ANALYSIS & PDE

Volume 16 No. 7 2023

T. J. CHRISTIANSEN

RESONANCES FOR SCHRÖDINGER OPERATORS ON INFINITE CYLINDERS AND OTHER PRODUCTS

msp

ANALYSIS AND PDE
Vol. 16 (2023), No. 7, pp. 1497–1546
DOI: 10.2140/apde.2023.16.1497

msp

RESONANCES FOR SCHRÖDINGER OPERATORS ON INFINITE CYLINDERS AND OTHER PRODUCTS

T. J. CHRISTIANSEN

We study the resonances of Schrödinger operators on the infinite product $X = \mathbb{R}^d \times \mathbb{S}^1$, where d is odd, $\mathbb{S}^1$ is the unit circle, and the potential V lies in $L^\infty_c(X)$. This paper shows that at high energy, resonances of the Schrödinger operator $-\Delta + V$ on $X = \mathbb{R}^d \times \mathbb{S}^1$ which are near the continuous spectrum are approximated by the resonances of $-\Delta + V_0$ on X, where the potential V_0 is given by averaging V over the unit circle. These resonances are, in turn, given in terms of the resonances of a Schrödinger operator on $\mathbb{R}^d$ which lie in a bounded set. If the potential is smooth, we obtain improved localization of the resonances, particularly in the case of simple, rank 1 poles of the corresponding scattering resolvent on $\mathbb{R}^d$. In that case, we obtain the leading order correction for the location of the corresponding high-energy resonances. In addition to direct results about the location of resonances, we show that at high energies away from the resonances, the resolvent of the model operator $-\Delta + V_0$ on X approximates that of $-\Delta + V$ on X. If $d = 1$, in certain cases this implies the existence of an asymptotic expansion of solutions of the wave equation. Again for the special case of $d = 1$, we obtain a resonant rigidity type result for the zero potential among all real-valued smooth potentials.

1. Introduction

We study the Schrödinger operator $-\Delta + V$ on the manifold $X = \mathbb{R}^d \times \mathbb{S}^1$ with the product metric, where d is odd, $\mathbb{S}^1$ is the unit circle, and $V \in L^\infty_c(X)$. In the special case $d = 1$, X is the infinite cylinder $\mathbb{R} \times \mathbb{S}^1$. We show that in the large energy limit, resonances near the continuous spectrum are well approximated by those of $-\Delta + V_0$, where V_0 is the average of V over $\mathbb{S}^1$: $V_0(x) = \frac{1}{2\pi}\int_0^{2\pi} V(x,\theta)\,d\theta$. By a separation of variables argument, these, in turn, are determined by the low energy resonances of the Schrödinger operator $-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0$ on $\mathbb{R}^d$. In the case of smooth potentials V, for simple rank 1 poles of the (scattering) resolvent of $-\sum_{j=0}^d \partial^2/\partial x_j^2 + V_0$, we find the leading-order corrections to the location of the corresponding poles of the resolvent of $-\Delta + V$ on X. Among other things, this allows us to prove that no other smooth real-valued potential on $\mathbb{R} \times \mathbb{S}^1$ has the same resonances as the zero potential. For potentials with $V_0 \equiv 0$, we show the existence of large resonance-free regions. When $d = 1$ and $V \in C^\infty_c(X; \mathbb{R})$, under certain hypotheses on the potential V_0 we are able to give an asymptotic expansion of solutions of the wave equation. For the case of $d = 1$ we study a simple example of a nontrivial potential V with $V_0 \equiv 0$ and locate some of the corresponding resonances. Some of these results are reminiscent of Drouot's results [2018] for rapidly oscillating potentials on $\mathbb{R}^d$.

MSC2020: primary 58J50, 81U24; secondary 35L05, 35P25.
Keywords: Schrödinger operator, resonance, infinite cylindrical end, scattering theory.

Let $\Delta \leq 0$ denote the Laplacian on $X = \mathbb{R}^d \times \mathbb{S}^1$ with the product metric. For $V \in L^\infty_c(X)$ the Schrödinger operator $-\Delta + V$ has continuous spectrum $[0, \infty)$, with multiplicity which increases at each *threshold* j^2, for $j \in \mathbb{N}_0$. For $\operatorname{Im} \zeta > 0$, set $R_V(\zeta) = (-\Delta + V - \zeta^2)^{-1}$. This (scattering) resolvent has a meromorphic continuation to $\hat{Z}$, the minimal Riemann surface for which $\tau_l(\zeta) \stackrel{\text{def}}{=} (\zeta^2 - l^2)^{1/2}$ is a single-valued analytic function for each $l \in \mathbb{N}_0$. The resonances are poles of the resolvent $R_V(\zeta)$. We refer to the portion of $\hat{Z}$ for which $\operatorname{Im} \tau_l(\zeta) > 0$ for all $l \in \mathbb{N}_0$ as the *physical space*. In this set R_V is a bounded operator on $L^2(X)$, away from a discrete set of points which correspond to (square roots of) eigenvalues. For $l \in \mathbb{N}_0$ and $\rho > 0$, denote by $B_l(\rho)$ the connected component of $\{\zeta \in \hat{Z} : |\tau_l(\zeta)| < \rho\}$ which nontrivially intersects both the physical space and the set $\{\zeta \in \hat{Z} : \operatorname{Re} \tau_0(\zeta) > 0\}$. Using as the coordinate $\tau_l(\zeta)$, $B_l(\rho)$ is identified with the disk of radius ρ in the complex plane, centered at the origin, and this identification is compatible with the complex structure of $\hat{Z}\restriction_{B_l(\rho)}$ if $\rho < \sqrt{2l-1}$. The point $\tau_l(\zeta) = 0$ in $B_l(\rho)$ corresponds to the l-th threshold. We study the resonances of $-\Delta + V$ in $B_l(\rho)$, or $B_l(\alpha \log l)$, as $l \to \infty$. Results of Section 6 show that these are the high-energy resonances "near" the continuous spectrum which have $\operatorname{Re} \tau_0 > 0$.

For a function $V \in L^\infty_c(X)$ and $m \in \mathbb{Z}$ define

$$V_m(x) = \frac{1}{2\pi} \int_0^{2\pi} V(x, \theta) e^{-im\theta}\, d\theta,$$

so that $V(x, \theta) = \sum_{m=-\infty}^{\infty} V_m(x) e^{im\theta}$. The minimal assumption on a potential V in most of this paper will be that

$$V \in L^\infty_c(X) \quad \text{and} \quad \|V_m\|_{L^\infty} = O(|m|^{-\delta}) \quad \text{for some } \delta \text{ with } 0 < \delta \leq \tfrac{1}{2}. \tag{1-1}$$

Note that this imposes an assumption on δ as well, which we shall include when we invoke hypothesis (1-1). We use the notation $\Delta_0 = \sum_{j=1}^d \partial^2 / \partial x_j^2$ for the Laplacian on $\mathbb{R}^d$,

$$R_{V_0,0}(\lambda) = (-\Delta_0 + V_0 - \lambda^2)^{-1}, \quad \text{if } \operatorname{Im} \lambda > 0 \tag{1-2}$$

with the same notation for its meromorphic continuation to the complex plane — see Section 3A. The poles of $R_{V_0,0}$ in $\mathbb{C}$ are the resonances of $-\Delta_0 + V_0$. The multiplicity $m_{V_0,0}(\lambda_0)$ of a resonance of $-\Delta_0 + V_0$ at λ_0 is given by the dimension of the range of the singular part of the resolvent at λ_0; this is discussed further in Section 4.

Theorem 1.1. *Let $X = \mathbb{R}^d \times \mathbb{S}^1$, d odd, and let $V \in L^\infty_c(X)$ satisfy $\|V_m\|_{L^\infty} = O(|m|^{-\delta})$ for some δ with $0 < \delta \leq \frac{1}{2}$. Suppose $\lambda_0 \in \mathbb{C}$, $\lambda_0 \neq 0$, is a resonance of $-\Delta_0 + V_0$ on $\mathbb{R}^d$, of multiplicity $m_{V_0,0}(\lambda_0)$. Let $\rho \in \mathbb{R}$, $\rho > |\lambda_0|$. Then there are $C_0 > 0$, $L > 0$ so that for $l > L$, $l \in \mathbb{N}$ there are exactly $2m_{V_0,0}(\lambda_0)$ resonances, when counted with multiplicity, of $-\Delta + V$ in the set*

$$\{\zeta \in B_l(\rho) : |\tau_l(\zeta) - \lambda_0| < C_0 l^{-\delta/(m_{V_0,0}(\lambda_0))}\}.$$

Here, and elsewhere in the paper, the apparent "doubling" of the number of poles (when counted with multiplicity) on X as compared with those on $\mathbb{R}^d$ is due to the fact that for $l \in \mathbb{N}$, l^2 is an eigenvalue of $-d^2/d\theta^2$ on $\mathbb{S}^1$ of multiplicity two. This can be seen immediately in the simplest case, $V \equiv V_0$, by separating variables.

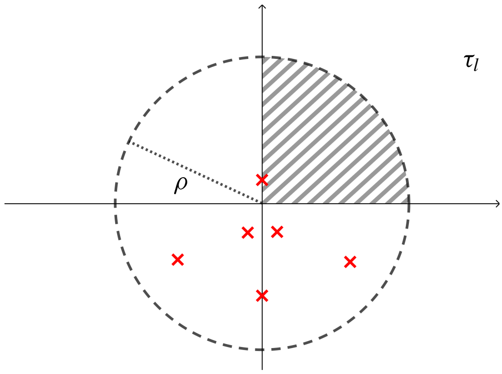

Figure 1. A schematic showing resonances of $-\Delta + V$ in $B_l(\rho)$, pictured in the τ_l-coordinate. Each red × indicates a single resonance of even multiplicity or a cluster of resonances. The hatched region indicates the portion of $B_l(\rho)$ which lies in the physical space. By comparing Figure 2, Section 3B one can see how this fits in the larger picture.

In this paper we refer to any pole of the resolvent as a resonance, including those which correspond to eigenvalues. The second part of Theorem 1.2, for which V is assumed to be smooth, implies an improved localization of the resonances for smooth potentials.

The next theorem shows that, other than possible poles near the threshold, the poles as described above are all the poles in $B_l(\rho)$ for sufficiently large l.

Theorem 1.2. *Let $X = \mathbb{R}^d \times \mathbb{S}^1$, d odd, and suppose V satisfies the hypothesis (1-1). Choose $\rho > 0$ so that if λ_j is a pole of $R_{V_0,0}(\lambda)$, then $|\lambda_j| \neq \rho$. Set*

$$\Lambda_\rho = \{\lambda_j \in \mathbb{C} : |\lambda_j| < \rho \text{ and } \lambda_j \text{ is a pole of } R_{V_0,0}(\lambda)\}.$$

Let $\epsilon' > 0$ be so that $\epsilon' < \min\{|\lambda_j| : \lambda_j \in \Lambda_\rho,\ \lambda_j \neq 0\}$. Then there are $\widetilde{C},\ L > 0$ so that for $l > L$, $l \in \mathbb{N}$, there are no resonances of $-\Delta + V$ in

$$\{\zeta \in B_l(\rho) : |\tau_l(\zeta)| > \epsilon' \text{ and } |\tau_l(\zeta) - \lambda_j| > \widetilde{C} l^{-\delta/m_{V_0,0}(\lambda_j)} \text{ for all } \lambda_j \in \Lambda_\rho\}.$$

Moreover, if V is smooth for perhaps larger L and $\widetilde{C}$, for $l > L$ there are no resonances in

$$\{\zeta \in B_l(\rho) : |\tau_l(\zeta)| > \epsilon' \text{ and } |\tau_l(\zeta) - \lambda_j| > \widetilde{C} l^{-2/(m_{V_0,0}(\lambda_j))} \text{ for all } \lambda_j \in \Lambda_\rho\}.$$

In addition, if $R_{V_0,0}(\lambda)$ is analytic in a neighborhood of the origin, then there are no poles in $B_l(\epsilon')$ for l sufficiently large.

We comment that smoothness of the potential V is more than is needed for the second part of Theorem 1.2. It would suffice to have $V \in C^k(X)$, for some k sufficiently large. In order to simplify the proofs, we have not tracked the value of k which is needed.

To help visualize these theorems, we include Figure 1, which is a schematic showing the resonances of $-\Delta + V$ in $B_l(\rho)$ for large l, using the τ_l-coordinate. This schematic is familiar from odd-dimensional scattering theory; that this should be so is a consequence of Theorems 1.1–1.3. One difference is that in this diagram, the only portion of $B_l(\rho)$ which lies in the physical space is the portion which is in the first quadrant, indicated by hatching. Another is that each × indicates either a single resonance of even multiplicity, or a cluster of resonances. See Figure 2 to see how B_l fits in a larger context.

For Schrödinger operators on $\mathbb{R}^d$, the behavior of the singularities of the resolvent at the origin is delicate. For example, notions of multiplicity of a resonance which agree at points away from the origin may differ at the origin; see [Dyatlov and Zworski 2019, Theorem 2.8]. These same sorts of issues arise at thresholds in the case under study here, and accounts for the fact that this next theorem, which concerns resonances very near the thresholds, is weaker than the previous ones.

Theorem 1.3. *Let V satisfy (1-1) and suppose the resolvent of $-\Delta_0 + V_0$ on $\mathbb{R}^d$ has a pole at 0 of order $r > 0$, and multiplicity $m_{V_0,0}(0)$ Then there are $C,\ L > 0$ so that $-\Delta + V$ on X has at least $2m_{V_0,0}(0)$ resonances, when counted with multiplicity, in $B_l(Cl^{-\delta/r})$ when $l > L$, $l \in \mathbb{N}$. Moreover, there is an $\epsilon > 0$ so that $-\Delta + V$ has no poles in $B_l(\epsilon) \setminus B_l(Cl^{-\delta/r})$ when $l > L$. If $V \in C_c^\infty(X)$, then this can be improved to show that there is a $C_1 > 0$ so that $-\Delta + V$ has no poles in $B_l(\epsilon) \setminus B_l(C_1 l^{-2/r})$ when $l > L$. Moreover, under the hypothesis (1-1), if $r = 1$ there are exactly $2m_{V_0,0}(0)$ resonances of $-\Delta + V$ in $B_l(Cl^{-\delta})$ for $l > L$.*

Suppose for the moment that V_0 is real-valued. In this case, it is well known that if $d = 1$ the order of the pole of the resolvent of $-d^2/dx^2 + V_0$ at 0 cannot exceed 1, and if it is 1, then $m_{V_0,0}(0) = 1$ [Dyatlov and Zworski 2019, Theorem 2.7]. If $d \geq 3$ is odd, then the order of the pole of the resolvent of $-\Delta_0 + V_0$ at 0 cannot exceed 2 [Dyatlov and Zworski 2019, Lemma 3.16]. For general V and r, the order of the pole at 0 can be bounded from above in terms of $m_{V_0,0}(0)$, and in the case $d = 1$, $m_{V_0,0}(0)$ can be bounded above by r.

It is of particular interest to understand poles of the resolvent R_V near the physical region. In Section 6 we show that there are large regions near the physical region that contain no resonances. A consequence of those results is that large energy resonances near the continuous spectrum and having $\operatorname{Re} \tau_0(\zeta) > 0$ are contained in regions of the form $B_l(\rho)$, where ρ depends on how near the continuous spectrum we wish to look. In Section 6 we further justify our focus on the resonances in sets $B_l(\rho)$.

Theorems 1.1–1.3 combined with results of Section 6 yield the following corollary. Here $d_{\hat{Z}}$ is a distance on $\hat{Z}$, defined in Section 6. The boundary of the physical region corresponds to the continuous spectrum. In the corollary, we use $\{\zeta_j^\flat\}$ to denote a sequence of points in $\hat{Z}$, to distinguish them from ζ_l which is used elsewhere to denote a particular mapping from an open subset of the complex plane into $\hat{Z}$.

Corollary 1.4. *Let $V \in L_c^\infty(X; \mathbb{R})$ satisfy (1-1). Then $R_V(\zeta)$ has a sequence $\{\zeta_j^\flat\}_{j=1}^\infty$ of poles satisfying both $|\tau_0(\zeta_j^\flat)| \to \infty$ as $j \to \infty$ and $d_{\hat{Z}}(\zeta_j^\flat, \text{physical region}) \to 0$ as $j \to \infty$ if and only if $R_{V_0,0}(\lambda)$ has at least one pole in $i[0, \infty)$.*

In particular, if $d = 1$, by [Reed and Simon 1978, Theorem XIII.110] if $\int_X V \leq 0$ then $R_V(\zeta)$ has such a sequence of poles approaching the physical space. In contrast, if $V_0(x) \geq 0$ for all x and V_0 is nontrivial, $R_V(\zeta)$ does not have such a sequence of poles. Note that for any fixed $k_0 \in \mathbb{N}$, we have $|\tau_0(\zeta_j^\flat)| \to \infty$ as $j \to \infty$ if and only if $|\tau_{k_0}(\zeta_j^\flat)| \to \infty$ as $j \to \infty$. We remark that we could prove an analog of Corollary 1.4 for complex-valued potentials as well.

If we enlarge the region centered at the threshold l^2 with increasing l, we have less fine localization of the resonances, see Theorem 7.1. However, when V_0, the average of the potential, is identically zero, we can get a larger resonance-free region. The difference in the next result for $d=1$ and $d\geq 3$ is due to the fact that the resolvent of $-d^2/dx^2$ on $\mathbb{R}$ has a pole at the origin, but that of $-\Delta_0$ on $\mathbb{R}^d$ for $d\geq 3$ odd does not.

Theorem 1.5. *Let $V\in L^\infty_c(X)$ satisfy (1-1), and suppose $V_0\equiv 0$. If $d=1$ there are $\alpha,\ c_0>0$ so that for $l\in\mathbb{N}$ sufficiently large there are no resonances of $-\Delta+V$ in the set $\{\zeta\in B_l(\alpha\log l):|\tau_l(\zeta)|>c_0/l^\delta\}$. If $d\geq 3$ is odd, there is an $\alpha>0$ so that for l sufficiently large there are no resonances of $-\Delta+V$ in the set $B_l(\alpha\log l)$.*

There is a sense in which this theorem is sharp; see Proposition 12.6 for a computation for the case $d=1$ with the potential $V(x,\theta)=2\chi_{I_0}(x)\cos\theta$, where χ_{I_0} is the characteristic function of the interval $[-1,1]$.

We can find the leading correction term for high-energy resonances of $-\Delta+V$ which correspond to simple resonances of $-\Delta_0+V_0$. In the next theorem, ∇_0 is the gradient on $\mathbb{R}^d$, so that

$$\nabla_0 f=\left(\frac{\partial}{\partial x_1}f,\frac{\partial}{\partial x_2}f,\dots,\frac{\partial}{\partial x_d}f\right).$$

Theorem 1.6. *Let $X=\mathbb{R}^d\times\mathbb{S}^1$, d odd, $V\in C^\infty_c(X)$, and suppose $\lambda_0\in\mathbb{C}$ is a simple pole of the scattering resolvent $R_{V_0,0}$ of $-\Delta_0+V_0$ on $\mathbb{R}^d$, and that the residue of $R_{V_0,0}$ at λ_0 has rank 1. Suppose for any $\chi\in C^\infty_c(\mathbb{R}^d)$,*

$$\chi\left(R_{V_0,0}(\lambda)-\frac{i}{\lambda-\lambda_0}u\otimes u\right)\chi \tag{1-3}$$

is analytic near $\lambda=\lambda_0$. Let $\rho>|\lambda_0|$. Then there are $\epsilon,\ L>0$ so that for $l>L$ there are exactly two poles of $R_V(\zeta)$, when counted with multiplicity, in $\{\zeta\in B_l(|\lambda_0|+1):|\tau_l(\zeta)-\lambda_0|<\epsilon\}$, and each pole of $R_V(\zeta)$ in this set satisfies

$$\tau_l(\zeta)=\lambda_0-\frac{i}{4l^2}\sum_{k\neq 0}\frac{1}{k^2}\int_{\mathbb{R}}(k^2V_{-k}V_ku^2+(\nabla_0V_{-k}\cdot\nabla_0V_k)u^2)(x)\,dx+O(l^{-3}).$$

We note that the normalization of the singularity in (1-3) is chosen so that if V is real-valued and $\lambda_0\in i[0,\infty)$, then u is real-valued. There is some further discussion of u at the beginning of Section 10. Proposition 12.3 shows that the leading correction may be rather different for a nonsmooth potential by considering the special case of the potential on $\mathbb{R}\times\mathbb{S}^1$ given by $V(x,\theta)=2\cos\theta\,\chi_{I_0}(x)$, where χ_{I_0} is the characteristic function of the interval $[-1,1]$. As for Theorem 1.2, the proof of Theorem 1.6 only needs V to be C^k for some k sufficiently large. Since Theorem 1.8 requires smoothness of the potential only for an application of Theorem 1.6, the same is true for it. Again, we have chosen not to track this value in the interest of simplifying proofs.

If $V_0\in L^\infty_c(\mathbb{R}^d;\mathbb{R})$ and the operator $-\Delta_0+V_0$ on $L^2(\mathbb{R}^d)$ has a simple negative eigenvalue $-\beta^2$, then this negative eigenvalue corresponds to a simple pole of $R_{V_0,0}$ on the positive imaginary axis at $i|\beta|$, and the residue has rank 1. By Theorem 1.1 (or Corollary 1.4), in this case R_V has a sequence of poles approaching the physical space. If $V\in C^\infty_c(X;\mathbb{R})$, the poles approach the physical space very rapidly.

Theorem 1.7. *Suppose $V \in C_c^\infty(X;\mathbb{R})$ and $\lambda_0 \in \mathbb{C}$ is a simple pole of $R_{V_0}(\lambda)$ with* $\operatorname{Re}\lambda_0 = 0$, *with residue of R_{V_0} at λ_0 having rank* 1. *Then there is an $\epsilon > 0$ so that if $\{\zeta_l^\flat\}_{l=L}^\infty \subset \hat{Z}$ is a sequence of poles of R_V with $\zeta_l^\flat \in B_l(|\lambda_0|+1)$ and $|\tau_l(\zeta_l^\flat)-\lambda_0| < \epsilon$, then* $\operatorname{Re}\tau_l(\zeta_l^\flat) = O(l^{-N})$ *for any N. In particular, this implies that if* $\operatorname{Im}\lambda_0 > 0$, *then $d_{\hat{Z}}(\zeta_l^\flat, \text{physical region}) = O(l^{-N})$.*

Proposition 12.3 demonstrates the necessity of assuming some regularity of the potential, at least for $d = 1$, by studying the resonances very near the l-th threshold for a certain real-valued potential with a jump singularity. These resonances in $B_l(1)$ arise from the pole of $R_{0,0}(\lambda)$ at $\lambda_0 = 0$. They have

$$|\tau_l(\zeta_l^\flat)| = O(l^{-3/2})$$

and, for a subsequence of l's tending to infinity,

$$|\operatorname{Im}(\tau_l(\zeta_l^\flat))| > \tfrac{1}{10} l^{-3/2}.$$

This paper was initially motivated by the case $d = 1$, as $\mathbb{R}\times\mathbb{S}^1$ provides a particularly simple example of a manifold with infinite cylindrical ends and as such provides a testing ground for studying resonances for Schrödinger operators on such manifolds. Most of the proofs of the preceding theorems are essentially identical for any odd dimension of the factor $\mathbb{R}^d$, so we have included the more general results. However, Theorems 1.8 and 1.9 are particular to the $d = 1$ case.

As a corollary of Theorems 1.1, 1.3, and 1.6, we get in the case $d = 1$ a uniqueness-type result for the zero potential among smooth real-valued potentials.

Theorem 1.8. *Let $V \in C_c^\infty(\mathbb{R}\times\mathbb{S}^1;\mathbb{R})$. Suppose for each $\rho > 0$ there is a sequence*

$$\{l_j\}_{j=1}^\infty = \{l_j(\rho)\}_{j=1}^\infty \subset \mathbb{N}$$

with $l_j \to \infty$ when $j \to \infty$ so that in $B_{l_j}(\rho)$ the resonances of $-\Delta + V$ and $-\Delta$ on $X = \mathbb{R}\times\mathbb{S}^1$ are the same. Then $V \equiv 0$.

This result is false if we omit the hypothesis that V is real-valued. For example, for $V_1 \in C_c^\infty(\mathbb{R})$ set $V(x,\theta) = V_1(x)e^{i\theta}$. Then the operators $-\Delta + V$ and $-\Delta$ have the same resonances; see [Autin 2011] or [Christiansen 2004, Section 4]. This example can be easily generalized.

As part of our study of the distribution of resonances, we prove that, in a suitable sense, near the physical region of $\hat{Z}$, R_V is well approximated by R_{V_0} away from the poles of R_{V_0}; see Proposition 5.4 and Lemma 5.5. In the case $d = 1$, this and results of [Christiansen and Datchev 2022] give a wave expansion; see Theorem 1.9.

Let $X = \mathbb{R}\times\mathbb{S}^1$, $V \in C_c^\infty(X;\mathbb{R})$, and suppose $-\Delta + V$ has finitely many eigenvalues $\mu_1, \mu_2, \dots, \mu_J$, repeated with multiplicity, with associated orthonormal eigenfunctions $\{\eta_j\}$, so that $(-\Delta + V)\eta_j = \mu_j\eta_j$. Let u satisfy

$$\frac{\partial^2}{\partial t^2}u - \Delta u + Vu = 0, \tag{1-4}$$
$$(u, u_t)\restriction_{t=0} = (f_1, f_2) \in C_c^\infty(X)\times C_c^\infty(X).$$

Theorem 1.9. *Let $X = \mathbb{R} \times \mathbb{S}^1$ and $V,\ f_1,\ f_2 \in C_c^\infty(X)$, with V real-valued, and suppose $-d^2/dx^2 + V_0$ on $\mathbb{R}$ has no negative eigenvalues and no resonance at 0. Let u be the solution of (1-4) on $[0,\infty) \times X$. Then for each $k_0 \in \mathbb{N}$ we can write $u(t) = u_e(t) + u_{\mathrm{thr},k_0}(t) + u_{r,k_0}(t)$, where*

$$u_e(t,x,\theta) = \sum_{\substack{\mu_j \in \sigma_p(-\Delta+V)\\ \mu_j \neq 0}} \eta_j(x,\theta)\left(\cos((\mu_j)^{1/2}t)\langle f_1,\eta_j\rangle + \frac{\sin((\mu_j)^{1/2}t)}{(\mu_j)^{1/2}}\langle f_2,\eta_j\rangle\right) + \sum_{\substack{\mu_j \in \sigma_p(-\Delta+V)\\ \mu_j = 0}} \eta_j(x,\theta)(\langle f_1,\eta_j\rangle + t\langle f_2,\eta_j\rangle) \tag{1-5}$$

and

$$u_{\mathrm{thr},k_0}(t,x,\theta) = b_{0,0,+}(x,\theta) + \sum_{k=0}^{k_0-1} t^{-1/2-k} \sum_{j=1}^{\infty} (e^{itj} b_{j,k,+}(x,\theta) + e^{-itj} b_{j,k,-}(x,\theta))$$

for some $b_{j,k,\pm} \in \langle x\rangle^{1/2+2k+\epsilon} L^2(X)$. For any $\chi \in C_c^\infty(X)$ there is a constant C so that

$$\sum_{j=1}^{\infty} \|\chi b_{j,k,\pm}\|_{L^2(X)} < C, \quad k = 0,1,2,\dots,k_0-1$$

and

$$\|\chi u_{r,k_0}(t)\|_{L^2(X)} \le C t^{-k_0} \quad \text{for } t \text{ sufficiently large.}$$

The assumption that $-d^2/dx^2 + V_0$ on $\mathbb{R}$ has no negative eigenvalues and no resonance at 0 means, by Theorem 1.2, that R_V has at most finitely many poles on the boundary of the physical space. In particular, this means at most finitely many eigenvalues of $-\Delta + V$, so that the sum in u_e is finite. Further, there are at most finitely many poles at thresholds, and this implies via results of [Christiansen and Datchev 2022] that at most finitely many of the $b_{j,0,\pm}$ are nonzero.

