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GLOBAL WELL-POSEDNESS OF THE MHD EQUATIONS IN A HOMOGENEOUS MAGNETIC FIELD

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We study the MHD equations with small viscosity and resistivity coefficients, which may be different. This is a typical setting in high temperature plasmas. It was proved that the MHD equations are globally well-posed if the initial velocity is close to 0 and the initial magnetic field is close to a homogeneous magnetic field in the weighted Hölder spaces. The main novelty is that the closeness is independent of the dissipation coefficients.

1. Introduction

We consider the incompressible magnetohydrodynamics (MHD) equations in $[0, T) \times \Omega$, with $\Omega \subseteq \mathbb{R}^d$,

$$\begin{cases} \partial_t v - \nu \,\Delta v + v \cdot \nabla v + \nabla p = b \cdot \nabla b, \\ \partial_t b - \mu \,\Delta b + v \cdot \nabla b = b \cdot \nabla v, \\ \operatorname{div} v = \operatorname{div} b = 0, \end{cases} \tag{1-1}$$

where v denotes the velocity field and b denotes the magnetic field, $\nu \geq 0$ is the viscosity coefficient, and $\mu \geq 0$ is the resistivity coefficient. If $\nu = \mu = 0$, (1-1) consists of the so-called ideal MHD equations; if $\nu > 0$ and $b = 0$, (1-1) is reduced to the Navier–Stokes equations. We refer to [Sermange and Temam 1983] for a mathematical introduction to the MHD equations.

It is well known that the 2-dimensional MHD equations with full viscosities (i.e., $\nu > 0$ and $\mu > 0$) have a global smooth solution. In the general case, the question of whether a smooth solution of the MHD equations develops a singularity in finite time is basically open [Sermange and Temam 1983; Cordoba and Fefferman 2001]. Recently, Cao and Wu [2011] studied the global regularity of the 2-dimensional MHD equations with partial dissipation and magnetic diffusion. We refer to [Cao et al. 2013; Chemin et al. 2016; Fefferman et al. 2014; He et al. 2014; Jiu et al. 2015; Lei 2015] for more relevant results.

In this paper, we are concerned with the global well-posedness of the MHD equations in a homogeneous magnetic field B_0. Recently, there have been a lot of works [Abidi and Zhang 2016; Lin et al. 2015; Ren et al. 2014; 2016; Zhang 2014] devoted to the case without resistivity (i.e, $\nu > 0$ and $\mu = 0$). Roughly speaking, it was proved that the MHD equations are globally well-posed and the solution decays in time if the initial velocity field is close to 0 and the initial magnetic field is close to B_0. These results especially justify the numerical observation [Califano and Chiuderi 1999]: the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity.

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In high temperature plasmas, both the viscosity coefficient ν and resistivity coefficient μ are usually very small [Califano and Chiuderi 1999]. Up to now, the heating mechanism of the solar corona is still an unsolved problem in physics [Priest et al. 1998], so it is very interesting to investigate the long-time dynamics of the MHD equations in the case when the dissipation coefficients are very small.

For simplicity, let us first look at the case $\mu = \nu$. Following [Bardos et al. 1988], we rewrite the system (1-1) in terms of the Elsässer variables

$$Z_+ = v + b, \quad Z_- = v - b.$$

Then the ideal MHD equations (1-1) can be written as

$$\begin{cases} \partial_t Z_+ + Z_- \cdot \nabla Z_+ = \nu \, \Delta Z_+ - \nabla p, \\ \partial_t Z_- + Z_+ \cdot \nabla Z_- = \nu \, \Delta Z_- - \nabla p, \\ \operatorname{div} Z_+ = \operatorname{div} Z_- = 0. \end{cases} \tag{1-2}$$

We introduce the fluctuations

$$z_+ = Z_+ - B_0, \quad z_- = Z_- + B_0.$$

Then the system (1-2) can be reformulated as

$$\begin{cases} \partial_t z_+ + Z_- \cdot \nabla z_+ = \nu \, \Delta z_+ - \nabla p, \\ \partial_t z_- + Z_+ \cdot \nabla z_- = \nu \, \Delta z_- - \nabla p, \\ \operatorname{div} z_+ = \operatorname{div} z_- = 0. \end{cases} \tag{1-3}$$

In the case of $\Omega = \mathbb{R}^d$ and $\nu = 0$, Bardos, Sulem and Sulem [Bardos et al. 1988] proved that for large time, the solution $z_\pm$ of (1-3) tends to linear Alfvén waves:

$$\partial_t w_\pm \mp B_0 \cdot \nabla w_\pm = 0.$$

Cai and Lei [2016] and He, Xu and Yu [He et al. 2016] studied the global well-posedness of (1-1) for any $\nu \geq 0$ and $\Omega = \mathbb{R}^3$. The result in [Cai and Lei 2016] also includes the case $\Omega = \mathbb{R}^2$. These works are based on an important observation: the nonlinear terms $z_- \cdot \nabla z_+$ and $z_+ \cdot \nabla z_-$ can be essentially neglected after a long time since $z_\pm$ are transported along the opposite direction. To justify this observation, the key point is to make weighted estimates for the fluctuations $z_\pm$. Due to the nonlocal pressure, the choice of weight function is very delicate. On the other hand, the viscosity gives rise to more technical troubles compared with the ideal case.

From the physical point of view, it is more natural to consider the MHD equations in a domain with boundary. One frequently used domain in physics is a slab bounded by two hyperplanes, i.e., $\Omega = \mathbb{R}^{d-1} \times [0, 1]$. More importantly, although both ν and μ are very small, they should be different in the real case. However, the proof in [Cai and Lei 2016; He et al. 2016] strongly relies on the facts that Ω is a whole space and $\nu = \mu$. In particular, the formulation (1-3) plays a crucial role.

The main goal of this paper is to prove the global well-posedness of (1-1) in the physical case when Ω is a slab and $\nu \neq \mu$. In this case, we need to impose suitable boundary conditions on $z_\pm$. Let $z_\pm$ be a function of (t, x, y), $(x, y) \in \Omega$. In the case when $\nu = \mu = 0$, we impose the nonpenetrating boundary condition

$$z_{\pm}^{d} = 0 \quad \text{on } y = 0, 1. \tag{1-4}$$

In the case when $\nu > 0$ and $\mu > 0$, we impose the Navier-slip boundary condition

$$z_{\pm}^{d} = 0, \qquad \partial_d z_{\pm}^{i} = 0, \quad i = 1, \dots, d-1, \qquad \text{on } y = 0, 1. \tag{1-5}$$

To deal with the boundary case, our idea is to use the symmetric extension and solve the MHD equations in the framework of Hölder spaces $C^{1,\alpha}$ for $0 < \alpha < 1$. In the ideal case, we give a representation formula of the pressure by using the symmetric extension. Although the extended solution does not have the same regularity as the original one under the nonpenetrating boundary condition, we have the important observation that ∇p still lies in $C^{1,\alpha}$ based on the representation formula. In the viscous case, we can reduce the slab domain to $\Omega = \mathbb{R}^{d-1} \times \mathbb{T}$ by using the symmetric extension, because the extended solution still keeps the $C^{1,\alpha}$ regularity under the Navier-slip boundary condition.

The most challenging task comes from the case $\nu \neq \mu$. To handle this case, we need to introduce some new ideas. First of all, we introduce a key decomposition: let $\mu_1 = \frac{1}{2}(\nu + \mu)$, $\mu_2 = \frac{1}{2}(\nu - \mu)$, and we have the decompositions $z_+ = z_+^{(1)} + z_+^{(2)}$ and $z_- = z_-^{(1)} + z_-^{(2)}$ such that

$$\begin{cases} \partial_t z_+^{(1)} + Z_- \cdot \nabla z_+^{(1)} = \mu_1 \, \Delta z_+^{(1)} - \nabla p_+^{(1)}, \\ \partial_t z_-^{(1)} + Z_+ \cdot \nabla z_-^{(1)} = \mu_1 \, \Delta z_-^{(1)} - \nabla p_-^{(1)}, \\ \partial_t z_+^{(2)} + Z_- \cdot \nabla z_+^{(2)} = \mu_1 \, \Delta z_+^{(2)} + \mu_2 \, \Delta z_- - \nabla p_+^{(2)}, \\ \partial_t z_-^{(2)} + Z_+ \cdot \nabla z_-^{(2)} = \mu_1 \, \Delta z_-^{(2)} + \mu_2 \, \Delta z_+ - \nabla p_-^{(2)}. \end{cases}$$

The next task is to establish a closed uniform estimate for the fluctuations $z_{\pm}^{(1)}$ and $z_{\pm}^{(2)}$ with respect to μ_1 and t. For this, we need the following key ingredients:

- The construction of the weighted Hölder spaces for the solution. Due to the appearance of the extra problematic terms $\Delta z_{\pm}$, we have to work in spaces with different regularity and weight for the solution $z_{\pm}^{(1)}$, $z_{\pm}^{(2)}$. Such inconsistencies give rise to the essential difficulties. In particular, the choice of the weight is very delicate. In [Bardos et al. 1988; Cai and Lei 2016; He et al. 2016], the weight has decay in all directions. For the slap domain, the weight is only allowed to decay in partial directions. Again, the weight has to be compatible with the estimate of the nonlocal pressure.
- Uniform estimates of the transport equation in the weighted Hölder spaces, which are very crucial to control the growth of the Lagrangian map.
- Uniform estimates for the parabolic equation with variable coefficients in the suitable weighted Hölder spaces. This is the most important step.
- Boundedness of the Riesz transform and its commutator in the weighted Hölder spaces, which is essentially used to handle the nonlocal pressure. To our knowledge, these results are new and may be independent of interest. The proof is highly nontrivial.

In this work, we require that μ_2/μ_1 is small. However, this cannot be handled as a perturbation of the case $\mu_2 = 0$ except when $|\mu_2| \leq \mu_1^{\alpha}$ for some $\alpha > 1$. In this case, the smallness of $z_{\pm}^{(2)}$ is not easily observed. If we directly use the energy method, we can only prove that $\|z_\pm^{(2)}(t)\|_{L^2} = O(|\mu_2|/\mu_1)$ for fixed $z_\pm(0)$. However, we can show that $|z_\pm^{(2)}(t)|_{0,\alpha} = O(\mu_1)$ for $t \sim 1/\mu_1$ and fixed $z_\pm(0)$.

In this paper, we consider the MHD equations in a homogeneous magnetic field. In the real case (for example, solar corona), it is more natural to consider the MHD equations in an inhomogeneous magnetic field. An important question is to consider the decay of Alfvén waves in an inhomogeneous magnetic field $B_0(y) = (b_1(y), b_2(y), 0)$. This is similar to the situation of Landau damping.

2. The weighted Hölder spaces and symmetric extension

Weighted Hölder spaces. Let $\Omega \subseteq \mathbb{R}^d$ be a domain and $\alpha \in (0, 1]$. We denote by $C^{k,\alpha}(\Omega)$, $(k = 0, 1)$ the Hölder space equipped with the norm

$$|u|_{0,\alpha;\Omega} := |u|_{0;\Omega} + [u]_{\alpha;\Omega}, \quad |u|_{1,\alpha;\Omega} := |u|_{0;\Omega} + |\nabla u|_{0,\alpha;\Omega},$$

where

$$|u|_{0;\Omega} = \sup_{X\in\Omega} |u(X)|, \quad [u]_{\alpha;\Omega} = \sup_{X,Y\in\Omega} \frac{|u(X)-u(Y)|}{|X-Y|^\alpha}.$$

Let $h(X) \in C(\mathbb{R}^d)$ be a positive bounded function. We introduce the weighted $C^{k,\alpha}$ norms

$$|u|_{0,\alpha;h,\Omega} := |u|_{0;h,\Omega} + [u]_{\alpha;h,\Omega}, \quad |u|_{1,\alpha;h,\Omega} := |u|_{0;h,\Omega} + |\nabla u|_{0,\alpha;h,\Omega},$$

where

$$|u|_{0;h,\Omega} = \left|\frac{u}{h}\right|_{0;\Omega}, \quad [u]_{\alpha;h,\Omega} = \sup_{X,Y\in\Omega} \frac{|u(X)-u(Y)|}{(h(X)+h(Y))|X-Y|^\alpha}.$$

We say that $u \in C_h^{k,\alpha}(\Omega)$ if $|u|_{k,\alpha;h,\Omega} < +\infty$. We also introduce

$$|u|_{k,\alpha;h,\Omega,T} := \sup_{0\le t\le T} |u(t)|_{k,\alpha;h(t),\Omega}.$$

When $\Omega = \mathbb{R}^d$, we will omit the subscript Ω in the norm of Hölder spaces.

The following two lemmas can be proved by using the definition of Hölder norm.

Lemma 2.1. *Let h, h_1, h_2 be the weight functions such that there exists a constant c_0 such that*

$$0 < c_0 h(X) \le h(Y) \quad \text{for any } X, Y \in \mathbb{R}^d, \ |X-Y| \le 2. \tag{2-1}$$

Then there exists a constant C depending only on c_0 such that, for $k = 0, 1$,

$$|u|_{0,\alpha;h,\Omega} \le C\big(|u|_{0;h,\Omega} + |\nabla u|_{0;h,\Omega}\big),$$

$$|uw|_{k,\alpha;h_1h_2,\Omega} \le C|u|_{k,\alpha;h_1,\Omega}|w|_{k,\alpha;h_2,\Omega},$$

$$\left|\int_t^s u(r)\,dr\right|_{k,\alpha;\int_t^s h(r)\,dr,\Omega} \le \sup_{t\le r\le s} |u(r)|_{k,\alpha;h(r),\Omega}.$$

Lemma 2.2. *Let Φ be a map from Ω to Ω with $\nabla\Phi \in C^{0,\alpha}(\Omega)$. It holds that*

$$|u\circ\Phi|_{0,\alpha;h\circ\Phi,\Omega} \le |u|_{0,\alpha;h,\Omega}\max(|\nabla\Phi|^{\alpha}_{0;\Omega},1),$$
$$|u\circ\Phi|_{1,\alpha;h\circ\Phi,\Omega} \le |u|_{1,\alpha;h,\Omega}\max(|\nabla\Phi|^{\alpha}_{0;\Omega},1)\max(|\nabla\Phi|_{0,\alpha;\Omega},1).$$

Here and in what follows, $|\nabla\Phi|$ denotes the matrix norm defined by

$$|A| := \sup_{|X|=1}|AX|. \tag{2-2}$$

To deal with the viscous case, we introduce the following scaled weighted Hölder space. Let $\alpha\in(0,1)$, $R\ge 0$ and define

$$|u|_{0,\alpha;h,R} := |u|_{0;h} + R^{\alpha}[u]_{\alpha;h},$$
$$|u|_{1,\alpha;h,R} := |u|_{0,\alpha;h} + \max(R,R^{1-\alpha})|\nabla u|_{0,\alpha;h,R}.$$

For these kinds of weighted spaces, we have analogues of Lemmas 2.1 and 2.2. For example, if $h(X)$ satisfies

$$0 < c_0 h(X) \le h(Y) \quad \text{for any } X,Y\in\mathbb{R}^d,\ |X-Y|\le 2R, \tag{2-3}$$

then for $R\ge 1$, we have

$$|u|_{0;h} + R|\nabla u|_{0,\alpha;h,R} \le |u|_{1,\alpha;h,R} \le |u|_{0,\alpha;h,R} + R|\nabla u|_{0,\alpha;h,R} \le C\big(|u|_{0;h} + R|\nabla u|_{0,\alpha;h,R}\big).$$

Here C is a constant depending only on c_0. In the following, we will fix $\alpha\in(0,1)$.

Lemma 2.3. *Let $\gamma>0$ and $h(X)>0$. Then there exists a constant C independent of h, γ, t such that*

$$\begin{aligned}\left|\int_0^t u(s)\,ds\right|_{1,\alpha;h,\sqrt{k+\gamma t}} &\le C\gamma^{-1}\sup_{0<s<t}\Big((\gamma s)^{\frac12}(\gamma(t-s))^{\frac12}|u(s)|_{0,\alpha;h} + \varphi_\alpha(\sqrt{k+\gamma s})(\gamma(t-s))^{1-\frac{\alpha}{2}}|\nabla u(s)|_{0;h} \\ &\qquad + \varphi_\alpha(\sqrt{k+\gamma s})(\gamma(t-s))^{\frac{3-\alpha}{2}}[\nabla u(s)]_{1;h}\Big),\end{aligned}$$

where $\varphi_\alpha(R) = \max(R,R^{1+\alpha})$.

Proof. We denote by $C\gamma^{-1}A$ the right-hand side of the inequality. Then we have

$$\left|\int_0^t u(s)\,ds\right|_{0,\alpha;h} \le \int_0^t |u(s)|_{0,\alpha;h}\,ds \le \int_0^t (\gamma s)^{-\frac12}(\gamma(t-s))^{-\frac12}\,ds A \le C\gamma^{-1}A,$$

$$\begin{aligned}\left|\nabla\int_0^t u(s)\,ds\right|_{0;h} \le \int_0^t |\nabla u(s)|_{0;h}\,ds &\le \int_0^t \varphi_\alpha(\sqrt{k+\gamma s})^{-1}(\gamma(t-s))^{-1+\frac{\alpha}{2}}\,ds A \\ &\le C\gamma^{-1}\min\big((k+\gamma t)^{-\frac12},(k+\gamma t)^{-\frac{1-\alpha}{2}}\big)A.\end{aligned}$$

For any $X,Y\in\mathbb{R}^d$, we have

$$|\nabla u(s,X)-\nabla u(s,Y)| \le |X-Y|(h(X)+h(Y))[\nabla u(s)]_{1;h},$$
$$|\nabla u(s,X)-\nabla u(s,Y)| \le |\nabla u(s,X)|+|\nabla u(s,Y)| \le (h(X)+h(Y))|\nabla u(s)|_{0;h}.$$

This gives

$$\begin{aligned}|\nabla u(s,X)-\nabla u(s,Y)| &\le \min\big((\gamma(t-s))^{\frac12},|X-Y|\big)(h(X)+h(Y))\big([\nabla u(s)]_{1;h}+(\gamma(t-s))^{-\frac12}|\nabla u(s)|_{0;h}\big)\\ &\le \min\big((\gamma(t-s))^{\frac12},|X-Y|\big)(h(X)+h(Y))\varphi_\alpha(\sqrt{k+\gamma s})^{-1}(\gamma(t-s))^{-\frac{3-\alpha}{2}}A.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|\nabla\int_0^t u(s)\,ds(X)-\nabla\int_0^t u(s)\,ds(Y)\right|\\ &\quad\le \int_0^t |\nabla u(s,X)-\nabla u(s,Y)|\,ds\\ &\quad\le \int_0^t \min\big((\gamma(t-s))^{\frac12},|X-Y|\big)(h(X)+h(Y))\varphi_\alpha(\sqrt{k+\gamma s})^{-1}(\gamma(t-s))^{-\frac{3-\alpha}{2}}A\,ds\\ &\quad\le C(h(X)+h(Y))A\bigg(\min\big((\gamma t)^{\frac12},|X-Y|\big)\int_0^{\frac t2}\varphi_\alpha(\sqrt{k+\gamma s})^{-1}\,ds(\gamma t)^{-\frac{3-\alpha}{2}}\\ &\qquad\qquad+\int_{\frac t2}^t \min\big((\gamma(t-s))^{\frac12},|X-Y|\big)(\gamma(t-s))^{-\frac{3-\alpha}{2}}\,ds\varphi_\alpha(\sqrt{k+\gamma t})^{-1}\bigg)\\ &\quad\le C(h(X)+h(Y))A\big((\gamma t)^{\frac{1-\alpha}{2}}|X-Y|^\alpha t\varphi_\alpha(\sqrt{k+\gamma t})^{-1}(\gamma t)^{-\frac{3-\alpha}{2}}+\gamma^{-1}|X-Y|^\alpha\varphi_\alpha(\sqrt{k+\gamma t})^{-1}\big)\\ &\quad\le C\gamma^{-1}(h(X)+h(Y))A|X-Y|^\alpha\varphi_\alpha(\sqrt{k+\gamma t})^{-1}.\end{aligned}$$

Hence, we deduce our result. □

Lemma 2.4. *Let Φ be a map from $\mathbb{R}^d$ to $\mathbb{R}^d$ with $\nabla\Phi\in C^{0,\alpha}(\mathbb{R}^d)$. It holds that*

$$\begin{aligned}|u\circ\Phi|_{0,\alpha;h\circ\Phi,R} &\le |u|_{0,\alpha;h,R}\max(|\nabla\Phi|_0^\alpha,1),\\ |u\circ\Phi|_{1,\alpha;h\circ\Phi,R} &\le |u|_{1,\alpha;h,R}\max(|\nabla\Phi|_0^\alpha,1)\max(|\nabla\Phi|_{0,\alpha;1,R},1).\end{aligned}$$

Symmetric extension. Let $\Omega=\mathbb{R}^{d-1}\times[0,1]$ be a strip and $X=(x,y)$, $x\in\mathbb{R}^{d-1}$, $y\in[0,1]$ be a point in Ω.

Let T_e be an even extension from $C(\Omega)$ to $C(\mathbb{R}^d)$ defined by

$$T_e f(x,2n+y)=T_e f(x,2n-y)=f(x,y)$$

for $x\in\mathbb{R}^{d-1}$, $y\in[0,1]$, $n\in\mathbb{Z}$. Let T_o be an odd extension from $C_0(\Omega)=\{u\in C(\Omega): u=0 \text{ on } \partial\Omega\}$ to $C(\mathbb{R}^d)$ defined by

$$T_o f(x,2n-y)=-f(x,y),\quad T_o f(x,2n+y)=f(x,y)$$

for $x\in\mathbb{R}^{d-1}$, $y\in[0,1]$, $n\in\mathbb{Z}$.

