplify the writing, we assume that $\Omega = \mathbb{R}^N$. We divide the proof into several steps:

Step 1: defining the 1-dimensional function $\tilde{\chi}(\,\cdot\,, \xi)$ for suitable directions $\xi \in \mathbb{S}^{N-1}$. Let $\mathcal{D}$ be the set of directions $\xi \in \mathbb{S}^{N-1}$ such that $\xi_N \neq 0$ and (20) holds true for the triple $(y, y + v, \xi)$ for a.e. $y \in \Omega$ and $\mathcal{H}^{N-1}$-a.e. $v \in \xi^\perp$ (with the segment $[y, y + v] \subset \Omega$, where y and $y + v$ are Lebesgue points of u). By Proposition 17, we know that $\mathcal{D}$ covers $\mathbb{S}^{N-1}$ up to a set of zero $\mathcal{H}^{N-1}$-measure. For such a direction $\xi \in \mathcal{D}$, we can choose a point $y_\xi \in \Omega$ (in a neighborhood of L) such that the map $\xi \in \mathcal{D} \mapsto y_\xi \in \Omega$ is Lebesgue measurable, the point $y_\xi + t\xi \in \Omega$ is a Lebesgue point of $\chi(\,\cdot\,, \xi)$ for $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$, the function $t \mapsto \chi(y_\xi + t\xi, \xi)$ is $\mathcal{H}^1$-measurable (by Fubini's theorem) and (20) holds true for the triple $(y_\xi + t\xi,\ y_\xi + t\xi + v,\ \xi)$ for $\mathcal{H}^{N-1}$-a.e. $v \in \xi^\perp$ and $\mathcal{H}^1$-a.e. t. Define the 1-dimensional function

$$s \mapsto \tilde{\chi}(s, \xi) := \chi\big(y_\xi + (s - y_\xi \cdot \xi)\xi,\ \xi\big) \in \{0, 1\}.$$

Then we have that for a.e. $x \in \Omega$ in a neighborhood of L,

$$\tilde{\chi}(x \cdot \xi, \xi) = \chi\big(y_\xi - y_\xi \cdot \xi\xi + x \cdot \xi\xi,\ \xi\big) \overset{(20)}{=} \chi(x, \xi), \tag{23}$$

because

$$v = y_\xi - y_\xi \cdot \xi\xi + x \cdot \xi\xi - x \in \xi^\perp.$$

Step 2: for $\xi \in \mathcal{D}$ and for every Lebesgue point $P = (0, \dots, 0, P_N) \in L$ of $\chi(\cdot, \xi)$ with $P_N \in (-1, 1)$, the point $P \cdot \xi$ is a Lebesgue point of $\tilde{\chi}(\cdot, \xi)$ and $\tilde{\chi}(P_N\xi_N, \xi) = \chi(P, \xi)$. Indeed, since $\xi_N \neq 0$, we have

$$\begin{aligned}
&\fint_{P_N\xi_N - r|\xi_N|}^{P_N\xi_N + r|\xi_N|} \big|\tilde{\chi}(t,\xi) - \chi(P,\xi)\big|\,dt \\
&\qquad = \fint_{(-r,r)^{N-1}} dx' \fint_{P_N - r}^{P_N + r} \big|\tilde{\chi}(\tilde{x}_N\xi_N, \xi) - \chi(P,\xi)\big|\,d\tilde{x}_N && (\text{since } t = \tilde{x}_N\xi_N) \\
&\qquad = \fint_{(-r,r)^{N-1}} dx' \fint_{P_N - x'\cdot\xi'/\xi_N - r}^{P_N - x'\cdot\xi'/\xi_N + r} \big|\underbrace{\tilde{\chi}(x'\cdot\xi' + x_N\xi_N, \xi)}_{\overset{(23)}{=}\chi(x,\xi)} - \chi(P,\xi)\big|\,dx_N && (\text{since } x'\cdot\xi' + x_N\xi_N = \tilde{x}_N\xi_N) \\
&\qquad \le \fint_{(-r,r)^{N-1}} dx' \frac{1}{2r} \int_{P_N - \tilde{r}}^{P_N + \tilde{r}} \big|\chi(x,\xi) - \chi(P,\xi)\big|\,dx_N \\
&\qquad \le C \fint_{P + (-\tilde{r},\tilde{r})^N} \big|\chi(x,\xi) - \chi(P,\xi)\big|\,dx \to 0 && \text{as } r \to 0,
\end{aligned}$$

where we used that $|x' \cdot \xi'| \le r\sqrt{N-1}$ for $x' \in (-r, r)^{N-1}$ and $\tilde{r} = (\sqrt{N-1}/|\xi_N| + 1)r$. Thus, $P_N\xi_N$ is a Lebesgue point of $\tilde{\chi}(\cdot, \xi)$. In particular, we have by Fubini's theorem, for every $\alpha > 0$,