If $-d^2/dx^2 + V_0$ on $\mathbb{R}$ has one or more negative eigenvalues, it seems plausible that there is an asymptotic expansion of solutions of the wave equation on compact sets. Since in this case by Theorem 1.7 the resolvent R_V may have a sequence of poles rapidly approaching, but not lying in, the continuous spectrum, such an expansion would need to take these into account and is more complicated — see for example [Tang and Zworski 2000] for an expansion in a Euclidean scattering setting with resonances approaching the continuous spectrum. In our setting proving the existence of such an expansion may use techniques similar to those of [Christiansen and Datchev 2022] but does not follow directly from the results of that work. Proving this is outside the scope of this paper.

In this paper we have, for simplicity, limited ourselves to the case of Schrödinger operators on $\mathbb{R}^d \times \mathbb{S}$. However, many of our results for L^∞ potentials hold as well for Schrödinger operators with Dirichlet or Neumann boundary conditions on $\mathbb{R}^{d-1} \times (0,\infty) \times \mathbb{S}$ or on $\mathbb{R}^d \times (0,\pi)$.

1A. ***Relation to previous work.*** This paper was inspired in part by two different sets of papers. The first are papers which study eigenvalues and resonances of Schrödinger operators on $\mathbb{R}^d$ with rapidly oscillating potentials, and includes [Borisov 2006; Borisov and Gadylshin 2006; Duchêne and Weinstein 2011; Duchêne et al. 2014; 2015; Dimassi 2016; Drouot 2018]. Of these the most closely related to this paper is that of Drouot [2018], which studies the distribution of resonances of Schrödinger operators $-\Delta_0 + V_\epsilon$ on $\mathbb{R}^d$ with d odd. Here

$$V_\epsilon(x) = V_0(x) + \sum_{k\in\mathbb{Z}^d, k\neq 0} V_k(x) e^{ik\cdot x/\epsilon}, \quad x \in \mathbb{R}^d.$$

Drouot shows in quantitative ways that in the limit $\epsilon \downarrow 0$, resonances of $-\Delta_0 + V_\epsilon$ near the continuous spectrum are well approximated by those of $-\Delta_0 + V_0$. In addition, he proves some refinements related, for example, to the leading order correction of the positions of the resonances. Theorems 1.1, 1.2, 1.3, 1.5, and 1.6, as well as some computations in Section 12, are inspired by results in [Drouot 2018]. However, the proofs are quite different. In part, this is because the different setting requires different techniques. Additionally, Drouot's results come mainly from studying regularized determinants. While this has the potential of producing in some instances more refined results than we obtain here, it requires a substantial amount of technical work. We have chosen instead to mostly avoid determinants, or to work only with determinants of operators of the type $I + F$, where F is finite rank. Instead, we use an operator Rouché theorem of Gohberg and Sigal [1971]. In some places this may allow for sharper results than could be obtained by using a regularized determinant. We note in addition that in the setting of [Drouot 2018], the resonances lie on the complex plane, while for us, the resonances lie on a Riemann surface which is a countable but infinite cover of the complex plane, with infinitely many branch points. This means that some of the techniques used in [Drouot 2018] cannot be applied here.

A less direct source of inspiration is work done on the distribution of eigenvalues of the Schrödinger operator $-\Delta_{\mathbb{S}^n} + W$ on the sphere $\mathbb{S}^n$ (and certain other compact manifolds), $n \geq 2$; see for example [Weinstein 1977; Widom 1979]. In this setting, eigenvalues of the Schrödinger operator occur in bands. Roughly speaking, these authors show that a suitable average of the potential W can be used to obtain information about the distribution of high-energy eigenvalues of the Schrödinger operator within these bands. This average is over closed geodesics, rather than over all of $\mathbb{S}^n$. Of course, our function $V_0(x)$ is the average of the potential V over the cross section of $\mathbb{S}^1$, the unique closed geodesic on $\mathbb{S}^1$.

This paper was originally motivated by the $d = 1$ case, which gives $X = \mathbb{R} \times \mathbb{S}^1$, a manifold with an infinite cylindrical end. The spectral and scattering theory of manifolds with infinite cylindrical ends has been studied in, for example, [Goldstein 1974; Guillopé 1989; Melrose 1993]. There is a large literature studying the existence of eigenvalues and, in certain settings, the locations of resonances for such manifolds and the related problems of waveguides which have a "one-dimensional infinity" as our $d = 1$ case does; see, e.g., [Levitin and Marletta 2008] or the monograph [Exner and Kovařík 2015]. This monograph also includes some results for manifolds with "higher-dimensional infinity". Many of these results focus on low-energy eigenvalues or resonances. We mention the papers [Christiansen 2002; 2004; Christiansen and Datchev 2021; Christiansen and Zworski 1995; Parnovski 1995; Edward 2002] which are more directly connected with high-energy behavior.

1B. *Comments regarding other product manifolds.* This paper studies only Schrödinger operators on $\mathbb{R}^d \times \mathbb{S}^1$, where d is odd. Here we comment on why we require that d be odd and on the choice of $\mathbb{S}^1$ for the second factor.

For Euclidean scattering, e.g., for the Schrödinger operator $-\Delta_{\mathbb{R}^d} + V_{\mathbb{R}^d}$ on $\mathbb{R}^d$ with $V_{\mathbb{R}^d} \in L^\infty_c(\mathbb{R}^d)$, the space to which the resolvent continues is determined by the dimension: for odd d the meromorphic continuation is to the complex plane, and for even d the meromorphic continuation is to Λ, the logarithmic cover of $\mathbb{C} \setminus \{0\}$. This means that certain questions related to the distribution of resonances are more difficult in even dimensional Euclidean scattering than in odd dimensional Euclidean scattering. For the problem we consider here, the Riemann surface on which the resonances live is a bit involved to describe when d is odd; see Section 3B. The Riemann surface when d is even is much more complicated, requiring as its building block Λ rather than $\mathbb{C}$. It is, however, clear that some of our results, appropriately interpreted, hold if d is even as long as we stay away from thresholds. In the interest of clarity we do not pursue this here.

Next we turn to the choice of the factor $\mathbb{S}^1$. There are three things that make this an especially nice choice:

(1) The spacing between the distinct eigenvalues grows as the eigenvalues grow.

(2) Upon averaging in $\mathbb{S}^1$, we get a model operator that we understand fairly well.

(3) There is a choice of eigenfunctions of the Laplacian on $\mathbb{S}^1$ so that a product of two eigenfunctions is again an eigenfunction: $e^{ij\theta}e^{ik\theta} = e^{i(j+k)\theta}$.

Not all of our results require this last property. In view of [Weinstein 1977; Widom 1979], it would be natural to think of replacing $\mathbb{S}^1$ with $\mathbb{S}^m$. Of course, the spacing of distinct eigenvalues of the Laplacian on $\mathbb{S}^m$ is similar to that for $\mathbb{S}^1$. However, when using a factor $\mathbb{S}^m$ with $m > 1$, obtaining a model operator is much more complicated, and it seems any results for general potentials would likely be substantially weaker.

1C. ***Ideas from the proofs.*** Our starting point for the study of resonances of $-\Delta + V$ is an identification of the resonances with the points ζ for which the operator $I + (V - V_0)R_{V_0}(\zeta)\chi$ has nontrivial null space. Here $R_W(\zeta)$ is the meromorphic continuation of the resolvent of $-\Delta + W$, and $\chi \in L^\infty_c(X)$ satisfies $\chi V = V$ and is, for convenience, chosen independent of θ. By separating variables, we can understand R_{V_0} in terms of the resolvent of $-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0(x)$ on $\mathbb{R}^d$.

We use two well-known and related properties of the resolvent of $-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0(x)$ on $\mathbb{R}^d$. One is the estimate

$$\left\| \tilde{\chi}\left(-\sum_{j=1}^{d} \partial^2/\partial x_j^2 + V_0 - (\lambda + i0)^2\right)^{-1} \tilde{\chi} \right\| = O(|\lambda|^{-1})$$

as $\lambda \to \infty$ for $\lambda \in \mathbb{R}$ and $\tilde{\chi} \in L^\infty_c(\mathbb{R}^d)$. The second is the existence of a logarithmic resonance-free neighborhood of the real axis.

An immediate consequence of this second fact and the fact that the distance between thresholds of our operator $-\Delta + V$ on X increases at high energy is that if $V = V_0$, at high energy near the thresholds the resonances of $-\Delta + V_0$ on X are determined by low-energy resonances of $-\sum_{j=1}^d \partial^2/\partial x_j^2 + V_0$ on $\mathbb{R}^d$. Moreover, using these facts and an operator Rouché theorem of Gohberg and Sigal [1971], we are able to show that at high energy near the thresholds the zeros of $I + (V - V_0)R_{V_0}\chi$ are approximated by the poles of $\chi R_{V_0}\chi$. These ideas underlie the proofs of the L^∞ results of Theorems 1.1–1.3 and 1.5. They are also central to the proofs of the smooth versions of these results and of Theorem 1.6, although these proofs require some additional study of the resolvent of $-\Delta + V_0$ when V_0 is smooth.

1D. *Organization.* In Section 3 we recall some results from Euclidean scattering and show that the resolvent of $-\Delta + V$ on X has a meromorphic continuation to $\hat{Z}$. (We note that this latter is known; see Section 3 for references.) We define the multiplicity of a pole of the resolvent, and give several useful identities involving it in Section 4. In addition, this section introduces some notation and results related to the operator Rouché theorem of Gohberg and Sigal [1971]. With these preliminaries we prove Theorems 1.1 and 1.2 in the case of an L^∞ potential V, using results from [Gohberg and Sigal 1971]. Section 6 contains more discussion of the Riemann surface $\hat{Z}$ and shows the existence of resonance-free regions which are, at high energy, near the physical region and away from thresholds. This provides the missing pieces of the proof of Corollary 1.4. Combining these with the resolvent estimates of Section 5 and results of [Christiansen and Datchev 2022] proves Theorem 1.9.

Section 8 contains preliminary computations which are needed to refine our results for smooth potentials. The smooth case of Theorem 1.2 is proved with techniques similar to that of the L^∞ result, but using in addition results of Section 8.

In Section 10 we prove Theorems 1.6 and 1.7. We do this using Fredholm determinants, but determinants of the form $\det(I + F)$, where F is a finite-rank operator. Theorem 1.8 follows rather directly from the earlier results. Finally, in Section 12, in the case $d = 1$ we give approximations of some of the high-energy resonances for a particularly simple potential which has $V_0 \equiv 0$ and which is not smooth.

2. Notation and conventions

On $X = \mathbb{R}^d \times \mathbb{S}^1$ we use the coordinates (x, θ) or (x', θ'), with $x,\ x' \in \mathbb{R}^d$ and $\theta,\ \theta' \in [0, 2\pi)$.

Throughout the paper, $V \in L^\infty_c(X)$ and $l \in \mathbb{N}_0$, and the dimension d of $\mathbb{R}^d$ is odd. We use C to stand for a positive constant, the value of which may change without comment.

Suppose A and B are linear operators on domains in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{S}^1)$, respectively, and are given by

$$(Af)(x) = \int_{\mathbb{R}^d} A(x, x') f(x')\, dx' \quad \text{and} \quad (Bg)(\theta) = \int_0^{2\pi} B(\theta, \theta') g(\theta')\, d\theta.$$

Then A and B give rise to linear operators on domains in $L^2(X)$, which we again denote by A and B, and which are given by

$$(Ah)(x, \theta) = \int_{\mathbb{R}^d} A(x, x') h(x', \theta)\, dx' \quad \text{and} \quad (Bh)(x, \theta) = \int_{\mathbb{R}^d} B(\theta, \theta') h(x, \theta')\, d\theta'.$$

For $f,\ g \in L^2(\mathbb{R}^d)$, the operator $f \otimes g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined via

$$((f \otimes g)h)(x) = f(x) \int_{\mathbb{R}^d} g(x') h(x')\, dx'.$$

If $f,\ g \in L^2(X)$, the operator $f \otimes g$ on $L^2(X)$ is defined analogously.

We list some repeatedly used notation for the convenience of the reader:

- The Laplacians on $\mathbb{R}^d$ and X are given, respectively, by
$$\Delta_0 = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \quad \text{and} \quad \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \theta^2}.$$
- $V_m(x) = \frac{1}{2\pi}\int_0^{2\pi} V(x,\theta)e^{-im\theta}\,d\theta$ for $m \in \mathbb{Z}$.
- $V^\# = V^\#(x,\theta) = V(x,\theta) - V_0(x)$.
- $B_l(\rho)$ and $D_l(\lambda_0,\rho)$ are open sets in $\hat{Z}$, defined in Sections 1 and 5, respectively.
- R_V is the (scattering) resolvent of $-\Delta + V$ on X; see Section 3B.
- $R_{V_0,0}$ is the (scattering) resolvent of $-\Delta_0 + V_0$ on $\mathbb{R}^d$; see Section 3.
- $m_V(\zeta_0)$ is the multiplicity of $\zeta_0 \in \hat{Z}$ as a pole of R_V; see (4-1).
- $m_{V_0,0}(\lambda_0)$ is the multiplicity of $\lambda_0 \in \mathbb{C}$ as a pole of $R_{V_0,0}$; see (4-2).
- $\zeta_l : \{z \in \mathbb{C} : |z| < \sqrt{2l-1}\} \to B_l(\sqrt{2l-1}) \subset \hat{Z}$ is the (local) inverse of
$$B_l(\sqrt{2l-1}) \ni \zeta \mapsto \tau_l(\zeta) \in \{z \in \mathbb{C} : |z| < \sqrt{2l-1}\} \subset \mathbb{C}.$$

3. Odd-dimensional Euclidean scattering and continuation of the resolvent

We begin by fixing notation and recalling some well-known facts from Euclidean scattering theory. We then use these to give a self-contained proof that the resolvent of $-\Delta + V$ on X has a meromorphic continuation to $\hat{Z}$.

3A. ***The Euclidean resolvent.*** Let $V_0 \in L^\infty_c(\mathbb{R}^d)$, d odd, and set
$$R_{V_0,0}(\lambda) = (-\Delta_0 + V_0 - \lambda^2)^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$
when $\operatorname{Im}\lambda > 0$. The 0 in the second place in the subscript is to help us think of this as a model operator, as we shall see. We shall later use the explicit expression for the resolvent as an integral when $d = 1$, $f \in L^2(\mathbb{R})$, and $\operatorname{Im}\lambda > 0$:
$$(R_{0,0}(\lambda)f)(x) = \frac{i}{2\lambda}\int e^{i\lambda|x-x'|} f(x')\,dx' \quad \text{for } d = 1. \tag{3-1}$$
From this we can see immediately that if $\chi \in L^\infty_c(\mathbb{R})$, then $\chi R_{0,0}(\lambda)\chi$ has a meromorphic continuation to $\mathbb{C} \setminus \{0\}$. The same is true when $d \geq 3$ is odd: if $\chi \in L^\infty_c(\mathbb{R}^d)$, then $\chi R_{0,0}(\lambda)\chi$ has an analytic continuation to the complex plane, see [Dyatlov and Zworski 2019, Theorem 3.3]. In higher dimensions, the Schwartz kernel is given in terms of a Hankel function. It is well known, see [Dyatlov and Zworski 2019, Theorem 3.8], that if V_0, $\chi \in L^\infty_c(\mathbb{R}^d)$, then $\chi R_{V_0,0}(\lambda)\chi$ has a meromorphic continuation to the complex plane. Alternatively, restricting the domain and enlarging the range, $R_{V_0,0}(\lambda) : L^2_c(\mathbb{R}^d) \to H^2_{\mathrm{loc}}(\mathbb{R}^d)$ has a meromorphic continuation to $\mathbb{C}$.

The following lemma is well known, but we include it for completeness, as it is crucial for our arguments.

Lemma 3.1. *Let $V_0,\ \chi \in L^\infty_c(\mathbb{R}^d)$. Then there are constants $C_0,\ C_1 > 0$ so that $\chi R_{V_0,0}(\lambda)\chi$ is analytic in $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| > C_0,\ \operatorname{Im}\lambda > -C_1 \log(1+|\operatorname{Re}\lambda|)\}$. Moreover, in this region $\|\chi R_{V_0,0}(\lambda)\chi\| = O(|\lambda|^{-1})$.*

Proof. Without loss of generality, we may assume $\chi V_0 = V_0$. Then

$$\chi R_{V_0,0}(\lambda)\chi = \chi R_{0,0}(\lambda)\chi (I + V_0 R_{0,0}(\lambda)\chi)^{-1}.$$

Since from (3-1) when $d = 1$ or [Dyatlov and Zworski 2019, Theorem 3.1] when $d \geq 3$, there is a $C > 0$ so that

$$\|V R_{0,0}(\lambda)\chi\| \leq C e^{C(\operatorname{Im}\lambda)_-}/|\lambda|,$$

where $(\operatorname{Im}\lambda)_- = \max(0, -\operatorname{Im}\lambda)$; the lemma follows immediately. □

3B. ***The resolvent of $-\Delta + V$ on X and the Riemann surface $\hat{Z}$.*** Recall that when $d = 1$, X is a manifold with infinite cylindrical ends. For a manifold with infinite cylindrical ends, the space to which the resolvent of a Schrödinger operator continues is determined by the distinct eigenvalues of the Laplacian on the cross-section of the end(s). Here that means $\{j^2\}_{j\in\mathbb{N}_0}$, since this is the set of (distinct) eigenvalues of $-d^2/d\theta^2$ on $\mathbb{S}^1$. As we show below, the resolvent for $-\Delta + V$ on $\mathbb{R}^d \times \mathbb{S}^1$ has a meromorphic continuation to the same space as that of the resolvent of $-\Delta + V$ on $\mathbb{R} \times \mathbb{S}^1$, provided d is odd.

For $j \in \mathbb{N}_0$ and $\zeta \in \mathbb{C}$, $\operatorname{Im}\zeta > 0$, set

$$\tau_j(\zeta) \stackrel{\text{def}}{=} (\zeta^2 - j^2)^{1/2}$$

with $\operatorname{Im}\tau_j(\zeta) > 0$. Set $\tau_{-j}(\zeta) = \tau_j(\zeta)$ if $j \in \mathbb{N}$.

The Riemann surface $\hat{Z}$ is defined to be the minimal Riemann surface on which, for each $j \in \mathbb{N}_0$, τ_j is a single-valued analytic function on $\hat{Z}$. We briefly describe its construction and some of its properties. Note that $\tau_0(\zeta) = \zeta$ for ζ in the upper half-plane, and this has, of course, an analytic continuation to $\mathbb{C}$. Now $\tau_1(\zeta) = \tau_{-1}(\zeta)$ is an analytic function of $\zeta \in \mathbb{C} \setminus ((-\infty, 1] \cup [1, \infty))$, and there is a minimal Riemann surface $\hat{Z}_1$ so that τ_1 extends analytically to $\hat{Z}_1$. This is a double cover of $\mathbb{C}$, ramified at the points ± 1. This process can be repeated for each $j \in \mathbb{N}$, resulting in a minimal Riemann surface $\hat{Z}$ on which τ_j is analytic for each $j \in \mathbb{N}_0$. We define a projection $p : \hat{Z} \to \mathbb{C}$ as follows. For ζ in the physical space, identified with the upper half-plane, $p(\zeta) = \zeta$, and p is in general the analytic continuation of this function. Then $\hat{Z}$ has infinitely many ramification points which project under p to $j \in \mathbb{Z} \setminus \{0\}$. We call the set $\{\zeta \in \hat{Z} : \operatorname{Im}\tau_j(\zeta) > 0 \text{ for all } j \in \mathbb{N}_0\}$ the *physical space*, or *physical region*. For further discussion of this Riemann surface; see [Melrose 1993, Section 6.6].

We shall say that a point $\zeta_0 \in \hat{Z}$ *corresponds to a threshold* if $\tau_0(\zeta_0) \in \mathbb{Z}$. Note that with this definition, all the ramification points of $\hat{Z}$ correspond to thresholds. In addition, the set of points corresponding to thresholds includes those points projecting to 0. These might naturally also be considered ramification points of $\hat{Z}$, as in some sense by choosing ζ^2 to originally be our spectral parameter we have already made the cuts corresponding to the zero threshold.

In order to separate variables below, we introduce the orthogonal projections $\mathcal{P}_k : L^2(X) \to L^2(X)$ defined for $k \in \mathbb{Z}$ by

$$(\mathcal{P}_k f)(x,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(x,\theta')(e^{ik(\theta-\theta')} + e^{-ik(\theta-\theta')})\, d\theta' \quad \text{if } k \in \mathbb{N},$$
$$(\mathcal{P}_0 f)(x,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(x,\theta')\, d\theta'.$$

We shall use these throughout the paper.

Let $V \in L^\infty_c(X)$. For $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta > 0$, set $R_V(\zeta) = (-\Delta + V - \zeta^2)^{-1}$. Consider first the special case where $V \in L^\infty_c(X)$ is independent of θ. Then $V = V_0$, and we can think of V_0 as an element of $L^\infty_c(X)$ or of $L^\infty_c(\mathbb{R}^d)$. In this special case we can separate variables to obtain

$$R_{V_0}(\zeta) = \sum_{k=0}^{\infty} R_{V_0,0}(\tau_k(\zeta))\mathcal{P}_k. \tag{3-2}$$

The explicit expression (3-2) for R_{V_0} using separation of variables shows that if $\chi \in L^\infty_c(X)$, then $\chi R_{V_0}\chi$ and $R_{V_0} : L^2_c(X) \to H^2_{\text{loc}}(X)$ have meromorphic continuations to $\hat{Z}$. In fact, the same is true for $\chi R_V \chi$ and R_V for general $V \in L^\infty_c(X)$. This is well known, at least when $d = 1$, see [Goldstein 1974; Guillopé 1989; Melrose 1993], though we sketch a proof below, valid for all odd d.

If $\zeta \in \mathbb{C}$, $\operatorname{Im}\zeta > 0$, then

$$(-\Delta + V - \zeta^2)R_0(\zeta) = I + VR_0(\zeta).$$

Multiplying by a function $\chi \in L^\infty_c(X)$ with $\chi V = V$,

$$(-\Delta + V - \zeta^2)R_0(\zeta)\chi = \chi(I + VR_0(\zeta)\chi),$$

implying that

$$\chi R_0(\zeta)\chi = \chi R_V(\zeta)\chi(I + VR_0(\zeta)\chi) \tag{3-3}$$

or

$$\chi R_V(\zeta)\chi = \chi R_0(\zeta)\chi(I + VR_0(\zeta)\chi)^{-1}. \tag{3-4}$$

Using $I - VR_0(\zeta)\chi(I + VR_0(\zeta)\chi)^{-1} = (I + VR_0(\zeta)\chi)^{-1}$ and (3-4) yields

$$(I + VR_0(\zeta)\chi)^{-1} = I - VR_V(\zeta)\chi; \tag{3-5}$$

compare [Dyatlov and Zworski 2019, (2.2.15)–(2.2.16)]. Likewise, writing

$$V^\# \overset{\text{def}}{=} V - V_0, \tag{3-6}$$

we find, making the additional hypothesis that $\chi V^\# = V^\#$,

$$\chi R_{V_0}(\zeta)\chi = \chi R_V(\zeta)\chi(I + V^\# R_{V_0}(\zeta)\chi) \quad \text{and} \quad (I + V^\# R_{V_0}(\zeta)\chi)^{-1} = I - V^\# R_V(\zeta)\chi. \tag{3-7}$$

Each of these is helpful. Since $VR_0(\zeta)\chi : L^2(X) \to L^2(X)$ is compact and has a meromorphic extension to $\hat{Z}$, and $I + VR_0(\zeta)\chi$ is invertible for ζ in the physical space with $\operatorname{Im}\zeta$ sufficiently large, meromorphic Fredholm theory ensures that $(I + VR_0(\zeta)\chi)^{-1}$ is a meromorphic operator-valued function on $\hat{Z}$, and each

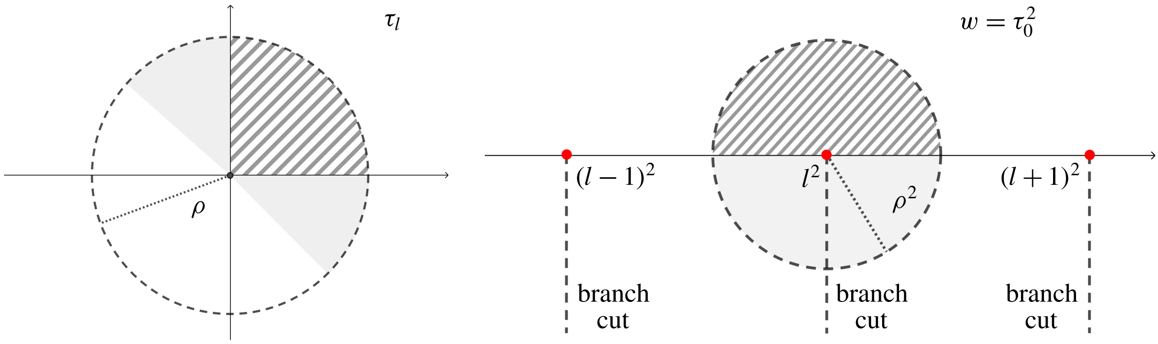

Figure 2. On the left, $B_l(\rho)$ in the τ_l-coordinate; on the right, a *portion* of $B_l(\rho)$ in the $w = (\tau_0(\zeta))^2$-coordinate for larger context. In the w diagram, $(-\Delta + V - w)^{-1}$ is bounded in the upper half-plane and the red dots on the horizontal axis indicate thresholds. The hatching denotes the portion of $B_l(\rho)$ in the physical region; the shaded region indicates the rest which is visible in the w plane diagram.

of (3-3)–(3-5) and (3-7) holds on all of $\hat{Z}$. Moreover, writing $I + VR_0 = (I + VR_0(I-\chi))(I + VR_0\chi)$ and noting that $(I + VR_0(I-\chi))^{-1} = I - VR_0(I-\chi)$, this shows that

$$R_V(\zeta) = R_0(\zeta)(I + VR_0(\zeta)\chi))^{-1}(I - VR_0(\zeta)(I-\chi)) : L^2_c(X) \to H^2_{\mathrm{loc}}(X)$$

has a meromorphic continuation to $\hat{Z}$.

We note from (3-2) that R_{V_0} is bounded on $L^2(X)$ when ζ is in the physical space and is away from a discrete set of poles (corresponding to eigenvalues). The same is true of R_V.