Lemma 2.5. *It holds that*

$$|T_e f|_{0,\alpha}=|f|_{0,\alpha,\Omega},\quad |f|_{0,\alpha;\Omega}\le|T_o f|_{0,\alpha}\le 2|f|_{0,\alpha;\Omega}.$$

The same result holds for the weighted Hölder norm $|\cdot|_{0,\alpha;h}$ if the weight function $h(X)$ depends only on x.

Proof. First of all, it is obvious that

$$|f|_{0,\alpha;\Omega} \le |T_e f|_{0,\alpha}, \quad |f|_{0,\alpha;\Omega} \le |T_o f|_{0,\alpha},$$

and the same is true for the weighted Hölder norm $|\cdot|_{0,\alpha;h}$. We define

$$\begin{aligned} \rho_0(y) &= \inf_{n\in\mathbb{Z}} |y-2n| \in [0,1] && \text{for } y \in \mathbb{R}, \\ \rho(X) &= (x, \rho_0(y)) \in \Omega && \text{for } X = (x,y) \in \mathbb{R}^d, \end{aligned}$$

and let

$$\Omega_+ = \bigcup_{n\in\mathbb{Z}} \mathbb{R}^{d-1} \times [2n, 2n+1], \quad \Omega_- = \bigcup_{n\in\mathbb{Z}} \mathbb{R}^{d-1} \times [2n-1, 2n].$$

Then it is easy to see that

$$\begin{gathered} T_e f = f \circ \rho, \\ T_o f = f \circ \rho \quad \text{in } \Omega_+, \quad T_o f = -f \circ \rho \quad \text{in } \Omega_-, \\ |\rho_0(y) - \rho_0(y')| \le |y-y'|, \quad |\rho(X) - \rho(Y)| \le |X-Y|, \end{gathered}$$

from which, it follows that

$$\begin{gathered} |T_e f|_{0,\alpha} \le |f|_{0,\alpha;\Omega}, \quad |T_e f|_{0,\alpha;h} \le |f|_{0,\alpha;h,\Omega}, \\ |T_o f|_0 \le |f|_{0;\Omega}, \qquad |T_o f|_{0;h} \le |f|_{0;h,\Omega}. \end{gathered}$$

Given $X = (x,y),\ Y = (x',y') \in \mathbb{R}^d$ with $y \le y'$, if $X, Y \in \Omega_+$ or $X, Y \in \Omega_-$, then

$$\begin{aligned} |T_o f(X) - T_o f(Y)| &= |f\circ\rho(X) - f\circ\rho(Y)| \\ &\le |f|_{0,\alpha;h,\Omega}\big(h\circ\rho(X) + h\circ\rho(Y)\big)|\rho(X)-\rho(Y)|^\alpha \\ &\le |f|_{0,\alpha;h,\Omega}(h(X)+h(Y))|X-Y|^\alpha. \end{aligned}$$

Here we used $h\circ\rho(X) = h(X)$. Otherwise, there exists $y_1, y_2 \in \mathbb{Z}$ so that $y_1 - 1 \le y \le y_1 \le y_2 \le y' \le y_2 + 1$. Let $X' = (x, y_1),\ Y' = (x', y_2)$. Then for $f \in C_0(\Omega)$, we have

$$\begin{aligned} |T_o f(X)| = |f\circ\rho(X)| &= |f\circ\rho(X) - f\circ\rho(X')| \\ &\le |f|_{0,\alpha;h,\Omega}\big(h\circ\rho(X) + h\circ\rho(X')\big)|\rho(X)-\rho(X')|^\alpha \\ &\le 2|f|_{0,\alpha;h,\Omega} h(X)|X-X'|^\alpha. \end{aligned}$$

Similarly, we have

$$|T_o f(Y)| \le 2|f|_{0,\alpha;h,\Omega} h(Y)|Y-Y'|^\alpha.$$

Then, using $|X-X'| + |Y-Y'| \le |X-Y|$, we get

$$|T_o f(X) - T_o f(Y)| \le 2|f|_{0,\alpha;h,\Omega}(h(X)+h(Y))|X-Y|^\alpha.$$

This shows $[T_o f]_{\alpha;h} \le 2[f]_{\alpha;h,\Omega}$. Similarly, $[T_o f]_\alpha \le 2[f]_{\alpha;\Omega}$. □

3. Global well-posedness for the ideal MHD equations

This section is devoted to the proof of the global well-posedness of the ideal MHD equations in $\mathbb{R}^{d-1}\times[0,1]$ with the boundary condition (1-4). Recall that in terms of the Elsässer variables $z_\pm = Z_\pm \pm B_0$, the ideal MHD equations take

$$\begin{cases} \partial_t z_+ + Z_- \cdot \nabla z_+ = -\nabla p, \\ \partial_t z_- + Z_+ \cdot \nabla z_- = -\nabla p, \\ \operatorname{div} z_+ = \operatorname{div} z_- = 0, \\ z_\pm^d(t,x,y) = 0 \quad \text{on } y = 0, 1. \end{cases} \tag{3-1}$$

Without loss of generality, we take the background magnetic field $B_0 = (1, 0, \dots, 0)$.

Main result. Let $f(x,y) = f_0(x_1)$, where $f_0 \in C^1(\mathbb{R})$ is chosen so that $|f_0'| < f_0 < 1$ and for some $C_1^* > 0$,

$$\begin{aligned} \delta(T) \triangleq \sup_{Y\in\mathbb{R}^d} \int_{-T}^{T} f(Y + 2B_0 t)\,dt \le C_1^* &\quad \text{for any } T > 0, \\ \int_{\mathbb{R}^d} \frac{f(Y)}{1+|X-Y|^{d+1}}\,dY \le C_1^* f(X) &\quad \text{for any } X \in \mathbb{R}^d, \\ f(X) \le 2f(Y) &\quad \text{for any } |X-Y| \le 2. \end{aligned} \tag{3-2}$$

In fact, $f_0(r) = (C_0 + r^2)^{-\frac{\delta+1}{2}}$ satisfies the above conditions for some $C_0 > 1$ and $0 < \delta < 1$.

Now we introduce the weight function $f_\pm(t, X)$ given by

$$f_\pm(t,X) \triangleq f(X \pm B_0 t),$$

which satisfies (2-1) with a uniform constant c_0 independent of t. Let

$$M_\pm(t) \triangleq \sup_{|s|\le t} |z_\pm(s)|_{1,\alpha;f_\pm(s),\Omega}.$$

The main result of this section is stated as follows.

Theorem 3.1. *Let $\alpha \in (0,1)$. There exists $\varepsilon > 0$ such that if $M_\pm(0) \le \varepsilon$, then there exists a global in time unique solution $(z_+, z_-) \in L^\infty\big(0,+\infty; C^{1,\alpha}(\Omega)\big)$, with the pressure p determined by* (3-10), *to the ideal MHD equations* (3-1), *which satisfies*

$$M_\pm(t) \le C\varepsilon \quad \text{for any } t \in [0,+\infty).$$

Remark 3.2. Since $M_\pm(0) \sim |z_\pm(0)\langle x_1\rangle^{1+\delta}|_{1,\alpha;\Omega}$ if $f_0(r) = (C_0 + r^2)^{-\frac{\delta+1}{2}}$, the initial data decays at infinity only in one direction. This is very crucial for the global well-posedness in the slap domain, especially in $\mathbb{R}\times[0,1]$.

We conclude this subsection by introducing some properties of weighted functions. Let

$$g(t,X) \triangleq \int_{\mathbb{R}^d} \frac{f(Y+B_0 t) f(Y - B_0 t)}{1+|X-Y|^{d+1}}\,dY.$$

We have the following important facts.

Lemma 3.3. *There exists a constant $C > 0$ such that for any $X \in \mathbb{R}^d$, $t \in \mathbb{R}$,*

$$f(X + B_0 t) f(X - B_0 t) \le C g(t, X),$$
$$g(t, X) \le C\bigl(1 + |X - Y|\bigr)^{d+1} g(t, Y),$$
$$\int_{-T}^{T} g(t, X \pm B_0 t)\, dt \le C\delta(T) f(X).$$

Proof. Thanks to $f(Y) \ge f(X)/2$ for $|X - Y| < 2$, we get

$$\begin{aligned} g(t, X) &\ge \int_{B(X,2)} \frac{f(Y + B_0 t) f(Y - B_0 t)}{1 + |X - Y|^{d+1}}\, dY \ge \frac{1}{4}\int_{B(X,2)} \frac{f(X + B_0 t) f(X - B_0 t)}{1 + |X - Y|^{d+1}}\, dY \\ &\ge C^{-1} f(X + B_0 t) f(X - B_0 t), \end{aligned}$$

which gives the first inequality.

Using the inequality

$$\frac{1}{1 + |X - Z|^{d+1}} \le C \frac{1 + |X - Y|^{d+1}}{1 + |Y - Z|^{d+1}},$$

we infer

$$\begin{aligned} g(t, X) &= \int_{\mathbb{R}^d} \frac{f(Z + B_0 t) f(Z - B_0 t)}{1 + |X - Z|^{d+1}}\, dZ \le C \int_{\mathbb{R}^d} \frac{f(Z + B_0 t) f(Z - B_0 t)}{1 + |Y - Z|^{d+1}} (1 + |X - Y|^{d+1})\, dY \\ &= C\bigl(1 + |X - Y|^{d+1}\bigr) g(t, Y), \end{aligned}$$

which gives the second inequality.

Make a change of variable

$$g(t, X + B_0 t) = \int_{\mathbb{R}^d} \frac{f(Y + B_0 t) f(Y - B_0 t)}{1 + |X + B_0 t - Y|^{d+1}}\, dY = \int_{\mathbb{R}^d} \frac{f(Y + 2B_0 t) f(Y)}{1 + |X - Y|^{d+1}}\, dY,$$

which along with (3-2) gives

$$\int_{-T}^{T} g(t, X + B_0 t) = \int_{\mathbb{R}^d} \frac{\int_{-T}^{T} f(Y + 2B_0 t) f(Y)\, dt}{1 + |X - Y|^{d+1}}\, dY \le C \int_{\mathbb{R}^d} \frac{\delta(T) f(Y)}{1 + |X - Y|^{d+1}}\, dY \le C\delta(T) f(X).$$

Similarly, we have

$$\int_{-T}^{T} g(t, X - B_0 t) \le C\delta(T) f(X). \qquad \square$$

Weighted $C^{1,\alpha}$ estimate for the transport equation. Let $Z \in C^1\bigl([0, T] \times \Omega\bigr)$ be a vector field with $Z^d = 0$ on $\partial\Omega$. We introduce the characteristic associated with Z:

$$\frac{d}{dt}\Phi(s, t, X) = Z(t, \Phi(s, t, X)), \quad \Phi(s, s, X) = X. \tag{3-3}$$

Then $\Phi(s, t, X) \in C^1\bigl([0, T] \times [0, T] \times \Omega\bigr)$ is a diffeomorphism from Ω to Ω and $\partial\Omega$ to $\partial\Omega$ having the property

$$\Phi(r, t) \circ \Phi(s, r) = \Phi(s, t), \quad \Phi(s, s) = \mathrm{Id}.$$

Lemma 3.4. *If $Z(t, X)$ satisfies the extra condition*

$$|\nabla Z|_{0,\alpha;h,\Omega,T} \int_{t_0}^{T} h(t, \Phi(T, t, X))\, dt \le A_0 \quad \textit{for any } X \in \Omega, \tag{3-4}$$

then it holds that for $0 \le t_0 \le t \le s < T$,

$$\begin{aligned}
|\nabla\Phi(s,t) - \mathrm{Id}|_{0;\Omega} &\le e^{A_0} - 1,\\
|\nabla\Phi(s,t)|_{0;\Omega} &\le e^{A_0},\\
[\nabla\Phi(s,t)]_{\alpha;\Omega} &\le 2A_0 e^{(2+\alpha)A_0}.
\end{aligned}$$

Proof. Thanks to the definition of $\Phi(s,t)$, we have

$$\begin{gathered}
\partial_t \nabla\Phi(s,t) = \nabla\Phi(s,t)\big((\nabla Z(t)) \circ \Phi(s,t)\big),\\
\Phi(s,s) = \mathrm{Id}, \quad \nabla\Phi(s,s) = \mathrm{Id},\\
|\nabla\Phi(s,t)| \le |\nabla\Phi(s,t) - \mathrm{Id}| + 1.
\end{gathered}$$

Here $|\nabla\Phi(s,t)|$ is the matrix norm defined by (2-2). Therefore,

$$\begin{aligned}
\big|\nabla\Phi(s,t) - \mathrm{Id}\big| &\le \int_t^s |\partial_r \nabla\Phi(s,r)|\, dr\\
&\le \int_t^s |\nabla\Phi(s,r)| \big|(\nabla Z(r)) \circ \Phi(s,r)\big|\, dr\\
&\le \int_t^s \big|(\nabla Z(r)) \circ \Phi(s,r)\big|\, dr + \int_t^s \big|\nabla\Phi(s,r) - \mathrm{Id}\big| \big|(\nabla Z(r)) \circ \Phi(s,r)\big|\, dr,
\end{aligned}$$

which implies

$$\big|\nabla\Phi(s,t) - \mathrm{Id}\big| \le \exp\bigg(\int_t^s \big|(\nabla Z(r)) \circ \Phi(s,r)\big|\, dr\bigg) - 1.$$

Thanks to

$$\big|(\nabla Z(r)) \circ \Phi(s,r)\big| \le |\nabla Z|_{0,\alpha;h,\Omega,T}\, h(r) \circ \Phi(s,r),$$

we get by (3-4) that

$$\begin{aligned}
\int_t^s \big|(\nabla Z(r)) \circ \Phi(s,r)(X)\big|\, dr &\le |\nabla Z|_{0,\alpha;h,\Omega,T} \int_t^s h(r) \circ \Phi(s,r)(X)\, dr\\
&= |\nabla Z|_{0,\alpha;h,\Omega,T} \int_t^s h\big(r, \Phi(T, r, \Phi(s,T)(X))\big)\, dr \le A_0.
\end{aligned}$$

Thus, we conclude that

$$\begin{gathered}
|\nabla\Phi(s,t) - \mathrm{Id}|_{0;\Omega} \le e^{A_0} - 1,\\
|\nabla\Phi(s,t)|_{0;\Omega} \le e^{A_0},\\
|\Phi(s,t,X) - \Phi(s,t,Y)| \le |\nabla\Phi(s,t)|_{0;\Omega} |X - Y| \le e^{A_0} |X - Y|.
\end{gathered}$$

Notice that

$$\big|\nabla\Phi(s,t,X)-\nabla\Phi(s,t,Y)\big| \le \int_t^s \big|\nabla\Phi(s,r,X)-\nabla\Phi(s,r,Y)\big|\big|(\nabla Z(r))\circ\Phi(s,r,X)\big|\,dr \\ + \int_t^s \big|\nabla\Phi(s,r,Y)\big|\big|(\nabla Z(r))\circ\Phi(s,r,X)-(\nabla Z(r))\circ\Phi(s,r,Y)\big|\,dr.$$

From this and Gronwall's inequality, we infer

$$\begin{aligned}
&\big|\nabla\Phi(s,t,X)-\nabla\Phi(s,t,Y)\big| \\
&\le \int_t^s \big|\nabla\Phi(s,r,Y)\big|\big|(\nabla Z(r))\circ\Phi(s,r,X)-(\nabla Z(r))\circ\Phi(s,r,Y)\big|\,dr\exp\Big(\int_t^s \big|(\nabla Z(r))\circ\Phi(s,r,X)\big|\,dr\Big) \\
&\le \int_t^s \big|\nabla\Phi(s,r,Y)\big| |\nabla Z|_{0,\alpha;h,\Omega,T}\big(h(r,\Phi(s,r,X))+h(r,\Phi(s,r,Y))\big)\big|\Phi(s,r,X)-\Phi(s,r,Y)\big|^\alpha\,dr e^{A_0} \\
&\le \int_t^s e^{A_0}|\nabla Z|_{0,\alpha;h,\Omega,T}\big(h(r,\Phi(s,r,X))+h(r,\Phi(s,r,Y))\big)e^{\alpha A_0}|X-Y|^\alpha\,dr e^{A_0} \\
&= e^{(2+\alpha)A_0}|X-Y|^\alpha|\nabla Z|_{0,\alpha;h,\Omega,T}\int_t^s \big(h(r,\Phi(s,r,X))+h(r,\Phi(s,r,Y))\big)\,dr \\
&\le 2A_0e^{(2+\alpha)A_0}|X-Y|^\alpha,
\end{aligned}$$

which shows the last inequality of the lemma. □

Next we consider the transport equation

$$\partial_t u + Z\cdot\nabla u = F, \quad u(0,X)=u_0(X). \tag{3-5}$$

Using the characteristic, the solution $u(t,X)$ is given by

$$u(t,X) = u_0(\Phi(t,0,X)) + \int_0^t F(s,\Phi(t,s,X))\,ds. \tag{3-6}$$

Lemma 3.5. *If Z satisfies* (3-4), *then we have*

$$|u(t)|_{0,\alpha;\Omega} \le e^{\alpha A_0}\Big(|u_0|_{0,\alpha;\Omega} + \int_0^t |F(s)|_{0,\alpha;\Omega}\,ds\Big),$$

$$|\mathrm{div}\ u(t)|_{0;\Omega} \le |\mathrm{div}\ u_0|_{0;\Omega} + \int_0^t \big|\big(\mathrm{tr}(\nabla Z\,\nabla u) - \mathrm{div}\,F\big)(s)\big|_{0;\Omega}\,ds.$$

Proof. Using (3-6) and Lemmas 2.2 and 3.4, we get

$$\begin{aligned}
|u(t)|_{0,\alpha;\Omega} &\le |u_0\circ\Phi(t,0)|_{0,\alpha;\Omega} + \int_0^t |F(s)\circ\Phi(t,s)|_{0,\alpha;\Omega}\,ds \\
&\le |u_0|_{0,\alpha;\Omega}\max\big(|\nabla\Phi(t,0)|_{0;\Omega}^\alpha, 1\big) + \int_0^t |F(s)|_{0,\alpha;\Omega}\max\big(|\nabla\Phi(t,s)|_{0;\Omega}^\alpha, 1\big)\,ds \\
&\le e^{\alpha A_0}\Big(|u_0|_{0,\alpha;\Omega} + \int_0^t |F(s)|_{0,\alpha;\Omega}\,ds\Big).
\end{aligned}$$

Taking the divergence of (3-5), we obtain

$$\partial_t \operatorname{div} u + Z \cdot \nabla \operatorname{div} u + \operatorname{tr}(\nabla Z\, \nabla u) = \operatorname{div} F, \quad u(0, X) = u_0(X).$$

So, we have

$$\operatorname{div} u(t) = \operatorname{div} u_0 \circ \Phi(t, 0) + \int_0^t \big(\operatorname{div} F - \operatorname{tr}(\nabla Z\, \nabla u)\big)(s) \circ \Phi(t, s)\, ds,$$

and then the second inequality follows easily. □

Proposition 3.6. *If* $|Z + B_0|_{1,\alpha;f_-,\Omega,T}\,\delta(T) < 1$, *then we have*

$$|u|_{1,\alpha;f_+,\Omega,T} \le C\big(|u_0|_{1,\alpha;f,\Omega} + \delta(T)|F|_{1,\alpha;g,\Omega,T}\big).$$

If $|Z - B_0|_{1,\alpha;f_+,\Omega,T}\,\delta(T) < 1$, *then we have*

$$|u|_{1,\alpha;f_-,\Omega,T} \le C\big(|u_0|_{1,\alpha;f,\Omega} + \delta(T)|F|_{1,\alpha;g,\Omega,T}\big).$$

Here C is a constant independent of T.

Proof. We only prove the first inequality; the proof of the second one is similar. Let us claim

$$\big|\Phi(s, t, X) + B_0(t - s) - X\big| < 2 \quad \text{for } 0 \le t \le s \le T. \tag{3-7}$$

Otherwise, there exists $t \in [0, s]$ such that $\big|\Phi(s,t,X)+B_0(t-s)-X\big| = 2$ and $\big|\Phi(s,r,X)+B_0(r-s)-X\big| \le 2$ for $r \in [t, s]$. Thus,

$$\begin{aligned}\big|\Phi(s, t, X) + B_0(t - s) - X\big| &\le \int_t^s |\partial_r \Phi(s, r, X) + B_0|\, dr\\ &= \int_t^s |Z(r, \Phi(s, r, X)) + B_0|\, dr\\ &\le \int_t^s |Z + B_0|_{1,\alpha;f_-,\Omega,T}\, f_-(r, \Phi(s, r, X))\, dr\\ &= |Z + B_0|_{1,\alpha;f_-,\Omega,T} \int_t^s f(\Phi(s, r, X) - B_0 r)\, dr,\end{aligned}$$

while, by (3-2),

$$\int_t^s f(\Phi(s, r, X) - B_0 r)\, dr \le 2\int_t^s f(X - B_0(r - s) - B_0 r)\, dr \le 2\delta(T).$$

This shows

$$\big|\Phi(s, t, X) + B_0(t - s) - X\big| \le 2|Z + B_0|_{1,\alpha;f_-,\Omega,T}\,\delta(T) < 2,$$

which is a contradiction; hence (3-7) is true.