$$\begin{aligned}
&\fint_{-\alpha r}^{\alpha r} d\tilde{t} \fint_{P_N\xi_N - r|\xi_N| + \tilde{t}}^{P_N\xi_N + r|\xi_N| + \tilde{t}} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt \\
&\qquad = \frac{1}{4\alpha|\xi_N|r^2} \int_{-\alpha r}^{\alpha r} \int_{P_N\xi_N - r(|\xi_N| + \alpha)}^{P_N\xi_N + r(|\xi_N| + \alpha)} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big| \mathbb{1}_{(P_N\xi_N - r|\xi_N| + \tilde{t},\, P_N\xi_N + r|\xi_N| + \tilde{t})}(t)\,dt\,d\tilde{t} \\
&\qquad = \frac{1}{4\alpha|\xi_N|r^2} \int_{P_N\xi_N - r(|\xi_N| + \alpha)}^{P_N\xi_N + r(|\xi_N| + \alpha)} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt \int_{-\alpha r}^{\alpha r} \mathbb{1}_{(-P_N\xi_N - r|\xi_N| + t,\, -P_N\xi_N + r|\xi_N| + t)}(\tilde{t})\,d\tilde{t} \\
&\qquad \le \frac{1}{2|\xi_N|r} \int_{P_N\xi_N - r(|\xi_N| + \alpha)}^{P_N\xi_N + r(|\xi_N| + \alpha)} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt \ \to 0 \quad \text{as } r \to 0.
\end{aligned} \tag{24}$$

Step 3: proof of (21). For $\xi \in \mathcal{D}$, we have, for small $r > 0$,

$$\begin{aligned}
&\fint_{(-r,r)^{N-1}} \int_{-1}^{1} \big|\chi(x,\xi) - \tilde{\chi}(x_N\xi_N,\xi)\big|\,dx'\,dx_N \\
&\qquad\qquad \overset{(23)}{=} \fint_{(-r,r)^{N-1}} \int_{-1}^{1} \big|\tilde{\chi}(x'\cdot\xi' + x_N\xi_N,\xi) - \tilde{\chi}(x_N\xi_N,\xi)\big|\,dx'\,dx_N \\
&\qquad\qquad \le \frac{1}{|\xi_N|} \sup_{|\tilde{t}| \le r\sqrt{N-1}} \int_{-|\xi_N|}^{|\xi_N|} \big|\tilde{\chi}(t+\tilde{t},\xi) - \tilde{\chi}(t,\xi)\big|\,dt \quad (\text{since } t = x_N\xi_N)
\end{aligned}$$

because $|x' \cdot \xi'| \le r\sqrt{N-1}$. Since the 1-dimensional function $t \mapsto \tilde{\chi}(t, \xi)$ belongs to L^∞, its L^1-modulus of continuity present in the above right-hand side tends to 0 as $r \to 0$, which leads to

$$\lim_{r\to 0} \fint_{(-r,r)^{N-1}} \int_{-1}^{1} \big|\chi(x,\xi) - \tilde{\chi}(x_N\xi_N,\xi)\big|\,dx'\,dx_N = 0.$$

This formula can be interpreted as the notion of the trace of $\chi(\,\cdot\,,\xi)$ on the segment L and yields (21). Indeed, due to (9), we define for a.e. $x_N \in (-1,1)$,

$$\tilde{u}(x_N) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi\, \tilde{\chi}(x_N \xi_N, \xi)\, d\mathcal{H}^{N-1}(\xi)$$

and we obtain, by Fubini's theorem,

$$\begin{aligned}
&\int_{(-r,r)^{N-1}} \int_{-1}^{1} \big|u(x', x_N) - \tilde{u}(x_N)\big|\, dx'\, dx_N \\
&\qquad \overset{(9)}{\le} \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \bigg(\int_{(-r,r)^{N-1}} \int_{-1}^{1} \big|\chi(x,\xi) - \tilde{\chi}(x_N\xi_N, \xi)\big|\, dx'\, dx_N \bigg)\, d\mathcal{H}^{N-1}(\xi) \xrightarrow{r\to 0} 0,
\end{aligned}$$

where we used the dominated convergence theorem.

Step 4: proof of (22). By Step 3, we deduce that

$$⨍_{(-r,r)^{N-1}} u(x', \cdot\,)\, dx' \xrightarrow{r\to 0} \tilde{u} \quad \text{in } L^1((-1,1));$$

therefore, the first statement in (22) follows immediately. Moreover,

$$\begin{aligned}
⨍_{-1}^{1} \big||\tilde{u}(x_N)| - 1\big|\, dx_N &= ⨍_{(-r,r)^{N-1}} \int_{-1}^{1} \big||\tilde{u}(x_N)| - |u(x', x_N)|\big|\, dx'\, dx_N \\
&\le ⨍_{(-r,r)^{N-1}} \int_{-1}^{1} \big|\tilde{u}(x_N) - u(x', x_N)\big|\, dx'\, dx_N \xrightarrow{(21)} 0 \quad \text{as } r \to 0;
\end{aligned}$$

thus, $|\tilde{u}(x_N)| = 1$ for $\mathcal{H}^1$-a.e. $x_N \in (-1,1)$.