Throughout this paper we shall mainly work with subsets of $B_l(\sqrt{2l-1}) \subset \hat{Z}$, for $l \in \mathbb{N}$. We recall $B_l(\rho)$ is defined to be the connected component of $\{\zeta \in \hat{Z} : |\tau_l(\zeta)| < \rho\}$ which has nonempty intersection with both the physical space and the portion of $\hat{Z}$ with $\operatorname{Re}\tau_0(\zeta) > 0$. The choice of $\sqrt{2l-1}$ in $B_l(\sqrt{2l-1})$ is made because then (for $l \geq 1$) $B_l(\sqrt{2l-1})$ contains only a single point of $\hat{Z}$ corresponding to a threshold, the one associated with the eigenvalue l^2 of $-d^2/d\theta^2$ on $\mathbb{S}^1$. If $\epsilon > 0$, then $z = \tau_l(\zeta)$ gives the complex structure of $\hat{Z}\restriction_{B_l(\sqrt{2l-1}-\epsilon)}$, and $B_l(\sqrt{2l-1}-\epsilon)$ is naturally identified with a disk $B_{\mathbb{C}}(\sqrt{2l-1}-\epsilon)$ of radius $\sqrt{2l-1}-\epsilon$ in $\mathbb{C}$, centered at the origin. In this coordinate z, we have that $z = 0$ corresponds to the threshold l^2 and the intersection of $B_{\mathbb{C}}(\sqrt{2l-1}-\epsilon)$ with the first quadrant corresponds to a region in physical space, and so has $\operatorname{Im}\tau_k > 0$ for all $k \in \mathbb{N}_0$. If z lies in the intersection of $B_{\mathbb{C}}(\sqrt{2l-1}-\epsilon)$ with the fourth quadrant, then $\operatorname{Im}\tau_k(\zeta(z)) < 0$ for $0 \leq k \leq l$ and $\operatorname{Im}\tau_k(\zeta(z)) > 0$ for $k > l$. On the other hand, if z lies in the intersection of $B_{\mathbb{C}}(\sqrt{2l-1}-\epsilon)$ with the second quadrant, then $\operatorname{Im}\tau_k(\zeta(z)) < 0$ for $0 \leq k \leq l-1$ and $\operatorname{Im}\tau_k(\zeta(z)) > 0$ for $k \geq l$. Figure 2 shows a schematic of $B_l(\rho)$ and, for context, the portion of $B_l(\rho)$ which is visible in the $w = (\tau_0(\zeta))^2$ plane. We note that while we have used the spectral parameter ζ^2 in the definition of $R_V(\zeta)$ to be consistent with the usual odd-dimensional Euclidean scattering resolvent, the diagram on the right in Figure 2 uses as spectral parameter $w = (\tau_0(\zeta))^2$ to make a more easily digested diagram. To put the diagram in context, think of $(-\Delta + V - w)^{-1}$ as having meromorphic continuation from the upper half-plane to $\left\{w \in \mathbb{C} \setminus \left(\bigcup_{j=0}^{\infty}(j^2 + i(-\infty, 0])\right)\right\}$ (which can, of course, be identified with a subset of $\hat{Z}$).

On the open set $B_l(\sqrt{2l-1}-\epsilon)$, $z=\tau_l(\zeta)$ is a coordinate compatible with the complex structure of $\hat{Z}$. Thus it is natural to use τ_l as a local coordinate. We write

$$\zeta_l : \{z \in \mathbb{C} : |z| < \sqrt{2l-1}-\epsilon\} \to B_l(\sqrt{2l-1}-\epsilon) \subset \hat{Z}$$

as the function satisfying

$$\zeta_l(\tau_l(\zeta)) = \zeta \quad \text{for all } \zeta \in B_l(\sqrt{2l-1}-\epsilon).$$

We note that if $\zeta \in B_l(\sqrt{2l-1}-\epsilon)$, then $\operatorname{Re}\tau_j(\zeta) > 0$ if $0 \le j < l$, and $\operatorname{Im}\tau_j(\zeta) > 0$ if $j > l$.

The next lemma follows easily from (3-2) and Lemma 3.1, but is fundamental to many of the results of this paper.

Lemma 3.2. *Let $V_0 \in L^\infty_c(\mathbb{R})$, $\alpha > 0$, and $\chi \in L^\infty_c(X)$. Then for l sufficiently large, uniformly for $\zeta \in B_l(\alpha \log l)$, we have $\|\chi(I-\mathcal{P}_l)R_{V_0}(\zeta)\chi\| = O(l^{-1/2})$.*

Proof. Set $\tau_l = z$ and $|z| < \alpha \log l$. Then using the identity

$$\tau_k^2 = \tau_l^2 + l^2 - k^2,$$

for l sufficiently large, $|\tau_k(\zeta_l(z))| > \sqrt{l}$ for $k \in \mathbb{N}_0$, $k \neq l$. Moreover, $\operatorname{Im}\tau_k(\zeta_l(z)) > 0$ if $k > l$, and $|\operatorname{Im}\tau_k(\zeta_l(z))| = O(1)$ if $k < l$. Then the lemma follows from Lemma 3.1 and the representation of $R_{V_0,0}$ given by (3-2). □

4. Multiplicities of poles and results of [Gohberg and Sigal 1971]

For an operator A depending meromorphically on $\zeta \in \mathbb{C}$ or $\zeta \in \hat{Z}$, let $\Xi(A,\zeta_0)$ denote the principal part of the Laurent expansion of A at ζ_0. For $V \in L^\infty_c(X)$ and $\zeta_0 \in \hat{Z}$, define

$$m_V(\zeta_0) \overset{\text{def}}{=} \operatorname{rank} \Xi(R_V,\zeta_0)(L^2_c(X)). \tag{4-1}$$

Suppose $\chi \in L^\infty_c(X)$ satisfies $\chi V = V$ (and, if $V \equiv 0$, χ is nontrivial). Then it follows from an expansion of R_V at its singularities as in [Dyatlov and Zworski 2019, Theorems 2.5, 2.7, 3.9, 3.17] and a unique continuation result, e.g., [Jerison and Kenig 1985, Remark 6.7], that $m_V(\zeta_0) = \operatorname{rank} \Xi(\chi R_V \chi, \zeta_0)$. Note that if R_V is analytic at ζ_0, then $m_V(\zeta_0) = 0$.

If $V_0 \in L^\infty_c(\mathbb{R}^d)$ and $\lambda_0 \in \mathbb{C}$ we define

$$m_{V_0,0}(\lambda_0) \overset{\text{def}}{=} \operatorname{rank} \Xi(R_{V_0,0},\lambda_0)(L^2_c(\mathbb{R}^d)). \tag{4-2}$$

Again, the second 0 in the subscript is meant to help us think of this as corresponding to a model. As for m_V, if $\chi \in L^\infty_c(\mathbb{R})$ satisfies $\chi V = V$ (and χ is nontrivial if $V_0 \equiv 0$), then $m_{V_0,0}(\lambda_0) = \operatorname{rank} \Xi(\chi R_{V_0,0}\chi, \lambda_0)$.

We recall some definitions and results of [Gohberg and Sigal 1971], adapted to our setting.

Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$, depending meromorphically on $z \in \Omega \subset \mathbb{C}$, where Ω is a domain. Near a point $z_0 \in \Omega$, we have $A(z) = \sum_{j=-n}^{\infty}(z-z_0)^j A_j$. If the operators $A_{-1}, \dots, A_{-n}$ are finite rank, then we say A is *finitely meromorphic* at z_0. If A is finitely meromorphic at each $z_0 \in \Omega$, then A is *finitely meromorphic on* Ω. Suppose that A is a compact operator on $\mathcal{H}$, A is finitely meromorphic on Ω, and $I + A(z_1)$ is invertible for some $z_1 \in \Omega$. Then by the meromorphic Fredholm theorem, $(I + A(z))^{-1}$ is finitely meromorphic on Ω.

Suppose A is a finitely meromorphic operator on a domain Ω, with $(I+A)^{-1}$ also finitely meromorphic on Ω. Below we denote the derivative of A with respect to z by $\dot{A}$. Then for $z_0 \in \Omega$, define

$$M(I + A, z_0) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{z_0}} \dot{A}(z)(I + A(z))^{-1}\, dz,$$

where γ_{z_0} is a positively oriented circle, centered at z_0 with radius ϵ. Here we choose ϵ small enough that $\{|z - z_0| \le \epsilon\} \subset \Omega$ and neither A nor $(I + A)^{-1}$ has poles in the set $\{z : 0 < |z - z_0| \le \epsilon\}$.

Our definition of finitely meromorphic is local, so it makes sense on domains in $\hat{Z}$ as well, using a local coordinate compatible with the complex structure of $\hat{Z}$. Likewise, we can define $M(I + A, \zeta_0)$ for such operators. (This requires the choice of a circle small enough that it has in its interior at most one ramification point of $\hat{Z}$.)

We will say the linear operator A on the Hilbert space $\mathcal{H}$ satisfies hypotheses (H1) on a domain $\Omega \subset \mathbb{C}$ if A is a finitely meromorphic, compact operator defined on Ω, and $I + A$ is invertible for at least one point in Ω and hence has a finitely meromorphic inverse in Ω.

The following lemma is a direct consequence of [Gohberg and Sigal 1971, Proposition 5].

Lemma 4.1 [Gohberg and Sigal 1971, Proposition 5]. *Suppose $A,\ B : \mathcal{H} \to \mathcal{H}$ satisfy hypotheses* (H1), *and suppose B and $(I + B)^{-1}$ are analytic on Ω. Then for $z_0 \in \Omega$,*

$$M(I + A, z_0) = M((I + A)(I + B), z_0).$$

Let $T : L^2(X) \to L^2(X)$ be a bounded linear operator. We shall repeatedly make use of the straightforward identities

$$I + T\mathcal{P}_l = (I + \mathcal{P}_l T\mathcal{P}_l)(I + (I - \mathcal{P}_l)T\mathcal{P}_l) \quad \text{and} \quad (I + (I - \mathcal{P}_l)T\mathcal{P}_l)^{-1} = I - (I - \mathcal{P}_l)T\mathcal{P}_l. \tag{4-3}$$

Lemma 4.2. *Let $A : L^2(X) \to L^2(X)$ satisfy hypotheses* (H1) *on a domain Ω. Then for $z_0 \in \Omega$,*

$$M(I + A\mathcal{P}_l, z_0) = M(I + \mathcal{P}_l A\mathcal{P}_l, z_0).$$

Proof. Using (4-3) implies that

$$M(I+A\mathcal{P}_l, z_0) = \frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{z_0}} \dot{A}(z)\mathcal{P}_l(I+A(z)\mathcal{P}_l)^{-1}\, dz = \frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{z_0}} \dot{A}(z)\mathcal{P}_l(I+\mathcal{P}_l A(z)\mathcal{P}_l)^{-1}\, dz, \tag{4-4}$$

where γ_{z_0} is a small circle centered at z_0 as in the definition of $M(I + A, z_0)$.

Because $\mathcal{P}_l$ is a projection, using the cyclicity of the trace, $\operatorname{tr}(B\mathcal{P}_l) = \operatorname{tr}(\mathcal{P}_l B\mathcal{P}_l)$ for a trace class operator $B : L^2(X) \to L^2(X)$. Using this in (4-4) gives

$$M(I + A\mathcal{P}_l, z_0) = \frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_{z_0}} \mathcal{P}_l \dot{A}(z)\mathcal{P}_l(I + \mathcal{P}_l A(z)\mathcal{P}_l)^{-1}\, dz = M(I + \mathcal{P}_l A\mathcal{P}_l, z_0). \qquad \square$$

The following proposition is a variant of a well-known result in the study of resonances of Schrödinger operators on $\mathbb{R}^d$; compare, e.g., [Dyatlov and Zworski 2019, Theorem 3.15].

Proposition 4.3. *Suppose $V \in L^\infty_c(X)$ is nontrivial, and let $\chi \in L^\infty_c(X)$ satisfy $\chi V = V$. Then the operator $R_V(\zeta)$ has a pole at $\zeta_0 \in \hat{Z}$ if and only if the operator $I + V R_0(\zeta)\chi$ has nontrivial null space at ζ_0. Moreover, if ζ_0 does not correspond to a threshold, then*

$$m_V(\zeta_0) = M(I + V R_0 \chi, \zeta_0).$$

Proof. A proof follows by essentially the same method as [Dyatlov and Zworski 2019, Theorem 3.15]. □

We recall the notation $V^\# = V - V_0$. Another useful identity is the following.

Lemma 4.4. *Let $\chi \in L^\infty_c(X)$ satisfy $\chi V = V$ and $\chi V_0 = V_0$. Then for $\zeta_0 \in \hat{Z}$ so that ζ_0 does not correspond to a threshold, we have*

$$m_V(\zeta_0) = M(I + V^\# R_{V_0}\chi, \zeta_0) + m_{V_0}(\zeta_0).$$

Proof. We first note that

$$I + V R_0\chi = (I + V^\# R_0\chi(I + V_0 R_0\chi)^{-1})(I + V_0 R_0\chi) = (I + V^\# R_{V_0}\chi)(I + V_0 R_0\chi). \tag{4-5}$$

Thus using Proposition 4.3 and [Gohberg and Sigal 1971, Theorem 5.2] gives

$$\begin{aligned} m_V(\zeta_0) = M(I + V R_0\chi, \zeta_0) &= M(I + V^\# R_{V_0}\chi, \zeta_0) + M(I + V_0 R_0\chi, \zeta_0) \\ &= M(I + V^\# R_{V_0}\chi, \zeta_0) + m_{V_0}(\zeta_0). \end{aligned}$$ □

Lemma 4.5. *Suppose $V,\ \chi \in L^\infty_c(X)$, with $\chi V = V$, and χ is independent of θ. Let $\alpha > 0$. Then there is an $L > 0$ so that for $l > L$*

$$M(I + V R_0\chi, \zeta_0) = M(I + \mathcal{P}_l(I + V R_0(I - \mathcal{P}_l)\chi)^{-1} V R_0 \mathcal{P}_l \chi, \zeta_0)$$

for any $\zeta_0 \in B_l(\alpha \log l)$.

Proof. We begin by writing

$$I + V R_0\chi = (I + V R_0(I - \mathcal{P}_l)\chi)(I + (I + V R_0(I - \mathcal{P}_l)\chi)^{-1} V R_0 \mathcal{P}_l \chi)$$

and noting that since by Lemma 3.2 $\|V R_0(I - \mathcal{P}_l)\chi\| = O(l^{-1/2})$ uniformly on $B_l(\alpha \log l)$ there is an $L > 0$ so that for $l > L$, $(I + V R_0(I - \mathcal{P}_l)\chi)^{-1}$ is analytic on $B_l(\alpha \log l)$. Thus for these l by Lemma 4.1 $M(I + V R_0\chi, \zeta_0) = M(I + (I + V R_0(I - \mathcal{P}_l)\chi)^{-1} V R_0 \mathcal{P}_l \chi, \zeta_0)$ for any $\zeta_0 \in B_l(\alpha \log l)$. An application of Lemma 4.2 completes the proof. □

Lemma 4.6. *Let $V,\ \chi \in L^\infty_c(X)$, with V satisfying (1-1), $\chi V = V$, and χ independent of θ. Set $A_{l,V} = (I + V R_0(I - \mathcal{P}_l)\chi)^{-1}$ and $B_{l,V} = V R_0 \mathcal{P}_l \chi$. Let $K \subset \mathbb{C}$ be a compact set such that $R_{V_0,0}$ is analytic on K, and suppose $0 \notin K$ if $d = 1$. Choose $\rho > 0$ so that $K \subset \{\lambda \in \mathbb{C} : |\lambda| < \rho\}$, and set $K_l = \{\zeta \in B_l(\rho) : \tau_l(\zeta) \in K\}$. Then for sufficiently large l,*

$$\|\mathcal{P}_l(A_{l,V} B_{l,V} - A_{l,V_0} B_{l,V_0})\| = O(l^{-\delta}) \tag{4-6}$$

and

$$\|(I + \mathcal{P}_l A_{l,V_0} B_{l,V_0})^{-1} \mathcal{P}_l (A_{l,V} B_{l,V} - A_{l,V_0} B_{l,V_0})\| = O(l^{-\delta}) \tag{4-7}$$

uniformly for $\zeta \in K_l$.

Proof. We write

$$\mathcal{P}_l(A_{l,V}B_{l,V} - A_{l,V_0}B_{l,V_0}) = \mathcal{P}_l(A_{l,V} - A_{l,V_0})B_{l,V} + \mathcal{P}_l A_{l,V_0}(B_{l,V} - B_{l,V_0}). \tag{4-8}$$

By Lemma 3.2, $\|A_{l,V} - I\| = O(l^{-1/2})$ and $\|A_{l,V_0} - I\| = O(l^{-1/2})$ uniformly on $B_l(\rho)$, so that the first term on the left-hand side is $O(l^{-1/2})$. Moreover,

$$\mathcal{P}_l A_{l,V_0}(B_{l,V} - B_{l,V_0}) = A_{l,V_0}\mathcal{P}_l(B_{l,V} - B_{l,V_0}) = A_{l,V_0}\mathcal{P}_l V^{\#} R_0 \mathcal{P}_l,$$

and $\|\mathcal{P}_l V^{\#}\mathcal{P}_l\| = O(l^{-\delta})$ by our assumption on V. Hence the norm of the second term on the right-hand side of (4-8) is $O(l^{-\delta})$. This proves (4-6).

On K_l,

$$I + \mathcal{P}_l A_{l,V_0}B_{l,V_0} = I + \mathcal{P}_l B_{l,V_0} + O(l^{-1/2}) = I + \mathcal{P}_l V_0 R_0 \chi + O(l^{-1/2}). \tag{4-9}$$

But

$$(I + \mathcal{P}_l V_0 R_0 \chi)^{-1} = I - \mathcal{P}_l + (I - V_0 R_{V_0,0}(\tau_l)\chi)\mathcal{P}_l = I - \mathcal{P}_l + T\mathcal{P}_l,$$

where T is given by $T = (I + V_0 R_{0,0}(\tau_l)\chi)^{-1} = I - V_0 R_{V_0,0}(\tau_l)\chi$. By our choice of K, we have that T is uniformly bounded for $\tau_l \in K$ or for $\zeta \in K_l$, and hence $(I + \mathcal{P}_l V_0 R_0 \chi)^{-1}$ is bounded on K_l. Using (4-9), this shows $(I + \mathcal{P}_l A_{l,V_0}B_{l,V_0})^{-1}$ is bounded on K_l, and thus, by (4-6), we get (4-7). □

5. A resolvent estimate and localizing the resonances in the L^∞ case: Proofs of Theorems 1.1, 1.2, and 1.3

In this section we prove Theorems 1.1–1.3 in the case of an L^∞ potential V, providing a high-energy localization of the resonances in sets $B_l(\rho)$. We also prove Proposition 5.4 and Lemma 5.5, which show that the resolvent for the potential V_0 is, at high energies, a good approximation of the resolvent for the potential V away from poles.

We shall use notation for a disk in the τ_l-coordinate in $B_l(\rho)$. For $\lambda_0 \in \mathbb{C}$ and $r_0 > 0$, set $\rho = |\lambda_0| + r_0 + 1$, and define, for $2l > \rho^2 + 1$, $D_l(\lambda_0, r_0) \subset B_l(\rho) \subset \hat{Z}$ by

$$D_l(\lambda_0, r_0) \overset{\text{def}}{=} \{\zeta \in B_l(\rho) : |\tau_l(\zeta) - \lambda_0| < r_0\}.$$

A preliminary step in the proof of Theorem 1.1 is the following proposition, which provides an initial localization of the resonances.

Proposition 5.1. *Let $V \in L_c^\infty(X)$ satisfy (1-1). Suppose $\lambda_0 \in \mathbb{C}$, $\lambda_0 \neq 0$ is a resonance of $-\Delta_0 + V_0$ on $\mathbb{R}^d$, of multiplicity $m_{V_0,0}(\lambda_0)$. Then there are L, $\epsilon > 0$ so that*

$$\sum_{\substack{\zeta \in D_l(\lambda_0,\epsilon) \\ m_V(\zeta) > 0}} m_V(\zeta) = 2m_{V_0,0}(\lambda_0)$$

when $l > L$.

Proof. Choose $\epsilon > 0$ so that $R_{V_0,0}(\lambda)$ is analytic on $0 < |\lambda - \lambda_0| \le \epsilon$ and $\epsilon < |\lambda_0|$. By our expression (3-2) for R_{V_0}, using separation of variables and Lemmas 3.1 and 3.2, $m_{V_0}(\zeta_l(\lambda_0)) = 2m_{V_0,0}(\lambda_0)$ for l sufficiently large. The factor of 2 on the right comes from the fact that the range of $\mathcal{P}_l$ (as an operator on $L^2(\mathbb{S}^1)$) has rank 2 for $l > 0$. Choose $\chi \in L^\infty_c(X)$ independent of θ so that $\chi V = V$. From Proposition 4.3 and our choice of ϵ, for l sufficiently large,

$$m_{V_0}(\zeta_l(\lambda_0)) = M(I + V_0 R_0 \chi, \zeta_l(\lambda_0)) = \sum_{\substack{\zeta \in D_l(\lambda_0,\epsilon) \\ M(I+V_0R_0\chi,\zeta)\neq 0}} M(I + V_0 R_0 \chi, \zeta).$$

Lemma 4.5 implies that if $W = V_0$ or $W = V$,

$$M(I + W R_0 \chi, \zeta') = M(I + \mathcal{P}_l (I + W R_0 (I - \mathcal{P}_l)\chi)^{-1} W R_0 \mathcal{P}_l \chi, \zeta') \quad \text{for } \zeta' \in D_l(\lambda_0, \epsilon) \tag{5-1}$$

if l is sufficiently large.

By Lemma 4.6 and an operator Rouché theorem [Gohberg and Sigal 1971, Theorem 2.2], for l sufficiently large,

$$\begin{aligned} &\sum_{\substack{\zeta \in D_l(\lambda_0,\epsilon) \\ M(I+\mathcal{P}_l(I+VR_0(I-\mathcal{P}_l)\chi)^{-1}VR_0\mathcal{P}_l\chi,\zeta)\neq 0}} M(I + \mathcal{P}_l (I + V R_0 (I - \mathcal{P}_l)\chi)^{-1} V R_0 \mathcal{P}_l \chi, \zeta) \\ &\qquad = \sum_{\substack{\zeta \in D_l(\lambda_0,\epsilon) \\ M(I+\mathcal{P}_l(I+V_0R_0(I-\mathcal{P}_l)\chi)^{-1}V_0R_0\mathcal{P}_l\chi,\zeta)\neq 0}} M(I + \mathcal{P}_l (I + V_0 R_0 (I - \mathcal{P}_l)\chi)^{-1} V_0 R_0 \mathcal{P}_l \chi, \zeta). \end{aligned} \tag{5-2}$$

Combining (5-1) (with $W = V$ and with $W = V_0$), (5-2), and another application of Proposition 4.3, this time with V, proves the proposition. ☐

5A. *Proofs of Theorems 1.1 and 1.2 for $V \in L^\infty_c(X)$.* Theorem 1.1 follows from combining the result of Theorem 1.2 for L^∞ potentials and Proposition 5.1. In this section we prove Theorem 1.2 for L^∞ potentials V.

Recall by the definition of $\Xi(R_{V_0,0}, \lambda_0)$, if $\lambda_0 \in \mathbb{C}$ is a pole of $R_{V_0,0}$, then $R_{V_0,0}(\lambda) - \Xi(R_{V_0,0}(\lambda), \lambda_0)$ is analytic at λ_0. Define

$$R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l) \overset{\text{def}}{=} R_{V_0}(\zeta) - \Xi(R_{V_0}, \zeta_l(\lambda_0)). \tag{5-3}$$

For l sufficiently large, by (3-2) and Lemma 3.2

$$R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l) = R_{V_0}(\zeta) - \Xi(R_{V_0,0}(\lambda), \lambda_0)|_{\lambda=\tau_l(\zeta)} \mathcal{P}_l. \tag{5-4}$$

Note that if R_{V_0} is analytic at $\zeta_l(\lambda_0)$, then $R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l) = R_{V_0}(\zeta)$.

Lemma 5.2. *Suppose V, $\chi \in L^\infty_c(X)$ and V satisfies (1-1). Let $\lambda_0 \in \mathbb{C}$ and $R^{\text{reg}}_{V_0} = R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l)$ be the operator defined in (5-3). If $R_{V_0,0}(\lambda)$ is analytic for $0 < |\lambda - \lambda_0| \le \epsilon$, then for l sufficiently large,*

$$V^\# R^{\text{reg}}_{V_0}(\zeta)\chi = V^\# R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l)\chi$$

is analytic on $\overline{D}_l(\lambda_0, \epsilon)$, and as $l \to \infty$ the estimate $\|\chi R^{\text{reg}}_{V_0}(\zeta) V^\# R^{\text{reg}}_{V_0}(\zeta)\chi\| = O(l^{-\delta})$ holds uniformly for $\zeta \in \overline{D}_l(\lambda_0, \epsilon)$.

Proof. Without loss of generality we can assume χ is independent of θ and $\chi V = V$. By (3-2) and Lemma 3.2, for l sufficiently large, $R^{\mathrm{reg}}_{V_0}(\zeta)$ is analytic and bounded in $\bar{D}_l(\lambda_0, \epsilon)$. We write

$$\chi R^{\mathrm{reg}}_{V_0}(\zeta) V^{\#} R^{\mathrm{reg}}_{V_0}(\zeta)\chi$$
$$= \chi R^{\mathrm{reg}}_{V_0} \chi (I - \mathcal{P}_l) V^{\#} R^{\mathrm{reg}}_{V_0} \chi + \chi R^{\mathrm{reg}}_{V_0} \chi \mathcal{P}_l V^{\#} R^{\mathrm{reg}}_{V_0} \chi (I - \mathcal{P}_l) + \chi R^{\mathrm{reg}}_{V_0} \chi \mathcal{P}_l V^{\#} R^{\mathrm{reg}}_{V_0} \chi \mathcal{P}_l. \quad (5\text{-}5)$$

Now for $\zeta \in \bar{D}_l(\lambda_0, \epsilon)$ and l sufficiently large, $\| \chi R^{\mathrm{reg}}_{V_0} \chi (I - \mathcal{P}_l)\| = O(l^{-1/2})$ uniformly in $\bar{D}_l(\lambda_0, \epsilon)$. Since $\|V_m\| = O(|m|^{-\delta})$ we have $\|\mathcal{P}_l V^{\#} \mathcal{P}_l\| = O(l^{-\delta})$, and so

$$\|\mathcal{P}_l V^{\#} R^{\mathrm{reg}}_{V_0} \chi \mathcal{P}_l\| = \|\mathcal{P}_l V^{\#} \mathcal{P}_l R^{\mathrm{reg}}_{V_0} \chi \mathcal{P}_l\| = O(l^{-\delta}). \qquad \square$$

A related lemma which we also need is the following.

Lemma 5.3. *Let $V, \chi \in L^\infty_c(X)$ with V satisfying (1-1). Let $K \subset \mathbb{C}$ be a compact set on which $R_{V_0,0}$ is analytic and suppose $K \subset \{\lambda \in \mathbb{C} : |\lambda| < \rho\}$. Set $K_l \overset{\mathrm{def}}{=} \{\zeta \in B_l(\rho) : \tau_l(\zeta) \in K\} \subset \hat{Z}$. Then for l sufficiently large, $\|\chi R_{V_0} V^{\#} R_{V_0} \chi\| = O(l^{-\delta})$ uniformly on K_l.*

Proof. This lemma can be proved by mimicking the proof of Lemma 5.2. Alternatively, it can be proved by covering K_l with a finite number of neighborhoods on which Lemma 5.2 holds. $\square$

Proof of Theorem 1.2 for $V \in L^\infty_c(X)$. We shall use the identities (3-7). Thus poles of R_V in $B_l(\rho)$ are the values of $\zeta \in B_l(\rho)$ such that $I + V^{\#} R_{V_0}(\zeta)\chi$ is not invertible. Here $\chi \in C^\infty_c(X)$ satisfies $\chi V = V$ and is independent of θ.

(1) For each $\lambda_j \in \Lambda_\rho$, $\lambda_j \neq 0$, let $\epsilon_j > 0$ be as guaranteed by Proposition 5.1, so that there are exactly $2m_{V_0,0}(\lambda_0)$ resonances (counted with multiplicity) of $-\Delta + V$ in $D_l(\lambda_j, \epsilon_j)$ for l sufficiently large. Set

$$K = \{\lambda \in \mathbb{C} : \epsilon' \le |\lambda| \le \rho \text{ and } |\lambda - \lambda_j| \ge \epsilon_j \text{ for all } \lambda_j \in \Lambda_\rho\},$$
$$K_l = \{\zeta \in B_l(\rho + 1) : \tau_l(\zeta) \in K\} = \bar{B}_l(\rho) \setminus \Big(D_l(0, \epsilon') \bigcup_{\lambda_j \in \Lambda_\rho} D_l(\lambda_j, \epsilon_j) \Big).$$

By an application of Lemma 5.3, for l sufficiently large, $I + V^{\#} R_{V_0}(\zeta)\chi$ is invertible by its Neumann series on K_l. Thus by (3-7) R_V has no poles on K_l for l sufficiently large.

