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A SPARSE DOMINATION PRINCIPLE FOR ROUGH SINGULAR INTEGRALS

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We prove that bilinear forms associated to the rough homogeneous singular integrals

$$T_\Omega f(x) = \mathrm{p.v.}\int_{\mathbb{R}^d} f(x-y)\Omega\left(\frac{y}{|y|}\right)\frac{\mathrm{d}y}{|y|^d},$$

where $\Omega \in L^q(S^{d-1})$ has vanishing average and $1 < q \le \infty$, and to Bochner–Riesz means at the critical index in $\mathbb{R}^d$ are dominated by sparse forms involving $(1, p)$ averages. This domination is stronger than the weak-L^1 estimates for T_Ω and for Bochner–Riesz means, respectively due to Seeger and Christ. Furthermore, our domination theorems entail as a corollary new sharp quantitative A_p-weighted estimates for Bochner–Riesz means and for homogeneous singular integrals with unbounded angular part, extending previous results of Hytönen, Roncal and Tapiola for T_Ω. Our results follow from a new abstract sparse domination principle which does not rely on weak endpoint estimates for maximal truncations.

1. Introduction and main results

Singular integral operators of Calderón–Zygmund type, which are a priori *signed* and *nonlocal*, can be dominated in norm [Lerner 2013], pointwise [Conde-Alonso and Rey 2016; Lacey 2017; Lerner and Nazarov 2015], or dually [Bernicot et al. 2016; Culiuc et al. 2016a; 2016b] by sparse averaging operators (forms), which are in contrast *positive* and *localized*. For $1 \le p_1, p_2 < \infty$, we define the *sparse* (p_1, p_2)*-averaging form* to be the bisublinear form

$$\mathsf{PSF}_{\mathcal{S};p_1,p_2}(f_1, f_2) := \sum_{Q\in\mathcal{S}} |Q|\langle f_1\rangle_{p_1,Q}\langle f_2\rangle_{p_2,Q}, \quad \langle f\rangle_{p,Q} := |Q|^{-\frac{1}{p}}\|f\mathbf{1}_Q\|_p,$$

associated to a (countable) sparse collection $\mathcal{S}$ of cubes of $\mathbb{R}^d$. The collection $\mathcal{S}$ is η-sparse if there exist $0 < \eta \le 1$ (a number which will not play a relevant role) and measurable, pairwise disjoint sets $\{E_I : I \in \mathcal{S}\}$ such that

$$E_I \subset I, \quad |E_I| \ge \eta|I|.$$

In this article, we prove a sparse domination principle of type

$$|\langle Tf_1, f_2\rangle| \lesssim \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};p_1,p_2}(f_1, f_2) \tag{1-1}$$

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for singular integral operators T whose (possible) lack of kernel smoothness forbids the avenue exploited in [Lacey 2017; Lerner 2016]. Our principle, summarized in Theorem C below, can be employed in a rather direct fashion to recover the best known, and sharp, sparse domination results for Dini- and Hörmander-type Calderón–Zygmund operators [Bui et al. 2017; Hytönen et al. 2017; Lacey 2017; Volberg and Zorin-Kranich 2016].

However, the main purpose of our work is to suitably extend (1-1) to the class of rough singular integrals introduced in the seminal paper of Calderón and Zygmund [1956], and further studied, notably, in [Duoandikoetxea and Rubio de Francia 1986; Christ 1988; Christ and Rubio de Francia 1988; Seeger 1996]. Prime examples from this class include the rough homogeneous singular integrals on $\mathbb{R}^d$

$$T_\Omega f(x) = \mathrm{p.v.}\int_{\mathbb{R}^d} f(x-y)\Omega\left(\frac{y}{|y|}\right)\frac{\mathrm{d}y}{|y|^d}, \tag{1-2}$$

with $\Omega \in L^q(S^{d-1})$ having zero average, as well as the critical Bochner–Riesz means in dimension d, defined by the multiplier operator

$$B_\delta f = \mathcal{F}^{-1}\big[\hat{f}(\cdot)(1-|\cdot|^2)_+^\delta\big], \quad \delta = \frac{d-1}{2}. \tag{1-3}$$

For the singular integrals (1-2) no sparse domination results were known prior to this article, although some quantitative weighted estimates were established in the recent works [Hytönen et al. 2017; Pérez et al. 2016]; see below for details. For the Bochner–Riesz means (1-3), the recent results of [Benea et al. 2017] and [Carro and Domingo-Salazar 2016] are far from being optimal at the critical exponent.

The main difficulty encountered by previous approaches in this setting is the following: first, notice that an estimate of the type (1-1) is already stronger than the weak-L^{p_1} bound for T. In particular, if $p_1 = 1$ then (1-1) recovers the weak-L^1 endpoint bound. On the other hand, the preexisting techniques for sparse domination [Benea et al. 2017; Bernicot et al. 2016; Hytönen et al. 2017; Lacey 2017; Lerner 2016] essentially rely on weak-L^p estimates for a grand maximal truncation of the singular integral operator T, but those do not seem attainable in the context, for instance, of [Seeger 1996], as observed in [Lerner 2016]. In fact, the rough singular integrals we consider below are not known to satisfy such an estimate for $p = 1$, and therefore a different approach is required in order to obtain the sparse bounds that we want.

As a corollary of our domination results, we obtain quantitative A_p-weighted estimates for homogeneous singular integrals (1-2) whose angular part belongs to $L^q(S^{d-1})$ for some $1 < q \le \infty$. These are novel, and sharp, when $q < \infty$, while in the case $q = \infty$ we recover the best known result recently proved in [Hytönen et al. 2017] by other methods. Although our result for the Bochner–Riesz means (1-3) seemingly yields the best known quantitative A_p estimates, we do not know whether our results are sharp in this case.

Main results. Our main results consist of estimates for the bilinear forms associated to T_Ω and B_δ by sparse operators involving L^p-averages. The formulation of our first theorem requires the Orlicz–Lorentz norms

$$\|\Omega\|_{L^{q,1}\log L(S^{d-1})} := q\int_0^\infty t\log(\mathrm{e}+t)\big|\{\theta \in S^{d-1} : |\Omega(\theta)| > t\}\big|^{\frac{1}{q}}\frac{\mathrm{d}t}{t}, \quad 1 \le q < \infty.$$

Theorem A. *There exists an absolute dimensional constant $C > 0$ such that the following holds. Let $\Omega \in L^1(S^{d-1})$ have zero average. Then for all $1 < t < \infty$, $f_1 \in L^t(\mathbb{R}^d)$, $f_2 \in L^{t'}(\mathbb{R}^d)$, we have*

$$|\langle T_\Omega f_1, f_2\rangle| \le \frac{Cp}{p-1} \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};1,p}(f_1, f_2) \begin{cases} \|\Omega\|_{L^{q,1}\log L(S^{d-1})}, & 1 < q < \infty,\ p \ge q', \\ \|\Omega\|_{L^\infty(S^{d-1})}, & 1 < p < \infty. \end{cases}$$

Remark 1.1. To avoid Lorentz norms in the statement, one may recall the continuous embeddings $L^{q+\varepsilon}(S^{d-1}) \hookrightarrow L^{q,1}\log L(S^{d-1}) \hookrightarrow L^q(S^{d-1})$ for all $1 \le q < \infty$ and $\varepsilon > 0$.

Theorem B. *There exists an absolute dimensional constant $C > 0$ such that the following holds. For all $1 < t < \infty$, $f_1 \in L^t(\mathbb{R}^d)$, $f_2 \in L^{t'}(\mathbb{R}^d)$, the critical Bochner–Riesz means (1-3) satisfy*

$$|\langle B_\delta f_1, f_2\rangle| \le \frac{Cp}{p-1} \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};1,p}(f_1, f_2), \quad 1 < p < \infty.$$

The weak-L^1 estimate for T_Ω is the main result of [Seeger 1996], while the same endpoint estimate for (1-3) has been established in [Christ 1988]. Theorems A and B recover such results; see Appendix B for a proof of this implication, which we include for future reference. This is not surprising as the localized estimates for (1-2), (1-3) which are needed to apply our abstract result are a distillation and an improvement of the microlocal techniques of [Seeger 1996] and of the previous works [Christ 1988; Christ and Rubio de Francia 1988], and of the oscillatory integral estimates of [Christ 1988] respectively.

We reiterate that the commonly used techniques for sparse domination, which rely on the weak-L^1 estimate for the maximal truncation of the singular integral operator, fail to be applicable in the context of Theorem A as the maximal truncations of T_Ω in (1-2) are not known to satisfy such an estimate even when $\Omega \in L^\infty(S^{d-1})$ [Grafakos and Stefanov 1999]. Our abstract result, Theorem C, whose statement is more technical and is postponed until Section 2, only relies on the uniform L^2-boundedness (or L^r-boundedness for any r) of the truncated operators, and thus might be considered stronger than the approaches of the mentioned references. See Remark 2.5 for additional discussion on this point.

Theorems A and B give as corollaries a family of quantitative weighted estimates.

Corollary A.1. *If Ω lies in the unit ball of $L^{q,1}\log L(S^{d-1})$ for some $1 < q < \infty$ and has zero average, we have the weighted norm inequalities*

$$\|T_\Omega\|_{L^t(w)\to L^t(w)} \le C_{t,q}[w]_{A_{\frac{t}{q'}}}^{\max\{1,\frac{1}{t-q'}\}}, \quad q' < t < \infty. \tag{1-4}$$

If furthermore $\|\Omega\|_{L^\infty(S^{d-1})} \le 1$,

$$\|T_\Omega\|_{L^t(w)\to L^t(w)} \le C_t[w]_{A_t}^{\frac{1}{t-1}\max\{t,2\}}, \quad 1 < t < \infty. \tag{1-5}$$

Corollary B.1. *Referring to (1-3), we have the weighted norm inequalities*

$$\|B_\delta\|_{L^t(w)\to L^t(w)} \le C_t[w]_{A_t}^{\frac{1}{t-1}\max\{t,2\}}, \quad 1 < t < \infty. \tag{1-6}$$

Proof of Corollaries A.1, B.1. To prove (1-4), applying Theorem A for $p = q'$ (strictly speaking, to the adjoint of T_Ω) yields that the bilinear form associated to T_Ω is dominated by

$$\sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};q',1}.$$

The proof of the weighted estimate can then be found, for instance, in [Bernicot et al. 2016, Proposition 6.4]. We prove (1-5), and (1-6) follows via the same argument: below, C denotes a positive absolute constant which may vary between occurrences. Combining the inequality [Di Plinio and Lerner 2014, Proposition 4.1]

$$\langle f\rangle_{1+\varepsilon,Q} \le \langle f\rangle_{1,Q} + C\varepsilon\langle \mathrm{M}_{1+\varepsilon} f\rangle_{1,Q},$$

which is valid for all $\varepsilon > 0$, with the estimate of Theorem A for $p = 1+\varepsilon$ we obtain

$$|\langle T_\Omega f_1, f_2\rangle| \le \frac{C}{\varepsilon} \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};1,1}(f_1, f_2) + C \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};1,1}(\mathrm{M}_{1+\varepsilon} f_1, f_2), \quad \varepsilon > 0.$$

The above display leads via standard reasoning [Cruz-Uribe et al. 2011; Hytönen et al. 2012; Moen 2012] to the chain of inequalities

$$\begin{aligned} \|T\|_{L^t(w)\to L^t(w)} &\le C_t [w]_{A_t}^{\max\{1,\frac{1}{t-1}\}} \inf_{0<\varepsilon<t-1} \left(\frac{1}{\varepsilon} + \|\mathrm{M}_{1+\varepsilon}\|_{L^t(w)\to L^t(w)}\right) \\ &\le C_t [w]_{A_t}^{\max\{1,\frac{1}{t-1}\}} \inf_{0<\varepsilon<t-1} \left(\frac{1}{\varepsilon} + [w]_{A_{\frac{t}{1+\varepsilon}}}^{\frac{1+\varepsilon}{t-(1+\varepsilon)}}\right) \le C_t [w]_{A_t}^{\frac{1}{t-1}\max\{t,2\}}. \qquad \square \end{aligned}$$

Corollary A.1 is a quantification of the weighted inequalities due to Watson [1990] and Duoandikoetxea [1993]: if $1 < q \le \infty$ and $\Omega \in L^q(S^{d-1})$ then

$$\left.\begin{array}{ll} w \in A_{\frac{t}{q'}}, & q' \le t < \infty,\ t \ne 1, \\ w^{\frac{1}{1-t}} \in A_{\frac{t'}{q'}}, & 1 < t \le q,\ t \ne \infty, \\ w^{q'} \in A_t, & 1 < t < \infty \end{array}\right\} \implies \|T_\Omega\|_{L^t(w)\to L^t(w)} < \infty.$$

Estimate (1-5) was first established by Hytönen, Roncal and Tapiola [Hytönen et al. 2017] via a different two-step technique involving sparse domination for Dini-type kernels, a Littlewood–Paley decomposition along the lines of [Christ and Rubio de Francia 1988] and interpolation with change of measure. In [Pérez et al. 2016], these ideas were extended to obtain A_1 estimates for T_Ω and commutators of T_Ω and BMO symbols. At this time, we do not know whether the power of the Muckenhoupt constant in (1-5) is sharp.

Qualitative A_p-bounds for critical Bochner–Riesz means are classical [Shi and Sun 1992]; see also [Vargas 1996]. On the other hand, Corollary B.1 seems to be the first quantitative A_p estimate for B_δ. We do not know whether the power of the A_p constant in (1-6) is sharp; the construction in [Luque et al. 2015, Corollary 3.1] shows that the optimal power α_p must obey $\alpha_p \ge \max\{1, 1/(p-1)\}$. The article [Benea et al. 2017] contains sparse domination estimates and weighted inequalities for the supercritical regime $0 < \delta' < \delta$ which are not informative in the critical case. An extension of our methods to the supercritical cases will appear in forthcoming work.

