ANALYSIS & PDE

Volume 10 No. 5 2017

NATHALIE AYI AND THIERRY GOUDON

REGULARITY OF VELOCITY AVERAGES FOR TRANSPORT EQUATIONS ON RANDOM DISCRETE VELOCITY GRIDS

msp

ANALYSIS AND PDE
Vol. 10, No. 5, 2017
dx.doi.org/10.2140/apde.2017.10.1201

REGULARITY OF VELOCITY AVERAGES FOR TRANSPORT EQUATIONS ON RANDOM DISCRETE VELOCITY GRIDS

NATHALIE AYI AND THIERRY GOUDON

We go back to the question of the regularity of the "velocity average" $\int f(x,v)\psi(v)\,\mathrm{d}\mu(v)$ when f and $v\cdot\nabla_x f$ both belong to L^2, and the variable v lies in a discrete subset of $\mathbb{R}^D$. First of all, we provide a rate, depending on the number of velocities, for the defect of $H^{1/2}$ regularity which is reached when v ranges over a continuous set. Second of all, we show that the $H^{1/2}$ regularity holds in expectation when the set of velocities is chosen randomly. We apply this statement to investigate the consistency with the diffusion asymptotics of a Monte Carlo-like discrete velocity model.

1. Introduction

The averaging lemma is now a classical tool for the analysis of kinetic equations. Roughly speaking it can be explained as follows. Let $\mathscr{V}\subset\mathbb{R}^D$, endowed with a measure $\mathrm{d}\mu$. We consider a sequence of functions $f_n:\mathbb{R}^D\times\mathscr{V}\to\mathbb{R}$. We assume that

(a) $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^D\times\mathscr{V})$,

(b) $(v\cdot\nabla_x f_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^D\times\mathscr{V})$.

Given $\psi\in C_c^\infty(\mathbb{R}^D)$, we are interested in the *velocity average*

$$\rho_n[\psi](x)=\int_{\mathscr{V}} f_n(x,v)\psi(v)\,\mathrm{d}\mu(v).$$

Of course, (a) already tells us that $(\rho_n[\psi])_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^D)$. We wish to obtain further regularity or compactness properties, as a consequence of the additional assumption (b), and the fact that we are averaging with respect to the variable v. The first result in that direction dates back to [Bardos et al. 1988] (see also [Agoshkov 1984]); it asserts that $(\rho_n[\psi])_{n\in\mathbb{N}}$ is bounded in the Sobolev space $H^{1/2}(\mathbb{R}^D)$ and it is thus relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}^D)$, by virtue of the standard Rellich's theorem. This basic result has been improved in many directions: L^2 can be replaced by the L^p framework, at least with $1<p<\infty$, and we can relax (b) by allowing derivatives with respect to v and certain loss of regularity with respect to x; see, among others, [DiPerna et al. 1991; Golse et al. 1988; Perthame and Souganidis 1998]. Time-derivative or force terms can be considered as well; see, in addition to the above-mentioned references, [Berthelin and Junca 2010]. Such an argument plays a crucial role in the stunning theory of "renormalized solutions" of the Boltzmann equation [DiPerna and Lions 1989b], and more generally in proving the existence of solutions to nonlinear kinetic models like in [DiPerna and

MSC2010: primary 35B65; secondary 35F05, 35Q20, 82C40.
Keywords: average lemma, discrete velocity models, random velocity grids, hydrodynamic limits.

Lions 1989a]. It is equally a crucial ingredient for the analysis of hydrodynamic regimes, which establish the connection between microscopic models and fluid mechanics systems, and for the asymptotic of the Boltzmann equation to the incompressible Navier–Stokes system, which needs a suitable L^1 version of the average lemma [Golse and Saint-Raymond 2002]; we refer the reader to [Golse and Saint-Raymond 2004; Saint-Raymond 2009; Villani 2002]. Finally, it is worth pointing out that the averaging lemma can be used to investigate the regularizing effects of certain PDEs (convection-diffusion and elliptic equations, nonlinear conservation laws, etc.) [Tadmor and Tao 2007].

In order to illustrate our purpose, let us consider the following simple model which can be motivated from radiative transfer theory:

$$\varepsilon\, \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon}\sigma(\rho_\varepsilon)(\rho_\varepsilon - f_\varepsilon), \tag{1-1}$$

where

$$\rho_\varepsilon(t, x) = \int_{\mathscr{V}} f_\varepsilon(t, x, v)\, \mathrm{d}\mu(v),$$

and $\sigma : [0, \infty) \to [0, \infty)$ is a given smooth function. The parameter $0 < \varepsilon \ll 1$ is defined from physical quantities. As it tends to 0, both $f_\varepsilon(t, x, v)$ and $\rho_\varepsilon(t, x)$ converge to $\rho(t, x)$, which satisfies the nonlinear diffusion equation

$$\partial_t \rho = \nabla_x \cdot (A \nabla_x F(\rho)), \quad A = \int_{\mathscr{V}} v \otimes v\, \mathrm{d}\mu(v), \quad F(\rho) = \int_0^\rho \frac{\mathrm{d}z}{\sigma(z)}. \tag{1-2}$$

The averaging lemma is an efficient tool to deal with the nonlinearity of such a problem, as discussed in [Bardos et al. 1988].

However the discussion above hides the fact that we need some assumptions on the measured set of velocities $(\mathscr{V}, \mathrm{d}\mu)$ in order to obtain the regularization property of the velocity averaging. Roughly speaking, we need "enough" directions v when we consider the derivatives in (b). More technically, the compactness statement holds provided for any $0 < R < \infty$ we can find $C_R > 0,\ \delta_0 > 0,\ \gamma > 0$ such that for $0 < \delta < \delta_0$ and $\xi \in \mathbb{S}^{N-1}$, we have

$$\operatorname{meas}\big(\{v \in \mathscr{V} \cap B(0, R) : |v \cdot \xi| \le \delta\}\big) \le C_R \delta^\gamma.$$

This assumption appears in many statements about regularity of the velocity averages; when we are only interested in the compactness issue, it can be replaced by a more intuitive assumption (see, e.g., [Golse 2000, Theorem 1 in Lecture 3]): for any $\xi \in \mathbb{S}^{N-1}$ we have

$$\operatorname{meas}\big(\{v \in \mathscr{V} \cap B(0, R) : v \cdot \xi = 0\}\big) = 0. \tag{1-3}$$

Clearly these assumptions are satisfied when the measure $\mathrm{d}\mu$ is absolutely continuous with respect to the Lebesgue measure (with, for the sake of concreteness, $\mathscr{V} = \mathbb{R}^D$ or $\mathscr{V} = \mathbb{S}^{D-1}$). However, they fail for models based on a discrete set of velocities. For instance let $\mathscr{V} = \{v_1, \dots, v_N\}$, with $v_j \in \mathbb{R}^D$, and $\mathrm{d}\mu(v) = \frac{1}{N}\sum_{j=1}^N \delta(v = v_j)$; it suffices to pick $\xi \in \mathbb{S}^{N-1}$ orthogonal to one of the v_j to contradict (1-3). (Note that alternative proofs based on compensated compactness techniques have been proposed to justify the asymptotic regime from (1-1) to (1-2) that apply to certain discrete velocity models; see [Degond et al.

2000; Goudon and Poupaud 2001; Lions and Toscani 1997].) Nevertheless, when the discrete velocities come from a discretization grid of the whole space, the averaging lemma can be recovered asymptotically letting the mesh step go to 0, as shown in [Mischler 1997], motivated by the convergence analysis of numerical schemes for the Boltzmann equation.

This paper aims at investigating further these issues. To be more specific, in Section 2 we revisit the averaging lemma for discrete velocities in two directions. First of all, we make more precise the analysis of [Mischler 1997], obtaining a rate on the defect to the $H^{1/2}$ regularity of the velocity average, depending on the mesh size. Second of all, we establish a stochastic version of the averaging lemma. We are still working with a finite number of velocities on bounded sets; however, choosing the velocities randomly, the "compactifying" property of assumption (b) can be restored by dealing with the expectation of $\rho_n[\psi]$. This is a natural way to involve "enough velocities", by looking at a large set of realizations of the discrete velocity grid. The analysis is completed in Section 3 by going back to the asymptotic problem $\varepsilon \to 0$ in (1-1), with a random discretization of the velocity variable, in the spirit of the Monte Carlo approach.

2. Discrete velocity averaging lemmas

Deterministic case: evaluation of the defect. As mentioned above, it is a well-known fact that, in the deterministic context, the averaging lemma fails for discrete velocity models. However, as established by S. Mischler [1997], the compactness of velocity averages is recovered asymptotically when we refine a velocity grid in order to recover a continuous velocity model. Here, we wish to quantify the defect of compactness when the number of velocities is finite and fixed. This is the aim of the following claim which shows that the macroscopic density $\rho[\psi]$ "belongs to $H^{1/2}(\mathbb{R}^D) + O(1/\sqrt{N})L^2(\mathbb{R}^D)$".

Proposition 2.1. *Let $N \in \mathbb{N} \setminus \{0\}$ and define*

$$A_N = \left(\frac{1}{N}\mathbb{Z}\right)^D \cap [-0.5, 0.5]^D.$$

Let $f, g \in L^2(\mathbb{R}^D \times A_N)$ satisfy, for all $k \in \mathbb{Z}^D$,

$$v_k \cdot \nabla_x f(x, v_k) = g(x, v_k). \tag{2-1}$$

We suppose that the L^2 norm of f and g is bounded uniformly with respect to N. Then, for all $\psi \in C_c^\infty(\mathbb{R}^D)$, the macroscopic quantity

$$\rho[\psi](x) = \frac{1}{(N+1)^D} \sum_k f(x, v_k)\psi(v_k)$$

can be split as $\rho[\psi](x) = \Theta[\psi](x) + (1/\sqrt{N})\widetilde{\Delta[\psi]}(x)$, where $\Theta[\psi]$ and $\widetilde{\Delta[\psi]}$ are bounded uniformly with respect to N in $H^{1/2}(\mathbb{R}^D)$ and $L^2(\mathbb{R}^D)$ respectively.

Remark 2.2. Note that in this statement N is the number of grid points per axis. Accordingly, there are $\mathcal{N} = (N+1)^D$ velocities in the set A_N. Therefore the defect of $H^{1/2}$ regularity decays like $\mathcal{N}^{1/2D}$, depending on the dimension.

Proof. As usual, we start by applying the Fourier transform to (2-1). Then for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^D$, we get

$$\xi \cdot v_k \hat{f}(\xi, v_k) = (-i)\hat{g}(\xi, v_k).$$

Let us set

$$F(\xi) := \left(\frac{1}{(N+1)^D}\sum_k |\hat{f}(\xi, v_k)|^2\right)^{1/2}, \quad G(\xi) := \left(\frac{1}{(N+1)^D}\sum_k |\hat{g}(\xi, v_k)|^2\right)^{1/2}.$$

By assumption, we have $F, G \in L^2_\xi$. Still following the standard arguments, we pick $\delta > 0$ and we split

$$\begin{aligned}\hat{\rho}[\psi](\xi) &= \frac{1}{(N+1)^D}\sum_k \hat{f}(\xi, v_k)\psi(v_k)\\ &= \frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|<\delta|\xi|} \hat{f}(\xi, v_k)\psi(v_k) + \frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|\geq\delta|\xi|} \hat{f}(\xi, v_k)\psi(v_k).\end{aligned}$$

The Cauchy–Schwarz inequality permits us to dominate the first term:

$$\left|\frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|<\delta|\xi|} \hat{f}(\xi, v_k)\psi(v_k)\right| \leq \|\psi\|_\infty \left(\frac{1}{(N+1)^D}\sum_k |\hat{f}(\xi, v_k)|^2\right)^{1/2}\left(\frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|<\delta|\xi|} 1\right)^{1/2}. \quad (2\text{-}2)$$

For the second term, we use the information in (2-1); it yields

$$\begin{aligned}&\left|\frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|\geq\delta|\xi|} \hat{f}(\xi, v_k)\psi(v_k)\right|\\ &\qquad = \left|\frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|\geq\delta|\xi|} \frac{(-i)\hat{g}(\xi, v_k)}{\xi\cdot v_k}\psi(v_k)\right|\\ &\qquad \leq \|\psi\|_\infty\left(\frac{1}{(N+1)^D}\sum_k |\hat{g}(\xi, v_k)|^2\right)^{1/2}\left(\frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|\geq\delta|\xi|}\frac{1}{|\xi\cdot v_k|^2}\right)^{1/2}. \quad (2\text{-}3)\end{aligned}$$

From now on we assume $\xi \neq 0$. Let $(e_1, \dots, e_D)$ stand for the canonical basis of $\mathbb{R}^D$ so that $\xi = \sum_{j=1}^D \alpha_j e_j$ with $\alpha_j \in \mathbb{R}$. We distinguish the following two cases:

(i) ξ is aligned with an axis, that is, all but one the α_j vanish, or

(ii) ξ is generated by at least two vectors of the basis.