(2) Now we work on $D_l(\lambda_j, \epsilon_j)$ and set $R^{\mathrm{reg}}_{V_0}(\zeta) = R^{\mathrm{reg}}_{V_0}(\zeta; l, \lambda_j)$, so that

$$R^{\mathrm{reg}}_{V_0}(\zeta) = R_{V_0}(\zeta) - \Xi(R_{V_0,0}(\lambda), \lambda_j)|_{\lambda = \tau_l(\zeta)} \mathcal{P}_l$$

for l sufficiently large. By our choice of ϵ_j this is analytic on $\bar{D}_l(\lambda_j, \epsilon_j)$ for large enough l. Then by Lemma 5.2 $I + V^{\#} R^{\mathrm{reg}}_{V_0}(\zeta)\chi$ is invertible in $D_l(\lambda_j, \epsilon_j)$, with

$$(I + V^{\#} R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1} = I - V^{\#} R^{\mathrm{reg}}_{V_0}(\zeta)\chi + O_{L^2(X)\to L^2(X)}(l^{-\delta})$$

for $\zeta \in \bar{D}_l(\lambda_j, \epsilon_j)$. Thus on $\bar{D}_l(\lambda_j, \epsilon_j)$,

$$I + V^{\#} R_{V_0} \chi = (I + V^{\#} R^{\mathrm{reg}}_{V_0}(\zeta)\chi)(I + (I + V^{\#} R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1} V^{\#} \Xi(R_{V_0,0}(\lambda), \lambda_0)|_{\lambda = \tau_l(\zeta)} \mathcal{P}_l \chi). \quad (5\text{-}6)$$

By (5-6) and (4-3), $I+V^{\#}R_{V_0}\chi$ is invertible at a point $\zeta\in D_l(\lambda_j,\epsilon_j)$ if and only if

$$I+\mathcal{P}_l(I+V^{\#}R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1}V^{\#}\Xi(R_{V_0,0}(\lambda),\lambda_0)|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi$$

is invertible at ζ. There is a C_j so that $\|\chi\,\Xi(R_{V_0,0},\lambda_j)\chi\|\le C_j|\lambda-\lambda_j|^{-m_{V_0,0}(\lambda_j)}$ on $\{\lambda\in\mathbb{C}:|\lambda-\lambda_j|\le\epsilon_j\}$; see [Dyatlov and Zworski 2019, Theorems 2.5 and 3.9]. Thus on $D_l(\lambda_j,\epsilon_j)$, using Lemma 5.2,

$$\begin{aligned}
&\|\mathcal{P}_l(I+V^{\#}R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1}V^{\#}\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi\|\\
&\qquad=\left\|\sum_{j=0}^{\infty}\mathcal{P}_l(-V^{\#}R_{V_0}^{\mathrm{reg}}(\zeta))^jV^{\#}\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi\right\|\\
&\qquad\le\|\mathcal{P}_l(I-V^{\#}R_{V_0}^{\mathrm{reg}}(\zeta))V^{\#}\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi\|+C_j'l^{-\delta}|\tau_l(\zeta)-\lambda_j|^{-m_{V_0,0}(\lambda_j)}.
\end{aligned}$$

Now we use Lemma 3.2, $\|V_m\|_{L^\infty}=O(|m|^{-\delta})$, and the fact that $\mathcal{P}_l$ commutes with $R_{V_0,0}$ so that

$$\|\mathcal{P}_l(I-V^{\#}R_{V_0}^{\mathrm{reg}}(\zeta))V^{\#}\mathcal{P}_l\|=O(l^{-\delta})$$

on $\bar{D}_l(\lambda_j,\epsilon_j)$. Thus there is a (new) C_j' so that

$$\|\mathcal{P}_l(I+V^{\#}R_{V_0}^{\mathrm{reg}}(\zeta))^{-1}V^{\#}\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi\|\le C_j'l^{-\delta}|\tau_l(\zeta)-\lambda_j|^{-m_{V_0,0}(\lambda_j)}$$

on $\bar{D}_l(\lambda_j,\epsilon_j)$. Therefore

$$I+\mathcal{P}_l(I+R_{V_0}^{\mathrm{reg}}(\zeta))^{-1}\Xi(R_{V_0,0}(\lambda,\lambda_0))|_{\lambda=\tau_l(\zeta)}\mathcal{P}_l\chi$$

is invertible in this region if $|\tau_l(\zeta)-\lambda_j|\ge C_jl^{-\delta/m_{V_0,0}(\lambda_j)}$, where we can take $C_j=(2C_j')^{1/m_{V_0,0}(\lambda_j)}$. Taking $\widetilde{C}=\max_{\lambda_j\in\Lambda_\rho}C_j$ finishes the proof of Theorem 1.2 away from $\tau_l=0$.

(3) If $R_{V_0,0}(\lambda)$ does not have a pole at the origin, then there is a $\delta>0$ so that for l sufficiently large, $R_{V_0}(\zeta)$ is analytic in $\bar{B}_l(\delta)$. Thus by Lemma 5.3, for l sufficiently large, $R_V(\zeta)$ is analytic in $\bar{B}_l(\delta)$. □

5B. *Approximating the resolvent R_V.* In a sense made precise below in Proposition 5.4 and Lemma 5.5, at high energies R_{V_0} approximates R_V well away from resonances. The first result is useful for neighborhoods of thresholds.

Proposition 5.4. *Let $V,\ \chi\in L^\infty_c(X)$, with V satisfying (1-1). Let $K\subset\mathbb{C}$ be a compact set on which $R_{V_0,0}$ is analytic and suppose $K\subset\{\lambda\in\mathbb{C}:|\lambda|<\rho\}$. Define $K_l\overset{\mathrm{def}}{=}\{\zeta\in B_l(\rho):\tau_l(\zeta)\in K\}\subset\hat{Z}$. Then for l sufficiently large, R_V is analytic on K_l. Moreover, if $\chi\in L^\infty_c(X)$, then $\|\chi(R_V(\zeta)-R_{V_0}(\zeta))\chi\|=O(l^{-\delta})$ uniformly for $\zeta\in K_l$.*

Proof. Without loss of generality we may assume χ is independent of θ and satisfies $\chi V=V$. Then $\chi R_{V_0}\chi=\chi R_V\chi(I+V^{\#}R_{V_0}\chi)$. Since by Lemma 5.3 $\|(V^{\#}R_{V_0}\chi)^2\|\le\frac{1}{2}$ on K_l for l sufficiently large, $I+V^{\#}R_{V_0}\chi$ is invertible as $(I+V^{\#}R_{V_0}\chi)^{-1}=\sum_{j=0}^{\infty}(-V^{\#}R_{V_0}\chi)^j$, and thus R_V is analytic on K_l. Moreover,

$$\chi(R_V(\zeta)-R_{V_0}(\zeta))\chi=\chi\sum_{j=1}^{\infty}R_{V_0}(\zeta)(-V^{\#}R_{V_0}(\zeta)\chi)^j.$$

By applying Lemma 5.3 twice, this becomes

$$\chi(R_V(\zeta)-R_{V_0}(\zeta))\chi=-\chi R_{V_0}(\zeta)V^{\#}R_{V_0}(\zeta)\chi+O_{L^2\to L^2}(l^{-\delta})=O_{L^2\to L^2}(l^{-\delta}).$$ □

A similar result with a similar proof is the following lemma. The points $\zeta \in \hat{Z}$ considered in this lemma lie on the boundary of the physical space, but are away from the thresholds.

Lemma 5.5. *Let $V,\ \chi \in L^\infty_c(X)$, with V satisfying (1-1). Then there are constants $M,\ L > 0$ so that*

$$\text{if } l > L,\ \zeta \in B_l(\sqrt{2l-1}),\ \tau_l(\zeta) \in i[0,\infty),\ \text{and } M < \frac{\tau_l(\zeta)}{i} < \sqrt{2l-1} - \frac{M}{\sqrt{l}},$$
$$\text{then } \|\chi(R_V(\zeta) - R_{V_0}(\zeta))\chi\| = O(l^{-\delta}). \tag{5-7}$$

Likewise, there are constants $M_1,\ L_1 > 0$ so that

$$\text{if } l > L_1,\ \zeta \in B_l(\sqrt{2l-1}),\ \tau_l(\zeta) \in [0,\infty),\ \text{and } M_1 < \tau_l(\zeta) < \sqrt{2l-1} - \frac{M_1}{\sqrt{l}},$$
$$\text{then } \|\chi(R_V(\zeta) - R_{V_0}(\zeta))\chi\| = O(l^{-\delta}). \tag{5-8}$$

Proof. This proof is very similar to the proof of Proposition 5.4. We outline the proof of the first statement only, as the proof of the second is analogous.

Without loss of generality, we may assume χ is independent of θ and satisfies $\chi V = V$.

We next note that if $\zeta \in B_l(\sqrt{2l-1})$, then for $l > 3$ either $|\tau_l(\zeta)| > \frac{1}{4}\sqrt{2l-1}$ or $|\tau_{l-1}(\zeta)| > \frac{1}{4}\sqrt{2l-1}$ or both are true. In either case, if $\tau_l(\zeta) \in i[0,\infty)$, then there is a $c_0 > 0$ so that $|\tau_j(\zeta)| > c_0 l^{1/2}$ for $j \neq l, l-1$. Moreover, again with $\tau_l(\zeta) \in i[0,\infty)$, $\operatorname{Im} \tau_j(\zeta) > 0$ if $j > l$ and $\operatorname{Im} \tau_j(\zeta) = 0$ if $0 \le j < l$.

Suppose $\zeta \in B_l(\sqrt{2l-1}),\ \tau_l(\zeta) \in i[0,\infty)$, and $|\tau_l(\zeta)| > \frac{1}{4}\sqrt{2l-1}$. Then using Lemma 3.1 and (3-2) we see that

$$\|\chi R_{V_0}(\zeta)\chi(I - \mathcal{P}_{l-1})\| = O(l^{-1/2}).$$

By Lemma 3.1 there is a $C > 0$ so that if $\lambda \in \mathbb{R},\ |\lambda| > C$, then

$$\|V^\#\|_{L^\infty} \|\chi R_{V_0,0}(\lambda)\chi\| \le \tfrac{1}{2}.$$

Choose $M > C+1$; then if $\tau_l(\zeta) \in i[0,\infty)$ with

$$\frac{\tau_l(\zeta)}{i} < \sqrt{2l-1} - \frac{M}{\sqrt{l}},$$

for l sufficiently large $|\tau_{l-1}(\zeta)| > C$. Now we restrict ourselves to $\tau_l(\zeta) \in i[0,\infty)$ with

$$\frac{1}{4}\sqrt{2l-1} < \frac{\tau_l(\zeta)}{i} < \sqrt{2l-1} - \frac{M}{\sqrt{l}}.$$

Since $\|\mathcal{P}_{l-1} V^\# \mathcal{P}_{l-1}\| = O(l^{-\delta})$ by our assumption on $\|V_m\|_{L^\infty}$,

$$\|\chi R_{V_0}(\zeta)\mathcal{P}_{l-1} V^\# R_{V_0}(\zeta)\mathcal{P}_{l-1}\chi\| = O(l^{-\delta}),$$

and we can follow the proof of Lemma 5.2 to show that $\|\chi R_{V_0}(\zeta) V^\# R_{V_0}(\zeta)\chi\| = O(l^{-\delta})$. Then

$$\begin{aligned}\|\chi(R_V(\zeta) - R_{V_0}(\zeta))\chi\| &= \|\chi R_{V_0}(\zeta)\chi((I + V^\# R_{V_0}(\zeta)\chi)^{-1} - I)\| \\ &= \|\chi R_{V_0}(\zeta) V^\# R_{V_0}(\zeta)\chi\| + O(l^{-\delta}) = O(l^{-\delta}),\end{aligned}$$

proving the lemma when $\tau_l(\zeta) \in i[0,\infty)$ with $\frac{1}{4}\sqrt{2l-1} < \frac{1}{i}\tau_l(\zeta) < \sqrt{2l-1} - M/\sqrt{l}$. A similar argument, singling out $\mathcal{P}_l$ rather than $\mathcal{P}_{l-1}$, handles the case when $\tau_l(\zeta) \in i[0,\infty)$ with $\frac{1}{4}\sqrt{2l-1} < |\tau_{l-1}(\zeta)|$. $\square$

5C. *Proof of Theorem 1.3*. Theorem 1.3 concerns poles of R_V arising as perturbations of threshold poles of $R_{V_0}(\zeta)$. Using separation of variables as in (3-2), these threshold poles, in turn, correspond to a pole of $R_{V_0,0}(\lambda)$ at $\lambda = 0$.

We begin with a lemma about poles of $R_{V_0}(\lambda)$ at the origin. This result is well known if V_0 is real-valued.

Lemma 5.6. *Suppose $V_0 \in L^\infty_c(\mathbb{R}^d)$, and near $\lambda = 0$*

$$R_{V_0,0}(\lambda) = \sum_{k=1}^{k_0} \frac{1}{\lambda^k} A_k + A(\lambda), \tag{5-9}$$

where A is analytic in a neighborhood of the origin. Then $m_{V_0,0}(0) = \max_{0\le t\le 1} \operatorname{rank}(A_1 + tA_2)$.

Since A_1, A_2 are finite-rank, the rank of $A_1 + tA_2$ is equal to its maximum for all but a finite number of values of t in [0, 1].

Proof. Using the expansion (5-9) and the identity $(-\Delta_0 + V_0 - \lambda^2)R_{V_0,0}(\lambda) = I$ shows that for $k > 0$, $(-\Delta_0+V_0)A_k = A_{k+2}$, where we use the convention $A_{k+2} = 0$ if $k+2 > k_0$. Just as in [Dyatlov and Zworski 2019, Theorem 2.5], one can use this and the fact that $-\Delta_0+V_0$ commutes with $R_{V_0,0}$ to show that for $j \in \mathbb{N}$, $\operatorname{Ran}(A_{2j}) \subset \operatorname{Ran}(A_2)$ and $\operatorname{Ran}(A_{2j+1}) \subset \operatorname{Ran}(A_1)$. Here $\operatorname{Ran}(A_k)$ denotes the range of the operator A_k on $L^2_c(\mathbb{R}^d)$. Since $m_{V_0,0}(0) = \dim\big(\bigcup_{k=1}^{k_0} \operatorname{Ran}(A_k)\big)$, this shows $m_{V_0,0}(0) = \dim(\operatorname{Ran} A_1 \cup \operatorname{Ran} A_2)$. But

$$\dim(\operatorname{Ran} A_1 \cup \operatorname{Ran} A_2) = \max_{t\in[0,1]} \dim \operatorname{Ran}(A_1 + tA_2) = \max_{t\in[0,1]} \operatorname{rank}(A_1 + tA_2),$$

proving the lemma. □

Lemma 5.7. *Let $V \in L^\infty_c(X)$ satisfy (1-1). Let $\epsilon > 0$ be chosen so that $R_{V_0,0}(\lambda)$ has no poles in $\{\lambda \in \mathbb{C} : 0 < |\lambda| < 2\epsilon\}$, and let $\gamma_l \subset B_l(2\epsilon) \subset \hat{Z}$ be the curve $\{|\tau_l| = \epsilon\}$ with positive orientation. Then for $t \in [0, 1]$ and l sufficiently large,*

$$\operatorname{rank} \int_{\gamma_l} (1 + t\tau_l(\zeta))R_V(\zeta)\, d\tau_l(\zeta) \ge \operatorname{rank} \int_{\gamma_l} (1 + t\tau_l(\zeta))R_{V_0}(\zeta)\, d\tau_l(\zeta).$$

Proof. We assume $V^\#$ is nontrivial, since otherwise there is nothing to prove.

We first point out that if $R_{V_0,0}(\lambda) = \sum_{k=1}^{k_0} \lambda^{-k} A_k + A(\lambda)$, with $A(\lambda)$ analytic near $\lambda = 0$, then for l sufficiently large

$$\int_{\gamma_l} (1 + t\tau_l(\zeta))R_{V_0}(\zeta)\, d\tau_l(\zeta) = \int_{\gamma_l} (1 + t\tau_l(\zeta))R_{V_0,0}(\tau_l(\zeta))\mathcal{P}_l\, d\tau_l(\zeta) = 2\pi i(A_1 + tA_2)\mathcal{P}_l.$$

Let $\chi \in L^\infty_c(X)$ satisfy $\chi V = V$, with χ independent of θ. Using Proposition 5.4, for l sufficiently large,

$$\left\| \int_{\gamma_l} (1 + t\tau_l(\zeta))\chi(R_V(\zeta) - R_{V_0}(\zeta))\chi\, d\tau_l(\zeta) \right\| = O(l^{-\delta}).$$

Thus

$$\left\| \int_{\gamma_l} (1 + t\tau_l(\zeta))\chi R_V(\zeta)\chi\, d\tau_l(\zeta) - 2\pi i \chi(A_1 + tA_2)\mathcal{P}_l\chi \right\| = O(l^{-\delta}),$$

and this implies that for l sufficiently large,

$$\operatorname{rank}\int_{\gamma_l}(1+t\tau_l(\zeta))\chi R_V(\zeta)\chi\,d\tau_l(\zeta)\geq 2\operatorname{rank}(\chi(A_1+tA_2)\chi). \tag{5-10}$$

But since $(-\Delta_0+V_0)^{k_0}A_j=0$ for $j=1,2$, a unique continuation theorem, e.g., [Jerison and Kenig 1985], ensures that $\operatorname{rank}(A_1+tA_2)=\operatorname{rank}(\chi(A_1+tA_2)\chi)$, and similarly

$$\operatorname{rank}\int_{\gamma_l}(1+t\tau_l(\zeta))\chi R_V(\zeta)\chi\,d\tau_l(\zeta)=\operatorname{rank}\int_{\gamma_l}(1+t\tau_l(\zeta))R_V(\zeta)\,d\tau_l(\zeta). \qquad \square$$

Lemma 5.8. *Let $V_0,\ \chi\in L^\infty_c(\mathbb{R}^d)$, with $\chi V_0=V_0$. Suppose $R_{V_0}(\lambda)$ has a pole of order 1 at the origin. Then for l sufficiently large, $2(m_{V_0,0}(0)-m_{0,0}(0))=M(I+V_0R_0\chi,\zeta_l(0))$.*

Proof. We note here that the requirement that l is sufficiently large is to ensure that, using (3-2), any poles of R_{V_0} at $\zeta_l(0)$ arise from poles of R_{V_0} at the origin. Then via separation of variables it suffices to show that

$$m_{V_0,0}(0)-m_{0,0}(0)=M(I+V_0R_{0,0}(\lambda)\chi,0).$$

For $d=1$, then $m_{V_0,0}(0)=1$ and if V_0 is real-valued, this follows immediately from [Dyatlov and Zworski 2019, (2.2.31)]. For complex-valued V_0, the proof is similar, if one uses the assumption that R_{V_0} has a simple pole at the origin. When $d\geq 3$ is odd, the lemma follows as in the proof of [Dyatlov and Zworski 2019, Theorem 3.15]. In each case, the assumption that the pole is of order 1 is important. $\square$

Lemma 5.9. *Let $V\in L^\infty_c(X)$ satisfy (1-1). Let $\epsilon>0$ be chosen so that $R_{V_0,0}(\lambda)$ has no poles in $\{\lambda\in\mathbb{C}:0<|\lambda|<2\epsilon\}$. Suppose $R_{V_0,0}(\lambda)$ has a pole of order 1 at the origin, with residue of rank $m_{V_0,0}(0)$. Then for l sufficiently large,*

$$\sum_{\substack{\zeta\in D_l(\epsilon)\\ m_V(\zeta)\neq 0}} m_V(\zeta)\leq 2m_{V_0,0}(0).$$

Proof. Let $\chi\in L^\infty_c(X)$ be independent of θ and satisfy $\chi V=V$. We first claim that for any $\zeta_0\in\hat{Z}$,

$$m_V(\zeta_0)\leq M(I+VR_0\chi,\zeta_0)+m_0(\zeta_0). \tag{5-11}$$

If ζ_0 does not correspond to a threshold, then $m_0(\zeta_0)=0$ and this follows from the stronger Proposition 4.3. If ζ_0 does correspond to a threshold, this follows from a simplified adaptation of the proof of [Dyatlov and Zworski 2019, Theorem 3.15].

Arguing as in the proof of Proposition 5.1, using Lemmas 4.5 and 4.6 and an operator Rouché theorem [Gohberg and Sigal 1971, Theorem 2.2], for l sufficiently large,

$$\sum_{\substack{\zeta\in B_l(\epsilon)\\ M(I+VR_0\chi,\zeta)\neq 0}} M(I+VR_0\chi,\zeta)=\sum_{\substack{\zeta\in B_l(\epsilon)\\ M(I+V_0R_0\chi,\zeta)\neq 0}} M(I+V_0R_0\chi,\zeta)=M(I+V_0R_0\chi,\zeta_l(0)). \tag{5-12}$$

But by our assumptions and Lemma 5.8, $M(I+V_0R_0\chi,\zeta_l(0))=2(m_{V_0,0}(0)-m_{0,0}(0))$ for l sufficiently large. Using this, (5-12), and applying (5-11) completes the proof. $\square$

Proof of Theorem 1.3 under the assumption $\|V_m\|_{L^\infty} = O(|m|^{-\delta})$. Let $\epsilon > 0$ be as in the statement of Lemma 5.7. By applying Lemmas 5.6 and 5.7, we see that for l sufficiently large,

$$\sum_{\substack{\zeta\in B_l(\epsilon)\\ m_V(\zeta)\neq 0}} m_V(\zeta) \geq \sum_{\substack{\zeta\in B_l(\epsilon)\\ m_{V_0}(\zeta)\neq 0}} m_{V_0}(\zeta) = 2m_{V_0,0}(0).$$

Thus for l sufficiently large R_V has at least $2m_{V_0,0}(0)$ poles in $B_l(\epsilon)$. If $R_{V_0,0}(\lambda)$ has a simple pole at the origin, then applying in addition Lemma 5.9 we see that R_V has at exactly $2m_{V_0,0}(0)$ poles in $B_l(\epsilon)$.

To finish the proof of the theorem for the L^∞ case we need to refine the estimate on the location of the resonances in $B_l(\epsilon)$. We do this by showing that there is a $C > 0$ so that there are no resonances in $B_l(\epsilon) \setminus B_l(Cl^{-\delta/r})$ for l sufficiently large. This follows almost exactly the proof of Theorem 1.2, point 2, with λ_j replaced by 0. The difference here is that the bound on the singular part of $\chi R_{V_0}\chi$ at the origin is given by $\|\chi\,\Xi(R_{V_0}, 0)\chi\| \leq C|\lambda|^{-r}$; that is, $m_{V_0,0}(\lambda_j)$ is replaced by r rather than $m_{V_0,0}(0)$. Having made this minor adaptation, the remainder of the proof follows without change. □

6. Resonance-free regions, poles of R_V and $R_{\overline{V}}$, and the proofs of Corollary 1.4 and Theorem 1.9

Thus far we have focused on resonances in the sets $B_l(\rho)$, for l large. In this section we justify this by showing that the high-energy resonances near the physical space which also have $\operatorname{Re}\tau_0(\zeta) > 0$ lie in $B_l(\rho)$, for ρ sufficiently large. We do this by showing the existence of large resonance-free regions in $B_l(\sqrt{2l-1})$. We discuss $\hat{Z}$ further, focusing on the region near the physical space. We describe the relationship between the resolvents R_V and $R_{\overline{V}}$, where $\overline{V}$ is the complex conjugate of V; see Lemma 6.2. This lemma shows that we can understand the poles of R_V which are near the physical space and have $\operatorname{Re}\tau_0(\zeta) < 0$ by understanding the poles of $R_{\overline{V}}$ which are near the physical space and have $\operatorname{Re}\tau_0(\zeta) > 0$.

Lemma 6.1. *Let $V \in L^\infty_c(X)$. Then for any $0 < \gamma < 1$ there are $M_+,\ c_+ > 0$ so that the region*

$$U_l^+ \stackrel{\text{def}}{=} \{\zeta \in B_l(\sqrt{2l-1}) : M_+ < \operatorname{Re}(\tau_l(\zeta)) < \gamma\sqrt{2l},\ \operatorname{Im}\tau_l(\zeta) > -c_+ \log \operatorname{Re}(\tau_l(\zeta))\}$$

contains no poles of R_V for l sufficiently large. Likewise, for any $\alpha > 0$ and $0 < \gamma < 1$, there is a constant $M_- > 0$ so that

$$U_l^- \stackrel{\text{def}}{=} \{\zeta \in B_l(\sqrt{2l-1}) : M_- < \operatorname{Im}(\tau_l(\zeta)) < \gamma\sqrt{2l},\ \operatorname{Re}\tau_l(\zeta) > -\alpha\}$$

contains no poles of R_V for l sufficiently large.

The region U_l^+ is reminiscent of the logarithmic resonance-free regions familiar from potential scattering on $\mathbb{R}^d$. We note that there is substantial overlap between U_l^+ and U_{l+1}^-.

Proof. Let $\chi \in L^\infty_c(X)$ be independent of θ and satisfy $\chi V = V$ and $0 \leq \chi \leq 1$. To prove the lemma, we use $\chi R_V(\zeta)\chi = \chi R_0(\zeta)(I + VR_0(\zeta)\chi)^{-1}$ and the representation (3-2) via separation of variables.

From (3-2) and the estimate $\|\chi R_{0,0}(\lambda)\chi\| \leq Ce^{(C\operatorname{Im}\lambda)_-}/|\lambda|$, there are constants $C_1,\ C_2$ so that

$$\|VR_0(\zeta)\chi\| \leq \sup_{j\in\mathbb{N}_0}\left(\frac{C_1 e^{C_2(\operatorname{Im}\tau_j(\zeta))_-}}{|\tau_j(\zeta)|}\right).$$

First consider U_l^+. Set $c_+ = 1/C_2 - \delta_+$, where $\delta_+ > 0$, $\delta_+ < 1/C_2$, and take $M_+ > (2C_1)^{1/(\delta_+ C_2)}$. Then if $\zeta \in U_l^+$,

$$\frac{C_1 e^{C_2(\operatorname{Im}\tau_l(\zeta))_-}}{|\tau_l(\zeta)|} < \tfrac{1}{2}.$$

If $j < l$ and $\zeta \in U_l^+$, then $|\tau_j(\zeta)| \geq |\tau_l(\zeta)|$ and a computation shows

$$\frac{e^{C_2(\operatorname{Im}\tau_j(\zeta))_-}}{|\tau_j(\zeta)|} < \frac{e^{C_2(\operatorname{Im}\tau_l(\zeta))_-}}{|\tau_l(\zeta)|}.$$

On the other hand, for $j > l$, if $\zeta \in U_l^+$, then

$$\operatorname{Re}(\tau_j(\zeta))^2 \leq \operatorname{Re}(\tau_{l+1}(\zeta))^2 = (\operatorname{Re}\tau_l(\zeta))^2 - 2l - (\operatorname{Im}\tau_l(\zeta))^2 - 1 \leq -2l(1-\gamma^2).$$

Since $\operatorname{Im}\tau_j(\zeta) > 0$ for $j > l$ and $\zeta \in B_l(\sqrt{2l-1})$, this is enough to show that

$$\frac{C_1 e^{C_2(\operatorname{Im}\tau_j(\zeta))_-}}{|\tau_j(\zeta)|} < \tfrac{1}{2}$$

for $\zeta \in U_l^+$ and l sufficiently large. Then $\|V R_0(\zeta)\chi\| < \frac{1}{2}$, and $I + V R_0(\zeta)\chi$ is invertible.