Finally, we mention that our argument for (1-5) and (1-6) shows that improvements of powers as those in Corollaries A.1 and B.1 are tied to the blowup rate as $p \to 1^+$ of the main estimate of Theorems A and B.

A remark on the proof and plan of the article. Theorems A and B fall under the scope of the same abstract result, Theorem C, which is stated and proved in Section 2. Theorem C is obtained by means of an iterative scheme reminiscent of the arguments used in [Culiuc et al. 2016a] by three of us to prove a sparse domination estimate for the bilinear Hilbert transform, and later adapted to dyadic and continuous Calderón–Zygmund singular integrals in [Culiuc et al. 2016b]. At each iteration, a decomposition of Calderón–Zygmund type is performed, and the operator itself is decomposed into small scales (scales falling within the exceptional set) which will be estimated at subsequent steps of the iteration, and large scales. The action of the large scales on the good parts is controlled by means of the uniform L^r-bound for the truncations of T. The contribution of the bad, mean zero part under the large scales of the operator is then controlled by means of suitably localized estimates relying on the cancellation of *constant-mean zero* type. We emphasize that the present work shares a perspective based on bilinear forms with other recent papers: [Krause and Lacey 2016; Lacey and Spencer 2017]. The notable difference is that these references, dealing with oscillatory and random discrete singular integrals, use (dilation) symmetry breaking and TT^*, rather than constant-mean zero, as the principal cancellation mechanisms, in accordance with the oscillatory nature of their objects of study.

Section 3 contains localized estimates for kernels of Dini- and Hörmander-type which, besides being of use in later arguments, allow us to reprove the optimal sparse domination results for these classes; see its last subsection for the statements. In Sections 4 and 5 we provide the necessary localized estimates for Theorems A and B respectively. The estimates of Section 4 are a delicate strengthening of the microlocal arguments of [Seeger 1996]. The proof of Theorem B, a re-elaboration along the same lines as the arguments of [Christ 1988], is carried out in Section 5. Although we find it hard to believe that these techniques can be sharpened towards the stronger localized $(1,1)$ estimate, we have no explicit counterexample for this possibility.

Notation. As is customary, $q' = \frac{q}{q-1}$ denotes the Lebesgue dual exponent to $q \in (1,\infty)$, with the usual extension $1' = \infty$, $\infty' = 1$. We denote the center of a cube $Q \in \mathbb{R}^d$ by c_Q and its sidelength by $\ell(Q)$. We will also adopt the shorthand $s_Q = \log_2 \ell(Q)$. We write

$$\mathrm{M}_p(f)(x) = \sup_{Q \subset \mathbb{R}^d} \langle f \rangle_{p,Q} \mathbf{1}_Q(x)$$

for the p-Hardy Littlewood maximal function. The positive constants implied by the almost inequality sign $\lesssim$ may depend (exponentially) on the dimension d only and may vary from line to line without explicit mention.

2. A sparse domination principle

This section is dedicated to the statement and proof of our sparse domination principle, Theorem C.

The main structural assumptions. Our structural assumptions in Theorem C will be the following. Let $1 < r < \infty$ and Λ be an $L^r(\mathbb{R}^d) \times L^{r'}(\mathbb{R}^d)$-bounded bilinear form whose kernel $K = K(x,y)$ coincides with a function away from the diagonal $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$. More precisely, whenever $f_1 \in L^r(\mathbb{R}^d)$,

$f_2 \in L^{r'}(\mathbb{R}^d)$ are compactly and disjointly supported

$$\Lambda(f_1, f_2) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y) f_1(y)\,\mathrm{d}y\, f_2(x)\,\mathrm{d}x$$

with absolute convergence of the integral. We assume that there exists $1 < q \le \infty$ such that the kernel K of Λ admits the decomposition

$$\begin{gathered} K(x,y) = \sum_{s\in\mathbb{Z}} K_s(x,y), \quad \operatorname{supp} K_s \subset \{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : x-y\in A_s\}, \\ A_s := \{z\in\mathbb{R}^d : 2^{s-2} < |z| < 2^s\}, \quad [K]_{0,q} := \sup_{s\in\mathbb{Z}} 2^{\frac{sd}{q'}} \sup_{x\in\mathbb{R}^d} \big(\|K_s(x, x+\cdot)\|_q + \|K_s(x+\cdot, x)\|_q\big) < \infty. \end{gathered} \tag{SS}$$

Further, we assume that the truncated forms associated to the above decomposition by

$$\Lambda_\mu^\nu(h_1,h_2) := \int \sum_{\mu<s\le\nu} K_s(x,y) h_1(y) h_2(x)\,\mathrm{d}y\,\mathrm{d}x \quad \mu,\nu\in\mathbb{Z}\cup\{-\infty,\infty\} \tag{2-1}$$

satisfy

$$C_{\mathsf{T}}(r) =: \sup_{\mu<\nu} \big(\|\Lambda_\mu^\nu\|_{L^r(\mathbb{R}^d)\times L^{r'}(\mathbb{R}^d)\to\mathbb{C}}\big) < \infty. \tag{T}$$

Remark 2.1. Under the assumptions (SS) and (T), a standard limiting argument [Stein 1993, Paragraph I.7.2] yields that

$$\Lambda(f_1, f_2) = \langle m f_1, \bar{f_2}\rangle + \lim_{\nu\to\infty} \Lambda_{-\nu}^{\nu}(f_1, f_2)$$

for some $m \in L^\infty(\mathbb{R}^d)$, whenever $f_1 \in L^r(\mathbb{R}^d)$, $f_2 \in L^{r'}(\mathbb{R}^d)$. It is not hard to see [Lacey and Mena Arias 2017, Lemma 4.7] that

$$|\langle m f_1, f_2\rangle| \lesssim \|m\|_\infty \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};1,1}(f_1, f_2)$$

so that for the purpose of our Theorem C below we may assume that $m = 0$ in the above equality. For this reason, when $\mu = -\infty$ or $\nu = \infty$ or both, we are allowed to omit the subscript or superscript in (2-1) and simply write Λ^ν or Λ_μ or Λ. Also, when $\mu \ge \nu$, the summation in (2-1) is void, so that $\Lambda_\mu^\nu \equiv 0$.

Localized spaces over stopping collections. A further condition in our abstract theorem will involve local norms associated to *stopping collections* of (dyadic) cubes. Throughout the article, by *dyadic cubes* we refer to the elements of any fixed dyadic lattice $\mathcal{D}$ in $\mathbb{R}^d$.

Let $Q \in \mathcal{D}$ be a fixed dyadic cube in $\mathbb{R}^d$. A collection $\mathcal{Q} \subset \mathcal{D}$ of dyadic cubes is a *stopping collection* with top Q if the elements of $\mathcal{Q}$ are pairwise disjoint and contained in $3Q$,

$$L, L' \in \mathcal{Q},\ L\cap L' \ne \varnothing \implies L = L', \qquad L\in\mathcal{Q} \implies L \subset 3Q, \tag{2-2}$$

and enjoy the further separation properties

$$L, L' \in \mathcal{Q},\ |s_L - s_{L'}| \ge 8 \implies 7L\cap 7L' = \varnothing, \qquad \bigcup_{L\in\mathcal{Q}: 3L\cap 2Q\ne\varnothing} 9L \subset \bigcup_{L\in\mathcal{Q}} L =: \mathsf{sh}\,\mathcal{Q}; \tag{2-3}$$

the notation $\mathsf{sh}\,\mathcal{Q}$ for the union of the cubes in $\mathcal{Q}$ will also be used below. For $1 \le p \le \infty$, define $\mathcal{Y}_p(\mathcal{Q})$ to be the subspace of $L^p(\mathbb{R}^d)$ of functions satisfying

$$\operatorname{supp} h \subset 3Q, \quad \infty > \|h\|_{\mathcal{Y}_p(\mathcal{Q})} := \begin{cases} \max\bigl\{\|h\mathbf{1}_{\mathbb{R}^d \setminus \mathsf{sh}\,\mathcal{Q}}\|_\infty,\ \sup_{L\in\mathcal{Q}} \inf_{x\in\widehat{L}} \mathrm{M}_p h(x)\bigr\}, & p<\infty,\\ \|h\|_\infty, & p=\infty,\end{cases}$$

where $\widehat{L}$ is the (nondyadic) 2^5-fold dilate of L. We also denote by $\mathcal{X}_p(\mathcal{Q})$ the subspace of $\mathcal{Y}_p(\mathcal{Q})$ of functions satisfying

$$b = \sum_{L\in\mathcal{Q}} b_L, \quad \operatorname{supp} b_L \subset L.$$

Furthermore, we write $b \in \dot{\mathcal{X}}_p(\mathcal{Q})$ if

$$b \in \mathcal{X}_p(\mathcal{Q}), \qquad \int_L b_L = 0 \quad \forall L \in \mathcal{Q}.$$

We will use the notation $\|b\|_{\mathcal{X}_p(\mathcal{Q})}$ for $\|b\|_{\mathcal{Y}_p(\mathcal{Q})}$ when $b \in \mathcal{X}_p(\mathcal{Q})$, and similar notation for $b \in \dot{\mathcal{X}}_p(\mathcal{Q})$. When the stopping collection $\mathcal{Q}$ is clear from the context or during proofs we may omit $(\mathcal{Q})$ from the subscript and simply write $\|\cdot\|_{\mathcal{Y}_p}$ or $\|\cdot\|_{\mathcal{X}_p}$.

Remark 2.2 (Calderón–Zygmund decomposition). There is a natural Calderón–Zygmund decomposition associated to stopping collections. Observe that if $\mathcal{Q}$ is a stopping collection, then

$$\sup_{L\in\mathcal{Q}} \langle h\rangle_{p,L} \le 2^{5d}\|h\|_{\mathcal{Y}_p(\mathcal{Q})}.$$

Therefore, we may decompose $h \in \mathcal{Y}_p(\mathcal{Q})$ as

$$h = g + b, \quad b = \sum_{L\in\mathcal{Q}} b_L \in \dot{\mathcal{X}}_p(\mathcal{Q}), \quad b_L = \left(h - \frac{1}{|L|}\int_L h(x)\,\mathrm{d}x\right)\mathbf{1}_L$$

such that

$$\|g\|_{\mathcal{Y}_\infty(\mathcal{Q})} \le 2^{5d}\|h\|_{\mathcal{Y}_p(\mathcal{Q})}, \quad \|b\|_{\dot{\mathcal{X}}_p(\mathcal{Q})} \le 2^{5d+1}\|h\|_{\mathcal{Y}_p(\mathcal{Q})}.$$

These are nothing but the usual properties of the Calderón-Zygmund decomposition rewritten in our context.

The statement. Before stating our result, we introduce the notation

$$\Lambda_{Q,\mu,\nu}(h_1,h_2) := \Lambda_\mu^{\min\{s_Q,\nu\}}(h_1\mathbf{1}_Q, h_2) = \Lambda_\mu^{\min\{s_Q,\nu\}}(h_1\mathbf{1}_Q, h_2\mathbf{1}_{3Q}) \tag{2-4}$$

for all dyadic cubes Q; the last equality in (2-4) is a consequence of the assumptions on the support of K_s in (SS). Furthermore, given a stopping collection $\mathcal{Q}$ with top Q, we define the truncated forms

$$\Lambda_{\mathcal{Q},\mu,\nu}(h_1,h_2) := \Lambda_{Q,\mu,\nu}(h_1,h_2) - \sum_{\substack{L\in\mathcal{Q}\\ L\subset Q}} \Lambda_{L,\mu,\nu}(h_1,h_2) = \Lambda_{\mathcal{Q},\mu,\nu}(h_1\mathbf{1}_Q, h_2\mathbf{1}_{3Q}). \tag{2-5}$$

Again, the last equality is due to the support of K_s in (SS). A further consequence of assumptions (SS) and (T) is that the forms $\Lambda_{\mathcal{Q},\mu,\nu}$ satisfy uniform bounds on $\mathcal{Y}_r(\mathcal{Q}) \times \mathcal{Y}_{r'}(\mathcal{Q})$.

Lemma 2.3. *There exists a positive absolute constant ϑ such that*

$$\left|\Lambda_{\mathcal{Q},\mu,\nu}(h_1,h_2)\right| \le 2^{\vartheta d} C_{\mathsf{T}}(r)|Q|\,\|h_1\|_{\mathcal{Y}_r(\mathcal{Q})}\|h_2\|_{\mathcal{Y}_{r'}(\mathcal{Q})}$$

uniformly over all μ, ν, all dyadic cubes Q and stopping collections $\mathcal{Q}$ with top Q.

Proof. We may estimate the first term in the definition (2-5) as follows:

$$|\Lambda_{Q,\mu,\nu}(h_1,h_2)| \le C_{\mathsf{T}}(r)\|h_1\mathbf{1}_Q\|_r\,\|h_2\mathbf{1}_{3Q}\|_{r'} \lesssim C_{\mathsf{T}}(r)|Q|\,\|h_1\|_{\mathcal{Y}_r}\|h_2\|_{\mathcal{Y}_{r'}}. \tag{2-6}$$

Further, using the support condition in (2-4) with L in place of Q and the disjointness property (2-2) in the last step, we obtain

$$\begin{aligned}\sum_{L\in\mathcal{Q}:L\subset Q}|\Lambda_{L,\mu,\nu}(h_1,h_2)| &= \sum_{L\in\mathcal{Q}:L\subset Q}|\Lambda_{L,\mu,\nu}(h_1\mathbf{1}_L,h_2\mathbf{1}_{3L})| \le C_{\mathsf{T}}(r)\sum_{L\in\mathcal{Q}:L\subset Q}\|h_1\mathbf{1}_L\|_r\,\|h_2\mathbf{1}_{3L}\|_{r'}\\ &\lesssim C_{\mathsf{T}}(r)\|h_1\|_{\mathcal{Y}_r}\|h_2\|_{\mathcal{Y}_{r'}}\sum_{L\in\mathcal{Q}}|L| \lesssim C_{\mathsf{T}}(r)|Q|\,\|h_1\|_{\mathcal{Y}_r}\|h_2\|_{\mathcal{Y}_{r'}}.\end{aligned}$$

The proof of the lemma is thus completed by combining (2-6) with the last display. □

Our main theorem hinges upon estimates which are modified versions of the one occurring in Lemma 2.3, when one of the two arguments of $\Lambda_{\mathcal{Q},\mu,\nu}$ belongs to $\mathcal{X}$-type localized spaces.

