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Curvewise characterizations of minimal upper gradients and the construction of a Sobolev differential

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CURVEWISE CHARACTERIZATIONS OF MINIMAL UPPER GRADIENTS AND THE CONSTRUCTION OF A SOBOLEV DIFFERENTIAL

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We represent minimal upper gradients of Newtonian functions, in the range $1 \le p < \infty$, by maximal directional derivatives along "generic" curves passing through a given point, using plan-modulus duality and disintegration techniques. As an application we introduce the notion of p-weak charts and prove that every Newtonian function admits a differential with respect to such charts, yielding a linear approximation along p-almost every curve. The differential can be computed curvewise, is linear, and satisfies the usual Leibniz and chain rules.

The arising p-weak differentiable structure exists for spaces with finite Hausdorff dimension and agrees with Cheeger's structure in the presence of a Poincaré inequality. In particular, it exists whenever the space is metrically doubling. It is moreover compatible with, and gives a geometric interpretation of, Gigli's abstract differentiable structure, whenever it exists. The p-weak charts give rise to a finite-dimensional p-weak cotangent bundle and pointwise norm, which recovers the minimal upper gradient of Newtonian functions and can be computed by a maximization process over generic curves. As a result we obtain new proofs of reflexivity and density of Lipschitz functions in Newtonian spaces, as well as a characterization of infinitesimal Hilbertianity in terms of the pointwise norm.

1. Introduction

1A. *Overview.* Minimal weak upper gradients of Sobolev-type functions on metric measure spaces were first introduced by Cheeger [1999], building on the notion of upper gradients from [Heinonen and Koskela 1998]. Shanmugalingam [2000] developed *Newtonian spaces* $N^{1,p}(X)$ using the modulus perspective of [Heinonen and Koskela 1998] and proved that they coincide with the Sobolev space defined by Cheeger up to modification of its elements on a set of measure zero. Further notions of Sobolev spaces, based on test plans, were developed by Ambrosio, Gigli and Savaré [Ambrosio et al. 2014], with a corresponding

notion of minimal gradient. Earlier, Hajłasz [1996] had introduced a Sobolev space whose associated minimal gradient, however, lacks suitable locality properties. While the various Sobolev spaces (with the exception of Hajłasz's definition) are equivalent for generic metric measure spaces, Newtonian spaces consist of representatives which are absolutely continuous along generic curves, a property central to the results in this paper.

The minimal p-weak upper gradient $g_f \in L^p(X)$ of a Newtonian function $f \in N^{1,p}(X)$ on a metric measure space X is a Borel function characterized (up to a null-set) as the minimal function satisfying

$$|(f \circ \gamma)_t'| \leq g_f(\gamma_t)|\gamma_t'| \quad \text{for a.e. } t \in I, \tag{1-1}$$

for all absolutely continuous $\gamma : I \to X$ outside a curve family of zero p-modulus. Here $|\gamma_t'|$ denotes the metric derivative of γ for a.e. t; see Section 2. When $X = \mathbb{R}^n$ and $f \in C_c^\infty(\mathbb{R}^n)$, g_f is given by $g_f = \|\nabla f\|$; in this case, for each $x \in X$, there exists a (smooth) gradient curve $\gamma : (-\varepsilon, \varepsilon) \to X$, with $\gamma_0 = x$, satisfying

$$(f \circ \gamma)_0' = g_f(x)|\gamma_0'|. \tag{1-2}$$

In general, however, despite the minimality of g_f, the equality in (1-2) is not always attained. For example the fat Sierpiński carpet (with the Hausdorff 2-measure and Euclidean metric) constructed in [Mackay et al. 2013] with a sequence in $\ell^2 \setminus \ell^1$, as pointed out in the introduction of that work, gives zero p-modulus ($p > 1$) to the family of curves parallel to the x-axis, and thus to the family of gradient curves of the function $f(x, y) = x$. We remark that the example above is measure doubling and supports a Poincaré inequality; in this context an approximate form of (1-2) for Lipschitz functions was proven in [Cheeger and Kleiner 2009, Theorem 4.2].

Towards a positive answer for generic spaces, an "integral formulation" of (1-2) given by [Gigli 2015, Theorem 3.14] states that, when $p > 1$ and $f \in N^{1,p}(X)$, there exist probability measures $\boldsymbol{\eta}$ on $C(I; X)$ (known as test plans representing the gradient of f) such that

$$\lim_{t \to 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, d\boldsymbol{\eta} = \lim_{t \to 0} \int \frac{1}{t} \int_0^t g_f(\gamma_s)|\gamma_s'| \, ds \, d\boldsymbol{\eta}.$$

In this paper we obtain a "pointwise" variant of (1-2) for general metric measure spaces using a combination of plan-modulus duality, developed in [Ambrosio et al. 2015b; Durand-Cartagena et al. 2021; Honzlová Exnerová et al. 2021], and disintegration techniques. (For $p > 1$, a pointwise variant also follows from Gigli's integral formulation; see Section 3C.) Theorem 1.1 below expresses the minimal weak upper gradient of a Newtonian function as the supremum of directional derivatives along generic curves passing through a given point. Here, it is crucial to use Newtonian functions, which are absolutely continuous along almost every curve.

This curvewise characterization of minimal upper gradients yields the existence of an abundance of curves in a given region of the space, provided the region supports nontrivial Newtonian functions. The idea of constructing an abundance of curves goes back to [Semmes 1996] in the presence of a Poincaré inequality. Under this assumption Cheeger showed that $g_f = \text{Lip } f$, where $\text{Lip } f$ denotes the pointwise Lipschitz constant of a Lipschitz function f. Note that inequality $\text{Lip } f \leq g$ for *continuous*

upper gradients g of a Lipschitz function f on a geodesic space is a direct, but central, observation made in [Cheeger 1999, pp. 432–433].

The work of Cheeger lead to many developments, including [Cheeger and Kleiner 2009] pioneering the idea of using directional derivatives along curves (and the early version of Theorem 1.1, appearing as Theorem 4.2 in that work) as well as the development and detailed analysis of *Lipschitz differentiability spaces*; see [Keith 2004a; Bate 2015; Bate and Speight 2013; Cheeger et al. 2016; Schioppa 2016a; 2016b]. In the latter, curves are replaced by *curve fragments* whose abundance is expressed using *Alberti representations*. Alberti representations are similar to *plans* used in this paper. The connection between such representations and the ideas in [Cheeger and Kleiner 2009] was first observed by Preiss, see [Bate 2015, p. 2], and can be used to prove the self-improvement of the Lip-lip inequality to the Lip-lip equality, see [Bate 2015; Schioppa 2016a; Cheeger et al. 2016].

Similarly the abundance of curves, obtained here using duality, yields geometric information on *Sobolev* functions on *general* metric measure spaces. (Indeed, duality, in the disguise of a minimax principle, was previously used to find Alberti representations in Lipschitz differentiability spaces; see [Bate 2015, Theorem 5.1] which uses [Rudin 1980, Lemma 9.4.3].) As an important first application, we use curvewise directional derivatives to define *p-weak charts* and a differential for Newtonian functions with respect to such charts. The arising *p-weak differentiable structure*, i.e., a covering by *p*-weak charts, exists far more generally than for Lipschitz differentiability spaces — indeed metric doubling and finite Hausdorff measure suffice; see Proposition 5.4. This existence result involves a new and crucial dimension bound for the charts and the induced differential structure; see Theorem 1.11(c) or Proposition 4.13. With the aid of Theorem 1.1 we adapt Cheeger's construction to produce a measurable L^∞-bundle, called the *p-weak cotangent bundle*, over spaces admitting a *p*-weak differentiable structure; differentials of Newtonian functions are sections over this bundle. While the Cheeger differential $d_C f$ yields a linearization of a Lipschitz function f, our *p*-weak differential df is given by a linearization along *p*-almost every curve, and the pointwise norm of df recovers the minimal weak upper gradient; see Theorem 1.7.

This definition of a weak differentiable structure seems to be the natural extension of the seminal work [Cheeger 1999] to settings without a Poincaré inequality and yields a "partial differentiable structure", which has been the aim of many authors previously; see [Lučić et al. 2021; Alberti and Marchese 2016; Schioppa 2016a; Cheeger et al. 2016]. Namely, the *p*-weak cotangent bundle measures and makes precise the set of accessible directions (for positive modulus) in the space. By constructing the differential using directional derivatives along curves, we give it a concrete geometric meaning. A sequence of recent work has sought such concrete descriptions; see, e.g., [Ikonen et al. 2022; Lučić et al. 2021]. Our approach yields a new unification of the concrete and abstract cotangent modules of [Cheeger 1999] and [Gigli 2018], respectively; the *p*-weak cotangent bundle is compatible with Gigli's cotangent module when the latter is locally finitely generated, and with Cheeger's cotangent bundle when the space satisfies a Poincaré inequality; see Theorems 1.11 and 1.8.

The geometric approach in this paper has many natural applications. We mention here the *tensorization of Cheeger energy*, pursued in [Ambrosio et al. 2014; 2015c; Lučić et al. 2021], and the identification of abstract tangent bundles with geometric tangent cones; see [Alberti and Marchese 2016; Lučić et al.

2021]. Our methods give tools to generalize and refine the results mentioned above, and moreover enable a blow-up analysis to study analogues of *generalized linearity* considered for example in [Cheeger 1999; Cheeger et al. 2016]. Indeed, blow-ups of plans that define the pointwise norm on a p-weak chart (see Lemmas 4.1, 4.3 and 4.2) along a sequence of rescaled spaces yield curves in the limiting space along which limiting maps of rescaled Newtonian maps behave linearly. In this context we highlight [Schioppa 2016a], which gives a similar geometric and blow-up analysis in the context of abstract Weaver derivations. We leave the detailed exploration and development of these ideas for future work.

1B. *Curvewise characterization of minimal upper gradients.* Throughout the paper, we fix a *metric measure space* $X = (X, d, \mu)$, that is, a complete separable metric space (X, d) together with a Radon measure μ which is finite on bounded sets. A *plan* is a finite measure η on $C(I; X)$ that is concentrated on the set $AC(I; X)$ of absolutely continuous curves. The natural evaluation map $e : C(I; X) \times I \to X$, $(\gamma, t) \mapsto \gamma_t$, gives rise to the *barycenter* $d\eta^\# := e_*(|\gamma_t'| \, dt \, d\eta)$ of η. If $d\eta^\# = \rho \, d\mu$ for some $\rho \in L^q(\mu)$, we say that η is a *q-plan* (not to be confused with q-test plans, see, e.g., Section 2 or [Ambrosio et al. 2015b]). Every finite measure π on $C(I; X) \times I$ admits a disintegration with respect to e: for $e_*\pi$-almost every $x \in X$, there exists a (unique) measure $\pi_x \in \mathcal{P}(C(I; X) \times I)$, concentrated on $e^{-1}(x)$, such that the collection $\{\pi_x\}$ satisfies

$$\pi(B) = \int \pi_x(B) \, d(e_*\pi)(x)$$

for all Borel sets $B \subset C(I; X) \times I$. We refer to [Ambrosio et al. 2008; Bogachev 2007] for more details.

We use these notions to define a "generic curve": if η is a q-plan on X and $\{\pi_x\}$ the disintegration of $d\pi := |\gamma_t'| \, dt \, d\eta$ with respect to e, then π_x-a.e.-curve passes through x, for $e_*\pi$-a.e. $x \in X$. In the forthcoming discussion, we omit the reference to e in the disintegration. We now formulate our first result, in which the equality in (1-2) is obtained as an essential supremum with respect to the disintegration for almost every point. In the statement below we write

$$\mathrm{Diff}(f) = \{(\gamma, t) \in AC(I; X) \times I : f \circ \gamma \in AC(I; \mathbb{R}), \ (f \circ \gamma)_t' \text{ and } |\gamma_t'| > 0 \text{ exist}\}$$

for a μ-measurable function $f : X \to [-\infty, \infty]$.

Theorem 1.1. *Let $1 \le p < \infty$, and let $1 < q \le \infty$ satisfy $1/p + 1/q = 1$. Suppose $f \in N^{1,p}(X)$, g_f is a Borel representative of the minimal p-weak upper gradient of f, and $D := \{g_f > 0\}$. There exists a q-plan η with $\mu|_D \ll \eta^\#$ so that the disintegration $\{\pi_x\}$ of $d\pi := |\gamma_t'| \, dt \, d\eta$ is concentrated on $e^{-1}(x) \cap \mathrm{Diff}(f)$ and*

$$g_f(x) = \left\| \frac{(f \circ \gamma)_t'}{|\gamma_t'|} \right\|_{L^\infty(\pi_x)} \tag{1-3}$$

for μ-almost every $x \in D$.

Remark 1.2. The statement also holds when $f \in N_{\mathrm{loc}}^{1,p}(X)$, that is, $f|_{B(x,r)} \in N^{1,p}(B(x,r))$ for each ball $B(x, r) \subset X$. Indeed, a localization argument, replacing f by $f\eta_n$ with η_n a sequence of Lipschitz functions with bounded support and $\eta_n|_{B(x_0, n-1)} = 1$ for some x_0, reduces the statement for $f \in N_{\mathrm{loc}}^{1,p}(X)$ to Theorem 1.1. Similarly, other notions in this paper, such as charts, could use a local Sobolev space, but to avoid technicalities we do not discuss this point further. A reader can see Lemma 4.5 and its proof for a prototypical form of such a localization argument.

In particular, we have the following corollary.

Corollary 1.3. *Let* p, q *and* f, g_f, D *be as in Theorem 1.1. There exists a* q*-plan* η *and, for every* $\varepsilon > 0$, *a Borel set* $B = B_\varepsilon \subset \text{Diff}(f)$ *with the following property*: *if* $\{\pi_x\}$ *denotes the disintegration of* $d\pi := |\gamma_t'| \, dt \, d\eta$, *then* $\pi_x(B) > 0$ *and*

$$(1 - \varepsilon) g_f(x) |\gamma_t'| \le (f \circ \gamma)_t' \le g_f(x) |\gamma_t'| \quad \text{for every } (\gamma, t) \in e^{-1}(x) \cap B,$$

for μ-*a.e.* $x \in D$.

Theorem 1.1 notably covers the case $p = 1$. In Section 3C we also prove a variant (Theorem 3.6) when $p > 1$, using test plans representing a gradient instead of plan-modulus duality.

1C. *Application*: *p-weak differentiable structure.* Cheeger [1999] showed that PI-spaces (metric measure spaces with a doubling measure supporting some Poincaré inequality) admit a countable cover by *Cheeger charts*, also called a *Lipschitz differentiable structure* (see [Keith 2004b]). Let $\text{LIP}(X)$ denote the collection of Lipschitz functions on X, and let $\text{LIP}_b(X)$ consist those Lipschitz functions with bounded support. A Cheeger chart (U, φ) of dimension n consists of a Borel set U with $\mu(U) > 0$, and a Lipschitz function $\varphi : X \to \mathbb{R}^n$ such that, for every $f \in \text{LIP}(X)$ and μ-a.e. $x \in U$, there exists a unique linear map $d_{C,x} f : \mathbb{R}^n \to \mathbb{R}$, called the Cheeger differential of f, such that

$$f(y) - f(x) = d_{C,x} f(\varphi(y) - \varphi(x)) + o(d(x, y)) \quad \text{as } y \to x. \tag{1-4}$$

Not every space admits Lipschitz differentiable structure, as shown by the so called *Rickman's rug* $X := [0, 1]^2$ equipped with the metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|^\alpha$, where $\alpha \in (0, 1)$ and $\mu = \mathcal{L}^2|_X$. Indeed, a Weierstrass-type function in the y-variable combined with [Schioppa 2016a, Theorem 1.14] would yield nonhorizontal rectifiable curves if the space were a differentiability space, contradicting the fact that all rectifiable curves in X are horizontal.

Here, we introduce *p-weak differentiable structures*, which exist in much more generality (including Rickman's rug, see the discussion after Definition 1.4), adapting Cheeger's construction by substituting (1-4) for a weaker curvewise control. To accomplish this, we replace the pointwise Lipschitz constant by the minimal p-weak upper gradient in the definition of "infinitesimal linear independence" (1-5) and use Theorem 1.1 to circumvent the difficulties arising from the fact that the latter is defined only up to a null-set.

In the remainder of the introduction, we use the notation $|Df|_p$ for the minimal p-weak upper gradient of $f \in N_{\text{loc}}^{1,p}(X)$ and refer to Section 2 for more discussion on this notation. Given $p \ge 1$ and $N \in \mathbb{N}$, we say that a Sobolev map $\varphi \in N_{\text{loc}}^{1,p}(X; \mathbb{R}^N)$ is *p-independent* in $U \subset X$ if

$$\underset{v \in S^{N-1}}{\text{ess inf}} |D(v \cdot \varphi)|_p > 0 \quad \mu\text{-a.e. in } U, \tag{1-5}$$

and *p-maximal* in U if no *Lipschitz* map into a higher-dimensional target is p-independent in a positive measure subset of U. Here, we use the essential infimum of an uncountable collection, which agrees μ-a.e. with the pointwise infimum over any countable dense collection of S^{N-1}; see Section A2. Note that p-maximality does not depend on the particular map φ but rather the dimension of its target space.