For U_l^-, choose $M_- > 0$ so that $16\|V\|_{L^\infty} < M_-^2$. Then using (3-2) and $\|R_{0,0}(\lambda)\| \leq 1/(\operatorname{dist}(\lambda^2, [0,\infty)))$ for $\operatorname{Im}\lambda > 0$, for $\zeta \in U_l^-$,

$$\Big\| V R_0(\zeta)\chi \sum_{j\geq l} \mathcal{P}_j \Big\| \leq \|V\|_{L^\infty} \sup_{j\geq l} \frac{1}{(\operatorname{dist}\tau_j^2, [0,\infty))} \leq \frac{8\|V\|_{L^\infty}}{M_-^2} \leq \tfrac{1}{2}.$$

Next we show that

$$\Big\| V R_0(\zeta) \sum_{0\leq j<l} \mathcal{P}_j \chi \Big\| \leq \tfrac{1}{2}$$

in U_l^- for sufficiently large l. Using the orthogonality of the projections $\sum_{j\geq l}\mathcal{P}_j$ and $\sum_{0\leq j<l}\mathcal{P}_j$ this will complete our proof that $I + V R_0\chi$ is invertible. Note that

$$\tau_{l-1}^2 = 2l - (\operatorname{Im}\tau_l)^2 + (\operatorname{Re}\tau_l)^2 - 1 + 2i\operatorname{Re}(\tau_l)\operatorname{Im}(\tau_l).$$

Thus $|\tau_{l-1}| \geq \sqrt{(1-\gamma^2)2l} + O(1)$ and $-\operatorname{Im}(\tau_{l-1}) \leq 2\alpha/\sqrt{1-\gamma^2} + O(l^{-1/2})$, so for l sufficiently large,

$$\frac{C_1 e^{C_2(\operatorname{Im}\tau_{l-1}(\zeta))_-}}{|\tau_{l-1}(\zeta)|} < \tfrac{1}{2}$$

for $\zeta \in U_l^-$. But if $0 \leq j < l-1$ and $\zeta \in U_l^-$,

$$\frac{C_1 e^{C_2(\operatorname{Im}\tau_j(\zeta))_-}}{|\tau_j(\zeta)|} < \frac{C_1 e^{C_2(\operatorname{Im}\tau_{l-1}(\zeta))_-}}{|\tau_{l-1}(\zeta)|}.$$

This ensures that

$$\Big\| V R_0(\zeta) \sum_{0\leq j<l} \mathcal{P}_l \chi \Big\| < \tfrac{1}{2}$$

so that $I + V R_0(\zeta)\chi$ is invertible on U_l^- for l sufficiently large. $\square$

We remark that we have not made an effort to optimize the results of Lemma 6.1, as in this paper we are concentrating instead on regions near the thresholds, where, as we have seen, resonances can occur.

Before proving Corollary 1.4, we discuss $\hat{Z}$ and the boundary of the physical space a bit more. To motivate the discussion, consider the simpler case of the Schrödinger operator $-\Delta_0 + V_0$ on $\mathbb{R}^d$, where we use λ^2 as the spectral parameter in defining the (scattering) resolvent. Thus, given a value $E > 0$, there are two points, $\pm\sqrt{E}$ corresponding to the spectral parameter E on the boundary of the physical space, with $R_{V_0,0}(\pm\sqrt{E}) = (-\Delta_0 + V_0 - (\sqrt{E} \pm i0))^{-1}$.

There is a similar phenomena in the case of $-\Delta + V$ on $\mathbb{R}^d \times \mathbb{S}^1$, but it is notationally harder to describe. Given $E > 0$, let $\sqrt{E} \pm i0 \in \hat{Z}$ be the points on $\hat{Z}$ with $R_V(\sqrt{E} \pm i0) = (-\Delta + V - E \mp i0)^{-1}$. Equivalently, we could define $\sqrt{E} \pm i0$ to be the point in $\hat{Z}$ with $\tau_j(\sqrt{E} \pm i0) = \pm\sqrt{E - j^2}$ if $j^2 \le E$, and $\tau_j(\sqrt{E} \pm i0) = i\sqrt{j^2 - E}$ if $j^2 > E$. By our definition of $B_l(\rho)$, if $l_E = \lfloor \sqrt{E} \rfloor$ and $l_E > 0$, then $\sqrt{E} + i0 \in B_{l_E}(\sqrt{2l_E - 1})$, but $\sqrt{E} - i0 \notin B_{l_E}(\sqrt{2l_E - 1})$. Thus there is some sense in which we have been studying only “half” of the boundary of the physical space. However, we shall see in Lemma 6.2 that this suffices for understanding the behavior of the resolvent, if we consider both the resolvent of $-\Delta + V$ and that of $-\Delta + \overline{V}$.

Thus, to fully cover points on the boundary of the physical space, we need to define another type of open set in $\hat{Z}$, analogous to $B_l(\rho)$. For $l \in \mathbb{N}$ and $\rho > 0$, denote by $B_l^{\pm}(\rho)$ the connected component of $\{\zeta \in \hat{Z} : |\tau_l(\zeta)| < \rho\}$ which intersects the physical space and includes a region with $\pm \operatorname{Re} \tau_0(\zeta) > 0$. With the $+$ sign, we get the set $B_l(\rho)$ defined in the introduction: $B_l^+(\rho) = B_l(\rho)$. If $l_E = \lfloor \sqrt{E} \rfloor$ and $\sqrt{E} - l_E < \rho$, then the point $\sqrt{E} - i0$ corresponding to E on the boundary of the physical space as defined above has $\sqrt{E} - i0 \in B_{l_E}^-(\rho)$. Hence any point on the boundary of the physical space lies in

$$B_0^+(1) \cup \left(\bigcup_{l=1}^{\infty} B_l^+(\sqrt{2l-1}) \right) \cup \left(\bigcup_{l=1}^{\infty} B_l^-(\sqrt{2l-1}) \right).$$

As before, we make the choice of $\sqrt{2l-1}$ for ρ as that is the largest value of ρ for which $B_l^{\pm}(\rho)$ contains only a single point corresponding to a threshold. For certain combinations of l and ρ it can happen that $B_l^+(\rho) = B_l^-(\rho)$.

Consider a Schrödinger operator on d-dimensional Euclidean space with potential $V_0 \in L_c^\infty(\mathbb{R}^d)$ and scattering resolvent $R_{V_0,0}(\lambda)$. When $\operatorname{Im} \lambda > 0$, that is λ is in the physical space,

$$R_{V_0,0}(\lambda) = (-\Delta_0 + V_0 - \lambda^2)^{-1} = ((-\Delta_0 + \overline{V}_0 - \bar{\lambda}^2)^{-1})^* = (R_{\overline{V}_0 0}(-\bar{\lambda}))^*.$$

Here $\overline{V}_0$ and $\bar{\lambda}$ denote the usual complex conjugates. For odd d the identity $R_{V_0,0}(\lambda) = (R_{\overline{V}_0 0}(-\bar{\lambda}))^*$ then holds by meromorphic continuation for all $\lambda \in \mathbb{C}$. In particular, this implies λ_0 is a pole of $R_{V_0,0}(\lambda)$ if and only if $-\bar{\lambda}_0$ is a pole of $R_{\overline{V}_0 0}(\lambda)$. For real-valued V, this is the well-known symmetry of resonances for symmetric Schrödinger operators in odd dimensions.

We turn to the analog of this result for R_V, which is shown in a similar way. Suppose ζ is in the physical space, here identified with the upper half-plane, so that $R_V(\zeta) = (-\Delta + V - \zeta^2)^{-1}$. Thus $(R_{\overline{V}}(-\bar{\zeta}))^* = R_V(\zeta)$. For general $\zeta \in \hat{Z}$, we define $-\zeta^\dagger \in \hat{Z}$ to be the point in $\hat{Z}$ which satisfies $\tau_j(-\zeta^\dagger) = -\overline{\tau_j}(\zeta)$ for all j. This is an antiholomorphic mapping, and if ζ is in the physical space, identified with the upper half-plane, the mapping $\zeta \mapsto -\zeta^\dagger$ agrees with the mapping $\zeta \mapsto -\bar{\zeta}$. Then the identity

$$(R_{\overline{V}}(-\zeta^\dagger))^* = R_V(\zeta), \quad \text{where } \tau_j(-\zeta^\dagger) = -\overline{\tau_j}(\zeta), \text{ for all } j \in \mathbb{N}_0 \tag{6-1}$$

holds for all $\zeta \in \hat{Z}$ by meromorphic continuation. In particular, this means that $\zeta_0 \in \hat{Z}$ is a pole of $R_V(\zeta)$ if and only if $-\zeta_0^\dagger$ is a pole of $R_{\overline{V}}(\zeta)$. Note that if $\zeta \in B_l^+(\rho) = B_l(\rho)$, then $-\zeta^\dagger \in B_l^-(\rho)$. Thus to study the poles of $R_V(\zeta)$ in $B_l^-(\rho)$ it suffices to study the poles of $R_{\overline{V}}(\zeta)$ in $B_l^+(\rho) = B_l(\rho)$. Likewise, an estimate on $R_{\overline{V}}$ in $B_l^+(\sqrt{2l-1})$ implies an estimate on R_V in $B_l^-(\sqrt{2l-1})$.

We summarize these results in the following lemma.

Lemma 6.2. *If $V_0 \in L_c^\infty(\mathbb{R}^d)$, then λ_0 is a pole of $R_{V_0,0}(\lambda)$ if and only if $-\bar{\lambda}_0$ is a pole of $R_{\overline{V}_0 0}(\lambda)$. Let $V \in L_c^\infty(X)$. Then $\zeta_0 \in \hat{Z}$ is a pole of $R_V(\zeta)$ if and only if $-\zeta_0^\dagger$ is a pole of $R_{\overline{V}}(\zeta)$. Here $\bar{\lambda}_0$, $\overline{V}$, and $\overline{V}_0$ are the complex conjugates of λ_0, V, and V_0, respectively, and $-\zeta^\dagger$ is as defined in (6-1).*

We define a distance on $\hat{Z}$ as follows: for $\zeta, \zeta' \in \hat{Z}$,

$$d_{\hat{Z}}(\zeta, \zeta') \overset{\text{def}}{=} \sup_j |\tau_j(\zeta) - \tau_j(\zeta')|. \tag{6-2}$$

That this is well defined and a metric is shown in [Christiansen and Datchev 2021, Section 5.1]. Note that if $\zeta, \zeta' \in \hat{Z}$ satisfy $\tau_j(\zeta) \neq -\tau_j(\zeta')$, then since $\tau_j(\zeta)^2 - \tau_j(\zeta')^2 = \tau_l(\zeta)^2 - \tau_l(\zeta')^2$,

$$|\tau_j(\zeta) - \tau_j(\zeta')| = |\tau_l(\zeta) - \tau_l(\zeta')| \left| \frac{\tau_l(\zeta) + \tau_l(\zeta')}{\tau_j(\zeta) + \tau_j(\zeta')} \right|.$$

In particular, this implies that for any $\rho > 0$ there is an $L = L(\rho)$ so that if $l \geq L$ and $\zeta, \zeta' \in B_l(\rho)$ then

$$d_{\hat{Z}}(\zeta, \zeta') = |\tau_l(\zeta) - \tau_l(\zeta')|.$$

Proof of Corollary 1.4. Recall our hypotheses include that V is real-valued, ensuring that V_0 is real-valued as well.

The operator-valued function $R_V(\zeta)$ has a sequence $\{\zeta_j^\flat\}$ of poles satisfying $|\tau_0(\zeta_j^\flat)| \to \infty$ as $j \to \infty$ and $d_{\hat{Z}}(\zeta_j^\flat, \text{physical space}) \to 0$ only if $R_V(\zeta)$ has infinitely many poles in $\bigcup_{l=1}^\infty B_l(\sqrt{2l-1})$ or infinitely many poles in $\bigcup_{l=1}^\infty B_l^-(\sqrt{2l-1})$ (or both). If $R_V(\zeta)$ has infinitely many poles in $\bigcup_{l=1}^\infty B_l^-(\sqrt{2l-1})$, then by Lemma 6.2, $R_{\overline{V}}(\zeta) = R_V(\zeta)$ has infinitely many poles in $\bigcup_{l=1}^\infty B_l(\sqrt{2l-1})$. Thus it suffices to study sequences of poles in $\bigcup_{l=1}^\infty B_l(\sqrt{2l-1})$.

Note that while $B_l(\sqrt{2l-1})$ contains only a single threshold, $B_l(\sqrt{2l-1})$ and $B_{l+1}(\sqrt{2l+1})$ are not disjoint and in fact have substantial overlap which contains an interval of the continuous spectrum. Moreover, for l sufficiently large the sets U_l^+ and U_{l+1}^- of Lemma 6.1 have nontrivial intersection. Applying Lemma 6.1 we see that in order to have a sequence of resonances contained in $\bigcup_{l=1}^\infty B_l(\sqrt{2l-1})$ and approaching the continuous spectrum (and with $|\tau_0| \to \infty$), the resonances must lie in $\bigcup_{l=1}^\infty B_l(M)$ for some M. But then the corollary follows from an application of Theorems 1.1–1.3. ☐

We now have the ingredients we need to prove Theorem 1.9.

Proof of Theorem 1.9. The hypotheses on $-d^2/dx^2 + V_0$ and the expression (3-2) mean that the resolvent $R_{V_0}(\zeta)$ has no poles on the boundary of the physical space. Moreover, since for any $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ there is a constant C so that $\|\tilde{\chi} R_{V_0,0}(\lambda)\tilde{\chi}\| \le C$ for all $\lambda \in \mathbb{R} \cup i[0,\infty)$, for any $\chi \in C_c^\infty(X)$ there is a $C_1 > 0$ so that $\|\chi R_{V_0}(\zeta)\chi\| \le C_1$ for all ζ in the boundary of the physical space.

Corollary 1.4 shows that there are no poles of the resolvent R_V in the continuous spectrum at high energy. Proposition 5.4 and Lemma 5.5 show that when ζ is in the boundary of the physical space and $\zeta \in B_l(\sqrt{2l-1})$, the cut-off resolvent of $-\Delta + V$ satisfies $\|\chi R_V(\zeta)\chi - \chi R_{V_0}(\zeta)\chi\| = O(l^{-1/2})$. Thus $\|\chi R_{V_0}(\zeta)\chi\|$ is uniformly bounded on the boundary of the physical space when $|\tau_0(\zeta)|$ is sufficiently large. Hence by [Christiansen and Datchev 2021, Theorem 5.6] the hypotheses of [Christiansen and Datchev 2022, Theorem 4.1] hold. Theorem 1.9 then follows directly. $\square$

7. Larger neighborhoods of the threshold l^2

In this section we consider poles of $R_V(\zeta)$ in neighborhoods $B_l(\alpha \log l)$ and $B_l(\alpha(\log l)^{1-\epsilon})$ of the l-th threshold. We prove Theorem 1.5 for potentials with $V_0 \equiv 0$ and the related, but weaker, Theorem 7.1 which holds for a general potential $V \in L_c^\infty(X)$.

The proof of Theorem 1.5 is similar to that of the proof of Theorem 1.2 for L^∞ potentials.

Proof of Theorem 1.5. Choose $\chi \in L_c^\infty(X)$, $\chi V = V$, and χ independent of θ. We write

$$\begin{aligned}\chi R_0 V R_0 \chi \\ = \chi R_0 \mathcal{P}_l V R_0 \mathcal{P}_l \chi + \chi R_0(1-\mathcal{P}_l) V R_0 \mathcal{P}_l \chi + \chi \mathcal{P}_l R_0 V R_0 (1-\mathcal{P}_l)\chi + \chi R_0 (1-\mathcal{P}_l) V R_0 (1-\mathcal{P}_l)\chi. \quad (7\text{-}1)\end{aligned}$$

Let $\alpha' > 0$, and let $\zeta \in B_l(\alpha' |\log l|)$, where l is large enough that $B_l(\alpha'|\log l|)$ contains only a single point of $\hat{Z}$ which corresponds to a threshold. Let $\zeta \in B_l(\alpha' \log l)$ satisfy $|\tau_l(\zeta)| \ge 1$. Then by Lemma 3.2,

$$\|\chi R_0(\zeta)(1-\mathcal{P}_l)\chi\| = O(l^{-1/2}),$$

and by (3-1) and [Dyatlov and Zworski 2019, Theorem 3.1],

$$\|\chi R_0(\zeta)\mathcal{P}_l\chi\| = O\left(\frac{e^{C(\operatorname{Im}\tau_l(\zeta))_-}}{|\tau_l(\zeta)|}\right)$$

for some $C > 0$. Using this estimate and $\mathcal{P}_l V \mathcal{P}_l = O(l^{-\delta})$ in (7-1) shows

$$\|\chi R_0(\zeta) V R_0(\zeta)\chi\| = O(l^{-\delta} e^{2C(\operatorname{Im}\tau_l(\zeta))_-}).$$

Thus from (7-1) there is a $C_1 > 0$ so that $I + V R_0(\zeta)\chi$ is invertible if l is sufficiently large, $\zeta \in B_l(\alpha' \log l)$, $|\tau_l(\zeta)| \ge 1$, and $e^{2C(\operatorname{Im}\tau_l(\zeta))_-} \le C_1 l^\delta$. This last item may be ensured by requiring $|\tau_l| \le \alpha \log l$, for suitably chosen $\alpha > 0$, $\alpha \le \alpha'$, and taking l sufficiently large. Recall that $-\Delta + V$ has no resonances in regions where $I + V R_0 \chi$ is invertible, see Proposition 4.3.

Applying Theorems 1.2 and 1.3 shows that if $d = 1$ there is a $c_0 > 0$ so that when l is sufficiently large the region $\{\zeta \in B_l(\alpha \log l) : 1 \ge |\tau_l(\zeta)| > c_0 l^{-\delta}\}$ contains no resonances, and if $d > 1$ there are no resonances in $B_l(1)$ for l sufficiently large. $\square$

A similar proof gives the next theorem.

Theorem 7.1. *Let $V \in L^\infty_c(X)$ satisfy (1-1), and let $\epsilon > 0$. Then there is a $c_0 = c_0(\epsilon, V) > 0$ so that for l sufficiently large, the region*

$$\{\zeta \in B_l(c_0(\log l)^{1/(d+\epsilon)}) : |\tau_l(\zeta) - \lambda'| \geq (1+|\lambda'|^2)^{-(d+\epsilon)/2} \text{ for every } \lambda' \in \mathbb{C} : m_{V_0,0}(\lambda') > 0\}$$

contains no poles of $R_V(\zeta)$.

Proof. We assume $V^\# = V - V_0 \not\equiv 0$, since otherwise there is nothing to prove.

Choose $\chi \in L^\infty_c(X)$ so that $\chi V = V$ and χ is independent of θ. We may think of χ as an element of $L^\infty_c(\mathbb{R}^d)$ as well.

Set

$$A_\epsilon \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} : |\lambda - \lambda'| \geq (1+|\lambda'|^2)^{-(d+\epsilon)/2} \text{ for every } \lambda' \in \mathbb{C} : m_{V_0,0}(\lambda') > 0\}.$$

We shall use, from the proof of [Dyatlov and Zworski 2019, Theorem 3.54], that there is a $C > 0$ so that

$$\|(I + V_0 R_{0,0}(\lambda))^{-1}\| \leq C \exp(C|\lambda|^{d+\epsilon}) \quad \text{if } \lambda \in A_\epsilon. \tag{7-2}$$

Choose $\alpha' > 0$. If $\zeta \in B_l(\alpha' \log l)$,

$$\begin{aligned} \chi R_{V_0}(\zeta)\mathcal{P}_l \chi &= \chi R_{V_0,0}(\tau_l(\zeta))\mathcal{P}_l \chi \\ &= \chi R_{0,0}(\tau_l(\zeta))\chi(I + V_0 R_{0,0}(\tau_l(\zeta))\chi)^{-1}\mathcal{P}_l. \end{aligned}$$

Thus, if $\zeta \in B_l(\alpha' \log l)$ with $\tau_l \in A_\epsilon$ and $|\tau_l(\zeta)| \geq 1$, then

$$\begin{aligned} \|\chi R_{V_0}(\zeta)\mathcal{P}_l \chi\| &\leq C \exp(C(\operatorname{Im} \tau_l(\zeta)_-)) \exp(C|\tau_l(\zeta)|^{d+\epsilon}) \\ &\leq C \exp(C|\tau_l(\zeta)|^{d+\epsilon}). \end{aligned} \tag{7-3}$$

Here and below we allow the constant C to change from line to line, and note that it depends on V, ϵ, and χ, but not l.

Let $\zeta \in B_l(\alpha' \log l)$ with $\tau_l \in A_\epsilon$ and $|\tau_l(\zeta)| \geq 1$. Writing $\chi R_{V_0}\chi$ as in (7-1) and applying Lemma 3.2 and (7-3), we find that for these ζ, if l is sufficiently large,

$$\|\chi R_{V_0}(\zeta) V^\# R_{V_0}(\zeta)\chi\| \leq C_1 l^{-\delta} \exp(C_1|\tau_l(\zeta)|^{d+\epsilon}) \tag{7-4}$$

for some C_1. Now we can choose $c_0 > 0$ sufficiently small and $L > 0$ sufficiently large so that

$$\text{if } |\tau_l(\zeta)| \leq c_0(\log l)^{1/(d+\epsilon)} \text{ and } l > L \quad \text{then } C_1 l^{-\delta} \exp(C_1|\tau_l(\zeta)|^{d+\epsilon}) \leq \tfrac{1}{2}$$

ensuring that $I + V^\# R_{V_0}(\zeta)\chi$ is invertible.

Recalling that with $V^\#$ nontrivial if $I + V^\# R_{V_0}(\zeta)\chi$ is invertible then ζ is not a resonance of $-\Delta + V$ proves the theorem. ☐

8. Expansion of $\mathcal{P}_l(I + V^\# R^{\mathrm{reg}}_{V_0}\chi)^{-1} V^\# \mathcal{P}_l$ for smooth V

This section contains preliminary computations which allow us to refine some of our results when V is smooth. We begin with a straightforward lemma about Schrödinger operators on $\mathbb{R}^d$.

Lemma 8.1. *Let $V_0,\ \chi \in C_c^\infty(\mathbb{R}^d)$ and $J \in \mathbb{N}$. Then as an operator from $H^s(\mathbb{R}^d)$ to $H^{s-2J}(\mathbb{R}^d)$,*

$$\chi R_{V_0,0}(\lambda)\chi = -\sum_{j=1}^{J} \frac{1}{\lambda^{2j}} \chi(-\Delta_0 + V_0)^{j-1}\chi + \frac{1}{\lambda^{2J}} \chi R_{V_0,0}(\lambda)(-\Delta_0 + V_0)^J \chi. \tag{8-1}$$

Proof. First assume λ is in the physical region, that is, $\operatorname{Im}\lambda > 0$. Then the $J = 1$ case follows from rearranging the equality

$$(-\Delta_0 + V_0 - \lambda^2) R_{V_0,0}(\lambda) = R_{V_0,0}(\lambda)(-\Delta_0 + V_0 - \lambda^2) = I$$

to get

$$R_{V_0,0}(\lambda) = \frac{1}{\lambda^2}(-I + R_{V_0,0}(\lambda)(-\Delta_0 + V_0)).$$

The general case follows by induction.

Since both sides of (8-1) have meromorphic continuations to the complex plane, the equality holds for all λ. □

We shall use the following Hilbert spaces: for $n \in \mathbb{N}_0$,

$$H_{(0,n)}(X) \overset{\text{def}}{=} \left\{ u \in L^2(X) : \frac{\partial^\alpha}{\partial x^\alpha} u \in L^2(X) \text{ if } |\alpha| \le n \right\} \quad \text{with } \|u\|^2_{H_{(0,n)}} = \sum_{|\alpha|\le n} \left\| \frac{\partial^\alpha}{\partial x^\alpha} u \right\|^2_{L^2(X)}.$$

Here we use the usual multi-index notation for $\alpha = (\alpha_1, \dots, \alpha_d)$. This allows us to indicate mapping properties of operators which act differently in the x and θ variables.

One of the main results of this section is the following proposition. Recall that $R^{\text{reg}}_{V_0}(\zeta) = R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l)$ is defined in (5-3).

Proposition 8.2. *Let $V,\ \chi \in C_c^\infty(X)$ satisfy $\chi V = V$. In addition, suppose χ is independent of θ. Let $\lambda_0 \in \mathbb{C}$, and suppose $R_{V_0,0}(\lambda)$ is analytic on $0 < |\lambda - \lambda_0| \le \epsilon$. Then, for $R^{\text{reg}}_{V_0}(\zeta) = R^{\text{reg}}_{V_0}(\zeta; \lambda_0, l)$ and $\zeta \in D_l(\lambda_0, \epsilon)$,*

$$\left\| \mathcal{P}_l (I + V^\# R^{\text{reg}}_{V_0}(\zeta)\chi)^{-1} V^\# \mathcal{P}_l + \frac{1}{l^2} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left(\frac{\tau_l^2 - k^2}{4k^2} V_{-k} V_k - \frac{V_{-k}}{4k^2}(-\Delta_0 + V_0) V_k \right) \mathcal{P}_l \right\|_{H_{(0,8)}(X)\to L^2(X)} = O(l^{-3}),$$

where the error is uniform on $D_l(\lambda_0, \epsilon)$ for l sufficiently large.

To prove this proposition we use Lemmas 8.3–8.6. In each of these, V, λ_0, $R^{\text{reg}}_{V_0}(\zeta)$, and ϵ are as in Proposition 8.2. Some of these computations rely on the identity $e^{\pm ik\theta} e^{\pm il\theta} = e^{\pm i(k+l)\theta}$ and hence use the structure of the eigenfunctions of the Laplacian on $\mathbb{S}^1$ in an essential way.

For $l \in \mathbb{N}$, let $\mathcal{P}_{l\pm} : L^2(X) \to L^2(X)$ denote orthogonal projection onto $L^2(\mathbb{R}^d_x) e^{\pm il\theta}$, so that

$$(\mathcal{P}_{l\pm} f)(x, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(x, \theta') e^{\pm il(\theta - \theta')}\, d\theta',$$

for $l > 0$ and $\mathcal{P}_l = \mathcal{P}_{l+} + \mathcal{P}_{l-}$.

Lemma 8.3. *Under the hypotheses of Proposition 8.2,*

$$\left\| \mathcal{P}_l V^\# R^{\mathrm{reg}}_{V_0}(\zeta) V^\# \mathcal{P}_l - \frac{1}{l^2} \sum_{k\in\mathbb{Z},\, k\neq 0} \left(\frac{\tau_l^2 - k^2}{4k^2} V_{-k} V_k - \frac{V_{-k}}{4k^2}(-\Delta_0 + V_0) V_k \right) \mathcal{P}_l \right\|_{H_{(0,n+6)} \to H_{(0,n)}} = O(l^{-3})$$

uniformly for $\zeta \in D_l(\lambda_0, \epsilon)$ when l is sufficiently large.