Now we verify (3-4) for $h = f_-$ and $A_0 = 2$. Indeed, by (3-2) and (3-7),

$$\int_0^T f_-(t, \Phi(T, t, X))\, dt = \int_0^T f(\Phi(T, t, X) - B_0 t)\, dt \le 2\int_0^T f(X - B_0(t - T) - B_0 t)\, dt \le 2\delta(T),$$

which implies (3-4). Then we infer from Lemma 3.4 that

$$|\nabla\Phi(t,s)|_{0,\alpha;\Omega} \le C. \tag{3-8}$$

It follows from Lemma 3.3 and (3-7) that

$$\int_0^t g(r,\Phi(t,r,X))\,dr \le C\int_0^t g(r,X-B_0(r-t))\,dr \le C\delta(T)f(X+B_0t),$$

which implies

$$|u(t)|_{1,\alpha;f_+(t),\Omega} \le |u_0\circ\Phi(t,0)|_{1,\alpha;f_+(t),\Omega} + C\delta(T)\sup_{0\le s\le t}|F(s)\circ\Phi(t,s)|_{0,\alpha;g(s)\circ\Phi(t,s),\Omega}.$$

Using the fact $f(\Phi(t,0,X)) \le 2f(X-B_0(0-t)) = 2f_+(t,X)$, we get

$$|u_0\circ\Phi(t,0)|_{1,\alpha;f_+(t),\Omega} \le 2|u_0\circ\Phi(t,0)|_{1,\alpha;f\circ\Phi(t,0),\Omega}.$$

Then by Lemma 2.2 and (3-8), we obtain

$$\begin{aligned}
&|u(t)|_{1,\alpha;f_+(t),\Omega}\\
&\quad\le C\big(|u_0\circ\Phi(t,0)|_{1,\alpha;f,\Omega}+\delta(T)\sup_{0\le s\le t}|F(s)|_{1,\alpha;g(s),\Omega}\big)\max\big(|\nabla\Phi(t,s)|^{\alpha}_{0;\Omega},1\big)\max\big(|\nabla\Phi(t,s)|_{0,\alpha;\Omega},1\big)\\
&\quad\le C|u_0|_{1,\alpha;f,\Omega}+C\delta(T)\sup_{0\le s\le t}|F(s)|_{1,\alpha;g(s),\Omega}.
\end{aligned}$$

This shows the first inequality of the lemma. □

Representation formula of the pressure. In this subsection, we give a representation formula of the pressure by using the symmetric extension.

Let (v,b,p) be a smooth solution of (1-1) in $[0,T]\times\Omega$ with the boundary condition (1-4). We make the following symmetric extension for the solution:

$$\bar v = Tv := \big(T_e v^1,\dots,T_e v^{d-1},T_o v^d\big),\quad \bar b = Tb,\quad \bar p = T_e p.$$

Then $(\bar v,\bar b,\bar p)$ satisfies (1-1) in $[0,T]\times\mathbb{R}^d$ in the weak sense. Although the solution after the symmetric extension does not have the same smoothness as the original one, we have the following important observation.

Lemma 3.7. *Let h be a weight satisfying* (2-1). *Let $u=(u^1,\dots,u^d)$, $w=(w^1,\dots,w^d)\in C_h^{1,\alpha}(\Omega)$ be two vector fields with $u^d=w^d=0$ on $\partial\Omega$. Let $\bar u = Tu$ and $\bar w = Tw$. Then it holds that for $i,j=1,\dots,d$,*

$$\begin{aligned}
|\partial_i\bar u^j\,\partial_j\bar w^i|_{0,\alpha;h}+|\partial_i\bar u^i\,\partial_j\bar w^j|_{0,\alpha;h} &\le C|\nabla u|_{0,\alpha;h,\Omega}|\nabla w|_{0,\alpha;h,\Omega},\\
|\bar u^j\,\partial_j\bar w^i|_{0,\alpha;h}+|\bar u^i\,\partial_j\bar w^j|_{0,\alpha;h} &\le C|u|_{0,\alpha;h,\Omega}|\nabla w|_{0,\alpha;h,\Omega}.
\end{aligned}$$

Proof. It is easy to verify that

$$\begin{aligned}
\partial_i \bar{u}^j\, \partial_j \bar{w}^i &= T_e(\partial_i u^j\, \partial_j w^i), & \partial_i \bar{u}^i\, \partial_j \bar{w}^j &= T_e(\partial_i u^i\, \partial_j w^j),\\
\bar{u}^j\, \partial_j \bar{w}^i &= T_e(u^j\, \partial_j w^i), & \bar{u}^i\, \partial_j \bar{w}^j &= T_e(u^i\, \partial_j w^j) \quad \text{for } i = 1, \dots, d-1,\\
\bar{u}^j\, \partial_j \bar{w}^d &= T_o(u^j\, \partial_j w^d), & \bar{u}^d\, \partial_j \bar{w}^j &= T_0(u^d\, \partial_j w^j).
\end{aligned}$$

Then the lemma follows easily from Lemma 2.5. □

Taking the divergence of the first equation of (1-1), we get

$$-\Delta \bar{p} = \partial_i (\bar{v}^j \partial_j \bar{v}^i - \bar{b}^j \partial_j \bar{b}^i).$$

Formally, we have

$$\nabla \bar{p}(t, X) = \nabla \int_{\mathbb{R}^d} N(X-Y)\partial_i (\bar{v}^j \partial_j \bar{v}^i - \bar{b}^j \partial_j \bar{b}^i)(t, Y)\, dY,$$

where $N(X)$ is the Newton potential. In terms of the Elsässer variables $\bar{z}_\pm(t, X)$, we have

$$\nabla \bar{p}(t, X) = \nabla \int_{\mathbb{R}^d} N(X-Y)\, \partial_i (\bar{z}_+^j \partial_j \bar{z}_-^i)(t, Y)\, dY.$$

However, this integral does not make sense for $\partial_i (\bar{z}_+^j \partial_j \bar{z}_-^i) \in C^{0,\alpha}$. To overcome this trouble, we introduce a smooth cut-off function $\theta(r)$ such that

$$\theta(r) = \begin{cases} 1 & \text{for } |r| \le 1, \\ 0 & \text{for } |r| \ge 2. \end{cases} \tag{3-9}$$

Integrating by parts, we can split $\nabla \bar{p}(t, X)$ as

$$\begin{aligned}
-\nabla \bar{p}(t, X) = \int_{\mathbb{R}^d} &\nabla N(X-Y)(\partial_i \bar{z}_+^j \partial_j \bar{z}_-^i)(t, Y)\, dY \\
&+ \int_{\mathbb{R}^d} \partial_i \partial_j \big(\nabla N(X-Y)(1 - \theta(|X-Y|))\big)(\bar{z}_+^j \bar{z}_-^i)(t, Y)\, dY.
\end{aligned} \tag{3-10}$$

It is easy to check that this representation makes sense for $\bar{z}_\pm \in W^{1,\infty}(\mathbb{R}^d)$.

We define

$$\begin{aligned}
T_1 u &\triangleq \int_{\mathbb{R}^d} \nabla N(X-Y)\theta(|X-Y|)u(Y)\, dY,\\
T_{ij} w &\triangleq \int_{\mathbb{R}^d} \partial_i \partial_j \big(\nabla N(X-Y)(1-\theta(|X-Y|))\big) w(Y)\, dY.
\end{aligned} \tag{3-11}$$

Let $u, w \in C^{1,\alpha}(\Omega)$ be two vector fields with $u^d = w^d = 0$ on $\partial\Omega$. Let $\bar{u} = Tu$ and $\bar{w} = Tw$ be the symmetric extension. We define

$$I(u, w) \triangleq T_1(\partial_i \bar{u}^j \partial_j \bar{w}^i - \partial_j \bar{u}^j \partial_i \bar{w}^i) + T_{ij}(\bar{u}^i \bar{w}^j). \tag{3-12}$$

Here and in what follows, the repeated index denotes the summation. Thanks to

$$\partial_i \bar{u}^j \partial_j \bar{w}^i - \partial_j \bar{u}^j \partial_i \bar{w}^i = \partial_i (\bar{u}^j \partial_j \bar{w}^i - \bar{u}^i \partial_j \bar{w}^j), \tag{3-13}$$

we infer from Lemmas A.1 and 3.7 that

$$|I(u,w)|_{0,\alpha;\Omega} \le C|u|_{0,\alpha;\Omega}|w|_{1,\alpha;\Omega}. \tag{3-14}$$

Using Lemma A.2 and (3-13), we calculate

$$\begin{aligned}
\operatorname{div} I(u,w) &+ (\partial_i u^j\,\partial_j w^i - \partial_i u^i\,\partial_j w^j)\\
&= \int_{\mathbb{R}^d} \nabla N(X-Y)\cdot\nabla\theta(|X-Y|)(\partial_i\bar u^j\,\partial_j\bar w^i - \partial_i\bar u^i\,\partial_j\bar w^j)(Y)\,dY\\
&\qquad\qquad - \int_{\mathbb{R}^d} \partial_i\partial_j\big(\nabla N(X-Y)\cdot\nabla\theta(|X-Y|)\big)(\bar u^j\bar w^i)(Y)\,dY\\
&= \int_{\mathbb{R}^d} \partial_i\big(\nabla N(X-Y)\cdot\nabla\theta(|X-Y|)\big)\big(-\bar u^j\partial_j\bar w^i + \bar u^i\,\partial_j\bar w^j + \partial_j(\bar u^j\bar w^i)\big)(Y)\,dY\\
&= \int_{\mathbb{R}^d} \partial_i\big(\nabla N(X-Y)\cdot\nabla\theta(|X-Y|)\big)(\bar u^i \operatorname{div}\bar w + \bar w^i\operatorname{div}\bar u)(Y)\,dY,
\end{aligned}$$

which implies

$$\big|\operatorname{div} I(u,w) - (\partial_i u^j\,\partial_j w^i - \partial_i u^i\,\partial_j w^j)\big|_{0;\Omega} \le C\big(|u|_{0;\Omega}|\operatorname{div} w|_{0;\Omega} + |w|_{0,\Omega}|\operatorname{div} u|_{0;\Omega}\big). \tag{3-15}$$

In the case of $\mathbb{R}^d$, the pressure $p(t,X)$ can also be expressed as

$$-\nabla p(t,X) = I(z_+,z_-), \tag{3-16}$$

where

$$\begin{aligned}
I(u,w) \triangleq \int_{\mathbb{R}^d} &\nabla N(X-Y)\theta(|X-Y|)(\partial_i u^j\,\partial_j v^i)(Y)\,dY\\
&+ \int_{\mathbb{R}^d} \partial_i\partial_j\big(\nabla N(X-Y)(1-\theta(|X-Y|))\big)(u^j v^i)(Y)\,dY.
\end{aligned} \tag{3-17}$$

Notice that the representation formula (3-16) is independent of the choice of θ in $I(u,w)$.

Proof of Theorem 3.1. Since we cannot find a well-posedness theory for the ideal MHD equations in the weighted Hölder spaces, we will present a complete proof of Theorem 3.1. In fact, we find that the proof of the existence part is very nontrivial.

Using the representation of the pressure (3-10), we rewrite the system (3-1) as

$$\begin{cases}
\partial_t z_+ + Z_-\cdot\nabla z_+ = -I(z_+,z_-),\\
\partial_t z_- + Z_+\cdot\nabla z_- = -I(z_+,z_-),\\
z_+(0,X) = z_{+0}(X), \quad z_-(0,X) = z_{-0}(X).
\end{cases} \tag{3-18}$$

Let $T>0$ be determined later and

$$A_1 = |z_{+0}|_{1,\alpha;f,\Omega} + |z_{-0}|_{1,\alpha;f,\Omega}.$$

When A_1 is sufficiently small, T can be taken to be $+\infty$. The system (3-18) is solved by the following iteration scheme:

$$z_+^{(0)} = z_-^{(0)} = 0, \quad Z_+^{(n)} = z_+^{(n)} + B_0, \quad Z_-^{(n)} = z_-^{(n)} - B_0.$$

Let us inductively assume that $z_\pm^{(n)}$ satisfies

$$|z_+^{(n)}|_{1,\alpha;f_+,\Omega,T} \le 2C_1A_1, \quad |z_-^{(n)}|_{1,\alpha;f_-,\Omega,T} \le 2C_1A_1,$$

where C_1 is the constant in Proposition 3.6.

Take $T>0$ so that $4C_1A_1\delta(T)<1$. Then we have

$$|z_+^{(n)}|_{1,\alpha;f_+,\Omega,T}\,\delta(T) < \tfrac{1}{2}, \quad |z_-^{(n)}|_{1,\alpha;f_-,\Omega,T}\,\delta(T) < \tfrac{1}{2}. \tag{3-19}$$

Now, the solution $z_+^{(n+1)}$, $z_-^{(n+1)}$ is determined by

$$\begin{cases} \partial_t z_+^{(n+1)} + Z_-^{(n)}\cdot\nabla z_+^{(n+1)} = -I(z_+^{(n)}, z_-^{(n)}),\\ \partial_t z_-^{(n+1)} + Z_+^{(n)}\cdot\nabla z_-^{(n+1)} = -I(z_+^{(n)}, z_-^{(n)}),\\ z_+^{(n+1)}(0,X) = z_{+0}(X), \quad z_-^{(n+1)}(0,X) = z_{-0}(X). \end{cases}$$

It follows from Proposition 3.6 that

$$\begin{aligned} |z_+^{(n+1)}|_{1,\alpha;f_+,\Omega,T} &\le C_1\big(|z_{+0}|_{1,\alpha;f,\Omega} + \delta(T)|z_+^{(n)}|_{1,\alpha;f_+,\Omega,T}|z_-^{(n)}|_{1,\alpha;f_-,\Omega,T}\big),\\ |z_-^{(n+1)}|_{1,\alpha;f_-,\Omega,T} &\le C_1\big(|z_{-0}|_{1,\alpha;f,\Omega} + \delta(T)|z_+^{(n)}|_{1,\alpha;f_+,\Omega,T}|z_-^{(n)}|_{1,\alpha;f_-,\Omega,T}\big). \end{aligned}$$

Here we used

$$|I(u,w)|_{1,\alpha;g,\Omega} \le C|\partial_i\bar u^j\partial_j\bar w^i - \partial_j\bar u^j\partial_i\bar w^i|_{0,\alpha;h} + C|\bar u\bar w|_{0;h} \le C|u|_{1,\alpha;h,\Omega}|w|_{1,\alpha;h,\Omega},$$

which follows from Lemma A.1 with $h(t,X) = f_+f_-(t,X)$ and Lemma 3.7.

Due to (3-19), we obtain

$$|z_+^{(n+1)}|_{1,\alpha;f_+,\Omega,T} \le 2C_1A_1, \quad |z_-^{(n+1)}|_{1,\alpha;f_-,\Omega,T} \le 2C_1A_1.$$

In particular, we show that for any n,

$$|z_+^{(n)}|_{1,\alpha;f_+,\Omega,T} \le C, \quad |z_-^{(n)}|_{1,\alpha;f_-,\Omega,T} \le C.$$

Next, we show that $\{z_\pm^{(n)}\}_{n\ge 0}$ are Cauchy sequences in $C^{0,\alpha}(\Omega)$. Indeed, we have

$$\begin{aligned} &\partial_t(z_+^{(n+1)} - z_+^{(n)}) + Z_-^{(n)}\cdot\nabla(z_+^{(n+1)} - z_+^{(n)}) + (z_-^{(n)} - z_-^{(n-1)})\cdot\nabla z_+^{(n)}\\ &\qquad\qquad + I(z_+^{(n)} - z_+^{(n-1)}, z_-^{(n)}) + I(z_+^{(n-1)}, z_-^{(n)} - z_-^{(n-1)}) = 0,\\ &\partial_t(z_-^{(n+1)} - z_-^{(n)}) + Z_+^{(n)}\cdot\nabla(z_-^{(n+1)} - z_-^{(n)}) + (z_+^{(n)} - z_+^{(n-1)})\cdot\nabla z_-^{(n)}\\ &\qquad\qquad + I(z_+^{(n)} - z_+^{(n-1)}, z_-^{(n)}) + I(z_+^{(n-1)}, z_-^{(n)} - z_-^{(n-1)}) = 0,\\ &\qquad (z_+^{(n+1)} - z_+^{(n)})(0,X) = 0, \quad (z_-^{(n+1)} - z_-^{(n)})(0,X) = 0. \end{aligned}$$

Then it follows from Lemma 3.5 and (3-14) that

$$\begin{aligned}\big|(z_+^{(n+1)}-z_+^{(n)})(t)\big|_{0,\alpha;\Omega} &\le C\int_0^t \big|(z_-^{(n)}-z_-^{(n-1)})(s)\big|_{0,\alpha;\Omega}\,|\nabla z_+^{(n)}(s)|_{0,\alpha;\Omega}\,ds\\ &\qquad + C\int_0^t \big|(z_+^{(n)}-z_+^{(n-1)})(s)\big|_{0,\alpha;\Omega}\,|z_-^{(n)}(s)|_{1,\alpha;\Omega}\,ds\\ &\qquad + C\int_0^t \big|(z_-^{(n)}-z_-^{(n-1)})(s)\big|_{0,\alpha;\Omega}\,|z_+^{(n-1)}(s)|_{1,\alpha;\Omega}\,ds\\ &\le C_2\int_0^t \Big(\big|(z_+^{(n)}-z_+^{(n-1)})(s)\big|_{0,\alpha;\Omega}+\big|(z_-^{(n)}-z_-^{(n-1)})(s)\big|_{0,\alpha;\Omega}\Big)\,ds.\end{aligned}$$

Similarly, we have

$$\big|(z_-^{(n+1)}-z_-^{(n)})(t)\big|_{0,\alpha;\Omega} \le C_2\int_0^t \Big(\big|(z_+^{(n)}-z_+^{(n-1)})(s)\big|_{0,\alpha;\Omega}+\big|(z_-^{(n)}-z_-^{(n-1)})(s)\big|_{0,\alpha;\Omega}\Big)\,ds.$$

This implies that

$$\big|(z_+^{(n+1)}-z_+^{(n)})(t)\big|_{0,\alpha;\Omega}+\big|(z_-^{(n+1)}-z_-^{(n)})(t)\big|_{0,\alpha;\Omega} \le C(2C_2t)^n/n!.$$

Therefore, $z_+^{(n)}, z_-^{(n)}$ converge to some z_+, z_- uniformly in $[0,t]\times\Omega$ for any $0<t<T$. As $z_+^{(n)}, z_-^{(n)}$ are uniformly bounded in $C^{1,\alpha}$, we have $z_+, z_-\in C^{1,\alpha}$. Then $\nabla z_+^{(n)}, \nabla z_-^{(n)}$ converge to $\nabla z_+, \nabla z_-$ uniformly in $[0,t]\times\Omega$ for any $0<t<T$. Using the equations of $z_+^{(n+1)}, z_-^{(n+1)}$, we have $\partial_t z_+^{(n)},\ \partial_t z_-^{(n)}$ also converge uniformly in $[0,t]\times\Omega$ for any $0<t<T$. Thus, $z_+,\ z_-\in C^1([0,t]\times\overline{\Omega})$ satisfies (3-18) and $z_+^d=z_-^d=0$ on $\partial\Omega$.

Finally, it remains to prove that if $\operatorname{div} z_{+0}=\operatorname{div} z_{-0}=0$, then $\operatorname{div} z_+=\operatorname{div} z_-=0$. It follows from Lemma 3.5 and (3-15) that

$$\begin{aligned}&|\operatorname{div} z_+(t)|_{0;\Omega}\\ &\quad\le \int_0^t \big|\big(\partial_i z_+^j\,\partial_j z_-^i-\operatorname{div} I(z_+,z_-)\big)(s)\big|_{0;\Omega}\,ds\\ &\quad\le C\int_0^t \big(|\operatorname{div} z_+(s)|_{0;\Omega}\,|\operatorname{div} z_-(s)|_{0;\Omega}+|z_+(s)|_{0;\Omega}\,|\operatorname{div} z_-(s)|_{0;\Omega}+|\operatorname{div} z_+(s)|_{0;\Omega}\,|z_-(s)|_{0;\Omega}\big)\,ds\\ &\quad\le C\int_0^t \big(|\operatorname{div} z_+(s)|_{0;\Omega}+|\operatorname{div} z_-(s)|_{0;\Omega}\big)\,ds.\end{aligned}$$

Similarly,

$$|\operatorname{div} z_-(t)|_{0;\Omega}\le C\int_0^t \big(|\operatorname{div} z_+(s)|_{0;\Omega}+|\operatorname{div} z_-(s)|_{0;\Omega}\big)\,ds.$$

This implies that $\operatorname{div} z_+=\operatorname{div} z_-=0$.

Let us remark that $I(z_+,z_-)$ can be expressed as ∇p. Indeed, we can find $\theta_1,\theta_2\in C^\infty(0,+\infty)$ such that $\theta_1'(r)=-\theta(r)N(r)$ and $\theta_2'(r)=(\theta(r)-1)N(r)$. Let $\theta_{ij}(X)=\partial_i\partial_j\theta_2(|X|)$ and

$$\begin{aligned}&I_*(u,w)(x)\\ &\quad=\int_{\mathbb{R}^d}\theta_1(|X-Y|)(\partial_i u^j\,\partial_j w^i-\partial_j u^j\,\partial_i w^i)(Y)\,dY+\int_{\mathbb{R}^d}\big(\theta_{ij}(X-Y)-\theta_{i,j}(-Y)\big)(u^j w^i)(Y)\,dY.\end{aligned}$$

Then we have $\nabla I_*(u,v)=I(u,v)$. Therefore, we can take $p=I_*(\bar z_+,\bar z_-)$, which satisfies $|p|\le C\ln(2+|x|)$. This completes the proof of Theorem 3.1.

4. Global well-posedness for the viscous MHD equations

In this section, we study the global well-posedness for the viscous MHD equations in the slab domain $\Omega=\mathbb{R}^{d-1}\times[0,1]$ with the Navier-slip boundary condition. Because we can reduce the slab domain $\Omega=\mathbb{R}^{d-1}\times[0,1]$ to $\mathbb{R}^{d-1}\times\mathbb{T}$ by using the symmetric extension, we will consider more general domain $\Omega=\mathbb{R}^k\times\mathbb{T}^{d-k}$ for $2\le k\le d$. The case $k=1$ is more difficult and will be dealt in the future work.

In fact, $\Omega=\mathbb{R}^k\times\mathbb{T}^{d-k}$ is a special case of $\mathbb{R}^d$ periodic in $d-k$ directions $e_1,\dots,e_{d-k}$. We will assume that $e_1,\dots,e_{d-k},B_0$ are linearly independent.