We start with the case (i), assuming for instance $\xi = \alpha e_1$. Then $\xi \cdot v_k = \alpha v_k^1$, where v_k^1 is the first component of the vector v_k.

We refer the reader to Figure 1 to complete the discussion. On each horizontal line we find $2\lfloor\delta N\rfloor + 1$ velocities such that $|\xi \cdot v_k| < \delta|\xi|$, where $\lfloor s\rfloor$ stands for the integer part of s. Thus, since there are $(N+1)^{D-1}$ such lines on the domain A_N, we obtain

$$\sum_{|\xi\cdot v_k|<\delta|\xi|} 1 = (2\lfloor\delta N\rfloor + 1)(N+1)^{D-1} = 2\Big(\delta + \frac{1}{N}\Big)(N+1)^D.$$

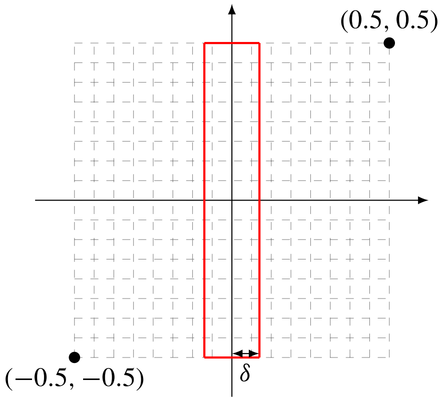

Figure 1. The delimited area corresponds to $|\xi \cdot v_k| < \delta|\xi|$ for ξ collinear to e_1.

Coming back to (2-2), we arrive at

$$\left|\frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|<\delta|\xi|}\hat{f}(\xi, v_k)\psi(v_k)\right| \le CF(\xi)\sqrt{\delta+\frac{1}{N}},$$

where $C > 0$ is a generic constant which does not depend on N and ξ.

Next, we cover the set of velocities such that $|v_k \cdot \xi| \ge \delta|\xi|$ by strips of width δ; see Figure 2 in dimension $D = 2$. We denote by S_p the p-th strip delimited by the straight lines $x = p\delta$ and $x = (p+1)\delta$. Each velocity on the strip S_p satisfies $p\delta \le v_k^1 \le (p+1)\delta$. Moreover, given a strip S_p, we cannot find more than $\lfloor \delta N\rfloor + 1$ abscissae in the strip and there are $(N+1)^{D-1}$ lines in the domain. It follows that

$$\begin{aligned}\sum_{|\xi\cdot v_k|\ge\delta|\xi|}\frac{1}{|\xi\cdot v_k|^2} &= \sum_{|\xi\cdot v_k|\ge\delta|\xi|}\frac{1}{|\xi|^2}\,\frac{1}{|\xi/|\xi|.v_k|^2}\\ &\le \frac{1}{|\xi|^2}\,2\Big(\sum_{p\ge1}\frac{1}{(p\delta)^2}\Big)(\delta N+1)(N+1)^{D-1} \le \frac{1}{|\xi|^2}\,2\Big(\sum_{p\ge1}\frac{1}{p^2}\Big)\frac{1}{\delta}\Big(1+\frac{1}{\delta N}\Big)(N+1)^D.\end{aligned}$$

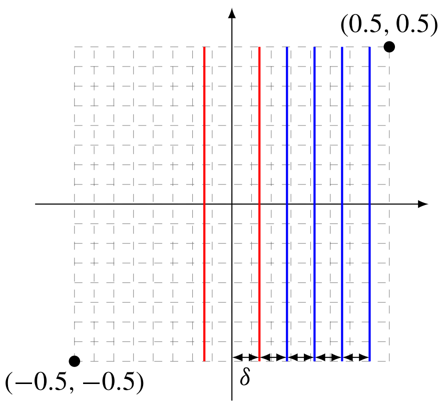

Figure 2. Splitting of the velocity space in strips of width δ. Since this space is symmetric, we only deal with the part corresponding to positive abscissae.

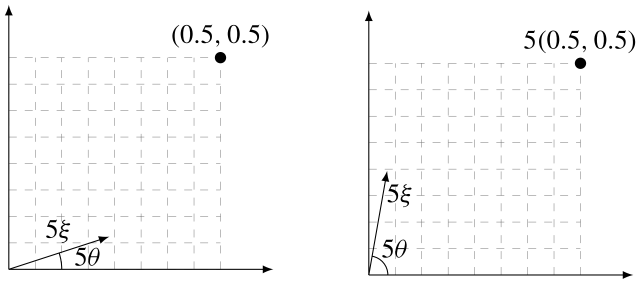

Figure 3. Representation of $\xi \in \mathbb{R}^2$ with $\theta \in \left]0, \frac{\pi}{4}\right]$ and $\theta \in \left]\frac{\pi}{4}, \frac{\pi}{2}\right]$ with $\cos\theta|\xi| = \xi \cdot e_1$.

Thus, we deduce from (2-3) that

$$\left|\frac{1}{(N+1)^D}\sum_{|\xi\cdot v_k|\geq\delta|\xi|}\hat{f}(\xi, v_k)\psi(v_k)\right| \leq CG(\xi)\frac{1}{|\xi|\sqrt{\delta}}\left(1+\frac{1}{\delta N}\right)^{1/2}.$$

We conclude that

$$\left|\hat{\rho}[\psi](\xi)\right| \leq C\left(F(\xi)\sqrt{\delta+\frac{1}{N}}+G(\xi)\frac{1}{|\xi|\sqrt{\delta}}\left(1+\frac{1}{\delta N}\right)^{1/2}\right) \tag{2-4}$$

holds when ξ is aligned with the axis.

We turn to the general case (ii). As illustrated in Figure 3, we can assume that the angle θ between ξ and one of the axes (say e_1) lies in $\left]0, \frac{\pi}{4}\right[$, the other cases follow by a symmetry argument.

The reasoning still consists in counting velocities in strips appropriately defined. As said above, without loss of generality we can assume that $\theta \in \left]0, \frac{\pi}{4}\right[$, where we have set $\cos\theta|\xi| = \xi \cdot e_1$. We set $\ell_1 := \delta/\cos\theta$. On a given strip, we can find at most $(\lfloor \ell_1 N \rfloor + 1) \times (N+1)^{D-1}$ velocities; see Figure 5.

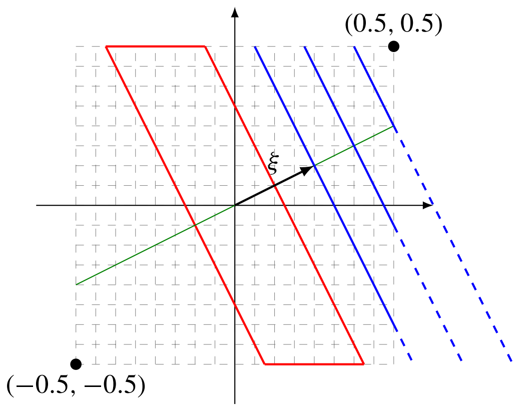

Figure 4. The area corresponding to $|\xi \cdot v_k| \leq \delta|\xi|$ is delimited as previously. The complementary set is split into strips of width δ.

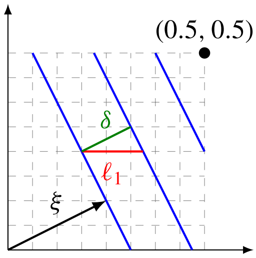

Figure 5. Representation of the parameter ℓ_1.

Therefore, bearing in mind that $0 < \theta < \frac{\pi}{4}$, we obtain

$$\sum_{|\xi\cdot v_k|\geq\delta|\xi|}\frac{1}{|\xi\cdot v_k|^2}=\sum_{|\xi\cdot v_k|\geq\delta|\xi|}\frac{1}{|\xi|^2}\frac{1}{|\xi/|\xi|\cdot v_k|^2}\leq\frac{1}{|\xi|^2}2\sum_{p\geq 1}\frac{1}{(p\delta)^2}\left(\frac{\delta}{\cos\theta}N+1\right)(N+1)^{D-1}$$
$$\leq\frac{1}{|\xi|^2}2\sum_{p\geq 1}\frac{1}{(p\delta)^2}\frac{1}{\delta\cos\theta}\left(1+\frac{1}{\delta N}\right)(N+1)^D\leq 2\sqrt{2}\frac{1}{|\xi|^2}\frac{1}{\delta}\left(1+\frac{1}{\delta N}\right)(N+1)^D$$

and

$$\sum_{|\xi\cdot v_k|<\delta|\xi|}1=(2\lfloor\ell_1 N\rfloor+1)(N+1)^{D-1}\leq 2\left(\frac{\delta}{\cos\theta}N+1\right)(N+1)^{D-1}\leq 2\sqrt{2}\left(\delta+\frac{1}{N}\right)(N+1)^D.$$

Thus, we deduce exactly like in case (i) that (2-4) holds for any $\xi\neq 0$.

Therefore, we have established that for all $\xi\neq 0$, we get (2-4) for all $\delta>0$. We take

$$\delta=\frac{1}{|\xi|}\mathbf{1}_{\{N\geq|\xi|\}}+\frac{1}{N}\mathbf{1}_{\{N<|\xi|\}}$$

and we define

$$\Theta_N(\xi):=\hat{\rho}[\psi](\xi)\mathbf{1}_{\{N\geq|\xi|\}},\qquad \Delta_N(\xi):=\hat{\rho}[\psi](\xi)\mathbf{1}_{\{N<|\xi|\}}.$$

Then, we have

$$\Theta_N(\xi)\leq C\left(F(\xi)\sqrt{\frac{1}{|\xi|}+\frac{1}{N}}+G(\xi)\frac{1}{|\xi|\sqrt{1/|\xi|}}\left(1+\frac{1}{N/|\xi|}\right)^{1/2}\right)\mathbf{1}_{\{N\geq|\xi|\}}\leq C(F(\xi)+G(\xi))\frac{1}{\sqrt{|\xi|}}.$$

It implies that

$$|\xi|\Theta_N(\xi)^2\leq C(G^2(\xi)+F^2(\xi)),$$

which equally holds true for $\xi=0$. Then by the assumption on f and g, we deduce that $\Theta_N\in H^{1/2}(\mathbb{R}^D)$. Finally, we evaluate the remainder:

$$\Delta_N(\xi)\leq C\left(F(\xi)\sqrt{\frac{2}{N}}+G(\xi)\frac{1}{|\xi|\sqrt{1/N}}\left(1+\frac{1}{(1/N)N}\right)\right)\mathbf{1}_{\{N<|\xi|\}}\leq\frac{C}{\sqrt{N}}(F(\xi)+G(\xi)).$$

We conclude that

$$\Delta_N^2(\xi)\leq\frac{C}{N}(F^2(\xi)+G^2(\xi)),$$

which is also satisfied when $\xi=0$. Thus, by the assumption on f and g, we know $\|\Delta_N\|_{L^2}$ is dominated by $1/\sqrt{N}$, an observation which finishes the proof. $\square$

A stochastic discrete velocity averaging lemma. Dealing with random discrete velocities we can expect to make the defect vanish when taking the expectation of the velocity averages. This is indeed the case as shown in the following statement.

Theorem 2.3. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $V_1, \dots, V_{\mathcal{N}}$ be i.i.d. random variables, distributed according to the continuous uniform distribution on $[-0.5, 0.5]^D$. We set*

$$\mathrm{d}\mu = \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \delta(v = V_k).$$

Let $f, g \in L^2(\mathbb{R}^D \times \mathbb{R}^D \times \Omega, \mathrm{d}x\, \mathrm{d}\mu(v)\, \mathrm{d}\mathbb{P})$ satisfy, for all $x \in \mathbb{R}^D$, $\omega \in \Omega$, and $k \in \{1, \dots, \mathcal{N}\}$,

$$V_k \cdot \nabla_x f(x, V_k) = g(x, V_k). \tag{2-5}$$

Then, for all $\psi \in C_c^\infty(\mathbb{R}^D)$, the macroscopic quantity

$$\rho[\psi](x) := \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} f(x, V_k)\psi(V_k) = \mathbb{R}^D f(x, v)\psi(v)\, \mathrm{d}\mu(v)$$

satisfies $\mathbb{E}\rho[\psi] \in H^{1/2}(\mathbb{R}^D)$ (and it is bounded in this space if the L^2 norm of f and g is bounded uniformly with respect to $\mathcal{N}$).

Remark 2.4. We point out that this statement has a different nature from the stochastic averaging lemma devised in [Debussche et al. 2015; 2016], where the velocity set still satisfies an assumption like (1-3) but the equation for $v \cdot \nabla_x f_n$ involves a stochastic term. Our analysis is closer in spirit to the results in [Lions et al. 2013], where the velocity variable is deterministic but is multiplied by a Brownian motion.

