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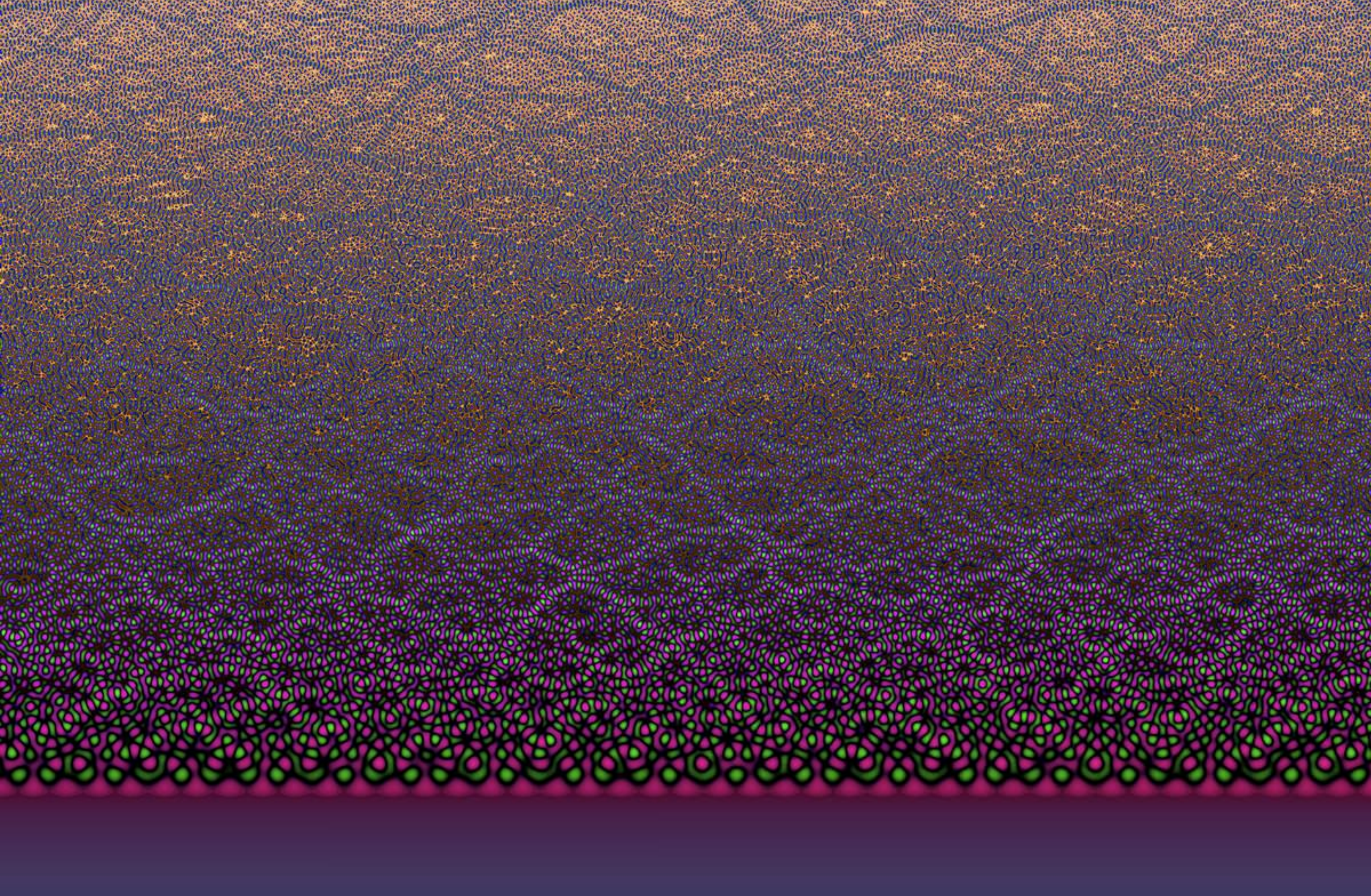

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DISCRETE VELOCITY BOLTZMANN EQUATIONS IN THE PLANE: STATIONARY SOLUTIONS

LEIF ARKERYD AND ANNE NOURI

We prove the existence of stationary mild solutions for normal discrete velocity Boltzmann equations in the plane with no pair of colinear interacting velocities and given ingoing boundary values. We remove an important restriction from a previous paper that all velocities point into the same half-space. A key property is L^1 compactness of integrated collision frequency for a sequence of approximations. This is proven using the Kolmogorov–Riesz theorem, which here replaces the L^1 compactness of velocity averages in the continuous velocity case, not available when the velocities are discrete.

1. Introduction

The Boltzmann equation is the fundamental mathematical model in the kinetic theory of gases. Replacing its continuum of velocities with a discrete set of velocities is a simplification, preserving the essential features of free flow and quadratic collision term. Besides this fundamental aspect, the discrete equations can approximate the Boltzmann equation with any given accuracy [Palczewski et al. 1997; Fainsilber et al. 2006; Mischler 1997], and are thereby useful for approximations and numerics. In the quantum realm they can also be more directly connected to microscopic quasi/particle models. A discrete velocity model of a kinetic gas is a system of partial differential equations having the form,

$$\frac{\partial f_i}{\partial t}(t,z)+v_i\cdot\nabla_z f_i(t,z)=Q_i(f,f)(t,z),\quad t>0,\ z\in\Omega,\ 1\le i\le p,$$

where $f_i(t,z)$, $1\le i\le p$, are phase space densities at time t, position z and velocities v_i. The spatial domain is Ω. The given discrete velocities are v_i, $1\le i\le p$. For $f=(f_i)_{1\le i\le p}$, the collision operator $Q=(Q_i)_{1\le i\le p}$ with gain part Q^+, loss part Q^-, and collision frequency ν, is given by

$$Q_i(f,f)=\sum_{j,l,m=1}^{p}\Gamma_{ij}^{lm}(f_l f_m-f_i f_j)=Q_i^+(f,f)-Q_i^-(f,f),$$

$$Q_i^+(f,f)=\sum_{j,l,m=1}^{p}\Gamma_{ij}^{lm}f_l f_m,\quad Q_i^-(f,f)=f_i\nu_i(f),\quad \nu_i(f)=\sum_{j,l,m=1}^{p}\Gamma_{ij}^{lm}f_j,\quad i=1,\dots,p.$$

The collision coefficients satisfy

$$\Gamma_{ij}^{lm}=\Gamma_{ji}^{lm}=\Gamma_{lm}^{ij}\ge 0. \tag{1-1}$$

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If a collision coefficient Γ_{ij}^{lm} is nonzero, then the conservation laws for momentum and energy,

$$v_i + v_j = v_l + v_m, \quad |v_i|^2 + |v_j|^2 = |v_l|^2 + |v_m|^2, \tag{1-2}$$

are satisfied. We call a pair of velocities (v_i, v_j) interacting if for some $(l, m) \in \{1, \dots, p\}^2$ we have $\Gamma_{ij}^{lm} > 0$. The discrete velocity model (DVM) is called normal (see [Cercignani 1985]) if any solution of the equations

$$\Psi(v_i) + \Psi(v_j) = \Psi(v_l) + \Psi(v_m),$$

where the indices $(i, j; l, m)$ take all possible values satisfying $\Gamma_{ij}^{lm} > 0$, is given by

$$\Psi(v) = a + b \cdot v + c|v|^2$$

for some constants $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^d$. We consider

the generic case of normal coplanar velocity sets
with no pair of colinear interacting velocities (v_i, v_j). (1-3)

The case is generic. Indeed, consider a normal velocity set such that, for some interacting velocities (v_i, v_j), v_i and v_j are colinear. Then there exists an arbitrary small vector v_0 such that the velocity set $(v_i + v_0)_{1 \le i \le p}$ is normal and with no colinear interacting velocities. The paper considers stationary solutions to normal coplanar discrete velocity models satisfying (1-3), in a strictly convex bounded open subset $\Omega \subset \mathbb{R}^2$, with C^2 boundary $\partial\Omega$ and given boundary inflow. Denote by $n(Z)$ the inward normal to $Z \in \partial\Omega$. Denote the v_i-ingoing (resp. v_i-outgoing) part of the boundary by

$$\partial\Omega_i^+ = \{Z \in \partial\Omega : v_i \cdot n(Z) > 0\} \quad (\text{resp. } \partial\Omega_i^- = \{Z \in \partial\Omega : v_i \cdot n(Z) < 0\}).$$

Let

$$s_i^+(z) = \inf\{s > 0 : z - s v_i \in \partial\Omega_i^+\}, \quad s_i^-(z) = \inf\{s > 0 : z + s v_i \in \partial\Omega_i^-\}, \quad z \in \Omega.$$

Write

$$z_i^+(z) = z - s_i^+(z) v_i \quad (\text{resp. } z_i^-(z) = z + s_i^-(z) v_i) \tag{1-4}$$

for the ingoing (resp. outgoing) point on $\partial\Omega$ of the characteristics through z in direction v_i.