Proof. Since $V \in C_c^\infty(X)$, we have $\|V_m\|_{L^\infty} = O(|m|^{-N})$ for any N, so $\|\mathcal{P}_l V^\# \mathcal{P}_l\| = O(l^{-N})$. Thus, choosing l sufficiently large so that (5-4) holds, it suffices to consider

$$\mathcal{P}_l V^\# R^{\mathrm{reg}}_{V_0}(\zeta)(I - \mathcal{P}_l) V^\# \mathcal{P}_l = \mathcal{P}_l V^\# R_{V_0}(\zeta)(I - \mathcal{P}_l) V^\# \mathcal{P}_l.$$

Then

$$\begin{aligned} \mathcal{P}_l V^\# R^{\mathrm{reg}}_{V_0}(I - \mathcal{P}_l) V^\# \mathcal{P}_l &= \sum_{\pm} \sum_{\substack{k\in\mathbb{Z} \\ 0<|k|,\, k\neq -l}} V_{\mp k} R_{V_0,0}(\tau_{l+k}) V_{\pm k} \mathcal{P}_{l\pm} \\ &= \sum_{\pm} \sum_{\substack{k\in\mathbb{Z} \\ 0<|k|<l^{1/2}}} V_{\mp k} R_{V_0,0}(\tau_{l+k}) V_{\pm k} \mathcal{P}_{l\pm} + O_{L^2\to L^2}(l^{-N}). \end{aligned}$$

Here we use the rapid decay of $\|V_m\|$ to bound the error obtained when we restrict the values of k in the sum. Using Lemma 8.1 with $J = 3$ gives

$$\mathcal{P}_l V^\# R^{\mathrm{reg}}_{V_0}(I - \mathcal{P}_l) V^\# \mathcal{P}_l = \sum_{\pm} \sum_{\substack{k\in\mathbb{Z} \\ 0<|k|<l^{1/2}}} V_{\mp k} \left(\frac{-1}{\tau_{l+k}^2} - \frac{1}{\tau_{l+k}^4}(-\Delta_0 + V_0) \right) V_{\pm k} \mathcal{P}_{l\pm} + O(l^{-3}), \tag{8-2}$$

where the error is as an operator from $H_{(0,n+6)}(X)$ to $H_{(0,n)}(X)$ and is uniform in $D_l(\lambda_0, \epsilon)$. Since we have restricted $|k|$ to be relatively small compared with l, we can expand $\tau_{l\pm k}$ asymptotically in l. Thus, with each sum over $k \in \mathbb{Z}$ with $0 < |k| < l^{1/2}$, using $\tau_{l\pm k}^2 = \tau_l^2 \mp 2lk - k^2$ gives

$$\begin{aligned} \sum_{0<|k|<l^{1/2}} \frac{1}{\tau_{l+k}^2} V_{-k} V_k = \frac{1}{2} \sum_{0<|k|<l^{1/2}} \left(\frac{1}{\tau_{l+k}^2} + \frac{1}{\tau_{l-k}^2} \right) V_{-k} V_k &= \sum_{0<|k|<l^{1/2}} \frac{\tau_l^2 - k^2}{(\tau_l^2 - k^2)^2 - 4k^2 l^2} V_{-k} V_k \\ &= \frac{-1}{4l^2} \sum_{0<|k|<l^{1/2}} \left(\frac{\tau_l^2 - k^2}{k^2} \right) V_{-k} V_k + O(l^{-4}). \end{aligned} \tag{8-3}$$

Here and below the error is uniform in $D_l(\lambda_0, \epsilon)$ when l is sufficiently large.

For the second term in (8-2), we write

$$\begin{aligned} \sum_{0<|k|<l^{1/2}} \frac{1}{\tau_{l+k}^4} V_{\mp k}(-\Delta_0 + V_0) V_{\pm k} &= \sum_{0<|k|<l^{1/2}} \frac{1}{(\tau_l^2 - 2lk - k^2)^2} V_{\mp k}(-\Delta_0 + V_0) V_{\pm k} \\ &= \frac{1}{4l^2} \sum_{0<|k|<l^{1/2}} \frac{1}{k^2} V_{\mp k}(-\Delta_0 + V_0) V_{\pm k} + O(l^{-3}). \end{aligned}$$

Note that

$$\sum_{0<|k|<l^{1/2}} \frac{1}{k^2} V_{\mp k}(-\Delta_0 + V_0) V_{\pm k} = \sum_{0<|k|<l^{1/2}} \frac{1}{k^2} V_{-k}(-\Delta_0 + V_0) V_k, \tag{8-4}$$

since the sum is over $k \in \mathbb{Z}$, with $0 < |k| < l^{1/2}$. The rapid decay in m of $\|V_m\|_{C^p}$ means we can replace the sums in (8-3) and (8-4) over $0 < |k| < l^{1/2}$ by sums over all nonzero $k \in \mathbb{Z}$, with an error which is $O(l^{-N})$. □

The next lemma is an algebraic identity.

Lemma 8.4. *For any $V \in C_c^\infty(X)$*

$$\sum_{\substack{m,j\in\mathbb{Z}\\ m,j\neq 0,\, m\neq -j}} \frac{1}{j(j+m)} V_m V_j V_{-m-j} = 0.$$

We give two different proofs.

Proof. For this proof, we show that for each $j_0 \neq 0$, $m_0 \neq 0$ the coefficient of $V_{j_0} V_{m_0} V_{-j_0-m_0}$ in the sum is zero. This proof is purely algebraic in nature.

If $m_0 \neq \pm j_0$, then there are six possibilities for the pair (j, m) which will give a term containing $V_{m_0} V_{j_0} V_{-m_0-j_0}$: (j_0, m_0), (m_0, j_0), $(-m_0-j_0, m_0)$, $(m_0, -j_0-m_0)$, $(j_0, -m_0-j_0)$, $(-m_0-j_0, j_0)$. Thus the sum of the coefficients of $V_{m_0} V_{j_0} V_{-m_0-j_0}$ is

$$\frac{1}{j_0(j_0+m_0)} + \frac{1}{m_0(j_0+m_0)} + \frac{1}{j_0(j_0+m_0)} - \frac{1}{j_0 m_0} - \frac{1}{j_0 m_0} + \frac{1}{m_0(j_0+m_0)} = 0.$$

A similar argument when $j_0 = m_0$ shows the coefficient of $V_{j_0}^2 V_{-2j_0}$ is zero as well. □

Alternate proof of Lemma 8.4. For this proof, we use that V_j is the j-th Fourier coefficient of V. Though in our applications V_j depends on x, that dependence is not important here so we will suppress it.

Set

$$W(\theta) = \sum_{j\neq 0} \frac{1}{j} V_j e^{ij\theta}$$

and note $d/d\theta\, W(\theta) = V(\theta) - V_0$. Then

$$\int_0^{2\pi} (V(\theta) - V_0)(W(\theta))^2 \, d\theta = \frac{1}{3}(W(\theta))^3\Big|_0^{2\pi} = 0 \tag{8-5}$$

by the fundamental theorem of calculus. But

$$\begin{aligned}\sum_{\substack{m,j\in\mathbb{Z}\\ m,j\neq 0\\ m\neq -j}} \frac{1}{j(j+m)} V_m V_j V_{-m-j} &= \sum_{\substack{m,j\in\mathbb{Z}\\ m,j\neq 0\\ m\neq -j}} \frac{-1}{jm} V_m V_j V_{-m-j} \\ &= -\int_0^{2\pi} (V(\theta) - V_0)(W(\theta))^2 \, d\theta, \end{aligned}\tag{8-6}$$

where the last equality uses $e^{ij\theta} e^{im\theta} = e^{i(j+m)\theta}$ and the fact that the integral of a function over a circle is its zeroth Fourier coefficient. Combining (8-5) and (8-6) proves the lemma. □

Lemma 8.5. *Under the hypotheses of Proposition 8.2, if l is sufficiently large*

$$\|\mathcal{P}_l (V^\# R_{V_0}^{\mathrm{reg}})^2 V^\# \mathcal{P}_l\|_{H_{(0,n+6)}(X) \to H_{(0,n)}(X)} = O(l^{-3}) \quad \textit{uniformly for } \zeta \in D_l(\lambda_0, \epsilon).$$

Proof. Again we use that $\|\mathcal{P}_l V^\# \mathcal{P}_l\| = O(l^{-N})$ for any N. This implies

$$\mathcal{P}_l (V^\# R_{V_0}^{\mathrm{reg}})^2 V^\# \mathcal{P}_l = \mathcal{P}_l (V^\# R_{V_0}^{\mathrm{reg}} (I - \mathcal{P}_l))^2 V^\# \mathcal{P}_l + O_{L^2 \to L^2}(l^{-N}).$$

Note that for $\zeta \in D_l(\lambda_0, \epsilon)$ and l sufficiently large, $R_{V_0}^{\mathrm{reg}}(\zeta)(I - \mathcal{P}_l) = R_{V_0}(\zeta)(I - \mathcal{P}_l)$. Then

$$\begin{aligned}
&\mathcal{P}_l (V^\# R_{V_0} (I - \mathcal{P}_l))^2 V^\# \mathcal{P}_l \\
&\qquad = \mathcal{P}_l \sum_{\pm} e^{\pm i(j+k+m)\theta} \sum_{\substack{k,m,j \in \mathbb{Z} \\ k, m+k \neq 0, -2l \\ m, j \neq 0}} V_{\pm j} R_{V_0,0}(\tau_{l+k+m}) V_{\pm m} R_{V_0,0}(\tau_{l+k}) V_{\pm k} \mathcal{P}_{l\pm} \\
&\qquad = \sum_{\pm} \sum_{\substack{k, m+k \neq 0, -2l \\ m \neq 0,\, k,m \in \mathbb{Z}}} V_{\mp(k+m)} R_{V_0,0}(\tau_{l+k+m}) V_{\pm m} R_{V_0,0}(\tau_{l+k}) V_{\pm k} \mathcal{P}_{l\pm} + O(l^{-N}). \qquad (8\text{-}7)
\end{aligned}$$

By Lemma 8.1, for $k, m+k \neq 0, -2l$,

$$\begin{aligned}
&\left\| \chi R_{V_0,0}(\tau_{l+k+m}) V_{\pm m} R_{V_0,0}(\tau_{l+k}) V_{\pm k} - \frac{1}{\tau_{l+k+m}^2 \tau_{l+k}^2} \chi V_{\pm m} V_{\pm k} \right\|_{H^{n+6}(\mathbb{R}^d) \to H^n(\mathbb{R}^d)} \\
&\qquad\qquad = O(l^{-3} \|V_{\pm k}\|_{C^{6+n}} \|V_{\pm m}\|_{C^{6+n}}).
\end{aligned}$$

This implies (with sums still over $\mathbb{Z}$), using $\|V_m\|_{C^p} = O(|m|^{-N})$, that

$$\begin{aligned}
\mathcal{P}_l (V^\# R_{V_0} (I - \mathcal{P}_l))^2 V^\# \mathcal{P}_l &= \sum_{\pm} \sum_{k,m,k+m \neq 0, -2l} \frac{1}{\tau_{l+k+m}^2 \tau_{l+k}^2} V_{\mp(k+m)} V_{\pm m} V_{\pm k} \mathcal{P}_{l\pm} + O(l^{-3}) \\
&= \sum_{\pm} \sum_{0<|k|,\ |k+m|<l^{1/2},\ m \neq 0} \frac{1}{\tau_{l+k+m}^2 \tau_{l+k}^2} V_{\mp(k+m)} V_{\pm m} V_{\pm k} \mathcal{P}_{l\pm} + O(l^{-3}) \\
&= \sum_{\pm} \sum_{0<|k|,\ |k+m|<l^{1/2},\ m \neq 0} \frac{1}{4l^2 k(k+m)} V_{\mp(k+m)} V_{\pm m} V_{\pm k} \mathcal{P}_{l\pm} + O(l^{-3}) \\
&= \sum_{\pm} \sum_{0 \neq k, k+m, m} \frac{1}{4l^2 k(k+m)} V_{\mp(k+m)} V_{\pm m} V_{\pm k} \mathcal{P}_{l\pm} + O(l^{-3}). \qquad (8\text{-}8)
\end{aligned}$$

Here errors are as operators from $H_{(0,n+6)}(X)$ to $H_{(0,n)}(X)$, and are uniform in $D_l(\lambda_0, \epsilon)$ when l is sufficiently large. But the final sum in (8-8) is zero by Lemma 8.4. □

Lemma 8.6. *Under the hypotheses of Proposition 8.2 for $j \geq 3$, $j \in \mathbb{N}$, and l sufficiently large,*

$$\|(V^\# R_{V_0}^{\mathrm{reg}}(\zeta))^j V^\# \mathcal{P}_l\|_{H_{(0,8)}(X) \to L^2(X)} = O(l^{-3})$$

uniformly for $\zeta \in D_l(\lambda_0, \epsilon)$.

Proof. By Lemma 8.5,

$$\|\mathcal{P}_l (V^\# R_{V_0}^{\mathrm{reg}})^2 V^\# \mathcal{P}_l\|_{H_{(0,n+6)} \to H_{(0,n)}} = O(l^{-3}).$$

This gives

$$\begin{aligned}
(V^\# R_{V_0}^{\mathrm{reg}})^3 V^\# \mathcal{P}_l &= V^\# R_{V_0}^{\mathrm{reg}} (I - \mathcal{P}_l)(V^\# R_{V_0}^{\mathrm{reg}})^2 V^\# \mathcal{P}_l + V^\# R_{V_0}^{\mathrm{reg}} \mathcal{P}_l (V^\# R_{V_0}^{\mathrm{reg}})^2 V^\# \mathcal{P}_l \\
&= V^\# R_{V_0}^{\mathrm{reg}} (I - \mathcal{P}_l)(V^\# R_{V_0}^{\mathrm{reg}})^2 V^\# \mathcal{P}_l + O(l^{-3}) \qquad (8\text{-}9)
\end{aligned}$$

as an operator from $H_{(0,n+6)}(X)$ to $H_{(0,n)}(X)$. Using that $\mathcal{P}_l$ commutes with R_{V_0} and $\|\mathcal{P}_l V^\# \mathcal{P}_l\| = O(l^{-N})$ for any N gives

$$(V^\# R_{V_0}^{\text{reg}})^2 V^\# \mathcal{P}_l = (V^\# R_{V_0}(I-\mathcal{P}_l))^2 V^\# \mathcal{P}_l + V^\# R_{V_0}^{\text{reg}} \mathcal{P}_l V^\# R_{V_0}(I-\mathcal{P}_l) V^\# \mathcal{P}_l + O_{L^2\to L^2}(l^{-N}).$$

Using this in (8-9) yields

$$\begin{aligned}(V^\# R_{V_0}^{\text{reg}})^3 V^\# \mathcal{P}_l &= (V^\# R_{V_0}(I-\mathcal{P}_l))^3 V^\# \mathcal{P}_l \\ &\quad + V^\# R_{V_0}(I-\mathcal{P}_l) V^\# R_{V_0}^{\text{reg}} \mathcal{P}_l V^\# R_{V_0}(I-\mathcal{P}_l) V^\# \mathcal{P}_l + O_{H_{(0,n+6)}\to H_{(0,n)}}(l^{-3}).\end{aligned} \tag{8-10}$$

For large l, Lemma 8.1 applied with $J=1$ shows that

$$\|(V^\# R_{V_0}(I-\mathcal{P}_l))^3 V^\# \mathcal{P}_l\|_{H_{(0,6)}(X)\to L^2(X)} = O(l^{-3}).$$

Choose $\chi \in C_c^\infty(X)$ independent of θ so that $V\chi = V$. We write the second term on the right in (8-10) as the composition of three operators, with the grouping indicated below by the brackets:

$$\begin{aligned}&V^\# R_{V_0}(I-\mathcal{P}_l) V^\# R_{V_0}^{\text{reg}} \mathcal{P}_l V^\# R_{V_0}(I-\mathcal{P}_l) V^\# \mathcal{P}_l \\ &\qquad = \big[V^\# R_{V_0}(I-\mathcal{P}_l) V^\#\big]\big[\chi R_{V_0}^{\text{reg}} \mathcal{P}_l \chi\big]\big[\mathcal{P}_l V^\# R_{V_0}(I-\mathcal{P}_l) V^\# \mathcal{P}_l\big].\end{aligned} \tag{8-11}$$

By Lemma 8.1,

$$\|V^\# R_{V_0}(I-\mathcal{P}_l) V^\#\|_{H_{(0,n+2)}\to H_{(0,n)}} = O(l^{-1}).$$

The second operator, $\chi R_{V_0}^{\text{reg}} \mathcal{P}_l \chi$, is bounded. By Lemma 8.3, the third is $O(l^{-2})$ as an operator from $H_{(0,n+6)}$ to $H_{(0,n)}$. Thus we have proved the lemma when $j=3$.

The case of $j>3$ follows from the $j=3$ case. □

We now can prove Proposition 8.2.

Proof of Proposition 8.2. For l sufficiently large, on $D_l(\lambda_0, \epsilon)$,

$$\mathcal{P}_l (I + V^\# R_{V_0}^{\text{reg}}(\zeta)\chi)^{-1} V^\# \mathcal{P}_l = \mathcal{P}_l \sum_{j=0}^{\infty} (-V^\# R_{V_0}^{\text{reg}}(\zeta)\chi)^j V^\# \mathcal{P}_l.$$

The proposition then follows from an application of Lemmas 8.3, 8.5, and 8.6, and recalling that $\|\mathcal{P}_l V^\# \mathcal{P}_l\| = O(l^{-N})$. □

The proof of Theorem 1.6 uses the next lemma, which computes an expression related to the leading term of

$$\mathcal{P}_l (I + V^\# R_{V_0}^{\text{reg}}(\zeta_l(z)))^{-1} V^\# \mathcal{P}_l.$$

Lemma 8.7. *Suppose $V \in C_c^\infty(X)$ and $u \in H^2(\mathbb{R}^d)$ satisfies $(-\Delta_0 + V_0 - \lambda_0^2)u = 0$. Then*

$$-\int_{\mathbb{R}^d} u((z^2-k^2)V_{-k}V_k u - V_{-k}(-\Delta_0+V_0)(V_k u))\,dx = \int_{\mathbb{R}^d} ((k^2+\lambda_0^2-z^2)u^2 V_{-k}V_k + u^2 \nabla_0 V_{-k}\cdot\nabla_0 V_k)\,dx.$$

Proof. We first compute $\int_{\mathbb{R}} uV_{-k}(-\Delta_0+V_0)(V_k u)\,dx$. Expanding and then integrating by parts yields

$$\begin{aligned}\int_{\mathbb{R}^d} & uV_{-k}(-\Delta_0+V_0)(V_k u)\,dx\\ &= -\int_{\mathbb{R}^d}(u^2V_{-k}\Delta_0 V_k+2V_{-k}u\nabla_0 V_k\cdot\nabla_0 u)\,dx+\int_{\mathbb{R}^d} uV_{-k}V_k(-\Delta_0+V_0)u\,dx\\ &= -\int_{\mathbb{R}^d} u^2V_{-k}\Delta_0 V_k\,dx+\int_{\mathbb{R}^d} u^2\sum_{j=1}^d \frac{\partial}{\partial x_j}\Big(V_{-k}\frac{\partial}{\partial x_j}V_k\Big)\,dx+\lambda_0^2\int_{\mathbb{R}^d} u^2V_{-k}V_k\,dx\\ &= \int_{\mathbb{R}^d} u^2\nabla_0 V_{-k}\cdot\nabla_0 V_k\,dx+\lambda_0^2\int_{\mathbb{R}^d} u^2V_{-k}V_k\,dx. \end{aligned}\tag{8-12}$$

Using this, we find

$$\int_{\mathbb{R}^d}((z^2-k^2)V_{-k}V_ku^2-uV_{-k}(-\Delta_0+V_0)(V_ku))\,dx=-\int_{\mathbb{R}^d}(((k^2+\lambda_0^2-z^2)V_{-k}V_k+\nabla_0V_{-k}\cdot\nabla_0V_k)u^2)\,dx,$$

completing the proof. □

The proof of the next lemma uses some of the same ideas as that of Proposition 8.2. This result will be used in the proof of Theorem 1.7.

Lemma 8.8. *Suppose $V\in C_c^\infty(X;\mathbb{R})$. Let $\lambda_0\in i\mathbb{R}$ be a simple pole of $R_{V_0,0}(\lambda)$ with residue of rank 1. Let $M>|\lambda_0|$ and $N\in\mathbb{N}$, and suppose $R_{V_0,0}(\lambda)-\Xi(R_{V_0,0}(\lambda),\lambda_0)$ is analytic for $|\lambda-\lambda_0|\le\epsilon$. Then if $\chi\in C_c^\infty(X;\mathbb{R})$ is independent of θ and satisfies $V\chi=V$, there exists an $s=s(N)\in\mathbb{N}$ and an $A_N=A_N(\tau_l,l):H_{(0,s)}(X)\to L^2(X)$ so that for l sufficiently large,*

$$\|\mathcal{P}_l(I+V^\#R^{\mathrm{reg}}_{V_0,0}(\zeta)\chi)^{-1}V^\#\mathcal{P}_l-A_N(\tau_l(\zeta),l)\|_{H_{(0,s)}(X)\to L^2(X)}=O(l^{-N})\tag{8-13}$$

uniformly for $\zeta\in\overline{D}_l(\lambda_0,\epsilon)$. Moreover, $A_N(z,l)$ depends analytically on z in the set $\{z\in\mathbb{C}:|z-\lambda_0|\le\epsilon\}$ and if $z\in i\mathbb{R}$, then $A_N(z,l)$ is symmetric on $C_c^\infty(X)\subset L^2(X)$. Furthermore,

$$\|\mathcal{P}_{l\pm}A_N\mathcal{P}_{l\mp}\|_{H_{(0,s)}(X)\to L^2(X)}=O(l^{-N})$$

for any N.

Proof. By Lemma 5.2, if $j>2N$, then on $\overline{D}_l(\lambda_0,\epsilon)$ we have

$$\|(V^\#R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^j\|_{L^2(X)\to L^2(X)}=O(l^{-N}).$$

Thus

$$\left\|(I+V^\#R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1}-\sum_{j=0}^{2N}(-V^\#R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^j\right\|_{L^2(X)\to L^2(X)}=O(l^{-N}).\tag{8-14}$$

Now we write, for l sufficiently large,

$$R^{\mathrm{reg}}_{V_0}=R^{\mathrm{reg}}_{V_0}\mathcal{P}_l+R_{V_0}(I-\mathcal{P}_l).\tag{8-15}$$

From our assumptions on V_0 and the pole of $R_{V_0,0}$ at λ_0, there is a $u \in C^\infty(\mathbb{R}^d;\mathbb{R})$ so that for $|\lambda-\lambda_0|\le\epsilon$, $R_{V_0,0}(\lambda)-i/(\lambda-\lambda_0)u\otimes u$ is analytic. Then for l sufficiently large

$$R^{\mathrm{reg}}_{V_0}(\zeta)\mathcal{P}_l = R^{\mathrm{reg}}_{V_0}(\zeta;\lambda_0,l)\mathcal{P}_l = R_{V_0,0}(\tau_l(\zeta))\mathcal{P}_l - \frac{i}{\tau_l(\zeta)-\lambda_0}(u\otimes u)\mathcal{P}_l.$$

If $\tau_l=\tau_l(\zeta)\in i\mathbb{R}$ and $\zeta\in\bar{D}_l(\lambda_0,\epsilon)$, the operator $\chi R^{\mathrm{reg}}_{V_0}(\zeta)\mathcal{P}_l\chi$ is symmetric on $C_c^\infty(X)$. On the other hand, for $k\ne l$, writing τ_k for $\tau_k(\zeta)$ and using Lemma 8.1 yields

$$\begin{aligned}\chi R_{V_0}\mathcal{P}_k\chi &= \chi R_{V_0,0}(\tau_k)\mathcal{P}_k\chi\\ &= -\chi\sum_{j=1}^{N}\frac{1}{(\tau_l^2+l^2-k^2)^j}(-\Delta_0+V_0)^{j-1}\mathcal{P}_k\chi\\ &\qquad + \chi\frac{1}{(\tau_l^2+l^2-k^2)^N}R_{V_0}(\tau_k)(-\Delta_0+V_0)^N\mathcal{P}_k\chi. \end{aligned}\tag{8-16}$$

If $\tau_l^2\in\mathbb{R}$, then

$$\chi\frac{1}{(\tau_l^2+l^2-k^2)^j}(-\Delta_0+V_0)^{j-1}\mathcal{P}_k\chi$$

is symmetric on $C_c^\infty(X)$. Set

$$T_N=T_N(\tau_l,l)=R^{\mathrm{reg}}_{V_0,0}(\tau_l)\mathcal{P}_l-\sum_{k\ne l}\sum_{j=1}^{N}\frac{1}{(\tau_l^2+l^2-k^2)^j}(-\Delta_0+V_0)^{j-1}\mathcal{P}_k. \tag{8-17}$$

Note that T_N is an analytic operator-valued function of τ_l for $\zeta\in\bar{D}_l(\lambda_0,\epsilon)$, where $|\tau_l-\lambda_0|\le\epsilon$. Using (8-16),

$$\|\chi(R^{\mathrm{reg}}_{V_0}-T_N)\chi\|_{H_{(0,2N+t)}(X)\to H_{(0,t)}(X)}=O(l^{-N}),$$

if $|\tau_l-\lambda_0|\le\epsilon$, and $\chi T_N(\tau_l,l)\chi$ is symmetric on $C_c^\infty(X)$ if $\tau_l\in i\mathbb{R}$. Moreover, by (8-14),

$$\left\|(I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta_l(\tau_l))\chi)^{-1}-\sum_{j=0}^{2N}(-V^\# T_N(\tau_l,l))^j\chi\right\|_{H_{(0,s(N))}\to L^2}=O(l^{-N})$$

if $s(N)\ge 4N^2$. Thus if we define

$$A_N=A_N(\tau_l,l)=\mathcal{P}_l\sum_{j=0}^{2N}(-V^\# T_N)^j V^\#\mathcal{P}_l \tag{8-18}$$

then A_N satisfies (8-13), A_N is an analytic function of τ_l if $|\tau_l-\lambda_0|\le\epsilon$, and $A_N(\tau_l,l)$ is symmetric on $C_c^\infty(X)$ if $\tau_l\in i\mathbb{R}$.

To show that $\|\mathcal{P}_{l\pm}A_N\mathcal{P}_{l\mp}\|_{H_{(0,s)}\to L^2}=O(l^{-N})$, consider a term $\mathcal{P}_{l+}(V^\# T_N)^j V^\#\mathcal{P}_{l-}$. We write

$$\mathcal{P}_{l+}(V^\# T_N)^j V^\#\mathcal{P}_{l-}=\sum_{\substack{m_1+m_2+\cdots+m_{j+1}=2l\\ m_k\ne 0}} V_{m_1}e^{im_1\theta}T_N V_{m_2}e^{im_2\theta}T_N\cdots V_{m_j}e^{im_j\theta}T_N V_{m_{j+1}}e^{im_{j+1}\theta}\mathcal{P}_{l-}.$$

Thus we see that at least one m_n must have absolute value at least $2l/(j+1)$. Since $\|V_m\|_{C^r} = O(|m|^{-p})$ for any fixed r, any p, we obtain

$$\|\mathcal{P}_{l+}(V^\# T_N)^j V^\# \mathcal{P}_{l-}\|_{H_{(0,s)}\to L^2} = O(l^{-N})$$

for some sufficiently large s. Thus the result for $\mathcal{P}_{l+} A_N \mathcal{P}_{l-}$ follows from our expression (8-18) for A_N. The result for $\mathcal{P}_{l-} A_N \mathcal{P}_{l+}$ follows similarly. □

9. Proofs of the smooth case of Theorem 1.2 and Theorem 1.3

The first application of our results in the previous section is to improve the localization of the resonances when $V \in C_c^\infty(X)$.

Proof of Theorem 1.2 for $V \in C_c^\infty(X)$. Let $\lambda_j \in \Lambda_\rho$ and choose $\epsilon > 0$ so that there are no poles of $R_{V_0,0}(\lambda)$ in $0 < |\lambda - \lambda_j| \le \epsilon$. We will show that there is a $C_j > 0$ so that there are no poles of $R_V(\zeta)$ in $\zeta \in D_l(\lambda_j, \epsilon)$ with $|\tau_l(\zeta) - \lambda_j| > C_j l^{-2/(m_{V_0,0}(\lambda_j))}$ when l is sufficiently large.