Definition 1.4. An N-dimensional p-weak chart (U, φ) of X consists of a Borel set $U \subset X$ with positive measure and a Lipschitz function $\varphi : X \to \mathbb{R}^N$ which is p-independent and p-maximal in U. We say that X admits a p-weak differentiable structure if it can be covered up to a null set by countably many p-weak charts.

By convention, zero-dimensional p-weak charts satisfy $\varphi \equiv 0$ and (1-5) is a vacuous condition, while maximality means that $|Df|_p = 0$ μ-a.e. on U for every $f \in \mathrm{LIP}_b(X)$ (see also Proposition 4.4). In Section 4F we briefly discuss a lower-regularity requirement in Definition 1.4 and the fact the resulting notion yields essentially the same p-weak differentiable structure. We also show that an N-dimensional p-weak chart (U, φ) satisfies $N \leq \dim_H U$, where $\dim_H U$ denotes the Hausdorff dimension of U; see Proposition 4.13. In particular, we have the following theorem.

Theorem 1.5. *A metric measure space of finite Hausdorff dimension admits a p-weak differentiable structure for any $p \geq 1$. In particular, this holds if the space is metrically doubling.*

We refer to Proposition 5.4 for a more technical statement, which immediately implies the theorem. Next, we give an analogue of the Cheeger differential (1-4) using p-weak charts.

Definition 1.6. Given an N-dimensional p-weak chart (U, φ) of X, a *p-weak differential* of a Newtonian function $f \in N^{1,p}(X)$ with respect to φ is a map $\mathrm{d}f : U \to (\mathbb{R}^N)^*$ (whose value at $x \in U$ is denoted by $\mathrm{d}_x f$) which satisfies

$$f(\gamma_s) - f(\gamma_t) = \mathrm{d}_{\gamma_t} f(\varphi(\gamma_s) - \varphi(\gamma_t)) + o(|t - s|) \quad \text{for a.e. } t \in \gamma^{-1}(U), \text{ as } s \to t, \qquad (1\text{-}6)$$

for p-a.e. absolutely continuous curve γ in X. We say that a function $f \in N^{1,p}(X)$ has a p-weak differential with respect to φ, if such a $\mathrm{d}f$ exists.

If the curve γ does not enter U, or only spends zero length in the set, then condition (1-6) becomes vacuously satisfied with both sides vanishing. The p-weak differential is uniquely determined up to almost everywhere equivalence by (1-6); see Lemma 4.3. Further, it is also local, i.e., if $g \in N^{1,p}(X)$ and $f|_A = g|_A$ on a positive measure subset $A \subset U$, then $\mathrm{d}f|_A = \mathrm{d}g|_A$. The differential satisfies various natural computation rules; see Propositions 4.10 and 5.7 for the most important ones.

Theorem 1.7. *Suppose $p \geq 1$, and $\varphi : X \to \mathbb{R}^N$ is a p-weak chart on U. Then any $f \in N^{1,p}(X)$ has a p-weak differential $\mathrm{d}f : U \to (\mathbb{R}^N)^*$ with respect to φ, which is μ-a.e. unique, and the map $f \mapsto \mathrm{d}f$ is linear.*

Moreover, for μ-a.e. $x \in U$, there is a norm $|\cdot|_x$ on $(\mathbb{R}^N)^*$ such that $x \mapsto |\xi|_x$ is Borel for every $\xi \in (\mathbb{R}^N)^*$ and

$$|\mathrm{d}f|_x = |Df|_p(x) \quad \text{for } \mu\text{-a.e. } x \in X,$$

for every $f \in N^{1,p}(X)$.

Whereas Lipschitz functions are differentiable with respect to Cheeger charts, (1-5) yields only the curvewise control (1-6). Indeed, if there are very few or no rectifiable curves, or if the curves only point into certain directions, then the p-weak differential vanishes, or measures only these directions, respectively. For example, given a fat Cantor set $K \subset \mathbb{R}^n$ with $\mathcal{L}^n(K) > 0$, $X := (K, d_{\mathrm{Eucl}}, \mathcal{L}^n|_K)$ is a

Lipschitz differentiability space but the minimal weak upper gradient of every Lipschitz function is zero. On the other hand, Rickman's rug admits nontrivial p-weak charts $\varphi(x, y) = x$. The p-weak differential in this case can be identified with the x-derivative, $df \equiv \partial_x f$, and the only curves with positive p-modulus are those which are horizontal. These examples demonstrate that p-weak differentiable structures might exist for spaces not admitting a Cheeger structure, but the two need not coincide even if both exist. However, if a Poincaré inequality is present, the two structures coincide.

Theorem 1.8. *Suppose X is a p-PI space for $p \geq 1$. Then any p-weak chart (U, φ) of X is a Cheeger chart.*

It follows from the discussion after Definition 1.4 that a p-PI space admits p-weak charts. In Section 1D, we obtain a precise statement on the relationship between the p-weak and Lipschitz differentiable structure, as well as a characterization of the existence of p-weak differentiable structures in terms of Gigli's cotangent module [2018]. Here we mention a noteworthy corollary of the existence of a p-weak differentiable structure.

Theorem 1.9. *Let $p \geq 1$. If X admits a p-weak differential structure, then $\mathrm{LIP}_b(X)$ is norm-dense in $N^{1,p}(X)$.*

Theorem 1.9 has been obtained by other methods for $p > 1$ in [Ambrosio et al. 2013] but is new in the case $p = 1$. In particular, we highlight that the density holds if X has finite Hausdorff dimension.

1D. *Connections to Cheeger's and Gigli's differentiable structures.* Together with the pointwise norm from Theorem 1.7, a p-weak differentiable structure gives rise to a *p-weak cotangent bundle $T_p^* X$* over X, analogous to the measurable L^∞-cotangent bundle $T_C^* X$ arising from the Lipschitz differentiable structure [Cheeger 1999; Keith 2004b], which is equipped with the pointwise norm

$$|\xi|_{C,x} := \mathrm{Lip}(\xi \circ \varphi)(x), \quad \xi \in (\mathbb{R}^N)^*,$$

for μ-a.e. $x \in U$, where (U, φ) is an N-dimensional Cheeger chart. For any $f \in \mathrm{LIP}_b(X)$, the differentials df and $d_C f$ are sections of the cotangent bundles $T_p^* X$ and $T_C^* X$, respectively. We refer to Section 5 for the precise definition of measurable L^∞-bundles and their sections.

In the next theorem we show that there is a submetric bundle map $T_C^* X \to T_p^* X$ and give a condition under which the bundle map is an isometric isomorphism. See Section 5 for the definition of bundle maps. In the statement, a modulus of continuity is an increasing continuous function $\omega : [0, \infty) \to [0, \infty)$, with $\omega(0) = 0$, and a linear submetry between normed spaces V and W is a surjective linear map $L : V \to W$, with $L(B_V(r)) = B_W(r)$.

Theorem 1.10. *Suppose X admits a Cheeger structure and let $p \geq 1$. There is a bundle map $\pi = \pi_{C,p} :$ $T_C^* X \to T_p^* X$ which is a linear submetry μ-a.e. and satisfies*

$$\pi_x(d_{C,x} f) = d_x f \quad \text{for } \mu\text{-a.e. } x \in X, \tag{1-7}$$

for every $f \in \mathrm{LIP}_b(X)$. If there exists a collection $\{\omega_x\}_{x \in X}$ of moduli of continuity satisfying

$$\mathrm{Lip} f(x) \leq \omega_x(|Df|_p(x)) \quad \text{for } \mu\text{-a.e. on } X,$$

for every $f \in \mathrm{LIP}_b(X)$, then $\pi_{C,p}$ is an isometric bijection μ-a.e.

Theorem 1.10 follows from [Ikonen et al. 2022, Theorem 1.1] and the following theorem, which identifies the space $\Gamma_p(T_p^*X)$ of p-integrable sections of the p-weak cotangent bundle T_p^*X with Gigli's cotangent module $L^p(T^*X)$. We refer to Section 6 for the relevant definitions, and remark here that Gigli's construction is the most general in the sense that $L^p(T^*X)$ can be defined for any metric measure space. It is a priori defined only as an abstract L^p-normed L^∞-module in the sense of [Gigli 2015; 2018].

We say that $L^p(T^*X)$ is *locally finitely generated* if X has a countable Borel partition $\mathscr{B}$ so that each $B \in \mathscr{B}$ admits a finite generating set in B. Here, a collection $\mathscr{V} \subset L^p(T^*X)$ is a generating set in B (or generates $L^p(T^*X)$ in B) if $\chi_B L^p(T^*X)$ is the smallest closed submodule of $L^p(T^*X)$ containing $\chi_B v$ for every $v \in \mathscr{V}$. Gigli's cotangent modules admit a *dimensional decomposition*, i.e., a Borel partition $\{A_N\}_{N \in \mathbb{N} \cup \{\infty\}}$ of X so that $L^p(T^*X)$ admits a generating set of cardinality N (and no smaller) in A_N for each N. For $N = \infty$, no finite set generates $L^p(T^*X)$ in A_N. The dimensional decomposition is uniquely determined up to μ-negligible sets.

Below we denote by $\mathrm{d}_G f$ and $|\cdot|_G$ the abstract differential and pointwise norm in the sense of Gigli; see Theorem 6.1. A morphism between L^p-normed L^∞-modules (i.e., a continuous L^∞-linear map) is said to be an isometric isomorphism if it preserves the pointwise norm and has an inverse that is a morphism.

Theorem 1.11. *Let X be a metric measure space and $p \geq 1$. Then X admits a p-weak differentiable structure if and only if $L^p(T^*X)$ is locally finitely generated. In this case,*

(a) *there exists an isometric isomorphism $\iota : \Gamma_p(T_p^*X) \to L^p(T^*X)$ of normed modules satisfying $\iota(\mathrm{d}f) = \mathrm{d}_G f$ for every $f \in N^{1,p}(X)$ and uniquely determined by this property,*

(b) *each set A_N in the dimensional decomposition of X can be covered up to a null-set by N-dimensional p-weak charts,*

(c) *$N \leq \dim_H(A_N)$ for each $N \in \mathbb{N}$.*

Theorem 1.11 gives a concrete interpretation of Gigli's cotangent module, and bounds the Hausdorff dimension of the sets in the dimensional decomposition. As corollaries we obtain the reflexivity of $N^{1,p}(X)$ when $p > 1$, and a characterization of infinitesimal Hilbertianity in terms of the pointwise norm of Theorem 1.7 when $p = 2$, for spaces admitting a p-weak differentiable structure; see Corollary 6.7. Reflexivity could also be obtained directly from Theorem 1.7 following the argument in [Cheeger 1999, Section 4].

2. Preliminaries

Throughout this paper $X = (X, d, \mu)$ will be a complete separable metric measure space equipped with a Radon measure μ finite on balls. We denote by $C(I; X)$ the space of continuous curves $\gamma : I \to X$ equipped with the metric of uniform convergence and by $\mathrm{AC}(I; X)$ the subset of absolutely continuous curves in X, where $I \subset \mathbb{R}$ is an interval. Mostly, we will be concerned with statements independent of parametrization; thus the choice of the interval I is immaterial. However, when we need to refer to the end points of the curve, then we will take $I = [0, 1]$.

If γ is a curve, its value at $t \in I$ is denoted by $\gamma_t := \gamma(t)$. If $f : X \to \mathbb{R}^N$ is a function, we also use this notation as $(f \circ \gamma)_t = f(\gamma_t)$. The derivative of f in the direction of γ at γ_t, when it exists, is denoted

by $(f \circ \gamma)'_t = (f \circ \gamma)'(t)$. The metric derivative of the curve, in the sense of say [Ambrosio et al. 2008, Section I.1], is defined as $|\gamma'_t| = \lim_{h \to 0} d(\gamma_{t+h}, \gamma_t)/h$, when it exists. The metric derivative is defined almost everywhere on I for $\gamma \in AC(I; X)$.

2A. *Plans and modulus.* A finite measure η on $C(I; X)$ is called a *plan* if it is concentrated on $AC(I; X)$, and a *q-plan* if the *barycenter* $d\eta^{\#} := e_*(|\gamma'_t| \, dt \, d\eta)$ satisfies $d\eta^{\#} = \rho \, d\mu$ for some $\rho \in L^q(\mu)$. We denote by $AC_q(I; X)$ the space of curves $\gamma \in AC(I; X)$ satisfying $\int_0^1 |\gamma'_t|^q \, dt < \infty$, and say that a *q*-plan $\eta \in \mathcal{P}(C(I; X))$ is a *q-test plan*, if it is concentrated on $AC_q(I; X)$ and

$$e_{t*}\eta \le C\mu \quad \text{for every } t \in I, \quad \text{and} \quad \iint_0^1 |\gamma'_t|^q \, dt \, d\eta < \infty$$

for some constant $C > 0$. Here $e_t : C(I; X) \to X$ is the map $e_t(\gamma) = \gamma_t$.

Remark 2.1. Every *q*-test plan is also a *q*-plan. However, the converse can fail for two reasons. A *q*-test plan fixes a given parametrization for curves (with an integrability condition on the speed) and insists on a compression bound $e_{t*}(\eta) \le C\mu$. However, for each *q*-plan supported on $\Gamma \subset AC(I; X)$, one can construct associated *q*-test plans supported on reparametrized curves, which are subcurves of curves in Γ.

The argument for this is a combination of two observations in [Ambrosio et al. 2015b]. First, for each *q*-plan one can reparametrize curves to get a plan with a good "parametric barycenter" [loc. cit., Definition 8.1 and Theorem 8.3]. The parametric barycenter depends on the parametrization, while the barycenter $\eta^{\#}$ does not. The second point concerns the compression bound, where given the previous plan, one can take subsegments of curves and average these over shifts to get a compression bound, which is explained as part of the proof of [loc. cit., Theorem 9.4].

This remark would allow, for example, to phrase Theorem 1.1 with test plans instead of plans, if one were so inclined.

If $\Gamma \subset C(I; X)$ is a family of curves, then a Borel function $\rho: X \to [0, \infty]$ is called admissible if $\int_\gamma \rho \, ds \ge 1$ for each rectifiable $\gamma \in \Gamma$. In particular, if there are no rectifiable curves, then this condition is vacuous. We define, for $p \in [1, \infty)$,

$$\text{Mod}_p(\Gamma) = \inf_\rho \int_X \rho^p \, d\mu,$$

where the infimum is over all admissible ρ. We remark, that due to Vitali–Carathéodory, such an infimum can always be taken with respect to lower semicontinuous functions. Notice that the modulus is supported on rectifiable curves and is independent of the parametrization of such curves. We say that a property holds for *p-almost every curve* if there is a family of curves Γ_B so that $\text{Mod}_p(\Gamma_B) = 0$ and the property holds for all $\gamma \in C(I; X) \setminus \Gamma_B$. Modulus is invariant of the parametrization of curves, but some of our statements depend on a parametrization. In those cases, we will say that the property holds for *p-almost every absolutely continuous curve in X* (or *p*-a.e. $\gamma \in AC(I; X)$) to emphasize that the property holds for each $\gamma \in AC(I; X) \setminus \Gamma_B$ with $\text{Mod}_p(\Gamma_B) = 0$. The reader may consult [Heinonen et al. 2015, Sections 4–7] for a more in-depth treatment of modulus, upper gradients and Vitali–Carathéodory.

Remark 2.2. A crucial fact we will use is that if Γ satisfies $\text{Mod}_p(\Gamma) = 0$, then for any *q*-plan η we have $\eta(\Gamma) = 0$ (which holds for $p \in [1, \infty)$ and q its dual exponent). The converse is also true for $p \in (1, \infty)$.

See the arguments and discussion in [Ambrosio et al. 2015b, Sections 4 and 9]. One point here is that if we used q-test plans, this relationship would be more complex, and we would need to consider "stable" families of curves; see [loc. cit., Theorem 9.4]. The case of $p = 1$ is also somewhat subtle, and we will deal with a special case of this issue in Section 3. The argument of Proposition 2.3 would give the converse for compact families of curves and $p = 1$. See also, [Honzlová Exnerová et al. 2021] for a much more detailed exploration of this borderline case.

The previous remark concerns an inequality relating modulus and q-plans. However, there is a closer connection, and in a sense these are dual to each other. Previously, this has been explored in [Ambrosio et al. 2015b, Theorem 5.1] for $p > 1$, and in [Honzlová Exnerová et al. 2021, Theorem 6.3] for $p = 1$. Due to its importance for us, we summarize one main consequence of these results. We further briefly describe the main steps of a direct proof from [David and Eriksson-Bique 2020, Proposition 4.5]. A similar argument appeared previously in a more specific context in [Durand-Cartagena et al. 2021, Theorem 3.7].