Theorem C. *There exists a positive absolute constant Θ such that the following holds. Let Λ be a bilinear form satisfying (SS) and (T) above. Assume that there exist $1\le p_1,p_2<\infty$ and a positive constant C_{L} such that the estimates*

$$\begin{aligned}\left|\Lambda_{\mathcal{Q},\mu,\nu}(b,h)\right| &\le C_{\mathsf{L}}|Q|\,\|b\|_{\dot{\mathcal{X}}_{p_1}(\mathcal{Q})}\|h\|_{\mathcal{Y}_{p_2}(\mathcal{Q})},\\ \left|\Lambda_{\mathcal{Q},\mu,\nu}(h,b)\right| &\le C_{\mathsf{L}}|Q|\,\|h\|_{\mathcal{Y}_\infty(\mathcal{Q})}\|b\|_{\dot{\mathcal{X}}_{p_2}(\mathcal{Q})}\end{aligned} \tag{L}$$

hold uniformly over all $\mu,\nu\in\mathbb{Z}$, all dyadic lattices $\mathcal{D}$, all $Q\in\mathcal{D}$ and all stopping collections $\mathcal{Q}\subset\mathcal{D}$ with top Q. Then the estimate

$$\sup_{\mu,\nu\in\mathbb{Z}}\left|\Lambda_\mu^\nu(f_1,f_2)\right| \le 2^{\Theta d}[C_{\mathsf{T}}(r)+C_{\mathsf{L}}]\sup_{\mathcal{S}}\mathsf{PSF}_{\mathcal{S};p_1,p_2}(f_1,f_2) \tag{2-7}$$

holds for all $f_j\in L^{p_j}(\mathbb{R}^d)$ with compact support, $j=1,2$.

Remark 2.4. By the limiting argument of Remark 2.1, the conclusion (2-7) gives that

$$\left|\Lambda(f_1,f_2)\right| \le 2^{\Theta d}[C_{\mathsf{T}}(r)+C_{\mathsf{L}}]\sup_{\mathcal{S}}\mathsf{PSF}_{\mathcal{S};p_1,p_2}(f_1,f_2) \tag{2-8}$$

when $f_1,f_2\in L^\infty(\mathbb{R}^d)$ with compact support. If we know that Λ extends boundedly to $L^t(\mathbb{R}^d)\times L^{t'}(\mathbb{R}^d)$ for some $1<t<\infty$, another simple limiting argument using the dominated convergence theorem extends (2-8) to all $f_1\in L^t(\mathbb{R}^d)$, $f_2\in L^{t'}(\mathbb{R}^d)$. It is in this last form that Theorem C will be applied to deduce Theorems A and B.

Remark 2.5 (a comparison between sparse domination principles). Theorem C identifies rather clearly the conditions needed for sparse domination of a kernel operator T, namely the adjoint of the bilinear form Λ. Condition (L) is a localized reformulation of the *constant-mean zero* cancellation around which L^p, $p \neq 2$, Calderón–Zygmund theory revolves, and it is essentially a strengthening of the weak-L^{p_j} estimate for T ($j = 1$) and its adjoint ($j = 2$). Further, our assumption of uniform L^r-boundedness of the truncations in (T) is much tamer than requiring L^r-boundedness of the maximal truncations of T. In fact, our theorem can be applied even when no estimates for maximal truncations of T are known.

Of course the exponents p_j enter the sparse domination estimate (2-7), while the exponent r occurring in (T) does not. This is in contrast with the other sparse domination principles occurring in the literature. For instance, in [Lerner 2016, Theorem 4.2], a sparse domination of type (1-1) with exponents $(r, 1)$ is obtained for operators T whose *grand maximal function*

$$\mathcal{M}_T f(x) := \sup_{Q \ni x} \sup_{y \in Q} \left| T(f \mathbf{1}_{\mathbb{R}^d \setminus 3Q})(y) \right|$$

has the weak-L^r-bound for some $r \geq 1$. Notice that $\mathcal{M}_T$ may be as large as the maximal truncation of T.

A further comparison can be drawn with the abstract result of [Bernicot et al. 2016], which is a sparse domination principle for nonintegral singular operators. The off-diagonal estimate assumption Theorem 1.1(b) of the work above is a clear counterpart of (SS), while the maximal truncation assumption of Theorem 1.1(c) in the same work is the nonkernel analogue of the grand maximal function from [Lerner 2016]. It would be interesting to investigate whether, in the nonkernel setting of [Bernicot et al. 2016], an assumption in the vein of (L) can be used instead.

Remark 2.6 (the essence of (L)). Let $\mathcal{Q}$ be a stopping collection with top Q. When b belongs to an $\mathcal{X}_\alpha(\mathcal{Q})$-type space, the forms

$$(b, h) \mapsto \Lambda_{\mathcal{Q},\mu,\nu}(b, h), \quad (b, h) \mapsto \Lambda_{\mathcal{Q},\mu,\nu}(h, b)$$

have a much more familiar representation, which is what allows verification of assumption (L) in practice. By rephrasing the definition, when $b \in \mathcal{X}_1(\mathcal{Q})$ is supported on Q (which we can assume with no restriction) we have the equality

$$\Lambda_{\mathcal{Q},\mu,\nu}(b, h) = \sum_{j \geq 1} \int \sum_{\mu < s \leq \min\{s_Q, \nu\}} K_s(x, y) b_{s-j}(y) h(x)\, \mathrm{d}y\, \mathrm{d}x, \quad \text{where } b_s := \sum_{\substack{L \in \mathcal{Q} \\ s_L = s}} b_L. \tag{2-9}$$

This notation will be used throughout the paper; see for instance (2-10) below. Furthermore, if q is the exponent occurring in (SS), $h \in \mathcal{Y}_{q'}(\mathcal{Q})$, and $b \in \mathcal{X}_{q'}(\mathcal{Q})$, then $\Lambda_{\mathcal{Q},\mu,\nu}(h, b)$ is essentially self-adjoint up to a tolerable error term. Namely, if h is supported on Q (which we can also always assume),

$$\Lambda_{\mathcal{Q},\mu,\nu}(h, b) = \left(\sum_{j \geq 1} \int \sum_{\mu < s \leq \min\{s_Q, \nu\}} K_s(y, x) b^{\mathrm{in}}_{s-j}(y) h(x)\, \mathrm{d}y\, \mathrm{d}x \right) + V_{\mathcal{Q}}(h, b), \tag{2-10}$$

where

$$b^{\mathrm{in}} = \sum_{\substack{L \in \mathcal{Q} \\ 3L \cap 2Q \neq \varnothing}} b_L$$

is a truncation of b and thus also belongs to $\mathcal{X}_{q'}(\mathcal{Q})$ with $\|b^{\mathrm{in}}\|_{\mathcal{X}_{q'}(\mathcal{Q})} \le \|b\|_{\mathcal{X}_{q'}(\mathcal{Q})}$, and the remainder $V_{\mathcal{Q}}(h,b)$ satisfies

$$|V_{\mathcal{Q}}(h,b)| \le 2^{\vartheta d}[K]_{0,q}|Q|\,\|h\|_{\mathcal{Y}_{q'}(\mathcal{Q})}\|b\|_{\mathcal{X}_{q'}(\mathcal{Q})} \tag{2-11}$$

for a suitable positive absolute constant ϑ. The representation (2-10)–(2-11) is a simple consequence of the structure of $b \in \mathcal{X}_{q'}(\mathcal{Q})$ and of the separation properties (2-2), (2-3). We provide the necessary details for (2-10)–(2-11) in Appendix A at the end.

Proof of Theorem C. Given a form Λ satisfying the assumptions of Theorem C, $\mu < \nu \in \mathbb{Z}$ and $f_j \in L^{p_j}(\mathbb{R}^d)$, $j = 1, 2$, with compact support, we will construct a sparse collection $\mathcal{S}$ of cubes of $\mathbb{R}^d$ such that

$$\left|\Lambda_\mu^\nu(f_1, f_2)\right| \le 2^{\Theta d}\boldsymbol{C}\sum_{Q\in\mathcal{S}}|Q|\langle f_1\rangle_{p_1,Q}\langle f_2\rangle_{p_2,Q}, \tag{2-12}$$

where $\boldsymbol{C}$ is the expression within the square brackets in the conclusion of Theorem C. Here and below, we denote by Θ a suitably large positive absolute constant which will be chosen during the course of the proof. Within this proof, we will also denote by ϑ positive absolute constants which belong to $[2^{-8}\Theta, 2^{-7}\Theta]$ and may differ at each occurrence. As the assumptions of Theorem C are stable if we replace Λ with Λ_μ^ν, we can work under the assumption that $K_s = 0$ for all $s \notin (\mu, \nu]$ and thus drop μ, ν from the notations (2-4), (2-5).

The proof of (2-12) is iterative and is carried out in the next subsection. Here, we give the main estimate for the form $\Lambda^{s_{\mathcal{Q}}}$ from (2-4) in terms of stopping collection norms.

Lemma 2.7. *Let Q be a fixed dyadic cube in $\mathbb{R}^d$ and $\mathcal{Q}$ be a stopping collection with top Q. Then*

$$\left|\Lambda^{s_{\mathcal{Q}}}(h_1\mathbf{1}_Q, h_2\mathbf{1}_{3Q})\right| \le 2^{\vartheta d}\boldsymbol{C}|Q|\,\|h_1\|_{\mathcal{Y}_{p_1}(\mathcal{Q})}\|h_2\|_{\mathcal{Y}_{p_2}(\mathcal{Q})} + \sum_{\substack{L\in\mathcal{Q}\\ L\subset Q}}\left|\Lambda^{s_L}(h_1\mathbf{1}_L, h_2\mathbf{1}_{3L})\right|. \tag{2-13}$$

Proof. We are free to assume that $\operatorname{supp} h_1 \subset Q$ and $\operatorname{supp} h_2 \subset 3Q$ for simplicity of notation. For $j = 1, 2$, construct the Calderón–Zygmund decomposition of h_j with respect to the family $\mathcal{Q}$ as described in Remark 2.2, that is,

$$h_j = g_j + b_j, \quad b_j = \sum_{L\in\mathcal{Q}} b_{jL}, \quad b_{jL} := \left(h_j - \frac{1}{|L|}\int_L h_j(x)\,\mathrm{d}x\right)\mathbf{1}_L.$$

The Calderón–Zygmund properties in this context are, for $j = 1, 2$,

$$\|g_j\|_{\mathcal{Y}_\infty} \lesssim \|h_j\|_{\mathcal{Y}_{p_j}}, \quad \|b_j\|_{\dot{\mathcal{X}}_{p_j}} \lesssim \|h_j\|_{\mathcal{Y}_{p_j}}.$$

Using the definition (2-5), we decompose on our way to (2-13):

$$\begin{aligned}\Lambda^{s_{\mathcal{Q}}}(h_1,h_2) &= \Lambda_{\mathcal{Q}}(h_1,h_2) + \sum_{\substack{L\in\mathcal{Q}\\ L\subset Q}}\Lambda^{s_L}(h_1\mathbf{1}_L,h_2)\\ &= \Lambda_{\mathcal{Q}}(g_1,g_2)+\Lambda_{\mathcal{Q}}(b_1,g_2)+\Lambda_{\mathcal{Q}}(g_1,b_2)+\Lambda_{\mathcal{Q}}(b_1,b_2)+\sum_{\substack{L\in\mathcal{Q}\\ L\subset Q}}\Lambda^{s_L}(h_1\mathbf{1}_L,h_2\mathbf{1}_{3L}).\end{aligned} \tag{2-14}$$

The last sum on the last right-hand side is estimated by the sum appearing on the right-hand side of (2-13). We are left with estimating the first four terms in the last line of (2-14). The leftmost is controlled by the estimate of Lemma 2.3:

$$|\Lambda_{\mathcal{Q}}(g_1,g_2)| \lesssim C_{\mathsf{T}}(r)|Q|\,\|g_1\|_{\mathcal{Y}_r}\|g_2\|_{\mathcal{Y}_{r'}} \lesssim \boldsymbol{C}|Q|\,\|h_1\|_{\mathcal{Y}_{p_1}}\|h_2\|_{\mathcal{Y}_{p_2}}.$$

The second term is handled by appealing to assumption (L), which yields

$$|\Lambda_{\mathcal{Q}}(b_1,g_2)| \le C_{\mathsf{L}}|Q|\,\|b_1\|_{\dot{\mathcal{X}}_{p_1}}\|g_2\|_{\mathcal{Y}_{p_2}} \lesssim \boldsymbol{C}|Q|\,\|h_1\|_{\mathcal{Y}_{p_1}}\|h_2\|_{\mathcal{Y}_{p_2}},$$

where the second estimate follows from the Calderón–Zygmund properties above. The third is also estimated by appealing to (L), as

$$|\Lambda_{\mathcal{Q}}(g_1,b_2)| \le C_{\mathsf{L}}|Q|\,\|g_1\|_{\mathcal{Y}_\infty}\|b_2\|_{\dot{\mathcal{X}}_{p_2}} \lesssim \boldsymbol{C}|Q|\,\|h_1\|_{\mathcal{Y}_{p_1}}\|h_2\|_{\mathcal{Y}_{p_2}}.$$

Finally, again by assumption (L),

$$|\Lambda_{\mathcal{Q}}(b_1,b_2)| \le C_{\mathsf{L}}|Q_0|\,\|b_1\|_{\dot{\mathcal{X}}_{p_1}}\|b_2\|_{\mathcal{Y}_{p_2}} \lesssim \boldsymbol{C}|Q|\,\|h_1\|_{\mathcal{Y}_{p_1}}\|h_2\|_{\mathcal{Y}_{p_2}},$$

where the final inequality follows again from the Calderón–Zygmund estimates. □

***Proof of* (2-12).** The proof is obtained by means of the iterative procedure described below.

Preliminaries: We will produce stopping collections iteratively, by suitable Whitney decompositions of unions of sets

$$E_Q = \left\{ x \in 3Q : \max_{j=1,2} \frac{\mathrm{M}_{p_j}(f_j \mathbf{1}_{3Q})(x)}{\langle f_j\rangle_{p_j,3Q}} > 2^{\frac{\Theta d}{4}} \right\} \tag{2-15}$$

associated to a cube Q and a pair of functions f_1, f_2. We notice that

$$E_Q \subset 3Q, \quad |E_Q| \le 2^{-\vartheta d}|Q|; \tag{2-16}$$

the measure estimate is a consequence of the maximal theorem, and holds provided Θ is chosen sufficiently large. In this proof, we say that two dyadic cubes L, L' are *neighbors*, and write $L \sim L'$, if

$$7L \cap 7L' \ne \varnothing, \quad |s_L - s_{L'}| < 8.$$

The separation condition (2-3) tells us that if the 7-fold dilates of two cubes L, L' belonging to the same stopping collection intersect nontrivially, then L, L' must be neighbors. We also recall the notation $\widehat{L}$ for the 2^5-fold dilate of L.