Step 5: conclusion. Let $P = (0, \dots, 0, P_N) \in \mathrm{Leb}$ be a Lebesgue point of u with $P_N \in (-1,1)$. We want to show that P_N is a Lebesgue point of $\tilde{u}$ and $\tilde{u}(P_N) = u(P)$. For that, we know by Lemma 15 that P is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ for every direction $\xi \in \mathbb{S}^{N-1}$ with $u(P)\cdot\xi \ne 0$. If in addition $\xi \in \mathcal{D}$, we know by Step 2 that $P\cdot\xi$ is also a Lebesgue point of $\tilde{\chi}(\,\cdot\,,\xi)$. By the same argument as in Step 3, we have

$$\begin{aligned}
&⨍_{P+(-r,r)^N} |u(x', x_N) - \tilde{u}(x_N)|\, dx'\, dx_N \\
&\quad \le \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} ⨍_{P+(-r,r)^N} \big| \underbrace{\chi(x,\xi)}_{\overset{(23)}{=} \tilde{\chi}(x'\cdot\xi' + x_N\xi_N, \xi)} - \tilde{\chi}(x_N\xi_N, \xi)\big|\, dx'\, dx_N\, d\mathcal{H}^{N-1}(\xi) \\
&\quad \le \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) \bigg[⨍_{P+(-r,r)^N} \big|\tilde{\chi}(x'\cdot\xi' + x_N\xi_N, \xi) - \tilde{\chi}(P_N\xi_N, \xi)\big|\, dx \\
&\qquad\qquad + ⨍_{P_N - r}^{P_N + r} \big|\tilde{\chi}(x_N\xi_N, \xi) - \tilde{\chi}(P_N\xi_N, \xi)\big|\, dx_N \bigg] \\
&\quad \le \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) ⨍_{(-r,r)^{N-1}} dx' ⨍_{P_N\xi_N - r|\xi_N| + x'\cdot\xi'}^{P_N\xi_N + r|\xi_N| + x'\cdot\xi'} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N, \xi)\big|\, dt \\
&\qquad\qquad + \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) ⨍_{P_N\xi_N - r|\xi_N|}^{P_N\xi_N + r|\xi_N|} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N, \xi)\big|\, dt.
\end{aligned}$$

Using the dominated convergence theorem twice, we conclude that the above right-hand side vanishes as $r \to 0$. Indeed, the second integrand converges to 0 as $r \to 0$ by Step 2 for a.e. $\xi \in \mathbb{S}^{N-1}$. For the first integrand, we proceed as follows: for $\mathcal{H}^{N-1}$-a.e. direction ξ, we may assume that $|\xi'| > 0$ and $\xi_N \neq 0$ so that there exists a rotation $R' \in \mathrm{SO}(N-1)$ with $R'\xi' = |\xi'|e_1$ and we have by the change of variables $\tilde{x}' = R'x'$ and $\hat{r} = r\sqrt{N-1}$,

$$\begin{aligned}
&\fint_{(-r,r)^{N-1}} dx' \fint_{P_N\xi_N - r|\xi_N| + x'\cdot\xi'}^{P_N\xi_N + r|\xi_N| + x'\cdot\xi'} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt \\
&\qquad \le C \fint_{\{|\tilde{x}'| < \hat{r}\}} d\tilde{x}' \fint_{P_N\xi_N - r|\xi_N| + \tilde{x}_1|\xi'|}^{P_N\xi_N + r|\xi_N| + \tilde{x}_1|\xi'|} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt \\
&\qquad \le C \fint_{-|\xi'|\hat{r}}^{|\xi'|\hat{r}} \fint_{P_N\xi_N - r|\xi_N| + \tilde{t}}^{P_N\xi_N + r|\xi_N| + \tilde{t}} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt\,d\tilde{t} \xrightarrow{(24)} 0 \quad \text{as } r \to 0. \qquad \square
\end{aligned}$$

6. Proof of Theorem 8

We start by showing some preliminary results that reveal the geometric consequences of the kinetic formulation (8). The following lemma is the first step in proving that u is constant along the characteristics and is reminiscent of the ideas presented in [Jabin, Otto, and Perthame 2002]:

Lemma 19. *Let $\Omega \subset \mathbb{R}^N$ be an open set such that $L = \{0\}^{N-1} \times [-1, 1] \subset \Omega$ and $u : \Omega \to \mathbb{S}^{N-1}$ be a Lebesgue-measurable vector field satisfying the kinetic formulation* (8)*. Assume that the origin $O \in \Omega \cap \mathrm{Leb}$ is a Lebesgue point of u and $u(O) = e_N$. Then for every Lebesgue point $x_N \in (-1, 1)$ of $\tilde{u}$, we have*

$$\tilde{u}(x_N) = \pm e_N,$$

where $\tilde{u}$ is the trace of u on L defined at Proposition 18.