The stationary boundary value problem

$$v_i \cdot \nabla f_i(z) = Q_i(f, f)(z), \quad z \in \Omega, \tag{1-5}$$

$$f_i(z) = f_{bi}(z), \quad z \in \partial\Omega_i^+, \ 1 \le i \le p, \tag{1-6}$$

is considered in L^1 in one of the following equivalent forms [DiPerna and Lions 1989]: the exponential multiplier form,

$$\begin{aligned} f_i(z) = f_{bi}(z_i^+(z)) e^{-\int_0^{s_i^+(z)} \nu_i(f)(z_i^+(z) + s v_i)\, ds} \\ + \int_0^{s_i^+(z)} Q_i^+(f, f)(z_i^+(z) + s v_i) e^{-\int_s^{s_i^+(z)} \nu_i(f)(z_i^+(z) + r v_i)\, dr}\, ds, \quad \text{a.a. } z \in \Omega, \ 1 \le i \le p, \end{aligned} \tag{1-7}$$

the mild form,

$$f_i(z) = f_{bi}(z_i^+(z)) + \int_0^{s_i^+(z)} Q_i(f, f)(z_i^+(z) + sv_i)\, ds, \quad \text{a.a. } z \in \Omega,\ 1 \le i \le p, \tag{1-8}$$

the renormalized form,

$$v_i \cdot \nabla \ln(1 + f_i)(z) = \frac{Q_i(f, f)}{1 + f_i}(z), \quad z \in \Omega, \qquad f_i(z) = f_{bi}(z), \quad z \in \partial\Omega_i^+,\ 1 \le i \le p, \tag{1-9}$$

in the sense of distributions. Denote by $L^1_+(\Omega)$ the set of nonnegative integrable functions on Ω. For a distribution function $f = (f_i)_{1 \le i \le p}$, define its entropy (resp. entropy dissipation) by

$$\sum_{i=1}^{p} \int_\Omega f_i \ln f_i(z)\, dz, \quad \left(\text{resp. } \sum_{i,j,l,m=1}^{p} \Gamma_{ij}^{lm} \int_\Omega (f_l f_m - f_i f_j) \ln \frac{f_l f_m}{f_i f_j}(z)\, dz\right).$$

The main result of the paper is:

Theorem 1.1. *Consider a coplanar normal discrete velocity model and a nonnegative ingoing boundary value* f_b *with mass and entropy inflows bounded,*

$$\int_{\partial\Omega_i^+} v_i \cdot n(z)\, f_{bi}(1 + \ln f_{bi})(z)\, d\sigma(z) < +\infty, \quad 1 \le i \le p.$$

For the boundary value problem (1-5)–(1-6) *satisfying* (1-3), *there exists a stationary mild solution in* $(L^1_+(\Omega))^p$ *with finite mass and entropy-dissipation.*

Given $i \in \{1, \dots, p\}$, if $\Gamma_{ij}^{lm} = 0$ for all j, l and m, then f_i equals its ingoing boundary value, and the rest of the system can be solved separately. Such i's are not present in the following discussion. Most mathematical results for stationary discrete velocity models of the Boltzmann equation have been obtained in one space dimension. An overview is given in [Płatkowski and Illner 1988]. Half-space problems [Bernhoff 2012] and weak shock waves [Bernhoff and Bobylev 2007] for discrete velocity models have also been studied. A discussion of normal discrete velocity models, i.e., conserving nothing but mass, momentum and energy, can be found in [Bobylev et al. 2010]. In two dimensions, special classes of solutions to the Broadwell model are given in [Bobylev and Toscani 1996; Bobylev 1996; Ilyin 2014]. The Broadwell model, not included in the present results, is a four-velocity model, with $v_1 + v_2 = v_3 + v_4 = 0$ and v_1, v_3 orthogonal. A detailed study of the stationary Broadwell equation in a rectangle with comparison to a Carleman-like system is given in [Bobylev 1996], as well as a discussion of (in-)compressibility aspects. A main result in [Cercignani et al. 1988] is the existence of continuous solutions to the two-dimensional stationary Broadwell model with continuous boundary data for a rectangle. The paper [Arkeryd and Nouri 2020b] solves that problem in an L^1-setting. The proof uses in an essential way the constancy of the sums $f_1 + f_2$ and $f_3 + f_4$ along characteristics, which no longer holds in the present paper. For every normal model, there is a priori control of entropy dissipation, mass and entropy flows through the boundary. From there, the main difficulties are to prove that for a sequence of approximations, weak L^1 compactness holds and the limit of the collision operator equals the collision operator of the limit. In [Arkeryd and Nouri 2020a], weak L^1 compactness of a sequence of approximations was obtained with assumption (1-3) together with the assumption that all velocities v_i point out into the same half-plane. In this paper we keep assumption (1-3), remove the second assumption and provide a new proof of weak L^1 compactness of approximations using (1-3). Assumption (1-3) is also crucial for proving L^1 compactness of the integrated collision frequencies, which is important for the convergence procedure. Our paper also differs from [Arkeryd and Nouri 2020a] in the limit procedure. The frame of the limit procedure in that paper is the splitting into "good" and "bad" characteristics following the approach in our earlier stationary continuous velocity papers [Arkeryd and Nouri 1995; 1999]. Here we have instead utilized sub- and supersolutions used in the classical evolutionary frame for renormalized solutions to the Boltzmann equation [DiPerna and Lions 1989]. For the continuous velocity evolutionary Boltzmann equation, the compactness properties of the collision frequency use in an essential way the averaging lemma, which is not available for the discrete velocity Boltzmann model. In the present paper, the compactness properties are proven by the Kolmogorov–Riesz theorem. Also the argument used in the stationary paper [Arkeryd and Nouri 1995] in the continuous velocity case for obtaining control of entropy, hence weak L^1 compactness of a sequence of approximations from the control of entropy dissipation, does not work in a discrete velocity case because the number of velocities is finite. The proof starts in Section 2 from bounded approximations. In Section 3, L^1 compactness properties of the approximations are proven. Section 4 is devoted to the proof of Theorem 1.1.

2. Approximations

Denote by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and by $a \wedge b$ the minimum of two real numbers a and b. Let μ_α be a smooth mollifier in $\mathbb{R}^2$ with support in the ball centered at the origin of radius α. Outside the boundary the function to be convolved with μ_α, is continued in the normal direction by its boundary value. Let $\tilde{\mu}_k$ be a smooth mollifier on $\partial\Omega$ in a ball of radius $1/k$. Define

$$f_{bi}^k = \Big(f_{bi}(\,\cdot\,) \wedge \frac{k}{2}\Big) * \tilde{\mu}_k, \quad 1 \le i \le p,\ k \in \mathbb{N}^*.$$

The lemma introduces a primary approximated boundary value problem with damping and convolutions.