Initialize: Let $f_j \in L^{p_j}(\mathbb{R}^d)$, $j = 1,2$, with compact support be fixed. By suitably choosing the dyadic lattice $\mathcal{D}$, we may find $Q_0 \in \mathcal{D}$ such that $\operatorname{supp} f_1 \subset Q_0$, $\operatorname{supp} f_2 \subset 3Q_0$ and s_{Q_0} is larger than the largest nonzero scale occurring in the kernel. Then set $\mathcal{S}_0 = \{Q_0\}$, $E_0 = 3Q_0$, and define referring to (2-15)

$$E_1 := E_{Q_0}, \quad \mathcal{S}_1 := \text{maximal cubes } L \in \mathcal{D} \text{ such that } 9L \subset E_1.$$

Notice that the following properties are satisfied:

$$L \in \mathcal{S}_1 \text{ are a pairwise disjoint collection,} \tag{2-17}$$

$$E_1 = \bigcup_{L\in\mathcal{S}_1} L = \bigcup_{L\in\mathcal{S}_1} 9L \subset E_0, \quad |Q_0 \setminus E_1| \ge (1-2^{-d\vartheta})|Q_0|, \tag{2-18}$$

$$L, L' \in \mathcal{S}_1,\ 7L \cap 7L' \ne \varnothing \implies L \sim L'. \tag{2-19}$$

Property (2-17) and the first part of (2-18) are by construction, while the second part of (2-18) follows from the estimate of (2-16). For (2-19) suppose instead that $7L \cap 7L'$ is not empty when $s_L \le s_{L'} - 8$. By the relation between the sidelengths it follows that $\widehat{L} \subset 9L'$, which implies that the 9-fold dilate of the dyadic parent of L is contained in $9L'$ as well, contradicting the maximality of L. By virtue of (2-17)–(2-19), $\mathcal{Q}_1(Q_0) := \mathcal{S}_1$ is a stopping collection with top Q_0; compare with (2-2), (2-3). The first property in (2-18) guarantees that

$$\sup_{x \notin \operatorname{sh} \mathcal{Q}_1(Q_0)} |f_j(x)| \le 2^{\frac{\Theta d}{4}} \langle f_j \rangle_{p_j, 3Q_0}.$$

Further, by the maximality condition on $L \in \mathcal{S}_1$, it follows that

$$\sup_{L\in\mathcal{Q}_1(Q_0)} \inf_{\widehat{L}} \mathrm{M}_{p_j}(f_j \mathbf{1}_{3Q_0}) \le 2^{\frac{\Theta d}{4}} \langle f_j \rangle_{p_j, 3Q_0}$$

for $j = 1, 2$. The last two inequalities tell us that

$$\|f_j\|_{\mathcal{Y}_{p_j}(\mathcal{Q}_1(Q_0))} \le 2^{\frac{\Theta d}{4}} \langle f_j \rangle_{p_j, 3Q_0}, \quad j = 1, 2.$$

Applying (2-13) to the stopping collection $\mathcal{Q}_1(Q_0)$, and $h_1 = f_1,\ h_2 = f_2$ we obtain

$$\begin{aligned} \bigl|\Lambda(f_1, f_2)\bigr| &= \bigl|\Lambda^{s_{Q_0}}(f_1 \mathbf{1}_{Q_0}, f_2 \mathbf{1}_{3Q_0})\bigr| \\ &\le 2^{\Theta d} \boldsymbol{C} |Q_0| \langle f_1 \rangle_{p_1, 3Q_0} \langle f_2 \rangle_{p_2, 3Q_0} + \sum_{\substack{L\in\mathcal{Q}_1(Q_0) \\ L \subset Q_0}} \bigl|\Lambda^{s_L}(f_1 \mathbf{1}_L, f_2 \mathbf{1}_{3L})\bigr|. \end{aligned} \tag{2-20}$$

The obtained properties (2-17)–(2-19) and estimate (2-20) are the $\ell = 1$ case of the induction assumption in the inductive step below.

Inductive step: Suppose inductively collections $\mathcal{S}_\ell$, $0 \le \ell \le k$, and sets E_ℓ, $1 \le \ell \le k$, have been constructed, with the properties that for all $1 \le \ell \le k$

$$L \in \mathcal{S}_\ell \text{ are a pairwise disjoint collection,} \tag{2-21}$$

$$E_\ell = \bigcup_{L\in\mathcal{S}_\ell} L = \bigcup_{L\in\mathcal{S}_\ell} 9L \subset E_{\ell-1}, \qquad |Q \setminus E_\ell| \ge (1-2^{-\vartheta d})|Q| \quad \forall Q \in \mathcal{S}_{\ell-1}, \tag{2-22}$$

$$L, L' \in \mathcal{S}_\ell,\ 7L \cap 7L' \ne \varnothing \implies L \sim L'. \tag{2-23}$$

Suppose also that if $\mathcal{T}_{k-1} = \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_{k-1}$, the estimate

$$\bigl|\Lambda(f_1, f_2)\bigr| \le 2^{\Theta d} \boldsymbol{C} \sum_{R\in\mathcal{T}_{k-1}} |R| \langle f_1 \rangle_{p_1, 3R} \langle f_2 \rangle_{p_2, 3R} + \sum_{Q\in\mathcal{S}_k} \bigl|\Lambda^{s_Q}(f_1 \mathbf{1}_Q, f_2 \mathbf{1}_{3Q})\bigr| \tag{2-24}$$

has been shown to hold. At this point define

$$E_{k+1} := \bigcup_{Q \in \mathcal{S}_k} E_Q, \quad \mathcal{S}_{k+1} := \text{maximal cubes } L \in \mathcal{D} \text{ such that } 9L \subset E_{k+1},$$
$$\mathcal{Q}_{k+1}(Q) = \{L \in \mathcal{S}_{k+1} : L \subset 3Q\}, \quad Q \in \mathcal{S}_k.$$

Property (2-21), together with the first property in (2-22), as $E_Q \subset 3Q \subset E_k$, and (2-23), via the same reasoning we used for (2-19), now hold for $\ell = k+1$ as well. Let now $Q \in \mathcal{S}_k$. Property (2-23) with $\ell = k$ implies that

$$3Q \cap E_{k+1} \subset \bigcup_{Q' \in \mathcal{S}_k : Q' \sim Q} E_{Q'}.$$

Therefore, we learn that

$$|Q \cap E_{k+1}| \le |3Q \cap E_{k+1}| \le \sum_{Q' \in \mathcal{S}_k : Q' \sim Q} |E_{Q'}| \le 2^{-\vartheta d} |Q| \tag{2-25}$$

by applying for each $Q' \in \mathcal{S}_k$ with $Q' \sim Q$ the estimate of (2-16), and observing that the cardinality of $\{Q' \in \mathcal{D} : Q' \sim Q\}$ is bounded by an absolute dimensional constant, and $|Q|, |Q'|$ are comparable, again up to an absolute dimensional constant. From the above display we obtain the second part of (2-22) for $\ell = k+1$. Moreover, one observes that if $L \in \mathcal{S}_{k+1}$ with $L \cap 3Q \neq \varnothing$, then by virtue of property (2-25), L must be significantly shorter than Q and thus contained in one of the 3^d translates of the dyadic cube Q whose union covers $3Q$. Namely, we have the equality

$$\mathcal{Q}_{k+1}(Q) = \{L \in \mathcal{S}_{k+1} : L \cap 3Q \neq \varnothing\},$$

which also gives the last equality in

$$\bigcup_{L \in \mathcal{Q}_{k+1}(Q) : 3L \cap 2Q \neq \varnothing} 9L \subset \bigcup_{L \in \mathcal{S}_{k+1} : L \cap 3Q \neq \varnothing} L = \bigcup_{L \in \mathcal{Q}_{k+1}(Q)} L = \operatorname{sh} \mathcal{Q}_{k+1}(Q),$$

as the set in the left-hand side of the last display is contained in $3Q$ and (2-22) holds for $\ell = k+1$. Comparing with (2-2), (2-3), the discussion above gives that $\mathcal{Q}_{k+1}(Q)$ is a stopping collection with top Q such that $E_Q \subset \operatorname{sh} \mathcal{Q}_{k+1}(Q)$, so that

$$\sup_{x \notin \operatorname{sh} \mathcal{Q}_{k+1}(Q)} |f_j \mathbf{1}_{3Q}(x)| \le 2^{\frac{\Theta d}{4}} \langle f_j \rangle_{p_j, 3Q}.$$

Furthermore, for $j = 1, 2$

$$\sup_{L \in \mathcal{Q}_{k+1}(Q)} \inf_{\hat{L}} \mathrm{M}_{p_j}(f_j \mathbf{1}_{3Q}) \le 2^{\frac{\Theta d}{4}} \langle f_j \rangle_{p_j, 3Q};$$

otherwise the 9-fold dilate of the dyadic parent of some $L \in \mathcal{Q}_{k+1}(Q)$ would be contained in E_Q and thus in E_{k+1}, contradicting the maximality of such an L. Therefore

$$\|f_j \mathbf{1}_{3Q}\|_{\mathcal{Y}_{p_j}(\mathcal{Q}_{k+1}(Q))} \le 2^{\frac{\Theta d}{4}} \langle f_j \rangle_{p_j, 3Q}, \quad j = 1, 2,$$

and we may apply (2-13) to each $Q \in \mathcal{S}_k$ summand in (2-24), with $h_1 = f_1$, $h_2 = f_2$ and obtain

$$\begin{aligned}\left|\Lambda^{s_Q}(f_1 \mathbf{1}_Q, f_2 \mathbf{1}_{3Q})\right| &\le 2^{\Theta d} \boldsymbol{C} |Q| \langle f_1 \rangle_{p_1,3Q} \langle f_2 \rangle_{p_2,3Q} + \sum_{L \in \mathcal{Q}_{k+1}(Q) : L \subset Q} \left|\Lambda^{s_L}(f_1 \mathbf{1}_L, f_2 \mathbf{1}_{3L})\right| \\ &= 2^{\Theta d} \boldsymbol{C} |Q| \langle f_1 \rangle_{p_1,3Q} \langle f_2 \rangle_{p_2,3Q} + \sum_{L \in \mathcal{S}_{k+1} : L \subset Q} \left|\Lambda^{s_L}(f_1 \mathbf{1}_L, f_2 \mathbf{1}_{3L})\right|.\end{aligned}$$

As $Q \in \mathcal{S}_k$ are pairwise disjoint, see (2-21), summing over $Q \in \mathcal{S}_k$, writing $\mathcal{T}_k = \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_k$ and combining the resulting estimate with (2-24), we arrive at

$$\left|\Lambda(f_1, f_2)\right| \le 2^{\Theta d} \boldsymbol{C} \sum_{Q \in \mathcal{T}_k} |Q| \langle f_1 \rangle_{p_1,3Q} \langle f_2 \rangle_{p_2,3Q} + \sum_{L \in \mathcal{S}_{k+1}} \left|\Lambda^{s_L}(f_1 \mathbf{1}_L, f_2 \mathbf{1}_{3L})\right|,$$

that is, (2-24) with k replaced by $k+1$. This, together with the previously obtained (2-21)–(2-23) for $\ell = k+1$, completes the current iteration.

<u>Termination</u>: A consequence of our construction is that $\sigma_k := \max\{s_Q : Q \in \mathcal{S}_k\} \le s_{Q_0} - \vartheta k$. The algorithm terminates when $k = K$, where K is such that σ_K is strictly less than the minimal nonzero scale in the kernel. For $k = K$ in (2-24) the second sum on the right-hand side vanishes identically and we have obtained the estimate (2-12) by setting $\mathcal{T} := \mathcal{T}_{K-1}$ and $\mathcal{S} := \{3Q : Q \in \mathcal{T}\}$. We see that the collection $\mathcal{T}$, and thus the collection of the dilates $\mathcal{S}$, are sparse by simply observing that the sets

$$F_Q := Q \setminus E_{k+1}, \quad Q \in \mathcal{S}_k,$$

are pairwise disjoint for $Q \in \mathcal{T}$ and have measure larger than $(1 - 2^{-d\vartheta})|Q|$, as can be seen from (2-22).