New formulation. Let $\mu_1=\frac12(\nu+\mu)$ and $\mu_2=\frac12(\nu-\mu)$. In terms of the Elsässer variables $Z_\pm=v\pm b$, the MHD equations (1-1) read

$$\begin{cases}\partial_t z_+ + Z_-\cdot\nabla z_+=\mu_1\,\Delta z_+ +\mu_2\,\Delta z_- -\nabla p,\\ \partial_t z_- + Z_+\cdot\nabla z_-=\mu_1\,\Delta z_- +\mu_2\,\Delta z_+ -\nabla p,\\ \operatorname{div} z_+=\operatorname{div} z_-=0,\end{cases}\tag{4-1}$$

where $z_\pm=Z_+\pm B_0$. In the case of $\nu=\mu$ (thus, $\mu_2=0$), the formulation (4-1) plays a crucial role in the proof of [Cai and Lei 2016; He et al. 2016]. To deal with the case of $\nu\ne\mu$, we need to introduce the key decomposition

$$z_+=z_+^{(1)}+z_+^{(2)},\quad z_-=z_-^{(1)}+z_-^{(2)},$$

where $z_\pm^{(1)}$ and $z_\pm^{(2)}$ are determined by

$$\begin{cases}\partial_t z_+^{(1)}+Z_-\cdot\nabla z_+^{(1)}=\mu_1\,\Delta z_+^{(1)}-\nabla p_+^{(1)},\\ \partial_t z_-^{(1)}+Z_+\cdot\nabla z_-^{(1)}=\mu_1\,\Delta z_-^{(1)}-\nabla p_-^{(1)},\\ \operatorname{div} z_+^{(1)}=\operatorname{div} z_-^{(1)}=0,\\ z_+^{(1)}(0)=z_+(0),\ z_-^{(1)}(0)=z_-(0),\end{cases}\tag{4-2}$$

and

$$\begin{cases}\partial_t z_+^{(2)}+Z_-\cdot\nabla z_+^{(2)}=\mu_1\,\Delta z_+^{(2)}+\mu_2\,\Delta z_- -\nabla p_+^{(2)},\\ \partial_t z_-^{(2)}+Z_+\cdot\nabla z_-^{(2)}=\mu_1\,\Delta z_-^{(2)}+\mu_2\,\Delta z_+ -\nabla p_-^{(2)},\\ \operatorname{div} z_+^{(2)}=\operatorname{div} z_-^{(2)}=0,\\ z_+^{(2)}(0)=z_-^{(2)}(0)=0.\end{cases}\tag{4-3}$$

To estimate $z_\pm^{(1)}$, we rewrite (4-2) as

$$\begin{cases}\partial_t z_+^{(1)}+Z_-^{(1)}\cdot\nabla z_+^{(1)}=\mu_1\,\Delta z_+^{(1)}-z_-^{(2)}\cdot\nabla z_+^{(1)}-I(z_-^{(2)},z_+^{(1)})-I(z_-^{(1)},z_+^{(1)}),\\ \partial_t z_-^{(1)}+Z_+^{(1)}\cdot\nabla z_-^{(1)}=\mu_1\,\Delta z_-^{(1)}-z_+^{(2)}\cdot\nabla z_-^{(1)}-I(z_+^{(2)},z_-^{(1)})-I(z_+^{(1)},z_-^{(1)}),\end{cases}\tag{4-4}$$

where $I(u,w)$ is defined by (3-17). We also need to use the equation of $J_\pm^{(1)} = \operatorname{curl} z_\pm^{(1)}$, which is given by

$$\begin{cases} \partial_t J_+^{(1)} + Z_-^{(1)}\cdot\nabla J_+^{(1)} + \nabla z_-^{(1)} \wedge \nabla z_+^{(1)} + \operatorname{curl}(z_-^{(2)}\cdot\nabla z_+^{(1)}) = \mu\,\Delta J_+^{(1)},\\ \partial_t J_-^{(1)} + Z_+^{(1)}\cdot\nabla J_-^{(1)} + \nabla z_+^{(1)} \wedge \nabla z_-^{(1)} + \operatorname{curl}(z_+^{(2)}\cdot\nabla z_-^{(1)}) = \mu\,\Delta J_-^{(1)}.\end{cases} \tag{4-5}$$

Here $A\wedge B = (AB)-(AB)^T$ is understood as matrix multiplication.

To estimate $z_\pm^{(2)}$, we need to introduce another formulation in terms of the stream function $\psi_\pm^{(2)} = \Delta^{-1}\operatorname{curl} z_\pm^{(2)}$, which satisfies

$$\begin{cases} \partial_t \psi_+^{(2)} + \Delta^{-1}\operatorname{curl}(Z_-\cdot\nabla z_+^{(2)}) = \mu_1\,\Delta\psi_+^{(2)} + \mu_2 J_-,\\ \partial_t \psi_-^{(2)} + \Delta^{-1}\operatorname{curl}(Z_+\cdot\nabla z_-^{(2)}) = \mu_1\,\Delta\psi_-^{(2)} + \mu_2 J_+,\end{cases}$$

where

$$J_\pm = \operatorname{curl} z_\pm = J_\pm^{(1)} + \operatorname{curl} z_\pm^{(2)}. \tag{4-6}$$

We introduce

$$\mathrm{II}_1(u,w) \triangleq \Delta^{-1}\operatorname{curl}\ \operatorname{div}(u\otimes w),$$
$$\mathrm{II}_2(u,w) \triangleq \Delta^{-1}\operatorname{curl}(u\cdot\nabla w) - u\cdot\nabla\,\Delta^{-1}\operatorname{curl} w.$$

So, we get

$$\Delta^{-1}\operatorname{curl}(Z_-\cdot\nabla z_+^{(2)}) = Z_-^{(1)}\cdot\nabla\psi_+^{(2)} + \mathrm{II}_1(z_-^{(2)}, z_+^{(2)}) + \mathrm{II}_2(z_-^{(1)}, z_+^{(2)}).$$

Then we deduce that

$$\begin{cases} \partial_t \psi_+^{(2)} + Z_-^{(1)}\cdot\nabla\psi_+^{(2)} + \mathrm{II}_2(z_-^{(1)}, z_+^{(2)}) + \mathrm{II}_1(z_-^{(2)}, z_+^{(2)}) = \mu_1\,\Delta\psi_+^{(2)} + \mu_2 J_-,\\ \partial_t \psi_-^{(2)} + Z_+^{(1)}\cdot\nabla\psi_-^{(2)} + \mathrm{II}_2(z_+^{(1)}, z_-^{(2)}) + \mathrm{II}_1(z_+^{(2)}, z_-^{(2)}) = \mu_1\,\Delta\psi_-^{(2)} + \mu_2 J_+.\end{cases} \tag{4-7}$$

A direct calculation shows

$$-\big(\Delta^{-1}\operatorname{curl}(u\cdot\nabla w)\big)^{jk} = \Delta^{-1}\big(\partial_k\partial_i(u^i w^j) - \partial_j\partial_i(u^i w^k)\big) = -R_k R_i(u^i w^j) + R_j R_i(u^i w^k),$$
$$-\big(u\cdot\nabla(\Delta^{-1}\operatorname{curl} w)\big)^{jk} = u^i\,\partial_i\Delta^{-1}(\partial_k w^j - \partial_j w^k) = u^i(-R_i R_k w^j + R_i R_j w^k),$$

where R_i is the Riesz transform defined by $R_i = \partial_i(-\Delta)^{-\frac{1}{2}}$. This gives

$$\mathrm{II}_2(u,w)^{jk} = [u^i, R_i R_j]w^k - [u^i, R_i R_k]w^j. \tag{4-8}$$

Weighted $C^{1,\alpha}$ estimates for the parabolic equation. We consider the parabolic equation with variable coefficients

$$\partial_t u - \gamma\,\partial_i(a_{ij}\,\partial_j u) + F_1 + F_2 + \partial_i G^i = 0, \tag{4-9}$$

where $\gamma>0$ and the coefficients $a_{ij}(t,X)$ satisfy

$$\sup_{t\in[0,T]}\big(|a_{ij}(t)-\delta_{ij}|_0 + (1+\gamma t)^{\frac{\alpha}{2}}[a_{ij}(t)]_\alpha\big) \le \varepsilon_0 \tag{4-10}$$

for some $\alpha\in(0,1)$, $\varepsilon_0>0$ and $T>0$.

Let $f(t,X)$ and $h(t,X)$ be two weight functions satisfying (2-1) with a uniform constant c_0 independent of t and

$$\int_0^t H(2\gamma(t-s))h(s,X)\,ds \le c_0^{-1} f(t,X), \quad H(2\gamma(t-s))f(s,X) \le c_0^{-1} f(t,X) \tag{4-11}$$

for all $0 \le s < t \le T,\ X \in \mathbb{R}^d$, where

$$H(t)\varphi(X) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|X-Y|^2}{4t}} \varphi(Y)\,dY.$$

Let $\delta > 0$. We introduce

$$\begin{aligned}\Lambda_1(T,F_1,F_2,G,f,h) \triangleq \sup_{0<t\le T} \Big(&|F_1(t)|_{1,\alpha;h(t),(1+\gamma t)^{1/2}} + \gamma^{-1}((\gamma t)^{\frac12} + (\gamma t)^{1+\frac{\delta}{2}})|F_2(t)|_{0,\alpha;f(t)} \\ &+ \gamma^{-1}(1+\gamma t)^{\frac12}|G(t)|_{0,\alpha;f(t),(1+\gamma t)^{1/2}}\Big),\end{aligned}$$

and

$$\begin{aligned}\Lambda_0(T,F_1,F_2,G,f,h) \triangleq \sup_{0<t\le T} \Big(&|F_1(t)|_{1,\alpha;h(t),(\gamma t)^{1/2}} + \gamma^{-1}((\gamma t)^{1-\frac{\alpha}{2}} + (\gamma t)^{1+\frac{\delta}{2}})|F_2(t)|_{0,\alpha;f(t)} \\ &+ \gamma^{-1}((\gamma t)^{\frac12} + (\gamma t)^{\frac{1-\alpha}{2}})|G(t)|_{0,\alpha;f(t),(\gamma t)^{1/2}}\Big).\end{aligned}$$

Proposition 4.1. *There exist $\varepsilon_0 > 0$ and $C > 0$ independent of γ and T such that*

$$\begin{aligned}\sup_{0<t\le T} |u(t)|_{1,\alpha;f(t),(1+\gamma t)^{1/2}} &\le C\big(|u(0)|_{1,\alpha;f(0),1} + \Lambda_1(T,F_1,F_2,G,f,h)\big),\\ \sup_{0<t\le T} |u(t)|_{1,\alpha;f(t),(\gamma t)^{1/2}} &\le C\big(|u(0)|_{0,\alpha;f(0)} + \Lambda_0(T,F_1,F_2,G,f,h)\big).\end{aligned}$$

Proof. Let us first consider the case $a_{ij} = \delta_{ij}$. Then we get

$$u(t) = H(\gamma t)u(0) + \int_0^t \big(H(\gamma(t-s))(F_1(s)+F_2(s)) + \partial_i H(\gamma(t-s))G^i(s)\big)\,ds.$$

Using $H(2\gamma t)f(0) \le c_0^{-1} f(0,X)$, we get by Lemma A.4 that

$$\begin{aligned}|H(\gamma t)u(0)|_{1,\alpha;f(t),(1+\gamma t)^{1/2}} &\le C|H(\gamma t)u(0)|_{1,\alpha;H(2\gamma t)f(0),(1+\gamma t)^{1/2}} \le C|u(0)|_{1,\alpha;f(0),1},\\ |H(\gamma t)u(0)|_{1,\alpha;f(t),(\gamma t)^{1/2}} &\le C|H(\gamma t)u(0)|_{1,\alpha;H(2\gamma t)f(0),(\gamma t)^{1/2}} \le C|u(0)|_{0,\alpha;f(0)}.\end{aligned}$$

By (4-11) and Lemma A.4, we have

$$\begin{aligned}\bigg|\int_0^t H(\gamma(t-s))F_1(s)\,ds\bigg|_{1,\alpha;f(t),(k+\gamma t)^{1/2}} &\le C \sup_{0<s<t} |H(\gamma(t-s))F_1(s)|_{1,\alpha;H(2\gamma(t-s))h(s),(k+\gamma t)^{1/2}}\\ &= C \sup_{0<s<t} |H(\gamma(t-s))F_1(s)|_{1,\alpha;H(2\gamma(t-s))h(s),(k+\gamma s+\gamma(t-s))^{1/2}}\\ &\le C \sup_{0<s<t} |F_1(s)|_{1,\alpha;h(s),(k+\gamma s)^{1/2}},\end{aligned}$$

and by Lemma A.4,

$$\begin{aligned}\left|\int_0^t H(\gamma(t-s))F_2(s)\,ds\right|_{1,\alpha;f(t),(k+\gamma t)^{1/2}} &\le C\int_0^t |H(\gamma(t-s))F_2(s)|_{1,\alpha;H(2\gamma(t-s))f(s),(k+\gamma t)^{1/2}}\,ds\\ &\le C\int_0^t \frac{\varphi_\alpha(\sqrt{k+\gamma t})}{\varphi_\alpha(\sqrt{\gamma(t-s)})}|F_2(s)|_{0,\alpha;f(s)}\,ds\end{aligned}$$

for $k=0,1$. Recall that $\varphi_\alpha(R)=\max(R,R^{1+\alpha})$ for $k=0,1$,

$$\begin{aligned}\int_0^t \varphi_\alpha(\sqrt{\gamma(t-s)})^{-1}\min\big((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}},(\gamma s)^{-1-\frac{\delta}{2}}\big)\,ds &\le \int_0^t (\gamma(t-s))^{-\frac12}(\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}}\,ds\\ &\le C\gamma^{-1}(\gamma t)^{-\frac{(1-k)(1-\alpha)}{2}},\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \varphi_\alpha(\sqrt{\gamma(t-s)})^{-1}\min\big((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}},(\gamma s)^{-1-\frac{\delta}{2}}\big)\,ds\\ &\qquad\le \int_0^t (\gamma(t-s))^{-\frac{1+\alpha}{2}}\min\big((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}},(\gamma s)^{-1-\frac{\delta}{2}}\big)\,ds\\ &\qquad\le C\int_0^{\frac t2}(\gamma t)^{-\frac{1+\alpha}{2}}\min\big((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}},(\gamma s)^{-1-\frac{\delta}{2}}\big)\,ds+\int_{\frac t2}^t(\gamma(t-s))^{-\frac{1+\alpha}{2}}(\gamma t)^{-1}\,ds\\ &\qquad\le C\gamma^{-1}(\gamma t)^{-\frac{1+\alpha}{2}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\int_0^t \frac{\varphi_\alpha(\sqrt{k+\gamma t})}{\varphi_\alpha(\sqrt{\gamma(t-s)})}\min\big((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}},(\gamma s)^{-1-\frac{\delta}{2}}\big)\,ds\\ &\qquad\le C\gamma^{-1}\max\big((k+\gamma t)^{\frac12},(k+\gamma t)^{\frac{1+\alpha}{2}}\big)\min\big((\gamma t)^{-\frac{(1-k)(1-\alpha)}{2}},(\gamma t)^{-\frac{1+\alpha}{2}}\big)\le C\gamma^{-1}.\end{aligned}$$

Therefore, we deduce that for $k=0,1$ and $j=1,2$,

$$\left|\int_0^t H(\gamma(t-s))F_j(s)\,ds\right|_{1,\alpha;f(t),(k+\gamma t)^{1/2}}\le C\Lambda_k(T,F_1,F_2,G,f,h).$$

It follows from Lemmas 2.3 and A.3 that for $k=0,1$,

$$\begin{aligned}&\left|\int_0^t \partial_i H(\gamma(t-s))G^i(s))\,ds\right|_{1,\alpha;f(t),(k+\gamma t)^{1/2}}\\ &\quad\le C\gamma^{-1}\sup_{0<s<t}\Big((\gamma s)^{\frac12}(\gamma(t-s))^{\frac12}\left|\partial_i H(\gamma(t-s))G^i(s)\right|_{0,\alpha;f(t)}\\ &\qquad\qquad+\varphi_\alpha(\sqrt{k+\gamma s})(\gamma(t-s))^{1-\frac{\alpha}{2}}\left|\nabla\partial_i H(\gamma(t-s))G^i(s)\right|_{0;f(t)}\\ &\qquad\qquad\qquad+\varphi_\alpha(\sqrt{k+\gamma s})(\gamma(t-s))^{\frac{3-\alpha}{2}}[\nabla\partial_i H(\gamma(t-s))G^i(s)]_{1;f(t)}\Big)\\ &\quad\le C\gamma^{-1}\sup_{0<s<t}\big((\gamma s)^{\frac12}|G(s)|_{0,\alpha;f(s)}+\varphi_\alpha(\sqrt{k+\gamma s})[G(s)]_{\alpha;f(s)}\big)\end{aligned}$$

$$\le C\gamma^{-1}\sup_{0<s<t}\big((k+\gamma s)^{\frac{1}{2}}+(k+\gamma s)^{\frac{1-\alpha}{2}}\big)|G(s)|_{0,\alpha;f(s),(k+\gamma s)^{1/2}}$$
$$\le C\Lambda_k(T,F_1,F_2,G,f,h).$$

Summing up, we conclude the proof for the case $a_{ij}=\delta_{ij}$.

To deal with the general case, we rewrite (4-9) as

$$\partial_t u-\gamma\,\Delta u+F_1+F_2+\partial_i\widehat{G}^i=0,$$

where $\widehat{G}^i=G^i-\gamma(a_{ij}-\delta_{ij})\,\partial_j u$. Thus, we have

$$\sup_{0<t\le T}|u(t)|_{1,\alpha;f(t),(k+\gamma t)^{1/2}}\le C\big(|u(0)|_{1,\alpha;f(0),k}+\Lambda_k(T,F_1,F_2,\widehat{G},f,h)\big)$$

for $k=0,1$, where

$$\Lambda_k(T,F_1,F_2,\widehat{G},f,h)$$
$$\le\Lambda_k(T,F_1,F_2,G,f,h)+\sup_{0\le t\le T}\sup_i\big((k+\gamma t)^{\frac{1}{2}}+(\gamma t)^{\frac{1-\alpha}{2}}\big)\big|(a_{ij}-\delta_{ij})\,\partial_j u(t)\big|_{0,\alpha;f(t),(k+\gamma t)^{1/2}},$$

and by (4-10),

$$\begin{aligned}\big|(a_{ij}-\delta_{ij})\,\partial_j u(t)\big|_{0,\alpha;f(t),(k+\gamma t)^{1/2}}&\le C|a_{ij}(t)-\delta_{ij}|_{0,\alpha;1,(k+\gamma t)^{1/2}}|\partial_j u(t)|_{0,\alpha;f(t),(k+\gamma t)^{1/2}}\\&\le C\varepsilon_0|\nabla u(t)|_{0,\alpha;f(t),(k+\gamma t)^{1/2}}\\&\le C\varepsilon_0\min\big((k+\gamma t)^{-\frac{1}{2}},(k+\gamma t)^{-\frac{1-\alpha}{2}}\big)|u(t)|_{1,\alpha;f(t),(k+\gamma t)^{1/2}}.\end{aligned}$$

This shows that

$$\sup_{0<t\le T}|u(t)|_{1,\alpha;f(t),(k+\gamma t)^{1/2}}$$
$$\le C\big(|u(0)|_{1,\alpha;f(0)}+\Lambda_k(T,F_1,F_2,G,f,h)+\varepsilon_0\sup_{0\le t\le T}|u(t)|_{1,\alpha;f(t),(k+\gamma t)^{1/2}}\big),$$

which gives the desired result by taking ε_0 such that $C\varepsilon_0\le\frac{1}{2}$. □

Weighted $C^{1,\alpha}$ estimates for the transport-diffusion equation. We consider the transport-diffusion equation with general form

$$\partial_t u+Z\cdot\nabla u-\gamma\,\Delta u+F_1+F_2+\partial_i G^i=0,\quad u(0,X)=u_0(X).\tag{4-12}$$

Given the divergence-free vector field $Z(t,X)\in C^1([0,T]\times\mathbb{R}^d)$ and $s\in[0,T]$, we define

$$\frac{d}{dt}\Phi(s,t,X)=Z(t,\Phi(s,t,X)),\quad \Phi(s,s,X)=X.$$

We denote by $D\Phi$ and $\nabla\Phi$ the matrix with the convention

$$(D\Phi)_{ij}=\partial_j\Phi^i,\quad (\nabla\Phi)_{ij}=\partial_i\Phi^j.$$

That is, $(D\Phi)=(\nabla\Phi)^T$. We introduce

$$b=(D\Phi)^{-1},\quad a=(D\Phi)^{-1}(\nabla\Phi)^{-1},\quad a_{ij}=b_{ik}b_{kj}.$$

For $v(t,X)$ defined in $[0,T]\times\mathbb{R}^d$, we define

$$v^*(t,X)\triangleq v(t,\Phi(s,t,X)).$$

Using the formulas

$$(\operatorname{div} G)\circ\Phi=\operatorname{div}\big((D\Phi)^{-1}G\circ\Phi\big),\quad (\Delta u)\circ\Phi=\operatorname{div}\big((D\Phi)^{-1}(\nabla\Phi)^{-1}\,\nabla u\circ\Phi\big),$$

we can transform (4-12) into the form

$$\partial_t u^*(t)-\gamma\partial_i(a_{ij}\partial_j u^*(t))+F_1^*+F_2^*+\partial_i G_*^i=0, \tag{4-13}$$

where $G_*^i=b_{ij}(G^*)^j$.