Lemma 2.1. *For any $\alpha > 0$ and $k \in \mathbb{N}^*$, there is a solution $F^{\alpha,k} \in (L^1_+(\Omega))^p$ to*

$$\alpha F_i^{\alpha,k} + v_i \cdot \nabla F_i^{\alpha,k} = \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm} \Big(\frac{F_l^{\alpha,k}}{1+F_l^{\alpha,k}/k} \, \frac{F_m^{\alpha,k} * \mu_\alpha}{1+F_m^{\alpha,k} * \mu_\alpha/k} - \frac{F_i^{\alpha,k}}{1+F_i^{\alpha,k}/k} \, \frac{F_j^{\alpha,k} * \mu_\alpha}{1+F_j^{\alpha,k} * \mu_\alpha/k} \Big), \tag{2-1}$$

$$F_i^{\alpha,k}(z) = f_{bi}^k(z), \quad z \in \partial\Omega_i^+,\ 1 \le i \le p. \tag{2-2}$$

Proof of Lemma 2.1. For a proof of Lemma 2.1 we refer to the second section in [Arkeryd and Nouri 2020a]. Let $k \in \mathbb{N}^*$ be given. Each component of $F^{\alpha,k}$ is bounded by a multiple of k^2. Therefore $(F^{\alpha,k})_{\alpha \in]0,1[}$ is weakly compact in $(L^1(\Omega))^p$. For a subsequence, the convergence is strong in $(L^1(\Omega))^p$ as stated in the following lemma. ☐

Lemma 2.2. *There is a sequence $(\beta(q))_{q \in \mathbb{N}}$ tending to zero when $q \to +\infty$ and a function $F^k \in L^1$ such that $(F^{\beta(q),k})_{q \in \mathbb{N}}$ strongly converges in $(L^1(\Omega))^p$ to F^k when $q \to +\infty$.*

Proof of Lemma 2.2. For a proof of Lemma 2.2 we refer to Lemma 3.1 in [Arkeryd and Nouri 2020a]. Define

$$Q_i^{+k} = \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm} \frac{F_l^k}{1+F_l^k/k} \frac{F_m^k}{1+F_m^k/k}, \quad \nu_i^k = \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm} \frac{F_j^k}{(1+F_i^k/k)(1+F_j^k/k)}, \tag{2-3}$$

$$Q_i^k = Q_i^{+k} - F_i^k \nu_i^k, \quad 1 \le i \le p, \tag{2-4}$$

and denote by $\widetilde{D}_k$ the entropy production term of the approximations,

$$\widetilde{D}_k = \sum_{i,j,l,m=1}^{p} \Gamma_{ij}^{lm} \left(\frac{F_l^k}{1+F_l^k/k} \frac{F_m^k}{1+F_m^k/k} - \frac{F_i^k}{1+F_i^k/k} \frac{F_j^k}{1+F_j^k/k} \right) \ln \frac{F_l^k F_m^k (1+F_i^k/k)(1+F_j^k/k)}{(1+F_l^k/k)(1+F_m^k/k) F_i^k F_j^k}. \tag{2-5}$$

All throughout the paper, c_b denotes constants that may vary from line to line but is independent of parameters tending to $+\infty$ or to zero. □

Lemma 2.3. *F^k is a nonnegative solution to*

$$v_i \cdot \nabla F_i^k = Q_i^{+k} - F_i^k \nu_i^k, \tag{2-6}$$

$$F_i^k(z) = f_{bi}^k(z), \quad z \in \partial\Omega_i^+, \ 1 \le i \le p. \tag{2-7}$$

Solutions $(F^k)_{k\in\mathbb{N}^*}$ to (2-6)–(2-7) *have mass and entropy dissipation bounded from above uniformly with respect to k. Moreover their outgoing flows at the boundary are controlled as follows*:

$$\sum_{i=1}^{p} \int_{\partial\Omega_i^-, F_i^k \le k} |v_i \cdot n(Z)| F_i^k \ln F_i^k(Z)\, d\sigma(Z) + \ln \frac{k}{2} \int_{\partial\Omega_i^-, F_i^k > k} |v_i \cdot n(Z)| F_i^k \, d\sigma(Z) \le c_b. \tag{2-8}$$

Proof of Lemma 2.3. Passing to the limit when $q \to +\infty$ in (2-1)–(2-2) written for $F^{\beta(q),k}$, implies that F^k is a solution in $(L_+^1(\Omega))^p$ to (2-6)–(2-7). For a proof of the rest of Lemma 2.3, we refer to Lemma 3.2 in [Arkeryd and Nouri 2020a]. □

3. On compactness of sequences of approximations

This section is devoted to proving L^1 compactness properties of the approximations. In Proposition 3.1, weak L^1 compactness of $(F^k)_{k\in\mathbb{N}^*}$ is proven. Lemma 3.2 splits Ω into a set of i-characteristics with arbitrary small measure and its complement, where both the approximations and their integrated collision frequencies are bounded. In Lemma 3.3, the strong L^1 compactness of integrated collision frequency is proven.

Proposition 3.1. *The sequence $(F^k)_{k\in\mathbb{N}^*}$ solution to* (2-6)–(2-7) *is weakly compact in L^1.*

Proof of Proposition 3.1. By Lemma 2.3, $(F^k)_{k\in\mathbb{N}^*}$ is uniformly bounded in $(L^1(\Omega))^p$. Given (2-8) and the bound

$$F_i^k(z) \le F_i^k(z + s_i^-(z) v_i) \exp\left(\Gamma \sum_{j \in J_i} \int_{-s_i^+(z)}^{s_i^-(z)} F_j(z + r v_i)\, dr \right), \quad z \in \Omega, \ i \in \{1, \dots, p\}, \tag{3-1}$$

on F^k, the weak L^1 compactness of $(F^k)_{k\in\mathbb{N}^*}$ will follow from the uniform boundedness in $L^\infty(\partial\Omega_i^+)$ of

$$\left(\int_0^{s_i^-(Z)} F_j(Z+rv_i)\,dr\right)_{j\in J_i,k\in\mathbb{N}}, \tag{3-2}$$

where J_i denotes the set $\{j\in\{1,\dots,p\} : (v_i,v_j)$ are interacting velocities$\}$. By (1-3), there exists $\eta>0$ such that, for all interacting velocities (v_i,v_j),

$$|\sin(\widehat{v_i,v_j})|>\eta. \tag{3-3}$$

Let $i\in\{1,\dots,p\}$ and $Z\in\partial\Omega_i^+$. Multiply the equation satisfied by F_j^k by $(v_i^\perp\cdot v_j)/|v_i|$ and integrate it on one of the half domains defined by the segment $[Z,Z+s_i^-(Z)v_i]$. Summing over $j\in\{1,\dots,p\}$ implies that

$$\sum_{j=1}^p \sin^2(\widehat{v_i,v_j})\int_0^{s_i^-(Z)} F_j^k(Z+sv_i)\,ds\le c_b,\qquad Z\in\partial\Omega_i^+. \tag{3-4}$$

Together with (3-3), this leads to the control of (3-2). □

Recall the exponential multiplier form for the approximations $(F^k)_{k\in\mathbb{N}^*}$,

$$F_i^k(z)=f_{bi}^k(z_i^+(z))e^{-\int_{-s_i^+(z)}^0 \nu_i^k(z+sv_i)\,ds} + \int_{-s_i^+(z)}^0 Q_i^{+k}(z+sv_i)e^{-\int_s^0 \nu_i^k(F^k)(z+rv_i)\,dr}\,ds,\qquad \text{a.a. } z\in\Omega,\ 1\le i\le p, \tag{3-5}$$

with ν_i^k and Q_i^{+k} defined in (2-3). An i-characteristics is a segment of points $[Z-s_i^+(Z)v_i,Z]$, where $Z\in\partial\Omega_i^-$. Define $\Gamma=\max_{i,j,l,m}\Gamma_{ij}^{lm}$.