Proof. Without loss of generality we assume that Ω is a convex open neighborhood of L. By Proposition 18, we know that O is also a Lebesgue point of $\tilde{u}$ and $\tilde{u}(0) = e_N$; moreover, $|\tilde{u}| = 1$ a.e. in $(-1, 1)$. Let $x_N \in (-1, 1) \setminus \{0\}$ be a Lebesgue point of $\tilde{u}$ such that $\mathcal{H}^{N-1}$-a.e. $z \in \Omega \cap (x_N e_N + e_N^{\perp})$ is a Lebesgue point of u and such that the following holds true (see Proposition 18):

$$\lim_{r\to 0} \fint_{(-r,r)^{N-1}} \big|u(x', x_N) - \tilde{u}(x_N)\big|\,dx' = 0. \tag{25}$$

Our goal is to prove that the component $\tilde{u}_i(x_N)$ of $\tilde{u}(x_N)$ in direction e_i vanishes for every $i = 1, \dots, N-1$. For that, we follow the ideas in [Jabin, Otto, and Perthame 2002]. Let $\varepsilon > 0$ be small and define the following subsets E_i^- and E_i^+ (depending on ε) of the hyperplane $(x_N e_N + e_N^{\perp})$ for $1 \le i \le N-1$:

$$E_i^{\pm} = \big\{z \in \Omega \cap \mathrm{Leb} : z_N = x_N,\ \varepsilon|x_N| \ge \pm z_i > 0\big\}.$$

By our assumption, the sets $E_i^{\pm}$ contain many points (e.g., for $i = 1$, the set E_1^+ covers the $(N-1)$-parallelepiped $(0, r) \times (-r, r)^{N-2} \times \{x_N\}$ up to a set of zero $\mathcal{H}^{N-1}$-measure for $r < \varepsilon$). For $z \in E_i^+$, we set $y = -z_i e_N + x_N e_i$ if $x_N > 0$ and $y = z_i e_N - x_N e_i$ if $x_N < 0$. Obviously, $z \cdot y = 0$; that is, $y \in z^{\perp}$.

By the convexity of Ω, the segment $[Oz]$ lies in Ω so that by Proposition 17 we have if $x_N > 0$ (resp. $x_N < 0$), then $u(O) \cdot y = -z_i < 0$ (resp. $u(O) \cdot y = z_i > 0$) so that $u(z) \cdot y \leq 0$ (resp. ≥ 0). It follows that

$$u_i(z) \leq \frac{z_i}{x_N} u_N(z) \leq \varepsilon \quad \left(\text{resp. } u_i(z) \geq \frac{-z_i}{|x_N|} u_N(z) \geq -\varepsilon\right),$$

because $|u_N(z)| \leq 1$. Similarly, for $z \in E_i^-$, one uses $y = z_i e_N - x_N e_i$ if $x_N > 0$ and $y = -z_i e_N + x_N e_i$ if $x_N < 0$ and deduces that $u_i(z) \geq -\varepsilon$ if $x_N > 0$ and $u_i(z) \leq \varepsilon$ if $x_N < 0$. We conclude that $\tilde{u}_i(x_N) \in [-\varepsilon, \varepsilon]$. Indeed, let us set $i = 1$ for simplicity of notation; by (25), we have

$$\tilde{u}_1(x_N) = \lim_{r \to 0} \fint_{(0,r)\times(-r,r)^{N-2}} u_1(x', x_N)\, dx' \leq \varepsilon \quad \text{if } x_N > 0 \quad (\text{resp. } \geq -\varepsilon \text{ if } x_N < 0)$$

and also,

$$\tilde{u}_1(x_N) = \lim_{r \to 0} \fint_{(-r,0)\times(-r,r)^{N-2}} u_1(x', x_N)\, dx' \geq -\varepsilon \quad \text{if } x_N > 0 \quad (\text{resp. } \leq \varepsilon \text{ if } x_N < 0).$$

Passing to the limit $\varepsilon \to 0$, we conclude that $\tilde{u}_i(x_N) = 0$ for $i = 1$ (similarly, for every $1 \leq i \leq N-1$). Obviously, $\mathcal{H}^1$-a.e. $x_N \in (-1, 1)$ satisfies this property. As a consequence, if $P_N \in (-1, 1)$ is a Lebesgue point of $\tilde{u}$ then for every $1 \leq i \leq N-1$,

$$\tilde{u}_i(P_N) = \lim_{r \to 0} \fint_{P_N - r}^{P_N + r} \tilde{u}_i(x_N)\, dx_N = 0.$$

Since $|\tilde{u}(P_N)| = 1$, we deduce that $\tilde{u}_N(P_N) = \pm 1$, that is, $\tilde{u}(P_N) = \pm e_N$. $\square$

We now prove the main result:

Proof of Theorem 8. We first treat the case where Ω is a ball and then the general case of a connected open set.