Proposition 2.3. *Let $p \in [1, \infty)$ and q its dual exponent with $p^{-1} + q^{-1} = 1$. If $K \subset C(I; X)$ is a compact family of curves, and $\mathrm{Mod}_p(K) \in (0, \infty)$, then there exists a q-plan $\boldsymbol{\eta}$ with $\mathrm{spt}(\boldsymbol{\eta}) \subset K$.*

Proof. A power of the modulus $\mathrm{Mod}_p(K)^{1/p}$ arises from a convex optimization problem on ρ with a constraint for every curve $\gamma \in K$. A dual formulation of this corresponds to a variable for each constraint, i.e., a measure ν supported on K. Thus, it is reasonable to consider a modified Lagrangian defined by

$$\Phi(\rho, \nu) = \|\rho\|_{L^p} - \mathrm{Mod}_p(K)^{1/p} \int_K \int_\gamma \rho \, ds \, d\nu_\gamma,$$

where $\rho : X \to [0, \infty]$ is a function and ν is a probability measure supported on K. Let $P(K)$ be the collection of these probability measures supported on K equipped with the topology of weak* convergence. In order to obtain the required continuity, we will restrict to $\rho \in G$, with

$$G := \{\rho : X \to [0, 1] : \rho \text{ compactly supported and continuous in } X\}.$$

The set G is equipped with the topology of uniform convergence. Then $\Phi : G \times P(K) \to \mathbb{R}$ is a functional with two properties: $\Phi(\cdot, \nu)$ is convex and continuous for each $\nu \in P(K)$, and $\Phi(\rho, \cdot)$ is concave and upper semicontinuous for each $\rho : X \to [0, 1]$. Further $P(K)$ is compact and convex in the weak* topology and G is a convex subset.

By Sion's minimax theorem, see, e.g., statement in [David and Eriksson-Bique 2020, Theorem 4.7], we have

$$\sup_{\nu \in P(K)} \inf_{\rho \in G} \Phi(\rho, \nu) = \inf_{\rho \in G} \sup_{\nu \in P(K)} \Phi(\rho, \nu).$$

We can compute $\inf_{\rho \in G} \sup_{\nu \in P(K)} \Phi(\rho, \nu) \geq 0$. Indeed, given any $\rho \in G$, we can use the definition of modulus to find a $\gamma \in K$ with $\int_\gamma \rho \, ds \leq \|g\|_p / \mathrm{Mod}_p(K)^{-1/p}$. If we choose $\nu = \delta_\gamma$, a Dirac measure on γ, the bound immediately follows.

Therefore, we have also $\sup_{\nu \in P(K)} \inf_{\rho \in G} \Phi(\rho, \nu) \geq 0$. But, up to showing that this supremum is attained, there must be some $\boldsymbol{\eta} \in P(K)$ for which we get $\inf_{\rho \in G} \Phi(\rho, \boldsymbol{\eta}) \geq 0$. After unwinding the definition of a q-plan, and an application of Radon–Nikodym on X, the measure $\boldsymbol{\eta}$ is our desired q-plan. $\qquad\square$

2B. *Sobolev spaces and functions.* A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is called Lipschitz if $\mathrm{LIP}(f) := \sup_{x,y \in X, x \neq y} d_Y(f(x), f(y))/d_X(x, y) < \infty$. A bijection $f : X \to Y$ is called bi-Lipschitz if f and f^{-1} are Lipschitz. Further, if $x \in X$, we define the local Lipschitz constant as

$$\mathrm{Lip}\, f(x) := \limsup_{y \to x, y \neq x} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

Let $\mathrm{LIP}_b(X)$ be the collection of Lipschitz maps $f : X \to \mathbb{R}$ with bounded support.

Definition 2.4. Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be measurable, $g : X \to [0, \infty]$ a Borel function, and $\gamma : I \to X$ a rectifiable path. We say that g is an upper gradient of f along γ, if $\int_\gamma g \, \mathrm{d}s < \infty$ and

$$|f(\gamma_t) - f(\gamma_s)| \leq \int_{\gamma|_{[s,t]}} g$$

for each $s < t$ with $s, t \in I$ with the convention $\infty - \infty = \infty$. We say that g is an upper gradient of f if it is an upper gradient along every rectifiable curve, and a p-weak upper gradient if g is an upper gradient of f along p-a.e. rectifiable curve.

The space $N^{1,p}(X)$ is defined as all μ-measurable functions $f \in L^p(X)$ which have an upper gradient g in $L^p(X)$. The (semi-)norm on this space is defined as

$$\|f\|_{N^{1,p}} = (\|f\|_{L^p}^p + \inf \|g\|_{L^p}^p)^{1/p},$$

where the infimum is taken over all L^p-integrable upper gradients g of f. The theory of these spaces was largely developed in [Shanmugalingam 2000]; see also [Heinonen et al. 2015] for most of the classical theory. By the results there combined with an observation of [Hajłasz 2003] in the case of $p = 1$, one can show that there always exists a unique minimal g_f, which is an upper gradient along p-almost every path, and for which $\|f\|_{N^{1,p}} = (\|f\|_{L^p}^p + \|g_f\|_{L^p}^p)^{1/p}$. We call g_f the *minimal p-upper gradient*. Similarly, we can define $f \in N_{\mathrm{loc}}^{1,p}(X)$ if $f\eta \in N^{1,p}$ whenever $\eta \in \mathrm{LIP}_b(X)$. In these cases we also can define a minimal p-upper gradient g_f, so that $\eta g_f \in L^p(X)$ for every $\eta \in \mathrm{LIP}_b(X)$. In other words, $g_f \in L_{\mathrm{loc}}^p(X)$.

We denote by $N^{1,p}(X; \mathbb{R}^N) \simeq N^{1,p}(X)^N$ the space of functions $\varphi \colon X \to \mathbb{R}^N$ so that each component is in $N^{1,p}$. Similarly, we define $\mathrm{LIP}_b(X; \mathbb{R}^N) \simeq \mathrm{LIP}_b(X)^N$.

Another notion of Sobolev space can be defined using *q-test plans* and we denote it by $W^{1,p}(X)$, with $|Df|_p$ denoting the minimal gradient of $f \in W^{1,p}(X)$. Namely, a function $f \in L^p(\mu)$ belongs to the Sobolev space $W^{1,p}(X)$ if there exists $g \in L^p(\mu)$ such that

$$\int |f(\gamma_1) - f(\gamma_0)| \, \mathrm{d}\eta \leq \iint_0^1 g(\gamma_t)|\gamma_t'| \, \mathrm{d}t \, \mathrm{d}\eta$$

for every q-test plan η on X. The space has a norm $\|f\|_{W^{1,p}} = (\|f\|_{L^p}^p + \inf_g \|g\|_{L^p}^p)^{1/p}$, where the infimum is over all such functions g. We refer to [Di Marino and Squassina 2019] for details.

Note that any representative of an element of $W^{1,p}(X)$ still belongs to $W^{1,p}(X)$, whilst a representative of an element in $N^{1,p}(X)$ belongs to $N^{1,p}(X)$ if and only if they agree outside a p-exceptional set. The next theorem says that up to this ambiguity of representatives, the two approaches produce the same

object. The measurability conclusion is also a corollary of [Eriksson-Bique 2023]. We refer to [Ambrosio et al. 2015a] for a proof.

Theorem 2.5. *Let $p \in (1, \infty)$. If $f \in N^{1,p}(X)$, then $f \in W^{1,p}(X)$ and $g_f = |Df|_p$ μ-a.e. Conversely, if $f \in W^{1,p}(X)$, then f has a Borel representative $\bar{f} \in N^{1,p}(X)$ with $g_{\bar{f}} = |Df|_p$ μ-a.e.*

3. Curvewise (almost) optimality of minimal upper gradients

3A. *Upper gradients with respect to plans.* Given a plan η, we can speak of a gradient along its curves.

Definition 3.1. If η is a q-plan and $f \in N^{1,p}(X)$, then a Borel function g is an η-upper gradient if g is an upper gradient of f along γ for η-almost every γ.

The following lemma gives a notion of a minimal η-upper gradient and shows how to compute it by using derivatives along curves.

Lemma 3.2. *Suppose g_f is a minimal upper gradient and η is any q-plan and $d\pi = d\eta |\gamma_t'| \, dt$, with disintegration π_x. Then:*

(1) $g_\eta = \|(f \circ \gamma)_t' / \gamma_t'\|_{L^\infty(\pi_x)}$ *is a η-upper gradient.*

(2) $g_\eta \leq g$ *for any other η-upper gradient for almost every $x \in X$.*

(3) $g_\eta \leq g_f$ *for almost every $x \in X$.*

(4) *Suppose η' is another q-plan and $\eta \ll \eta'$. Then $g_\eta \leq g_{\eta'}$.*

Proof. Let g_f be the minimal p-upper gradient for f. By Lemma A.2 there is a Borel family $\Gamma_0 \subset C(I; X)$, so that f is absolutely continuous on each curve $\gamma \notin \Gamma_0^c$ with upper gradient g_f and so that $\eta(\Gamma_0) = 0$. By Corollary A.3 and Lemma A.1 there is a set $N \subset C(I; X) \times I$ so that for $\pi(N) = 0$, and for each $(\gamma, t) \notin N$, both $(f \circ \gamma)'(t)$ and $|\gamma_t'|$ are defined and measurable. Let $M_0 = \Gamma_0 \times I \cup N$. We get $\pi(M_0) = 0$. For each curve $\gamma \notin \Gamma_0$ the function f is absolutely continuous with upper gradient $(f \circ \gamma)_t' / |\gamma_t'|$. Since $g_\eta(\gamma_t) \geq (f \circ \gamma)_t' / |\gamma_t'|$ for π-almost every $(\gamma, t) \in M_0$, we have that g_η is an η upper gradient.

If g is any other Borel η-upper gradient, then the set of $(\gamma, t) \in \mathrm{Diff}(f) \setminus M_0$ with $(f \circ \gamma)_t' / |\gamma_t'| > g(\gamma_t)$ must have null measure, and thus the claim follows by Fubini and the definition in (1).

The function g_f is an upper gradient for f on curves in Γ_0^c, and thus the claim follows again from curvewise absolute continuity and by showing that the set of (γ, t) with $(f \circ \gamma)_t' / |\gamma_t'| > g_f(\gamma_t)$ must have null π-measure. The final claim follows since $g_{\eta'}$ must be a η-upper gradient for f. $\qquad\square$

3B. *Proof of Theorem 1.1.* In this subsection we prove Theorem 1.1. The idea is that for each q-plan η we can associate a gradient "along" the curves of such a plan. Each such gradient must be less than the minimal upper gradient, and thus the task is to show that by varying over different plans η we can obtain the minimal upper gradient through maximization. In order to show equality of the result of this maximization, we argue by contradiction, that if it were not a minimal upper gradient, then we could witness this by a given plan. This is the core of the following result. It should be compared to [Ambrosio et al. 2015b, Sections 9–11], where a similar analysis is done, but with different terminology and only for $p > 1$. In the following statement we will need to refer to end points of curves, and thus choose the domain of curves as $I = [0, 1]$.

Lemma 3.3. *Let $p \in [1, \infty)$, and q be its dual exponent. Let $f \in N^{1,p}(X)$. Suppose g is any nonnegative Borel function so that $A = \{g < g_f\}$ has positive measure. Then there exists a q-plan η, so that for η-almost every curve $\gamma : [0, 1] \to X$ we have*

$$|f(\gamma_1) - f(\gamma_0)| > \int_\gamma g \, ds. \tag{3-1}$$

Proof. By Vitali–Carathéodory we may find a lower semicontinuous $\tilde{g} \geq g$ which is integrable and so that $\tilde{A} = \{\tilde{g} < g_f\}$ has positive measure. We will suppress the tildes below to simplify notation and thus only consider the case of g lower semicontinuous. Since $g < g_f$ on a positive measure subset, g cannot be a minimal upper gradient, and thus there must exist a family $\Gamma \subset C(I; X)$ of curves with $\mathrm{Mod}_p(\Gamma) > 0$, so that (3-1) holds for each $\gamma \in \Gamma$. Modulus is invariant under reparametrization of curves and so we may consider the subset of those $\gamma \in \Gamma$ which are Lipschitz. We want to find a plan supported on Γ. However, the issue with this is that since $p = 1$ is allowed the family Γ may not be compact, the duality of modulus and q-plans may fail. So, we seek to "cover" Γ, up to a null modulus family by compact families. This covering is done in an iterative way.

Fix an R so that the modulus of Γ_R of those curves in Γ, which are contained in a ball $B(x_0, R)$ for some fixed $x_0 \in X$, is positive. Since f is measurable and X is complete and separable, Egorov's theorem implies the existence of an increasing sequence of compact sets K_n satisfying $\mu(B(x_0, R) \setminus \bigcup K_n) = 0$ for which $f|_{K_n}$ is continuous for each n. Define $\mu(B(x_0, R) \setminus K_n) = \varepsilon_n$. By passing to a subsequence of n, we may assume that $\sum_n \sqrt{\varepsilon_n} < 1$.

Define $\bar{\Gamma}$ as the collection of $\gamma \in \Gamma_R$ so that f is absolutely continuous on γ and $\mathcal{H}^1(\gamma \setminus (\bigcup_{n=1}^\infty K_n)) = 0$. This holds for Mod_p-almost every curve, since $f \in N^{1,p}(X)$ and since p-almost every curve spends measure zero in the null set $X \setminus \bigcup_{n=1}^\infty K_n$. Thus, $\mathrm{Mod}_p(\bar{\Gamma}) > 0$.

Next, let Γ^m be those curves $\gamma : I \to X$, which are m-Lipschitz, so that $\mathrm{Len}(\gamma) \leq m|b - a|$, $\mathrm{diam}(\gamma) \geq 1/m$, $\gamma_0, \gamma_1 \in K_m$ and (3-1) holds. We will show that every $\gamma \in \bar{\Gamma}$ contains a subcurve, up to reparametrization, in $\bigcup_{m=1}^\infty \Gamma^m$. From this, and [Björn and Björn 2011, Lemma 1.34], it follows that $\mathrm{Mod}_p(\bigcup_{m=1}^\infty \Gamma^m) > 0$, and thus there is some $M > 0$ so that $\mathrm{Mod}_p(\Gamma^M) > 0$. It is easy to show that Γ^m is a closed family of curves in $C(I; X)$ with respect to uniform convergence, since g is taken to be lower semicontinuous (see, e.g., [Keith 2003, Proposition 4]).

To obtain the previous fact, consider a nonconstant curve $\gamma \in \bar{\Gamma}$. We have

$$|f(\gamma_1) - f(\gamma_0)| > \int_\gamma g \, ds.$$

We may also parametrize γ by constant speed as the claim is invariant under reparametrizations.

Since γ has constant speed, we know $|I \setminus \bigcup_{n=1}^\infty \gamma^{-1}(K_n)| = 0$ and $f \circ \gamma$ is continuous. Since $\int_\gamma g \, ds < \infty$ and $f \circ \gamma$ is continuous, we can find (for all $n \geq N$ for some $N \in \mathbb{N}$) sequences $a_n, b_n \in [0, 1]$ so that $\lim_{n \to \infty} a_n = a$, $\gamma_{a_n} \in K_n$, $\gamma_{b_n} \in K_n$ and $\lim_{n \to \infty} b_n = b$. Then, for sufficiently large n

$$|f(\gamma_{b_n}) - f(\gamma_{a_n})| > \int_{\gamma|_{[a_n, b_n]}} g \, ds.$$

For n large enough we also have $\mathrm{Len}(\gamma_{[a_n,b_n]}) \leq n|b-a|$, $\mathrm{diam}(\gamma_{[a_n,b_n]})) \geq 1/n$. Since the curves are parametrized by constant speed, they are n-Lipschitz. So $\gamma' = \gamma_{[a_n,b_n]}$ is, up to a reparametrization, in Γ^n for n large enough, and the claim follows.

Fix $M > 0$ so that $\mathrm{Mod}_p(\Gamma^M) > 0$. Next, choose $\delta < \min(\mathrm{Mod}_p(\Gamma^M), 1)$. Define $\delta_n = \varepsilon_n^{1/2p}$. Choose N so that $\sum_{n=N}^\infty \sqrt{\varepsilon_n} < \delta^{1+p}/2$. Let Γ_t^M be the family of curves $\gamma \in \Gamma^M$ so that $\int_\gamma 1_{X \setminus K_n}\, ds \leq \delta\delta_n$ for each $n \geq N$. Since $\left(\sum_{n \geq N}(1_{X \setminus K_n}/(\delta\delta_n))^p\right)^{1/p}$ is a function admissible for $\Gamma^M \setminus \Gamma_t^M$, we have

$$\mathrm{Mod}_p(\Gamma^M \setminus \Gamma_t^M) \leq \sum_{n \geq N} \frac{\varepsilon_n}{\delta^p \delta_n^p} < \delta/2.$$

Thus, by subadditivity of modulus, see, e.g., [Fuglede 1957, Theorem 1],

$$\mathrm{Mod}_p(\Gamma_t^M) \geq \mathrm{Mod}_p(\Gamma^M) - \mathrm{Mod}_p(\Gamma^M \setminus \Gamma_t^M) > \delta/2.$$

By Lemma 3.4, since Γ^M is closed, the family $\Gamma_t^M \subset \Gamma^M$ is a compact family of curves in a complete space. Then, by Proposition 2.3 there exists a q-plan η supported on Γ_t^M. Each curve $\gamma \in \Gamma_t^M$ satisfies (3-1), and thus the claim follows. □

For the following proof, recall that if $A, B \subset X$, then $d(A, B) := \inf_{a \in A} \inf_{b \in B} d(a, b)$, and $N_\varepsilon(A) := \bigcup_{a \in A} B(a, \varepsilon)$ for $\varepsilon > 0$.