3. Localized estimates for Dini- and Hörmander-type kernels

In the first part of this section, we state and prove a family of localized estimates, of the type occurring in condition (L) of Theorem C, for kernels falling within the scope of (SS) and possessing additional smoothness properties, of Dini or Hörmander type. These estimates and their proof are a reformulation of the classical inequalities intervening in the proof of the weak-L^1-bound for Calderón–Zygmund operators (see, for example, [Stein 1993, Chapter I]). We choose to provide details as we believe the arguments to be rather explanatory of the driving philosophy behind Theorem C.

As we mentioned in the Introduction, our abstract Theorem C, coupled with the localized estimates that follow, can be employed to reprove the optimal sparse domination estimates for Calderón–Zygmund kernels of Dini and Hörmander type, thus recovering the results (among others) of [Bui et al. 2017; Hytönen et al. 2017; Lacey 2017; Lerner 2016; Volberg and Zorin-Kranich 2016]. We provide a summary of the statements of such domination theorems in the second part of this section.

Localized estimates and kernel norms. Throughout these estimates, we assume that a stopping collection $\mathcal{Q}$ with top Q as in Section 2 has been fixed, and the notations $\Lambda_{\mathcal{Q},\mu,\nu}$ refer to (2-5). It is understood that the constants implied by the almost inequality signs depend on dimension only and are in particular are uniform over the choice of $\mathcal{Q}$. We begin with the single-scale localized estimate where no cancellation is exploited.

Lemma 3.1 (trivial estimate). *Let* $1 < \beta \le \infty$ *and* $\alpha = \beta'$. *Then for all* $j \ge 1$,

$$\sum_s \int |K_s(x,y)|\,|b_{s-j}(y)|\,|h(x)| \,\mathrm{d}y\,\mathrm{d}x \lesssim [K]_{0,\beta}\, |Q|\, \|b\|_{\mathcal{X}_1} \|h\|_{\mathcal{Y}_\alpha}.$$

Proof. As $\|b_L\|_1 \lesssim |L|\,\|b\|_{\mathcal{X}_1}$ for $L \in \mathcal{Q}$, it suffices to prove that for each $L \in \mathcal{Q}$ and $s = s_L + j$,

$$\int |K_s(x,y)|\,|b_L(y)|\,|h(x)| \,\mathrm{d}y\,\mathrm{d}x \lesssim [K]_{0,\beta}\, \|b_L\|_1 \|h\|_{\mathcal{Y}_\alpha}. \tag{3-1}$$

In turn, it then suffices to prove that

$$s \ge s_L \implies \sup_{y\in L} \int |K_s(y+u,y)|\,|h(y+u)| \,\mathrm{d}u \lesssim [K]_{0,\beta}\, \|h\|_{\mathcal{Y}_\alpha},$$

which readily follows from

$$\begin{aligned}\int |K_s(y+u,y)|\,|h(y+u)|\,\mathrm{d}u &\le \|K_s(y+\,\cdot\,,y)\|_\beta \left(\int_{B(y,2^{s+10})} |h(z)|^\alpha \,\mathrm{d}z\right)^{\frac{1}{\alpha}} \\ &\lesssim [K]_{0,\beta} \big(\inf_{\widehat{L}} \mathrm{M}_\alpha h\big) \le [K]_{0,\beta} \|h\|_{\mathcal{Y}_\alpha}\end{aligned}$$

when $y \in L$. Above, we used the support condition (SS) and Hölder's inequality for the first step, and subsequently that the ball $B(y, 2^{s+10}) = \{z \in \mathbb{R}^d : |z-y| < 2^{s+10}\}$ contains the dilate $\widehat{L}$. $\square$

We introduce a further family of kernel norms in addition to the one of (SS), to which we refer for notation. For $1 < \beta \le \infty$ set

$$[K]_{1,\beta} := \sum_{j=1}^{\infty} \varpi_{j,\beta}(K), \tag{3-2}$$

where

$$\varpi_{j,\beta}(K) := \sup_{s\in\mathbb{Z}} 2^{\frac{sd}{\beta'}} \sup_{x\in\mathbb{R}^d} \sup_{\substack{h\in\mathbb{R}^d\\ \|h\|_\infty < 2^{s-j-1}}} \Big(\big\|K_s(x,x+\cdot)-K_s(x+h,x+\cdot)\big\|_\beta + \big\|K_s(x+\cdot,x)-K_s(x+\cdot,x+h)\big\|_\beta\Big).$$

The second localized estimate we consider uses the finiteness of $[K]_{1,\beta}$ to incorporate the constant-mean zero cancellation effect.

Lemma 3.2 (cancellation estimate). *Let* $1 < \beta \le \infty$ *and* $\alpha = \beta'$. *Then for all* $\mu, \nu \in \mathbb{Z}$,

$$\big|\Lambda_{\mathcal{Q},\mu,\nu}(b,h)\big| + \big|\Lambda_{\mathcal{Q},\mu,\nu}(h,b)\big| \lesssim \big([K]_{0,\infty} + [K]_{1,\beta}\big) |Q|\, \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_\alpha}. \tag{3-3}$$

Proof. It will suffice to prove the estimate

$$\sum_{L\in\mathcal{Q}} \sum_{j=1}^{\infty} \left| \int K_{s_L+j}(x,y) \tilde{b}_L(y) \tilde{h}(x) \,\mathrm{d}y\,\mathrm{d}x \right| \lesssim [K]_{1,\beta}\, |Q|\, \|\tilde{b}\|_{\dot{\mathcal{X}}_1} \|\tilde{h}\|_{\mathcal{Y}_\alpha}. \tag{3-4}$$

In fact, by using the representations in (2-9), (2-10) we see that for all $\mu, \nu \in \mathbb{Z}$ and each pair $b \in \dot{\mathcal{X}}_1$, $h \in \mathcal{Y}_\alpha$, the forms $|\Lambda_{\mathcal{Q},\mu,\nu}(b,h)|$, $|\Lambda_{\mathcal{Q},\mu,\nu}(h,b)|$ are both bounded above by the left-hand side of (3-4) for suitable $\tilde{b} \in \dot{\mathcal{X}}_1$, $\tilde{h} \in \mathcal{Y}_\alpha$ whose norms are dominated by $\|b\|_{\dot{\mathcal{X}}_1}$, $\|h\|_{\mathcal{Y}_\alpha}$ respectively, up to possibly replacing K_s with its transpose and controlling the remainder term $V_{\mathcal{Q}}(h,b)$ in the case of $\Lambda_{\mathcal{Q},\mu,\nu}(h,b)$. This remainder is estimated in (2-11) for $q=\infty$, which is acceptable for the right-hand side of (3-3).

We will obtain estimate (3-4) from the bound

$$\sum_{j=1}^{\infty}\left|\int K_{s_L+j}(x,y)b_L(y)\tilde{h}(x)\,\mathrm{d}y\,\mathrm{d}x\right| \lesssim [K]_{1,\beta}\,|L|\,\|\tilde{b}\|_{\dot{\mathcal{X}}_1}\|\tilde{h}\|_{\mathcal{Y}_\alpha}, \quad L\in\mathcal{Q} \tag{3-5}$$

by summing over $L\in\mathcal{Q}$ in and using their disjointness, given in (2-2). Fix $L\in\mathcal{Q}$ and $j\geq 1$. Using the cancellation of $\tilde{b}_L$ and then arguing as in the proof of (3-1) above we obtain

$$\begin{aligned}\left|\int K_{s_L+j}(x,y)b_L(y)\tilde{h}(x)\,\mathrm{d}y\,\mathrm{d}x\right| &\leq \|\tilde{b}_L\|_1 \sup_{y\in L}\int |K_{s_L+j}(y+u,y)-K_{s_L+j}(y+u,c_L)|\,|\tilde{h}(y+u)|\,\mathrm{d}u\\ &\lesssim \|\tilde{b}_L\|_1\,\omega_{j,\beta}(K)\big(\inf_{\hat{L}}\mathrm{M}_\alpha\tilde{h}\big) \lesssim \omega_{j,\beta}(K)\,|L|\,\|\tilde{b}\|_{\dot{\mathcal{X}}_1}\|\tilde{h}\|_{\mathcal{Y}_\alpha},\end{aligned}$$

and (3-5) follows by summing over $j\geq 1$. □

Sparse domination of Calderón–Zygmund kernels. We now briefly mention how our abstract result, Theorem C, can be employed to recover sparse domination, and thus weighted bounds, for Calderón–Zygmund kernels with minimal smoothness assumptions. Let T be an $L^2(\mathbb{R}^d)$-bounded operator whose kernel K satisfies the usual size normalization

$$\sup_{x\neq y}|x-y|^d\,|K(x,y)|\leq 1.$$

Let ψ be a fixed Schwartz function supported in $A_1=\{x\in\mathbb{R}^d : 2^{-2}<|x|<1\}$ such that

$$\sum_{s\in\mathbb{Z}}\psi(2^{-s}x)=1, \quad x\neq 0.$$

It is immediate to see that (SS) holds, and in particular $[K]_{0,\infty}\leq C$, for the decomposition

$$K_s(x,y):=K(x,y)\,\psi\left(\frac{x-y}{2^s}\right), \quad s\in\mathbb{Z}.$$

We further assume that $[K]_{1,\beta}<\infty$ for some $1<\beta\leq\infty$, where the kernel norm has been defined in (3-2). When $\beta=\infty$, this is exactly the Dini condition [Hytönen et al. 2017; Lacey 2017; Lerner 2016]. For $\beta<\infty$, the above condition is equivalent to the assumptions of [Volberg and Zorin-Kranich 2016], where in fact a multilinear version is presented.

The assumptions of Theorem C then hold for the dual form

$$\Lambda(f_1,f_2)=\langle Tf_1,\bar{f_2}\rangle.$$

We have already observed that (SS) is verified with $q=\infty$. It is well known that the L^2-boundedness of Λ together with $[K]_{1,\beta}<\infty$ yields that the truncation forms Λ_μ^ν, see (2-1), are uniformly bounded on $L^t(\mathbb{R}^d)\times L^{t'}(\mathbb{R}^d)$ [Stein 1993, Chapter I.7] for all $1<t<\infty$; thus we have condition (T) with, for instance, $r=2$. Furthermore, Lemma 3.2 is exactly (L) for the corresponding $\Lambda_{\mathcal{Q},\mu,\nu}$, with $p_1=1$, $p_2=\alpha=\beta'$.

Applying Theorem C in the form given in Remark 2.4, we obtain the following sparse domination result, which recovers (the dual form of) the domination theorems from the above-mentioned references. We cite the same references for the sharp weighted norm inequalities that descend from this result.

Theorem D (Calderón–Zygmund theory). *Let T be as above and $1 \le \beta < \infty$. For all $1 < t < \infty$ and all pairs $f_1 \in L^t(\mathbb{R}^d)$, $f_2 \in L^{t'}(\mathbb{R}^d)$,*

$$|\langle Tf_1, f_2\rangle| \le C_\beta [K]_{1,\beta} \sup_{\mathcal{S}} \mathsf{PSF}_{\mathcal{S};1,\beta'}(f_1, f_2),$$

where C_β is a positive constant depending on β and on the dimension d only.

4. Proof of Theorem A

Let $1 < q \le \infty$ and suppose that $\Omega \in L^q(S^{d-1})$ has unit norm and vanishing integral. Set $x' = x/|x|$. We decompose for $x \neq 0$ the kernel of T_Ω in (1-2) as

$$\frac{\Omega(x')}{|x|^d} = \sum_s K_s(x), \quad K_s(x) = \Omega(x') 2^{-sd} \phi(2^{-s}x),$$

where ϕ is a suitable smooth radial function supported in $A_1 = \{2^{-2} \le |x| \le 1\}$. The main result of this section is the following proposition: again, we assume that a stopping collection $\mathcal{Q}$ with top the dyadic cube Q as in Section 2 has been fixed and the notations $\mathcal{Y}_t$ and similar refer to that fixed setting.

Proposition 4.1. *Let $\Omega \in L^q(S^{d-1})$ be of unit norm and vanishing integral. Let $\{\varepsilon_s\} \in \{-1, 0, 1\}^{\mathbb{Z}}$ be a choice of signs, $b \in \dot{\mathcal{X}}_1$ and define*

$$\mathsf{K}(b, h) := \sum_{j \ge 1} \sum_s \varepsilon_s \langle K_s * b_{s-j}, \bar{h}\rangle$$

where

$$b_s = \sum_{\substack{L \in \mathcal{Q} \\ s_L = s}} b_L.$$

There exists an absolute constant C, in particular uniform over all $\{\varepsilon_s\} \in \{-1, 0, 1\}^{\mathbb{Z}}$, such that

$$|\mathsf{K}(b, h)| \le \frac{Cp}{p-1} |Q| \, \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_p} \begin{cases} \|\Omega\|_{L^{q,1}\log L(S^{d-1})}, & q < \infty, \ p \ge q', \\ \|\Omega\|_{L^\infty(S^{d-1})}, & q = \infty, \ p > 1. \end{cases} \tag{4-1}$$

With the above proposition in hand, we may now give the proof of Theorem A. The structural assumptions (SS), (T) of the abstract result Theorem C applied to the above decomposition of (the dual form of) T_Ω are respectively verified with $q = q$ and with $r = 2$ (this is the classical L^2-boundedness of the truncations of T_Ω [Calderón and Zygmund 1956; Grafakos and Stefanov 1999]).