Case I: Ω is a ball. Since u is not a constant vector field, there exist two Lebesgue points $P_0, P_1 \in \Omega \cap \mathrm{Leb}$ of u such that

$$u(P_0) \neq u(P_1).$$

Let D_0 (resp. D_1) be the line directed by $u(P_0)$ (resp. $u(P_1)$) that passes through P_0 (resp. P_1).

Step 1: we show that D_0 and D_1 are coplanar. Assume by contradiction that D_0 and D_1 are not coplanar; in particular $|u(P_0) \cdot u(P_1)| < 1$. Set $A \in D_0$ and $B \in D_1$ such that

$$0 < |A - B| = \min_{x \in D_0, y \in D_1} |x - y|.$$

Obviously, the segment $[AB]$ is orthogonal to D_0 and D_1. Let O be the middle point of the segment $[AB]$ (see Figure 2). Let

$$w_1 = u(P_0), \quad w_2 = \frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} \quad \text{and} \quad w_3 = \alpha u(P_0) + \beta u(P_1),$$

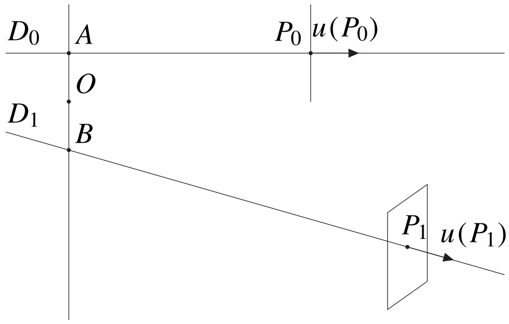

Figure 2. Two noncoplanar lines D_0 and D_1.

where

$$\alpha = \frac{-u(P_0)\cdot u(P_1)}{\sqrt{1-\big(u(P_0)\cdot u(P_1)\big)^2}} \quad \text{and} \quad \beta = \frac{1}{\sqrt{1-\big(u(P_0)\cdot u(P_1)\big)^2}} > 0. \tag{26}$$

The choice of α and β is done in order to ensure that $w_1 \cdot w_3 = 0$ and $|w_3|^2 = 1$, which finally yields the orthonormal basis w_1, w_2 and w_3. Note now that the vectors $u(P_0)$ and $u(P_1)$ have the following components in the basis (w_1, w_2, w_3):

$$u(P_0) = (1, 0, 0) \quad \text{and} \quad u(P_1) = \left(-\frac{\alpha}{\beta}, 0, \frac{1}{\beta}\right).$$

We want to find the expression of $\overrightarrow{P_0P_1}$ in that basis, too. For that, we have

$$\overrightarrow{P_0P_1} = \overrightarrow{P_0A} + \overrightarrow{AB} + \overrightarrow{BP_1},$$

which implies the existence of three real numbers $\lambda, \tilde{\lambda}, \hat{\lambda} \in \mathbb{R}$ with $\tilde{\lambda} \neq 0$ such that

$$\overrightarrow{P_0P_1} = \lambda w_1 + \tilde{\lambda} w_2 + \hat{\lambda} u(P_1) = \lambda w_1 + \tilde{\lambda} w_2 + \hat{\lambda}\left(\frac{1}{\beta} w_3 - \frac{\alpha}{\beta} w_1\right).$$

Thus, $\overrightarrow{P_0P_1}$ has the following components in the basis (w_1, w_2, w_3):

$$\overrightarrow{P_0P_1} = \left(\lambda - \frac{\alpha}{\beta}\hat{\lambda}, \tilde{\lambda}, \frac{\hat{\lambda}}{\beta}\right).$$

Define the vector $\xi := (1, s, -\beta) \neq 0$, written in our basis where

$$s := \frac{\hat{\lambda}(\alpha+\beta)}{\beta\tilde{\lambda}} - \frac{\lambda}{\tilde{\lambda}}.$$

Then $[P_0P_1] \subset \Omega$ (since Ω is a ball) and

$$\overrightarrow{P_0P_1} \cdot \xi = 0, \quad \text{i.e., } \xi \in P_0P_1^{\perp},$$
$$u(P_0)\cdot \xi = 1 > 0, \quad u(P_1)\cdot\xi = u(P_0)u(P_1) - 1 < 0,$$

which contradicts Proposition 17. Thus, D_0 and D_1 are coplanar.