Lemma 3.4. *Suppose that K_n are compact sets, $\eta_n > 0$ constants with $\lim_{n \to \infty} \eta_n = 0$, $L > 0$ and let $\Gamma \subset C(I; X)$ be a closed family of curves in a complete space X. Let $\Gamma^{n,L}$ be the family of curves $\gamma \in \Gamma$ for which $\mathrm{Len}(\gamma) \leq L$, $\mathrm{diam}(\gamma) \geq 1/L$ and which are L-Lipschitz, with $\int_\gamma 1_{X \setminus K_n}\, ds \leq \eta_n$ for each $n \in \mathbb{N}$. Then $\Gamma^{n,L}$ is compact.*

Proof. Let $I = [a, b]$. Since Γ and $\Gamma^{n,L}$ are closed, it suffices to show precompactness.

Let $\gamma \in \Gamma^{n,L}$. We may suppose that $\eta_n < 1/(2L)$ by restricting to large enough n. Then, we have for each n

$$\int_\gamma 1_{K_n}\, ds = \int_\gamma 1\, ds - \int_\gamma 1_{X \setminus K_n}\, ds \geq \mathrm{diam}(\gamma) - \eta_n > \frac{1}{L} - \eta_n.$$

Thus $\gamma \cap K_n \neq \varnothing$. Moreover, if $t \in I$, and $d(\gamma_t, K_n) = s$, then there will be a subsegment of length at least $\min(s, \mathrm{diam}(\gamma_k)/2)$ in $X \setminus K_n$. This gives $\min(s, \mathrm{diam}(\gamma)/2) \leq \eta_n < 1/(2L)$. This is only possible if $s \leq \eta_n$, since $\mathrm{diam}(\gamma)/2 \geq 1/L$. Indeed, we have $d(\gamma, K_n) \leq \eta_n$.

To run the usual proof of Arzelà–Ascoli, since we have equicontinuity with the Lipschitz bound, we only need to show that for each fixed $t \in I$ the set $A_t = \{\gamma_t : \gamma \in \Gamma^{n,L}\}$ is precompact. However, since X is complete, it suffices to show that A_t is totally bounded. Fix $\varepsilon > 0$. We concluded that $d(\gamma, K_n) \leq \eta_n$ for all $n \in \mathbb{N}$. Thus, we have for some large n that $\eta_n \leq \varepsilon/4$ and that $A_t \subset N_{\eta_n}(K_n) \subset N_{\varepsilon/4}(K_n)$. Since K_n is compact, it is totally bounded, and the claim follows by covering K_n by finitely many $\varepsilon/4$ balls and noting that $\varepsilon > 0$ is arbitrary. □

Proof of Theorem 1.1. Let Π_q be the set of all q-plans, and for each $\eta \in \Pi_q$, with its disintegration being given by π_x, define

$$g_\eta(x) = \left\| \frac{(f \circ \gamma)_t'}{|\gamma_t'|} \right\|_{L^\infty(\pi_x)}.$$

Finally, define

$$|D_\pi f| = \operatorname*{ess\,sup}_{\eta \in \Pi_\infty} g_\pi(x).$$

Claim 1. *There is a q-plan $\tilde{\eta}$ so that $|D_\pi f| = g_{\tilde{\eta}}$.*

By Lemma A.5, we can find a sequence η_n so that

$$g_{\eta_n} \to |D_\pi f|$$

almost everywhere. Consider the measures $d\pi^n := |\gamma_t'| \, d\eta_n \, dt$ on $AC(I; X) \times I$. Set

$$a_n = 1 + \eta_n(C(I; X)) + \left\| \frac{d\eta^\#}{d\mu} \right\|_{L^q} + \pi^n(AC(I; X) \times I),$$

where $\eta^\#$ is the barycenter of η, which is absolutely continuous with respect to μ. Let $\tilde{\eta} = \sum_{n=1}^\infty a_n^{-1} 2^{-n} \eta_n$. This will be a plan with $g_{\tilde{\eta}} \geq g_{\eta_n}$ for each n by Lemma 3.2. For μ-almost every x, we have $g_{\tilde{\eta}} \geq |D_\pi f|$. Then, by Lemma 3.3 we have $\|(f \circ \gamma)_t' / |\gamma_t'|\|_{L^\infty(\pi_x)} = |D_\pi f|$, as stated.

Claim 2. *We have $|D_\pi f| = g_f$ almost everywhere.*

Since g_f is a p-weak upper gradient, Lemma 3.2 gives $|D_\pi f| \leq g_f$. Suppose for the sake of contradiction then that $|D_\pi f| < g_f$ on a positive measure subset. Then, by Lemma 3.2, there exists a plan η' so that

$$|f(\gamma_1) - f(\gamma_0)| > \int_\gamma |D_\pi f| \, ds$$

for η'-almost every γ.

However, by the definition of a plan upper gradient, we have for η' almost every curve that

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_\gamma g_{\eta'} \, ds.$$

Now, as $g_{\eta'} \leq |D_\pi f|$ almost everywhere and as η' is a q-plan, we have for η'-almost every curve γ that

$$\int_\gamma g_{\eta'} \, ds \leq \int_\gamma |D_\pi f| \, ds,$$

which contradicts the above inequalities.

Finally, since $|D_\pi f| = g_{\tilde{\eta}} = g_f$, we must have $\mu|_D \ll \tilde{\eta}^\#$. Indeed, otherwise there would be a non-null Borel set $E \subset D$ for which $\mu(E) > 0$ and $\tilde{\eta}^\#(E) = 0$. However, then $g_{\tilde{\eta}}|_E = 0$, contradicting the equality μ-almost everywhere. $\qquad\square$

We now prove Corollary 1.3.

Proof. Let $f \in N^{1,p}$ and consider the plan η' obtained from Theorem 1.1. Let $\eta'' = r_*(\eta')$, where $r : C(I; X) \to C(I; X)$ is the reversal-map which reverses the orientation of every path. Define $\eta = \eta'' + \eta'$. Fix $\varepsilon > 0$, and define $B = \{(\gamma, t) \in \operatorname{Diff}(f) : g_f(x) \geq (f \circ \gamma)_t' / |\gamma_t'| \geq (1 - \varepsilon) g_f(x)\}$. Since $\|(f \circ \gamma)_t' / |\gamma_t'|\|_{L^\infty(\pi_x')} = g_f(x)$ for μ-almost every $x \in D$, where π_x' is the disintegration for η', we have $\pi_x(B) > 0$ for μ-almost every $x \in D$ where π_x is the disintegration corresponding to η. Note that, we can remove the absolute values from the supremum norm since for each path γ in the support of η' we include also its reversal, and r preserves η. $\qquad\square$

3C. *Alternative curvewise characterizations of upper gradients when $p > 1$.* In this subsection we
assume that $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$ and prove a variant Theorem 1.1 using test plans
representing gradients, introduced by Gigli.

Given $f \in N^{1,p}(X)$, a q-test plan $\boldsymbol{\eta}$ *represents* g_f if

$$\frac{f \circ e_t - f \circ e_0}{t \widetilde{E}_t^{1/q}} \to g_f \circ e_0 \qquad \text{and} \qquad \widetilde{E}_t^{1/p} \to g_f \circ e_0 \quad \text{in } L^p(\boldsymbol{\eta}),$$

where

$$\widetilde{E}_t(\gamma) = \frac{1}{t} \int_0^t |\gamma_s'|^q \, ds, \quad \gamma \in \mathrm{AC}(I; X), \ \widetilde{E}_t(\gamma) = +\infty \text{ otherwise.}$$

A test plan $\boldsymbol{\eta}$ representing the gradient of a Sobolev map $f \in N^{1,p}(X)$ is concentrated on "gradient curves"
of f in an asymptotic and integrated sense. We refer to [Gigli 2015; Pasqualetto 2022] for discussion
of the definition we are using here. The following result of Gigli states that Sobolev functions always
possess test plans representing their gradient. In the statement, $\mathcal{P}_q(X)$ denotes probability measures ν
on X with $\int d(x_0, x)^q \, d\nu(x) < \infty$ for some and thus any $x_0 \in X$.

Theorem 3.5 [Gigli 2015, Theorem 3.14]. *If $f \in N^{1,p}(X)$ and $\nu \in \mathcal{P}_q(X)$ satisfies $\nu \leq C\mu$ for some
$C > 0$, there exists a q-test plan $\boldsymbol{\eta}$ **representing** g_f, with $e_{0*}\boldsymbol{\eta} = \nu$.*

We now state the main result of this subsection.

Theorem 3.6. *Let $f \in N^{1,p}(X)$ and g_f be a Borel representative of the minimal p-weak upper gradient
of f, with $D := \{g_f > 0\}$ of positive μ-measure. Let $\boldsymbol{\eta}$ be a q-test plan representing g_f with $\mu|_D \ll
e_{0*}\boldsymbol{\eta} \ll \mu|_D$.*

*For every $\varepsilon > 0$ there exists a Borel set $B \subset \mathrm{Diff}(f)$ such that $d\boldsymbol{\pi} := \chi_B |\gamma_t'| \, dt \, d\boldsymbol{\eta}$ is a positive (finite)
measure with $\mu|_D \ll e_*\boldsymbol{\pi} \ll \mu|_D$, whose disintegration $\{\boldsymbol{\pi}_x\}$ with respect to e satisfies*

$$(1 - \varepsilon)g_f(x) \leq \frac{(f \circ \gamma)_t'}{|\gamma_t'|} \leq g_f(x) \quad \text{and} \quad (1 - \varepsilon)g_f(x)^{p/q} \leq |\gamma_t'| \leq (1 + \varepsilon)g_f(x)^{p/q} \quad \text{for } \boldsymbol{\pi}_x\text{-a.e. } (\gamma, t),$$

for μ-almost every $x \in D$.

For the proof, we present the following three elementary lemmas. Define

$$D_t(\gamma) = \frac{f(\gamma_t) - f(\gamma_0)}{t} \qquad \text{and} \qquad G_t(\gamma) = \frac{1}{t} \int_0^t g_f(\gamma_s)^p \, ds, \quad \gamma \in \mathrm{AC}(I; X),$$

and $+\infty$ otherwise. The following observation is essentially made in [Pasqualetto 2022, Lemma 1.19]
(we are using different notation for our purposes). See Lemma A.1(3) for the Borel measurability of the
functionals in the claim.

Lemma 3.7. *Suppose $f \in N^{1,p}(X)$ and suppose $\boldsymbol{\eta}$ is a q-test plan representing g_f. Then*

$$D_t, G_t, \widetilde{E}_t \to g_f^p \circ e_0 \quad \text{in } L^1(\boldsymbol{\eta}).$$

Proof. Since $\widetilde{E}_t^{1/p} \to g_f^p \circ e_0$ in $L^p(\boldsymbol{\eta})$, it follows that $\widetilde{E}_t \to g_f^p \circ e_0$ in $L^1(\boldsymbol{\eta})$. The convergence $D_t \to g_f^p \circ e_0$
is proven in [Pasqualetto 2022, Lemma 1.19], while $G_t \to g_f^p \circ e_0$ in $L^1(\boldsymbol{\eta})$ follows from [Gigli 2015,
Proposition 2.11]. $\qquad \square$

Lemma 3.8. *For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $a, b > 0$ and $a^p/p + b^q/q \leq ab/(1-\delta)$, then $|a^{p/q}/b - 1| < \varepsilon$.*

Proof. The function $h : (0, \infty) \to (0, \infty)$, given by $h(t) = t/p + t^{-q/p}/q$, has a global minimum at $t = 1$, with $h(1) = 1$. Thus $h|_{(0,1]}$ and $h|_{[1,\infty)}$ have continuous inverses and it follows that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|1 - h(t)| < \delta$ then $|1 - t| < \varepsilon$ (expressing the fact that both inverses are continuous at 1). The claim follows from this by noting that if $a^p/p + b^q/q \leq ab/(1-\delta)$ then $0 \leq h(t) - 1 < \delta$, where $t := a^{p/q}/b$. $\qquad\square$

Lemma 3.9. *Let $h \leq g$ be two integrable functions on an interval $I = [0, T]$, with*

$$\liminf_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} g \, ds =: A > 0 \quad and \quad \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} [g - h] \, ds = 0$$

for some sequence $T_n \to 0$. Then, for every $\varepsilon > 0$ and n, the set $\{(1 - \varepsilon)g < h\} \cap [0, T_n]$ has positive $\mathcal{L}^1$-measure.

Proof. For large enough n we have $0 < A/2 < (1/T_n) \int_0^{T_n} g \, ds$ and $0 \leq (1/T_n) \int_0^{T_n} [g - h] \, ds < \varepsilon A/2$. Thus, we may find some n_0 for which $(1/T_n) \int_0^{T_n} [g - h] \, ds < (\varepsilon/T_n) \int_0^{T_n} g \, ds$ for each $n > n_0$. It follows that $\int_0^{T_n} [(1 - \varepsilon)g - h] \, ds < 0$ for $n > n_0$, and the claim follows from this. $\qquad\square$

We will also need the following technical result; compare Lemma 3.7.

Lemma 3.10. *Let $E \subset X$ be a Borel set, $t > 0$, and let*

$$D_{E,t}(\gamma) := \frac{1}{t} \int_0^t \chi_E(\gamma_s)(f \circ \gamma)'_s \, ds, \quad \gamma \in \Gamma(f).$$

Then $D_{E,t} \to (\chi_E g_f^p) \circ e_0$ in $L^1(\eta)$.

Proof. Define

$$F_t(\gamma) := \frac{1}{t} \int_0^t g_f(\gamma_s)|\gamma'_s| \, ds.$$

Since $D_t \leq F_t \leq (1/p)G_t + (1/q)\widetilde{E}_t$ η-almost everywhere, Lemma 3.7 implies $F_t \to g_f^p \circ e_0$ and thus $(\chi_E \circ e_0)F_t \to (\chi_E g_f^p) \circ e_0$ in $L^1(\eta)$. We show that $(\chi_E \circ e_0)F_t - D_{E,t} \to 0$ in $L^1(\eta)$.

For η-almost every γ we have

$$|\chi_E(\gamma_0)F_t(\gamma) - D_{E,t}(\gamma)|$$

$$= \left| \frac{1}{t} \int_0^t [\chi_E(\gamma_0)g_f(\gamma_s)|\gamma'_s| - \chi_E(\gamma_s)(f \circ \gamma)'_s] \, ds \right|$$

$$\leq \frac{1}{t} \int_0^t \big(|(\chi_E g_f)(\gamma_s) - (\chi_E g_f)(\gamma_0)||\gamma'_s| + \chi_E(\gamma_0)|g_f(\gamma_s) - g_f(\gamma_0)||\gamma'_s| $$
$$+ \chi_E(\gamma_s)[g_f(\gamma_s)|\gamma'_s| - (f \circ \gamma)'_s] \big) \, ds$$

$$\leq \left[\left(\frac{1}{t} \int_0^t |(\chi_E g_f)(\gamma_s) - (\chi_E g_f)(\gamma_0)|^p \, ds \right)^{1/p} + \left(\frac{1}{t} \int_0^t |g_f(\gamma_s) - g_f(\gamma_0)|^p \, ds \right)^{1/p} \right] \left(\frac{1}{t} \int_0^t |\gamma'_s|^q \, ds \right)^{1/q}$$
$$+ F_t(\gamma) - D_t(\gamma).$$

This estimate, together with the Hölder inequality and Lemma 3.7, yields

$$\limsup_{t \to 0} \int |(\chi_E \circ e_0)F_t - D_{E,t}| \, d\eta$$

$$\leq \limsup_{t \to 0} \left[\left(\int \frac{1}{t} \int_0^t |g_f(\gamma_s) - g_f(\gamma_0)|^p \, ds \, d\eta \right)^{1/p} \right.$$

$$\left. + \left(\int \frac{1}{t} \int_0^t |(\chi_E g_f)(\gamma_s) - (\chi_E g_f)(\gamma_0)|^p \, ds \, d\eta \right)^{1/p} \right] \times \left(\int g_f^p \circ e_0 \, d\eta \right)^{1/q}$$

$$= \limsup_{t \to 0} \left[\left(\frac{1}{t} \int_0^t \|g_f \circ e_s - g_f \circ e_0\|_{L^p(\eta)}^p \, ds \right)^{1/p} \right.$$

$$\left. + \left(\frac{1}{t} \int_0^t \|(\chi_E g_f) \circ e_s - (\chi_E g_f) \circ e_0\|_{L^p(\eta)}^p \, ds \right)^{1/p} \right] \times \left(\int g_f^p \circ e_0 \, d\eta \right)^{1/q}.$$

Since $s \mapsto h \circ e_s$ is continuous in $L^p(\eta)$ whenever $h \in L^p(\mu)$ (see [Gigli and Pasqualetto 2020, Proposition 2.1.4]) all terms above tend to zero, proving the claimed convergence. $\square$

Proof of Theorem 3.6. Let N be the negligible set is as in Corollary A.3. The function

$$A(\gamma, t) = \frac{1}{p} g_f(\gamma_t)^p + \frac{1}{q} |\gamma_t'|^q, \quad (\gamma, t) \notin N, \ A(\gamma, t) = +\infty, \ (\gamma, t) \in N,$$

is Borel. Let η represent g_f and satisfy $\mu|_D \ll e_{0*}\eta \ll \mu|_D$. Fix $\varepsilon > 0$, let $\delta > 0$ be as in Lemma 3.8, and set $\delta_0 = \min\{\varepsilon, \delta\}$. We define the Borel function

$$H(\gamma, t) = (1 - \delta_0)A(\gamma, t) - (f \circ \gamma)_t', \quad (\gamma, t) \notin N, \ H = +\infty \text{ otherwise};$$

see Corollary A.3. The set $B := \{H \leq 0\}$ is Borel and, for $(\gamma, t) \notin N$, we have

$$(f \circ \gamma)_t' \leq g_f(\gamma_t)|\gamma_t'| \leq A(\gamma, t). \tag{3-2}$$

Note that

$$H(\gamma, t) \leq 0 \quad \text{implies} \quad (1 - \varepsilon)g_f(\gamma_t)|\gamma_t'| \leq (f \circ \gamma)_t' \text{ and } \left| 1 - \frac{g_f(\gamma_t)^{p/q}}{|\gamma_t'|} \right| < \varepsilon; \tag{3-3}$$

see (3-2) and Lemma 3.8. Once we show that $d\boldsymbol{\pi} := \chi_B |\gamma_t'| \, dt \, d\eta$ satisfies

$$\mu|_D \ll e_*\boldsymbol{\pi} \ll \mu|_D,$$

it follows from (3-2) and (3-3) that $\boldsymbol{\pi}' := \boldsymbol{\pi}/\boldsymbol{\pi}(C(I; X) \times I) \in \mathcal{P}(C(I; X) \times I)$ satisfies

$$(1 - \varepsilon)g_f(\gamma_t)|\gamma_t'| \leq (f \circ \gamma)_t' \leq g_f(\gamma_t) \quad \text{and} \quad \frac{g(\gamma_t)^{p/q}}{1 + \varepsilon} \leq |\gamma_t'| \leq \frac{g(\gamma_t)^{p/q}}{1 - \varepsilon}$$

for $\boldsymbol{\pi}'$-almost every (γ, t), which readily implies the inequalities in the theorem.