Lemma 3.2. *For $i\in\{1,\dots,p\}$, $k\in\mathbb{N}^*$ and $\epsilon>0$, there is a subset $\Omega_i^{k,\epsilon}$ of i-characteristics of Ω with measure smaller than $c_b\epsilon$ such that for any $z\in\Omega\setminus\Omega_i^{k,\epsilon}$*

$$F_i^k(z)\le\frac{1}{\epsilon^2}\exp\left(\frac{p\Gamma}{\epsilon^2}\right),\qquad \int_{-s_i^+(z)}^{s_i^-(z)}\nu_i^k(z+sv_i)\,ds\le\frac{p\Gamma}{\epsilon^2}. \tag{3-6}$$

Proof of Lemma 3.2. By the strict convexity of Ω, there are for every $i\in\{1,\dots,p\}$ two points of $\partial\Omega$, denoted by $\widetilde{Z}_i$ and $\bar{Z}_i$, such that

$$v_i\cdot n(\widetilde{Z}_i)=v_i\cdot n(\bar{Z}_i)=0.$$

Let $\tilde{l}_i$ (resp. $\bar{l}_i$) be the largest boundary arc included in $\partial\Omega_i^-$ with one endpoint $\widetilde{Z}_i$ (resp. $\bar{Z}_i$) such that

$$-\epsilon\le v_i\cdot n(Z)\le 0,\qquad Z\in\tilde{l}_i\cup\bar{l}_i. \tag{3-7}$$

Let J_i be the subset of $\{1,\dots,p\}$ such that,

$$\text{for some } (l,m)\in\{1,\dots,p\}^2,\quad \Gamma_{ij}^{lm}>0,\quad j\in J_i. \tag{3-8}$$

It follows from the exponential form of F_i^k that

$$F_i^k(z)\le F_i^k(z+s_i^-(z)v_i)\exp\left(\Gamma\sum_{j\in J_i}\int_{-s_i^+(z)}^{s_i^-(z)}F_j(z+rv_i)\,dr\right),\qquad z\in\Omega. \tag{3-9}$$

The boundedness of the mass flow of $(F_i^k)_{k\in\mathbb{N}^*}$ across $\partial\Omega_i^-$ is

$$\int_{\partial\Omega_i^-} |v_i\cdot n(Z)|F_i^k(Z)\,d\sigma(Z)\le c_b,\quad k\in\mathbb{N}^*. \tag{3-10}$$

It follows from (3-7)–(3-10) that the measure of the set

$$\Bigl\{Z\in\partial\Omega_i^-\cap\tilde{l}_i^c\cap\bar{l}_i^c : F_i^k(Z)>\frac{1}{\epsilon^2}\Bigr\}$$

is smaller than $c_b\epsilon$. The boundedness of the mass of $(F_j^k)_{k\in\mathbb{N}^*}$ can be written

$$\int_\Omega F_j^k(z)\,dz=\int_{\partial\Omega_i^-}|v_i\cdot n(Z)|\biggl(\int_{-s_i^+(Z)}^0 F_j^k(Z+rv_i)\,dr\biggr)\,d\sigma(Z)\le c_b,\quad j\in J_i.$$

Hence the measure of the set

$$\Bigl\{Z\in\partial\Omega_i^-\cap\tilde{l}_i^c\cap\bar{l}_i^c : \int_{-s_i^+(Z)}^0 F_j^k(Z+rv_i)\,dr>\frac{1}{\epsilon^2}\Bigr\},\quad j\in J_i,$$

is smaller than $c_b\epsilon$. Consequently, the measure of the set of $Z\in\partial\Omega_i^-\cap\tilde{l}_i^c\cap\bar{l}_i^c$ outside of which

$$F_i^k(Z)\le\frac{1}{\epsilon^2}\quad\text{and}\quad\int_{-s_i^+(Z)}^0 F_j^k(Z+rv_i)\,dr\le\frac{1}{\epsilon^2},\quad j\in J_i,$$

is bounded by $c_b\epsilon$. Together with (3-9), this implies that the measure of the complement of the set of $Z\in\partial\Omega_i^-$ such that

$$F_i^k(z)\le\frac{1}{\epsilon^2}\exp\Bigl(\frac{p\Gamma}{\epsilon^2}\Bigr)\quad\text{and}\quad\int_{-s_i^+(z)}^{s_i^-(z)}\nu_i^k(z+rv_i)\,dr\le\frac{p\Gamma}{\epsilon^2}$$

for $z=Z+sv_i$, $s\in[-s_i^+(Z),0]$, is bounded by $c_b\epsilon$. With it $c_b\epsilon$ is a bound for the measure of the complement, denoted by $\Omega_i^{k,\epsilon}$, of the set of i-characteristics in Ω such that for all points z on the i-characteristics, (3-6) holds. ☐

Given $i\in\{1,\dots,p\}$ and $\epsilon>0$, let $\chi_i^{k,\epsilon}$ denote the characteristic function of the complement of $\Omega_i^{k,\epsilon}$. The following lemma proves the compactness in $L^1(\Omega)$ of the k-sequence of integrated collision frequencies.

Lemma 3.3. *The sequences*

$$\biggl(\int_{-s_i^+(z)}^0 \nu_i^k(z+sv_i)\,ds\biggr)_{k\in\mathbb{N}^*},\quad 1\le i\le p,$$

are strongly compact in $L^1(\Omega)$.

Proof of Lemma 3.3. Take $\Gamma_{ij}^{lm}>0$. By (1-3), v_i and v_j span $\mathbb{R}^2$. Denote by (a,b) the corresponding coordinate system, (a^-,a^+) defined by

$$a^-=\min\{a\in\mathbb{R}:(a,b)\in\Omega\text{ for some }b\},\quad a^+=\max\{a\in\mathbb{R}:(a,b)\in\Omega\text{ for some }b\},$$

and by D the Jacobian of the change of variables $z \to (a, b)$. The uniform bound for the mass of $(F^k)_{k\in\mathbb{N}^*}$ proven in Lemma 2.3, implies

$$\left(\int_\Omega \int_{-s_i^+(z)}^0 \nu_i^k(z+sv_i)\,ds\,dz\right)_{k\in\mathbb{N}^*}$$

is bounded in L^1 uniformly with respect to k. Indeed, for some $(b^-(a), b^+(a))$, $a \in [a^-, a^+]$,

$$\begin{aligned}\int_\Omega \int_{-s_i^+(z)}^0 F_j^k(z+sv_i)\,ds\,dz &= D\int_{a^-}^{a^+}\int_{b^-(a)}^{b^+(a)}\int_{-s_i^+(bv_j)}^{a} F_j^k(bv_j+sv_i)\,ds\,db\,da\\ &\le D\int_{a^-}^{a^+}\int_{b^-(a)}^{b^+(a)}\int_{-s_i^+(bv_j)}^{s_i^-(bv_j)} F_j^k(bv_j+sv_i)\,ds\,db\,da\\ &\le c\int_\Omega F_j^k(z)dz, \qquad j\in J_i.\end{aligned}$$

By the Kolmogorov–Riesz theorem [Kolmogorov 1931; Riesz 1933], the compactness of

$$\left(\int_{-s_i^+(z)}^0 \nu_i^k(z+sv_i)\,ds\right)_{k\in\mathbb{N}^*}$$

will follow from its translational equicontinuity in $L^1(\Omega)$. Equicontinuity in the direction v_i, and in the direction v_j with the mild form (1-8) for F_j^k, come naturally. Here the assumption (1-3) becomes crucial. The sequence

$$\left(\int_{-s_i^+(z)}^0 F_j^k(z+sv_i)\,ds\right)_{k\in\mathbb{N}^*}, \qquad j\in J_i, \tag{3-11}$$

is translationally equicontinuous in the v_i-direction. Indeed, $s_i^+(z+hv_i) = s_i^+(z)+h$ so that, denoting by $I(0,h)$ the interval with endpoints 0 and h and using the uniform bound on the mass of $(F_j^k)_{k\in\mathbb{N}^*}$,

$$\begin{aligned}\int_\Omega\left|\int_{-s_i^+(z+hv_i)}^0 F_j^k(z+hv_i+sv_i)\,ds - \int_{-s_i^+(z)}^0 F_j^k(z+sv_i)\,ds\right|dz &= \int_\Omega\int_{s\in I(0,h)} F_j^k(z+sv_i)\,ds\,dz\\ &\le c|h|.\end{aligned}$$