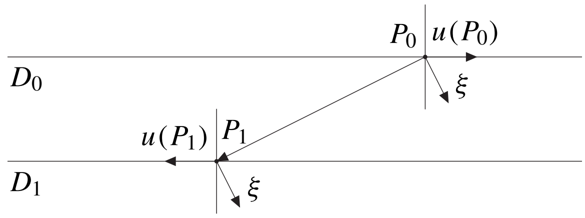

Figure 3. Two parallel lines D_0 and D_1.

Step 2: we show that D_0 and D_1 must intersect (D_0 *might coincide with* D_1). Assume by contradiction that D_0 and D_1 are parallel and $D_0 \neq D_1$. This means that $u(P_0) = -u(P_1)$ (because of our choice $u(P_0) \neq u(P_1)$). Set (w_1, w_2) to be an orthonormal basis in the 2-dimensional plane Π determined by D_0 and D_1 with $w_1 = u(P_0)$. In the basis (w_1, w_2), we write $\overrightarrow{P_0P_1} = (\lambda, \tilde{\lambda})$, where $\tilde{\lambda} \neq 0$ (since $D_0 \neq D_1$), and set $\xi = (-\tilde{\lambda}, \lambda)$ to be an orthogonal vector to $\overrightarrow{P_0P_1}$ in Π (see Figure 3). Then one checks that $u(P_0) \cdot \xi = -\tilde{\lambda}$ and $u(P_1) \cdot \xi = \tilde{\lambda}$ have different signs, which again contradicts Proposition 17.

Step 3: there exists a point $O \in D_0$ with $O \neq P_0, P_1$ and a sign $\gamma \in \{\pm 1\}$ such that

$$u(P_i) = \gamma \frac{\overrightarrow{OP_i}}{|\overrightarrow{OP_i}|}, \quad i = 0, 1.$$

If $D_0 = D_1$, then $u(P_0) = -u(P_1)$, so any point $O \in D_0$ located between P_0 and P_1 leads to the conclusion. Otherwise, $D_0 \neq D_1$ and we define $\{O\} = D_0 \cap D_1$. First, we prove that $O \neq P_0, P_1$. Assume by contradiction that $O = P_0 \in D_0 \cap D_1$. Then by Proposition 18 we know that P_0 and P_1 are Lebesgue points of the trace $\tilde{u}$ of u on the segment $D_1 \cap \Omega$ (directed by $u(P_1)$) with $\tilde{u}(P_0) = u(P_0)$ and $\tilde{u}(P_1) = u(P_1)$ so that by Lemma 19, we should have $u(P_0)$ is parallel with $u(P_1)$, which is a contradiction with $D_0 \neq D_1$. So, $O \neq P_0, P_1$. Next, note that for any orthogonal vector ξ to $\overrightarrow{P_0P_1}$ in the plane determined by D_0 and D_1, we have by Proposition 17 that $u(P_0) \cdot \xi$ and $u(P_1) \cdot \xi$ have the same sign, i.e.,

$$(u(P_0) \cdot \xi) \cdot (u(P_1) \cdot \xi) \geq 0. \tag{27}$$

Write now

$$\overrightarrow{OP_0} = \lambda u(P_0) \quad \text{and} \quad \overrightarrow{OP_1} = \tilde{\lambda} u(P_1)$$

with $\lambda, \tilde{\lambda}$ nonzero real numbers. The conclusion of Step 3 is equivalent to proving that λ and $\tilde{\lambda}$ have the same sign. For that, as in Step 1, we choose the orthonormal basis $w_1 = u(P_0)$ and $w_2 = \alpha u(P_0) + \beta u(P_1)$ with $\alpha \in \mathbb{R}$ and $\beta > 0$ given in (26) (recall that $|u(P_0) \cdot u(P_1)| < 1$ because of the assumption $D_0 \neq D_1$). Since $\overrightarrow{P_0P_1} = \overrightarrow{OP_1} - \overrightarrow{OP_0} = \tilde{\lambda} u(P_1) - \lambda u(P_0)$, we write, in the basis (w_1, w_2),

$$u(P_0) = (1, 0), \quad u(P_1) = \left(-\frac{\alpha}{\beta}, \frac{1}{\beta}\right), \quad \overrightarrow{P_0P_1} = \left(-\lambda - \frac{\alpha}{\beta}\tilde{\lambda}, \frac{\tilde{\lambda}}{\beta}\right).$$

Then for the orthogonal vector $\xi = (\tilde{\lambda}, \lambda\beta + \alpha\tilde{\lambda}) \neq 0$ to $\overrightarrow{P_0P_1}$, we have by (27) that

$$0 \leq (u(P_0) \cdot \xi) \cdot (u(P_1) \cdot \xi) = \tilde{\lambda} \cdot \lambda.$$