We still need to verify (L) for the values $p_1 = 1$ and $p_2 = p$ for each p in the claimed range (depending on whether $q = \infty$ or not). It is immediate from the representations (2-9) that in this setting $\Lambda_{\mathcal{Q},\mu,\nu}(b, h) = \mathsf{K}(b\mathbf{1}_Q, h)$ for a suitable choice of signs $\{\varepsilon_s\}$ depending on μ, ν. So Proposition 4.1 yields the first condition in (L) with $p_1 = 1$, $p_2 = p$. On the other hand, we get from (2-10) that $\Lambda_{\mathcal{Q},\mu,\nu}(h, b)$ is equal to $\mathsf{K}(b^{\mathrm{in}}, h\mathbf{1}_Q)$, again for a suitable choice of signs $\{\varepsilon_s\}$ depending on μ, ν, up to replacing K_s

by $K_s(-\cdot)$, and up to subtracting the remainder term from (2-11), which is estimated in this case by an absolute constant times

$$|Q|\,\|h\|_{\mathcal{Y}_\infty}\|b\|_{\mathcal{Y}_{q'}} \le |Q|\,\|h\|_{\mathcal{Y}_\infty}\|b\|_{\mathcal{Y}_p},$$

which is acceptable for the right-hand side of the second condition in (L) when $p_2 = p$. These considerations and another application of Proposition 4.1 finally yield Theorem A, via our abstract result in the form described in Remark 2.4.

Proof of Proposition 4.1. Throughout this proof, C is a positive absolute dimensional constant which may vary at each occurrence without explicit mention. We assume $\{\varepsilon_s\} \in \{-1,0,1\}^{\mathbb{Z}}$ is given. For the sake of simplicity, we redefine $K_s := \varepsilon_s K_s$; it will be clear from the proof below that the signs of K_s play no role. Fix a positive integer j. For $\delta > 0$ to be fixed at the end of the argument define

$$O_j = \{\theta \in S^{d-1} : |\Omega(\theta)| > 2^{\delta j}\}, \quad \Omega_j = \Omega\,\mathbf{1}_{S^{d-1}\setminus O_j}, \quad \Delta_j = \Omega\,\mathbf{1}_{O_j}. \tag{4-2}$$

We now have the decomposition

$$K_s = H_s^j + V_s^j, \quad H_s^j = K_s\mathbf{1}_{\operatorname{supp}\Omega_j}, \quad V_s^j = K_s\mathbf{1}_{O_j}. \tag{4-3}$$

The first localized form we treat, namely the contribution of the unbounded part of Ω, is dealt with by means of a trivial estimate.

Lemma 4.2. $$\mathsf{V}^j(b,h) := \sum_s |\langle V_s^j * b_{s-j}, \bar{h}\rangle| \le C\|\Delta_j\|_q|Q|\,\|b\|_{\mathcal{X}_1}\|h\|_{\mathcal{Y}_p}, \quad p \ge q'.$$

Proof. It suffices of course to prove the estimate above with q' in place of p. This is actually a particular case of Lemma 3.1 applied with $K = \{V_s^j\}$ and $\beta = q$, as it is immediate to see that for this kernel one has $[K]_{0,q} \le C\|\Delta_j\|_q$. □

The contribution of the bounded part of K_s in (4-3) is more delicate, and we postpone the proof of the following lemma to the next subsection.

Lemma 4.3. *There exist absolute constants $C, c > 0$ such that for all $1 < p \le \infty$*

$$\mathsf{H}^j(b,h) := \left|\sum_s \langle H_s^j * b_{s-j}, \bar{h}\rangle\right| \le C2^{-cj\frac{p-1}{p}}\|\Omega_j\|_\infty|Q|\,\|b\|_{\dot{\mathcal{X}}_1}\|h\|_{\mathcal{Y}_p}.$$

We may now complete the proof of Proposition 4.1. We assume $q < \infty$. The remaining case is actually simpler as V^j is identically zero. Our decomposition (4-3) yields that

$$|\mathsf{K}(b,h)| \le \sum_{j\ge1}|\mathsf{H}^j(b,h)| + \sum_{j\ge1}|\mathsf{V}^j(b,h)|.$$

Choosing $\delta = c(p-1)/(2p)$ in (4-2) and using Lemma 4.3, we estimate

$$\begin{aligned}\sum_{j\ge1}|\mathsf{H}^j(b,h)| &\le C|Q|\,\|b\|_{\dot{\mathcal{X}}_1}\|h\|_{\mathcal{Y}_p}\sum_{j\ge1}2^{-cj\frac{p-1}{p}}\|\Omega_j\|_\infty \\ &\le C|Q|\,\|b\|_{\dot{\mathcal{X}}_1}\|h\|_{\mathcal{Y}_p}\sum_{j\ge1}2^{-cj\frac{p-1}{2p}} \le \frac{Cp}{p-1}|Q|\,\|b\|_{\dot{\mathcal{X}}_1}\|h\|_{\mathcal{Y}_p},\end{aligned}$$

which is smaller than the right-hand side of (4-1). Using Lemma 4.2, the latter sum involving V_j is then estimated by

$$\Big(\sum_{j\geq 1}\|\Delta_j\|_q\Big)|Q|\,\|b\|_{\mathcal{X}_1}\|h\|_{\mathcal{Y}_p}\leq \frac{Cp}{p-1}\|\Omega\|_{L^{q,1}\log L(S^{d-1})}|Q|\,\|b\|_{\mathcal{X}_1}\|h\|_{\mathcal{Y}_p},$$

which also complies with the right-hand side of (4-1); here we have used that

$$\sum_{j\geq 1}\|\Delta_j\|_q\leq \sum_{j\geq 1}\sum_{k\geq j}2^{\delta k}|O_k\setminus O_{k+1}|^{\frac{1}{q}}\leq \sum_{k\geq 1}k2^{\delta k}|O_k\setminus O_{k+1}|^{\frac{1}{q}}\leq \frac{C}{\delta}\|\Omega\|_{L^{q,1}\log L(S^{d-1})}.$$

The proposition is thus proved up to establishing Lemma 4.3.

Proof of Lemma 4.3. Our first observation is actually another trivial estimate.

Lemma 4.4. *There exists $C>0$ such that $|\mathsf{H}^j(b,h)|\leq C\|\Omega_j\|_\infty|Q|\,\|b\|_{\mathcal{X}_1}\|h\|_{\mathcal{Y}_1}$.*

Proof. This is an application of Lemma 3.1 to $K=\{H_s^j\}$ with $\beta=\infty$, as it is immediate to see that for this kernel one has $[K]_{0,\infty}\leq C\|\Omega_j\|_\infty$. □

The second step is an estimate with decay, but involving $\mathcal{Y}_\infty$ norms.

Lemma 4.5. *There exist $C,c>0$ such that $|\mathsf{H}^j(b,h)|\leq C2^{-cj}\|\Omega_j\|_\infty|Q|\,\|b\|_{\dot{\mathcal{X}}_1}\|h\|_{\mathcal{Y}_\infty}$.*

Before the proof of Lemma 4.5, which is given in the next subsection, we observe that the estimate of Lemma 4.3 is obtained by Riesz–Thorin (for instance) interpolation in h of the last two lemmata.

Proof of Lemma 4.5. The techniques of this subsection are an elaboration of the arguments of [Seeger 1996]. In particular Lemma 4.6 below is a stronger version of Lemma 2.1 of that work, while Lemma 4.7 is essentially the dual form of its Lemma 2.2.

We perform a further decomposition of H_s^j. Let $\Xi=\{e_\nu\}$ be a maximal 2^{-j-10d}-separated set contained in $\operatorname{supp}\Omega_j$. We may partition $\operatorname{supp}\Omega_j$ into $\#\Xi\lesssim 2^{j(d-1)}$ subsets E_ν each containing e_ν and such that $\operatorname{diam}|E_\nu|\lesssim 2^{-j}$. Set

$$H_{s\nu}^j(x)=H_s^j(x)\mathbf{1}_{E_\nu}(x').$$

Also, let ψ be a smooth function on $\mathbb{R}$ with $\mathbf{1}_{[-2,2]}\leq\psi\leq\mathbf{1}_{[-4,4]}$. Let $\kappa\in[0,1)$ and define the multiplier operator

$$\widehat{P}_\nu^j(\xi)=\psi(2^{j(1-\kappa)}\xi'\cdot e_\nu).$$

We now have the decomposition

$$H_s^j:=\Gamma_s^j+\Upsilon_s^j,\quad \Gamma_s^j:=\sum_\nu P_\nu^j*H_{s\nu}^j,\quad \Upsilon_s^j:=H_s^j-\Gamma_s^j$$

so that H^j is the sum of the single-scale bilinear forms

$$\mathsf{G}_j(b,h)=\Big\langle\sum_s\Gamma_s^j*b_{s-j},\bar{h}\Big\rangle,\quad \mathsf{U}_j(b,h)=\Big\langle\sum_s\Upsilon_s^j*b_{s-j},\bar{h}\Big\rangle$$

satisfying the estimates below.

Lemma 4.6. *Let $\tau > 1$. Then*

$$|\mathsf{G}_j(b,h)| \le C_\tau 2^{-j\frac{(1-\kappa)}{2}} \|\Omega_j\|_\infty |Q| \|b\|_{\mathcal{X}_1} \|h\|_{\mathcal{Y}_\tau}, \quad C_\tau = \frac{C\tau}{\tau-1}.$$

Lemma 4.7. *Let $b \in \dot{\mathcal{X}}_1$. For all $\varepsilon > 0$ there exists a constant $C_{\kappa,\varepsilon}$ depending on κ, ε only such that*

$$|\mathsf{U}_j(b,h)| \le C_{\kappa,\varepsilon} 2^{-\varepsilon j} \|\Omega_j\|_\infty |Q| \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_\infty}.$$

Notice that the combination of Lemma 4.6 with $\tau = 2$ and $\kappa = \frac{1}{2}$ and Lemma 4.7 with $\varepsilon = \frac{1}{4}$ yields the required estimate for Lemma 4.5, with $c = \frac{1}{4}$. Lemma 4.5 is thus proved up to the arguments for Lemmata 4.6 and 4.7.

Proof of Lemma 4.6. We may factor out $\|\Omega_j\|_\infty$ and assume that the angular part in the definition of Γ_j is bounded by 1. We can also assume that $H^j_{s\nu}$ and b are positive as cancellation plays no role in this argument; this is just a matter of saving space in the notation. Using interpolation and duality with t below being the dual exponent of τ, the estimate of the lemma follows if we show that for each integer $r \ge 1$ and $t = 2r$

$$\frac{1}{|Q|^{\frac{1}{t}}} \Big\| \sum_s \Gamma^j_s * b_{s-j} \Big\|_t \lesssim t 2^{-\frac{j(1-\kappa)}{2}} \|b\|_{\mathcal{X}_1} \tag{4-4}$$

with an implicit constant that does not depend on r. Setting

$$M_\nu = \sum_s P^j_\nu * H^j_{s\nu} * b_{j-s}, \quad D_\nu = \sum_s H^j_{s\nu} * b_{s-j},$$

we rewrite the left-hand side of (4-4) raised to t-th power and subsequently estimate

$$\begin{aligned} \Big\| \sum_{\nu_1,\dots,\nu_r} \prod_{k=1}^r M_{\nu_k} \Big\|_2^2 = \Big\| \sum_{\nu_1,\dots,\nu_r} \widehat{M}_{\nu_1} * \cdots * \widehat{M}_{\nu_r} \Big\|_2^2 &\lesssim 2^{rj(d-2+\kappa)} \sum_{\nu_1,\dots,\nu_r} \Big\| \prod_{k=1}^r D_{\nu_k} \Big\|_2^2 \\ &\lesssim 2^{tj(d-1)} 2^{-rj(1-\kappa)} \sup_\nu \|D_\nu\|_t^t. \end{aligned} \tag{4-5}$$

We have used Plancherel for the first equality, followed by the observation that $\widehat{P}^j_{\nu_k}(\xi)$ is uniformly bounded and nonzero only if $|\xi' - e_{\nu_k}| < 2^{-j(1-\kappa)}$. Thus there are at most $C 2^{rj(d-2+\kappa)}$ r-tuples such that the r-fold convolution is nonzero, whence the first bound. Another usage of Plancherel, the observation that there are at most $2^{rj(d-1)}$ tuples in the summation, and finally Hölder's inequality yield the second bound. We are thus done if we estimate for each fixed ν

$$\sum_{s_1 \ge \cdots \ge s_t} \int \Big(\prod_{k=1}^t H^j_{s_k\nu}(x - y_k) b_{s_k-j}(y_k) \Big) \,\mathrm{d}y_1 \cdots \mathrm{d}y_t \,\mathrm{d}x \lesssim C^t 2^{-tj(d-1)} |Q| \|b\|^t_{\mathcal{X}_1} \tag{4-6}$$

as $\|D_\nu\|_t^t$ is at most t^t times the above integral. Notice that if $\sigma \le s$ then supp $H^j_{\sigma\nu}$ is contained in a box R_s centered at zero and having one long side of length $\lesssim 2^s$ and $d-1$ short sides of length 2^{s-j}. If $z \in \mathbb{R}^d$, $R_s(z) = z + R_s$ and

$$\mathcal{Q}_s(z) = \{L \in \mathcal{Q} : s_L \le s - j,\ L \subset 100 R_s(z)\}, \quad \mathsf{b}_{R_s(z)} := \sum_{L \in \mathcal{Q}_s(z)} b_L,$$

we have, by the disjointness of $L \in \mathcal{Q}$,

$$2^{-sd}\|\mathsf{b}_{R_s(z)}\|_1 \lesssim 2^{-sd}|R_s(z)|\,\|b\|_{\mathcal{X}_1} \le C2^{-j(d-1)}\|b\|_{\mathcal{X}_1} := \alpha. \tag{4-7}$$