Proof. We apply the Fourier transform to (2-5). Then, for all k, we get

$$\xi \cdot V_k \hat{f}(\xi, V_k) = (-i)\hat{g}(\xi, V_k).$$

We set

$$F(\xi) := \left(\frac{1}{\mathcal{N}} \mathbb{E} \sum_k |\hat{f}(\xi, V_k)|^2\right)^{1/2}, \quad G(\xi) := \left(\frac{1}{\mathcal{N}} \mathbb{E} \sum_k |\hat{g}(\xi, V_k)|^2\right)^{1/2}.$$

Let us split

$$\begin{aligned}\mathbb{E}\hat{\rho}[\psi](\xi) &= \mathbb{E}\left[\frac{1}{\mathcal{N}} \sum_k \hat{f}(\xi, V_k)\psi(V_k)\right] \\ &= \mathbb{E}\left[\frac{1}{\mathcal{N}} \sum_{|\xi \cdot V_k| < \delta|\xi|} \hat{f}(\xi, V_k)\psi(V_k)\right] + \mathbb{E}\left[\frac{1}{\mathcal{N}} \sum_{|\xi \cdot V_k| \geq \delta|\xi|} \hat{f}(\xi, V_k)\psi(V_k)\right]\end{aligned}$$

for $\delta > 0$. The Cauchy–Schwarz inequality leads to the following estimates: on the one hand,

$$\left|\mathbb{E}\left[\frac{1}{\mathcal{N}} \sum_{|\xi \cdot V_k| < \delta|\xi|} \hat{f}(\xi, V_k)\psi(V_k)\right]\right| \leq \|\psi\|_\infty \left(\frac{1}{\mathcal{N}} \mathbb{E} \sum_k |\hat{f}(\xi, V_k)|^2\right)^{1/2} \left(\frac{1}{\mathcal{N}} \mathbb{E} \sum_{|\xi \cdot V_k| < \delta|\xi|} 1\right)^{1/2},$$

and, on the other hand,

$$\left|\mathbb{E}\left[\frac{1}{\mathcal{N}}\sum_{|\xi\cdot v_k|\geq\delta|\xi|}\hat{f}(\xi,V_k)\psi(V_k)\right]\right| = \left|\mathbb{E}\left[\frac{1}{\mathcal{N}}\sum_{|\xi\cdot V_k|\geq\delta|\xi|}\frac{(-i)\hat{g}(\xi,V_k)}{\xi\cdot V_k}\,\psi(V_k)\right]\right|$$
$$\leq \|\psi\|_\infty\left(\frac{1}{\mathcal{N}}\,\mathbb{E}\sum_k|\hat{g}(\xi,V_k)|^2\right)^{1/2}\left(\frac{1}{\mathcal{N}}\,\mathbb{E}\sum_{|\xi\cdot v_k|\geq\delta|\xi|}\frac{1}{|\xi\cdot V_k|^2}\right)^{1/2}.$$

We only detail the case where $\xi=\alpha e_1$, $\alpha\in\mathbb{R}$, the other cases being deduced by adapting the reasoning of the proof of Proposition 2.1. We have

$$\mathbb{E}\left[\sum_{|\xi\cdot V_k|\geq\delta|\xi|}\frac{1}{|\xi\cdot V_k|^2}\right]=\mathbb{E}\left[\sum_{|\xi\cdot V_k|\geq\delta|\xi|}\frac{1}{|\xi|^2}\frac{1}{|\xi/|\xi|\cdot V_k|^2}\right]\leq\mathbb{E}\left[\frac{1}{|\xi|^2}2\left(\sum_{p\geq1}\frac{1}{(p\delta)^2}\right)M_p\right],$$

where M_p is the number of velocities in the p-th strip (see Figure 2). We bear in mind that M_p is a random variable: since the V_i are distributed according to the uniform law, we have

$$\mathbb{P}(V_i\in S_p)=\delta$$

and, since the variables $V_1,\dots,V_{\mathcal{N}}$ are independent, M_p follows a binomial distribution of parameters $\mathcal{N}$ and δ. Therefore, we are led to

$$\mathbb{E}\left[\sum_{|\xi\cdot V_k|\geq\delta|\xi|}\frac{1}{|\xi\cdot V_k|^2}\right]\leq\frac{1}{|\xi|^2}2\left(\sum_{p\geq1}\frac{1}{(p\delta)^2}\right)\mathbb{E}[M_p]\leq C\frac{1}{|\xi|^2\delta}\mathcal{N}, \tag{2-6}$$

which yields

$$\left|\mathbb{E}\left[\frac{1}{\mathcal{N}}\sum_{|\xi\cdot V_k|\geq\delta|\xi|}\hat{f}(\xi,V_k)\psi(V_k)\right]\right|\leq CG(\xi)\frac{1}{|\xi|\sqrt{\delta}}.$$

By the same token, we get

$$\mathbb{E}\left[\sum_{|\xi\cdot V_k|<\delta|\xi|}1\right]=2\delta\mathcal{N} \tag{2-7}$$

so that

$$\left|\mathbb{E}\left[\frac{1}{\mathcal{N}}\sum_{|\xi\cdot V_k|<\delta|\xi|}\hat{f}(\xi,V_k)\psi(V_k)\right]\right|\leq CF(\xi)\sqrt{\delta}.$$

Finally, we arrive at

$$\left|\mathbb{E}\hat{\rho}[\psi](\xi)\right|\leq C\left(F(\xi)\sqrt{\delta}+\frac{G(\xi)}{|\xi|\sqrt{\delta}}\right).$$

We apply this inequality with $\delta=G(\xi)/(|\xi|F(\xi))$, which leads to

$$\left|\mathbb{E}\hat{\rho}[\psi](\xi)\right|\leq C\sqrt{F(\xi)G(\xi)}\frac{1}{\sqrt{|\xi|}}.$$

This concludes the proof by using the assumptions on f and g. □

Remark 2.5. We can readily extend the result to nonuniform laws: we assume that the V_i are identically and independently distributed in $\mathbb{R}^D$ according to a continuous and bounded density of probability Φ. The number M_p of velocities in the strip S_p still follows a binomial law but now the expectation value depends on Φ and M_p can be shown to be dominated by $\mathscr{N}\|\Phi\|_\infty\delta$.

For certain applications, the variable v lies on the sphere. This is the case for the kinetic models arising in radiative transfer theory, where v represents the *direction* of flight of photons, which, of course, all travel with the speed of light. We can adapt the stochastic averaging lemma to this situation.

Theorem 2.6. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $V_1, \ldots, V_{\mathscr{N}}$ be i.i.d. random variables, distributed according to the continuous uniform distribution on $\mathbb{S}^{D-1}$. We set*

$$\mathrm{d}\mu = \frac{1}{\mathscr{N}} \sum_{k=1}^{\mathscr{N}} \delta(v = V_k).$$

Let $f, g \in L^2(\mathbb{R}^D \times \mathbb{R}^D \times \Omega, \mathrm{d}x\, \mathrm{d}\mu(v)\, \mathrm{d}\mathbb{P})$ satisfy, for all $x \in \mathbb{R}^D$, $\omega \in \Omega$, and $k \in \{1, \ldots, \mathscr{N}\}$,

$$V_k \cdot \nabla_x f(x, V_k) = g(x, V_k).$$

Then, for all $\psi \in C_c^\infty(\mathbb{S}^{D-1})$, the macroscopic quantity

$$\rho[\psi](x) := \frac{1}{\mathscr{N}} \sum_{k=1}^{\mathscr{N}} f(x, V_k)\psi(V_k) = \mathbb{R}^D f(x, v)\psi(v)\, \mathrm{d}\mu(v)$$

satisfies $\mathbb{E}\rho[\psi] \in H^{1/2}(\mathbb{R}^D)$.

Proof. The proof follows the same arguments as those for Theorem 2.3; we only indicate the main changes. The proof still relies on counting the velocities produced by the random sampling in the domain

$$S_p = \left\{ v \in \mathbb{S}^{D-1} : \delta p|\xi| \leq |v \cdot \xi| \leq \delta(p+1)|\xi| \right\}$$

for given $\xi \in \mathbb{R}^D \backslash \{0\}$, $\delta > 0$ and $p \in \mathbb{Z}$. We define $\theta \in [0, 2\pi]$ such that

$$v \cdot \xi|\xi| = \cos\theta \in [-1, +1].$$

Considering the random vectors V_k, the associated variable θ_k is randomly distributed on $[0, 2\pi]$. For symmetry reasons, $\mathbb{P}(V_k \in S_p)$ is thus proportional to

$$\mathbb{P}\big(\delta|p| \leq \cos\theta_k \leq \delta(|p|+1)\big).$$

We start with the specific case of dimension $D = 2$, and we refer the reader to Figure 6. In this case, θ is uniformly distributed on $[0, 2\pi]$. Therefore, for any $p \in \mathbb{N}$, we know $\mathbb{P}(\delta p \leq \cos\theta \leq \delta(p+1))$ is proportional to

$$\Pi_{\delta,p} = \arccos(\delta(p+1))^{\arccos(\delta p)}\, \mathrm{d}\theta = \arccos(\delta p) - \arccos(\delta(p+1))$$

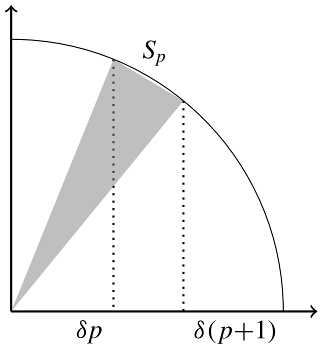

Figure 6. Velocities on the sphere $\mathbb{S}^1$, with domain S_p.

and $M_p = \#\{V_k \in S_p\}$ is driven by the binomial law with parameters $\mathscr{N}$ and $\alpha \Pi_{\delta,p}$ for a certain constant $\alpha > 0$. Hence, the analog of (2-7) is dominated, up to some constant, by

$$\mathscr{N} \Pi_{\delta,0} = \mathscr{N}\big(\tfrac{1}{2}\pi - \arccos\delta\big) = \mathscr{N}\delta \frac{\mathrm{d}x}{\sqrt{1-x^2}} \le C\mathscr{N}\delta$$

as far as $0 < \delta \le \delta_0 < 1$. Similarly, the analog of (2-6) involves the sum

$$\sum_{p\ge 1} \frac{\mathscr{N}}{\delta^2 p^2} \Pi_{\delta,p},$$

which we split into

$$\mathrm{I} = \sum_{1\le p\le 1/2\delta} \frac{\mathscr{N}}{\delta^2 p^2} \Pi_{\delta,p}, \quad \mathrm{II} = \sum_{1/2\delta < p \le 1/\delta} \frac{\mathscr{N}}{\delta^2 p^2} \Pi_{\delta,p}.$$

For I, we can still use the fact that $x \mapsto 1/\sqrt{1-x^2}$ is nonincreasing and bounded far away from $x = 1$ and we are led to the estimate

$$\mathrm{I} = \sum_{1\le p\le 1/2\delta} \frac{\mathscr{N}}{\delta^2 p^2} \int_{\delta(p+1)}^{\delta p} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \le \sum_{1\le p\le 1/2\delta} \frac{\mathscr{N}}{\delta^2 p^2} \frac{\delta}{\sqrt{1-\delta^2(p+1)^2}} \le C\frac{\mathscr{N}}{\delta}.$$

For II, we use a summation by parts which yields

$$\begin{aligned} \mathrm{II} &= \sum_{1/2\delta < p \le 1/\delta} \frac{\mathscr{N} \arccos(\delta p)}{\delta^2} \left(\frac{1}{(p-1)^2} - \frac{1}{p^2} \right) \\ &\le \sum_{1/2\delta < p \le 1/\delta} \frac{\mathscr{N} \arccos(\delta p)}{\delta^2} \frac{2}{p(p-1)^2} \le \frac{4\delta}{\delta^2} \pi \mathscr{N} \sum_{p\ge 1} \frac{1}{p^2} \le C\frac{\mathscr{N}}{\delta}. \end{aligned}$$

Having these estimates at hand, we can repeat the same arguments as in the proof of Theorem 2.3.

For higher dimensions, the situation is actually simpler since θ is now distributed on $\big[0, \frac{\pi}{2}\big]$ according to the law with density $(\sin\theta)^{D-2}\,\mathrm{d}\theta$. Thus (with the simple estimate $0 \le (\sin\theta)^{D-2} \le \sin\theta$) we obtain directly the analog of estimates (2-6) and (2-7). $\square$

The result can be extended to the L^p cases for $1 < p < \infty$ by using an interpolation argument as in [Golse et al. 1988, Theorem 2].