Let us prove the translational equicontinuity of (3-11) in the v_j-direction. By the weak L^1 compactness of $(F_j^k)_{k\in\mathbb{N}^*}$, it is sufficient to prove the translational equicontinuity in the v_j-direction of

$$\left(\int_{s_i^+(z)}^0 \chi_j^{k,\epsilon}F_j^k(z+sv_i)\,ds\right)_{k\in\mathbb{N}^*}.$$

Expressing $F_j^k(z+hv_j+sv_i)$ (resp. $F_j^k(z+sv_i)$) as integral along its v_j-characteristics, it holds that

$$\left|\int_{-s_i^+(z+hv_j)}^0 \chi_j^{k,\epsilon}F_j^k(z+hv_j+sv_i)\,ds - \int_{-s_i^+(z)}^0 \chi_j^{k,\epsilon}F_j^k(z+sv_i)\,ds\right| \le |A_{ij}^k(z,h)| + |B_{ij}^k(z,h)|,$$

where

$$A_{ij}^k(z,h) = \int_{-s_i^+(z+hv_j)}^0 \chi_j^{k,\epsilon}f_{bj}^k(z_j^+(z+hv_j+sv_i))\,ds - \int_{-s_i^+(z)}^0 \chi_j^{k,\epsilon}f_{bj}^k(z_j^+(z+sv_i))\,ds,$$

and

$$B_{ij}^k(z,h)=\int_{-s_i^+(z+hv_j)}^0\int_{-s_j^+(z+hv_j+sv_i)}^0 \chi_j^{k,\epsilon}\,Q_j^k(z+hv_j+sv_i+rv_j)\,dr\,ds \\ -\int_{-s_i^+(z)}^0\int_{-s_j^+(z+sv_i)}^0 \chi_j^{k,\epsilon}\,Q_j^k(z+sv_i+rv_j)\,dr\,ds,$$

with Q_i^k defined in (2-3). Denote by $(z_j^+(z_i^+(z)), z_j^+(z_i^+(z+hv_j)))$ the boundary arc with endpoints $z_j^+(z_i^+(z))$ and $z_j^+(z_i^+(z+hv_j))$ and of length tending to zero with h. Performing the change of variables $s\to Z=z_j^+(z+hv_j+sv_i)$ (resp. $s\to Z=z_j^+(z+sv_i)$) in the first (resp. second) term of $A_{ij}^k(z,h)$, and using that the sequence $(f_{bi}^k)_{k\in\mathbb{N}^*}$ is bounded by f_{bi}, it holds that

$$\lim_{h\to 0}\int_\Omega |A_{ij}^k(z,h)|\,dz=0, \tag{3-12}$$

uniformly with respect to k. Moreover, for some $\omega_h(z)\subset\Omega$ of measure or order $|h|$ uniformly with respect to $z\in\Omega$,

$$B_{ij}^k(z,h)=\int_{\omega_h(z)}\chi_j^{k,\epsilon}\,Q_j^k(Z)\,dZ. \tag{3-13}$$

The sequence $(\chi_j^{k,\epsilon}Q_j^k)_{k\in\mathbb{N}^*}$ is weakly compact in L^1. Indeed,

$$\begin{aligned}\chi_j^{k,\epsilon}Q_j^k&\le\frac{1}{\ln\Lambda}\widetilde{D}_k+\Gamma\Lambda\Big(\sum_{i\in J_j}F_i^k\Big)(\chi_j^{k,\epsilon}F_j^k)\\&\le\frac{1}{\ln\Lambda}\widetilde{D}_k+\frac{\Gamma\Lambda}{\epsilon^2}\exp\Big(\frac{p\Gamma}{\epsilon^2}\Big)\Big(\sum_{i\in J_j}F_i^k\Big),\qquad\Lambda>1,\end{aligned}\tag{3-14}$$

with $(\widetilde{D}_k)_{k\in\mathbb{N}^*}$ uniformly bounded in L^1 and $(F_i^k)_{k\in\mathbb{N}^*}$ weakly compact in L^1. Hence,

$$\lim_{h\to 0}\int_\Omega |B_{ij}^k(z,h)|\,dz=0,\quad\text{uniformly with respect to } k.\qquad\square$$

4. The passage to the limit in the approximations

Let f be the weak L^1 limit of a subsequence of the solutions $(F^k)_{k\in\mathbb{N}^*}$ to (2-6)–(2-7), still denoted by $(F^k)_{k\in\mathbb{N}^*}$. For proving that f is a mild solution of (1-5)–(1-6), it is sufficient to prove that, for any $\eta>0$ and $i\in\{1,\dots,p\}$, there is a set X_i^η of i-characteristics with complementary set of measure smaller than $c\eta$, such that

$$\int_\Omega\varphi\chi_i^\eta f_i(z)\,dz=\int_\Omega\varphi\chi_i^\eta f_{bi}(z_i^+(z))\,dz \\ +\int_\Omega\int_{-s_i^+(z)}^0(\varphi\chi_i^\eta Q_i(f,f)+\chi_i^\eta f_i v_i\cdot\nabla\varphi)(z+sv_i)\,ds\,dz,\quad\varphi\in C^1(\bar{\Omega}), \tag{4-1}$$

where χ_i^η denotes the characteristic function of X_i^η. Define the set X_i^η as follows. For every $\epsilon>0$, pass to the limit when $k\to+\infty$ in

$$\chi_i^{k,\epsilon}F_i^k(z)\le\chi_i^{k,\epsilon}F_i^k(z_i^-(z))\,\exp\Big(\int_{-s_i^+(z)}^{s_i^-(z)}\nu_i^k(z+sv_i)\,ds\Big),\quad\text{a.a. } z\in\Omega,\ k\in\mathbb{N}^*, \tag{4-2}$$

and use the weak L^1 compactness of $(\chi_i^{k,\epsilon} F_i^k)_{k\in\mathbb{N}^*}$, the weak L^1 compactness and the uniform boundedness in L^∞ of $(\chi_i^{k,\epsilon} F_i^k(z_i^-(z)))_{k\in\mathbb{N}^*}$, and the strong L^1 compactness of

$$\left(\int_{-s_i^+(z)}^{s_i^-(z)} \nu_i^k(z+s v_i)\,ds\right)_{k\in\mathbb{N}^*}.$$

It implies

$$F_i^\epsilon(z) \le F_i^\epsilon(z_i^-(z)) \exp\left(\int_{-s_i^+(z)}^{s_i^-(z)} \nu_i(f)(z+s v_i)\,ds\right), \quad \text{a.a. } z\in\Omega,\ \epsilon\in\,]0,1[,$$

where F_i^ϵ is the limit of a subsequence of $(\chi_i^{k,\epsilon} F_i^k)_{k\in\mathbb{N}^*}$ and $\nu_i(f)=\sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm} f_j$. By the monotonicity in ϵ of $(F^\epsilon)_{\epsilon\in]0,1[}$ (resp. $(F^\epsilon(z_i^-(z)))_{\epsilon\in]0,1[}$) and the uniform boundedness of their masses, it holds that

$$f_i(z) \le f_i(z_i^-(z)) \exp\left(\int_{-s_i^+(z)}^{s_i^-(z)} \nu_i(f)(z+s v_i)\,ds\right), \quad \text{a.a. } z\in\Omega.$$

From here the proof follows the lines of the proof of Lemma 3.2, so that given $\eta>0$, there is a set X_i^η of i-characteristics, with complementary set of measure smaller than $c\eta$, such that

$$f_i(z)\le \frac{1}{\eta} e^{p\Gamma/\eta} \quad\text{and}\quad \int_{-s_i^+(z)}^{s_i^-(z)} \nu_i(f)(z+s v_i)\,ds \le \frac{p\Gamma}{\eta}, \quad \text{a.a. } z\in X_\eta. \tag{4-3}$$

Denote by $C_+^1(\bar\Omega)$ the subspace of nonnegative functions of $C^1(\bar\Omega)$.