To prove $e_*\boldsymbol{\pi} \ll \mu|_D$ observe that (3-3) implies $\chi_B |\gamma_t'| \, dt \, d\eta \leq (1 + \varepsilon)g(\gamma_t)^{p/q} \, dt \, d\eta$ and thus

$$\iint_0^1 \chi_B(\gamma, t)\chi_E(\gamma_t)|\gamma_t'| \, dt \, d\eta \leq (1 + \varepsilon) \int_0^1 \int_X \chi_E g_f^{p/q} e_{t*}(d\eta) \, dt \leq C \int_E g_f^{p/q} \, d\mu$$

for any Borel set $E \subset X$.

It remains to prove that $\mu|_D \ll e_*\boldsymbol{\pi}$. Let $E \subset D$ be a Borel set with $\mu(E) > 0$. Then $e_{0*}\boldsymbol{\eta}(E) = \boldsymbol{\eta}(\{\gamma : \gamma_0 \in E\}) > 0$. Since

$$0 \leq \frac{1}{t} \int_0^t \chi_E(\gamma_s) A(\gamma, s)\, ds - D_{E,t}(\gamma) \leq \frac{1}{p} G_t(\gamma) + \frac{1}{q} \widetilde{E}_t(\gamma) - D_t(\gamma) \xrightarrow{t \to 0} 0,$$

$$D_{E,t} \xrightarrow{t \to 0} \chi_E g_f^p \circ e_0$$

in $L^1(\boldsymbol{\eta})$, see Lemmas 3.7 and 3.10 respectively, there exists a sequence $T_n \to 0$ such that for $\boldsymbol{\eta}$-almost every $\gamma \in e_0^{-1}(E)$ the functions

$$h_\gamma(s) := \chi_E(\gamma_s)(f \circ \gamma)_s', \quad g_\gamma(s) := \chi_E(\gamma_s) A(\gamma, s)$$

satisfy the hypotheses of Lemma 3.9. It follows that for $\boldsymbol{\eta}$-almost every $\gamma \in e_0^{-1}(E)$ the sets

$$I_\gamma^n := \{s \in [0, T_n] : (1 - \delta_0) g_\gamma(s) < h_\gamma(s)\} = \{s \in [0, T_n] : \gamma_s \in E, \ H(\gamma, s) \leq 0\}$$

have positive measure for all n. Notice that, for $\boldsymbol{\eta}$-almost every γ, if $s \in I_\gamma^n$ then $\gamma_s \in E$ and $|\gamma_s'| > 0$, $g_f(\gamma_s) > 0$ (since $0 < (f \circ \gamma)_s' \leq g_f(\gamma_t)|\gamma_s'|$). Consequently

$$\int_0^1 \chi_B(\gamma, s) \chi_E(\gamma_s) |\gamma_s'|\, ds \geq \int_{I_\gamma^n} |\gamma_s'|\, ds > 0$$

for $\boldsymbol{\eta}$-almost every $\gamma \in e_0^{-1}(E)$, which in turn implies $e_*\boldsymbol{\pi}(E) > 0$. Since $E \subset D$ is an arbitrary Borel set with positive μ-measure, this completes the proof. $\qquad\square$

4. Charts and differentials

4A. *Notational remarks.* In what follows, define for any set $U \subset X$ the set of curves which spend positive *length* in U:

$$\Gamma_U^+ = \left\{\gamma \in \mathrm{AC}(I; X) : \int_\gamma \chi_U\, ds > 0\right\}.$$

Having positive length in U is more restrictive than assuming that $\gamma^{-1}(U)$ has positive measure. We will also discuss p-weak differentials and covector fields of the form $df : U \to (\mathbb{R}^N)^*$ or $\boldsymbol{\xi} : U \to (\mathbb{R}^N)^*$ for measurable subsets $U \subset X$. The values of such a map at $x \in U$ are denoted by $d_x f, \boldsymbol{\xi}_x$, respectively.

4B. *Canonical minimal gradients.* Let $p \geq 1$ and $N \geq 0$ be given. For the next three lemmas we fix $\varphi \in N_{\mathrm{loc}}^{1,p}(X; \mathbb{R}^N) \simeq N_{\mathrm{loc}}^{1,p}(X)^N$, with the convention $N_{\mathrm{loc}}^{1,p}(X; \mathbb{R}^N) = N_{\mathrm{loc}}^{1,p}(X)^N = \{0\}$ when $N = 0$. Our aim is to construct a "canonical" representative of the minimal weak upper gradients $|D(\xi \circ \varphi)|_p$ of the functions $\xi \circ \varphi$. We will use a plan to represent it.

Lemma 4.1. *There exists a q-plan $\boldsymbol{\eta}$ and a Borel set D with $\mu|_D \ll \boldsymbol{\eta}^{\#}$ such that*

$$\Phi_\xi(x) := \chi_D(x) \left\| \frac{\xi((\varphi \circ \gamma)_t')}{|\gamma_t'|} \right\|_{L^\infty(\boldsymbol{\pi}_x)} \tag{4-1}$$

is a representative of $|D(\xi \circ \varphi)|_p$ for every $\xi \in (\mathbb{R}^N)^*$. Here $\{\boldsymbol{\pi}_x\}$ is the disintegration of $d\boldsymbol{\pi} := |\gamma_t'|\, d\boldsymbol{\eta}\, dt$ with respect to the evaluation map e.

Proof. Let $\{\xi_0, \xi_1, \dots\} \subset (\mathbb{R}^N)^*$ be a countable dense set and, for each $n \in \mathbb{N}$, choose Borel representatives ρ_n of $|D(\xi_n \circ \varphi)|_p$ and define $D_n := \{\rho_n > 0\}$. By Theorem 1.1 and the Borel regularity of μ, for each $n \in \mathbb{N}$ there exists a q-plan $\boldsymbol{\eta}_n$ and a Borel set $B_n \subset D_n$ with $\mu(D_n \setminus B_n) = 0$ such that the disintegration $\{\pi_x^n\}$ of $d\pi^n := |\gamma_t'| \, d\boldsymbol{\eta}_n \, dt$ satisfies

$$\left\| \frac{\xi_n((\varphi \circ \gamma)_t')}{|\gamma_t'|} \right\|_{L^\infty(\pi_x^n)} = \rho_n(x)$$

for every $x \in B_\xi$.

Define $D := \bigcup_{n \in \mathbb{N}} B_n$ and $\boldsymbol{\eta} = \sum_n 2^{-n} a_n^{-1} \boldsymbol{\eta}_n$, where $a_n = 1 + \boldsymbol{\eta}_n(C(I; X)) + \|d\boldsymbol{\eta}_n^\# / d\mu\|_{L^q} + \pi^n(\mathrm{AC}(I; X) \times I)$. Then $\mu|_D \ll \boldsymbol{\eta}^\#$. Define $\Phi_\xi(x)$ as in (4-1). By Lemma 3.2 we have $\rho_n = \Phi_{\xi_n}$ μ-a.e. on X and thus the claim holds for every $\xi_n \in A$.

We prove the claim in the statement for arbitrary $\xi \in (\mathbb{R}^N)^*$. Let $(\xi_{n_l})_l \subset A$ be a sequence with $|\xi_{n_l} - \xi| < 2^{-l}$ and denote by $\varphi_1, \dots, \varphi_N \in N^{1,p}(X)$ the component functions of φ. Since

$$||D(\xi_{n_l} \circ \varphi)|_p - |D(\xi \circ \varphi)|_p| \leq |D((\xi_{n_l} - \xi) \circ \varphi)|_p \leq |\xi_{n_l} - \xi| \sum_k^N |D\varphi_k|_p$$

μ-a.e., we have $|D(\xi \circ \varphi)|_p = \lim_{l \to \infty} \Phi_{\xi_{n_l}}$ μ-a.e. on X. In particular, $|D(\xi \circ \varphi)|_p = 0$ μ-a.e. on $X \setminus D$. On the other hand, for p-a.e. curve γ, we have

$$|\xi_{n_l}((\varphi \circ \gamma)_t') - \xi((\varphi \circ \gamma)_t')| \leq |\xi_{n_l} - \xi| \sum_k^N |D\varphi_k|_p(\gamma_t)|\gamma_t'| \quad \text{for a.e. } t.$$

Since $\boldsymbol{\eta}$ is a q-plan with $\mu|_D \ll \boldsymbol{\eta}^\#$, this implies

$$\limsup_{l \to \infty} \left| \frac{\xi_{n_l}((\varphi \circ \gamma)_t')}{|\gamma_t'|} - \frac{\xi((\varphi \circ \gamma)_t')}{|\gamma_t'|} \right| \leq \limsup_{l \to \infty} |\xi_{n_l} - \xi| \sum_k^N |D\varphi_k|_p(x) = 0 \quad \text{for } \pi_x\text{-a.e. } (\gamma, t),$$

for μ-a.e. $x \in D$. Thus $\Phi_\xi(x) = \lim_{l \to \infty} \Phi_{\xi_{n_l}}(x)$ for μ-a.e. $x \in D$. Since $\Phi_\xi = 0 = |D(\xi \circ \varphi)|_p$ μ-a.e. on $X \setminus D$, the proof is completed. $\qquad \square$

In the next two lemmas we collect the properties of the Borel function constructed above.

Lemma 4.2. *The map* $\Phi : (\mathbb{R}^N)^* \times X \to \mathbb{R}$ *given by* (4-1) *is Borel and satisfies the following*:

(1) *For every* $\xi \in (\mathbb{R}^N)^*$, $\Phi_\xi := \Phi(\xi, \cdot)$ *is a representative of* $|D(\xi \circ \varphi)|_p$.

(2) *For every* $x \in X$, $\Phi^x := \Phi(\cdot, x)$ *is a seminorm in* $(\mathbb{R}^N)^*$.

Moreover, there exists a path family Γ_B *with* $\mathrm{Mod}_p(\Gamma_B) = 0$ *and for each* $\gamma \in \mathrm{AC}(I; X) \setminus \Gamma_B$ *a null-set* $E_\gamma \subset I$ *so that, for* **every** $\xi \in (\mathbb{R}^N)^*$, *we have*:

(3) Φ_ξ *is an upper gradient of* $\xi \circ \varphi$ *along* γ.

(4) $|(\xi \circ \varphi \circ \gamma)_t'| \leq \Phi_\xi(\gamma_t)|\gamma_t'|$ *for* $t \notin E_\gamma$.

Proof of Lemma 4.2. Borel measurability follows from Lemma A.1 and Corollary A.3, and property (1) follows from Lemma 4.1, while (2) follows from (4-1).

Fix a countable dense set $A \subset (\mathbb{R}^N)^*$ and one $\xi \in A$. We have that $\Phi(\xi, x)$ is a weak upper gradient for $\xi \circ \varphi$, so there is family of curves Γ_ξ so that $\xi \circ \varphi$ is absolutely continuous with upper gradient $|D(\xi \circ \varphi)|_p$ on each $\gamma \in \Gamma_i$, and so that $\mathrm{Mod}_p(\Gamma \setminus \Gamma_i) = 0$. Let $\Gamma' = \bigcap_{\xi \in A} \Gamma_\xi$, whose complement $\Gamma_B = \mathrm{AC}(I; X) \setminus \Gamma'$ has null p-modulus.

Since $\zeta \circ \varphi$ has as upper gradient $\Phi_\zeta(x)$ on γ for each $\zeta \in A$, by considering a sequence ξ_l in A converging to $\xi \notin A$ we obtain the same conclusion.

Finally, fixing an absolutely continuous curve $\gamma \notin \Gamma_B$ there is a full measure set F_γ^1, where the components of $\varphi \circ \gamma_t$ are differentiable at $t \in F_\gamma^1$. Both sides of (4) are continuous and defined in ξ on the set F_γ^1. Since $\Phi_\xi(x)$ is an upper gradient for $\xi \circ \varphi$ along γ, there is a full measure subset $F_\gamma \subset F_\gamma^1$, where the inequality holds for $\xi \in A$. Continuity then extends it for all $\xi \in (\mathbb{R}^N)^*$ and $t \in F_\gamma$ and the claim follows by setting $E_\gamma = I \setminus F_\gamma$. $\qquad\square$

Next, we collect some basic properties of the canonical minimal gradient. Let Φ be the map given by (4-1).

Lemma 4.3. *Set* $I(\varphi)(x) := \inf_{\|\xi\|_*=1} \Phi^x(\xi)$ *for* μ-*a.e.* $x \in X$. *Then*:

(1) $I(\varphi) = \mathrm{ess\ inf}_{\|\xi\|_*=1} |D(\xi \circ \varphi)|_p$ μ-*a.e. in* X.

(2) *If* $U \subset X$ *and* $\boldsymbol{\xi} : U \to (\mathbb{R}^N)^*$ *are Borel, then* $\Phi^x(\boldsymbol{\xi}_x) = 0$ μ-*a.e.* $x \in U$ *if and only if* $\boldsymbol{\xi}_{\gamma_t}((\varphi \circ \gamma)'_t) = 0$ *a.e.* $t \in \gamma^{-1}(U)$ *for* p-*a.e. absolutely continuous* γ *in* X.

(3) *If* φ *is* p-*independent on* U *and* $f \in N^{1,p}(X)$, *then the* p-*weak differential* $\mathrm{d}f$ *with respect to* (U, φ), *if it exists, must be unique.*

Proof of Lemma 4.3. First, we show (1). For any ξ in the unit sphere of $(\mathbb{R}^N)^*$, we have $\Phi_\xi(x) = |D(\xi \circ \varphi)|_p$ almost everywhere by Lemma 4.1. Taking an infimum on the left then gives

$$\inf_{\|\zeta\|_*=1} \Phi_\zeta(x) \le |D(\xi \circ \varphi)|_p,$$

i.e., $\inf_{\|\zeta\|_*=1} \Phi_\zeta(x) \le \mathrm{ess\ inf}_{\|\xi\|_*=1} |D(\xi \circ \varphi)|_p$ almost everywhere by the definition of an essential infimum; see Definition A.4.

On the other hand, if ξ_n, for $n \in \mathbb{N}$, is a countably dense collection in the unit sphere of $(\mathbb{R}^N)^*$, then we have $\Phi_{\xi_n}(x) = |D(\xi_n \circ \varphi)|_p \ge \mathrm{ess\ inf}_{\|\xi\|_*=1} |D(\xi \circ \varphi)|_p$ almost everywhere. By intersecting the sets where this holds for different ξ_n and since the collection is countable, we have that these hold simultaneously on a full-measure set. Specifically, $\inf_{n \in \mathbb{N}} \Phi_{\xi_n}(x) \ge \mathrm{ess\ inf}_{\|\xi\|_*=1} |D(\xi \circ \varphi)|_p$. By Lemma 4.2, we have that $\xi \to \Phi_\xi(x)$ is Lipschitz. Thus, almost everywhere,

$$\inf_{\|\xi\|_*=1} \Phi(\xi, x) = \inf_{n \in \mathbb{N}} \Phi(\xi_n, x) \ge \mathrm{ess\ inf}_{\|\xi\|_*=1} |D(\xi \circ \varphi)|_p,$$

which gives the claim.