Corollary 2.7. *In Theorems 2.3 and 2.6, we assume that f and g belong to $L^p(\mathbb{R}^D \times \mathscr{V} \times \Omega, \mathrm{d}x\, \mathrm{d}\mu(v)\, \mathrm{d}\mathbb{P})$ for some $1 < p < \infty$, with $\mathscr{V}$ either $\mathbb{R}^D$ or $\mathbb{S}^{D-1}$. Then $\mathbb{E}\rho[\psi]$ lies in the Sobolev space $W^{s,p}(\mathbb{R}^D)$ with $0 < s < \min(1/p, 1 - 1/p) < 1$.*

Proof. We readily adapt the interpolation argument in [Golse et al. 1988]. Let $\mathscr{T}$ be the operator

$$\mathscr{T} : h \mapsto \mathbb{E} \int f(x, v)\psi(v)\, \mathrm{d}\mu(v),$$

where

$$f(x, V_k) + V_k \cdot \nabla_x f(x, V_k) = h(x, V_k).$$

Clearly $\mathscr{T}$ maps continuously $L^r(\mathbb{R}^D \times \mathscr{V} \times \Omega, \mathrm{d}x\, \mathrm{d}\mu(v)\, \mathrm{d}\mathbb{P})$ into $L^r(\mathbb{R}^D)$ for any $1 < r < \infty$. Moreover, Theorems 2.3 and 2.6 tell us that $\mathscr{T}$ is a continuous operator from $L^2(\mathbb{R}^D \times \mathscr{V} \times \Omega, \mathrm{d}x\, \mathrm{d}\mu(v)\, \mathrm{d}\mathbb{P})$ to $H^{1/2}(\mathbb{R}^D)$. We conclude by interpreting the Sobolev space $W^{s,p}$ by interpolation, as being an intermediate space between $L^r = W^{0,r}$ and $H^{1/2} = W^{1/2,2}$ [Bergh and Löfström 1976, Theorem 6.4.5, relation (7)], and L^p as being interpolated between L^r and L^2. □

We can equally extend the compactness statement to the L^1 framework by following [Golse and Saint-Raymond 2002].

Corollary 2.8. *We consider a random set of velocities defined as in Theorem 2.3 orTheorem 2.6. Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences of functions defined on $\mathbb{R}^D \times \mathscr{V} \times \Omega$ such that*

(i) *$\{f_n : n \in \mathbb{N}\}$ is a relatively weakly compact set in $L^1(\mathbb{R}^D \times \mathscr{V} \times \Omega, \mathrm{d}x\, \mathrm{d}\mu(v)\, \mathrm{d}\mathbb{P})$,*

(ii) *$\{g_n : n \in \mathbb{N}\}$ is bounded in $L^1(\mathbb{R}^D \times \mathscr{V} \times \Omega, \mathrm{d}x\, \mathrm{d}\mu(v)\, \mathrm{d}\mathbb{P})$,*

(iii) *we have $V_k \cdot \nabla_x f_n(x, V_k) = g_n(x, V_k)$.*

Then $\mathbb{E}\rho_n[\psi](x) = \mathbb{E}\int f_n(x, v)\psi(v)\, \mathrm{d}\mu(v)$ lies in a relatively compact set of $L^1(B(0, R))$ for any $0 < R < \infty$ (for the strong topology).

Proof. The proof follows closely [Golse and Saint-Raymond 2002]; we sketch the arguments for the sake of completeness. For $\psi \in C_c^\infty(\mathscr{V})$, we denote by $\mathscr{A}$ the operator

$$\mathscr{A} : f \mapsto \mathbb{E} \int f(x, v)\psi(v)\, \mathrm{d}\mu(v).$$

For $\lambda > 0$, we also introduce the operator

$$R_\lambda : h \mapsto \int_0^\infty e^{-\lambda t} h(x - vt, v)\, \mathrm{d}t,$$

which returns the solution $f = R_\lambda h$ of $(\lambda + v \cdot \nabla_x) f = h$. It is a continuous operator on $L^p(\mathbb{R}^D \times \mathscr{V}, \mathrm{d}x\, \mathrm{d}\mu(v))$ spaces and we have

$$\|R_\lambda h\|_{L^p} \le \frac{\|h\|_{L^p}}{\lambda}. \tag{2-8}$$

Let us temporarily assume that the compactness statement holds for $\mathscr{A} R_\lambda g_n$, for any $\lambda > 0$, when (i)–(ii) is strengthened to

(ii′) $\{g_n : n \in \mathbb{N}\}$ is a relatively weakly compact set in $L^1(\mathbb{R}^D \times \mathscr{V} \times \Omega, \mathrm{d}x\, \mathrm{d}\mu(v)\, \mathrm{d}\mathbb{P})$.

Therefore, writing $(\lambda + v \cdot \nabla_x) R_\lambda f_n = f_n$, we deduce from (i) that $(\mathscr{A} R_\lambda f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^1(B(0, R))$ for any $\lambda > 0$ and $0 < R < \infty$. Next, we write $f_n = \lambda R_\lambda f_n + R_\lambda(v \cdot \nabla_x f_n)$ so that, owing to (2-8), $\mathscr{A} f_n = \lambda \mathscr{A} R_\lambda f_n + \mathscr{A} R_\lambda(v \cdot \nabla_x f_n)$ appears as the sum of a sequence which is compact in $L^1(B(0, R))$ and a sequence whose norm is dominated by $1/\lambda$, uniformly with respect to n. Consequently, $(\mathscr{A} f_n)_{b\in\mathbb{N}}$ is relatively compact in $L^1(B(0, R))$.

We are thus left with the task of justifying the gain of compactness for $\mathscr{A} R_\lambda g_n$ when (i)–(ii) is replaced by (ii′); see [Golse et al. 1988, Proposition 3]. To this end, for $\lambda, M > 0$ we set $R_\lambda g_n = \gamma_n$ and we split

$$\gamma_n = \gamma_{n,M} + \gamma_n^M,$$

where

$$\begin{aligned}(\lambda + V_k \cdot \nabla_x)\gamma_{n,M}(x, V_k) &= g_n(x, V_k)\mathbf{1}_{g_n(x,V_k)\le M},\\ (\lambda + V_k \cdot \nabla_x)\gamma_n^M(x, V_k) &= g_n(x, V_k)\mathbf{1}_{g_n(x,V_k)>M}.\end{aligned}$$

Since for any fixed $M > 0$, the set $\{g_n \mathbf{1}_{h_n\le M} : n \in \mathbb{N}\}$ is bounded in $L^1 \cap L^\infty \subset L^2$, we can apply Theorem 2.3 or Theorem 2.6, which imply that $(\mathscr{A}\gamma_{n,M})_{n\in\mathbb{N}}$ is compact in $L^1(B(0, R))$ for any finite R. We can conclude by showing that γ_n^M can be made arbitrarily small, in L^1 norm, uniformly with respect to $n \in \mathbb{N}$, for a suitable choice of $M > 0$. This is indeed the case because (ii′) implies

$$\lim_{M\to\infty}\left\{\sup_n \int |g_n| \mathbf{1}_{g_n>M}\, \mathrm{d}\mu(v)\, \mathrm{d}x\, \mathrm{d}\mathbb{P}(\omega)\right\} = 0$$

by virtue of the Dunford–Pettis theorem; see [Goudon 2011, §7.3.2]. Going back to (2-8) finishes the proof. □

3. Application to the Rosseland approximation

Let us go back to the asymptotic behavior of the solutions of (1-1). The problem (1-1) is completed with the initial condition

$$f_\varepsilon|_{t=0} = f_\varepsilon^0.$$

It satisfies $f_\varepsilon^0 \ge 0$ and $f_\varepsilon^0 \in L^1(\mathbb{R}^D \times \mathscr{V})$, as it is physically relevant, f_ε being a particle density. For the set $(\mathscr{V}, \mathrm{d}\mu)$, in what follows we suppose at least that $\mathscr{V}$ is a bounded subset in $\mathbb{R}^D$ and

$$\int_{\mathscr{V}} \mathrm{d}\mu(v) = 1, \qquad \int_{\mathscr{V}} v\, \mathrm{d}\mu(v) = 0.$$

These assumptions are crucial for the analysis of the diffusion regime. Then, the connection to (1-2) can be established as follows.

Theorem 3.1. *We assume that (1-3) is fulfilled. Let σ be a function such that $\sigma(\rho) = \rho^\gamma \Sigma(\rho)$ with $|\gamma| < 1$ and $0 < \sigma_* \le \Sigma(\rho) \le \sigma^* < \infty$. Let $(f_\varepsilon^0)_{\varepsilon>0}$ satisfy*

$$\sup_{\varepsilon>0}\left(\int_{\mathbb{R}^d}\int_{\mathscr{V}}\big(1+\varphi(x)+|\ln f_\varepsilon^0|f_\varepsilon^0\big)\,\mathrm{d}\mu(v)\,\mathrm{d}x + \|f_\varepsilon^0\|_{L^\infty(\mathbb{R}^d\times\mathscr{V})}\right) = M_0 < +\infty$$

for a certain weight function such that $\lim_{|x|\to+\infty}\varphi(x) = +\infty$. Then (up to a subsequence) the solution f_ε of (1-1) and ρ_ε converge to $\rho(t,x)$ in $L^p((0,T)\times\mathbb{R}^d\times\mathscr{V})$ and $L^p((0,T)\times\mathbb{R}^d)$ respectively, for any $1 \le p < \infty$, $0 < T < \infty$, where ρ is a solution to (1-2) with the initial data $\rho|_{t=0}$ given by the weak limit in $L^p(\mathbb{R}^d)$ of $\int_{\mathscr{V}} f_\varepsilon^0\,\mathrm{d}\mu(v)$ as $\varepsilon \to 0$.

For instance this statement holds with $\mathscr{V} = \mathbb{S}^{D-1}$ endowed with the Lebesgue measure. We refer the reader to [Bardos et al. 1988] for a detailed proof, where the velocity averaging lemma is used to manage the passage to the limit in the nonlinearity. Assumption (1-3) can be replaced by

$$\text{for any } \xi \ne 0, \quad \operatorname{meas}\big(\{v \in \mathscr{V}\cap B(0,R) : v\cdot\xi \ne 0\}\big) > 0,$$

which allows us to deal with certain discrete velocity models. Then, the asymptotic regime can be analyzed with a *compensated compactness* argument, which relies on the structure of the system satisfied by the zeroth and first moments of f_ε, as pointed out in [Degond et al. 2000; Goudon and Poupaud 2001; Lions and Toscani 1997]; see also [Marcati and Milani 1990]. The question of the relation between the diffusion equation that corresponds to a discretization of the velocity set (discrete ordinate equation) and the diffusion equation that corresponds to the continuous model can be addressed. For the simple collision operator in (1-1), velocity grids, which differ from the simplest uniform mesh, can be constructed that lead to the *exact* diffusion coefficient, namely

$$\frac{1}{\mathscr{N}}\sum_{k=1}^{\mathscr{N}} v_k\otimes v_k = \int_{\mathbb{S}^{D-1}} v\otimes v\,\mathrm{d}v = \frac{1}{D}\mathbb{I};$$

we refer the reader to [Buet et al. 2002; Golse et al. 1999; Jin and Levermore 1991] for further discussion on this issue. However, for more general collision operators, it might happen that the equilibrium functions that make the collision operator vanish or the diffusion coefficient are not explicitly known; see [Bonnaillie-Noël et al. 2016; Degond et al. 2000].

We wish to revisit this question by means of a Monte Carlo approach: instead of the discrete ordinate viewpoint where a discrete velocity grid is adopted once and for all, we deal with a random set of velocities and we wonder whether it can provide, in expectation, a consistent approximation of the diffusion regime. The consistency analysis we propose uses Theorem 2.3 or Theorem 2.6 to justify the following claim.

Theorem 3.2. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $V_1, \dots, V_{\mathscr{N}}$ be i.i.d. random variables distributed according to the continuous uniform law on $\mathscr{V}$. Then, we obtain a set $\mathscr{V}_{\mathscr{N}}$ of $2\mathscr{N}$ velocities in $\mathscr{V}$ by setting $V_{\mathscr{N}+j} = -V_j$ for all $j \in \{1, \dots, \mathscr{N}\}$. We denote the associated discrete measure on $\mathscr{V}$ by*

$$\mathrm{d}\mu_{\mathscr{N}}(v) = \frac{1}{2\mathscr{N}}\sum_{k=1}^{2\mathscr{N}}\delta(v = V_k).$$

Let $f_\varepsilon|_{t=0} = f_\varepsilon^0 \geq 0$ *satisfy*

$$\sup_{\varepsilon>0,\ \mathscr{N}\in\mathbb{N}} \left(\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \big(1+\varphi(x)+|\ln f_\varepsilon^0|\big) f_\varepsilon^0 \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x + \|f_\varepsilon^0\|_{L^\infty(\Omega\times\mathbb{R}^d\times\mathscr{V})} \right) = M_0 < +\infty. \tag{3-1}$$

Let f_ε *be a solution of the equation*

$$\partial_t f_\varepsilon(t,x,V_j) + \frac{1}{\varepsilon} V_j\cdot\nabla_x f_\varepsilon(t,x,V_j) = \frac{1}{\varepsilon^2}\sigma(\rho_{\varepsilon,\mathscr{N}})[\rho_{\varepsilon,\mathscr{N}}(t,x) - f_\varepsilon(t,x,V_j)], \tag{3-2}$$

with

$$\rho_{\varepsilon,\mathscr{N}}(t,x) := \frac{1}{2\mathscr{N}}\sum_{i=1}^{2\mathscr{N}} f_\varepsilon(t,x,V_j).$$

We suppose that $\rho\in[0,\infty)\mapsto\sigma(\rho)$ *is a nonnegative function such that for any* $0<R<\infty$, *there exists* $\sigma_\star(R)>0$ *satisfying* $0<1/\sigma_\star(R)\leq\sigma(\rho)\leq\sigma_\star(R)$ *and* $|\sigma'(\rho)|\leq\sigma_\star(R)$ *for any* $0\leq\rho\leq R$. *Then* $\mathbb{E}\rho_{\varepsilon,\mathscr{N}}$ *converges to* $\mathbb{E}\rho_{\mathscr{N}}$ *in* $L^2((0,T)\times\mathbb{R}^D)$ *as* ε *goes to* 0 *with* $0<T<\infty$, *where* $\mathbb{E}\rho_{\mathscr{N}}$ *is solution of*

$$\partial_t\mathbb{E}\rho_{\mathscr{N}} + \operatorname{div}(\mathscr{J}_{\mathscr{N}}) = 0, \qquad \sigma(\mathbb{E}\rho_{\mathscr{N}})\mathscr{J}_{\mathscr{N}} = -\mathbb{E}A_{\mathscr{N}}\nabla_x\mathbb{E}\rho_{\mathscr{N}} + O\left(\frac{1}{\sqrt{\mathscr{N}}}\right),$$

with $A_{\mathscr{N}}$ *the* $D\times D$ *matrix with random components defined by*

$$A_{\mathscr{N}} := \frac{1}{2\mathscr{N}}\sum_{j=1}^{2\mathscr{N}} V_j\otimes V_j,$$

and $\mathbb{E}\rho_{\mathscr{N}}|_{t=0}$ *is the weak limit of* $\int\mathbb{E}f_\varepsilon^0\,\mathrm{d}\mu(v)$.