Lemma 4.1. *The function f is a subsolution of* (1-5)–(1-6), *i.e.*,

$$\int_\Omega \varphi\chi_i^\eta f_i(z)\,dz \le \int_\Omega \varphi f_{bi}(z_i^+(z))\,dz + \int_\Omega\int_{-s_i^+(z)}^0 \chi_i^\eta f_i\, v_i\cdot\nabla\varphi(z+s v_i)\,ds\,dz \\ + \int_\Omega\int_{-s_i^+(z)}^0 \varphi Q_i(f,f)(z+s v_i)\,ds\,dz, \quad 1\le i\le p,\ \varphi\in C_+^1(\bar\Omega). \tag{4-4}$$

Proof of Lemma 4.1. Let $i\in\{1,\dots,p\}$ and $\varphi\in C_+^1(\bar\Omega)$ be given. Write the mild form of $\varphi\chi_i^\eta\chi_i^{k,\epsilon}F_i^k$ and integrate it on Ω. This yields

$$\int_\Omega \varphi\chi_i^\eta\chi_i^{k,\epsilon}F_i^k(z)\,dz = \int_\Omega \varphi\chi_i^\eta\chi_i^{k,\epsilon} f_{bi}^k(z_i^+(z))\,dz + \int_\Omega\int_{-s_i^+(z)}^0 \chi_i^\eta\chi_i^{k,\epsilon}F_i^k v_i\cdot\nabla\varphi(z+s v_i)\,ds\,dz \\ + \int_\Omega\int_{-s_i^+(z)}^0 \varphi\chi_i^\eta\chi_i^{k,\epsilon}(Q_i^{+k}-F_i^k\nu_i^k)(z+s v_i)\,ds\,dz. \tag{4-5}$$

By the weak L^1 compactness of $(F_i^k)_{k\in\mathbb{N}^*}$ and the linearity with respect to $\chi_i^{k,\epsilon}F_i^k$ of the first line of (4-5), its passage to the limit when $k\to+\infty$ is straightforward. Let us pass to the limit when $k\to+\infty$ in any term of the loss term of (4-5), denoted by $\Gamma_{ij}^{lm}L^k$, where

$$L^k := \int_\Omega \chi_i^\eta\chi_i^{k,\epsilon}(z)\int_{-s_i^+(z)}^0 \varphi\frac{F_i^k}{1+F_i^k/k}\,\frac{F_j^k}{1+F_j^k/k}(z+s v_i)\,ds\,dz, \quad j\in J_i, \tag{4-6}$$

and J_i is defined in (3-8). By integration by parts, L_k equals

$$\begin{aligned}\int_\Omega\int_{-s_i^+(z)}^0 \chi_i^\eta\chi_i^{k,\epsilon}(\varphi(Q_i^{+k}-F_i^k\nu_i^k)+(v_i\cdot\nabla\varphi)F_i^k)(z+sv_i)&\\ \times\left(\int_s^0\chi_i^{k,\epsilon}\frac{F_j^k}{(1+F_i^k/k)(1+F_j^k/k)}(z+rv_i)\,dr\right)ds\,dz&\\ +\int_\Omega\chi_i^\eta\chi_i^{k,\epsilon}\varphi\frac{f_{bi}^k}{1+f_{bi}^k/k}(z_i^+(z))\int_{-s_i^+(z)}^0\frac{F_j^k}{1+F_j^k/k}(z+sv_i)\,ds\,dz.&\end{aligned}\tag{4-7}$$

Denote by (a,b) the coordinate system in the (v_i,v_j) basis, $(a^-,a^+)\in\mathbb{R}^2$ and $(b^-(a),b^+(a))\in\mathbb{R}^2$ for every $a\in\,]a^-,a^+[$, such that

$$\Omega=\{av_i+bv_j : a\in\,]a^-,a^+[,\ b\in\,]b^-(a),b^+(a)[\}.\tag{4-8}$$

The first term in L^k can be written as $\int_{a^-}^{a^+}l^k(a)\,da$ with l^k defined as

$$\begin{aligned}l^k(a)=\int_{b^-(a)}^{b^+(a)}\int_{-s_i(bv_j)}^a\chi_i^\eta\chi_i^{k,\epsilon}\big(\varphi(Q_i^{+k}-F_i^k\nu_i^k)+(v_i\cdot\nabla\varphi)F_i^k\big)(sv_i+bv_j)&\\ \times\left(\int_s^a\chi_i^{k,\epsilon}\frac{F_j^k}{(1+F_i^k/k)(1+F_j^k/k)}(rv_i+bv_j)\,dr\right)ds\,db.&\end{aligned}\tag{4-9}$$

For each rational number a, the sequence of functions

$$(b,s)\in[b^-(a),b^+(a)]\times[-s_i^+(bv_j),a]\to\chi_i^\eta\chi_i^{k,\epsilon}\big(\varphi(Q_i^{+k}-F_i^k\nu_i^k)+(v_i\cdot\nabla\varphi)F_i^k\big)(sv_i+bv_j)$$

is weakly compact in L^1, whereas

$$(b,s)\to\int_s^a\chi_i^{k,\epsilon}\frac{F_j^k}{(1+F_i^k/k)(1+F_j^k/k)}(rv_i+bv_j)\,dr$$

is by Lemma 3.3 strongly compact in L^1, and by Lemma 3.2 uniformly bounded in L^∞. The convergence follows for any rational number a. With a diagonal process, there is a subsequence of (l^k), still denoted by (l^k), converging for any rational a. Moreover,

$$\lim_{h\to0}(l^k(a+h)-l^k(a))=0,\tag{4-10}$$

uniformly with respect to k and a, by the weak L^1 compactness of

$$\big(\chi_i^\eta\chi_i^{k,\epsilon}(\varphi(Q_i^{+k}-F_i^k\nu_i^k)+(v_i\cdot\nabla\varphi)F_i^k)\big)_{k\in\mathbb{N}^*}\quad\text{and}\quad(F_j^k)_{k\in\mathbb{N}^*}.$$

Thus (l^k) is a uniform converging sequence on $[a^-,a^+]$. The second term in L^k can be treated analogously, $(\chi_i^{k,\epsilon}f_{bi}^k)_{k\in\mathbb{N}^*}$ being uniformly bounded in L^∞. The convergence follows. In order to determine the limit of L^k when $k\to+\infty$, note that

$$\chi_i^\eta\chi_i^{k,\epsilon}(\varphi(Q_i^{+k}-F_i^k\nu_i^k)+(v_i\cdot\nabla\varphi)F_i^k)=v_i\cdot\nabla(\chi_i^\eta\chi_i^{k,\epsilon}\varphi F_i^k),$$

which weakly converges in L^1 to $v_i \cdot \nabla(\chi_i^\eta \varphi F_i^\epsilon)$ when $k \to +\infty$. Hence

$$\begin{aligned}\lim_{k\to+\infty} L^k = \int_\Omega \int_{-s_i^+(z)}^0 v_i \cdot \nabla(\chi_i^\eta \varphi F_i^\epsilon)(z+sv_i)\Big(\int_s^0 f_j(z+rv_i)\,dr\Big)\,ds\,dz& \\ + \int_\Omega \chi_i^\eta \varphi f_{bi}(z_i^+(z))\Big(\int_{-s_i^+(z)}^0 f_j(z+sv_i)\,ds\Big)\,dz&.\end{aligned}$$

By a backwards integration by parts,

$$\lim_{k\to+\infty} L^k = \int_\Omega \int_{-s_i^+(z)}^0 \varphi \chi_i^\eta F_i^\epsilon f_j(z+sv_i)\,ds\,dz. \tag{4-11}$$