We introduce the weight functions $f(t,X)$, $\hat f(t,X)$, $h(t,X)$, which satisfy (2-1) with a uniform constant c_0 and

$$\begin{gathered}\int_0^t H(2\gamma(t-s))h_\pm(s,X)\,ds\le c_0^{-1}\hat f(t,X)\quad\text{for all }0\le t\le T,\ X\in\mathbb{R}^d,\\ \int_0^T f_\pm(t,X\pm B_0t)\,dt=\int_0^T f(t,X\pm 2B_0t)\,dt\le c_0^{-1},\\ H(2\gamma(t-s))\hat f(s,X)\le c_0^{-1}\hat f(t,X)\quad\text{for all }0\le s\le t\le T,\ X\in\mathbb{R}^d,\end{gathered} \tag{4-14}$$

where we set

$$f_\pm(t,X)=U(\pm t)f(t,X),\quad U(t)f(s,X)=f(s,X+B_0t).$$

Proposition 4.2. *There exist $\varepsilon_1>0$ and $C>0$ independent of γ and T such that if*

$$|Z(t)+B_0|_{1,\alpha;f_-(t),(1+\gamma t)^{1/2}}<\varepsilon_1$$

and (4-14) *holds for the minus sign, then it holds that for $k=0,1$,*

$$\sup_{0\le t\le T}|u(t)|_{1,\alpha;\hat f_+(t),(k+\gamma t)^{1/2}}\le C\big(|u_0|_{1,\alpha;\hat f(0),k}+\Lambda_k(T,F_1,F_2,G,\hat f_+,h)\big).$$

Similarly, if

$$|Z(t)-B_0|_{1,\alpha;f_+(t),(1+\gamma t)^{1/2}}<\varepsilon_1,$$

and (4-14) *holds for the plus sign, then it holds that for $k=0,1$,*

$$\sup_{0\le t\le T}|u(t)|_{1,\alpha;\hat f_-(t),(k+\gamma t)^{1/2}}\le C\big(|u_0|_{1,\alpha;\hat f(0),k}+\Lambda_k(T,F_1,F_2,G,\hat f_-,h)\big).$$

Proof. We only consider the case $|Z(t)+B_0|_{1,\alpha;f_-(t),(1+\gamma t)^{1/2}}<\varepsilon_1$. In this case, similar to (3-7), we have

$$|\Phi(s,t,X)+B_0(t-s)-X|<2\quad\text{for }0\le t\le s\le T.$$

Then we get by (2-1) and (4-14) that

$$\sup_{t\le s\le T} |\nabla Z(s)|_{0,\alpha;f_-(s)} \int_0^T f_-\big(s,\Phi(T,s,X)\big)\,ds \\ \le \varepsilon_1(1+\gamma t)^{-\frac12} c_0^{-1}\int_0^T f_-\big(s,X-B_0(s-T)\big)\,ds \le \varepsilon_1(1+\gamma t)^{-\frac12}c_0^{-1}, \tag{4-15}$$

and by (2-1),

$$U(s-t)h(t) = h(t,X+B_0(s-t)) \ge c_0 h(t)\circ\Phi(s,t), \tag{4-16}$$

$$U(s)\hat f(t) = U(s-t)\hat f_+(t) \ge c_0 \hat f_+(t)\circ\Phi(s,t). \tag{4-17}$$

Now we fix $s\ge 0$ and assume $0\le t\le s\le T$. With (4-15), we infer from Lemma 3.4 that

$$|\nabla\Phi(s,t)-\mathrm{Id}|_{0,\alpha} \le C\varepsilon_1(1+\gamma t)^{-\frac12}. \tag{4-18}$$

This implies that

$$|a_{ij}(t)-\delta_{ij}|_{0,\alpha} \le C\varepsilon_1(1+\gamma t)^{-\frac12}, \quad |b_{ij}(t)|_{0,\alpha;1,(1+\gamma t)^{1/2}} \le C.$$

Using (4-14), it is easy to verify that

$$H(2\gamma(t-\tau))U(s)\hat f(\tau,X) = U(s)H(2\gamma(t-\tau))\hat f(\tau,X) \le c_0^{-1}U(s)\hat f(t),$$

and

$$\int_0^t H(2\gamma(t-\tau))U(s-\tau)h(\tau,X)\,d\tau = \int_0^t H(2\gamma(t-\tau))U(s)h_-(\tau,X)\,d\tau \le c_0^{-1}U(s)\hat f(t).$$

Therefore, if we take $\varepsilon_1>0$ so that $C\varepsilon_1\le\varepsilon_0$, then we can apply Proposition 4.1 to obtain

$$\sup_{0<t\le s} |u^*(t)|_{1,\alpha;U(s)\hat f(t),(k+\gamma t)^{1/2}} \\ \le C\big(|u_0\circ\Phi(s,0)|_{1,\alpha;U(s)\hat f(0),k} + \Lambda_k(s,F_1^*,F_2^*,G_*,U(s)\hat f,U(s-\cdot)h)\big).$$

Thanks to (4-18), we get by Lemma 2.4, (4-16) and (4-17) that

$$|u_0\circ\Phi(s,0)|_{1,\alpha;U(s)\hat f(0),k} \le C|u_0\circ\Phi(s,0)|_{1,\alpha;\hat f(0)\circ\Phi(s,0),k} \le C|u_0|_{1,\alpha;\hat f(0),k},$$

$$|F_2^*(t)|_{0,\alpha;U(s)\hat f(t)} \le C|F_2^*(t)|_{0,\alpha;\hat f_+(t)\circ\Phi(s,t)} \le C|F_2(t)|_{0,\alpha;\hat f_+(t)},$$

$$|F_1^*(t)|_{1,\alpha;U(s-t)h(t),(k+\gamma t)^{1/2}} \le C|F_1^*(t)|_{1,\alpha;h(t)\circ\Phi(s,t),(k+\gamma t)^{1/2}} \le C|F_1(t)|_{1,\alpha;h(t),(k+\gamma t)^{1/2}},$$

and

$$\begin{aligned}|G_*(t)|_{0,\alpha;U(s)\hat f(t),(k+\gamma t)^{1/2}} &\le C|G_*(t)|_{0,\alpha;\hat f_+(t)\circ\Phi(s,t),(k+\gamma t)^{1/2}}\\ &\le C|b(t)|_{0,\alpha;1,(1+\gamma t)^{1/2}}|G(t)\circ\Phi(s,t)|_{0,\alpha;\hat f_+(t)\circ\Phi(s,t),(k+\gamma t)^{1/2}}\\ &\le C|G(t)|_{0,\alpha;\hat f_+(t),(k+\gamma t)^{1/2}}.\end{aligned}$$

This proves

$$\Lambda_k\big(s,F_1^*,F_2^*,G_*,U(s)\hat f,U(s-t)h\big) \le C\Lambda_k(s,F_1,F_2,G,\hat f_+,h).$$

Therefore, we conclude

$$\sup_{0<t\le s} |u^*(t)|_{1,\alpha;U(s)\hat{f}(t),(k+\gamma t)^{1/2}} \le C\big(|u_0|_{1,\alpha;\hat{f}(0),k} + \Lambda_k(s, F_1, F_2, G, \hat{f}_+, h)\big).$$

Thanks to $u^*(s) = u(s)$ and $U(s)\hat{f}(s) = \hat{f}_+(s)$, we have

$$|u(s)|_{1,\alpha;\hat{f}_+(s),(k+\gamma s)^{1/2}} \le C\big(|u_0|_{1,\alpha;\hat{f}(0),k} + \Lambda_k(s, F_1, F_2, G, \hat{f}_+, h)\big)$$

for all $0 < s \le T$. The case $s = 0$ is trivial. □

Main result. Let us first introduce the weight functions

$$f(t) = H(1+2\mu_1 t)\varphi_1, \quad f_1(t) = H(1+2\mu_1 t)\varphi_0,$$

where if $B_0 = (1, 0, \dots, 0)$, we may take

$$\varphi_1(X) = |x_1^2 + x_2^2|^{-\frac{1+\delta}{2}}, \quad \varphi_0(X) = |x_2|^{-\delta} \tag{4-19}$$

for some $0 < \delta < \frac{1}{2}$. Let

$$g(t, X) \triangleq \int_{\mathbb{R}^d} \frac{f_+(t, Y) f_-(t, Y)}{1 + |X - Y|^{d+1}}\, dY.$$

We introduce

$$M_\pm(t) \triangleq \sup_{0\le\tau\le t} \big(|z_\pm^{(1)}(\tau)|_{1,\alpha;f_\pm(\tau),(1+\mu_1\tau)^{1/2}} + |J_\pm^{(1)}(\tau)|_{1,\alpha;f_\pm(\tau),(\mu_1\tau)^{1/2}} + \mu_1^{-1}|\psi_\pm^{(2)}(\tau)|_{1,\alpha;f_1(\tau),(\mu_1\tau)^{1/2}}\big).$$

The main result of this section is stated as follows.

Theorem 4.3. *Let $\alpha \in (0, 1)$. There exists $\varepsilon_2 > 0$ such that if $M_\pm(0) + |\mu_2|/\mu_1 \le \varepsilon \le \varepsilon_2$, then there exists a global in time unique solution $(z_+, z_-) \in L^\infty((0, +\infty) \times \Omega)$, with the pressure p determined by* (3-16), *to the viscous MHD equations* (4-1) *satisfying*

$$M_\pm(t) \le C\varepsilon \quad \textit{for any } t \in [0, +\infty).$$

Remark 4.4. Since $M_\pm(0) \sim |z_\pm(0)\langle(x_1, x_2)\rangle^{1+\delta}|_{1,\alpha}$, the initial data decays at infinity only in two directions. This is crucial for the global well-posedness in domains like $\mathbb{R}^2$ and $\mathbb{R}^2 \times [0, 1]$.

Remark 4.5. The condition $|\mu_2| \le \varepsilon\mu_1$ is crucial to our proof. Although μ_2/μ_1 is small, the smallness is independent of μ_1. It remains open whether one can prove a similar result for any $\mu > 0$, $\nu > 0$.

Remark 4.6. In numerical simulation, μ_2 is usually taken to be zero, although it is unreasonable in physics. However, our result provides a theoretical base for the validity of such a choice, because our result shows that a small discrepancy between the dissipation coefficients does not change the dynamics of the system.

To proceed, we need to verify that the weight functions introduced here satisfy some key properties, (2-3) and (4-14).

With the choice of (4-19), it is easy to check that for $k = 0, 1$,

$$C^{-1} R^d \min(\varphi_k(X), R^{-k-\delta}) \le \int_{B(X,R)} \varphi_k(Y)\, dY \le C R^d \min(\varphi_k(X), R^{-k-\delta}),$$

$$\int_{\mathbb{R}} \varphi_1(X + B_0 t)\, dt \le C \varphi_0(X),$$

which imply

$$C^{-1} \min\bigl(\varphi_1(X), (1+\mu_1 t)^{-\frac{1+\delta}{2}}\bigr) \le f(t, X) \le C \min\bigl(\varphi_1(X), (1+\mu_1 t)^{-\frac{1+\delta}{2}}\bigr), \tag{4-20}$$

$$C^{-1} \min\bigl(\varphi_0(X), (1+\mu_1 t)^{-\frac{\delta}{2}}\bigr) \le f_1(t, X) \le C \min\bigl(\varphi_0(X), (1+\mu_1 t)^{-\frac{\delta}{2}}\bigr), \tag{4-21}$$

$$\int_{\mathbb{R}} f(t, X + B_0 s)\, ds \le C f_1(t, X). \tag{4-22}$$

Therefore,

$$\int_{B(X,R)} f_1(t, Y)\, dY \le C R^d \min\bigl(R^{-\delta}, (1+\mu_1 t)^{-\frac{1+\delta}{2}}\bigr) \tag{4-23}$$

and

$$\int_{B(X,R)} h(Y)\, dY \le C h(X), \tag{4-24}$$

which is true for $h = 1$, $f(t)$, $f_1(t)$, and $f_\pm(t)$ by translation. Thus,

$$\int_{\mathbb{R}^d} \frac{f_\pm(t, Y)\, dY}{R^{d+1} + |X - Y|^{d+1}} \le C R^{-1} f_\pm(t, X). \tag{4-25}$$

Lemma 4.7. (1) *The weight functions $f(t, X)$, $f_1(t, X)$, $g(t, X)$ satisfy* (2-3) *with $R = (1+\mu_1 t)^{\frac{1}{2}}$ and a uniform constant c_0 independent of t.*

(2) *Property* (4-14) *with $\gamma = \mu_1$ holds true for $(\hat{f}, h) = (f, g)$ and $(\hat{f}, h) = (f_1, f_-)$ for the minus sign or $(\hat{f}, h) = (f_1, f_+)$ for the plus sign.*

Proof. We deduce from (4-20) and (4-21) that $f(t)$ and $f_1(t)$ satisfy (2-3) with $R = (1+\mu_1 t)^{\frac{1}{2}}$. So do $f_\pm(t)$ and $f_+(t) f_-(t)$, and thus $g(t)$. This also implies

$$g(t, X) \ge C^{-1} f_+(t, X) f_-(t, X).$$

By definition, we have

$$H(2\mu_1(t-s)) f(s, X) = f(t, X), \quad H(2\mu_1(t-s)) f_1(s, X) = f_1(t, X),$$

which give the third inequality of (4-14).

By

$$\int_0^T f_\pm(t, X \pm B_0 t)\, dt = \int_0^T f(t, X \pm 2B_0 t)\, dt \le C f_1(t, X) \le C,$$

we get the second inequality of (4-14).

Thanks to

$$\begin{aligned}\int_0^t H(2\mu_1(t-s)) f_-(s,X)\,ds &= \int_0^t H(2\mu_1(t-s))U(-2s)f(s,X)\,ds\\ &= \int_0^t f(t,X-2B_0s)\,ds \le Cf_1(t,X),\end{aligned}$$

we get the first inequality of (4-14) with minus sign for $(\hat{f},h)=(f_1,f_-)$. Similarly, the first inequality of (4-14) with plus sign for $(\hat{f},h)=(f_1,f_+)$ is true.

Notice that

$$\begin{aligned}H(2\mu_1(t-s))g_\pm(s,X) &= \int_{\mathbb{R}^d}\frac{H(2\mu_1(t-s))(f_+(s)f_-(s))(Y\pm B_0s)}{1+|X-Y|^{d+1}}\,dY\\ &= \int_{\mathbb{R}^d}\frac{H(2\mu_1(t-s))\big(f(s)U(\pm 2s)f(s)\big)(Y)}{1+|X-Y|^{d+1}}\,dY.\end{aligned}$$

By (4-20), we have

$$f(t,X)\le C\bigg(1+\frac{|Y-X|}{\sqrt{1+\mu_1 t}}\bigg)^{1+\delta} f(t,Y),$$

which gives

$$f(s)U(\pm 2s)f(s)(X)\le C\bigg(1+\frac{|Y-X|}{\sqrt{1+\mu_1 s}}\bigg)^{2+2\delta} f(s)U(\pm 2s)f(s)(Y).$$

Therefore, for $t/2\le s<t$,

$$\begin{aligned}&H(2\mu_1(t-s))\big(f(s)U(\pm 2s)f(s)\big)(Y)\\ &\qquad= \int_{\mathbb{R}^d} K\big(2\mu_1(t-s),X-Y\big)\big(f(s)U(\pm 2s)f(s)\big)(X)\,dX\\ &\qquad\le C\int_{\mathbb{R}^d} K\big(2\mu_1(t-s),X-Y\big)\bigg(1+\frac{|Y-X|}{\sqrt{1+\mu_1 s}}\bigg)^{2+2\delta} f(s)U(\pm 2s)f(s)(Y)\,dX\\ &\qquad\le Cf(s)U(\pm 2s)f(s))(Y)\le Cf(t)U(\pm 2s)f(0))(Y),\end{aligned}$$

and for $0\le s\le t/2$,

$$H(2\mu_1(t-s))\big(f(s)U(\pm 2s)f(s)\big)\le CH(2\mu_1 t)\big(f(s)U(\pm 2s)f(s)\big)\le CH(2\mu_1 t)\big(f(0)U(\pm 2s)f(0)\big).$$

Therefore,

$$\begin{aligned}&\int_0^t H(2\mu_1(t-s))\big(f(s)U(\pm 2s)f(s)\big)\,ds\\ &\qquad\le C\int_0^{\frac{t}{2}} H(2\mu_1 t)\big(f(0)U(\pm 2s)f(0)\big)\,ds + C\int_{\frac{t}{2}}^t \big(f(t)U(\pm 2s)f(0)\big)\,ds\\ &\qquad\le CH(2\mu_1 t)(f(0)f_1(0)) + Cf(t)f_1(0)\le Cf(t).\end{aligned}$$

This shows that

$$\int_0^t H(2\mu_1(t-s))g_\pm(s,X)\,ds \le C\int_{\mathbb{R}^d} \frac{f(t,Y)}{1+|X-Y|^{d+1}}\,dY \le Cf(t,X),$$

which gives the first inequality of (4-14) for $(\hat{f},h)=(f,g)$. □

Proof of Theorem 4.3. The following lemma gives the relation between the Hölder norms of $z_\pm^{(i)}(t)$, $i=1,2$, and $M_\pm(t)$.

Lemma 4.8. *It holds that*

$$|z_\pm^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}} \le C\mu_1 \min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1-\alpha}{2}}\big)M_\pm(t),$$
$$|z_\pm^{(2)}(t)|_{0,\alpha;1,(1+\mu_1 t)^{1/2}} \le C\mu_1 \min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1+\delta}{2}}\big)M_\pm(t),$$
$$|\nabla z_\pm^{(1)}|_{1,\alpha;f_\pm(t),(\mu_1 t)^{1/2}} \le CM_\pm(t).$$

Proof. As $z_\pm^{(2)}=\operatorname{div}\psi_\pm^{(2)}$, we have

$$\begin{aligned}|z_\pm^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}} &\le C|\nabla\psi_\pm^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}}\\ &\le C|\psi_\pm^{(2)}(t)|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}}\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1-\alpha}{2}}\big)\\ &\le C\mu_1\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1-\alpha}{2}}\big)M_\pm(t),\end{aligned}$$

which along with (4-21) gives

$$\begin{aligned}|z_\pm^{(2)}(t)|_{0,\alpha;1,(1+\mu_1 t)^{1/2}} &\le |z_\pm^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}}\Big(1+\frac{1}{\mu_1 t}\Big)^{\frac{\alpha}{2}}|f_1(t)|_0\\ &\le C\mu_1\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1-\alpha}{2}}\big)M_\pm(t)\Big(1+\frac{1}{\mu_1 t}\Big)^{\frac{\alpha}{2}}(1+\mu_1 t)^{-\frac{\delta}{2}}\\ &\le C\mu_1\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1+\delta}{2}}\big)M_\pm(t).\end{aligned}$$

Obviously, we have

$$|\nabla z_\pm^{(1)}|_{0,\alpha;f_\pm(t)} \le |z_\pm^{(1)}|_{1,\alpha;f_\pm(t),(1+\mu_1 t)^{1/2}} \le M_\pm(s).$$

Thanks to $\Delta z_\pm^{(1)}=\operatorname{div} J_\pm^{(1)}$, we have

$$\begin{aligned}|\Delta z_\pm^{(1)}|_{0,\alpha;f_\pm(t),(\mu_1 t)^{1/2}} &\le C|J_\pm^{(1)}|_{1,\alpha;f_\pm(t),(\mu_1 t)^{1/2}}\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1-\alpha}{2}}\big)\\ &\le CM_\pm(s)\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1-\alpha}{2}}\big).\end{aligned}$$

Notice that by Lemma 4.7,

$$f_\pm(t,X)\le Cf_\pm(t,Y)\quad \text{if } |X-Y|\le(1+\mu_1 t)^{\frac12}.$$

Then we infer from Lemma A.10 that

$$\begin{aligned}|\nabla^2 z_\pm^{(1)}|_{0,\alpha;f_\pm(t),(\mu_1 t)^{1/2}} &\le C\big(|\nabla z_\pm^{(1)}|_{0,\alpha;f_\pm(t)}\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1-\alpha}{2}}\big)+|\Delta z_\pm^{(1)}|_{0,\alpha;f_\pm(t),(\mu_1 t)^{1/2}}\big)\\ &\le CM_\pm(s)\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1-\alpha}{2}}\big).\end{aligned}$$

This proves the third inequality. □

Proof of Theorem 4.3. For fixed $\nu > 0$ and $\mu > 0$, the local well-posedness of the MHD equations in the weighted Hölder spaces can be proved by using the semigroup method and the estimates of the heat operator in the weighted Hölder spaces (see the subsection on page 1394). Here we omit the details. The local well-posedness of the linear equations (4-2)–(4-7) in the weighted Hölder spaces is also true.

The proof of global well-posedness is based on a continuity argument. Let us first assume

$$M_\pm(s) < \varepsilon_1 \tag{4-26}$$

for $\varepsilon_1 > 0$ given by Proposition 4.2. This in particular gives

$$|Z_\pm^{(1)}(t) \pm B_0|_{1,\alpha;f_\mp(t),(1+\mu_1 t)^{1/2}} < \varepsilon_1.$$

Our next goal is to show that

$$M_+(s) \le C\big(M_+(0) + (M_+(s) + |\mu_2|/\mu_1)M_-(s)\big), \tag{4-27}$$

$$M_-(s) \le C\big(M_-(0) + (M_-(s) + |\mu_2|/\mu_1)M_+(s)\big). \tag{4-28}$$

With the above estimates, we can deduce our result if ε_2 is taken small enough that

$$CM_\pm(0) \le C\varepsilon_2 < \tfrac{1}{2}\varepsilon_1, \quad C^2\varepsilon_2 < \tfrac{1}{2}.$$

This condition on ε_2 implies that if $M_\pm(s) < \varepsilon_1$ then $M_\pm(s) \le 2CM_\pm(0) < \varepsilon_1$.

The proof of (4-27) and (4-28) is split into three steps.