Note that the construction of the set $\mathscr{V}_{\mathscr{N}}$ ensures that the null flux condition $\int v\,\mathrm{d}\mu_{\mathscr{N}}(v)=0$ is fulfilled, but the elements of $\mathscr{V}_{\mathscr{N}}$ are not independent. Nevertheless, the stochastic averaging lemma still applies to this situation, with a straightforward adaptation of the proof. It is likely that the assumptions on σ can be substantially weakened, but it not our aim here to seek refinements in this direction. We will make precise in the proof in which sense the consistency error $O(1/\sqrt{\mathscr{N}})$ should be understood.

Entropy estimates. In order to prove Theorem 3.2, the first step consists in establishing some a priori estimates, uniform with respect to the parameters ε and $\mathscr{N}$. We will then deduce the compactness needed to obtain the result. These estimates are quite classical; the proof that we sketch for the sake of completeness follows directly from [Bardos et al. 1988; Goudon and Poupaud 2001; Lions and Toscani 1997].

Proposition 3.3. *Let* f_ε^0 *satisfy* (3-1) *with* $\varphi(x)=(1+x^2)^\beta$, $0<\beta<1$. *Let* $0<T<\infty$. *There exists a constant* $C(T)$ *which only depends on* T *such that*

$$\sup_{\varepsilon>0,\mathscr{N}\in\mathbb{N}}\left\{\sup_{0\leq t\leq T}\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}}\big(1+\varphi(x)+|\ln f_\varepsilon|\big)f_\varepsilon\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x + \|f_\varepsilon\|_{L^\infty(\Omega\times(0,T)\times\mathbb{R}^D\times\mathscr{V})}\right\} = C(T) < +\infty \tag{3-3}$$

and, furthermore,

$$\sup_{\varepsilon>0,\mathscr{N}\in\mathbb{N}}\mathbb{E}\int_0^T\int_{\mathbb{R}^D}\int_{\mathscr{V}}\frac{\sigma(\rho_{\varepsilon,\mathscr{N}})}{\varepsilon^2}(f_\varepsilon-\rho_{\varepsilon,\mathscr{N}})\ln\left(\frac{f_\varepsilon}{\rho_{\varepsilon,\mathscr{N}}}\right)\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\,\mathrm{d}t \leq C(T). \tag{3-4}$$

Proof. As said above we crucially use the fact that

$$\int_{\mathscr{V}} \mathrm{d}\mu_{\mathscr{N}}(v) = 1, \qquad \int_{\mathscr{V}} v\,\mathrm{d}\mu_{\mathscr{N}}(v) = 0.$$

As a matter of fact, the collision operator is mass-conserving in the sense that

$$\int_{\mathscr{V}} \sigma(\rho)(f-\rho)\,\mathrm{d}\mu_{\mathscr{N}}(v) = 0.$$

Accordingly, integrating immediately leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x = 0. \tag{3-5}$$

More generally, let $G : [0, \infty) \to \mathbb{R}$ be a convex function. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} G(f_\varepsilon)\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x = -\frac{1}{\varepsilon^2}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \sigma(\rho_{\varepsilon,\mathscr{N}})(\rho_{\varepsilon,\mathscr{N}}-f_\varepsilon)(G'(\rho_{\varepsilon,\mathscr{N}})-G'(f_\varepsilon))\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x \le 0.$$

With $G(z) = z^p,\ p \ge 1$, it gives an estimate on the L^p norm of the solution. Similarly, with $G(z) = [z - \|f_\varepsilon^0\|_\infty]_+^2$, we conclude that

$$\|f_\varepsilon\|_{L^\infty(\Omega\times(0,T)\times\mathbb{R}^D\times\mathscr{V})} \le \|f_\varepsilon^0\|_\infty.$$

Finally, with $G(z) = z\ln(z)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon \ln f_\varepsilon\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x = -\frac{1}{\varepsilon^2}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \sigma(\rho_{\varepsilon,\mathscr{N}})[\rho_{\varepsilon,\mathscr{N}}-f_\varepsilon]\ln\!\left(\frac{f_\varepsilon}{\rho_{\varepsilon,\mathscr{N}}}\right)\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x \le 0. \tag{3-6}$$

Let us focus on the following quantity obtained by multiplying (3-2) by φ and integrating

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \varphi(x) f_\varepsilon\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x &= -\frac{1}{\varepsilon}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \varphi(x) v\cdot\nabla_x f_\varepsilon\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\\ &= \frac{1}{\varepsilon}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon v\cdot\nabla_x\varphi(x)\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\\ &= \mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} v\cdot\nabla_x\varphi(x)\frac{f_\varepsilon-\rho_{\varepsilon,\mathscr{N}}}{\varepsilon}\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x.\end{aligned}$$

Note that we have used $\int v\,\mathrm{d}_{\mathscr{N}}(v) = 0$. By the Cauchy–Schwarz inequality, we know that

$$|\sqrt{b}-\sqrt{a}|^2 = \left|\int_a^b \frac{\mathrm{d}s}{2\sqrt{s}}\right|^2 \le \left|\int_a^b \frac{\mathrm{d}s}{4s}\right|\left|\int_a^b \mathrm{d}s\right| = \tfrac{1}{4}(b-a)\ln(b/a).$$

Thus, we get

$$\begin{aligned}\int_{\mathscr{V}} |f_\varepsilon-\rho_{\varepsilon,\mathscr{N}}|\,\mathrm{d}\mu_{\mathscr{N}}(v) &= \int_{\mathscr{V}} (\sqrt{f_\varepsilon}+\sqrt{\rho_{\varepsilon,\mathscr{N}}})\big|\sqrt{f_\varepsilon}-\sqrt{\rho_{\varepsilon,\mathscr{N}}}\big|\,\mathrm{d}\mu_{\mathscr{N}}(v)\\ &\le \left(\int_{\mathscr{V}} (\sqrt{f_\varepsilon}+\sqrt{\rho_{\varepsilon,\mathscr{N}}})^2\,\mathrm{d}\mu_{\mathscr{N}}(v)\right)^{1/2}\left(\int_{\mathscr{V}} (\sqrt{f_\varepsilon}-\sqrt{\rho_{\varepsilon,\mathscr{N}}})^2\,\mathrm{d}\mu_{\mathscr{N}}(v)\right)^{1/2}\\ &\le C\sqrt{\rho_{\varepsilon,\mathscr{N}}}\left(\int_{\mathscr{V}} (f_\varepsilon-\rho_{\varepsilon,\mathscr{N}})\ln(f_\varepsilon/\rho_{\varepsilon,\mathscr{N}})\,\mathrm{d}\mu_{\mathscr{N}}(v)\right)^{1/2},\end{aligned}$$

and we finally obtain the bound

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \varphi f_\varepsilon \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x
&\le \|v\|_{L^\infty(\Omega\times S)}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} |\nabla_x\varphi\frac{|f_\varepsilon-\rho_{\varepsilon,\mathscr{N}}|}{\varepsilon}\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\\
&\le C\,\mathbb{E}\int_{\mathbb{R}^D} |\nabla_x\varphi|\sqrt{\frac{\rho_{\varepsilon,\mathscr{N}}}{\sigma(\rho_{\varepsilon,\mathscr{N}})}}\biggl(\int_{\mathscr{V}}\frac{\sigma(\rho_{\varepsilon,\mathscr{N}})}{\varepsilon^2}(f_\varepsilon-\rho_{\varepsilon,\mathscr{N}})\ln(f_\varepsilon/\rho_{\varepsilon,\mathscr{N}})\,\mathrm{d}\mu_{\mathscr{N}}(v)\biggr)^{1/2}\mathrm{d}x\\
&\le C\,\mathbb{E}\biggl(\int_{\mathbb{R}^D}|\nabla_x\varphi|^2\frac{\rho_{\varepsilon,\mathscr{N}}}{\sigma(\rho_{\varepsilon,\mathscr{N}})}\,\mathrm{d}x\biggr)^{1/2}\biggl(\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}}\frac{\sigma(\rho_{\varepsilon,\mathscr{N}})}{\varepsilon^2}(f_\varepsilon-\rho_{\varepsilon,\mathscr{N}})\ln(f_\varepsilon/\rho_{\varepsilon,\mathscr{N}})\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\biggr)^{1/2}.
\end{aligned}$$

By assumption, $1/\sigma(\rho_{\varepsilon,\mathscr{N}})$ is uniformly bounded. It follows that

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^D}|\nabla_x\varphi|^2\frac{\rho_{\varepsilon,\mathscr{N}}}{\sigma(\rho_{\varepsilon,\mathscr{N}})}\,\mathrm{d}x &\le C\biggl(\mathbb{E}\int_{\mathbb{R}^D}|\nabla_x\varphi|^{2q}\,\mathrm{d}x\biggr)^{1/q}\biggl(\mathbb{E}\int_{\mathbb{R}^D}\rho_{\varepsilon,\mathscr{N}}^p\,\mathrm{d}x\biggr)^{1/p}\\
&\le C\biggl(\mathbb{E}\int_{\mathbb{R}^D}|\nabla_x\varphi|^{2q}\,\mathrm{d}x\biggr)^{1/q}\biggl(\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}}|f_\varepsilon|^p\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\biggr)^{1/p}\le C
\end{aligned}$$

holds provided the Hölder conjugate q of $p\ge 1$ satisfies $\beta\le 1/2-D/(4q)$.

The Young inequality

$$ab\le\frac{a^2}{4\theta}+\theta b^2$$

yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}}\varphi(x)f_\varepsilon(t,x,v)\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\le C+\frac{1}{2}\,\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}}\frac{\sigma(\rho_{\varepsilon,\mathscr{N}})}{\varepsilon^2}(f_\varepsilon-\rho_{\varepsilon,\mathscr{N}})\ln(f_\varepsilon/\rho_{\varepsilon,\mathscr{N}})\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x.$$

Let us set

$$D_\varepsilon:=\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}}\frac{\sigma(\rho_{\varepsilon,\mathscr{N}})}{\varepsilon^2}(f_\varepsilon-\rho_{\varepsilon,\mathscr{N}})\ln(f_\varepsilon/\rho_{\varepsilon,\mathscr{N}})\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\ge 0.$$

Coming back to (3-6), we get

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon(t,x,v)\ln f_\varepsilon(t,x,v)\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x+\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}}\varphi(x)f_\varepsilon(t,x,v)\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x+\frac{1}{2}\int_0^t D_\varepsilon(s)\,\mathrm{d}s\\
&\qquad\le Ct+\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon^{\omega,0}(x,v)\ln f_\varepsilon^{\omega,0}(x,v)\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x+\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}}\varphi(x)f_\varepsilon^{\omega,0}(x,v)\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x.
\end{aligned}$$

Since $z|\ln z|=z\ln z-2z\ln z\,\mathbf{1}_{\{0\le z\le 1\}}$, we have

$$0\le-\int_{0\le f\le 1} f\ln f\,\mathrm{d}y=-\int_{0\le f\le e^{-\varphi}} f\ln f\,\mathrm{d}y-\int_{e^{-\varphi}\le f\le 1} f\ln f\,\mathrm{d}y\le\int\varphi f\,\mathrm{d}y+\int e^{-\varphi/2}\,\mathrm{d}y.$$

Then, we are led to

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon |\ln f_\varepsilon| \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x + \frac{1}{2}\int_0^t D_\varepsilon(s)\,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \varphi f_\varepsilon \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x \\
&\quad = \mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon \ln f_\varepsilon \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x - 2\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon \ln f_\varepsilon \mathbf{1}_{\{0\le f_\varepsilon \le 1\}} \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x \\
&\qquad\qquad + \frac{1}{2}\int_0^t D_\varepsilon(s)\,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \varphi f_\varepsilon \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x \\
&\quad \le \mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} f_\varepsilon \ln f_\varepsilon \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x + 2\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \frac{\varphi}{4} f_\varepsilon \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x \\
&\qquad\qquad + 2\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} e^{-\varphi/8} \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x + \frac{1}{2}\int_0^t D_\varepsilon(s)\,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_{\mathbb{R}^D}\int_{\mathscr{V}} \varphi f_\varepsilon \,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x \\
&\quad \le C(T).
\end{aligned}$$
□

Moreover, we can deduce from above that f_ε behaves like its macroscopic part $\rho_{\varepsilon,\mathscr{N}}$ for small ε.