Next fix $\boldsymbol{\xi} : U \to (\mathbb{R}^N)^*$ as in (2). Assume first that $\Phi^x(\boldsymbol{\xi}_x) = 0$ for μ-a.e. $x \in U$. Set $C = \{x : \Phi^x(\boldsymbol{\xi}_{\gamma_x}) \ne 0\}$ with $\mu(C) = 0$. Since $\mu(C) = 0$, we have $\mathrm{Mod}_p(\Gamma_C^+) = 0$. Let Γ_B be the family of curves from Lemma 4.2. We will show the claim for $\gamma \in \mathrm{AC}(I; X) \setminus (\Gamma_B \cup \Gamma_C^+)$. By Lemma 4.2(4), we obtain a null set E_γ so that for any $\xi \in (\mathbb{R}^N)^*$ we have $|(\xi \circ \varphi \circ \gamma)'_t| \le \Phi_\xi(\gamma_t)|\gamma'_t|$ and $t \notin E_\gamma$. Let F_γ be the set of $t \notin E_\gamma$ so that

$|\gamma'_t| > 0$ and $\Phi^{\gamma_t}(\boldsymbol{\xi}_{\gamma_t}) \neq 0$. Since $0 = \int_\gamma 1_C \, \mathrm{d}s \geq \int_{F_\gamma} |\gamma'_t| \, \mathrm{d}t$, we have that the measure of F_γ is null. Now, if $t \notin E_\gamma \cup F_\gamma$, then either $|\gamma'_t| = 0$ (and the condition is vacuously satisfied), or the claim follows from $\Phi^{\gamma_t}(\boldsymbol{\xi}_{\gamma_t}) = 0$.

On the other hand, suppose that $\boldsymbol{\xi}_{\gamma_t}((\varphi \circ \gamma)'_t) = 0$ for a.e. $t \in \gamma^{-1}(U)$ and p-a.e. absolutely continuous curve γ. Let η be the q-plan from Lemma 4.1 and $\{\pi_x\}$ the disintegration given there. The equality $\boldsymbol{\xi}_{\gamma_t}((\varphi \circ \gamma)'_t) = 0$ holds then for η-a.e. curve and a.e. $t \in \gamma^{-1}(U)$, since η is a q-plan (recall Remark 2.2). Then for μ-a.e. x we have $\Phi_\xi(x) = 0$ or we have $\Phi_{\boldsymbol{\xi}_x}(x) = \|\boldsymbol{\xi}_x((\varphi \circ \gamma)'_t)/|\gamma'_t|\|_{L^\infty(\pi_x)}$. In the latter case, since η is a q-plan, we have for μ-a.e. such x and π_x-a.e. $(\gamma, t) \in \mathrm{Diff}(f) \cap e^{-1}(x)$ that $\boldsymbol{\xi}_x((\varphi \circ \gamma)'_t) = 0$. Thus, the claim follows together with the properties of disintegrations and Corollary A.3, since the essential supremum then vanishes.

The final claim about uniqueness follows since, if $\mathrm{d}_i f$ were two p-weak differentials for $i = 1, 2$, then we could define $\boldsymbol{\xi}_x = (\mathrm{d}_1 f - \mathrm{d}_2 f)/\|\mathrm{d}_1 f - \mathrm{d}_2\|_{x,*}$ when $\mathrm{d}_1 f \neq \mathrm{d}_2 f$ and otherwise $\boldsymbol{\xi}_x = 0$. We then get immediately from the definition and the second part that $\Phi^x(\boldsymbol{\xi}) = 0$ for μ-a.e. $x \in U$. This would contradict independence. $\qquad\square$

4C. _Charts._ The presentation here should be compared to [Cheeger 1999, Section 4], and specifically to the proof of Theorem 4.38 there, where similar arguments are employed. We first consider 0-dimensional p-weak charts. These correspond to regions of the space where no curve spends positive time.

Proposition 4.4. _Suppose (U, φ) is a 0-dimensional p-weak chart. Then_

$$\mathrm{Mod}_p(\Gamma_U^+) = 0. \tag{4-2}$$

Conversely, if $U \subset X$ is Borel and satisfies (4-2), then $(U, 0)$ is a 0-dimensional p-weak chart of X.

Proof. Since (U, φ) is a 0-dimensional p-weak chart, we have

$$|Df|_p = 0 \quad \text{for } \mu\text{-a.e. in } U, \tag{4-3}$$

for every $f \in \mathrm{LIP}_b(X)$. Let $\{x_n\} \subset X$ be a countable dense subset, and $f_n := \max\{1 - d(x_n, \cdot), 0\}$. By [Ambrosio et al. 2008, Theorem 1.1.2] (see also its proof) and (4-3) we have

$$|\gamma'_t| = \sup_n |(f_n \circ \gamma)'_t| \leq \sup_n |Df_n|_p(\gamma_t)|\gamma'_t| = 0 \quad \text{for a.e. } t \in \gamma^{-1}(U),$$

for p-a.e. $\gamma \in \mathrm{AC}(I; X)$. It follows that $\int_\gamma \chi_U \, \mathrm{d}s = 0$ for p-a.e. $\gamma \in \mathrm{AC}(I; X)$, proving (4-2).

In the converse direction, (4-2) implies, for any $f \in \mathrm{LIP}_b(X)$, that

$$\int_0^1 \chi_U(\gamma_t)|(f \circ \gamma)'_t| \, \mathrm{d}t \leq \mathrm{LIP}(f) \int_0^1 \chi_U(\gamma_t)|\gamma'_t| \, \mathrm{d}t = 0$$

for p-a.e. $\gamma \in \mathrm{AC}(I; X)$. Thus $|(f \circ \gamma)'_t| = 0$ for p-a.e. $\gamma \in \mathrm{AC}(I; X)$ and a.e. $t \in \gamma^{-1}(U)$. Then, by Theorem 1.1, together with measurability considerations from Corollary A.3, this gives $|Df|_p = 0$ μ-a.e. on U for every $f \in \mathrm{LIP}_b(X)$, showing that $(U, 0)$ is a 0-dimensional p-weak chart. $\qquad\square$

For the remainder of this subsection we assume that $N \geq 1$ and that (U, φ) is an N-dimensional chart of X. Denote by Φ the canonical minimal gradient of φ (see Lemma 4.1).

Lemma 4.5. *The function $\xi \mapsto \Phi^x(\xi)$ is a norm on $(\mathbb{R}^N)^*$ for μ-a.e. $x \in U$. Moreover, for every $f \in \mathrm{LIP}(X)$ there exists a p-weak differential $\mathrm{d}f$. That is, a Borel measurable map $\mathrm{d}f : U \to (\mathbb{R}^N)^*$ satisfying*

$$(f \circ \gamma)'_t = \mathrm{d}_{\gamma_t} f((\varphi \circ \gamma)'_t) \quad \text{for a.e. } t \in \gamma^{-1}(U),$$

for p-a.e. absolutely continuous curves γ in X. The map $\mathrm{d}f$ is uniquely determined a.e. in U and satisfies $|Df|_p(x) = \Phi^x(\mathrm{d}f)$ μ-a.e. in U.

Remark 4.6. The equation in the statement is an equivalent formulation of the definition of the p-weak differential in Definition 1.6. Indeed, the latter follows by integration of the first, and conversely, the first follows by Lebesgue differentiation. Further, it would be enough to consider only p-a.e. curve $\gamma \in \Gamma_U^+$. Indeed, if a curve γ does not spend positive length in the set U, then $|\gamma'_t| = 0$ for a.e. $t \in \gamma^{-1}(U)$ and both sides of the equation vanish.

Proof. First, consider $f \in \mathrm{LIP}_b(X)$. Since Φ^x is a norm if and only if $I(\varphi)(x) > 0$, Lemma 4.3(1) and (1-5) imply that Φ^x is a norm for μ-a.e. $x \in U$.

Next, let $f \in \mathrm{LIP}_b(X)$ and consider the map $\psi = (\varphi, f) : X \to \mathbb{R}^{N+1}$. Let Ψ be the canonical minimal gradient of ψ. Given $\xi \in (\mathbb{R}^N)^*$ and $a \in \mathbb{R}$, we use the notation

$$(\xi, a) \in (\mathbb{R}^{N+1})^*, \quad v = (v', v_{N+1}) \mapsto \xi(v') + a v_{N+1}.$$

For μ-a.e. $x \in U$, we have $\Psi^x(\xi, 0) = \Phi^x(\xi)$ and $\Psi^x(0, a) = |a| |Df|_p(x)$ for every $\xi \in (\mathbb{R}^N)^*$, $a \in \mathbb{R}$ (see Lemma 4.2(3) and (4)). Since φ is a chart, we have $I(\psi) = 0$ almost everywhere. Thus, given that $I(\varphi) > 0$, $\ker \Psi^x$ is a 1-dimensional subspace of $(\mathbb{R}^{N+1})^*$. Thus for μ-a.e. $x \in U$ there exists a unique $\xi := \mathrm{d}_x f \in (\mathbb{R}^N)^*$ such that $\Psi^x(\mathrm{d}_x f, -1) = 0$, and the map $x \mapsto \mathrm{d}_x f$ is Borel; see, e.g., [Bogachev 2007, Lemma 6.7.1]. By Lemma 4.3(2), $\mathrm{d}f : U \to (\mathbb{R}^N)^*$ satisfies

$$0 = (\mathrm{d}_{\gamma_t} f, -1)((\psi \circ \gamma)'_t) = \mathrm{d}_{\gamma_t} f((\varphi \circ \gamma)'_t) - (f \circ \gamma)'_t \quad \text{for a.e. } t \in \gamma^{-1}(U),$$

for p-a.e. γ. Moreover, we have

$$||Df|_p(x) - \Phi^x(\mathrm{d}_x f)| \leq |\Psi^x(0, -1) - \Psi^x(\mathrm{d}_x f, 0)| \leq \Psi^x(\mathrm{d}_x f, -1) = 0$$

for μ-a.e. $x \in U$, completing the proof in the case $f \in \mathrm{LIP}_b(X)$.

The case of $f \in \mathrm{LIP}(X)$ follows through localization. Indeed, let $x_0 \in X$ be arbitrary, and consider the functions $\eta_n(x) := \min\{\max\{n - d(x_0, d), 0\}, 1\}$ for $n \in \mathbb{N}$. Then, define $f_n = \eta_n f$ so that $f_n|_{B(x_0, n-1)} = f|_{B(x_0, n-1)}$. For each f_n we can define a differential $\mathrm{d}f_n$, and $\mathrm{d}f_n|_{B(x_0, \min(m,n)-1)} = \mathrm{d}f_m|_{B(x_0, \min(m,n)-1)}$ (a.e.) for each $n, m \in \mathbb{N}$. Thus, we can define $\mathrm{d}f(x) = \mathrm{d}f_n(x)$ for $x \in B(x_0, n-1)$ with only an ambiguity on a null set. It is easy to check that $\mathrm{d}f$ is a differential. $\square$

4D. *Differential and pointwise norm.* Let $|\cdot|_x := \Phi^x$ and define

$$\Gamma_p(T^*U) = \{\boldsymbol{\xi} : U \to (\mathbb{R}^N)^* \text{ Borel} : \|\boldsymbol{\xi}\|_{\Gamma_p(T^*U)} < \infty\}, \quad \|\boldsymbol{\xi}\|_{\Gamma_p(T^*U)} := \left(\int_U |\boldsymbol{\xi}|_x^p \, \mathrm{d}\mu \right)^{1/p}$$

(with the usual identification of elements that agree μ-a.e.). Then $(\Gamma_p(T^*U), \|\cdot\|_{\Gamma_p(T^*U)})$ is a normed space. Observe that, if $V_j := U \cap \{I(\varphi) \geq 1/j\}$, the sets $U_j := V_j \setminus \bigcup_{i<j} V_i$ partition U up to a null-set

and we have an isometric identification

$$\Gamma_p(T^*U) \simeq \bigoplus_{l^p} \Gamma_p(T^*U_j), \quad \text{where } \Gamma_p(T^*U_j) \simeq L^p(U_j; (\mathbb{R}^N)^*). \tag{4-4}$$

Thus $(\Gamma_p(T^*U), \|\cdot\|_{\Gamma_p(T^*U)})$ is a Banach space. Recall, that an ℓ_p-direct sum of Banach spaces B_i with norms $\|\cdot\|_{B_i}$ with countable index set I is defined by

$$\bigoplus_{\ell^p} B_i := \{(v_i)_{i \in I} : \|(v_i)_{i \in I}\| = (\|v_i\|_{B_i}^p)^{1/p}, v_i \in B_i\}.$$

Lemma 4.7. *Suppose* $(f_n) \subset \mathrm{LIP}_b(X)$ *is a sequence such that* $f_n \to f$ *in* $L^p(X)$ *and* $\mathrm{d}f_n \to \boldsymbol{\xi}$ *in* $\Gamma_p(T^*U)$ *for some* $f \in N^{1,p}(X)$ *and* $\boldsymbol{\xi} \in \Gamma_p(T^*U)$. *Then* $\boldsymbol{\xi}$ *is the* (*uniquely defined*) *differential of* f *in* U, *and*

$$\lim_{n\to\infty} \int_U |D(f_n - f)|_p^p \, \mathrm{d}\mu = 0.$$

In particular, $\Phi(\boldsymbol{\xi}, \cdot) = |Df|_p$ μ-*a.e. in* U.

Proof. By Lemma 4.5 and Fuglede's theorem [1957, Theorem 3(f)] (applied to the sequence of functions $h_n = \chi_U(\gamma_t)|\mathrm{d}_{\gamma_t} f_n - \boldsymbol{\xi}_{\gamma_t}|_{\gamma_t}$ and f_n) we can pass to a subsequence so that

$$\lim_{n\to\infty} \int_0^1 \chi_U(\gamma_t)|(f_n \circ \gamma)'_t - \boldsymbol{\xi}_{\gamma_t}((\varphi \circ \gamma)'_t)| \, \mathrm{d}t \le \lim_{n\to\infty} \int_0^1 \chi_U(\gamma_t)|\mathrm{d}_{\gamma_t} f_n - \boldsymbol{\xi}_{\gamma_t}|_{\gamma_t}|\gamma'_t| \, \mathrm{d}t = 0,$$

$$\lim_{n\to\infty} \int_0^1 |f_n(\gamma_t) - f(\gamma_t)||\gamma'_t| \, \mathrm{d}t = 0 \tag{4-5}$$

for p-a.e. $\gamma \in \mathrm{AC}(I; X)$. Fix a curve γ where (4-5) holds and $f_n \circ \gamma$, $f \circ \gamma$ are absolutely continuous. We may assume that γ is constant-speed parametrized. By (4-5), $f_n \circ \gamma \to f \circ \gamma$ in $L^1([0,1])$ and $(f_n \circ \gamma)' \to g$ in $L^1(\gamma^{-1}(U))$, where $g(t) := \chi_U(\gamma_t)\boldsymbol{\xi}_{\gamma_t}((\varphi \circ \gamma)'_t)$. It follows that

$$(f \circ \gamma)'_t = \boldsymbol{\xi}_{\gamma_t}((\varphi \circ \gamma)'_t) \quad \text{a.e. } t \in \gamma^{-1}(U).$$

This shows that $\boldsymbol{\xi}$ is the differential of f, and uniqueness follows from Lemma 4.3(3). The identity $((f - f_n) \circ \gamma)'_t = (\boldsymbol{\xi}_{\gamma_t} - \mathrm{d}f_n)((\varphi \circ \gamma)'_t)$ for a.e. $t \in \gamma^{-1}(U)$, for p-a.e. $\gamma \in \mathrm{AC}(I; X)$, together with Lemma 3.2(3), implies $\Phi^x(\boldsymbol{\xi} - \mathrm{d}f_n) \le |D(f - f_n)|_p$ for μ-a.e. $x \in U$. By the convergence $\mathrm{d}(f_m - f_n) \to \boldsymbol{\xi} - \mathrm{d}f_n$ (as $m \to \infty$) we have $|D(f_m - f_n)|_p \to_{m\to\infty} \Phi^x(\boldsymbol{\xi} - \mathrm{d}f_n)$ in $L^p(U)$, and thus $|D(f - f_n)|_p \le \Phi^x(\boldsymbol{\xi} - \mathrm{d}f_n)$ μ-a.e. in U. Thus $|D(f - f_n)|_p = \Phi^x(\boldsymbol{\xi} - \mathrm{d}f_n)$ converges to zero in $L^p(U)$. The equality $\Phi_{\boldsymbol{\xi}} = |Df|_p$ follows, completing the proof. □

We say that a sequence $(\boldsymbol{\xi}_n)_n \subset \Gamma_p(T^*U)$ is equi-integrable if the sequence $\{|\boldsymbol{\xi}_n|_x\}_n \subset L^p(U)$ is equi-integrable. Recall, that a collection of integrable functions $\mathcal{F}$ is called equi-integrable, if there is a M so that $\int_X |f|^p \, \mathrm{d}\mu \le M$ for every $f \in \mathcal{F}$ and if for every $\varepsilon > 0$, there is an $\delta > 0$ and a positive measure subset Ω_ε, so that for any measurable set E with $\mu(E) \le \delta$, we have $\int_{\Omega_\varepsilon^c \cup E} |f|^p \, \mathrm{d}\mu \le \varepsilon$ for each $f \in \mathcal{F}$. By the Dunford–Pettis theorem a set of L^1 functions is equi-integrable if and only if it is sequentially compact; see for example [Dunford and Schwartz 1958, Theorem IV.8.9].