<u>Step 1</u>: $C^{1,\alpha}$ estimate for $z_\pm^{(1)}$. For the system (4-4), we apply Proposition 4.2 to obtain

$$\begin{aligned}\sup_{0\le t\le s} &|z_+^{(1)}(t)|_{1,\alpha;f_+(t),(1+\mu_1 t)^{1/2}} \\ &\le C\big(|z_+(0)|_{1,\alpha;f(0),1} + \Lambda_1\big(s, I(z_-^{(1)}, z_+^{(1)}), z_-^{(2)}\cdot\nabla z_+^{(1)} + I(z_-^{(2)}, z_+^{(1)}), 0, f_+, g\big)\big).\end{aligned}$$

By (A-5), we have

$$\begin{aligned}\big|I(z_-^{(1)}(t), z_+^{(1)}(t))\big|_{1,\alpha;g(t),(1+\mu_1 t)^{1/2}} &\le C|z_-^{(1)}(t)|_{1,\alpha;f_-(t),(1+\mu_1 t)^{1/2}}|z_+^{(1)}(t)|_{1,\alpha;f_+(t),(1+\mu_1 t)^{1/2}} \\ &\le CM_+(s)M_-(s).\end{aligned}$$

By (A-6) and Lemma 4.8,

$$\begin{aligned}\big|z_-^{(2)}\cdot\nabla z_+^{(1)}(t) + I(z_-^{(2)}(t), z_+^{(1)}(t))\big|_{0,\alpha;f_+(t)} \\ &\le C|z_-^{(2)}(t)|_{0,\alpha;1,(1+\mu_1 t)^{1/2}}|z_+^{(1)}(t)|_{1,\alpha;f_+(t),(1+\mu_1 t)^{1/2}}(1+\mu_1 t)^{-\frac{1}{2}} \\ &\le C\mu_1 M_-(s)\min\big((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1+\delta}{2}}\big)M_+(s)(1+\mu_1 t)^{-\frac{1}{2}} \\ &\le C\mu_1 M_+(s)M_-(s)\min\big((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-1-\frac{\delta}{2}}\big),\end{aligned}$$

and obviously,

$$|z_+(0)|_{1,\alpha;f(0),1} \le M_+(0).$$

Therefore, we obtain

$$\sup_{0\le t\le s} |z_+^{(1)}(t)|_{1,\alpha;f_+(t),(1+\mu_1 t)^{1/2}} \le C\big(M_+(0)+M_+(s)M_-(s)\big).$$

Similarly, we have

$$\sup_{0\le t\le s} |z_-^{(1)}(t)|_{1,\alpha;f_-(t),(1+\mu_1 t)^{1/2}} \le C\big(M_-(0)+M_+(s)M_-(s)\big).$$

Step 2: $C^{1,\alpha}$ estimate for $J_\pm^{(1)}$. For the system (4-5), we apply Proposition 4.2 to obtain

$$\sup_{0\le t\le s} |J_+^{(1)}(t)|_{1,\alpha;f_+(t),(\mu_1 t)^{1/2}} \le C\big(|J_+(0)|_{0,\alpha;\hat f(0)} + \Lambda_0\big(s,\nabla z_-^{(1)}\wedge\nabla z_+^{(1)},0,z_-^{(2)}\cdot\nabla z_+^{(1)},f_+,g\big)\big).$$

Thanks to the choice of weight functions, we have

$$f_-(t,X)f_+(t,X)\le Cg(t,X).$$

Then by Lemma 4.8 and the analogue of Lemma 2.1, we have

$$\begin{aligned}
\big|\nabla z_-^{(1)}\wedge\nabla z_+^{(1)}(t)\big|_{1,\alpha;g(t),(\mu_1 t)^{1/2}} &\le C|\nabla z_-^{(1)}|_{1,\alpha;f_-(t),(\mu_1 t)^{1/2}}|\nabla z_+^{(1)}(t)|_{1,\alpha;f_+(t),(\mu_1 t)^{1/2}}\\
&\le CM_+(s)M_-(s),\\
\big|z_-^{(2)}\cdot\nabla z_+^{(1)}(t)\big|_{0,\alpha;f_+(t),(\mu_1 t)^{1/2}} &\le C|z_-^{(2)}(t)|_{0,\alpha;1,(\mu_1 t)^{1/2}}|\nabla z_+^{(1)}(t)|_{0,\alpha;f_+(t),(\mu_1 t)^{1/2}}\\
&\le C\mu_1\min\big((\mu_1 t)^{-\frac{1-\alpha}{2}},(\mu_1 t)^{-\frac12}\big)M_-(s)M_+(s),
\end{aligned}$$

and $|J_+(0)|_{0,\alpha;f(0)}\le M_+(0)$. Therefore, we obtain

$$\sup_{0\le t\le s} |J_+^{(1)}(t)|_{1,\alpha;f_+(t),(\mu_1 t)^{1/2}} \le C\big(M_+(0)+M_+(s)M_-(s)\big).$$

Similarly, we have

$$\sup_{0\le t\le s} |J_-^{(1)}(t)|_{1,\alpha;f_-(t),(\mu_1 t)^{1/2}} \le C\big(M_-(0)+M_+(s)M_-(s)\big).$$

Step 3: $C^{1,\alpha}$ estimate for $\psi_\pm^{(2)}$. For the system (4-7), we apply Proposition 4.2 to obtain

$$\sup_{0\le t\le s} |\psi_+^{(2)}(t)|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}} \le C\Lambda_0\big(s,\mathrm{II}_2(z_-^{(1)},z_+^{(2)})-\mu_2 J_-^{(1)},\mathrm{II}_1(z_-^{(2)},z_+^{(2)}),\mu_2 z_-^{(2)},f_1,f_-\big),$$

where we used the facts that $\psi_\pm^{(2)}(0)=0$ and $f_{1\pm}=f_1$, and the decomposition of $J_\pm$ in (4-6). We get by Proposition A.6 and Lemma 4.8 that

$$\begin{aligned}
&\big|\mathrm{II}_2(z_-^{(1)}(t),z_+^{(2)}(t))-\mu_2 J_-^{(1)}(t)\big|_{1,\alpha;f_-(t),(\mu_1 t)^{1/2}}\\
&\qquad\le C|z_-^{(1)}(t)|_{1,\alpha;f_-(t),(\mu_1 t)^{1/2}}|\nabla\psi_+^{(2)}(t)|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}}+|\mu_2||J_-^{(1)}(t)|_{1,\alpha;f(t),(\mu_1 t)^{1/2}}\\
&\qquad\le C\mu_1 M_+(s)M_-(s)+|\mu_2|M_-(s),
\end{aligned}$$

and

$$\begin{aligned}\left|\mathrm{II}_1(z_-^{(2)}(t), z_+^{(2)}(t))\right|_{0,\alpha;f_1(t)} &\le C|z_-^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}}|z_-^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}}(1+\mu_1 t)^{-\frac{\delta}{2}}(1+(\mu_1 t)^{-\frac{\alpha}{2}})\\ &\le C\mu_1^2 \min\bigl((\mu_1 t)^{-1+\frac{\alpha}{2}}, (\mu_1 t)^{-1-\frac{\delta}{2}}\bigr) M_-(s) M_+(s),\end{aligned}$$

and

$$|\mu_2 z_-^{(2)}(t)|_{0,\alpha;f_1(t),(\gamma t)^{1/2}} \le C\mu_1|\mu_2| \min\bigl((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1-\alpha}{2}}\bigr) M_-(s).$$

This shows that

$$\sup_{0\le t\le s} |\psi_+^{(2)}(t)|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}} \le C\bigl(\mu_1 M_+(s) + |\mu_2|\bigr) M_-(s).$$

Similarly, we have

$$\sup_{0\le t\le s} |\psi_-^{(2)}(t)|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}} \le C\bigl(\mu_1 M_-(s) + |\mu_2|\bigr) M_+(s).$$

Summing up the estimates in Steps 1–3, we conclude (4-27) and (4-28). □

Appendix

Weighted $C^{1,\alpha}$ estimate for the integral operator. Recall that

$$T_1 u \triangleq \int_{\mathbb{R}^d} \nabla N(X-Y)\theta(|X-Y|)u(Y)\,dY, \quad T_{ij} w \triangleq \int_{\mathbb{R}^d} \partial_i\partial_j\bigl(\nabla N(X-Y)(1-\theta(|X-Y|))\bigr)w(Y)\,dY,$$

where the cut-off function θ is given by (3-9).

Lemma A.1. *Let $u, w \in C_h^{0,\alpha}(\mathbb{R}^d)$, with the weight h satisfying* (2-1). *Then there exists a constant $C > 0$ depending only on c_0 such that*

$$|T_1 u|_{1,\alpha;h} \le C|u|_{0,\alpha;h}, \quad |T_{ij} w|_{1,\alpha;g} \le C|w|_{0;h},$$

where

$$g(X) = \int_{\mathbb{R}^d} \frac{h(Y)}{1+|X-Y|^{d+1}}\,dy.$$

In particular, we have

$$|T_1 u + T_{ij} w|_{1,\alpha;g} \le C\bigl(|u|_{0,\alpha;h} + |w|_{0,h}\bigr).$$

Proof. Thanks to

$$\left|\nabla^k \partial_i\partial_j\bigl(\nabla N(X-Y)\cdot(1-\theta(|x-y|))\bigr)\right| \le \frac{C}{1+|x-y|^{d+1}}, \quad k = 0, 1, 2,$$

and $h(X) \le Cg(X)$, we get

$$|\nabla^k T_{ij} w(X)| \le Cg(X)\left|\frac{w}{h}\right|_0,$$

which in particular implies

$$|T_{ij}w|_{1,\alpha;g} \le C\left|\frac{w}{h}\right|_0. \tag{A-1}$$

To deal with $T_1 u$, we decompose it as

$$T_1 u = \sum_{k=0}^{+\infty} B_k(u),$$

where

$$B_k(u) = \int_{\mathbb{R}^d} \varphi_k(X-Y)u(Y)\,dY, \quad \varphi_k(X) = \nabla N(X)\cdot\big(\theta(2^k|X|) - \theta(2^{k+1}|X|)\big).$$

To proceed, we need to use the simple facts

$$\int_{\mathbb{R}^d} |\varphi_k(X)|\,dX \le C2^{-k}, \quad \int_{\mathbb{R}^d} |\nabla\varphi_k(X)||X|^\alpha\,dX \le C2^{-k\alpha}, \quad \int_{\mathbb{R}^d} |\nabla^2\varphi_k(X)||X|^\alpha\,dX \le C2^{k(1-\alpha)},$$
$$\varphi_k(X) = 0 \quad \text{for } |X| > 2,\ k \ge 0.$$

Then we have

$$|B_k(u)(X)| \le \int_{\mathbb{R}^d} |\varphi_k(X-Y)||h(Y)|\,dY\left|\frac{u}{h}\right|_0 \le C2^{-k}h(X)\left|\frac{u}{h}\right|_0. \tag{A-2}$$

Notice that

$$\nabla B_k(u)(X) = \int_{\mathbb{R}^d} \nabla\varphi_k(X-Y)(u(Y)-u(X))\,dY,$$

from which, we deduce

$$|\nabla B_k(u)(X)| \le \int_{\mathbb{R}^d} |\nabla\varphi_k(X-Y)||X-Y|^\alpha(h(X)+h(Y))\,dY\,|u|_{0,\alpha;h} \le C2^{-k\alpha}h(X)|u|_{0,\alpha;h}. \tag{A-3}$$

Similarly, we have

$$|\nabla^2 B_k(u)(X)| \le C2^{k(1-\alpha)}h(X)|u|_{0,\alpha;h}. \tag{A-4}$$

It follows from (A-2) and (A-3) that

$$\sum_{k=0}^{+\infty} |B_k(u)(X)| \le \sum_{k=0}^{+\infty} C2^{-k}h(X)\left|\frac{u}{h}\right|_0 \le Ch(X)\left|\frac{u}{h}\right|_0,$$
$$\sum_{k=0}^{+\infty} |\nabla B_k(u)(X)| \le \sum_{k=0}^{+\infty} C2^{-k\alpha}h(X)|u|_{0,\alpha} \le Ch(X)|u|_{0,\alpha;h}.$$

It follows from (A-3) and (A-4) that

$$\big|\nabla B_k(u)(X) - \nabla B_k(u)(Y)\big| \le C2^{-k\alpha}(h(X)+h(Y))|u|_{0,\alpha;h}\min(1, 2^k|X-Y|),$$

which gives

$$\begin{aligned}\left|\sum_{k=0}^{+\infty} \nabla\big(B_k(u)(X) - B_k(u)(Y)\big)\right| &\le C(h(X)+h(Y))|u|_{0,\alpha;h}\sum_{k=0}^{+\infty} 2^{-k\alpha}\min(1, 2^k|X-Y|)\\ &\le C\big(h(X)+h(Y)\big)|u|_{0,\alpha;h}|X-Y|^\alpha.\end{aligned}$$

Now we can conclude that

$$|T_1 u|_{1,\alpha;h} \le \Big|\sum_{k=0}^{+\infty} B_k(u)\Big|_{1,\alpha;h} \le C|u|_{0,\alpha;h}. \qquad \square$$

Lemma A.2. *It holds that*

$$\begin{aligned}&\operatorname{div}(T_1 u + T_{ij} w^{ij}) + u\\ &\quad= \int_{\mathbb{R}^d} \nabla N(X-Y)\cdot\nabla\theta(|X-Y|)u(Y)\,dY - \int_{\mathbb{R}^d} \partial_i\partial_j\big(\nabla N(X-Y)\cdot\nabla\theta(|X-Y|)\big)w^{ij}(Y)\,dY.\end{aligned}$$

Proof. With the notations in Lemma A.1, a direct calculation gives

$$\begin{aligned}\operatorname{div}\ T_{ij}(w^{ij}) &= -\int_{\mathbb{R}^d} \partial_i\partial_j\big(\nabla N(X-Y)\cdot\nabla\theta(|X-Y|)\big)w^{ij}(Y)\,dY,\\ \operatorname{div} B_k(u) &= \int_{\mathbb{R}^d} \operatorname{div}\varphi_k(X-Y)u(Y)\,dY,\end{aligned}$$

where

$$\begin{aligned}\operatorname{div}\varphi_k(X) &= \nabla N(X)\cdot\nabla\big(\theta(2^k|X|)-\theta(2^{k+1}|X|)\big) = \varphi_k^*(X)-\varphi_{k+1}^*(X),\\ \varphi_k^*(X) &= \nabla N(X)\cdot\nabla\theta(2^k|X|) = -c_d\frac{2^k\theta'(2^k|X|)}{|X|^{d-1}} \ge 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{div}\sum_{k=0}^{N} B_k(u)+u &= \int_{\mathbb{R}^d}\big(\varphi_0^*(X-Y)-\varphi_{N+1}^*(X-Y)\big)u(Y)\,dY+u(X)\\ &= \int_{\mathbb{R}^d}\varphi_0^*(|X-Y|)u(Y)\,dY-\int_{\mathbb{R}^d}\varphi_{N+1}^*(X-Y)(u(Y)-u(X))\,dY \triangleq I_0^*-I_{N+1}^*,\end{aligned}$$

where we used $\int_{\mathbb{R}^d}\varphi_k^*(X)\,dX = 1$. Now,

$$|I_{N+1}^*| \le [u]_\alpha\int_{\mathbb{R}^d}\varphi_{N+1}^*(X-Y)|X-Y|^\alpha\,dY = C[u]_\alpha 2^{-N\alpha}\to 0$$

as $N\to+\infty$. This proves the lemma. $\square$

We also introduce

$$\begin{aligned}T_1(u,R) &\triangleq \int_{\mathbb{R}^d}\nabla N(X-Y)\theta(|X-Y|/R)u(Y)\,dY,\\ T_{ij}(w,R) &\triangleq \int_{\mathbb{R}^d}\partial_i\partial_j\big(\nabla N(X-Y)(1-\theta(|X-Y|/R))\big)w(Y)\,dY,\end{aligned}$$

where $N(X)$ is the Newton potential. Let $R\ge 1$. If $h(X)\le C_0 h(Y)$ for $|X-Y|\le 2R$, then we can deduce by following the proof of Lemma A.1 that

$$|T_1(u,R)|_{1,\alpha;g,R} + |T_{ij}(w,R)|_{1,\alpha;g,R} \le C\big(R^2|u|_{0,\alpha;h,R}+|w|_{0;h}\big),$$

where

$$g(X)=\int_{\mathbb{R}^d}\frac{h(Y)}{R^{d+1}+|X-Y|^{d+1}}\,dy.$$

Due to (4-25), we also have

$$R^{-1}|T_1(u,R)|_{1,\alpha;f_\pm(t),R}+|T_{ij}(w,R)|_{0,\alpha;f_\pm(t),R}\le C\big(|u|_{0,\alpha;f_\pm(t),R}+R^{-1}|w|_{0;f_\pm(t)}\big)$$

for $R=\sqrt{1+\mu_1 t}$.

In particular, we have

$$\begin{aligned}|I(u,w)|_{1,\alpha;g(t),(1+\mu_1t)^{1/2}}&\\ &\le C\big((1+\mu_1t)|\nabla u|_{0,\alpha;f_+(t),(1+\mu_1t)^{1/2}}|\nabla w|_{0,\alpha;f_-(t),(1+\mu_1t)^{1/2}}+|u|_{0;f_+(t)}|w|_{0;f_-(t)}\big)\\ &\le C|u|_{1,\alpha;f_+(t),(1+\mu_1t)^{1/2}}|w|_{1,\alpha;f_-(t),(1+\mu_1t)^{1/2}},\end{aligned}\tag{A-5}$$

where g, $f_\pm$ are defined as in the subsection on page 1385.

For $\operatorname{div} u=\operatorname{div} w=0$, we have

$$I(u,w)\triangleq T_1(\partial_i u^j\partial_j w^i,R)+T_{ij}(u^iw^j,R)=\partial_i T_1(u^j\partial_j w^i,R)+T_{ij}(u^iw^j,R).$$

Therefore, we deduce

$$|I(u,w)|_{0,\alpha;f_\pm(t)}\le C|u|_{0,\alpha;1,(1+\gamma t)^{1/2}}|w|_{1,\alpha;f_\pm(t),(1+\gamma t)^{1/2}}(1+\gamma t)^{-\frac12}.\tag{A-6}$$

Weighted Hölder estimates for the heat operator. Let $H(t)$ be the heat operator given by

$$H(t)f(X):=\frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|X-Y|^2}{4t}}f(Y)\,dY=\int_{\mathbb{R}^d}K(t,X-Y)f(Y)\,dY,$$

where $K(t,X)=(4\pi t)^{-\frac d2}e^{-\frac{|X|^2}{4t}}$. Let $\alpha\ge 0$ and $k\in\mathbb{N}$. It is easy to verify the properties

$$\begin{gathered}|\nabla^kK(t,X)|\le Ct^{-\frac k2}K(2t,X),\\ |\nabla^kK(t,X)||X'|^\alpha\le Ct^{-\frac{k-\alpha}{2}}K(2t,X),\\ |\nabla^kK(t,X)-\nabla^kK(t,Y)|\le Ct^{-\frac{k+1}{2}}K(2t,X)|X-Y|,\\ |\nabla^kK(t,X)-\nabla^kK(t,Y)||X'|^\alpha\le Ct^{-\frac{k+1-\alpha}{2}}K(2t,X)|X-Y|\end{gathered}\tag{A-7}$$

for any $X',Y\in B(X,\sqrt t)$. Here C is a constant independent of t.

We introduce the seminorm

$$[u]_{1;h}:=\sup_{X,Y\in\mathbb{R}^d}\frac{|u(X)-u(Y)|}{(h(X)+h(Y))|X-Y|}.$$

Then it is easy to check that

$$[u]_{\alpha;h}\le[u]_{1;h}^\alpha|u|_{0;h}^{1-\alpha},\quad |\nabla u|_{0;h}\le 2[u]_{1;h}.\tag{A-8}$$

Lemma A.3. *Let $u \in C_h^{0,\alpha}(\mathbb{R}^d)$, with $0 < h < C_0$ and $\alpha \in (0,1)$. Then there exists a constant $C > 0$ depending only on d, α, k such that, for $k \in \mathbb{N}$,*

$$|\nabla^k H(t)u|_{0;H(2t)h} \le Ct^{-\frac{k}{2}}|u|_{0;h}, \quad [\nabla^k H(t)u]_{1;H(2t)h} \le Ct^{-\frac{k+1}{2}}|u|_{0;h},$$
$$[\nabla^k H(t)u]_{\alpha;H(2t)h} \le Ct^{-\frac{k}{2}}[u]_{\alpha;h}, \quad [\nabla^k H(t)u]_{1;H(2t)h} \le Ct^{-\frac{k+1-\alpha}{2}}[u]_{\alpha;h}.$$

Proof. Thanks to (A-7), we have

$$\begin{aligned}|\nabla^k H(t)u(X)| &= \left|\int_{\mathbb{R}^d} \nabla^k K(t, X-Y)u(Y)\,dY\right| \\ &\le \int_{\mathbb{R}^d} |\nabla^k K(t,X-Y)||u(Y)|\,dY \\ &\le Ct^{-\frac{k}{2}} \int_{\mathbb{R}^d} K(2t, X-Y)h(Y)\,dY|u|_{0;h} \\ &\le Ct^{-\frac{k}{2}} H(2t)h(X)|u|_{0;h},\end{aligned}$$

which gives the first inequality.