Corollary 3.4. *We set* $g_{\varepsilon,\mathscr{N}} := (f_\varepsilon - \rho_{\varepsilon,\mathscr{N}})/\varepsilon$. *Then, we have*

$$\sup_{\varepsilon>0,\ \mathscr{N}} \mathbb{E}\int_0^T\int_{\mathbb{R}^D}\left|\int_{\mathscr{V}} g_{\varepsilon,\mathscr{N}}\,\mathrm{d}\mu_{\mathscr{N}}(v)\right|^2 \mathrm{d}x\,\mathrm{d}t \le C(T).$$

Proof. We write

$$\begin{aligned}
\mathbb{E}\int_0^T\int_{\mathbb{R}^D}\left|\int_{\mathscr{V}} g_{\varepsilon,\mathscr{N}}\,\mathrm{d}\mu_{\mathscr{N}}(v)\right|^2 \mathrm{d}x\,\mathrm{d}t &= \mathbb{E}\int_0^T\int_{\mathbb{R}^D}\left(\int_{\mathscr{V}} \frac{|f_\varepsilon-\rho_{\varepsilon,\mathscr{N}}|}{\varepsilon}\,\mathrm{d}\mu_{\mathscr{N}}(v)\right)^2 \mathrm{d}x\,\mathrm{d}t \\
&\le C\,\mathbb{E}\int_0^T\int_{\mathbb{R}^D} \rho_{\varepsilon,\mathscr{N}}\int_{\mathscr{V}} (f_\varepsilon-\rho_{\varepsilon,\mathscr{N}})\ln(f_\varepsilon/\rho_{\varepsilon,\mathscr{N}})\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\,\mathbb{E}\int_0^T\int_{\mathbb{R}^D} \frac{\rho_{\varepsilon,\mathscr{N}}}{\sigma(\rho_{\varepsilon,\mathscr{N}})}\int_{\mathscr{V}} \sigma(\rho_{\varepsilon,\mathscr{N}})(f_\varepsilon-\rho_{\varepsilon,\mathscr{N}})\ln(f_\varepsilon/\rho_{\varepsilon,\mathscr{N}})\,\mathrm{d}\mu_{\mathscr{N}}(v)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since by the assumption on σ we know that $z \mapsto z/\sigma(z)$ is bounded on bounded sets and since $\rho_{\varepsilon,\mathscr{N}}$ is bounded in $L^\infty(\Omega \times (0,T) \times \mathbb{R}^D)$, we can conclude by using (3-4). □

Diffusive limit. We can now discuss how to pass to the limit $\varepsilon \to 0$.

Proof of Theorem 3.2. Applying the Dunford–Pettis theorem (see [Goudon 2011, §7.3.2]) we deduce from Proposition 3.3 that, possibly at the price of extracting a subsequence,

$$f_\varepsilon \rightharpoonup f_{\mathscr{N}} \quad \text{weakly in } L^1(\Omega \times (0,T) \times \mathbb{R}^D \times \mathscr{V}_{\mathscr{N}}).$$

Consequently, we also have

$$\rho_{\varepsilon,\mathscr{N}} = \int_{\mathscr{V}} f_\varepsilon\,\mathrm{d}\mu_{\mathscr{N}}(v) \rightharpoonup \rho_{\mathscr{N}} = \int_{\mathscr{V}} f_{\mathscr{N}}\,\mathrm{d}\mu_{\mathscr{N}}(v) \quad \text{weakly in } L^1(\Omega \times (0,T) \times \mathbb{R}^D)$$

and

$$\mathbb{E}\rho_{\varepsilon,\mathscr{N}} \rightharpoonup \mathbb{E}\rho_{\mathscr{N}} \quad \text{weakly in } L^1((0,T) \times \mathbb{R}^D).$$

Next, we consider the equations satisfied by the moments of f_ε. To this end, let us set

$$J_{\varepsilon,\mathcal{N}}(t,x) := \frac{1}{2\mathcal{N}}\sum_{i=1}^{2\mathcal{N}} \frac{V_i}{\varepsilon} f_\varepsilon(t,x,V_i), \quad \mathbb{P}_{\varepsilon,\mathcal{N}}(t,x) := \frac{1}{2\mathcal{N}}\sum_{i=1}^{2\mathcal{N}} V_i \otimes V_i f_\varepsilon(t,x,V_i).$$

Integrating (3-2) with respect to the velocity variable v yields

$$\partial_t \rho_{\varepsilon,\mathcal{N}} + \operatorname{div}(J_{\varepsilon,\mathcal{N}}) = 0. \tag{3-7}$$

Similarly, multiplying (3-2) by v and integrating leads to

$$\varepsilon^2 \partial_t J_{\varepsilon,\mathcal{N}} + \operatorname{div}(\mathbb{P}_{\varepsilon,\mathcal{N}}) = -\sigma(\rho_{\varepsilon,\mathcal{N}}) J_{\varepsilon,\mathcal{N}}. \tag{3-8}$$

Lemma 3.5. *The sequence $(J_{\varepsilon,\mathcal{N}})_{\varepsilon>0}$ is bounded in $L^2(\Omega\times(0,T)\times\mathbb{R}^D)$ and we can write $\mathbb{P}_{\varepsilon,\mathcal{N}} = A_{\mathcal{N}}\rho_{\varepsilon,\mathcal{N}} + \varepsilon\mathbb{K}_{\varepsilon,\mathcal{N}}$ with $A_{\mathcal{N}} = \frac{1}{2\mathcal{N}}\sum_{j=1}^{2\mathcal{N}} V_j\otimes V_j$ and the components of $(\mathbb{K}_{\varepsilon,\mathcal{N}})_{\varepsilon>0}$ are bounded in $L^2(\Omega\times(0,T)\times\mathbb{R}^D)$.*

Proof. The proof is based on the fact that $f_\varepsilon = \rho_{\varepsilon,\mathcal{N}} + \varepsilon g_{\varepsilon,\mathcal{N}}$. Since $\sum_{j=1}^{2\mathcal{N}} V_j = 0$, it allows us to write

$$J_{\varepsilon,\mathcal{N}} = \int v g_{\varepsilon,\mathcal{N}} \,\mathrm{d}\mu_{\mathcal{N}}(v),$$

and we deduce the bound on $J_{\varepsilon,\mathcal{N}}$ from Corollary 3.4 since $\|v\|_{L^\infty(\Omega\times S)} \le C$. In addition, we have

$$\mathbb{P}_{\varepsilon,\mathcal{N}} = \int v\otimes v \,\mathrm{d}\mu_{\mathcal{N}}(v)\rho_{\varepsilon,\mathcal{N}} + \varepsilon \int v\otimes v g_{\varepsilon,\mathcal{N}} \,\mathrm{d}\mu_{\mathcal{N}}(v).$$

We set

$$\mathbb{K}_{\varepsilon,\mathcal{N}}(t,x) := \int v\otimes v g_{\varepsilon,\mathcal{N}}(t,x,v)\,\mathrm{d}\mu_{\mathcal{N}}(v).$$

We conclude by using the estimates in Corollary 3.4 again. □

Owing to Lemma 3.5, (3-8) can be recast as

$$\varepsilon\big(\varepsilon\partial_t J_{\varepsilon,\mathcal{N}} + \operatorname{div}(\mathbb{K}_{\varepsilon,\mathcal{N}})\big) + A_{\mathcal{N}}\nabla_x \rho_{\varepsilon,\mathcal{N}} = -\nu_{\varepsilon,\mathcal{N}},$$

with $\nu_{\varepsilon,\mathcal{N}} := \sigma(\rho_{\varepsilon,\mathcal{N}})J_{\varepsilon,\mathcal{N}}$. Passing to the limit, up to subsequences, we are led to

$$\begin{cases} \partial_t \rho_{\mathcal{N}} + \operatorname{div}(J_{\mathcal{N}}) = 0, \\ A_{\mathcal{N}}\nabla\rho_{\mathcal{N}} = -\nu_{\mathcal{N}}, \end{cases} \tag{3-9}$$

where $\nu_{\mathcal{N}}$ is the weak limit as $\varepsilon\to 0$ of $\nu_{\varepsilon,\mathcal{N}}$, which is a bounded sequence in $L^2(\Omega\times(0,T)\times\mathbb{R}^D)$. It remains to establish a relation between $\nu_{\mathcal{N}}$, $\rho_{\mathcal{N}}$ and $J_{\mathcal{N}}$, or more precisely the expectation of these quantities. To this end, we are going to use the strong compactness of $\mathbb{E}\rho_{\varepsilon,\mathcal{N}}$ by using the averaging lemma. Indeed, we know that $\mathbb{E}\rho_{\varepsilon,\mathcal{N}}$ belongs to a bounded set in $L^2(0,T;H^{1/2}(\mathbb{R}^D))$; the proof follows exactly the same argument as for Theorem 2.3, taking the Fourier transform with respect to both the time and space variables t, x. However, because of the ε in front of the time derivative, we cannot expect a gain of regularity with respect to the time variable. Then, we need to combine this estimate with another argument as follows:

(i) By using the Weil–Kolmogorov–Fréchet theorem, see [Goudon 2011, Théorème 7.56], we deduce from the averaging lemma that

$$\lim_{|h|\to 0}\left(\sup_{\varepsilon}\int_0^T\int_{\mathbb{R}^D}\left|\mathbb{E}\rho_{\varepsilon,\mathscr{N}}(t,x+h)-\mathbb{E}\rho_{\varepsilon,\mathscr{N}}(t,x)\right|^2\,\mathrm{d}x\,\mathrm{d}t\right)=0.$$

(ii) Going back to (3-7), Lemma 3.5 tells us that $\partial_t\mathbb{E}\rho_{\varepsilon,\mathscr{N}}=-\mathrm{div}(\mathbb{E}J_{\varepsilon,\mathscr{N}})$ is bounded, uniformly with respect to ε, in $L^2(0,T;H^{-1}(\mathbb{R}^D))$.

Then, this is enough to deduce that $\mathbb{E}\rho_{\varepsilon,\mathscr{N}}$ strongly converges to $\mathbb{E}\rho_{\mathscr{N}}$ in $L^2((0,T)\times\mathbb{R}^D)$ (see, e.g., [Alonso et al. 2017, Appendix B] for a detailed proof).