In order to prove (4-4), let us prove that each

$$\Gamma_{ij}^{lm} \int_\Omega \int_{-s_i^+(z)}^0 \varphi \chi_i^\eta \chi_i^{k,\epsilon} \frac{F_l^k}{1+F_l^k/k} \frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz, \qquad j \in J_i, \tag{4-12}$$

term from Q_i^{+k} in (4-5) converges when $k \to +\infty$ to a limit smaller than

$$\Gamma_{ij}^{lm} \int_\Omega \int_{-s_i^+(z)}^0 \varphi \chi_i^\eta F_l^{\epsilon'} f_m(z+sv_i)\,ds\,dz + \alpha(\epsilon'), \quad \epsilon' \in\,]0,1[, \text{ with } \lim_{\epsilon'\to 0} \alpha(\epsilon') = 0. \tag{4-13}$$

Take $\Gamma_{ij}^{lm} = 1$, $j \in J_i$, for simplicity. Let $(\mu_{1/n})_{n\in\mathbb{N}^*}$ be the sequence of mollifiers defined at the beginning of Section 2 for $\alpha = 1/n$, and split (4-12) into

$$\begin{aligned}
&\int_\Omega \int_{-s_i^+(z)}^0 \varphi(\chi_i^\eta * \mu_{1/n}) \chi_l^{k,\epsilon'} \chi_i^{k,\epsilon} \frac{F_l^k}{1+F_l^k/k} \frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz \\
&\qquad + \int_\Omega \int_{-s_i^+(z)}^0 \varphi(\chi_i^\eta * \mu_{1/n})(1-\chi_l^{k,\epsilon'}) \chi_i^{k,\epsilon} \frac{F_l^k}{1+F_l^k/k} \frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz \\
&\qquad\qquad + \int_\Omega \int_{-s_i^+(z)}^0 \varphi(\chi_i^\eta - (\chi_i^\eta * \mu_{1/n})) \chi_i^{k,\epsilon} \frac{F_l^k}{1+F_l^k/k} \frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz \\
&\leq \int_\Omega \int_{-s_i^+(z)}^0 \varphi(\chi_i^\eta * \mu_{1/n}) \chi_l^{k,\epsilon'} \frac{F_l^k}{1+F_l^k/k} \frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz \\
&\qquad + \frac{c}{\ln\Lambda} + \frac{c\Lambda}{\epsilon^2} e^{p\Gamma/\epsilon^2} \sum_{j\in J_i} \Big(\int_{\Omega_l^{k,\epsilon'}} F_j^k(z)\,dz + \int_\Omega \varphi |\chi_i^\eta - (\chi_i^\eta * \mu_{1/n})| F_j^k(z)\,dz \Big) \\
&\leq \int_\Omega \int_{-s_i^+(z)}^0 \varphi(\chi_i^\eta * \mu_{1/n}) \chi_l^{k,\epsilon'} \frac{F_l^k}{1+F_l^k/k} \frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz \\
&\qquad + \frac{c}{\ln\Lambda} + \frac{c\Lambda}{\epsilon^2} e^{p\Gamma/\epsilon^2} \Big(\Lambda'\epsilon' + \frac{1}{\ln\Lambda'} + \frac{1}{\ln(k/2)} + \widetilde{\Lambda} \|\chi_i^\eta - (\chi_i^\eta * \mu_{1/n})\|_{L^1} + \frac{1}{\ln\widetilde{\Lambda}} \Big) \quad \text{by (2-8)–(3-1)}, \\
&\qquad \Lambda > 1, \quad \Lambda' > 1, \quad \widetilde{\Lambda} > 1, \quad \epsilon' > 0.
\end{aligned} \tag{4-14}$$

Denote by D the Jacobian of the change of variables $z \to (a, b)$. For some smooth function A, and any integrable function g,

$$\begin{aligned}\int_\Omega \int_{-s_i^+(z)}^0 g(z+sv_i)ds dz &= D \int_{b^-}^{b^+} \int_{a^-(b)}^{a^+(b)} \int_{-s_i^+(bv_j)}^{a} g(sv_i+bv_j)ds\, da\, db \\ &= D \int_{b^-}^{b^+} \int_{-s_i^+(bv_j)}^{a^+(b)} (a^+(b)-\max\{a^-(b), s\})g(sv_i+bv_j)\, ds\, db \\ &= \int_\Omega A(\alpha, \gamma) g(\alpha v_l + \gamma v_m)\, d\alpha\, d\gamma.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{k\to+\infty} \iint_{-s_i^+(z)}^0 &\varphi(\chi_i^\eta * \mu_{1/n})\chi_l^{k,\epsilon'} \frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\, ds\, dz \\ &= \int_\Omega \int_{-s_i^+(z)}^0 \varphi(\chi_i^\eta * \mu_{1/n}) F_l^{\epsilon'} f_m(z+sv_i)\, ds\, dz, \quad \epsilon' \in \,]0, 1[. \end{aligned} \tag{4-15}$$

For $\widetilde{\Lambda}$ large enough, pass to the limit when $k \to +\infty$ and $n \to +\infty$ in (4-14). Up to subsequences, the weak L^1 limits F_i^ϵ and $F_i^{\epsilon'}$ of $(\chi_i^{k,\epsilon} F_i^k)_{k\in\mathbb{N}^*}$ and $(\chi_i^{k,\epsilon'} F_i^k)_{k\in\mathbb{N}^*}$ when $k \to +\infty$ satisfy

$$\begin{aligned}\int_\Omega \varphi\chi_i^\eta F_i^\epsilon(z)\, dz \le \int_\Omega \varphi\chi_i^\eta f_{bi}^k(z_i^+(z))\, dz &+ \int_\Omega \int_{-s_i^+(z)}^0 \chi_i^\eta F_i^\epsilon v_i \cdot \nabla\varphi(z+sv_i)\, ds\, dz \\ &+ \int_\Omega \int_{-s_i^+(z)}^0 \varphi\chi_i^\eta (Q_i^+(F^{\epsilon'}, f) - F_i^\epsilon \nu_i(f))(z+sv_i)\, ds\, dz \\ &+ \frac{c}{\ln\Lambda} + \frac{c\Lambda}{\epsilon^2} e^{p\Gamma/\epsilon^2}\left(\Lambda'\epsilon' + \frac{1}{\ln \Lambda'}\right), \quad (\epsilon, \epsilon') \in \,]0, 1[^2, \ \Lambda > 1, \ \Lambda' > 1. \end{aligned} \tag{4-16}$$

Choose Λ large enough, ϵ small enough, Λ' large enough, ϵ' small enough, in this order. The passage to the limit when $\epsilon \to 0$ and $\epsilon' \to 0$ in (4-16) results from the monotone convergence theorem, the family $(F^\epsilon)_{\epsilon\in]0,1[}$ being nondecreasing, with mass uniformly bounded, together with the mass of $(\chi_i^\eta Q_i^+(F^{\epsilon'}, f))_{\epsilon'\in]0,1[}$ and $(\chi_i^\eta F_i^{\epsilon'} \nu_i(f))_{\epsilon'\in]0,1[}$. Consequently, (4-4) holds. □

Lemma 4.2. *The function* f *is a solution to* (1-5)–(1-6).