Step 4: conclusion. For every Lebesgue point $P \in \mathrm{Leb} \cap \Omega$ of u, we consider the line D passing through P and directed by $u(P)$. Call $\mathcal{D}$ the set of these lines. Obviously, $\mathcal{D}$ covers $\mathcal{H}^N$-almost all of the ball Ω (since $\mathcal{H}^N(\Omega \setminus \mathrm{Leb}) = 0$); in particular, $\mathcal{D}$ is not planar. By Step 1, we know that every two lines in $\mathcal{D}$ are coplanar. Then Proposition 9 (whose proof is presented below) implies that either all these lines are parallel, or they pass through the same point O. Since u is nonconstant, we deduce by Step 2 that only the last situation holds true. By Step 3, we conclude that $u = \gamma u^\star(\,\cdot\, - O)$ a.e. in Ω.

Case II: Ω is a connected open set. By Case I, we know that in every open ball $B \subset \Omega$ around a Lebesgue point of u, the vector field u is either a vortex-type vector field in B, or u is constant in B. Since u is nonconstant in Ω, there exists a Lebesgue point P_0 of u and a ball $B_0 \subset \Omega$ around P_0 such that u is a vortex-type vector field in B_0; say for simplicity $u = u^\star$. Let $P \neq P_0$ be any other Lebesgue point of u. Since Ω is path-connected, there exists a path $\Gamma \subset \Omega$ from P_0 to P. Then we can cover the path Γ by a finite number of open balls $\{B_j\}_{0 \le j \le n}$ such that $P \in B_n$, $B_j \cap B_{j+1} \neq \varnothing$ for $0 \le j \le n-1$ and u is either constant or a vortex-type vector field in any B_j. Since $u = u^\star$ in B_0 and $B_0 \cap B_1$ is a nonempty open set, the analysis in Case I yields $u = u^\star$ in B_1 and by induction, $u = u^\star$ in B_n, which is a neighborhood of P. □

Let us now present the proof of the geometric result in Proposition 9, which is independent of the previous results:

Proof of Proposition 9. Assume that there are two lines $D_0, D_1 \in \mathcal{D}$ that are not collinear. Since D_0 and D_1 are coplanar, they intersect at a point P. Call Π the plane determined by D_0 and D_1. We show that all the lines in $\mathcal{D}$ pass through P. Let $D_2 \in \mathcal{D}$ be any line not included in Π (such a line exists because $\mathcal{D}$ is not planar). We know that D_2 is coplanar with D_0 and D_1, respectively. Then D_2 cannot be parallel with D_0 (otherwise, $D_2 \parallel D_0$ and $D_2 \cap D_1 \neq \varnothing$ imply that $D_2 \subset \Pi$, which is a contradiction with our assumption). Similarly, D_2 cannot be parallel with D_1. Therefore, D_2 intersects both D_0 and D_1. Since D_2 is not included in Π, the intersection points coincide with P. Let now $D_3 \in \mathcal{D}$ be any line included in Π (different than D_0 and D_1). Then D_3 is not included in the plane determined by D_1 and D_2. The previous argument leads again to $P \in D_3$, which concludes our proof. □

7. Vector fields of vortex-line type

We will prove the characterization of the weakened kinetic formulation (10) in Theorem 10. This result is in the spirit of Corollary 14 and leads to vector fields that have vortex-line singularities.

Proof of Theorem 10. For $x \in \mathbb{R}^N$, recall the notation $x = (x', x_N)$ with $x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$. As the result is local in the set $\{u_N \neq \pm 1\}$, we will assume that $\omega = B' \times (-1, 1)$ is included in that set, where B' is the unit ball in $\mathbb{R}^{N-1}$. Let $\xi' \in \mathbb{S}^{N-2}$ and $\xi = (\xi', 0) \in \mathcal{E}$. Since $e_N \in \xi^\perp$, we deduce by (10) that

$$e_N \cdot \nabla_x \chi(\,\cdot\,, \xi) = \partial_N \chi(\,\cdot\,, \xi) = 0 \quad \text{in } \mathcal{D}'(\omega). \tag{28}$$

We know that the point (x', t) is a Lebesgue point of $\chi(\,\cdot\,, \xi)$ for $\mathcal{H}^{N-1}$-a.e. $x' \in B'$ and $\mathcal{H}^1$-a.e. $t \in (-1, 1)$ and the convolution argument in the proof of Proposition 17 yields

$$\chi(x, \xi) = \chi(x + te_N, \xi) \quad \text{for } \mathcal{H}^N\text{-a.e. } x \in \omega \text{ and } \mathcal{H}^1\text{-a.e. } t.$$