Remark 4.8. It follows from (4-4) that, if $(\boldsymbol{\xi}_n)_n \subset \Gamma_p(T^*U)$ is equi-integrable, then there exists $\boldsymbol{\xi} \in \Gamma_p(T^*U)$ such that $\boldsymbol{\xi}_n \rightharpoonup \boldsymbol{\xi}$ weakly in $\Gamma_p(T^*U)$ up to a subsequence and, by Mazur's lemma, that a

convex combination of $\boldsymbol{\xi}_n$'s converges to $\boldsymbol{\xi}$ in $\Gamma_p(T^*U)$. Indeed, the $p > 1$ case is direct and the $p = 1$ case uses the Dunford–Pettis argument above.

Next, we show that any Sobolev function $f \in N^{1,p}$ has a uniquely defined differential with respect to a chart. Note, however, that here we still postulate the existence of charts.

Proof of Theorem 1.7. The measurable norm $|\cdot|_x$ is given by Lemma 4.5. Let $f \in N^{1,p}(X)$. Lemma 4.3(3) implies that df, if it exists, is a.e. uniquely determined on U. Let $(f_n) \subset \mathrm{LIP}_b(X)$ be such that $f_n \to f$ and $|Df_n|_p \to |Df|_p$ in $L^p(\mu)$ as $n \to \infty$, which exists by [Eriksson-Bique 2023, Theorem 1.1]. By Lemma 4.5, $(df_n)_n \subset \Gamma_p(T^*U)$ is equi-integrable. It follows that there exists $\boldsymbol{\xi} \in \Gamma_p(T^*U)$ such that $df_n \rightharpoonup \boldsymbol{\xi}$ weakly in $L^p(T^*U)$; see Remark 4.8. By Mazur's lemma, a sequence $(g_n) \subset \mathrm{LIP}_b(X)$ of convex combinations of the f_n's converges to f in $L^p(\mu)$ and $dg_n \to \boldsymbol{\xi}$ in $\Gamma_p(T^*U)$. By Lemma 4.7, $\boldsymbol{\xi} =: df$ is the differential of f. The linearity of $f \mapsto df$ follows from the uniqueness of differentials; see Lemma 4.3(3). $\qquad\square$

The proof above also yields the following corollary. Note that, while the claim initially holds only after passing to a subsequence, since the limit is unique, the convergence holds along the full sequence.

Corollary 4.9. *Let (U, φ) be a p-weak chart of X. Suppose that $f \in N^{1,p}(X)$ and $(f_n) \subset \mathrm{LIP}_b(X)$ converges to f in energy, that is, $f_n \to_{L^p} f$ and $|Df_n|_p \to_{L^p} |Df|_p$. Then we have that $df_n \rightharpoonup df$ weakly in $\Gamma_p(T^*U)$.*

Using Lemma 4.3 we prove that the differential satisfies natural rules of calculation. The following properties are stated for $f, g \in N^{1,p}(X)$, but they would equivalently hold if we assumed only we have the local assumption $f, g \in N^{1,p}_{\mathrm{loc}}(X)$.

For the following, recall that if $A \subset \mathbb{R}$ is a measurable set, then t is a *density point* of A if

$$\lim_{h \to 0} \frac{|A \cap [t - h, t + h]|}{2h} = 1.$$

Here $|\cdot|$ denotes the Lebesgue measure of the set.

Proposition 4.10. *Let (U, φ) be an N-dimensional p-weak chart of X, $f, g \in N^{1,p}(X)$, and $F : X \to Y$ be a Lipschitz map into a metric measure space (Y, d, ν) with $F_*\mu \leq C\nu$ for some $C > 0$:*

(1) *If (V, ψ) is a p-weak chart with $\varphi|_{U \cap V} = \psi|_{U \cap V}$ then the p-weak differentials of f with respect to both charts agree μ-a.e. on $U \cap V$.*

(2) *If $f|_A = g|_A$ for some A, then $df = dg$ μ-a.e. on $A \cap U$.*

(3) *If $f, g \in L^\infty(X) \cap N^{1,p}(X)$, then $d(fg) = f\, dg + g\, df$ μ-a.e. on U.*

(4) *If $h \in C^1(\mathbb{R})$ and if $h \circ f \in N^{1,p}(X)$, then $d(h \circ f) = h'(f(x))\, df(x)$ holds μ-a.e. on U.*

(5) *Let (V, ψ) be an M-dimensional p-weak chart of Y with $\mu(U \cap F^{-1}(V)) > 0$. For μ-a.e. $U \cap F^{-1}(V)$ there exists a unique linear map $D_x F : \mathbb{R}^N \to \mathbb{R}^M$ satisfying the following: if $h \in N^{1,p}(Y)$ and E is the set of $y \in V$ where the differential $d_y h$ does not exist, then $\mu(U \cap F^{-1}(E)) = 0$ and*

$$d_x(h \circ F) = d_{F(x)}h \circ D_x F \quad \text{for } \mu\text{-a.e. } x \in U \cap F^{-1}(V \setminus E).$$

Proof. Claim (1) follows from Lemma 4.3(2) and the fact that $(\varphi \circ \gamma)'_t = (\psi \circ \gamma)'_t$ for a.e. $t \in \gamma^{-1}(U \cap V)$, for p-a.e. $\gamma \in \mathrm{AC}(I; X)$. Indeed, for p-a.e. curve and a.e. $t \in \gamma^{-1}(U \cap V)$ both derivatives agree since a generic such t will satisfy either $|\gamma'_t| = 0$ or that t is a density point of $\gamma^{-1}(U \cap V)$. In both cases the equality follows.

Claim (2) is similar. Define $\mathrm{d}'f = \mathrm{d}f$ if $x \in U \setminus A$ (when defined) and $\mathrm{d}'f = \mathrm{d}g$ for $x \in A$. Now, suppose for p-almost every absolutely continuous γ we have $(f \circ \gamma)'_t = \mathrm{d}f_{\gamma_t}(\varphi \circ \gamma)'_t$ for a.e. $t \in \gamma^{-1}(U)$ and $(g \circ \gamma)'_t = \mathrm{d}g_{\gamma_t}(\varphi \circ \gamma)'_t$. We will verify for almost every $t \in \gamma^{-1}(U)$ that $(f \circ \gamma)'_t = \mathrm{d}'f_{\gamma_t}(\varphi \circ \gamma)'_t$ so that $\mathrm{d}'f$ is a differential. Then, by uniqueness it agrees with $\mathrm{d}f$. Now, almost every $t \in \gamma^{-1}(U)$ will satisfy that $(f \circ \gamma)'_t$ and $(g \circ \gamma)'_t$ exist and one (or more) of the following: $|\gamma'_t| = 0$, t is a density point of $\gamma^{-1}(A)$, or t is a density point of $\gamma^{-1}(U \setminus A)$. In the first and last cases the equality $(f \circ \gamma)'_t = \mathrm{d}'f_{\gamma_t}(\varphi \circ \gamma)'_t$ is obvious. In the second case $(f \circ \gamma)'_t = (g \circ \gamma)'_t$ because t is a density point.

To prove (3) note that, since we have $(fg \circ \gamma)'_t = g(\gamma_t)(f \circ \gamma)'_t + f(\gamma_t)(g \circ \gamma)'_t$ for a.e. t for p-a.e. curve $\gamma \in \mathrm{AC}(I; X)$, it follows from (1-6) that

$$\mathrm{d}_{\gamma_t}(fg)((\varphi \circ \gamma)'_t) = g(\gamma_t)\, \mathrm{d}f_{\gamma_t}((\varphi \circ \gamma)'_t) + f(\gamma_t)\, \mathrm{d}g_{\gamma_t}((\varphi \circ \gamma)'_t) \quad \text{for a.e. } t \in \gamma^{-1}(U),$$

for p-a.e. $\gamma \in \mathrm{AC}(I; X)$. By Lemma 4.3(2) and (3) the claimed equality holds.

The argument is similar to before. Indeed, for p-a.e. absolutely continuous γ we have that $f \circ \gamma$ is absolutely continuous and $(f \circ \gamma)'_t = \mathrm{d}f_{\gamma_t}(\varphi \circ \gamma)'_t$. Then $h \circ f \circ \gamma$ is differentiable whenever $f \circ \gamma$ is, with derivative $(h \circ f \circ \gamma)'_t = h'(f(\gamma_t))(f \circ \gamma)'_t$. Therefore, $h'(f(x))\mathrm{d}f_x$ is a p-weak differential, and by uniqueness it is the p-weak differential.

Finally, for (5), let $G = (G_1, \ldots, G_M) = \psi \circ F \in \mathrm{LIP}(X; \mathbb{R}^M)$ and define the expression $D_x F := (\mathrm{d}_x G_1, \ldots, \mathrm{d}_x G_M) : \mathbb{R}^N \to \mathbb{R}^M$ for μ-a.e. $x \in U \cap F^{-1}(V)$. We have that

$$(\psi \circ F \circ \gamma)'_t = D_{\gamma_t} F((\varphi \circ \gamma)'_t) \quad \text{for a.e. } t \in \gamma^{-1}(U),$$

for p-a.e. $\gamma \in \mathrm{AC}(I; X)$. Note that if h and E are as in the claim, then $\mu(U \cap F^{-1}(E)) \leq C\nu(E) = 0$. To show the claimed identity, let $\Gamma_0 \subset C(I; Y)$ be a path family with $\mathrm{Mod}_p \Gamma_0 = 0$ such that

$$(h \circ \alpha)'_t = \mathrm{d}_{\alpha_t} h((\psi \circ \alpha)'_t) \quad \text{for a.e. } t \in \alpha^{-1}(V),$$

for every absolutely continuous $\alpha \notin \Gamma_0$, and set $\Gamma_1 = F^{-1}\Gamma_0 := \{\gamma \in C(I; X) : F \circ \gamma \in \Gamma_0\}$. Since $\mathrm{Mod}_p \Gamma_1 \leq C \mathrm{LIP}(F)^p \mathrm{Mod}_p(\Gamma_0) = 0$ it follows from the two identities above that

$$(h \circ F \circ \gamma)'_t = \mathrm{d}_{F(\gamma_t)} h((\psi \circ F \circ \gamma)'_t) = \mathrm{d}_{F(\gamma_t)} h(D_{\gamma_t}((\varphi \circ \gamma)'_t)) \quad \text{for a.e. } t \in \gamma^{-1}(U \cap F^{-1}(V)),$$

for p-a.e. $\gamma \in \mathrm{AC}(I; X)$. Lemma 4.3(2) and (3) imply the claim. $\square$

4E. *Dimension bound.*

In this section we give a geometric condition which guarantees that finite dimensional weak p-charts exist. This involves a bound on the size of p-independent Lipschitz maps.

As a technical tool we need the notion of a decomposability bundle $V(\nu)$ of a Radon measure ν on $\mathbb{R}^m$; see [Alberti and Marchese 2016]. We will not fully define this here, as we only need some of its properties. Firstly, let $\mathrm{Gr}(m)$ be the set of linear subspaces of $\mathbb{R}^m$ equipped with a metric $d(V, V')$ defined as the Hausdorff distance of $V \cap \overline{B(0, 1)}$ to $V' \cap \overline{B(0, 1)}$. The linear dimension of a subspace V is denoted

by $\dim(V)$. The decomposability bundle is then a certain Borel measurable map $\mathbb{R}^m \to \mathrm{Gr}(m)$, which associates to every $x \in \mathbb{R}^m$ a subspace $V(\nu)_x \in \mathrm{Gr}(m)$. In a sense, this bundle measures the directions in which a Lipschitz function must be differentiable in (at almost every point). We collect the main properties we need for this bundle and briefly cite where the proofs of these claims can be found.

Theorem 4.11. *Suppose that ν is a Radon measure on $\mathbb{R}^m$. Then there exists a decomposability bundle $V(\nu)$ with the following properties:*

(1) *If $\dim(V(\nu)_x) = m$ for ν-a.e. $x \in \mathbb{R}^m$, then $\nu \ll \lambda$.*

(2) *There is a Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$ so that for ν-a.e. $x \in \mathbb{R}^m$ we have that the directional derivative of f does not exist in the direction v for any $v \notin V(\nu)_x$.*

(3) *If $\nu' \ll \nu$, then $V(\nu')_x = V(\nu)_x$ for ν'-a.e. $x \in \mathbb{R}^m$.*

Proof. The first follows from [De Philippis and Rindler 2016, Theorem 1.14] when combined with [Alberti and Marchese 2016, Theorem 1.1(i)]. The second claim follows from [loc. cit., Theorem 1.1(ii)]. Note that the second claim is vacuous for those points $x \in \mathbb{R}^m$ where the decomposability bundle has dimension m. The third claim is [loc. cit., Proposition 2.9(i)]. $\square$

The following lemma gives a modulus perspective to the decomposability bundle.

Lemma 4.12. *Assume $N \geq 1$, $\varphi : X \to \mathbb{R}^N$ is Lipschitz, $U \subset X$ is a Borel set of bounded measure and $\nu = \varphi_*(\mu|_U)$. Then, for p-a.e. curve γ and almost every $t \in \gamma^{-1}(U)$ we have that $(\varphi \circ \gamma)'_t$ exists and $(\varphi \circ \gamma)'_t \in V(\nu)_{\varphi(\gamma_t)}$.*

Proof. By part (ii) of Theorem 4.11, there is a Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$, so that for ν-almost every $x \in \mathbb{R}^N$ and any $v \notin V(\nu)_x$ we have that the directional derivative $D_v(f) = \lim_{h \to 0} (f(x + hv) - f(x))/h$ does not exist. Let $A \subset \mathbb{R}^N$ be a full ν-measure Borel set so that this claim holds.

Let $B = \varphi^{-1}(\mathbb{R}^N \setminus A) \cap U$, which is μ-null. The family Γ_B^+ has null p-modulus. We will show that the claim holds for p-a.e. $\gamma \in \mathrm{AC}(I; X) \setminus \Gamma_B^+$. The derivatives $(\varphi \circ \gamma)'_t$ and $(f \circ \varphi \circ \gamma)'_t$ exist for almost every $t \in \gamma^{-1}(U)$. Also, for a.e. $t \in I$ we can either take $|\gamma'_t| = 0$ or $\gamma_t \notin B$ and so $(\varphi \circ \gamma)_t \notin A$, since $\gamma \notin \Gamma_B^+$. If $|\gamma'_t| = 0$, then $(\varphi \circ \gamma)'_t = 0 \in V(\nu)_{\varphi(\gamma_t)}$. In the other case, when $\gamma_t \notin B$, the function f does not have a directional derivative for $v \notin V(\nu)_{(\varphi \circ \gamma)_t}$. The only way for both $(\varphi \circ \gamma)'_t$ and $(f \circ \varphi \circ \gamma)'_t$ to exist then is if $(\varphi \circ \gamma)'_t \in V(\nu)_{\varphi(\gamma_t)}$, which gives the claim. $\square$

The following should be compared to [Cheeger 1999, Lemma 4.37].

Proposition 4.13. *Suppose $\varphi \in \mathrm{LIP}(X; \mathbb{R}^N)$ is p-independent on U. Then $N \leq \dim_H U$.*

Proof. By restriction to a subset of the form $U \cap B(x_0, R)$ for $x_0 \in X$, $R > 0$, of positive measure, it suffices to assume that U has finite measure. The claim is automatic, if $\dim_H U = \infty$. Thus, assume that the Hausdorff dimension is finite. Set $\nu = \varphi_*(\mu|_U)$ and let $V(\nu)$ be the decomposability bundle of ν. If $V(\nu)_x$ has dimension N for almost every x with respect to ν, then $\nu \ll \lambda$ by Theorem 4.11(1) and thus $\mathcal{H}^N(\varphi(U)) > 0$, since ν is concentrated on $\varphi(U)$. Then $N \leq \dim_H(\varphi(U)) \leq \dim_H(U)$.

Suppose then to the contrary, that there exists a subset $A \subset U$ with positive ν-measure where $V(\nu)_x$ has dimension less than $\dim_H(U)$ for each $x \in A$. We can take A to be Borel. Consider $\mu' = \mu|_{\varphi^{-1}(A)}$, which has

push-forward $\nu' = \nu|_A = \varphi_*(\mu')$. By the third part in Theorem 4.11 we have that $V(\nu')_x = V(\nu)_x$ for ν'-a.e. $x \in A$. Further $\varphi^{-1}(A) \subset U$, so φ is still p-independent on $\varphi^{-1}(A) = U'$. Now, by considering U' instead of U and ν' instead of ν, we have that $V(\nu')_{\varphi(x)}$ has dimension less than N for ν'-almost every $x \in U$. In the following, we simplify notation by dropping the primes, and restricting to the positive measure subset U' so constructed. For ν-almost every $x \in U$, we have that $V(\nu)_{\varphi(x)}$ is a strict subspace of $\mathbb{R}^N$, and thus there are vectors perpendicular to these. Since $x \to V(\nu)_{\varphi(x)}$ is Borel, we can choose a Borel map $x \to \boldsymbol{\xi}_x \in (\mathbb{R}^N)^*$ so that $\boldsymbol{\xi}_x$ is a unit vector that vanishes on $V(\nu)_{\varphi(x)}$ for μ-a.e. $x \in U$ (see, e.g., [Bogachev 2007, Theorem 6.9.1], which is an instance of a Borel selection theorem). Let $\widetilde{U} \subset U$ be the full measure subset where these properties hold for every $x \in \widetilde{U}$. Now, by Lemma 4.12 we have for p-a.e. curve γ that $(\varphi \circ \gamma)'_t \in V(\nu)_{\varphi(\gamma_t)}$ for almost every $t \in \gamma^{-1}(U)$. The set $U \setminus \widetilde{U}$ has null measure, and thus $\Gamma^+_{U \setminus \widetilde{U}}$ has null modulus.