Then, we rewrite

$$\mathbb{E}J_{\varepsilon,\mathscr{N}}=\mathbb{E}\left(\frac{\nu_{\varepsilon,\mathscr{N}}}{\sigma(\rho_{\varepsilon,\mathscr{N}})}\right)=\frac{\mathbb{E}\nu_{\varepsilon,\mathscr{N}}}{\sigma(\mathbb{E}\rho_{\varepsilon,\mathscr{N}})}+\mathbb{E}r_{\varepsilon,\mathscr{N}},\qquad r_{\varepsilon,\mathscr{N}}=\left[\nu_{\varepsilon,\mathscr{N}}\left(\frac{1}{\sigma(\rho_{\varepsilon,\mathscr{N}})}-\frac{1}{\sigma(\mathbb{E}\rho_{\varepsilon,\mathscr{N}})}\right)\right].\tag{3-10}$$

From the previous discussion, extracting further subsequences if necessary, we know that $\mathbb{E}\nu_{\varepsilon,\mathscr{N}}$ converges weakly to $\mathbb{E}\nu_N$ in $L^2((0,T)\times\mathbb{R}^D)$, while $\mathbb{E}\rho_{\varepsilon,\mathscr{N}}$ converges strongly in $L^2((0,T)\times\mathbb{R}^D)$ and a.e. to $\mathbb{E}\rho_{\mathscr{N}}$. Since σ is continuous and bounded from below, $1/\sigma(\mathbb{E}\rho_{\varepsilon,\mathscr{N}})$ converges to $1/\sigma(\mathbb{E}\rho_{\mathscr{N}})$ a.e. too, and it is bounded in $L^\infty((0,T)\times\mathbb{R}^D)$. We deduce that

$$\frac{\mathbb{E}\nu_{\varepsilon,\mathscr{N}}}{\sigma(\mathbb{E}\rho_{\varepsilon,\mathscr{N}})}\rightharpoonup\frac{\mathbb{E}\nu_{\mathscr{N}}}{\sigma(\mathbb{E}\rho_{\mathscr{N}})}\quad\text{weakly in } L^2((0,T)\times\mathbb{R}^D).$$

We are left with the task of proving that the last term in the right hand side of (3-10) tends to 0 as $\mathscr{N}\to\infty$, uniformly with respect to ε. The Cauchy–Schwarz inequality yields

$$\begin{aligned}|\mathbb{E}\,r_{\varepsilon,\mathscr{N}}|&\le(\mathbb{E}[(\nu_{\varepsilon,\mathscr{N}})^2])^{1/2}\left(\mathbb{E}\left[\left(\frac{1}{\sigma(\rho_{\varepsilon,\mathscr{N}})}-\frac{1}{\sigma(\mathbb{E}\rho_{\varepsilon,\mathscr{N}})}\right)^2\right]\right)^{1/2}\\&\le(\mathbb{E}\,[(\nu_{\varepsilon,\mathscr{N}})^2])^{1/2}\left(\mathbb{E}\left[\left(\int_{\mathbb{E}\rho_{\varepsilon,\mathscr{N}}}^{\rho_{\varepsilon,\mathscr{N}}}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\sigma(z)}\right]\mathrm{d}z\right)\right]^2\right)^{1/2}\\&\le(\mathbb{E}\,[(\nu_{\varepsilon,\mathscr{N}})^2])^{1/2}\left(\mathbb{E}\,[(\rho_{\varepsilon,\mathscr{N}}-\mathbb{E}\rho_{\varepsilon,\mathscr{N}}^2])\right)^{1/2}\\&\le(\mathbb{E}\,[(\nu_{\varepsilon,\mathscr{N}})^2])^{1/2}\left(\mathbb{E}\left[\left(\frac{1}{2\mathscr{N}}\sum_{i=1}^{2\mathscr{N}}f_\varepsilon(V_i)-\mathbb{E}\rho_{\varepsilon,\mathscr{N}}\right)^2\right]\right)^{1/2}.\end{aligned}\tag{3-11}$$

We remind the reader that the $2\mathscr{N}$ velocities are constructed by symmetry from $V_1,\dots,V_{\mathscr{N}}$, which are i.i.d. velocities in $[-0.5,0.5]^D$, and we write

$$\begin{aligned}&\mathbb{E}\left[12\mathscr{N}\sum_{i=1}^{2\mathscr{N}}f_\varepsilon(V_i)-\mathbb{E}\rho_{\varepsilon,\mathscr{N}}\right]^2\\&\qquad=\mathbb{E}\left[\frac{1}{4\mathscr{N}^2}\sum_{i,j=1}^{\mathscr{N}}\left\{\left(f_\varepsilon(V_i)+f_\varepsilon(-V_i)-2\mathbb{E}\rho_{\varepsilon,\mathscr{N}}\right)\left(f_\varepsilon(V_j)+f_\varepsilon(-V_j)-2\mathbb{E}\rho_{\varepsilon,\mathscr{N}}\right)\right\}\right].\end{aligned}\tag{3-12}$$

When $i \neq j$, we know V_i and V_j are independent, which implies

$$\mathbb{E}\big[(f_\varepsilon(V_i)+f_\varepsilon(-V_i)-2\mathbb{E}\rho_{\varepsilon,\mathcal{N}})(f_\varepsilon(V_j)+f_\varepsilon(-V_j)-2\mathbb{E}\rho_{\varepsilon,\mathcal{N}})\big]$$
$$=\mathbb{E}\big[f_\varepsilon(V_i)+f_\varepsilon(-V_i)-2\mathbb{E}\rho_{\varepsilon,\mathcal{N}}\big]\,\mathbb{E}\big[f_\varepsilon(V_j)+f_\varepsilon(-V_j)-2\mathbb{E}\rho_{\varepsilon,\mathcal{N}}\big].$$

Now, we use the fact that the V_i are identically distributed so that

$$\begin{aligned}2\mathbb{E}\rho_{\varepsilon,\mathcal{N}} &= 2\mathbb{E}\left(\frac{1}{2\mathcal{N}}\sum_{k=1}^{2\mathcal{N}} f_\varepsilon(V_k)\right)=\mathbb{E}\left(\frac{1}{\mathcal{N}}\sum_{k=1}^{\mathcal{N}}\big(f_\varepsilon(V_k)+f_\varepsilon(-V_k)\big)\right)\\ &=\frac{1}{\mathcal{N}}\sum_{k=1}^{\mathcal{N}}\big(\mathbb{E}f_\varepsilon(V_k)+\mathbb{E}f_\varepsilon(-V_k)\big)=\mathbb{E}f_\varepsilon(V_j)+\mathbb{E}f_\varepsilon(-V_j)\end{aligned}$$

for any $j\in\{1,\dots,\mathcal{N}\}$. It follows that

$$\mathbb{E}\big[f_\varepsilon(V_i)+f_\varepsilon(-V_i)-2\mathbb{E}\rho_{\varepsilon,\mathcal{N}})(f_\varepsilon(V_j)+f_\varepsilon(-V_j)-2\mathbb{E}\rho_{\varepsilon,\mathcal{N}})\big]=0\quad\text{when } i\neq j.$$

Going back to (3-12), we obtain

$$\mathbb{E}\left[\frac{1}{2\mathcal{N}}\sum_{i=1}^{2\mathcal{N}} f_\varepsilon(V_i)-\mathbb{E}\rho_{\varepsilon,\mathcal{N}}\right]^2=\mathbb{E}\left[\frac{1}{4\mathcal{N}^2}\sum_{i=1}^{\mathcal{N}}\big(f_\varepsilon(V_i)+f_\varepsilon(-V_i)-2\mathbb{E}\rho_{\varepsilon,\mathcal{N}}\big)^2\right].$$

Since f_ε and $\rho_{\varepsilon,\mathcal{N}}$ are uniformly bounded, we conclude that the estimate

$$\mathbb{E}\left[\frac{1}{2\mathcal{N}}\sum_{i=1}^{2\mathcal{N}} f_\varepsilon(V_i)-\mathbb{E}\rho_{\varepsilon,\mathcal{N}}\right]^2\leq\frac{C}{\mathcal{N}}$$

holds. Inserting this information in (3-11), we arrive at

$$\int_0^T\int_{\mathbb{R}^D}|\mathbb{E}r_{\varepsilon,\mathcal{N}}|^2\,\mathrm{d}x\,\mathrm{d}t\leq\frac{C}{\mathcal{N}}\,\mathbb{E}\int_0^T\int_{\mathbb{R}^D}\nu_{\varepsilon,\mathcal{N}}^2\,\mathrm{d}x\,\mathrm{d}t,$$

which is thus of order $O(1/\mathcal{N})$, uniformly with respect to ε.

Therefore, we can let ε run to 0 in (3-10) and, for a suitable subsequence, we are led to

$$\mathbb{E}J_{\varepsilon,\mathcal{N}}\rightharpoonup\mathbb{E}J_{\mathcal{N}}=\frac{\mathbb{E}\nu_{\mathcal{N}}}{\sigma(\mathbb{E}\rho_{\mathcal{N}})}+r_{\mathcal{N}}\quad\text{weakly in } L^2((0,T)\times\mathbb{R}^D)\text{ with }\|r_{\mathcal{N}}\|_{L^2((0,T)\times\mathbb{R}^D)}\leq\frac{C}{\sqrt{\mathcal{N}}}.$$

Finally, we take the expectation in (3-9) and we get

$$\mathbb{E}(A_{\mathcal{N}}\nabla_x\rho_{\mathcal{N}})=-\mathbb{E}\nu_{\mathcal{N}}=-\sigma(\mathbb{E}\rho_{\mathcal{N}})\mathbb{E}J_{\mathcal{N}}+\sigma(\mathbb{E}\rho_{\mathcal{N}})r_{\mathcal{N}}.$$

Note that the last term is still of order $O(1/\sqrt{\mathcal{N}})$ in the $L^2((0,T)\times\mathbb{R}^D)$ norm. By reasoning similar to that above, we check that, for any $i,j\in\{1,\dots,D\}$,

$$\sqrt{\mathbb{E}\big[\big([A_{\mathcal{N}}]_{ij}-\mathbb{E}[A_{\mathcal{N}}]_{ij}\big)^2\big]}=O\left(\frac{1}{\sqrt{\mathcal{N}}}\right)$$

(this is the standard result about Monte Carlo integration). It implies that we can find a constant $C > 0$, which only depends on the dimension D, such that for any $\xi \in \mathbb{R}^D$,

$$\mathbb{E}\big[\big|A_{\mathscr{N}}\xi - \mathbb{E}[A_{\mathscr{N}}\xi]\big|^2\big] \le \frac{C|\xi|^2}{\mathscr{N}}.$$

Then we get

$$\mathbb{E}(A_{\mathscr{N}}\nabla_x\rho_{\mathscr{N}}) = \mathbb{E}A_{\mathscr{N}}\nabla_x\mathbb{E}\rho_{\mathscr{N}} + s_{\mathscr{N}}, \quad s_{\mathscr{N}} = \mathbb{E}\big[(A_{\mathscr{N}} - \mathbb{E}A_{\mathscr{N}})\nabla_x\rho_{\mathscr{N}}\big].$$

The remainder term should be analyzed in a weak sense, due to a lack of a priori regularity of $\nabla_x\rho_{\mathscr{N}}$ (we only know that the product $A_{\mathscr{N}}\nabla_x\rho_{\mathscr{N}}$ lies in L^2, but the invertibility of $A_{\mathscr{N}}$ is not guaranteed). We have, for any $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^D)$,

$$\begin{aligned}\big|\langle\mathbb{E}s_{\mathscr{N}}|\varphi\rangle\big| &= \left|-\mathbb{E}\int_0^T\int_{\mathbb{R}^D}\rho_{\mathscr{N}}(A_{\mathscr{N}} - \mathbb{E}A_{\mathscr{N}})\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\le \left(\mathbb{E}\int_0^T\int_{\mathbb{R}^D}\rho_{\mathscr{N}}^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_{\mathbb{R}^D}|\nabla_x\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\frac{C}{\sqrt{\mathscr{N}}}.\end{aligned}$$

Owing to the estimates (3-3) in Proposition 3.3, it means that $s_{\mathscr{N}}$ is therefore of order $O(1/\sqrt{\mathscr{N}})$ in the $L^2(0, T; H^{-1}(\mathbb{R}^D))$-norm. □

Remark 3.6. The random matrix $A_{\mathscr{N}}$ might be singular. However $\mathbb{E}A_{\mathscr{N}}$ is invertible. Indeed for any $\xi \ne 0$, we have

$$\mathbb{E}A_{\mathscr{N}}\xi\cdot\xi = \frac{1}{2\mathscr{N}}\sum_{j=1}^{2\mathscr{N}}\mathbb{E}[|V_j\cdot\xi|^2] \ge 0.$$

This quantity is actually positive since $\mathbb{P}(v\cdot\xi = 0) = 0$ for the continuous laws we are dealing with.

4. Comments and perspectives

The Monte Carlo procedure is widely used to numerically evaluate multidimensional integrals, precisely because, evaluating the numerical effort by the number $\mathscr{N}$ of quadrature points, it provides a result with an accuracy of order $O(1/\sqrt{\mathscr{N}})$, independently of the space dimension, in contrast to the deterministic quadrature methods where the error is $O(\mathscr{N}^{-k/D})$, k being the order of the method; see [Caflisch 1998; Lapeyre et al. 1998, Chapitre 1]. Application of such stochastic quadrature approaches to the numerical treatment of kinetic models for neutron transport dates back to the Manhattan project [Metropolis and Ulam 1949]. For applications to radiative transfer computations we refer the reader, e.g., to [Campbell 1967] and for a more recent overview to [Whitney 2011]. After the pioneering works by K. Nanbu [1980] and G. A. Bird [1970], Monte Carlo techniques are at the basis of the simulation of the Boltzmann equation for rarefied gases. (By the way, note that the construction of a suitable deterministic quadrature formula for approximating the Boltzmann operator can be a bit tricky, with unexpected connections to subtle number theory arguments [Michel and Schneider 2000].) Very comprehensive introductions can be found in [Graham and Méléard 1999; Pareschi 2005; Pareschi and Russo 1999] and in the textbook [Lapeyre et al. 1998]. The method can naturally be presented as a particulate method; roughly speaking, it works according to a splitting approach [Lapeyre et al. 1998, Chapter 3]: first, particles (which, here, are "test" particles intended to actually represent a set of real particles) are displaced according to free transport over the time step Δt, and, second, the effects of the interaction between particles during the time step are evaluated by using a random sampling. Convergence of the method for the Boltzmann equation as the number of particles tends to ∞ is analyzed in [Graham and Méléard 1997; Pulvirenti et al. 1994; Wagner 1992; 2004]. However, the performance of Monte Carlo algorithms is known to degrade in near-continuum regimes, where the number of collision events per time unit increases; see [Caflisch 1998, §7; Lapeyre et al. 1998, §3.7.1 and §4.5]. This observation has motivated the development of hybrid methods [Dimarco and Pareschi 2008; Pareschi 2005].