Then one can define the measurable function $\tilde{\chi}(\,\cdot\,,\xi') : B' \to \{0,1\}$ by

$$\tilde{\chi}(x',\xi') := \chi(x,\xi) = \mathbb{1}_{\{x\in\omega : u'(x)\cdot\xi'>0\}} \quad \text{for } \mathcal{H}^N\text{-a.e. } x=(x',t)\in\omega.$$

Set

$$\tilde{u}(x') = \frac{1}{V_{N-2}} \int_{\mathbb{S}^{N-2}} \xi' \tilde{\chi}(x',\xi')\, d\mathcal{H}^{N-2}(\xi'), \quad x'\in B'.$$

Thanks to (9),

$$\tilde{u}(x') = \frac{u'(x)}{|u'(x)|} \quad \text{for } \mathcal{H}^N\text{-a.e. } x=(x',t)\in\omega\subset\{|u'|>0\}.$$

In particular, $\tilde{\chi}(x',\xi') = \mathbb{1}_{\{x'\in B' : \tilde{u}(x')\cdot\xi'>0\}}$ in B' for every $\xi'\in\mathbb{S}^{N-2}$. Therefore, we deduce by (10) that $\tilde{u} : B' \to \mathbb{S}^{N-2}$ satisfies

$$\forall \xi'\in\mathbb{S}^{N-2},\ \forall v'\in(\xi')^{\perp}, \quad v'\cdot\nabla'_{x'}\tilde{\chi}(x',\xi') = 0 \text{ in } B',$$

where $\nabla'_{x'} = (\partial_1,\dots,\partial_{N-1})$. As $N-1\geq 3$, Theorem 8 yields either $\tilde{u}(x') = w'$ for almost every $x'\in B'$, where $w'\in\mathbb{S}^{N-2}$ is a constant vector, or $\tilde{u}(x') = \gamma(x'-P')/|x'-P'|$ for almost every $x'\in B'$, where $\gamma\in\{\pm1\}$ and $P'\in\mathbb{R}^{N-1}$ is some point. This means that for a.e. $x\in\omega$,

$$\text{either} \quad u'(x) = |u'(x)|w' \quad \text{or} \quad u'(x) = \gamma|u'(x)|\frac{x'-P'}{|x'-P'|}.$$

<u>Case 1</u>. Let $u'(x) = |u'(x)|w'$ for a.e. $x\in\omega$. By (11), we have for $k\in\{1,\dots,N-1\}$,

$$\partial_k u_N = \partial_N u_k = w_k \partial_N(|u'|) \quad \text{in } \omega, \tag{29}$$

which yields, for all $k, j\in\{1,\dots,N-1\}$,

$$w_j\partial_k u_N = w_k\partial_j u_N \quad \text{in } \omega.$$

Therefore, $u_N(x) = g(\alpha, x_N)$ in ω for some 2-dimensional function g with the new variable $\alpha := \alpha(x) = x'\cdot w'$. Moreover, by (29), the function g satisfies the following: since $w_k\neq 0$ for some $k\in\{1,\dots,N-1\}$ (because $w\in\mathbb{S}^{N-1}$), the equation $|u'|^2+u_N^2 = 1$ a.e. in ω implies

$$w_k\partial_\alpha g = \partial_k u_N \overset{(29)}{=} w_k\partial_N(|u'|) = w_k\partial_N(\sqrt{1-g^2}).$$

The Poincaré lemma yields the existence of a stream function $\psi(\alpha, x_N)$ such that $g = \partial_N\psi$ and $\sqrt{1-g^2} = \partial_\alpha\psi$ so that $u(x) = \nabla_x[\psi(\alpha, x_N)]$ and therefore, ψ satisfies the 2-dimensional eikonal equation

$$(\partial_\alpha\psi)^2 + (\partial_N\psi)^2 = 1.$$

<u>Case 2</u>. Let $u'(x) = \gamma|u'(x)|(x'-P')/|x'-P'|$ for a.e. $x\in\omega$. As above, we have, for $k\in\{1,\dots,N-1\}$,

$$\partial_k u_N = \partial_N u_k = \gamma\frac{x_k-P_k}{|x'-P'|}\partial_N(|u'|) \quad \text{in } \omega \tag{30}$$

and we deduce that, for all $k, j\in\{1,\dots,N-1\}$,

$$(x_j-P_j)\partial_k u_N = (x_k-P_k)\partial_j u_N \quad \text{in } \omega.$$

Therefore, $u_N(x) = g(\alpha, x_N)$ in ω for some 2-dimensional function g with the new variable $\alpha := \alpha(x) = |x'|$. By (30), we conclude as above that there exists a stream function ψ solving the eikonal equation in the variables (α, x_N) such that

$$u(x) = \nabla_x[\psi(\alpha, x_N)]. \qquad \square$$

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