Also notice that for all fixed $y_1, \dots, y_t$ and for all $s_1 \ge \cdots \ge s_t$,

$$I_{s_1,\dots,s_t}(y_1,\dots,y_t) := \int \biggl(\prod_{k=1}^{t} H^j_{s_k\nu}(x-y_k)\biggr)\,\mathrm{d}x \le \|H^j_{s_t\nu}\|_1 \prod_{k=1}^{t-1}\|H^j_{s_k\nu}\|_\infty \lesssim 2^{-j(d-1)}2^{-d\mathsf{s}_{t-1}},$$

where, here and in what follows, we set

$$\mathsf{s}_n = \sum_{k=1}^{n} s_k, \quad n = 1, \dots, t.$$

Furthermore, $I_{s_1,\dots,s_t}(y_1,\dots,y_t)$ is nonzero only if $y_k \in 2R_{s_{k-1}}(y_{k-1})$ for $k = t, t-1, \dots, 2$. Now, writing b_{s_k} in place of b_{s_k-j} for reasons of space as j is kept fixed throughout and using (4-7) repeatedly, the sum in (4-6) is equal to

$$\begin{aligned}
&\sum_{s_1\ge\cdots\ge s_t}\int I_{s_1,\dots,s_t}(y_1,\dots,y_t)\biggl(\prod_{k=1}^{t}\mathsf{b}_{s_k}(y_k)\biggr)\mathrm{d}y_1\cdots\mathrm{d}y_t\\
&\lesssim 2^{-j(d-1)}\sum_{s_1\ge\cdots\ge s_{t-1}}2^{-d\mathsf{s}_{t-2}}\int \mathsf{b}_{s_1}(y_1)\biggl(\prod_{k=2}^{t-1}\mathsf{b}_{s_k}(y_k)\mathbf{1}_{2R_{s_{k-1}}(y_{k-1})}(y_k)\biggr)\frac{\|\mathsf{b}_{R_{s_{t-1}}(y_{t-1})}\|_1}{2^{ds_{t-1}}}\,\mathrm{d}y_1\cdots\mathrm{d}y_{t-1}\\
&\lesssim \alpha 2^{-j(d-1)}\sum_{s_1\ge\cdots\ge s_{t-2}}2^{-d\mathsf{s}_{t-3}}\int \mathsf{b}_{s_1}(y_1)\biggl(\prod_{k=2}^{t-2}\mathsf{b}_{s_k}(y_k)\mathbf{1}_{2R_{s_{k-1}}(y_{k-1})}(y_k)\biggr)\frac{\|\mathsf{b}_{R_{s_{t-2}}(y_{t-2})}\|_1}{2^{ds_{t-2}}}\,\mathrm{d}y_1\cdots\mathrm{d}y_{t-2}\\
&\lesssim\cdots\lesssim \alpha^{t-1}2^{-j(d-1)}|Q|\,\|b\|_{\mathcal{X}_1} \le C^t 2^{-tj(d-1)}|Q|\,\|b\|^t_{\mathcal{X}_1}
\end{aligned}$$

as claimed, and this completes the proof. □

Proof of Lemma 4.7. Again we factor out $\|\Omega_j\|_\infty$ and work under the assumption that the angular part is bounded by 1. In this proof, M is a large integer whose value may differ at each occurrence and the constants implied by the almost inequality sign are allowed to depend on M only. Let β be a smooth function supported in $A_1 = \{2^{-1} \le |\xi| \le 2\}$ and satisfying

$$\sum_{k\in\mathbb{Z}}\beta^2(2^k\xi) = 1, \quad \xi \ne 0.$$

Set $B_k = \mathcal{F}^{-1}\{\beta(2^k\,\cdot)\}$. Defining

$$\widehat{R}^{jk}_{s\nu}(\xi) = \beta(2^k\xi)(1-\widehat{P}^j_\nu(\xi))\widehat{H}^j_{s\nu}(\xi),$$

we recall from [Seeger 1996, equations (2.6), (2.7)] the estimate

$$\|R^{jk}_{s\nu}\|_1 \lesssim_M 2^{-j(d-1)}\min\{1, 2^{-M\kappa j}2^{-M(s-j-k)}\}.$$

Now, fix s and $L \in \mathcal{Q}$ with $\ell(L) = 2^{s-j}$ for the moment. Recalling the definition of Υ_s^j, we have the decomposition

$$|\langle \Upsilon_s^j * b_L, \bar{h}\rangle| \le \sum_\nu \sum_k |\langle R_{s\nu}^{jk} * B_k * b_L, \bar{h}\rangle|,$$

and the cancellation estimate (cf. [Seeger 1996, equation (2.5)], a simpler version of Lemma 3.2)

$$\begin{aligned} |\langle R_{s\nu}^{jk} * B_k * b_L, \bar{h}\rangle| &\lesssim \min\{1, 2^{(s-j)-k}\} \|R_{s\nu}^{jk}\|_1 \|b_L\|_1 \|h\|_\infty \\ &\lesssim 2^{-j(d-1)} \min\{2^{(s-j)-k}, 2^{-M\kappa j - M(s-j-k)}\} |L| \, \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_\infty}. \end{aligned} \tag{4-8}$$

Note that $\#\Xi \lesssim 2^{j(d-1)}$. So for each $\varepsilon > 0$ we can use the left estimate in (4-8) for $k \ge s - j(1-\varepsilon)$ and the right estimate otherwise, and obtain

$$|\langle \Upsilon_s^j * b_L, \bar{h}\rangle| \le \sum_\nu \sum_k |\langle R_{s\nu}^{jk} * B_k * b_L, \bar{h}\rangle| \lesssim 2^{-\varepsilon j} |L| \, \|b\|_{\dot{\mathcal{X}}_1} \|h\|_{\mathcal{Y}_\infty} \tag{4-9}$$

provided that M is chosen large enough to have $2\varepsilon < M\kappa$. The proof is thus completed by summing (4-9) over $L \in \mathcal{Q}$ with $\ell(L) = 2^{s-j}$ and later over s. □

5. Proof of Theorem B

Throughout this proof, C is a positive absolute dimensional constant which may vary at each occurrence without explicit mention. Most of the arguments in this section are contained in [Christ 1988, Section 3]; we reproduce the details for clarity.

Let $\psi(x) = \cos(2\pi(|x| - \delta/4))$. From the asymptotic expansion of the inverse Fourier transform of the multiplier of B_δ [Christ 1988, Section 3], which is $\mathcal{C}^\infty$ and radial, we obtain the kernel representation

$$B_\delta(x) = \sum_{s \ge 1} \sum_\nu K_{s,\nu}(x) + L(x).$$

Here

$$K_{s,\nu}(x) = \Omega_\nu(x')\psi(x) 2^{-sd} \phi(2^{-s}x),$$

with Ω_ν a finite smooth partition of unity on the unit sphere S^{d-1} with sufficiently small support which is introduced for technical reasons, and ϕ a suitable smooth radial function supported in $A_1 = \{2^{-2} \le |x| \le 1\}$, while $L(x)$ is an integrable kernel with $L(x) \le C(1+|x|)^{-(d+1)}$, so that

$$Lf(x) \le C\mathrm{M}_1 f(x),$$

which can be ignored for our purposes. We can also think of ν as fixed and omit it from the notation, and consider the kernel $K = \{K_s\}$ as above. We are going to verify that conditions in Theorem C are satisfied by (the dual form to) B_δ. First of all, condition (SS) is obvious from the above discussion as $[K]_{0,\infty} < \infty$. Second, the (T) condition follows from the well-known estimate

$$\sup_{\mu,\nu} \|\Lambda_\mu^\nu\|_{L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} \le C;$$

see for instance [Duoandikoetxea and Rubio de Francia 1986, Theorem E]. In order to verify condition (L), let $\mathcal{Q}$ be a stopping collection with top Q. Let $b \in \mathcal{X}_1(\mathcal{Q})$; we change a bit the notation for b_s in this context by redefining

$$b_s := \sum_{s_L = s} b_L, \quad s \geq 1, \qquad b_0 := \sum_{s_L \leq 0} b_L.$$

It is easy to see that in this context if $b \in \mathcal{X}_1$ supported on Q and $h \in \mathcal{Y}_1$, one has

$$\Lambda_{\mathcal{Q},\mu,\nu}(b,h) = \left\langle \sum_{j \geq 1} \sum_{s \geq j} \varepsilon_s K_s * b_{s-j}, \bar{h} \right\rangle$$

for a suitable choice of signs $\{\varepsilon_s\} \in \{-1,0,1\}^{\mathbb{Z}}$, and the same for $\Lambda_{\mathcal{Q},\mu,\nu}(h,b)$ up to replacing b by b^{in}, restricting h to be supported on Q, transposing K_s, and subtracting the remainder terms, which are estimated by

$$|Q| \, \|b\|_{\mathcal{X}_1} \|h\|_{\mathcal{Y}_1}.$$

Theorem B is thus obtained from the next proposition via an application of Theorem C.

Proposition 5.1. *Let $\{\varepsilon_s\} \in \{-1,0,1\}^{\mathbb{Z}}$ be a choice of signs, $b \in \mathcal{X}_1$ and define*

$$\mathsf{K}(b,h) := \left\langle \sum_{j \geq 1} \sum_{s \geq j} \varepsilon_s K_s * b_{s-j}, \bar{h} \right\rangle.$$

There exists an absolute constant C, in particular uniform over $\{\varepsilon_s\} \in \{-1,0,1\}^{\mathbb{Z}}$, such that

$$|\mathsf{K}(b,h)| \leq \frac{Cp}{p-1} |Q| \, \|b\|_{\mathcal{X}_1} \|h\|_{\mathcal{Y}_p}.$$

Notice that here we do not need to require $b \in \dot{\mathcal{X}}_1$ as per the oscillatory nature of the problem.

Proof of Proposition 5.1. Given our choice of $\{\varepsilon_s\} \in \{-1,0,1\}^{\mathbb{Z}}$, we relabel $K_s := \varepsilon_s K_s$. It will be clear from the proof that the signs ε_s play no role. We split

$$\mathsf{K}(b,h) = \sum_{j \geq 1} \mathsf{K}^j(b,h), \quad \mathsf{K}^j(b,h) := \sum_{s \geq j} \langle K_s * b_{s-j}, \bar{h} \rangle.$$

The first estimate is a trivial one.

Lemma 5.2. *There exists $C > 0$ such that $|\mathsf{K}^j(b,h)| \leq C|Q| \, \|b\|_{\mathcal{X}_1} \|h\|_{\mathcal{Y}_1}$.*

Proof. This follows from applying Lemma 3.1 with $\beta = \infty$ to $K = \{K_s\}$, as it is immediate to see that for this kernel one has $[K]_{0,\infty} \leq C$ as already remarked. □

The second estimate, which is essentially contained in [Christ 1988, Section 3], is the one providing decay.

Lemma 5.3. *There exists $C, c > 0$ such that $|\mathsf{K}^j(b,h)| \leq C 2^{-cj} |Q| \, \|b\|_{\mathcal{X}_1} \|h\|_{\mathcal{Y}_2}$.*

It is easy to see that interpolating the above estimates yields

$$|\mathsf{K}^j(b,h)| \le C2^{-j\frac{c(p-1)}{p}}|Q|\,\|b\|_{\mathcal{X}_1}\|h\|_{\mathcal{Y}_p},$$

the summation of which yields Proposition 5.1.

Proof of Lemma 5.3. Let $\widetilde{K}_s(\cdot) = \overline{K_s(-\cdot)}$. We recall from [Christ 1988, Lemma 3.1] the estimates

$$\begin{aligned} &|K_s * \widetilde{K}_s(x)| \le C2^{-ds}(1+|x|)^{-\delta}, \\ &\|K_s * \widetilde{K}_t\|_\infty \le C2^{-dt}2^{-\delta s}, \quad \forall s < t-1. \end{aligned} \tag{5-1}$$

By duality, it suffices to prove that

$$\|K_j * b_0\|_2^2 + \Big\|\sum_{s>j} K_s * b_{s-j}\Big\|_2^2 \le C2^{-cj}|Q|\,\|b\|_{\mathcal{X}_1}^2. \tag{5-2}$$

For the first term we use the first estimate in (5-1):

$$\|K_j * b_0\|_2^2 = |\langle b_0, K_j * \widetilde{K}_j * b_0\rangle| \le \|b_0\|_1\|K_j * \widetilde{K}_j * b_0\|_\infty \le C2^{-\min(\delta,d)j}|Q|\,\|b\|_{\mathcal{X}_1}^2.$$

The last inequality above follows from

$$\|K_j * \widetilde{K}_j * b_0\|_\infty \le 2^{-jd}\sum_{m=0}^{j} 2^{-m\delta}\sup_{x\in\mathbb{R}^d}\|b_0\|_{L^1(B(x,C2^m))} \le C2^{-\min(\delta,d)j}\|b\|_{\mathcal{X}_1},$$

where $B(x, C2^m)$ denotes a ball centered at x with radius $C2^m$. For the second term, we begin by quoting from [Christ 1988, inequality (3.2)] that

$$\|K_s * b_{s-j}\|_2^2 \le C2^{-\delta j}\|b\|_{\mathcal{X}_1}\|b_{s-j}\|_1. \tag{5-3}$$

Observe that

$$\begin{aligned} &\Big\|\sum_{s>j} K_s * b_{s-j}\Big\|_2^2 \\ &\quad\le \sum_{s>j}\|K_s*b_{s-j}\|_2^2 + 2\sum_s|\langle K_s*b_{s-j}, K_{s-1}*b_{s-1-j}\rangle| + 2\sum_t\sum_{j<s<t-1}|\langle \widetilde{K}_t*K_s*b_{s-j}, b_{t-j}\rangle|. \end{aligned} \tag{5-4}$$

The first two terms are bounded by

$$C2^{-\delta j}\|b\|_{\mathcal{X}_1}\sum_s\|b_{s-j}\|_1 \le C2^{-\delta j}|Q|\,\|b\|_{\mathcal{X}_1}^2,$$

according to (5-3) for the first one and Cauchy–Schwarz followed by (5-3) for the second. For the third term, from the second estimate of (5-1) and support considerations one has

$$\|\widetilde{K}_t * K_s * b_{s-j}\|_\infty \le C\big(\sup_{x\in\mathbb{R}^d}\|b_{s-j}\|_{L^1(B(x,C2^t))}\big)\|\widetilde{K}_t * K_s\|_\infty \le C2^{-\delta s}\|b\|_{\mathcal{X}_1}.$$