If $|X-Y| < \sqrt{t}$, then we get by (A-7) that

$$\begin{aligned}\left|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)\right| &= \left|\int_{\mathbb{R}^d} (\nabla^k K(t, X-X') - \nabla^k K(t, Y-X'))u(X')\,dX'\right| \\ &\le \int_{\mathbb{R}^d} \left|\nabla^k K(t,X-X') - \nabla^k K(t,Y-X')\right||u(X')|\,dX' \\ &\le Ct^{-\frac{k+1}{2}}|X-Y|\int_{\mathbb{R}^d} K(2t,X-X')h(X')\,dX'|u|_{0;h} \\ &\le Ct^{-\frac{k+1}{2}}|X-Y|H(2t)h(X)|u|_{0;h},\end{aligned}$$

and if $|X-Y| \ge \sqrt{t}$, then

$$\begin{aligned}\left|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)\right| &\le |\nabla^k H(t)u(X)| + |\nabla^k H(t)u(Y)| \\ &\le Ct^{-\frac{k}{2}}H(2t)h(X)|u|_{0;h} + Ct^{-\frac{k}{2}}H(2t)h(Y)|u|_{0;h} \\ &\le Ct^{-\frac{k+1}{2}}|X-Y|\big(H(2t)h(X) + H(2t)h(Y)\big)|u|_{0;h},\end{aligned}$$

which imply the second inequality.

For any $X, Y \in \mathbb{R}^d$, we have

$$\begin{aligned}\left|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)\right| &= \left|\int_{\mathbb{R}^d} \nabla^k K(t,X')u(X-X')\,dX' - \int_{\mathbb{R}^d} \nabla^k K(t,X')u(Y-X')\,dX'\right| \\ &\le \int_{\mathbb{R}^d} |\nabla^k K(t,X')|\left|u(X-X') - u(Y-X')\right|dX' \\ &\le Ct^{-\frac{k}{2}}\int_{\mathbb{R}^d} K(2t,X')\big(h(X-X') + h(Y-X')\big)\,dX'|X-Y|^\alpha[u]_{\alpha;h} \\ &\le Ct^{-\frac{k}{2}}\big(H(2t)h(X) + H(2t)h(Y)\big)|X-Y|^\alpha[u]_{\alpha;h},\end{aligned}$$

which gives the third inequality.

For any $X, Y \in \mathbb{R}^d$, if $|X-Y| < \sqrt{t}$, we take $Y' \in B(X, \sqrt{t})$ so that

$$h(Y') \int_{B(X,\sqrt{t})} K(2t, X-X')\,dX' \le \int_{B(X,\sqrt{t})} K(2t, X-X')h(X')\,dX' \le H(2t)h(X),$$

which gives $h(Y') \le CH(2t)h(X)$. Then we deduce, for $|X-Y| < \sqrt{t}$,

$$\begin{aligned}
&|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)| \\
&\qquad = \left| \int_{\mathbb{R}^d} \big(\nabla^k K(t, X-X') - \nabla^k K(t, Y-X')\big)\big(u(X') - u(Y')\big)\,dX' \right| \\
&\qquad \le \int_{\mathbb{R}^d} \big|\nabla^k K(t, X-X') - \nabla^k K(t, Y-X')\big|\,\big|u(X') - u(Y')\big|\,dX' \\
&\qquad \le \int_{\mathbb{R}^d} \big|\nabla^k K(t, X-X') - \nabla^k K(t, Y-X')\big|\,|X'-Y'|^\alpha \big(h(X') + h(Y')\big)\,dX' [u]_{\alpha;h} \\
&\qquad \le Ct^{-\frac{k+1-\alpha}{2}} |X-Y| \int_{\mathbb{R}^d} K(2t, X-X')\big(h(X') + h(Y')\big)\,dX' [u]_{\alpha;h} \\
&\qquad \le Ct^{-\frac{k+1-\alpha}{2}} |X-Y| \big(H(2t)h(X) + h(Y')\big)[u]_{\alpha;h} \\
&\qquad \le Ct^{-\frac{k+1-\alpha}{2}} |X-Y| H(2t)h(X)[u]_{\alpha;h}.
\end{aligned}$$

While, if $|X-Y| \ge \sqrt{t}$, then

$$\begin{aligned}
\big|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)\big| &\le Ct^{-\frac{k}{2}} \big(H(2t)h(X) + H(2t)h(Y)\big)|X-Y|^\alpha [u]_{\alpha;h} \\
&\le Ct^{-\frac{k+1-\alpha}{2}} \big(H(2t)h(X) + H(2t)h(Y)\big)|X-Y|\,[u]_{\alpha;h}.
\end{aligned}$$

This proves the fourth inequality. □

Lemma A.4. *Let $\gamma > 0$, $k \ge 0$, and $u \in C_h^{0,\alpha}(\mathbb{R}^d)$, with $0 < h < C_0$. Let $R \ge \sqrt{t} > 0$. Then there exists a constant $C > 0$ depending only on d, α such that*

$$|H(t)u|_{1,\alpha;H(2t)h,\sqrt{k+t}} \le C|u|_{1,\alpha;h,\sqrt{k}}, \quad |H(t)u|_{1,\alpha;H(2t)h,R} \le C\frac{\varphi_\alpha(R)}{\varphi_\alpha(\sqrt{t})}|u|_{0,\alpha;h},$$

where $\varphi_\alpha(R) = \max(R, R^{1+\alpha})$.

Proof. By Lemma A.3 and (A-8), we have

$$\begin{gathered}
|H(t)u|_{0;H(2t)h} \le C|u|_{0;h}, \quad [H(t)u]_{\alpha;H(2t)h} \le C[u]_{\alpha;h}, \quad |H(t)u|_{0,\alpha;H(2t)h} \le C|u|_{0,\alpha;h}, \\
|\nabla H(t)u|_{0;H(2t)h} \le \min\big(Ct^{-\frac{1}{2}}|u|_{0;h}, Ct^{-\frac{1-\alpha}{2}}[u]_{\alpha;h}\big) \le C\min(t^{-\frac{1}{2}}, t^{-\frac{1-\alpha}{2}})|u|_{0,\alpha;h}, \\
[\nabla H(t)u]_{\alpha;H(2t)h} \le \min\big(Ct^{-\frac{1+\alpha}{2}}|u|_{0;h}, Ct^{-\frac{1}{2}}[u]_{\alpha;h}\big) \le C\min(t^{-\frac{1+\alpha}{2}}, t^{-\frac{1}{2}})|u|_{0,\alpha;h}.
\end{gathered}$$

Due to $\nabla H(t)u = H(t)\nabla u$, we have

$$|\nabla H(t)u|_{0;H(2t)h} \le C|\nabla u|_{0;h}, \quad [\nabla H(t)u]_{\alpha;H(2t)h} \le C[\nabla u]_{\alpha;h}.$$

Therefore,

$$
\begin{aligned}
|H(t)u|_{1,\alpha;H(2t)h,\sqrt{k+t}} &= |H(t)u|_{0,\alpha;H(2t)h} + \max((k+t)^{\frac{1-\alpha}{2}}, (k+t)^{\frac{1}{2}})|\nabla H(t)u|_{0;H(2t)h} \\
&\quad + \max((k+t)^{\frac{1}{2}}, (k+t)^{\frac{1+\alpha}{2}})[\nabla H(t)u]_{\alpha;H(2t)h} \\
&\le C|u|_{0,\alpha;h} + \max(k^{\frac{1-\alpha}{2}}, k^{\frac{1}{2}})|\nabla H(t)u|_{0;H(2t)h} \\
&\quad + \max(t^{\frac{1-\alpha}{2}}, t^{\frac{1}{2}})|\nabla H(t)u|_{0;H(2t)h} + \max(k^{\frac{1}{2}}, k^{\frac{1+\alpha}{2}})[\nabla H(t)u]_{\alpha;H(2t)h} \\
&\qquad + \max(t^{\frac{1}{2}}, t^{\frac{1+\alpha}{2}})[\nabla H(t)u]_{\alpha;H(2t)h} \\
&\le C|u|_{0,\alpha;h} + C\max(k^{\frac{1-\alpha}{2}}, k^{\frac{1}{2}})|\nabla u|_{0;h} + C|u|_{0,\alpha;h} \\
&\quad + C\max(k^{\frac{1}{2}}, k^{\frac{1+\alpha}{2}})[\nabla u]_{\alpha;h} + C|u|_{0,\alpha;h} \\
&\le C|u|_{1,\alpha;h,\sqrt{k}},
\end{aligned}
$$

which gives the first inequality. Also,

$$
\begin{aligned}
|H(t)u|_{1,\alpha;H(2t)h,R} &= |H(t)u|_{0,\alpha;H(2t)h} + \max(R^{1-\alpha}, R)\big(|\nabla H(t)u|_{0;H(2t)h} + R^{\alpha}[\nabla H(t)u]_{\alpha;H(2t)h}\big) \\
&\le C|u|_{0,\alpha;h} + \max(R, R^{1+\alpha})\big(t^{-\frac{\alpha}{2}}|\nabla H(t)u|_{0;H(2t)h} + [\nabla H(t)u]_{\alpha;H(2t)h}\big) \\
&\le C|u|_{0,\alpha;h} + C\varphi_\alpha(R)\min(t^{-\frac{1+\alpha}{2}}, t^{-\frac{1}{2}})|u|_{0,\alpha;h} \\
&\le C\frac{\varphi_\alpha(R)}{\varphi_\alpha(\sqrt{t})}|u|_{0,\alpha;h},
\end{aligned}
$$

which gives the second inequality. ☐

Riesz transform in the weighted Hölder spaces. Throughout this subsection, we take f, f_1, $f_\pm$ to be as in the subsection on page 1385. We need the following property for the weight functions.

Lemma A.5. *For $h = 1$, $f_1(t)$, $f(t)$, and $f_\pm(t)$, we have*

$$
R^{-d}\int_{B(X,R)} h(Y) f_1(t,Y)\,dY \le Ch(X)\min(R^{-\delta}, (1+\mu_1 t)^{-\frac{\delta}{2}}). \tag{A-9}
$$

Proof. The case of $h = 1$ follows from (4-23). We define

$$
\rho_1(X) = |x_2|, \quad \rho_2(X) = |(x_1, x_2)| \quad \text{for } X = (x_1, \dots, x_d) \in \mathbb{R}^d.
$$

Then by (4-21), for $h = f_1(t)$ if $\rho_1(X) \ge 2R$ or $\rho_1(X) \le 2\sqrt{1+\mu_1 t}$, we have

$$
h(Y) \le Ch(X) \quad \text{for } |Y - X| \le R,
$$

which gives,

$$
R^{-d}\int_{B(X,R)} h(Y) f_1(t,Y)\,dY \le CR^{-d}\int_{B(X,R)} h(X) f_1(t,Y)\,dY \le Ch(X)\min(R^{-\delta}, (1+\mu_1 t)^{-\frac{\delta}{2}}).
$$

Using (4-20), the above inequality holds for $h = f(t)$ if $\rho_2(X) \ge 2R$ or $\rho_2(X) \le 2\sqrt{1+\mu_1 t}$.

For the case $h = f_1(t)$, if $2\sqrt{1+\mu_1 t} \le \rho_1(X) \le 2R$, then by (4-21),

$$
h(X) \ge C^{-1}\varphi_1(X) \ge C^{-1}R^{-\delta}, \quad h(Y) f_1(t,Y) \le C\varphi_1(Y)^2 = C\rho_1(Y)^{-2\delta},
$$

which imply

$$R^{-d}\int_{B(X,R)} h(Y)f_1(t,Y)\,dY \le CR^{-d}\int_{B(X,R)} \rho_1(Y)^{-2\delta}\,dY \le CR^{-2\delta} \le Ch(X)R^{-\delta}.$$

For the case $h=f(t)$, if $2\sqrt{1+\mu_1 t} \le \rho_2(X) \le 2R$, then by (4-20),

$$h(X) \ge C^{-1}\varphi_2(X) \ge C^{-1}R^{-1-\delta}, \quad h(Y)f_1(t,Y) \le C\varphi_1(Y)\varphi_2(Y) = C|y_1|^{-\frac{1}{2}-\delta}|y_2|^{-\frac{1}{2}-\delta},$$

which imply

$$R^{-d}\int_{B(X,R)} h(Y)f_1(t,Y)\,dY \le CR^{-1-2\delta} \le Ch(X)R^{-\delta}.$$

Thus (A-9) is true for $h=f_1(t), f(t)$. The case $h=f_\pm(t)$ follows from the case $h=f(t)$ by translation. $\square$

Proposition A.6. *It holds that*

$$\big|[u,R_iR_j]\,\partial_k w\big|_{1,\alpha;f_\pm(t),(\mu_1 t)^{1/2}} \le C|u|_{1,\alpha;f_\pm(t),(1+\mu_1 t)^{1/2}}|w|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}},$$

$$|R_iR_j(uw)|_{0,\alpha;f_1(t)} \le C(1+\mu_1 t)^{-\frac{\delta}{2}}(1+(\mu_1 t)^{-\frac{\alpha}{2}})|u|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}}|w|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}}.$$

The proof of the proposition is very complicated. Let us begin with some reductions. For fixed i,j,

$$R_iR_jw(X)+\frac{\delta_{ij}}{d}w(X) = -\text{p.v.}\int_{\mathbb{R}^d}\partial_i\partial_j N(X-Y)w(Y)\,dY \triangleq \sum_{n=-\infty}^{\infty} R_{ij}^n(w),$$

where

$$R_{ij}^n(u) = -\int_{\mathbb{R}^d}\varphi_n(X-Y)u(Y)\,dY,$$

with $\varphi_n(X)=\partial_i\partial_j N(X)\big(\theta(2^n|X|)-\theta(2^{n+1}|X|)\big)$. Therefore,

$$[u,R_iR_j]\,\partial_k w = \sum_{n=-\infty}^{\infty}[u,R_{ij}^n]\,\partial_k w. \tag{A-10}$$

Lemma A.7. *For $h=1, f_1(t), f(t)$ and $f_\pm(t)$, it holds that*

$$|R_iR_j(u)|_{0,\alpha;h,(1+\mu_1 t)^{1/2}} \le C(1+\mu_1 t)^{-\frac{\delta}{2}}|u|_{0,\alpha;hf_1(t),(1+\mu_1 t)^{1/2}}.$$

Proof. Notice that

$$\int_{R^d}\varphi_n(X)\,dX=0, \quad \operatorname{supp}\varphi_n\subset B(0,2^{1-n})\setminus B(0,2^{-1-n}), \ \ |\nabla^l\varphi_n| \le C2^{n(d+l)},\ l=0,1,2.$$

We deduce from Lemma A.5 that

$$\begin{aligned}|R_{ij}^n(u)(X)| &\le \int_{\mathbb{R}^d}|\varphi_n(X-Y)|h(Y)f_1(t,Y)\,dY|u|_{0;hf_1(t)}\\ &\le C2^{nd}\int_{B(X,2^{1-n})}h(Y)f_1(t,Y)\,dY|u|_{0;hf_1(t)} \le C2^{n\delta}h(X)|u|_{0;hf_1(t)}.\end{aligned}$$

For $X\in\mathbb{R}^d$,

$$R_{ij}^n(u)(X) = -\int_{\mathbb{R}^d}\varphi_n(X-Y)(u(Y)-u(X))\,dY,$$

which along with (4-24) gives

$$|R^n_{ij}(u)(X)| \le \int_{\mathbb{R}^d} |\varphi_n(X-Y)|(h(X)+h(Y))|X-Y|^\alpha \, dY [u]_{\alpha;h}$$
$$\le C 2^{n(d-\alpha)} \int_{B(X,2^{1-n})} (h(X)+h(Y))\, dY [u]_{\alpha;h} \le C 2^{-n\alpha} h(X)[u]_{\alpha;h}.$$

By (4-21), we have

$$[u]_{\alpha;h} \le C(1+\mu_1 t)^{-\frac{\delta}{2}} [u]_{\alpha;hf_1(t)} \le C(1+\mu_1 t)^{-\frac{\alpha+\delta}{2}} |u|_{0,\alpha;hf_1(t),(1+\mu_1 t)^{1/2}}.$$

Thus, we can conclude

$$|R_i R_j(u)(X)| \le \sum_{n=-\infty}^{\infty} |R^n_{ij}(u)(X)|$$
$$\le \sum_{n=-\infty}^{\infty} C \min\bigl(2^{n\delta}, 2^{-n\alpha}(1+\mu_1 t)^{-\frac{\alpha+\delta}{2}}\bigr) h(X)|u|_{0,\alpha;hf_1(t),(1+\mu_1 t)^{1/2}}$$
$$\le C(1+\mu_1 t)^{-\frac{\delta}{2}} h(X)|u|_{0,\alpha;hf_1(t),(1+\mu_1 t)^{1/2}}.$$

For any $X, X' \in \mathbb{R}^d$, with $|X-X'| \le 2^{-n}$,

$$\bigl|R^n_{ij}(u)(X) - R^n_{ij}(u)(X')\bigr| \le \int_{\mathbb{R}^d} \bigl|\varphi_n(X-Y) - \varphi_n(X'-Y)\bigr|(h(X)+h(Y))|X-Y|^\alpha \, dY [u]_{\alpha;h}$$
$$\le C 2^{n(d+1-\alpha)} |X-X'| \int_{B(X,2^{1-n})} (h(X)+h(Y))\, dY [u]_{\alpha;h}$$
$$\le C 2^{n(1-\alpha)} |X-X'| h(X)[u]_{\alpha;h},$$

which gives, for any $X, X' \in \mathbb{R}^d$,

$$\bigl|R^n_{ij}(u)(X) - R^n_{ij}(u)(X')\bigr| \le C 2^{-n\alpha} \min(1, 2^n|X-X'|)(h(X)+h(X'))[u]_{\alpha;h}.$$

Then we have

$$\bigl|R_i R_j(u)(X) - R_i R_j(u)(X')\bigr| \le \sum_{n=-\infty}^{\infty} \bigl|R^n_{ij}(u)(X) - R^n_{ij}(u)(X')\bigr|$$
$$\le \sum_{n=-\infty}^{\infty} C 2^{-n\alpha} \min(1, 2^n|X-X'|)(h(X)+h(X'))[u]_{\alpha;h}$$
$$\le C|X-X'|^\alpha (h(X)+h(X'))[u]_{\alpha;h},$$

which implies $[R_i R_j u]_{\alpha;h} \le C[u]_{\alpha;h}$. Thus,

$$|R_i R_j(u)|_{0,\alpha;h,(1+\mu_1 t)^{1/2}} = |R_i R_j(u)|_{0;h} + (1+\mu_1 t)^{\frac{\alpha}{2}} [R_i R_j u]_{\alpha;h}$$
$$\le C(1+\mu_1 t)^{-\frac{\delta}{2}} |u|_{0,\alpha;hf_1(t),(1+\mu_1 t)^{1/2}} + (1+\mu_1 t)^{\frac{\alpha}{2}} [u]_{\alpha;h}$$
$$\le C(1+\mu_1 t)^{-\frac{\delta}{2}} |u|_{0,\alpha;hf_1(t),(1+\mu_1 t)^{1/2}},$$

which gives our result. □

Lemma A.8. *For $l=0,1$, it holds that*

$$\big|\nabla^l[u,R_{ij}^n]\,\partial_k w(X)\big|\le C2^{n(l-\alpha)}|\nabla u|_{0;B(X,2^{1-n})}[w]_\alpha.$$

Proof. Thanks to

$$\begin{aligned}[u,R_{ij}^n]\,\partial_k w(X)&=\int_{\mathbb{R}^d}\varphi_n(X-Y)(u(Y)-u(X))\,\partial_k w(Y)\,dY\\&=\int_{\mathbb{R}^d}\partial_k\varphi_n(X-Y)(u(Y)-u(X))w(Y)dY-\int_{\mathbb{R}^d}\varphi_n(X-Y)\,\partial_k u(Y)w(Y)\,dY,\end{aligned}\tag{A-11}$$

we deduce that

$$\begin{aligned}\big|[u,R_{ij}^n]\,\partial_k w(X)\big|\le&\int_{\mathbb{R}^d}|\partial_k\varphi_n(X-Y)|\,|X-Y|\,dY|\nabla u|_{0;B(X,2^{1-n})}|w|_{0;B(X,2^{1-n})}\\&+\int_{\mathbb{R}^d}|\varphi_n(X-Y)|\,dY|\nabla u|_{0;B(X,2^{1-n})}|w|_{0;B(X,2^{1-n})}\\\le&\, C|\nabla u|_{0;B(X,2^{1-n})}|w|_{0;B(X,2^{1-n})},\end{aligned}$$

Thanks to

$$\begin{aligned}\nabla[u,R_{ij}^n]\,\partial_k w(X)=&\int_{\mathbb{R}^d}\nabla\partial_k\varphi_n(X-Y)(u(Y)-u(X))w(Y)\,dY\\&-\nabla u(X)\int_{\mathbb{R}^d}\partial_k\varphi_n(X-Y)w(Y)dY-\int_{\mathbb{R}^d}\nabla\varphi_n(X-Y)\,\partial_k u(Y)w(Y)\,dY,\end{aligned}\tag{A-12}$$

we can similarly deduce that

$$\big|\nabla[u,R_{ij}^n]\,\partial_k w(X)\big|\le C2^n|\nabla u|_{0;B(X,2^{1-n})}|w|_{0;B(X,2^{1-n})}.$$

As $[u,R_{ij}^n]\,\partial_k w=[u,R_{ij}^n]\,\partial_k(w-w(X))$, we have, for $l=0,1$,

$$\big|\nabla^l[u,R_{ij}^n]\,\partial_k w(X)\big|\le C2^{nl}|\nabla u|_{0;B(X,2^{1-n})}|w-w(X)|_{0;B(X,2^{1-n})}\le C2^{n(l-\alpha)}|\nabla u|_{0;B(X,2^{1-n})}[w]_\alpha.$$

□

Lemma A.9. *If $|u|_{1,\alpha;h,(1+\mu_1 t)^{1/2}}=|w|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}}=1$ for $h=1$, $f_1(t)$, $f(t)$ and $f_\pm(t)$, then we have*

$$\begin{aligned}\big|[u,R_{ij}^n]\,\partial_k w(X)\big|&\le Ch(X)\min\big(2^{n\delta}(1+\mu_1 t)^{-\frac12},2^{-n\alpha}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}\big),\\\big|\partial_l[u,R_{ij}^n]\,\partial_k w(X)\big|&\le Ch(X)(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}\min\big(2^{n(1-\alpha)},2^{-n\alpha}(\mu_1 t)^{-\frac12}\big).\end{aligned}$$