Choose $\chi \in C_c^\infty(X)$ so that $\chi V = V$ and χ is independent of θ. As previously, if l is sufficiently large,

$$R_{V_0}^{\mathrm{reg}}(\zeta) = R_{V_0}^{\mathrm{reg}}(\zeta; \lambda_j, l) = R_{V_0}(\zeta) - \Xi(R_{V_0,0}, \lambda_j)|_{\lambda=\tau_l(\zeta)} \mathcal{P}_l$$

and note that $R_{V_0}^{\mathrm{reg}}(\zeta; \lambda_j, l)$ is analytic on $\overline{D}_l(\lambda_j, \epsilon)$. By (3-7), any poles of $R_V(\zeta)$ in $D_l(\lambda_j, \epsilon)$ are points at which $I + \mathcal{P}_l(I + V^\# R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}, \lambda_j)\chi\mathcal{P}_l$ has nontrivial null space.

Using the smoothness of V, for any fixed $s \in \mathbb{N}$ there is a constant $C > 0$ (depending on s, V_0, λ_j) with

$$\|V^\# \Xi(R_{V_0,0}, \lambda_j)|_{\lambda=\tau_l(\zeta)} \chi \mathcal{P}_l\|_{L^2(X)\to H_{(0,s)}(X)} \le \frac{C}{|\tau_l(\zeta) - \lambda_j|^{m_{V_0,0}(\lambda_j)}}, \tag{9-1}$$

[Dyatlov and Zworski 2019, Theorems 2.5, 2.7, 3.9, and 3.17]. Thus on $D_l(\lambda_j, \epsilon)$, for l sufficiently large by Proposition 8.2,

$$\|\mathcal{P}_l(I + V^\# R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}, \lambda_j)|_{\lambda=\tau_l(\zeta)} \chi \mathcal{P}_l\|_{L^2(X)\to L^2(X)} \le \frac{C}{l^2|\tau_l(\zeta) - \lambda_j|^{m_{V_0,0}(\lambda_j)}},$$

for some C. Thus there is a $C_j > 0$ so that if $\zeta \in D_l(\lambda_j, \epsilon)$ and $|\tau_l(\zeta) - \lambda_j| > C_j l^{-2/m_{V_0,0}(\lambda_j)}$, then $I + \mathcal{P}_l(I + V^\# R_{V_0}^{\mathrm{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}, \lambda_j)\mathcal{P}_l$ is invertible, and ζ is not a resonance.

Since $\lambda_j \in \Lambda_\rho$ is arbitrary, Λ_ρ contains only finitely many elements and we have already proved the theorem for the case of an L^∞ potential V, this suffices to prove the smooth version of the theorem. □

The proof of the smooth case of Theorem 1.3 is almost identical, given our earlier results.

Proof of Theorem 1.3 for $V \in C_c^\infty(X)$. Recall that we have already proved the L^∞ case of this theorem. Thus, the proof follows just as in the proof of the smooth case of Theorem 1.2, except that estimate (9-1) is replaced by

$$\|V^\# \Xi(R_{V_0,0}, 0)|_{\lambda=\tau_l(\zeta)} \chi \mathcal{P}_l\|_{L^2(X)\to H_{(0,s)}(X)} \le \frac{C}{|\tau_l(\zeta)|^r}. \qquad \square$$

10. Proofs of Theorems 1.6 and 1.7

We prove Theorems 1.6 and 1.7 in this section, using results of Section 8.

Before turning to the proofs of the theorems, we say something more about the function u of (1-3). The mapping properties of the resolvent mean that for any $\epsilon > 0$ away from its poles, we have the map

$$R_{V_0,0}(\lambda) : e^{-(\epsilon+\max(0,-\operatorname{Im}\lambda))|x|}L^2(\mathbb{R}^d) \to e^{(\epsilon+\max(0,-\operatorname{Im}\lambda))|x|}L^2(\mathbb{R}^d).$$

With $R_{V_0,0}(\lambda)^t$ denoting the transpose, we have the symmetry $R_{V_0,0}(\lambda)^t = R_{V_0,0}(\lambda)$, checked first for $\operatorname{Im}\lambda > 0$ and then holding by analytic continuation for all λ. This implies that if $R_{V_0,0}(\lambda)$ has a simple pole of rank 1 at λ_0, then there is a $u \in e^{(\epsilon+\max(0,-\operatorname{Im}\lambda))|x|}L^2(\mathbb{R}^d)$ so that (1-3) holds, where the operator $u\otimes u$ is understood as an operator between weighted L^2 spaces.

Now we turn more directly to the proofs, beginning with a preliminary lemma.

Lemma 10.1. *Let λ_0 be a pole of $R_{V_0,0}$ and set $R^{\mathrm{reg}}_{V_0}(\zeta) = R^{\mathrm{reg}}_{V_0}(\zeta;\lambda_0,l)$. Let $\chi \in C_c^\infty(X)$ be independent of θ and satisfy $\chi V = V$, with χ nontrivial. Suppose $R_{V_0,0}(\lambda)$ is analytic for $0 < |\lambda-\lambda_0| \le \epsilon$. Then there is an $L > 0$ so that for $l > L$, if $\zeta_0 \in D_l(\lambda_0,\epsilon)$, then*

$$M(I + V^\# R_{V_0}(\zeta)\chi, \zeta_0) = M(I + (I + V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}(\lambda),\lambda_0)\restriction_{\lambda=\tau_l(\zeta)}\mathcal{P}_l, \zeta_0).$$

Proof. By Lemma 5.2, there is an $L > 0$ so that $I + V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi$ is invertible on $D_l(\lambda_0,\epsilon)$ for $l > L$. Then if $l > L$ and $\zeta_0 \in D_l(\lambda_0,\epsilon)$,

$$\begin{aligned} M(I+V^\# R_{V_0}\chi,\zeta_0) &= M((I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)(I+(I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1}V^\#\Xi(R_{V_0,0}(\lambda),\lambda_0)\restriction_{\lambda=\tau_l(\zeta)}\mathcal{P}_l),\zeta_0) \\ &= M(I+(I+V^\# R^{\mathrm{reg}}_{V_0}(\zeta)\chi)^{-1}V^\#\Xi(R_{V_0,0}(\lambda),\lambda_0)\restriction_{\lambda=\tau_l(\zeta)}\mathcal{P}_l,\zeta_0), \end{aligned}$$

where the second equality uses Lemma 4.1. □

Given $f \in C_c^\infty(\mathbb{R}^d)$, define $h_{\pm l} \in C_c^\infty(X)$ by $h_{\pm l}(x,\theta) = f(x)e^{\pm il\theta}/\sqrt{2\pi}$. For $z_0 \in \mathbb{C}$ and an operator $A : H_{(0,s)}(X) \to L^2(X)$ set

$$\mathcal{D}_A(z) = \det\left(I + \frac{i}{z-z_0}(Ah_l\otimes h_{-l} + Ah_{-l}\otimes h_l)\right). \tag{10-1}$$

Here "det" is the Fredholm determinant. In this special case it is easily calculated to be

$$\mathcal{D}_A(z) = \frac{1}{(z-z_0)^2}\left\{\left(z-z_0+i\int_X h_{-l}(Ah_l)\right)\left(z-z_0+i\int_X h_l(Ah_{-l})\right)+\int_X h_{-l}(Ah_{-l})\int_X h_l(Ah_l)\right\}. \tag{10-2}$$

Proposition 10.2. *Let $z_0 \in \mathbb{C}$, $\epsilon > 0$, and set $U_\epsilon = \{z \in \mathbb{C} : |z-z_0| < \epsilon\}$. Suppose there are L_1, $m_0 \ge \frac{1}{2}$ and $s \in \mathbb{N}$ so that for $l > L_1$, $l \in \mathbb{N}$ and $z \in U_\epsilon$ there are linear operators $S_l = S_l(z)$ and $T_l = T_l(z)$ mapping $H_{(0,s)}(X)$ to $L^2(X)$ which are operator-valued functions analytic on U_ϵ satisfying:*

- $\sup_{z\in U_\epsilon}\|\mathcal{P}_l S_l(z)\mathcal{P}_l - T_l(z)\mathcal{P}_l\|_{H_{(0,s)}(X)\to L^2(X)} = O(l^{-m_0})$,
- $T_l(z)\mathcal{P}_l = \mathcal{P}_{l+}T_l(z)\mathcal{P}_{l+} + \mathcal{P}_{l-}T_l(z)\mathcal{P}_{l-}$ *and* $\sup_{z\in U_\epsilon}\|T_l(z)\|_{H_{(0,s)}(X)\to L^2(X)} = O(l^{-1/2})$.

Then given $f \in C_c^\infty(\mathbb{R}^d)$, for l sufficiently large the functions $(z-z_0)^2\mathcal{D}_{S_l}(z)$ and $(z-z_0)^2\mathcal{D}_{T_l}(z)$ have exactly two zeros, counted with multiplicity, in U_ϵ, and they lie in $U_{\epsilon/2}$. Moreover, there is a labeling of these two sets of zeros as $z_{S_l\pm}$ and $z_{T_l\pm}$, so that $|z_{S_l\pm} - z_{T_l\pm}| = O(l^{-m_0})$.

Proof. By translating if necessary, we may assume $z_0 = 0$.

Our assumptions on T_l imply that $F_\pm(z) = F_\pm(z; l) \stackrel{\text{def}}{=} z + i \int_X h_{\mp l}(T_l(z)h_{\pm l})$ is analytic on U_ϵ and satisfies $F_\pm(z) = z + O(l^{-1/2})$ uniformly on U_ϵ. Applying Rouché's theorem to the pair $F_\pm(z)$ and the function z, we see that $F_\pm$ has, for l sufficiently large, exactly one zero in the set $U_{\epsilon/4}$ and no zeros in $U_\epsilon \setminus U_{\epsilon/4}$. We label this zero as $z_{T_l\pm}$. Since $\int_X h_{\pm l}(T_l h_{\pm l}) = 0$, we have that $z^2 \mathcal{D}_{T_l}(z) = F_+(z)F_-(z)$ and $z_{T_l\pm}$ are the zeros of $z^2\mathcal{D}_{T_l}$.

We write

$$F_\pm(z; l) = z + i \int_X h_{\mp l}(T_l(z)h_{\pm l}) = (z - z_{T_l\pm})\varphi_\pm(z; l), \tag{10-3}$$

with $\varphi_\pm$ analytic on U_ϵ for l sufficiently large. An application of the maximum principle shows that there is a $C > 0$ independent of l so that for l sufficiently large,

$$\frac{1}{C} \le |\varphi_\pm(z; l)| \le C \quad \text{for all } z \in U_{3\epsilon/4}. \tag{10-4}$$

Next consider the intermediary

$$G_\pm(z) = G_\pm(z; l) \stackrel{\text{def}}{=} z + i \int_X h_{\mp l}(S_l(z)h_{\pm l}) = z + i \int_X h_{\mp l}(T_l(z)h_{\pm l}) + O(l^{-m_0}).$$

Our estimate $G_\pm - F_\pm = O(l^{-m_0})$, (10-3), and (10-4) allow an application of Rouché's theorem to the pair $F_\pm$, $G_\pm$ on a disk with center $z_{T_l\pm}$ and radius $c_0 l^{-m_0}$ for an appropriate choice of $c_0 > 0$ and for l sufficiently large. This shows that for l sufficiently large, $G_\pm$ has exactly one zero (counting multiplicity) in $U_{\epsilon/3}$. We label this zero $z_{I,l,\pm}$ (the "I" here stands for intermediate, as this is an intermediate step). We have shown $|z_{I,l,\pm} - z_{T_l,\pm}| = O(l^{-m_0})$. As before, by the maximum principle we may write

$$G_\pm(z; l) = (z - z_{I,l,\pm})\varphi_{I\pm}(z; l), \quad \text{with } \frac{1}{C} \le |\varphi_{I\pm}(z; l)| \le C, \quad \text{for all } z \in U_{3\epsilon/4} \tag{10-5}$$

for some constant C independent of l, and for l sufficiently large.

Now consider $z^2\mathcal{D}_{S_l}(z)$. By our assumptions on S_l and T_l,

$$z^2\mathcal{D}_{S_l}(z) = G_+(z)G_-(z) + O(l^{-2m_0}) = (z - z_{I,l,+})(z - z_{I,l,-})\varphi_{I+}(z)\varphi_{I-}(z) + O(l^{-2m_0}).$$

Thus we can apply Rouché's theorem again, this time to the pair $z^2\mathcal{D}_{S_l}(z)$ and $G_+(z; l)G_-(z; l)$ at a distance proportional to l^{-m_0} of $z_{I,l,\pm}$, proving the proposition. □

We apply this proposition in the proof of Theorem 1.6.

Proof of Theorem 1.6. We assume that $V^\# \not\equiv 0$, since otherwise there is nothing to prove. Choose $\chi \in C_c^\infty(X)$ with $\chi V = V$, and χ independent of θ.

Let $R_{V_0}^{\text{reg}}(\zeta) = R_{V_0}^{\text{reg}}(\zeta; \lambda_0, l)$, and let ϵ, $L > 0$ be as in Lemma 10.1. For $l > L$ the function

$$F_l(\zeta) \stackrel{\text{def}}{=} (\tau_l(\zeta) - \lambda_0)^2 \det(I + (I + V^\# R_{V_0}^{\text{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}, \lambda_0)|_{\lambda=\tau_l(\zeta)} \mathcal{P}_l)$$

is analytic on $D_l(\lambda_0, \epsilon)$. Moreover, the order of vanishing of F_l at $\zeta_0 \in D_l(\lambda_0, \epsilon)$ is given by

$$M(I + (I + V^\# R_{V_0}^{\text{reg}}(\zeta)\chi)^{-1} V^\# \Xi(R_{V_0,0}, \lambda_0)|_{\lambda=\tau_l(\zeta)} \mathcal{P}_l, \zeta_0) + m_{V_0}(\zeta_0),$$

see [Gohberg and Sigal 1971, Theorem 5.1]. Note that for $\zeta_0 \in D_l(\lambda_0, \epsilon)$ and l sufficiently large, $m_{V_0}(\zeta_0) \neq 0$ if and only if $\tau_l(\zeta_0) = \lambda_0$. For $\lambda_0 \neq 0$, combining this with Lemmas 10.1 and 4.4, we see that the poles of R_V in $D_l(\lambda_0, \epsilon)$ are, for $l > L$, given by the zeros of F_l, and the multiplicities agree. If $\lambda_0 = 0$, the same is true, but as in the proof of Theorem 1.3 we use Lemmas 5.6, 5.7, and 5.9.

To prove the theorem, we will apply Proposition 10.2 with the following choices: $z = \tau_l(\zeta)$, $z_0 = \lambda_0$, $f(x) = \chi(x)u(x)$ so that $h_{\pm l}(x, \theta) = \chi(x)u(x)e^{\pm il\theta}/\sqrt{2\pi}$,

$$S_l = S_l(z) = (I + V^\# R_V^{\text{reg}}(\zeta_l(z)))^{-1} V^\# \mathcal{P}_l,$$

$$T_l = T_l(z) = \frac{-1}{l^2} \sum_{k\neq 0} \left(\frac{z^2 - k^2}{4k^2} V_{-k} V_k - \frac{1}{4k^2} V_{-k}(-\Delta_0 + V_0) V_k \right) \mathcal{P}_l,$$

and $s = 8$. By Proposition 8.2 we have, in the notation of Proposition 10.2, $m_0 = 3$. Note that using the coordinate $z = \tau_l(\zeta)$, we have $F_l(\zeta_l(z)) = (z - \lambda_0)^2 \mathcal{D}_{S_l}(z)$, where $\mathcal{D}_{S_l}$ is as defined via (10-1).

The function $(z - \lambda_0)^2 \mathcal{D}_{T_l}(z)$ has a single zero of multiplicity 2 in U_ϵ, and by Lemma 8.7 this is the zero of

$$z - \lambda_0 + \frac{i}{4l^2} \sum_{k\neq 0} \int_{\mathbb{R}} \left(\frac{k^2 + \lambda_0^2 - z^2}{k^2} u^2 V_{-k} V_k + \frac{u^2 \nabla_0 V_{-k} \cdot \nabla_0 V_k}{k^2} \right)$$

near $z = \lambda_0$. This zero is given by

$$z_{T_l\pm} = \lambda_0 - \frac{i}{4l^2} \sum_{k\neq 0} \int_{\mathbb{R}} \left(u^2 V_{-k} V_k + \frac{u^2 \nabla_0 V_{-k} \cdot \nabla_0 V_k}{k^2} \right) + O(l^{-4}).$$

By Proposition 10.2, the zeros of $(z - \lambda_0)^2 \mathcal{D}_{S_l}(z)$ in U_ϵ are within $O(l^{-m_0}) = O(l^{-3})$ of the zero (of multiplicity 2) of $(z - \lambda_0)^2 \mathcal{D}_{T_l}(z)$ in U_ϵ, thus completing the proof. □

The proof of Theorem 1.7 is similar.

Proof of Theorem 1.7. We prove the theorem by showing that for any $N \in \mathbb{N}$ there is an $\epsilon > 0$ so that for $l \in \mathbb{N}$ sufficiently large if $\zeta_l^\flat \in D_l(\lambda_0, \epsilon)$ and $\zeta_l^\flat$ is a pole of $R_V(\zeta)$, then $\operatorname{Re} \tau_l(\zeta_l^\flat) = O(l^{-N})$.

Choose $\chi \in C_c^\infty(X; \mathbb{R})$ so that $\chi V = V$ and χ is independent of θ. Choose $\epsilon,\ L > 0$, as in Lemma 10.1. Let $u \in C^\infty(\mathbb{R}^d)$ be such that $R_{V_0,0}(\lambda) - i/(\lambda - \lambda_0) u \otimes u$ is analytic for λ near λ_0. Our assumptions on V and λ_0 imply that u is real-valued. We apply Proposition 10.2 in a way very similar to the proof of Theorem 1.6. We make the following choices: $z = \tau_l(\zeta)$, $z_0 = \lambda_0$, $h_{\pm l}(x, \theta) = \chi(x)u(x)e^{\pm il\theta}/\sqrt{2\pi}$, and $S_l = S_l(z) = (I + V^\# R_V^{\text{reg}}(\zeta_l(z))\chi)^{-1} V^\# \mathcal{P}_l$, where $R_V^{\text{reg}}(\zeta) = R_V^{\text{reg}}(\zeta; \lambda_0, l)$. For l sufficiently large, S_l is analytic on U_ϵ. Let $A_N = A_N(z, l)$ be the operator from Lemma 8.8, and set

$$T_l = T_l(z; N) = \mathcal{P}_{l+} A_N \mathcal{P}_{l+} + \mathcal{P}_{l-} A_N \mathcal{P}_{l-}.$$

By Lemma 8.8, there is an $s \in \mathbb{N}$ so that

$$\|\mathcal{P}_l S_l(z) \mathcal{P}_l - T_l(z)\|_{H_{(0,s)}(X) \to L^2(X)} = O(l^{-N})$$

uniformly for $z \in U_\epsilon$. Thus for our application of Proposition 10.2 we have $m_0 = N$.

Following the proof of Theorem 1.6, the poles of R_V in $D_l(\lambda_0, \epsilon)$ are determined by the zeros of $(z - \lambda_0)^2 \mathcal{D}_{S_l}(z)$ in U_ϵ, using $U_\epsilon \ni z = \tau_l(\zeta)$. By Proposition 10.2, these zeros are approximated by those of $(z-\lambda_0)^2\mathcal{D}_{T_l}(z)$ in U_ϵ, with an error which is $O(l^{-N})$. We compete the proof by showing that for l sufficiently large the zeros of $\mathcal{D}_{T_l}(z)$ in U_ϵ lie on the imaginary axis.

Set $a_\pm(z;l) \overset{\text{def}}{=} \int_X h_{\mp l}(T_l(z)h_{\pm l}) = \int_X \overline{h_{\pm l}}(T_l(z)h_{\pm l})$. From Lemma 8.8 and the definition of T_l, if $z \in U_\epsilon \cap i\mathbb{R}$, then $T_l(iz)$ is symmetric on $C_c^\infty(X) \subset L^2(X)$. In particular, this implies that if $z \in i\mathbb{R} \cap U_\epsilon$ then $a_\pm(z;l) \in \mathbb{R}$. Since $a_\pm(z;l)$ is analytic for $z \in U_\epsilon$ and is real-valued for $z \in i\mathbb{R} \cap U_\epsilon$, we must have

$$a_\pm(z;l) = \bar{a}_\pm(-\bar{z};l) \quad \text{for } z \in U_\epsilon. \tag{10-6}$$

We remark that since $\lambda_0 \in i\mathbb{R}$, we have $z \in U_\epsilon$ if and only if $-\bar{z} \in U_\epsilon$.

From the proof of Proposition 10.2, the zeros of $(z-\lambda_0)^2\mathcal{D}_{T_l}(z)$ in U_ϵ are given by the zeros of $z-\lambda_0+ia_\pm(z,l)$ in U_ϵ, and there is, for l sufficiently large, exactly one such zero for each choice of $\pm$. We denote these zeros by $z_{T_l\pm}$ and focus on the zero for the "+" sign, z_{T_l+}. Using $\lambda_0 \in i\mathbb{R}$,

$$\begin{aligned} z_{T_l+} - \lambda_0 + ia_+(z_{T_l+};l) = 0 = \overline{z_{T_l+} - \lambda_0 + ia_+(z_{T_l+};l)} &= -(-\overline{z_{T_l+}} - \lambda_0 + i\bar{a}_+(z_{T_l+};l)) \\ &= -(-\overline{z_{T_l+}} - \lambda_0 + ia_+(-\overline{z_{T_l+}};l)), \end{aligned}$$

where the last equality uses (10-6). Hence $-\overline{z_{T_l+}}$ is also a zero of $z-\lambda_0+ia_+(z;l)$ in U_ϵ, and since there is exactly one such zero, it must be that $-\overline{z_{T_l+}} = z_{T_l+}$, and thus $z_{T_l+} \in i\mathbb{R}$. The same argument shows $z_{T_l-} \in i\mathbb{R}$. □

11. Proof of Theorem 1.8, the resonant uniqueness of $V \equiv 0$ when $d = 1$

Theorem 1.8, a result on the resonant rigidity of the zero potential on $\mathbb{R} \times \mathbb{S}^1$, follows rather directly from Theorems 1.1, 1.3, and 1.6.

Proof of Theorem 1.8. Suppose $X = \mathbb{R} \times \mathbb{S}^1$ and V is as in Theorem 1.8. Then by Theorems 1.1 and 1.3, the one-dimensional operator $-d^2/dx^2 + V_0$ on $\mathbb{R}$ must have a resonance at the origin and nowhere else, and this resonance must have multiplicity 1. But since $V_0 \in L_c^\infty(\mathbb{R})$, by well-known results for one-dimensional Schrödinger operators, $V_0 \equiv 0$; see for example [Zworski 1987].

The operator $R_{0,0}(\lambda) - i/(2\lambda)1 \otimes 1$ is analytic at the origin. Using this in Theorem 1.6 along with the fact that R_V has poles at a sequence of thresholds tending to infinity, we find

$$\sum_{k\neq 0} \frac{1}{k^2} \int_{\mathbb{R}} (k^2 V_k V_{-k} + V_k' V_{-k}')(x)\,dx = 0.$$

But since $V_{-k}(x) = \overline{V}_k(x)$ for a real-valued potential V, this implies $V_k \equiv 0$ for all k, and hence $V \equiv 0$. □

12. The potential $V(x,\theta) = 2\chi_{I_0}(x)\cos\theta$ on $\mathbb{R} \times \mathbb{S}^1$

In this section we investigate the resonances near the l-th threshold of the Schrödinger operator with potential $V(x,\theta) = 2\chi_{I_0}(x)\cos\theta$ on $X = \mathbb{R} \times \mathbb{S}^1$. Here $\chi_{I_0}(x)$ is the characteristic function of the interval $I_0 = [-1,1]$, so $\chi_{I_0}(x) = 1$ if $|x| \leq 1$ and $\chi_{I_0}(x) = 0$ if $|x| > 1$. This potential has $V_0 \equiv 0$ so that $V^\# = V$. Proposition 12.3 shows that the resonances nearest the threshold, which correspond to perturbations of the pole at the origin for $R_{0,0}(\lambda)$, are, for this potential, localized in a different way than for smooth potentials;

compare Theorem 1.6. By Proposition 12.6, there is a sense in which Theorem 1.5 is sharp. We remark that some of the computations of this section are reminiscent of those found in [Drouot 2018, Section 2].

In all of this section,

$$V(x,\theta)=2\chi_{I_0}(x)\cos\theta \quad \text{and} \quad X=\mathbb{R}\times\mathbb{S}^1.$$

We will use this preliminary lemma.

Lemma 12.1. *For $\lambda,\ \lambda'\in\mathbb{C},\ \lambda\neq\pm\lambda'$,*

$$\chi_{I_0}R_{0,0}(\lambda)\chi_{I_0}R_{0,0}(\lambda')\chi_{I_0} = \frac{1}{(\lambda')^2-\lambda^2}\chi_{I_0}(R_{0,0}(\lambda')-R_{0,0}(\lambda))\chi_{I_0}+\frac{i}{4\lambda\lambda'(\lambda+\lambda')}e^{i(\lambda+\lambda')}(\phi_\lambda\otimes\phi_{\lambda'}+\phi_{-\lambda}\otimes\phi_{-\lambda'}), \tag{12-1}$$

where

$$\phi_{\pm\lambda}(x)=e^{\pm i\lambda x}\chi_{I_0}(x).$$

Moreover, if $\tau\in\mathbb{C}$, $\tau\neq\pm\lambda$, applying the operator $\chi_{I_0}R_{0,0}(\tau)$ to the function $\chi_{I_0}(x)e^{i\lambda x}$ yields

$$(\chi_{I_0}R_{0,0}(\tau)\chi_{I_0}e^{i\lambda\star})(x)=\chi_{I_0}(x)\left(\frac{1}{\lambda^2-\tau^2}e^{i\lambda x}+\frac{1}{2\tau(\lambda-\tau)}e^{-i\lambda}e^{i\tau(1+x)}+\frac{1}{2\tau(\tau+\lambda)}e^{i\lambda}e^{i\tau(1-x)}\right). \tag{12-2}$$

Proof. The first can be seen, for example, by using (3-1), the explicit expression for the Schwartz kernel of $R_{0,0}$, and evaluating

$$\int_{-1}^{1}e^{i\lambda|x-x''|+i\lambda'|x''-x'|}\,dx''$$

for $|x|,\ |x'|\le 1$. Likewise, (12-2) follows from an explicit computation using (3-1). □

12A. *Resonances near the threshold $\tau_l=0$ for $V(x,\theta)=2\chi_{I_0}(x)\cos\theta$.* Since in this section we concentrate on the resonance near the threshold, we work on $B_l(1)$. A preliminary step is the following.

Lemma 12.2. *Let $R_0^{\mathrm{reg}}(\zeta)=R_0^{\mathrm{reg}}(\zeta;0,l)$. Then for l sufficiently large, uniformly on $B_l(1)$,*

$$\|\mathcal{P}_l((I+VR_0^{\mathrm{reg}}(\zeta)\chi_{I_0})^{-1}V+VR_0^{\mathrm{reg}}(\zeta)V+(VR_0^{\mathrm{reg}}(\zeta))^3V)\mathcal{P}_l\|=O(l^{-2}).$$

Proof. Using the Neumann series,

$$(I+VR_0^{\mathrm{reg}}(\zeta)\chi_{I_0})^{-1}V=\sum_{j=0}^{\infty}(-VR_0^{\mathrm{reg}}(\zeta))^jV.$$

By Lemma 5.2, $\|(-VR_0^{\mathrm{reg}}(\zeta))^j\|=O(l^{-2})$ on $B_l(1)$ if $j\ge 4$ and l is sufficiently large. This ensures the Neumann series for $(I+VR_0^{\mathrm{reg}}(\zeta)\chi_{I_0})^{-1}$ converges, and

$$\left\|(I+VR_0^{\mathrm{reg}}(\zeta)\chi_{I_0})^{-1}V-\sum_{j=0}^{3}(-VR_0^{\mathrm{reg}}(\zeta))^jV\right\|=O(l^{-2})$$

on $B_l(1)$.