Therefore, the third summand in (5-4) is dominated by

$$C\|b\|_{\mathcal{X}_1}\sum_{t>j}\|b_{t-j}\|_1\sum_{j<s<t-1}2^{-\delta s} \le C2^{-\delta j}|Q|\,\|b\|_{\mathcal{X}_1}^2,$$

and collecting all the above estimates (5-2) follows. □

Appendix A: Verification of (2-10)–(2-11)

Let $\mathcal{Q}$ be a stopping collection with top Q, $h \in \mathcal{Y}_{q'}$, $b \in \mathcal{X}_{q'}$. Clearly we can assume $\operatorname{supp} h \subset Q$. By possibly replacing K_s by zero when $s \notin (\mu, \nu]$ we can ignore the truncations μ, ν in what follows and omit them from the notation. Recall the definitions (2-4), (2-5)

$$\Lambda_{\mathcal{Q}}(h,b) = \Lambda_Q(h,b) - \sum_{\substack{R\in\mathcal{Q}\\ R\subset Q}} \Lambda_R(h,b) = \Lambda^{s_Q}(h,b) - \sum_{\substack{R\in\mathcal{Q}\\ R\subset Q}} \Lambda^{s_R}(h\mathbf{1}_R, b)$$

and the decomposition

$$b = b^{\mathrm{in}} + b^{\mathrm{out}}, \quad b^{\mathrm{in}} = \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 2Q\neq\varnothing}} b_L, \quad b^{\mathrm{out}} = \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 2Q=\varnothing}} b_L.$$

We first estimate

$$|\Lambda_{\mathcal{Q}}(h, b^{\mathrm{out}})| \lesssim [K]_{0,q}|Q|\,\|h\|_{\mathcal{Y}_1}\|b\|_{\mathcal{X}_{q'}}, \tag{A-1}$$

which is a single-scale estimate. In fact, since $\operatorname{dist}(R, \operatorname{supp} b^{\mathrm{out}}) \ge \ell(R)/2$ for all $R \subset Q$, by virtue of the support restriction in (SS),

$$s < s_R \implies \int K_s(x,y)h(y)\mathbf{1}_R(y)b^{\mathrm{out}}(x)\,\mathrm{d}y\,\mathrm{d}x = 0.$$

Therefore, by the same argument used in (3-1),

$$|\Lambda^{s_Q}(h, b^{\mathrm{out}})| \le \int |K_{s_Q}(x,y)|\,|h(y)|\,|b^{\mathrm{out}}(x)|\,\mathrm{d}y\,\mathrm{d}x \lesssim [K]_{0,q}|Q|\,\|h\|_{\mathcal{Y}_1}\|b\|_{\mathcal{X}_{q'}}. \tag{A-2}$$

Proceeding similarly, if $R \in \mathcal{Q}$, $R \subset Q$

$$|\Lambda^{s_R}(h\mathbf{1}_R, b^{\mathrm{out}})| \le \int |K_{s_R}(x,y)|\,|h\mathbf{1}_R(y)|\,|b^{\mathrm{out}}(x)|\,\mathrm{d}y\,\mathrm{d}x \lesssim [K]_{0,q}|R|\,\|h\|_{\mathcal{Y}_1}\|b\|_{\mathcal{X}_{q'}}.$$

and the claimed (A-1) follows by summing the last display over $R \in \mathcal{Q}, R \subset Q$, which are pairwise disjoint, and combining the result with (A-2). The representation (2-10) will then be a simple consequence of the equality

$$\Lambda_{\mathcal{Q}}(h, b^{\mathrm{in}}) = \Bigg(\Lambda^{s_Q}(h, b^{\mathrm{in}}) - \sum_{L\in\mathcal{Q}:3L\cap 2Q\neq\varnothing} \Lambda^{s_L}(h, b_L)\Bigg) + V_{\mathcal{Q}}(h,b), \tag{A-3}$$

where the remainder $V_{\mathcal{Q}}$ satisfies

$$|V_{\mathcal{Q}}(h,b)| \lesssim [K]_{0,q}|Q|\,\|h\|_{\mathcal{Y}_{q'}}\|b\|_{\mathcal{X}_{q'}}. \tag{A-4}$$

We turn to the proof of (A-3). We will use below without explicit mention that whenever $L, R \in \mathcal{Q}$ with $3R \cap 3L \neq \varnothing$, we have $|s_L - s_R| < 8$, a consequence of the separation property (2-3). First of all, the restriction on the support (SS) gives that

$$\sum_{R\in\mathcal{Q}} \Lambda^{s_R}(h\mathbf{1}_R, b^{\mathrm{in}}) = \sum_{R\in\mathcal{Q}} \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 3R\neq\varnothing\\ 3L\cap 2Q\neq\varnothing}} \Lambda^{s_R}(h\mathbf{1}_R, b_L), \tag{A-5}$$

as $\Lambda^{s_R}(h\,\mathbf{1}_R, b_L) = 0$ unless $3L \cap 3R$ is nonempty. As there are at most 16 s-scales in each difference $\Lambda^{s_L} - \Lambda^{s_R}$, using the trivial estimate (3-1) with $\beta = q$ for each such scale yields

$$\begin{aligned}\sum_{R\in\mathcal{Q}} \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 3R\neq\varnothing\\ 3L\cap 2Q\neq\varnothing}} &\left|\Lambda^{s_L}(h\,\mathbf{1}_R, b_L) - \Lambda^{s_R}(h\,\mathbf{1}_R, b_L)\right| \\ &\lesssim [K]_{0,q}\|h\|_{\mathcal{Y}_{q'}} \sum_{R\in\mathcal{Q}} \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 3R\neq\varnothing\\ 3L\cap 2Q\neq\varnothing}} \|b_L\|_1 \\ &\lesssim [K]_{0,q}\,\|h\|_{\mathcal{Y}_{q'}}\|b\|_{\mathcal{X}_1} \sum_{R\in\mathcal{Q}} |R| \lesssim [K]_{0,q}\,|Q|\,\|h\|_{\mathcal{Y}_{q'}}\|b\|_{\mathcal{X}_1}. \end{aligned} \tag{A-6}$$

Recalling the second property of stopping collections in (2-3), we have the decomposition

$$h = h^{\text{in}} + h^{\text{out}}, \quad h^{\text{in}} := h\,\mathbf{1}_{\bigcup_{R\in\mathcal{Q}} R}, \quad \operatorname{supp} h^{\text{out}} \cap \Bigg(\bigcup_{\substack{L\in\mathcal{Q}\\ 3L\cap 2Q\neq\varnothing}} 9L\Bigg) = \varnothing.$$

Therefore, up to including the error term of (A-6) in (A-4), (A-5) can be rewritten as

$$\begin{aligned}\sum_{R\in\mathcal{Q}} \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 3R\neq\varnothing\\ 3L\cap 2Q\neq\varnothing}} \Lambda^{s_L}(h\,\mathbf{1}_R, b_L) &= \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 2Q\neq\varnothing}} \Lambda^{s_L}(h^{\text{in}}, b_L) - \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 2Q\neq\varnothing}} \Lambda^{s_L}(\tilde{h}_L, b_L), \\ \tilde{h}_L = \sum_{\substack{R\in\mathcal{Q}\\ 3L\cap 3R=\varnothing}} h\,\mathbf{1}_R, &\quad \operatorname{supp} \tilde{h}_L \subset \mathbb{R}^d \setminus 3L.\end{aligned} \tag{A-7}$$

We note that all the terms in the second sum on the right-hand side of the first line of (A-7) vanish due to the support restriction on K_s, as all the scales appearing are less than or equal to s_L and $\operatorname{supp} b_L \subset L$. The reasoning beginning with decomposition (A-5) leads thus to the equality, up to tolerable error terms,

$$\sum_{R\in\mathcal{Q}} \Lambda^{s_R}(h\,\mathbf{1}_R, b^{\text{in}}) = \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 2Q\neq\varnothing}} \Lambda^{s_L}(h, b_L) - \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 2Q\neq\varnothing}} \Lambda^{s_L}(h^{\text{out}}, b_L). \tag{A-8}$$

Finally the second term on the right-hand side of (A-8) also vanishes, by virtue of the restriction on the support of h^{out}, which does not intersect $9L$ for any L in the sum. Therefore, (A-8) is actually the equality

$$\sum_{\substack{R\in\mathcal{Q}\\ R\subset Q}} \Lambda^{s_R}(h\,\mathbf{1}_R, b^{\text{in}}) = \sum_{R\in\mathcal{Q}} \Lambda^{s_R}(h\,\mathbf{1}_R, b^{\text{in}}) = \sum_{\substack{L\in\mathcal{Q}\\ 3L\cap 2Q\neq\varnothing}} \Lambda^{s_L}(h, b_L) + V_Q(h, b),$$

where $V_Q(h, b)$ satisfies (A-4); the first equality in the above display is due to $\operatorname{supp} h \subset Q$. This equality clearly implies the sought after (A-3).

Appendix B: Sparse domination implies weak L^1 estimate

We show that if a sublinear operator T satisfies the sparse estimate (1-1) for $p_1 = 1$, $p_2 = r$ for some $1 \le r < \infty$ then T is of weak type $(1, 1)$. In particular, as mentioned in the Introduction, together with

Theorem A, this yields the weak L^1 estimate of T_Ω, which is the main result of [Seeger 1996]. The proof that follows is a simplified version of the arguments in [Culiuc et al. 2016a, Appendix A]; we are sure these arguments are well known but were unable to locate a precise reference.

Theorem E. *Suppose that the sublinear operator T has the following property: there exists $C > 0$ and $1 \le r < \infty$ such that for every f_1, f_2 bounded with compact support there exists a sparse collection $\mathcal{S}$ such that*

$$|\langle Tf_1, f_2\rangle| \le C \sum_{Q\in\mathcal{S}} |Q|\langle f_1\rangle_{1,Q}\langle f_2\rangle_{r,Q}. \tag{B-1}$$

Then $T : L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ boundedly.

Proof. By standard arguments it suffices to verify that

$$\sup_{\|f_1\|_1=1}\ \sup_{G\subset\mathbb{R}^d}\ \inf_{\substack{G'\subset G\\ |G|\le 2|G'|}}\ \sup_{|f_2|\le \mathbf{1}_{G'}} |\langle Tf_1, f_2\rangle| \le C,$$

where f_1, f_2 are bounded and compactly supported and G has finite measure. Given such f_1 with $\|f_1\|_1 = 1$ and G of finite measure, define the sets

$$H := \{x\in\mathbb{R}^d : \mathrm{M}_1 f_1(x) > C|G|^{-1}\}, \quad \widetilde{H} := \bigcup_{Q\in\mathcal{Q}} 3Q, \quad \mathcal{Q} = \big\{\text{max. dyad. cube } Q : |Q\cap H| \ge 2^{-5}|Q|\big\}.$$

It is easy to see that $|\widetilde{H}| \le 2^{-10}|G|$ for suitable choice of C. Therefore the set $G' : G\backslash\widetilde{H}$ satisfies $|G| \le 2|G'|$. We make the preliminary observation that

$$\sup_{x\in H^c} \mathrm{M}_1 f_1(x) \le C|G|^{-1},$$

so that by interpolation

$$\|\mathrm{M}_1 f_1\|_{L^{p'}(H^c)} \le \big(\sup_{x\in H^c} \mathrm{M}_1 f_1(x)\big)^{1-\frac{1}{p'}} \|\mathrm{M}_1 f_1\|_{1,\infty}^{\frac{1}{p'}} \le C|G|^{-(1-\frac{1}{p'})}, \tag{B-2}$$

where $p' > 1$ is chosen such that $p > r$. Fixing now any f_2 restricted to G', we apply the domination estimate, yielding the existence of a sparse collection $\mathcal{S}$ for which we have the estimate

$$|\langle Tf_1, f_2\rangle| \le C \sum_{Q\in\mathcal{S}} |Q|\langle f_1\rangle_{1,Q}\langle f_2\rangle_{r,Q}.$$

We claim that

$$|Q\cap H| \le 2^{-5}|Q| \quad \forall Q\in\mathcal{S}. \tag{B-3}$$

This is because if (B-3) fails for Q, we know Q must be contained in $3Q'$ for some $Q'\in\mathcal{Q}$. But the support of f_2 is contained in $\widetilde{H}^c$, which does not intersect $3Q'$, whence $\langle f_2\rangle_{r,Q} = 0$. Relation (B-3) has the consequence that if $\{E_Q : Q\in\mathcal{S}\}$ denote the distinguished pairwise disjoint subsets of $Q\in\mathcal{S}$

with $|E_Q| \geq 2^{-2}|Q|$, the sets $\widetilde{E}_Q := E_Q \cap H^c$ are also pairwise disjoint and $|\widetilde{E}_Q| \geq 2^{-3}|Q|$. Therefore, since the union of $\widetilde{E}_Q$ is contained in H^c, by standard arguments we arrive at

$$|\langle Tf_1, f_2\rangle| \leq C \sum_{Q\in\mathcal{S}} |Q| \langle f_1\rangle_{1,Q} \langle f_2\rangle_{r,Q} \leq C \sum_{Q\in\mathcal{S}} |\widetilde{E}_Q| \langle f_1\rangle_{1,Q} \langle f_2\rangle_{r,Q} \leq C \int_{H^c} \mathrm{M}_1 f(x) \mathrm{M}_r f_2(x)\,\mathrm{d}x$$
$$\leq C \|\mathrm{M}_1 f_1\|_{L^{p'}(H^c)} \|\mathrm{M}_r f_2\|_{L^p(\mathbb{R}^d)} \leq C|G|^{-(1-\frac{1}{p'})}|G|^{\frac{1}{p}} \leq C,$$

using (B-2) in the last step. □

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