As pointed out in the Introduction, the average lemma plays a central role in the analysis of nonlinear kinetic models and their hydrodynamic limits, with fundamental obstructions in extending to discrete velocity models. We expect that the stochastic average lemma established here might help in analyzing stochastic algorithms for kinetic models. Our first attempt remains at the level of space-time continuous models for the simplest radiative transfer equation: it is just a consistency result with the diffusion approximation. It is remarkable that the consistency error preserves the typical feature of the Monte Carlo error estimate in $O(1/\sqrt{\mathscr{N}})$, independently of the space dimension. A next step, likely inspired by the "time-discretized" version of the averaging lemma in [Bouchut and Desvillettes 1999; Horsin et al. 2003], would be to consider time-discretized models, where the random velocity grid is reconstructed at each time step.

References

[Agoshkov 1984] V. I. Agoshkov, "Spaces of functions with differential-difference characteristics and the smoothness of solutions of the transport equation", *Dokl. Akad. Nauk SSSR* **276**:6 (1984), 1289–1293. In Russian; translated in *Sov. Math. Dokl.* **29** (1984), 662–666. MR Zbl

[Alonso et al. 2017] A. Alonso, T. Goudon, and A. Vavasseur, "Damping of particles interacting with a vibrating medium", *Ann. Inst. H. Poincaré Anal. Non Linéaire* (online publication January 2017).

[Bardos et al. 1988] C. Bardos, F. Golse, B. Perthame, and R. Sentis, "The nonaccretive radiative transfer equations: existence of solutions and Rosseland approximation", *J. Funct. Anal.* **77**:2 (1988), 434–460. MR Zbl

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer, 1976. MR Zbl

[Berthelin and Junca 2010] F. Berthelin and S. Junca, "Averaging lemmas with a force term in the transport equation", *J. Math. Pures Appl.* (9) **93**:2 (2010), 113–131. MR Zbl

[Bird 1970] G. A. Bird, "Direct simulation and the Boltzmann equation", *Phys. Fluids* **13**:11 (1970), 2676–2687. Zbl

[Bonnaillie-Noël et al. 2016] V. Bonnaillie-Noël, J. A. Carrillo, T. Goudon, and G. A. Pavliotis, "Efficient numerical calculation of drift and diffusion coefficients in the diffusion approximation of kinetic equations", *IMA J. Numer. Anal.* **36**:4 (2016), 1536–1569. MR

[Bouchut and Desvillettes 1999] F. Bouchut and L. Desvillettes, "Averaging lemmas without time Fourier transform and application to discretized kinetic equations", *Proc. Roy. Soc. Edinburgh Sect. A* **129**:1 (1999), 19–36. MR Zbl

[Buet et al. 2002] C. Buet, S. Cordier, B. Lucquin-Desreux, and S. Mancini, "Diffusion limit of the Lorentz model: asymptotic preserving schemes", *M2AN Math. Model. Numer. Anal.* **36**:4 (2002), 631–655. MR Zbl

[Caflisch 1998] R. E. Caflisch, "Monte Carlo and quasi-Monte Carlo methods", *Acta Numer.* **7** (1998), 1–49. MR Zbl

[Campbell 1967] P. M. Campbell, "Monte Carlo method for radiative transfer", *Int. J. Heat and Mass Transfer* **10**:4 (1967), 519–527.

[Debussche et al. 2015] A. Debussche, S. De Moor, and J. Vovelle, "Diffusion limit for the radiative transfer equation perturbed by a Wiener process", *Kinet. Relat. Models* **8**:3 (2015), 467–492. MR Zbl

[Debussche et al. 2016] A. Debussche, S. De Moor, and J. Vovelle, "Diffusion limit for the radiative transfer equation perturbed by a Markovian process", *Asymptot. Anal.* **98**:1-2 (2016), 31–58. MR Zbl

[Degond et al. 2000] P. Degond, T. Goudon, and F. Poupaud, "Diffusion limit for nonhomogeneous and non-micro-reversible processes", *Indiana Univ. Math. J.* **49**:3 (2000), 1175–1198. MR Zbl

[Dimarco and Pareschi 2008] G. Dimarco and L. Pareschi, "Hybrid multiscale methods, II: Kinetic equations", *Multiscale Model. Simul.* **6**:4 (2008), 1169–1197. MR Zbl

[DiPerna and Lions 1989a] R. J. DiPerna and P.-L. Lions, "Global weak solutions of Vlasov–Maxwell systems", *Comm. Pure Appl. Math.* **42**:6 (1989), 729–757. MR Zbl

[DiPerna and Lions 1989b] R. J. DiPerna and P.-L. Lions, "On the Cauchy problem for Boltzmann equations: global existence and weak stability", *Ann. of Math.* (2) **130**:2 (1989), 321–366. MR Zbl

[DiPerna et al. 1991] R. J. DiPerna, P.-L. Lions, and Y. Meyer, "L^p regularity of velocity averages", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8**:3-4 (1991), 271–287. MR Zbl

[Golse 2000] F. Golse, "From kinetic to macroscopic models", pp. 41–121 in *Kinetic equations and asymptotic theory*, edited by B. Perthame and L. Desvillettes, Series in Applied Mathematics **4**, Gauthier-Villars, Paris, 2000. MR Zbl

[Golse and Saint-Raymond 2002] F. Golse and L. Saint-Raymond, "Velocity averaging in L^1 for the transport equation", *C. R. Math. Acad. Sci. Paris* **334**:7 (2002), 557–562. MR Zbl

[Golse and Saint-Raymond 2004] F. Golse and L. Saint-Raymond, "The Navier–Stokes limit of the Boltzmann equation for bounded collision kernels", *Invent. Math.* **155**:1 (2004), 81–161. MR Zbl

[Golse et al. 1988] F. Golse, P.-L. Lions, B. t. Perthame, and R. Sentis, "Regularity of the moments of the solution of a transport equation", *J. Funct. Anal.* **76**:1 (1988), 110–125. MR Zbl

[Golse et al. 1999] F. Golse, S. Jin, and C. D. Levermore, "The convergence of numerical transfer schemes in diffusive regimes, I: Discrete-ordinate method", *SIAM J. Numer. Anal.* **36**:5 (1999), 1333–1369. MR Zbl

[Goudon 2011] T. Goudon, *Intégration: intégrale de Lebesgue et introduction à l'analyse fonctionnelle*, Ellipses, Paris, 2011. Zbl

[Goudon and Poupaud 2001] T. Goudon and F. Poupaud, "Approximation by homogenization and diffusion of kinetic equations", *Comm. Partial Differential Equations* **26**:3-4 (2001), 537–569. MR Zbl

[Graham and Méléard 1997] C. Graham and S. Méléard, "Stochastic particle approximations for generalized Boltzmann models and convergence estimates", *Ann. Probab.* **25**:1 (1997), 115–132. MR Zbl

[Graham and Méléard 1999] C. Graham and S. Méléard, "Probabilistic tools and Monte-Carlo approximations for some Boltzmann equations", pp. 77–126 in *CEMRACS 1999* (Orsay, 1999), edited by F. Coquel and S. Cordier, ESAIM Proc. **10**, Soc. Math. Appl. Indust., Paris, 1999. MR Zbl

[Horsin et al. 2003] T. Horsin, S. Mischler, and A. Vasseur, "On the convergence of numerical schemes for the Boltzmann equation", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20**:5 (2003), 731–758. MR Zbl

[Jin and Levermore 1991] S. Jin and D. Levermore, "The discrete-ordinate method in diffusive regimes", *Transport Theory Statist. Phys.* **20**:5-6 (1991), 413–439. MR Zbl

[Lapeyre et al. 1998] B. Lapeyre, E. Pardoux, and R. Sentis, *Méthodes de Monte-Carlo pour les équations de transport et de diffusion*, Mathématiques & Applications **29**, Springer, 1998. MR Zbl

[Lions and Toscani 1997] P. L. Lions and G. Toscani, "Diffusive limit for finite velocity Boltzmann kinetic models", *Rev. Mat. Iberoamericana* **13**:3 (1997), 473–513. MR Zbl

[Lions et al. 2013] P.-L. Lions, B. t. Perthame, and P. E. Souganidis, "Stochastic averaging lemmas for kinetic equations", exposé 26 , 17 p. in *Séminaire Laurent Schwartz: équations aux dérivées partielles et applications,* 2011–2012, École Polytech., Palaiseau, 2013. MR Zbl

[Marcati and Milani 1990] P. Marcati and A. Milani, "The one-dimensional Darcy's law as the limit of a compressible Euler flow", *J. Differential Equations* **84**:1 (1990), 129–147. MR Zbl

[Metropolis and Ulam 1949] N. Metropolis and S. Ulam, "The Monte Carlo method", *J. Amer. Statist. Assoc.* **44** (1949), 335–341. MR Zbl

[Michel and Schneider 2000] P. Michel and J. Schneider, "Approximation simultanée de réels par des nombres rationnels et noyau de collision de l'équation de Boltzmann", *C. R. Acad. Sci. Paris Sér. I Math.* **330**:9 (2000), 857–862. MR Zbl

[Mischler 1997] S. Mischler, "Convergence of discrete-velocity schemes for the Boltzmann equation", *Arch. Rational Mech. Anal.* **140**:1 (1997), 53–77. MR Zbl

[Nanbu 1980] K. Nanbu, "Direct simulation scheme derived from the Boltzmann equation, I: Monocomponent gases", *J. Phys. Soc. Jpn.* **49**:5 (1980), 2042–2049.

[Pareschi 2005] L. Pareschi, "Hybrid multiscale methods for hyperbolic and kinetic problems", pp. 87–120 in *GRIP—Research Group on Particle Interactions* (Sophia Antipolis, France, 2005), edited by T. Goudon et al., ESAIM Proc. **15**, EDP Sci., Les Ulis, 2005. MR Zbl

[Pareschi and Russo 1999] L. Pareschi and G. Russo, "An introduction to Monte Carlo methods for the Boltzmann equation", pp. 35–76 in *CEMRACS 1999* (Orsay, 1999), edited by F. Coquel and S. Cordier, ESAIM Proc. **10**, Soc. Math. Appl. Indust., Paris, 1999. MR Zbl

[Perthame and Souganidis 1998] B. Perthame and P. E. Souganidis, "A limiting case for velocity averaging", *Ann. Sci. École Norm. Sup.* (4) **31**:4 (1998), 591–598. MR Zbl

[Pulvirenti et al. 1994] M. Pulvirenti, W. Wagner, and M. B. Zavelani Rossi, "Convergence of particle schemes for the Boltzmann equation", *European J. Mech. B Fluids* **13**:3 (1994), 339–351. MR Zbl

[Saint-Raymond 2009] L. Saint-Raymond, *Hydrodynamic limits of the Boltzmann equation*, Lecture Notes in Mathematics **1971**, Springer, 2009. MR Zbl

[Tadmor and Tao 2007] E. Tadmor and T. Tao, "Velocity averaging, kinetic formulations, and regularizing effects in quasi-linear PDEs", *Comm. Pure Appl. Math.* **60**:10 (2007), 1488–1521. MR Zbl

[Villani 2002] C. Villani, "Limites hydrodynamiques de l'équation de Boltzmann", exposé 893 , pp. 365–405 in *Séminaire Bourbaki,* 2000/2001, Astérisque **282**, Soc. Mat. de France, Paris, 2002. MR Zbl

[Wagner 1992] W. Wagner, "A convergence proof for Bird's direct simulation Monte Carlo method for the Boltzmann equation", *J. Statist. Phys.* **66**:3-4 (1992), 1011–1044. MR Zbl

[Wagner 2004] W. Wagner, "Stochastic models and Monte Carlo algorithms for Boltzmann type equations", pp. 129–153 in *Monte Carlo and quasi-Monte Carlo methods 2002* (Singapore, 2002), edited by H. Niederreiter, Springer, 2004. MR Zbl

[Whitney 2011] B. A. Whitney, "Monte Carlo radiative transfer", *Bull. Astr. Soc. India* **39**:1 (2011), 101–127.

Received 7 Nov 2016. Revised 12 Feb 2017. Accepted 28 Mar 2017.

NATHALIE AYI: nathalie.ayi@inria.fr
Inria Rennes-Bretagne Atlantique, IPSO, Research team, IRMAR, UMR CNRS 6625, Campus de Beaulieu, Bâtiment 22/23, 263 Avenue du Général Leclerc, 35042 Rennes, France

THIERRY GOUDON: thierry.goudon@inria.fr
Université Côte d'Azur, Inria, CNRS, LJAD, Parc Valrose, 06108 Nice, France

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2017 is US $265/year for the electronic version, and $470/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 10 No. 5 2017