Now we note that our explicit expression for V means that $\mathcal{P}_lV\mathcal{P}_l=0$. Likewise, it implies that $\mathcal{P}_l(VR_0^{\mathrm{reg}}(\zeta))^2V\mathcal{P}_l=0$, completing the proof. □

Proposition 12.3. *For l sufficiently large, the poles of $R_V(\zeta)$ in $B_l(1)$ satisfy*

$$\tau_l(\zeta) = \frac{1}{4l\sqrt{2l}}(-1-i+e^{i2\sqrt{2l}}) + O(l^{-2}).$$

Proof. We give a proof similar to that of Theorem 1.6 using Proposition 10.2.

Let R_0^{reg} be as in Lemma 12.2, and restrict ζ to $\zeta \in B_l(1)$. Note

$$R_{0,0}(\lambda) - \frac{i}{2\lambda} 1 \otimes 1$$

is regular at $\lambda = 0$. Set $z = \tau_l(\zeta)$,

$$S_l(z) = (I + V R_0^{\mathrm{reg}}(\zeta_l(z))\chi_{I_0})^{-1} V \mathcal{P}_l, \quad \text{and} \quad h_{\pm l}(x,\theta) = \frac{1}{\sqrt{2\pi}} \chi_{I_0}(x) e^{\pm il\theta}$$

We use $\mathcal{D}_{S_l}$ as is defined by (10-1) and U_ϵ as in Proposition 10.2. Then just as in the proof of Theorem 1.6, the poles of R_V in $B_l(1)$ are identified via $z = \tau_l(\zeta)$ with the zeros of $z^2 \mathcal{D}_{S_l}(z)$ in U_1. Set $z_0 = 0$ and $T_l = \mathcal{P}_l(-V R_0^{\mathrm{reg}}(\zeta) V - (V R_0^{\mathrm{reg}}(\zeta))^3 V)\mathcal{P}_l$. Then by Lemma 12.2, in our application of Proposition 10.2 we can take $s = 0$ and $m_0 = 2$. We claim that uniformly for $z \in U_1$,

$$z^2 \mathcal{D}_{T_l}(z) = \left(z + \frac{1}{2(2l)^{3/2}}(1 - e^{2i\sqrt{2l}} + i) + O(l^{-2}) \right)^2. \tag{12-3}$$

Assuming for the moment that (12-3) holds, this shows that the two zeros (when counted with multiplicity) of $z^2 \mathcal{D}_{T_l}(z)$ in U_1 satisfy

$$z = \frac{-1 - i + e^{2i\sqrt{2l}}}{2(2l)^{3/2}} + O(l^{-2}).$$

An application of Proposition 10.2 and Lemma 12.2 then proves the proposition.

We now turn to showing (12-3). We use

$$R_0^{\mathrm{reg}}(\zeta_l(z)) V \mathcal{P}_l = \sum_{\pm} (e^{\pm i\theta} R_{0,0}(\tau_{l+1}) + e^{\mp i\theta} R_{0,0}(\tau_{l-1}))\chi_{I_0} \mathcal{P}_{l\pm}, \tag{12-4}$$

where $\tau_{l\pm 1} = \tau_{l\pm 1}(\zeta_l(z))$, so that

$$\mathcal{P}_l V R_0^{\mathrm{reg}}(\zeta_l(z)) V \mathcal{P}_l = \chi_{I_0}(R_{0,0}(\tau_{l-1}) + R_{0,0}(\tau_{l+1}))\chi_{I_0} \mathcal{P}_l. \tag{12-5}$$

Then using (12-2) gives

$$\int_X h_{\mp l} V R_0^{\mathrm{reg}}(\zeta_l(z)) V h_{\pm l} = \frac{-i}{2(2l)^{3/2}}(1 - e^{2i\sqrt{2l}}) + \frac{1}{2(2l)^{3/2}} + O(l^{-2}) \tag{12-6}$$

uniformly on U_1. Now note

$$\int_X h_{\mp l}(V R_0^{\mathrm{reg}})^3 V h_{\pm l} = \int_X (V R_0^{\mathrm{reg}} V h_{\mp l})(\chi_{I_0}(R_0^{\mathrm{reg}} V)^2 h_{\pm l}). \tag{12-7}$$

By (12-2),

$$\|(V R_0^{\mathrm{reg}} V h_{\mp l})\| = O(l^{-1}) \quad \text{and} \quad \|\chi_{I_0}(R_0^{\mathrm{reg}} V)^2 h_{\pm l}\| = O(l^{-1}).$$

Using the expression for $\mathcal{D}_{T_l}$ as in (10-2) and equations (12-5)–(12-7) completes the proof of (12-3). ☐

12B. ***Existence of poles of R_V within $\approx \log l$ of the l-th threshold, for $V(x,\theta) = 2\chi_{I_0}(x)\cos\theta$.*** As a point of comparison with Theorem 1.5, for the special case $V(x,\theta) = 2\chi_{I_0}(x)\cos\theta$ on $X = \mathbb{R}\times\mathbb{S}^1$ we consider the existence of poles of $R_V(\zeta)$ in $D_l(\alpha\log l)$ with $|\tau_l(\zeta)| > 1$.

Again, we use the coordinate $z = \tau_l(\zeta)$ on $B_l(\alpha\log l)$, and the functions ϕ_λ are as defined in Lemma 12.1.

Lemma 12.4. *Let $\alpha > 0$ be fixed, and set $z = \tau_l(\zeta)$. For l sufficiently large, uniformly on $B_l(\alpha\log l)\setminus B_l(1)$ we have*

$$\left\| \mathcal{P}_l(I + VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1} VR_0(\zeta)\chi_{I_0}\mathcal{P}_l + (f_+\otimes\phi_z + f_-\otimes\phi_{-z})\mathcal{P}_l - \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l \right\| \\ = O\left(\frac{1}{l^{5/2}}e^{2(\operatorname{Im} z)_-}\right) + O(l^{-3/2}), \quad (12\text{-}8)$$

where

$$f_\pm(x) = f_\pm(x,z,l) = \frac{ie^{iz}}{4z}\chi_{I_0}(x)\left(\frac{e^{i\tau_{l+1}}}{\tau_{l+1}(z+\tau_{l+1})}\phi_{\pm\tau_{l+1}} + \frac{e^{i\tau_{l-1}}}{\tau_{l-1}(z+\tau_{l-1})}\phi_{\pm\tau_{l-1}}\right).$$

For notational simplicity, we have written $\tau_{l\pm1}$ for $\tau_{l\pm1}(\zeta_l(z))$.

Proof. We use

$$(I + VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1} = \sum_{j=0}^{\infty}(-VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^j$$

since $\|VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l)\| = O(l^{-1/2})$. This estimate, along with others in this proof, are uniform for $\zeta\in B_l(\alpha\log l)\setminus B_l(1)$. By Lemma 12.1, (3-1), and the explicit expression for V, we see that

$$\|\chi_{I_0}R_0(\zeta)(I-\mathcal{P}_l)VR_0(\zeta)\chi_{I_0}\mathcal{P}_l\| = O(e^{2(\operatorname{Im} z)_-}/(l|z|)) \quad \text{for } \zeta\in B_l(\alpha\log l)$$

for l sufficiently large. Moreover, this same lemma implies that if $|j-l|\le 2$, then

$$\|\chi_{I_0}(VR_0(\zeta)(I-\mathcal{P}_l))^2\chi_{I_0}\mathcal{P}_j\| = O(l^{-3/2})$$

uniformly on $B_l(\alpha\log l)$. This ensures that

$$\left\| \left((I + VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1} - \sum_{j=0}^{2}(-VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^j\right) VR_0(\zeta)\chi_{I_0}\mathcal{P}_l \right\| \\ = O\left(\frac{1}{l^{5/2}|z|}e^{2(\operatorname{Im} z)_-}\right). \quad (12\text{-}9)$$

Since, as in the proof of Proposition 12.3, $\mathcal{P}_lV\mathcal{P}_l = 0$ and $\mathcal{P}_l(VR_0(I-\mathcal{P}_l))^2V\mathcal{P}_l = 0$, it suffices to use $-\mathcal{P}_lVR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l)VR_0(\zeta)\mathcal{P}_l$ to approximate $\mathcal{P}_l(I + VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1}VR_0(\zeta)\chi_{I_0}\mathcal{P}_l$ with the desired accuracy.

Using Lemma 12.1 and its notation,

$$\begin{aligned}
&\mathcal{P}_l V R_0(\zeta_l(z))(I-\mathcal{P}_l) V R_0(\zeta_l(z))\chi_{I_0}\mathcal{P}_l \\
&\quad = \chi_{I_0}(R_{0,0}(\tau_{l+1})\chi_{I_0}R_{0,0}(z) + R_{0,0}(\tau_{l-1})\chi_{I_0}R_{0,0}(z))\chi_{I_0}\mathcal{P}_l \\
&\quad = \frac{1}{\tau_{l+1}^2 - z^2}\chi_{I_0}(R_{0,0}(\tau_{l+1}) - R_{0,0}(z))\chi_{I_0}\mathcal{P}_l + \frac{i e^{i(z+\tau_{l+1})}}{4z\tau_{l+1}(z+\tau_{l+1})}(\phi_{\tau_{l+1}}\otimes\phi_z + \phi_{-\tau_{l+1}}\otimes\phi_{-z})\mathcal{P}_l \\
&\qquad\quad + \frac{1}{\tau_{l-1}^2 - z^2}\chi_{I_0}(R_{0,0}(\tau_{l-1}) - R_{0,0}(z))\chi_{I_0}\mathcal{P}_l + \frac{i e^{i(z+\tau_{l-1})}}{4z\tau_{l-1}(z+\tau_{l-1})}(\phi_{\tau_{l-1}}\otimes\phi_z + \phi_{-\tau_{l-1}}\otimes\phi_{-z})\mathcal{P}_l.
\end{aligned}$$

Note that

$$\left\| \frac{1}{\tau_{l\pm1}^2 - z^2}\chi_{I_0}R_{0,0}(\tau_{l\pm1})\chi_{I_0}\right\| = O(l^{-3/2})$$

and

$$\left\| \left(\frac{1}{\tau_{l+1}^2 - z^2} + \frac{1}{\tau_{l-1}^2 - z^2}\right)\chi_{I_0}R_{0,0}(z)\chi_{I_0} - \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\right\| = O(l^{-4}|z|^{-1}e^{2(\operatorname{Im} z)_-}).$$

This gives

$$\begin{aligned}
&\mathcal{P}_l(V R_0(\zeta_l(z))(I-\mathcal{P}_l) V R_{0,0}(\zeta_l(z))\chi_{I_0}\mathcal{P}_l \\
&= \frac{i e^{i(z+\tau_{l+1})}}{4z\tau_{l+1}(z+\tau_{l+1})}(\phi_{\tau_{l+1}}\otimes\phi_z + \phi_{-\tau_{l+1}}\otimes\phi_{-z})\mathcal{P}_l + \frac{i e^{i(z+\tau_{l-1})}}{4z\tau_{l-1}(z+\tau_{l-1})}(\phi_{\tau_{l-1}}\otimes\phi_z + \phi_{-\tau_{l-1}}\otimes\phi_{-z})\mathcal{P}_l \\
&\qquad - \frac{1}{2l^2}R_{0,0}(z)\mathcal{P}_l + O_{L^2\to L^2}\left(\frac{1}{l^{5/2}|z|}e^{2(\operatorname{Im} z)_-}\right) + O_{L^2\to L^2}(l^{-3/2}), \qquad (12\text{-}10)
\end{aligned}$$

and completes the proof. □

Note that the functions $f_\pm$ and $\phi_\pm$ in Lemma 12.4 depend holomorphically on z in the set

$$\{z \in \mathbb{C} : 1 \le z \le \alpha \log l\}.$$

The function g_l of the next lemma appears in the proof of Proposition 12.6, as its zeros approximate the locations of the poles of $R_V(\zeta)$ away from the threshold in $B_l(\alpha \log l)$, if $\alpha < 1$. A discussion of the Lambert W function can be found, for example, in [Corless et al. 1996]. This next lemma is very similar to [Drouot 2018, Lemma 2.4].

Lemma 12.5. *The zeros of*

$$g_l(z) \stackrel{\text{def}}{=} \left(1 - \frac{1}{z8l\sqrt{2l}}e^{2i(\sqrt{2l}+z)}\right)^2 - \left(\frac{1}{8lz\sqrt{2l}}(ie^{2iz} + e^{2iz})\right)^2$$

are given by $z_\nu^\pm = z_\nu^\pm(l) = \frac{i}{2}\mathcal{W}_\nu((-ie^{2i\sqrt{2l}} \mp i \pm 1)/(4l\sqrt{2l}))$, *where* $\mathcal{W}_\nu$ *is the* ν*-th branch of the Lambert W function. In particular, we have* $z_1^+ \sim -\frac{3i}{4}\log l$. *Moreover, for* l *sufficiently large there is an* $r_0 > 0$ *independent of* l *so that if* $w \in \mathbb{C}$ *and* $|w| < r_0$, *then*

$$|g_l(z_1^+(l) + w)| \ge \tfrac{2}{3}|w|. \qquad (12\text{-}11)$$

Proof. The zeros of g_l are solutions of

$$1-\frac{1}{z8l\sqrt{2l}}e^{2i(\sqrt{2l}+z)}=\pm\frac{1}{8lz\sqrt{2l}}(ie^{2iz}+e^{2iz})$$

and so satisfy

$$ze^{-2iz}=\frac{1}{8l\sqrt{2l}}(e^{2i\sqrt{2l}}\pm 1\pm i).$$

Solutions of this equation are given by

$$z_\nu^{\pm}=\frac{i}{2}\mathcal{W}_\nu\biggl(\frac{1}{4l\sqrt{2l}}(-ie^{2i\sqrt{2l}}\mp i\pm 1)\biggr).$$

From [Corless et al. 1996, (4.20)], we have $z_1^+\sim -\frac{3i}{4}\log l$ as $l\to\infty$.

To finish the proof, we set $\gamma=1/(8l\sqrt{2l})$ and write

$$g_l(z)=\biggl(1+\frac{\gamma}{z}e^{2iz}(-e^{2i\sqrt{2l}}-1-i)\biggr)\biggl(1+\frac{\gamma}{z}e^{2iz}(-e^{2i\sqrt{2l}}+1+i)\biggr).$$

Now we evaluate at $z=z_1^++w$, with $w\in\mathbb{C}$, $|w|$ small, to find

$$\begin{aligned}g_l(z_1^++w)&=\biggl(1+\frac{z_1^+e^{2iw}}{z_1^++w}\frac{\gamma}{z_1^+e^{-2iz_1^+}}(-e^{2i\sqrt{2l}}-1-i)\biggr)\biggl(1+\frac{z_1^+e^{2iw}}{z_1^++w}\frac{\gamma}{z_1^+e^{-2iz_1^+}}(-e^{2i\sqrt{2l}}+1+i)\biggr)\\&=\biggl(1-\frac{z_1^+e^{2iw}}{z_1^++w}\biggr)\biggl(1+\frac{z_1^+e^{2iw}}{z_1^++w}\frac{-e^{2i\sqrt{2l}}+1+i}{e^{2i\sqrt{2l}}+1+i}\biggr),\end{aligned}$$

where for the second equality we have used $z_1^+e^{-2iz_1^+}=\gamma(e^{2i\sqrt{2l}}+1+i)$. This gives, then, recalling $|z_1^+|\to\infty$ as $l\to\infty$,

$$g_l(z_1^++w)=(-2iw+O(|w|/|z_1^+|)+O(|w|^2))\biggl(\frac{2(i+1)}{e^{2i\sqrt{2l}}+1+i}+O(|w|)\biggr)$$

for $|w|$ small. Then there is a $r_0>0$ independent of l so that for l sufficiently large and $|w|<r_0$, $|g_l(z_1^++w)|>\frac{2}{3}|w|$. $\square$

Proposition 12.6. *For $V(x,\theta)=2\chi_{I_0}(x)\cos\theta$ and l sufficiently large, $R_V(\zeta)$ has a pole at a point $\zeta_l^+\in B_l\bigl(\frac{7}{8}\log l\bigr)$ with ζ_l^+ satisfying*

$$\tau_l(\zeta_l^+)=\frac{i}{2}\mathcal{W}_1\biggl(\frac{1}{4l\sqrt{2l}}(ie^{2i\sqrt{2l}}-i+1)\biggr)+O(l^{-1/2+\epsilon})$$

for any $\epsilon>0$.

Proof. We continue to use $z=\tau_l(\zeta)$ and work in a region with $1<|z|<\frac{7}{8}\log l$.

Using Lemma 12.4,

$$\mathcal{P}_l(I+VR_0(\zeta)\chi_{I_0}(I-\mathcal{P}_l))^{-1}VR_0(\zeta)\chi_{I_0}\mathcal{P}_l=F\mathcal{P}_l+\frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l+A,$$

where, with notation from Lemma 12.4,

$$F=F(z,l)=-f_+\otimes\phi_z-f_-\otimes\phi_{-z}$$

and $\|A\| = O(l^{-5/2}e^{(2\operatorname{Im} z)_-}) + O(l^{-3/2})$ on $B_l\big(\frac{7}{8}\log l\big) \setminus B_1(l)$. We recall that the poles of R_V in $B_l\big(\frac{7}{8}\log l\big) \setminus B_1(l)$ are the zeros of $I + \mathcal{P}_l(I + V R_0(\zeta)\chi_{I_0}(I - \mathcal{P}_l))^{-1} V R_0(\zeta)\chi_{I_0}\mathcal{P}_l$ in $B_l\big(\frac{7}{8}\log l\big) \setminus B_1(l)$. We write

$$I + \mathcal{P}_l(I + V R_0(\zeta)\chi_{I_0}(I - \mathcal{P}_l))^{-1} V R_0(\zeta)\chi_{I_0}\mathcal{P}_l = \left(I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l\right)\left(I + \left(I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l\right)^{-1}(F\mathcal{P}_l + A)\right) \tag{12-12}$$

since

$$I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l$$

is invertible here. For notational convenience, set

$$S = S_l = \left(I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l\right)^{-1},$$

and note that

$$S = I - \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\mathcal{P}_l + O_{L^2\to L^2}(l^{-4}e^{4(\operatorname{Im} z)_-}).$$

We first consider the poles of $I + SF\mathcal{P}_l$. These poles are given by the zeros of the function

$$\widetilde{\mathcal{D}}_l(z) \overset{\text{def}}{=} \det(I + SF\mathcal{P}_{l\pm}) = \left(1 - \int_{\mathbb{R}}(Sf_+)\phi_z\right)\left(1 - \int_{\mathbb{R}}(Sf_-)\phi_{-z}\right) - \left(\int_{\mathbb{R}}(Sf_-)\phi_z\right)\left(\int_{\mathbb{R}}(Sf_+)\phi_{-z}\right)$$

with twice the multiplicity. A computation and use of the approximations $\tau_{l+1} = i\sqrt{2l} + O(l^{-1/2})$ and $\tau_{l-1} = \sqrt{2l} + O(l^{-1/2})$ show that

$$\widetilde{\mathcal{D}}_l(z) = g_l(z) + O(l^{-3/2}) + O(l^{-2}\log l\, e^{2(\operatorname{Im} z)_-}),$$

where g_l is the function of Lemma 12.5. We note that both g_l and $\widetilde{\mathcal{D}}_l$ are analytic in z if $1 < |z| < \frac{7}{8}\log l$. We use $z_1^+(l)$ as in Lemma 12.5. Recalling that $\operatorname{Im} z_1^+ \sim -\frac{3}{4}\log l$, the estimate (12-11) combined with Rouché's theorem shows that $\widetilde{\mathcal{D}}_l(z)$ has a zero within $O(l^{-1/2+\epsilon})$, for any $\epsilon > 0$, of $z_1^+(l)$. This, in turn, means that

$$(I + SF\mathcal{P}_l)^{-1} = \left(I + \left(I + \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\right)^{-1}F\mathcal{P}_l\right)^{-1}$$

has a single pole of multiplicity two at a point satisfying $z = z_1^+(l) + O(l^{-1/2+\epsilon})$. Moreover, we can find a $c_0 = c_0(\epsilon)$ so that

$$\left\|\left(I + \left(I - \frac{1}{2l^2}\chi_{I_0}R_{0,0}(z)\chi_{I_0}\right)F\mathcal{P}_l\right)^{-1}\right\| = O(l^{1+\epsilon})$$

when the distance from z to the pole is given by $c_0 l^{-1/2+\epsilon}$.

Now using our estimate on $\|A\|$ we can apply the operator Rouché theorem to the pair $I + SF\mathcal{P}_l$ and $I + SF\mathcal{P}_l + SA$, to find that $I + SF\mathcal{P}_l + SA$ has two poles (when counted with multiplicity) which are, using the z-coordinate, within $O(l^{-1/2+\epsilon})$ of $z_1^+(l)$. □

Acknowledgments

The author gratefully acknowledges the partial support of a University of Missouri Research Leave and a Simons Foundation Collaboration Grant for Mathematicians. Thank you to Kiril Datchev, Alexis Drouot, Adam Helfer, and Alejandro Uribe for helpful conversations. The author is also grateful to an anonymous referee whose comments and questions improved the exposition of the paper.

References

[Autin 2011] A. Autin, "Isoresonant complex-valued potentials and symmetries", *Canad. J. Math.* **63**:4 (2011), 721–754. MR Zbl

[Borisov 2006] D. I. Borisov, "On the spectrum of the Schrödinger operator perturbed by a rapidly oscillating potential", *J. Math. Sci.* (*N.Y.*) **139**:1 (2006), 6243–6322. MR Zbl

[Borisov and Gadylshin 2006] D. I. Borisov and R. R. Gadylshin, "On the spectrum of the Schrödinger operator with a rapidly oscillating compactly supported potential", *Teoret. Mat. Fiz.* **147**:1 (2006), 58–63. In Russian; translated in *Theoret. and Math. Phys.* **147**:1 (2006), 496–500. MR Zbl

[Christiansen 2002] T. Christiansen, "Some upper bounds on the number of resonances for manifolds with infinite cylindrical ends", *Ann. Henri Poincaré* **3**:5 (2002), 895–920. MR Zbl

[Christiansen 2004] T. Christiansen, "Asymptotics for a resonance-counting function for potential scattering on cylinders", *J. Funct. Anal.* **216**:1 (2004), 172–190. MR Zbl

[Christiansen and Datchev 2021] T. J. Christiansen and K. Datchev, "Resolvent estimates on asymptotically cylindrical manifolds and on the half line", *Ann. Sci. École Norm. Sup.* (4) **54**:4 (2021), 1051–1088. MR Zbl

[Christiansen and Datchev 2022] T. J. Christiansen and K. Datchev, "Wave asymptotics for waveguides and manifolds with infinite cylindrical ends", *Int. Math. Res. Not.* **2022**:24 (2022), 19431–19500. MR

[Christiansen and Zworski 1995] T. Christiansen and M. Zworski, "Spectral asymptotics for manifolds with cylindrical ends", *Ann. Inst. Fourier* (*Grenoble*) **45**:1 (1995), 251–263. MR Zbl

[Corless et al. 1996] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function", *Adv. Comput. Math.* **5**:4 (1996), 329–359. MR Zbl

[Dimassi 2016] M. Dimassi, "Semi-classical asymptotics for the Schrödinger operator with oscillating decaying potential", *Canad. Math. Bull.* **59**:4 (2016), 734–747. MR Zbl

[Drouot 2018] A. Drouot, "Scattering resonances for highly oscillatory potentials", *Ann. Sci. École Norm. Sup.* (4) **51**:4 (2018), 865–925. MR Zbl

[Duchêne and Weinstein 2011] V. Duchêne and M. I. Weinstein, "Scattering, homogenization, and interface effects for oscillatory potentials with strong singularities", *Multiscale Model. Simul.* **9**:3 (2011), 1017–1063. MR Zbl

[Duchêne et al. 2014] V. Duchêne, I. Vukićević, and M. I. Weinstein, "Scattering and localization properties of highly oscillatory potentials", *Comm. Pure Appl. Math.* **67**:1 (2014), 83–128. MR Zbl

[Duchêne et al. 2015] V. Duchêne, I. Vukićević, and M. I. Weinstein, "Homogenized description of defect modes in periodic structures with localized defects", *Commun. Math. Sci.* **13**:3 (2015), 777–823. MR Zbl

[Dyatlov and Zworski 2019] S. Dyatlov and M. Zworski, *Mathematical theory of scattering resonances*, Grad. Stud. in Math. **200**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Edward 2002] J. Edward, "On the resonances of the Laplacian on waveguides", *J. Math. Anal. Appl.* **272**:1 (2002), 89–116. MR Zbl

[Exner and Kovařík 2015] P. Exner and H. Kovařík, *Quantum waveguides*, Springer, 2015. MR Zbl

[Gohberg and Sigal 1971] I. C. Gohberg and E. I. Sigal, "An operator generalization of the logarithmic residue theorem and Rouché's theorem", *Mat. Sb.* (*N.S.*) **84(126)** (1971), 607–629. In Russian; translated in *Math. USSR-Sb.* **13** (1971), 603–625. MR Zbl

[Goldstein 1974] C. I. Goldstein, "Meromorphic continuation of the $\mathcal{S}$-matrix for the operator $-\Delta$ acting in a cylinder", *Proc. Amer. Math. Soc.* **42**:2 (1974), 555–562. MR Zbl

[Guillopé 1989] L. Guillopé, "Théorie spectrale de quelques variétés à bouts", *Ann. Sci. École Norm. Sup.* (4) **22**:1 (1989), 137–160. MR Zbl

[Jerison and Kenig 1985] D. Jerison and C. E. Kenig, "Unique continuation and absence of positive eigenvalues for Schrödinger operators", *Ann. of Math.* (2) **121**:3 (1985), 463–494. MR Zbl

[Levitin and Marletta 2008] M. Levitin and M. Marletta, "A simple method of calculating eigenvalues and resonances in domains with infinite regular ends", *Proc. Roy. Soc. Edinburgh Sect. A* **138**:5 (2008), 1043–1065. MR Zbl

[Melrose 1993] R. B. Melrose, *The Atiyah–Patodi–Singer index theorem*, Res. Notes Math. **4**, Peters, Wellesley, MA, 1993. MR Zbl

[Parnovski 1995] L. B. Parnovski, "Spectral asymptotics of the Laplace operator on manifolds with cylindrical ends", *Int. J. Math.* **6**:6 (1995), 911–920. MR Zbl

[Reed and Simon 1978] M. Reed and B. Simon, *Methods of modern mathematical physics*, *IV*: *Analysis of operators*, Academic Press, New York, 1978. MR Zbl

[Tang and Zworski 2000] S.-H. Tang and M. Zworski, "Resonance expansions of scattered waves", *Comm. Pure Appl. Math.* **53**:10 (2000), 1305–1334. MR Zbl

[Weinstein 1977] A. Weinstein, "Asymptotics of eigenvalue clusters for the Laplacian plus a potential", *Duke Math. J.* **44**:4 (1977), 883–892. MR Zbl

[Widom 1979] H. Widom, "Eigenvalue distribution theorems for certain homogeneous spaces", *J. Funct. Anal.* **32**:2 (1979), 139–147. MR Zbl

[Zworski 1987] M. Zworski, "Distribution of poles for scattering on the real line", *J. Funct. Anal.* **73**:2 (1987), 277–296. MR Zbl

Received 11 Dec 2020. Revised 19 Oct 2021. Accepted 4 Feb 2022.

T. J. CHRISTIANSEN: christiansent@missouri.edu
Department of Mathematics, University of Missouri, Columbia, MO, United States

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2023 is US $405/year for the electronic version, and $630/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 16 No. 7 2023