Proof. As $|u|_{1,\alpha;h,(1+\mu_1 t)^{1/2}}=|w|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}}=1$, we have

$$|u(X)|\le h(X),\quad|\nabla u(X)|\le h(X)(1+\mu_1 t)^{-\frac12},\quad|w(X)|\le f_1(t,X).$$

Using $f_1(t,X)\le C(1+\mu_1 t)^{-\frac{\delta}{2}}$, we also have

$$\begin{aligned}&|w|_0\le C(1+\mu_1 t)^{-\frac{\delta}{2}},\quad[w]_\alpha\le C(1+\mu_1 t)^{-\frac{\delta}{2}},\quad|\nabla w|_0\le C(1+\mu_1 t)^{-\frac{\delta}{2}}(\mu_1 t)^{-\frac12},\\&[\nabla w]_\alpha\le C(1+\mu_1 t)^{-\frac{\delta}{2}}\min\big((\mu_1 t)^{-\frac12},(\mu_1 t)^{-\frac{1+\alpha}{2}}\big)\le C(1+\mu_1 t)^{-\frac{\delta+\alpha}{2}}(\mu_1 t)^{-\frac12},\end{aligned}$$

and

$$[w]_\alpha \le C|w|_0^{1-\alpha}|\nabla w|_0^\alpha \le C(1+\mu_1 t)^{-\frac{\delta}{2}}(\mu_1 t)^{-\frac{\alpha}{2}}.$$

Therefore

$$[w]_\alpha \le C(1+\mu_1 t)^{-\frac{\delta}{2}}\min(1,(\mu_1 t)^{-\frac{\alpha}{2}}) \le C(1+\mu_1 t)^{-\frac{\delta+\alpha}{2}}.$$

Then we deduce from (A-11) and Lemma A.5 that, for $2^{-n} \ge \sqrt{1+\mu_1 t}$,

$$\begin{aligned}
&\big|[u, R_{ij}^n]\,\partial_k w(X)\big| \\
&\quad\le \int_{\mathbb{R}^d} |\partial_k\varphi_n(X-Y)|(h(X)+h(Y)) f_1(t,Y)\,dY + (1+\mu_1 t)^{-\frac{1}{2}}\int_{\mathbb{R}^d} |\varphi_n(X-Y)| h(Y) f_1(t,Y)\,dY \\
&\quad\le C2^{n(d+1)}\int_{B(X,2^{1-n})}(h(X)+h(Y)) f_1(t,Y)\,dY + C(1+\mu_1 t)^{-\frac{1}{2}}2^{nd}\int_{B(X,2^{1-n})} h(Y) f_1(t,Y)\,dY \\
&\quad\le C2^{n(1+\delta)}h(X) + C(1+\mu_1 t)^{-\frac{1}{2}}2^{n\delta}h(X) \\
&\quad\le C(1+\mu_1 t)^{-\frac{1}{2}}2^{n\delta}h(X).
\end{aligned}$$

For $2^{-n} \le \sqrt{1+\mu_1 t}$, we have

$$|\nabla u|_{0;B(X,2^{1-n})} \le |\nabla u|_{0;h,B(X,2^{1-n})}|h|_{0;B(X,2^{1-n})} \le Ch(X)(1+\mu_1 t)^{-\frac{1}{2}},$$

where we used the fact that h satisfies (2-3) with $R=\sqrt{1+\mu_1 t}$. Similarly, we have

$$[\nabla u]_{\alpha;B(X,2^{1-n})} \le [\nabla u]_{\alpha;h,B(X,2^{1-n})}|h|_{0;B(X,2^{1-n})} \le Ch(X)(1+\mu_1 t)^{-\frac{1+\alpha}{2}}.$$

Then we get by Lemma A.8 that

$$\big|[u, R_{ij}^n]\,\partial_k w(X)\big| \le C2^{n\alpha}h(X)|\nabla u|_{0;B(X,2^{1-n})}[w]_\alpha \le C2^{n\alpha}h(X)(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}},$$

which gives the first inequality of the lemma.

Similarly, by (A-12) and Lemmas A.5 and A.8, we can deduce

$$\begin{aligned}
|\partial_l[u, R_{ij}^n]\,\partial_k w(X)| &\le Ch(X)2^n\min\big(2^{n\delta}(1+\mu_1 t)^{-\frac{1}{2}}, 2^{-n\alpha}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}\big) \\
&\le Ch(X)2^{n(1-\alpha)}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\partial_l[u, R_{ij}^n]\,\partial_k w(X) &= \int_{\mathbb{R}^d}\partial_l\varphi_n(X-Y)(u(Y)-u(X))\,\partial_k w(Y)\,dY - \partial_l u(X)\int_{\mathbb{R}^d}\varphi_n(X-Y)\,\partial_k w(Y)\,dY \\
&\triangleq [u, \partial_l R_{ij}^n]\,\partial_k w(X) + \partial_l u(X) R_{ij}^n\,\partial_k w(X).
\end{aligned}$$

From the proof of Lemma A.7, we can see that

$$|R_{ij}^n\,\partial_k w(X)| \le C2^{-n\alpha}[\partial_k w]_\alpha \le C2^{-n\alpha}(1+\mu_1 t)^{-\frac{\delta+\alpha}{2}}(\mu_1 t)^{-\frac{1}{2}}.$$

By (4-24), we deduce that, for $2^{-n} \geq \sqrt{1+\mu_1 t}$,

$$\begin{aligned}
\left|[u, \partial_l R_{ij}^n]\, \partial_k w(X)\right| &\leq \int_{\mathbb{R}^d} |\partial_l \varphi_n(X-Y)| \big(|u(Y)| + |u(X)|\big) |\partial_k w(Y)|\, dY \\
&\leq C 2^{n(d+1)} (1+\mu_1 t)^{-\frac{\delta}{2}} (\mu_1 t)^{-\frac{1}{2}} \int_{B(X,2^{1-n})} (h(Y)+h(X))\, dY \\
&\leq C 2^{n} (1+\mu_1 t)^{-\frac{\delta}{2}} (\mu_1 t)^{-\frac{1}{2}} h(X) \\
&\leq C 2^{-n\alpha} (1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}} (\mu_1 t)^{-\frac{1}{2}} h(X).
\end{aligned}$$

For $2^{-n} \leq \sqrt{1+\mu_1 t}$, using the formula

$$\begin{aligned}
[u, \partial_l R_{ij}^n]\, \partial_k w(X) = \int_{\mathbb{R}^d} \partial_l \varphi_n(X-Y)(u(Y)-u(X))(\partial_k w(Y) - \partial_k w(X))\, dY \\
+ \partial_k w(X) \int_{\mathbb{R}^d} \varphi_n(X-Y)(\partial_l u(Y) - \partial_l u(X))\, dY,
\end{aligned}$$

we deduce that

$$\begin{aligned}
\left|[u, \partial_l R_{ij}^n]\, \partial_k w(X)\right| &\leq \int_{\mathbb{R}^d} |\partial_l \varphi_n(X-Y)|\,|X-Y|^{1+\alpha}\, dY\, |\nabla u|_{0;B(X,2^{1-n})} [\partial_k w]_\alpha \\
&\qquad + |\partial_k w|_0 \int_{\mathbb{R}^d} \varphi_n(X-Y)|X-Y|^{\alpha}\, dY\, [\nabla u]_{\alpha; B(X,2^{1-n})} \\
&\leq C 2^{-n\alpha} h(X) (1+\mu_1 t)^{-\frac{1}{2}} (1+\mu_1 t)^{-\frac{\delta+\alpha}{2}} (\mu_1 t)^{-\frac{1}{2}} \\
&\qquad + C (1+\mu_1 t)^{-\frac{\delta}{2}} (\mu_1 t)^{-\frac{1}{2}} 2^{-n\alpha} h(X) (1+\mu_1 t)^{-\frac{1+\alpha}{2}} \\
&\leq C 2^{-n\alpha} h(X) (1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}} (\mu_1 t)^{-\frac{1}{2}}.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\left|\partial_l [u, R_{ij}^n]\, \partial_k w(X)\right| \\
&\quad\leq \left|[u, \partial_l R_{ij}^n]\, \partial_k w(X)\right| + \left|\partial_l u(X) R_{ij}^n\, \partial_k w(X)\right| \\
&\quad\leq C 2^{-n\alpha} h(X) (1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}} (\mu_1 t)^{-\frac{1}{2}} + C h(X) (1+\mu_1 t)^{-\frac{1}{2}} 2^{-n\alpha} (1+\mu_1 t)^{-\frac{\delta+\alpha}{2}} (\mu_1 t)^{-\frac{1}{2}} \\
&\quad\leq C 2^{-n\alpha} h(X) (1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}} (\mu_1 t)^{-\frac{1}{2}},
\end{aligned}$$

which gives the second inequality of the lemma. □

Using the formula

$$\begin{aligned}
&\partial_m [u, \partial_l R_{ij}^n]\, \partial_k w(X) \\
&\quad= \int_{\mathbb{R}^d} \partial_m \partial_l \varphi_n(X-Y)(u(Y)-u(X))\, \partial_k w(Y)\, dY - \partial_m u(X) \int_{\mathbb{R}^d} \partial_l \varphi_n(X-Y)\, \partial_k w(Y)\, dY,
\end{aligned}$$

we can also deduce that

$$\left|\partial_m [u, \partial_l R_{ij}^n]\, \partial_k w(X)\right| \leq C 2^{n(1-\alpha)} h(X) (1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}} (\mu_1 t)^{-\frac{1}{2}}. \tag{A-13}$$

Now we are in position to prove Proposition A.6.

Proof. We get by Lemma A.7 with $h = f_1(t)$ that

$$\begin{aligned}|R_i R_j(uw)|_{0,\alpha;f_1(t)} &\le C(1+\mu_1 t)^{-\frac{\delta}{2}}|uw|_{0,\alpha;f_1(t)^2,(1+\mu_1 t)^{1/2}}\\ &\le C(1+\mu_1 t)^{-\frac{\delta}{2}}(1+(\mu_1 t)^{-\frac{\alpha}{2}})|uw|_{0,\alpha;f_1(t)^2,(\mu_1 t)^{1/2}}\\ &\le C(1+\mu_1 t)^{-\frac{\delta}{2}}(1+(\mu_1 t)^{-\frac{\alpha}{2}})|u|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}}|w|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}},\end{aligned}$$

which gives the second inequality of the proposition.

For the first inequality, without loss of generality, we can assume

$$|u|_{1,\alpha;h,(1+\mu_1 t)^{1/2}} = |w|_{1,\alpha;f_1(t),(\mu_1 t)^{1/2}} = 1,$$

where $h = f_\pm(t)$.

First of all, by Lemma A.9, we have

$$\begin{aligned}\big|[u, R_i R_j]\,\partial_k w(X)\big| &\le \sum_{n=-\infty}^{\infty}\big|[u, R_{ij}^n]\,\partial_k w(X)\big|\\ &\le C\sum_{n=-\infty}^{\infty} h(X)\min\big(2^{n\delta}(1+\mu_1 t)^{-\frac12}, 2^{-n\alpha}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}\big) \le Ch(X)(1+\mu_1 t)^{-\frac{1+\delta}{2}},\end{aligned}$$

and

$$\begin{aligned}\big|\partial_l[u, R_i R_j]\,\partial_k w(X)\big| &\le \sum_{n=-\infty}^{\infty}\big|\partial_l[u, R_{ij}^n]\,\partial_k w(X)\big|\\ &\le C\sum_{n=-\infty}^{\infty} h(X)(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}\min\big(2^{n(1-\alpha)}, 2^{-n\alpha}(\mu_1 t)^{-\frac12}\big)\\ &\le Ch(X)(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac{1-\alpha}{2}}.\end{aligned}$$

Now we consider $X, Y \in \mathbb{R}^d$, with $|X-Y| \le \sqrt{1+\mu_1 t}$. It follows from Lemma A.9 that

$$\big|[u, R_{ij}^n]\,\partial_k w(X) - [u, R_{ij}^n]\,\partial_k w(Y)\big| \le Ch(X)2^{-n\alpha}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}\min(1, 2^n|X-Y|),$$

where we used the fact that h satisfies (2-3) with $R = \sqrt{1+\mu_1 t}$. Therefore,

$$\begin{aligned}\big|[u, R_i R_j]\,\partial_k w(X) - [u, R_i R_j]\,\partial_k w(Y)\big| &\le \sum_{n=-\infty}^{\infty}\big|[u, R_{ij}^n]\,\partial_k w(X) - [u, R_{ij}^n]\,\partial_k w(Y)\big|\\ &\le C\sum_{n=-\infty}^{\infty} h(X)2^{-n\alpha}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}\min(1, 2^n|X-Y|)\\ &\le Ch(X)(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}|X-Y|^\alpha.\end{aligned}$$

We write

$$\partial_l[u, R_i R_j]\,\partial_k w = [u, \partial_l R_i R_j]\,\partial_k w + \partial_l u\, R_i R_j\, \partial_k w,$$

where

$$[u, \partial_l R_i R_j]\,\partial_k w = \sum_{n=-\infty}^{\infty}[u, \partial_l R_{ij}^n]\,\partial_k w.$$

We get by Lemma A.9 and (A-13) that

$$\big|[u, \partial_l R_{ij}^n]\,\partial_k w(X) - [u, \partial_l R_{ij}^n]\,\partial_k w(Y)\big| \le Ch(X)2^{-n\alpha}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac12}\min(1, 2^n|X-Y|),$$

which gives

$$\begin{aligned}
&\big|[u,\partial_l R_i R_j]\,\partial_k w(X)-[u,\partial_l R_i R_j]\,\partial_k w(Y)\big|\\
&\qquad\le \sum_{n=-\infty}^{\infty}\big|[u,\partial_l R^n_{ij}]\,\partial_k w(X)-[u,\partial_l R^n_{ij}]\,\partial_k w(Y)\big|\\
&\qquad\le C\sum_{n=-\infty}^{\infty} h(X)2^{-n\alpha}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac12}\min(1,2^n|X-Y|)\\
&\qquad\le Ch(X)(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac12}|X-Y|^{\alpha}.
\end{aligned}$$

By

$$\begin{aligned}
|\partial_k w|_{0,\alpha;f_1(t),(1+\mu_1 t)^{1/2}} &\le (1+(\mu_1 t)^{-\frac{\alpha}{2}})|\partial_k w|_{0,\alpha;f_1(t),(\mu_1 t)^{1/2}}\\
&\le (1+(\mu_1 t)^{-\frac{\alpha}{2}})\min\big((\mu_1 t)^{-\frac{1-\alpha}{2}},(\mu_1 t)^{-\frac12}\big)\le 2(\mu_1 t)^{-\frac12},
\end{aligned}$$

we infer from Lemma A.7 that

$$\begin{aligned}
|\partial_l u\ R_i R_j\ \partial_k w|_{0,\alpha;h,(1+\mu_1 t)^{1/2}} &\le C|\partial_l u|_{0,\alpha;h,(1+\mu_1 t)^{1/2}}|R_i R_j\ \partial_k w|_{0,\alpha;1,(1+\mu_1 t)^{1/2}}\\
&\le C(1+\mu_1 t)^{-\frac12}(1+\mu_1 t)^{-\frac{\delta}{2}}|\partial_k w|_{0,\alpha;h,(1+\mu_1 t)^{1/2}}\\
&\le C(1+\mu_1 t)^{-\frac{1+\delta}{2}}(\mu_1 t)^{-\frac12},
\end{aligned}$$

and

$$[\partial_l u\ R_i R_j\ \partial_k w]_{\alpha;h}\le C(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac12}.$$

This shows that

$$\big|\partial_l[u,R_i R_j]\,\partial_k w(X)-\partial_l[u,R_i R_j]\,\partial_k w(Y)\big|\le Ch(X)(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac12}|X-Y|^{\alpha}.$$

For the case $X,Y\in\mathbb{R}^d$, with $|X-Y|\ge\sqrt{1+\mu_1 t}$, we have

$$\begin{aligned}
\big|[u,R_i R_j]\,\partial_k w(X)-[u,R_i R_j]\,\partial_k w(Y)\big| &\le C(h(X)+h(Y))(1+\mu_1 t)^{-\frac{1+\delta}{2}}\\
&\le C(h(X)+h(Y))(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}|X-Y|^{\alpha},
\end{aligned}$$

and

$$\begin{aligned}
\big|\partial_l[u,R_i R_j]\,\partial_k w(X)-\partial_l[u,R_i R_j]\,\partial_k w(Y)\big| &\le C(h(X)+h(Y))(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac{1-\alpha}{2}}\\
&\le C(h(X)+h(Y))(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac12}|X-Y|^{\alpha}.
\end{aligned}$$

In summary, we conclude

$$\begin{aligned}
&\big|[u,R_i R_j]\,\partial_k w\big|_{1,\alpha;h,(\mu_1 t)^{1/2}}\\
&\quad=\big|[u,R_i R_j]\,\partial_k w\big|_{0;h}+\big[[u,R_i R_j]\,\partial_k w\big]_{\alpha;h}\\
&\qquad\qquad+\max((\mu_1 t)^{\frac{1-\alpha}{2}},(\mu_1 t)^{\frac12})\big(\big|\nabla[u,R_i R_j]\,\partial_k w\big|_{0;h}+(\mu_1 t)^{\frac{\alpha}{2}}\big[\nabla[u,R_i R_j]\,\partial_k w\big]_{\alpha;h}\big)\\
&\quad\le C(1+\mu_1 t)^{-\frac{1+\delta}{2}}+C(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}\\
&\qquad+C\max\big((\mu_1 t)^{\frac{1-\alpha}{2}},(\mu_1 t)^{\frac12}\big)\big((1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac{1-\alpha}{2}}+(\mu_1 t)^{\frac{\alpha}{2}}(1+\mu_1 t)^{-\frac{1+\delta+\alpha}{2}}(\mu_1 t)^{-\frac12}\big)\le C,
\end{aligned}$$

which gives the first inequality of the proposition. $\square$

Weighted Schauder estimate. Let $h(X)$ be a positive bounded weight satisfying

$$h(X) \le C_0 h(Y) \quad \text{for } |X-Y| \le 2R,\ R>0.$$

Lemma A.10. *Let $u \in C_h^{2,\alpha}(\mathbb{R}^d)$. Then we have*

$$|\nabla^2 u|_{0,\alpha;h,R} \le C\big(|\nabla u|_{0,\alpha;h} \min(R^{-1+\alpha}, R^{-1}) + |\Delta u|_{0,\alpha;h,R}\big).$$

Here C is a constant depending only on C_0.

Proof. Fix $X \in \mathbb{R}^d$ and consider the function $w(Y) = u(Y) - u(X) - (Y-X)\cdot\nabla u(X)$. So,

$$\nabla^2 w = \nabla^2 u, \quad \Delta w = \Delta u, \quad |\Delta u|_{0,\alpha;B(X,2R),R} \le 2C_0 h(X)|\Delta u|_{0,\alpha;h,R},$$

where

$$|u|_{0,\alpha;B(X,2R),R} \triangleq |u|_{0;B(X,2R)} + R^\alpha [u]_{\alpha;B(X,2R)}.$$

As $\nabla w(Y) = \nabla u(Y) - \nabla u(X)$, we have for $|X-Y| \le 2R$,

$$|\nabla w(Y)| = |\nabla u(Y) - \nabla u(X)| \le (h(X)+h(Y))|X-Y|^\alpha |\nabla u|_{0,\alpha;h} \le 4C_0 h(X) R^\alpha |\nabla u|_{0,\alpha;h},$$

$$|\nabla w(Y)| \le |\nabla u(Y)| + |\nabla u(X)| \le (h(X)+h(Y))|\nabla u|_{0,\alpha;h} \le 2C_0 h(X)|\nabla u|_{0,\alpha;h}.$$

This shows that

$$|\nabla w|_{0;B(X,2R)} \le 4C_0 h(X) \min(R^\alpha, 1)|\nabla u|_{0,\alpha;h},$$

from which and $w(X)=0$, we infer

$$|w|_{0;B(X,2R)} \le 2R|\nabla w|_{0;B(X,2R)} \le 8C_0 h(X)\min(R^{1+\alpha}, R)|\nabla u|_{0,\alpha;h}.$$

Then by the (scaled) Schauder estimate, we obtain

$$\begin{aligned} |\nabla^2 w|_{0,\alpha;B(X,R),R} &\le C\big(R^{-2}|w|_{0;B(X,2R)} + |\Delta w|_{0,\alpha;B(X,2R),R}\big) \\ &\le C h(X)\big(\min(R^{-1+\alpha}, R^{-1})|\nabla u|_{0,\alpha;h} + |\Delta u|_{0,\alpha;h,R}\big) \triangleq C h(X) A, \end{aligned}$$

which in particular shows

$$|\nabla^2 u(X)| = |\nabla^2 w(X)| \le |\nabla^2 w|_{0,\alpha;B(X,R),R} \le C h(X) A.$$

On the other hand, if $|Y-X| < R$, then

$$\big|\nabla^2 u(X) - \nabla^2 u(Y)\big| \le |X-Y|^\alpha R^{-\alpha} |\nabla^2 w|_{0,\alpha;B(X,R),R} \le C h(X) A |X-Y|^\alpha R^{-\alpha},$$

and if $|Y-X| \ge R$, then

$$\big|\nabla^2 u(X) - \nabla^2 u(Y)\big| \le |\nabla^2 u(X)| + |\nabla^2 u(Y)| \le C h(X) A + C h(Y) A \le C(h(X)+h(Y)) A |X-Y|^\alpha R^{-\alpha}.$$

This gives

$$|\nabla^2 u|_{0,\alpha;h,R} = |\nabla^2 u|_{0;h} + R^\alpha [\nabla^2 u]_{\alpha;h} \le CA. \qquad \square$$

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