Proof of Lemma 4.2. For proving Lemma 4.2, it remains to prove that

$$\begin{aligned}\int_\Omega \varphi\chi_i^\eta f_i(z)\, dz \ge \int_\Omega \varphi\chi_i^\eta f_{bi}(z_i^+(z))\, dz &+ \int_\Omega \int_{-s_i^+(z)}^0 \chi_i^\eta f_i v_i \cdot \nabla\varphi(z+sv_i)\, ds\, dz \\ &+ \int_\Omega \int_{-s_i^+(z)}^0 \varphi\chi_i^\eta Q_i(f, f)(z+sv_i)\, ds\, dz, \quad 1 \le i \le p, \ \varphi \in C_+^1(\bar{\Omega}). \end{aligned} \tag{4-17}$$

For $\beta > 0$, start from the equation for $\varphi\chi_i^\eta F_i^k$ written in renormalized form,

$$\begin{aligned}&\beta^{-1}\varphi\chi_i^\eta \ln(1+\beta F_i^k)(z) - \beta^{-1}\varphi\chi_i^\eta \ln(1+\beta f_{bi}^k)(z_i^+(z)) \\ &\quad + \int_{-s_i^+(z)}^0 \beta^{-1}\chi_i^\eta \ln(1+\beta F_i^k)\, v_i \cdot \nabla\varphi(z+sv_i)\, ds = \int_{-s_i^+(z)}^0 \frac{\varphi\chi_i^\eta (Q_i^{+k} - F_i^k \nu_i^k)}{1+\beta F_i^k}(z+sv_i)\, ds. \end{aligned} \tag{4-18}$$

It holds

$$\beta^{-1}\ln(1+\beta x) < x, \quad \beta\in\,]0,1[\qquad \text{and} \qquad \lim_{\beta\to 0}\beta^{-1}\ln(1+\beta x) = x, \quad x>0.$$

Hence in weak L^1 the sequence $(\beta^{-1}\ln(1+\beta F_i^k))_{k\in\mathbb{N}^*}$ converges modulo a subsequence to a function $F^\beta \le f$ when $k\to+\infty$. The mass of the limit increases to the mass of f, when $\beta\to 0$. This gives in the final limit $\beta\to 0$ for the left-hand side of (4-18)

$$\varphi\chi_i^\eta f_i(z) - \varphi\chi_i^\eta f_{bi}(z_i^+(z)) - \int_{-s_i^+(z)}^0 \chi_i^\eta f_i\, v_i\cdot\nabla\varphi(z+sv_i)\,ds. \tag{4-19}$$

Using analogous arguments as for the limit of the loss term in Lemma 4.1, it holds that

$$\begin{aligned}\lim_{k\to+\infty}\Gamma_{ij}^{lm}\int_\Omega\int_{-s_i^+(z)}^0 &\frac{\varphi\chi_i^\eta F_i^k F_j^k}{1+\beta F_i^k}(z+sv_i)\,ds\,dz\\ &= \Gamma_{ij}^{lm}\int_\Omega\int_{-s_i^+(z)}^0 \varphi\chi_i^\eta\Big(\operatorname*{weak\,}L^1\lim_{k\to+\infty}\frac{F_i^k}{1+\beta F_i^k}\Big) f_j(z+sv_i)\,ds\,dz, \quad j\in J_i.\end{aligned}$$

But

$$\text{weak } L^1\lim_{k\to+\infty}\frac{F_i^k}{1+\beta F_i^k} \le \text{weak } L^1\lim_{k\to+\infty} F_i^k,$$

and

$$\int_\Omega \text{weak } L^1\lim_{k\to+\infty}\frac{F_i^k}{1+\beta F_i^k}(z)\,dz \quad \text{increases to} \quad \int_\Omega \text{weak } L^1\lim_{k\to+\infty} F_i^k(z)\,dz$$

when $\beta\to 0$. Hence

$$\lim_{\beta\to 0}\lim_{k\to+\infty}\Gamma_{ij}^{lm}\int_\Omega\int_{-s_i^+(z)}^0 \frac{\varphi\chi_i^\eta F_i^k F_j^k}{1+\beta F_i^k}(z+sv_i)\,ds\,dz = \Gamma_{ij}^{lm}\int_\Omega\int_{-s_i^+(z)}^0 \varphi\chi_i^\eta f_i f_j(z+sv_i)\,ds\,dz. \tag{4-20}$$

For the gain term and any $(l,m)\in\{1,\dots,p\}^2$ such that $\Gamma_{ij}^{lm}>0$ for some $j\in\{1,\dots,p\}$,

$$\begin{aligned}&\int_\Omega\int_{-s_i^+(z)}^0 \frac{\varphi\chi_i^\eta}{1+\beta F_i^k}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz\\ &\quad\ge \int_\Omega\int_{-s_i^+(z)}^0 \frac{\varphi\chi_i^\eta\chi_l^{k,\epsilon}}{1+\beta F_i^k}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz\\ &\quad= \int_\Omega\int_{-s_i^+(z)}^0 \varphi\chi_i^\eta\chi_l^{k,\epsilon}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz\\ &\qquad\qquad - \int_\Omega\int_{-s_i^+(z)}^0 \varphi\chi_i^\eta\chi_l^{k,\epsilon}\frac{\beta F_i^k}{1+\beta F_i^k}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz\\ &\quad\ge \int_\Omega\int_{-s_i^+(z)}^0 \varphi\chi_i^\eta\chi_l^{k,\epsilon}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz\\ &\qquad - c\Lambda\sum_{j\in J_i}\int_\Omega\int_{-s_i^+(z)}^0 \varphi\chi_i^\eta\chi_l^{k,\epsilon}\frac{\beta(F_i^k)^2F_j^k}{1+\beta F_i^k}(z+sv_i)\,ds\,dz - \frac{c}{\ln\Lambda}, \quad \Lambda>1,\ \epsilon\in\,]0,1[.\end{aligned} \tag{4-21}$$

It holds

$$\lim_{k\to+\infty}\int_\Omega\int_{-s_i^+(z)}^0 \varphi\chi_i^\eta\chi_l^{k,\epsilon}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz=\int_\Omega\int_{-s_i^+(z)}^0\varphi\chi_i^\eta F_l^\epsilon f_m^k(z+sv_i)\,ds\,dz. \tag{4-22}$$

Choose Λ large enough and split the domain of integration of every $j\in J_i$ term in (4-21) into

$$\{F_i^k\le\Lambda'\}\cup\left\{F_i^k>\Lambda' \text{ and } F_i^kF_j^k>\widetilde{\Lambda}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}\right\}$$
$$\cup\left\{F_i^k>\Lambda' \text{ and } F_i^kF_j^k\le\widetilde{\Lambda}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}\right\},\quad \Lambda'>1,\ \widetilde{\Lambda}>1.$$

It holds that

$$\int_\Omega\int_{-s_i^+(z)}^0\varphi\chi_i^\eta\chi_l^{k,\epsilon}\frac{\beta(F_i^k)^2F_j^k}{1+\beta F_i^k}(z+sv_i)\,ds\,dz$$
$$\le c\left(\beta(\Lambda')^2+\frac{1}{\ln\widetilde{\Lambda}}+\frac{\widetilde{\Lambda}}{\epsilon^2}e^{p\Gamma/\epsilon^2}\int_{F_i^k>\Lambda'}F_m^k(z)\,dz\right),\quad \beta\in\,]0,1[,\ \Lambda'>0,\ \widetilde{\Lambda}>1. \tag{4-23}$$

The last term in (4-23) tends to zero when $\widetilde{\Lambda}\to+\infty$, $\Lambda'\to+\infty$, $\beta\to0$ in this order, uniformly with respect to k. Consequently,

$$\lim_{\beta\to0}\lim_{k\to+\infty}\int_\Omega\int_{-s_i^+(z)}^0\frac{\varphi\chi_i^\eta}{1+\beta F_i^k}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz\ge\int_\Omega\int_{-s_i^+(z)}^0\varphi\chi_i^\eta F_l^\epsilon f_m(z+sv_i)\,ds\,dz.$$

This holds for every $\epsilon>0$. Hence

$$\lim_{\beta\to0}\lim_{k\to+\infty}\int_\Omega\int_{-s_i^+(z)}^0\frac{\varphi\chi_i^\eta}{1+\beta F_i^k}\frac{F_l^k}{1+F_l^k/k}\frac{F_m^k}{1+F_m^k/k}(z+sv_i)\,ds\,dz\ge\int_\Omega\int_{-s_i^+(z)}^0\varphi\chi_i^\eta f_lf_m(z+sv_i)\,ds\,dz.$$

And so, (4-17) holds. Together with (4-4), this proves (4-1). □

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