Thus, for p-a.e. curve $\gamma \in \mathrm{AC}(I; X)$ and a.e. $t \in \gamma^{-1}(U)$ we can further assume $\gamma_t \in \widetilde{U}$ or $|\gamma'_t| = 0$. Therefore, $\boldsymbol{\xi}_{\gamma_t}((\varphi \circ \gamma)'_t) = 0$ for almost every $t \in \gamma^{-1}(U)$ and such curves γ. By part (2) of Lemma 4.3, we have that $I(\varphi) \leq \Phi^x(\boldsymbol{\xi}_x) = 0$ for μ-a.e. $x \in U$. This contradicts p-independence and proves the claim. $\square$

4F. Sobolev charts. By definition, a p-weak chart is a Lipschitz map which has target of maximal dimension with respect to Lipschitz maps. The notions of p-independence and maximality, however, are well-defined for any Sobolev map, and in fact p-weak charts *could* be required to have Sobolev (instead of Lipschitz) regularity. Despite the apparent difference of the alternative definition, the existence of maximal p-independent Sobolev maps also guarantees the existence of p-weak chart of the same dimension. This follows from the energy density of Lipschitz functions, see [Eriksson-Bique 2023], together with results of the previous subsection.

Proposition 4.14. *Suppose $p \geq 1$, and $\varphi \in N^{1,p}(X; \mathbb{R}^N)$ is p-independent and p-maximal in a bounded Borel set $U \subset X$. For any $\varepsilon > 0$ there exists $V \subset U$ with $\mu(U \setminus V) < \varepsilon$, and a Lipschitz function $\psi : X \to \mathbb{R}^N$ such that (V, ψ) is an N-dimensional p-weak chart.*

Proof. For any $V \subset U$ with $\mu(V) > 0$, let n_V be the supremum of numbers n so that there exists $\psi \in \mathrm{LIP}_b(X; \mathbb{R}^n)$ which is p-independent on a positive measure subset of V. By the maximality of N we have that $n_V \leq N$. Thus n_V is attained for every such V and, by [Keith 2004a, Proposition 3.1], there is a partition of U up to a null-set by p-weak charts V_i, $i \in \mathbb{N}$, of dimension $\leq N$. By [Eriksson-Bique 2023, Theorem 1.1], Corollary 4.9 (with a diagonal argument) and Mazur's lemma we have that, for each component $\varphi_k \in N^{1,p}(X)$ of φ, there exists a sequence $(\psi_k^n) \subset \mathrm{LIP}_b(X)$ with $|D(\varphi_k - \psi_k^n)|_p \to 0$ in $L^p(V_i)$. Thus, $|D(\varphi_k - \psi_k^n)|_p \to 0$ in $L^p(U)$. Here, we use that $|D(\varphi_k - \psi_k^n)|_p \leq |D\varphi_k|_p + |D\psi_k^n|_p$ and the L^p-convergence of the right-hand side from [Eriksson-Bique 2023].

If Φ and Ψ_n denote the canonical minimal gradients associated to φ and $\psi^n := (\psi_1^n, \ldots, \psi_N^n)$, we have

$$\sup_{\|\xi\|_*=1} |\Phi(\xi, \cdot) - \Psi_n(\xi, \cdot)| \leq \underset{\|\xi\|_*=1}{\mathrm{ess\,sup}} |D(\xi \circ (\varphi - \psi^n))|_p \leq \sum_{k=1}^N |D(\varphi_k - \psi_k^n)|_p \quad \mu\text{-a.e. in } U.$$

It follows that

$$\lim_{n \to \infty} \mu(U \setminus \{I(\psi^n) > 0\}) = 0,$$

completing the proof, since ψ^n is p-independent and maximal on the set $\{I(\psi^n) > 0\}$. $\square$

Another condition in this context is strong maximality: a map $\varphi \in N^{1,p}(X; \mathbb{R}^N)$ is *strongly maximal* in $U \subset X$ if no positive measure subset $V \subset U$ admits a p-independent Sobolev map into a higher-dimensional Euclidean space. This condition excludes not only Lipschitz, but also Sobolev functions into higher-dimensional targets, and is thus a priori stronger than maximality. However, it follows from Proposition 4.14 that a maximal p-independent Sobolev map is also strongly maximal. Conversely, if one has a Lipschitz chart, then the Lipschitz chart is also strongly maximal.

4G. *p-weak charts in Poincaré spaces.* Recall that a metric measure space $X = (X, d, \mu)$ is said to be a p-PI space if μ is doubling, and X supports a weak $(1, p)$-Poincaré inequality: there exist constants $C, \sigma > 0$ so that, for any $f \in L^1(X)$ with upper gradient g, we have

$$\fint_B |f - f_B| \, d\mu \le Cr \left(\fint_{\sigma B} g^p \, d\mu \right)^{1/p}$$

for all balls $B \subset X$ of radius r. Here $h_B = \fint_B h \, d\mu = (1/\mu(B)) \int_B h \, d\mu$ for a ball $B \subset X$ and $h \in L^1(B)$. The celebrated result from [Cheeger 1999] states that a PI-space admits a Lipschitz differentiable structure. We will return to this structure in Section 6B, but here recall the constructions from [Cheeger 1999, Section 4]. Cheeger's paper does not employ the following terminology, but it simplifies and clarifies our presentation.

Given a Lipschitz map $\varphi : X \to \mathbb{R}^N$ and a positive measure subset $U \subset X$ the pair (U, φ) is called *a Cheeger chart* if for every Lipschitz map $f : X \to \mathbb{R}$ and a.e. $x \in U$ there is a unique element $d_{C,x} f \in (\mathbb{R}^N)^*$ satisfying

$$\text{Lip}(d_{C,x} f \circ \varphi - f)(x) = 0. \tag{4-6}$$

This equality is equivalent to (1-4).

Proof of Theorem 1.8. Let (U, φ) be a p-weak chart of dimension N and let $f \in \text{LIP}(X)$. Denote by Φ the canonical minimal gradient of $(\varphi, f) : X \to \mathbb{R}^{N+1}$; see Lemma 4.1. Since X is a p-PI space, it follows that $\text{Lip } h = |Dh|_p$ μ-a.e. for any $h \in \text{LIP}(X)$; see [Cheeger 1999, Theorem 6.1]. (In fact, the slightly easier comparability from Lemma 4.35 of that work suffices for the following.) Then, for any $\xi \in (\mathbb{R}^N)^*$ and for μ-a.e. $x \in U$, we have

$$\text{Lip}(\xi \circ \varphi - f)(x) = \Phi^x(\xi, -1), \quad \xi \in (\mathbb{R}^N)^*.$$

Arguing using in the proof of Lemmas 4.1 and 4.5 we obtain this equality, simultaneously, for a.e. $x \in U$ and for any $\xi \in A$ for a dense subset of $A \subset (\mathbb{R}^N)^*$. From this, and the continuity of both sides in ξ, we obtain that for μ-a.e. $x \in U$, the equality holds simultaneously for all $\xi \in (\mathbb{R}^N)^*$.

Since the p-weak differential df is characterized by the property $\Phi^x(df, -1) = 0$ for μ-a.e. $x \in U$, it follows that, for μ-a.e. $x \in U$, $d_x f \in (\mathbb{R}^N)^*$ satisfies (4-6). Thus (U, φ) is a Cheeger chart. The uniqueness follows from the equality in a similar way. $\square$

Remark 4.15. The proof of Theorem 1.8 also yields the claim under the weaker assumption $\text{Lip } f \le \omega(|Df|_p)$ for some collection of moduli of continuity ω (compare Theorem 1.10) since the equality $\text{Lip } f = |Df|_p$ follows from this by [Ikonen et al. 2022, Theorem 1.1].

5. The *p*-weak differentiable structure

5A. *The p-weak cotangent bundle.* A *measurable L^∞-bundle* $\mathcal{T}$ over X consists of a collection $(\{U_i, V_{i,x}\})_{i \in I}$ together with a collection $(\{\phi_{i,j,x}\})$ of transformations with a countable index set I, where:

(1) $U_i \subset X$ are Borel sets for each $i \in I$, and cover X up to a μ-null set.

(2) For any $i \in I$ and μ-a.e. $x \in U_i$, $V_{i,x} = (V_i, |\cdot|_{i,x})$ is a finite-dimensional normed space so that $x \mapsto |v|_{i,x}$ is Borel for any $v \in V_i$.

(3) For any $i, j \in I$ and μ-a.e. $x \in U_i \cap U_j$, $\phi_{i,j,x} : V_{i,x} \to V_{j,x}$ is an isometric bijective linear map satisfying the *cocycle condition*: for any $i, j, k \in I$ and μ-a.e. $x \in U_i \cap U_j \cap U_k$, we have $\phi_{j,k,x} \circ \phi_{i,j,x} = \phi_{i,k,x}$.

For each $i \in I$ and μ-a.e. $x \in U_i$, we denote by $\mathcal{T}_x$ the equivalence class of the normed vector space $V_{i,x}$ under identification by isometric isomorphisms. By (3), $\mathcal{T}_x$ is well-defined for μ-a.e. $x \in X$.

We now show that a p-weak differentiable structure $\mathscr{A}$ on X gives rise to a measurable bundle.

Proposition 5.1. *Let $p \geq 1$, and let $\{(U_i, \varphi_i)\}$ be an atlas of p-weak charts on X. The collection $\{(U_i, (\mathbb{R}^{N_i})^*, |\cdot|_{i,x})\}$ forms a measurable bundle over X, the transformations given by the collection $\{D\Phi_{i,j,x}\}$ constructed in Lemma 5.2.*

First, we construct the transformation maps.

Lemma 5.2. *Let (U_i, φ^i) be N_i-dimensional p-weak charts on X, with corresponding differentials d^i and norms $|\cdot|_{i,x}$ for $i = 1, 2$. If $\mu(U_1 \cap U_2) > 0$, then $N_1 = N_2 := N$ and, for μ-a.e. $x \in U_1 \cap U_2$, there exists a unique bijective isometric isomorphism $D\Phi_{1,2,x} : ((\mathbb{R}^N)^*, |\cdot|_{1,x}) \to ((\mathbb{R}^N)^*, |\cdot|_{2,x})$ such that $d^1 f = d^2 f \circ D\Phi_{1,2,x}$. Further $D\Phi_{1,2,x}$ satisfies the measurability constraint (2).*

In the proof, we denote by $\varphi^i_1, \ldots, \varphi^i_{N_i}$ the components of φ^i.

Proof. For μ-a.e. $x \in U_1 \cap U_2$, define

$$D_x = D = (d^1 \varphi^1_1, \ldots, d^2 \varphi^1_{N_1}) : \mathbb{R}^{N_2} \to \mathbb{R}^{N_1}.$$

D is a linear map satisfying, for all $\xi \in (\mathbb{R}^{N_1})^*$,

$$\xi \circ D((\varphi^2 \circ \gamma)'_t) = \xi((\varphi^1 \circ \gamma)'_t) \quad \text{for a.e. } t \in \gamma^{-1}(U_1 \cap U_2), \tag{5-1}$$

for p-a.e. $\gamma \in \Gamma^+_{U_1 \cap U_2}$. Note that, by the uniqueness of differentials, D is the unique linear map satisfying (5-1) for p-a.e. curve. By Lemma 4.3(2) it follows that

$$|\xi \circ D|_{2,x} = |\xi|_{1,x}, \quad \xi \in (\mathbb{R}^{N_1})^*,$$

for μ-a.e. $x \in U_1 \cap U_2$. Thus D^* is an isometric embedding and in particular $N_1 \leq N_2$. Reversing the roles of φ^1 and φ^2 we obtain that $N_1 = N_2$ and consequently $D\Phi_{1,2,x} := D^*_x : ((\mathbb{R}^{N_1})^*, |\cdot|_{1,x}) \to ((\mathbb{R}^{N_2})^*, |\cdot|_{2,x})$ is an isometric isomorphism for μ-a.e. $x \in U_i \cap U_j$.

For any $f \in N^{1,p}(X)$, the identity $d^1_x f = d^2_x f \circ D\Phi_{1,2,x}$ for μ-a.e. $x \in U_1 \cap U_2$ follows from (5-1) and (1-6). $\qquad\square$

Proof of Proposition 5.1. Conditions (1) and (2) are satisfied by Lemma 4.2. The cocycle condition follows from Lemma 5.2. □

Definition 5.3. We call the measurable bundle given by Proposition 5.1 the p-weak cotangent bundle and denote it by $T_p^* X$. We define $T_{p,x}^* X = ((\mathbb{R}^N)^*, |\cdot|_x)$ and $T_{p,x} X = (\mathbb{R}^N, |\cdot|_{*,x})$ for almost every $x \in U$, where (U, φ) is an N-dimensional p-weak chart and $|\cdot|_x$ the norm given by the canonical minimal gradient Φ; see Lemmas 4.1 and 4.5. The spaces $T_{p,x}$ are here defined pointwise almost everywhere. By considering the adjoints of transition maps in the definition above, one can patch these together to form a measurable L^∞ tangent bundle, which is dual to $T_p^* X$, whose fibers are $T_{p,x} X$.

The next proposition establishes the existence of a p-weak differentiable structure under a mild finite dimensionality condition.

Proposition 5.4. *Suppose X is a metric measure space and $\{X_i\}_{i \in \mathbb{N}}$ a covering of X with $\dim_H X_i < \infty$. Then, for any $p \geq 1$, X admits a p-weak differentiable structure. Moreover, $N \leq \dim_H X_i$ whenever (U, φ) is an N-dimensional p-weak chart with $\mu(U \cap X_i) > 0$.*

Proof. For any Borel set $U \subset X$ with $\mu(U) > 0$ there exists $i \in \mathbb{N}$ such that $\mu(U \cap X_i) > 0$. By Proposition 4.13 we have that $N \leq \dim_H(U \cap X_i)$ whenever $\varphi \in \mathrm{LIP}_b(X; \mathbb{R}^N)$ is p-independent in a positive measure subset of $U \cap X_i$. Using [Keith 2004b, Proposition 3.1] we can cover X up to a null-set by Borel sets U_k for which there exist $\varphi_k \in \mathrm{LIP}_b(X; \mathbb{R}^{N_k})$ that are p-independent and p-maximal on U_k. The collection $\{(U_k, \varphi_k)\}_{k \in \mathbb{N}}$ is a p-weak differentiable structure on X. The last claim follows by the argument above. □

5B. *Sections of measurable bundles.* A measurable bundle $\mathcal{T}$ over X comes with a projection map $\pi : \mathcal{T} \to X$, $(x, v) \mapsto x$, and a *section* of $\mathcal{T}$ is a collection $\omega = \{\omega_i : U_i \to V_i\}$ of Borel measurable maps satisfying $\pi \circ \omega_i = \mathrm{id}_{U_i}$ μ-a.e. and $\phi_{i,j,x}(\omega_i) = \omega_j$ for each $i, j \in I$ and almost every $x \in U_i \cap U_j$. Observe that the map $x \mapsto |\omega(x)|_x$ given by

$$|\omega(x)|_x := |\omega_i(x)|_{i,x} \quad \text{for } \mu\text{-a.e. } x \in U_i \tag{5-2}$$

is well-defined up to negligible sets by the cocycle condition and the fact that $\phi_{i,j,x}$ is isometric.

Definition 5.5. For $p \in [1, \infty]$, let $\Gamma_p(\mathcal{T})$ be the space of sections ω of $\mathcal{T}$ with

$$\|\omega\|_p := \|x \mapsto |\omega(x)|_x\|_{L^p(\mu)} < \infty.$$

We call $\Gamma_p(\mathcal{T})$ the space of p-integrable sections of $\mathcal{T}$. The space $\Gamma_p(T_p^* X)$ is called the p-weak cotangent module.

Note that $\Gamma_p(\mathcal{T})$, equipped with the pointwise norm (5-2) and the natural addition and multiplication operations, is a normed module in the sense of [Gigli 2015]. Recall that an L^p-normed L^∞-module over X is a Banach module $(\mathscr{M}, \|\cdot\|)$ over $L^\infty(X)$, equipped with a pointwise norm $|\cdot| : X \to \mathbb{R}$ that satisfies

$$|gm| = |g||m| \quad \text{and} \quad \|m\| = \left(\int_X |m|_x^p \, \mathrm{d}\mu(x)\right)^{1/p}$$

for all $m \in \mathscr{M}$ and $g \in L^\infty(X)$. We refer to [Gigli 2015; 2018] for a detailed account of the theory of normed modules.

Next we consider the p-weak cotangent module $\Gamma_p(T_p^*X)$. For a p-weak chart (U, φ) of X and $f \in N^{1,p}(X)$, denote by $\mathrm{d}_{(U,\varphi)}f$ the differential of f with respect to (U, φ). Lemma 5.2 implies that the collection of differentials with respect to different charts satisfies the compatibility condition above.

Definition 5.6. Let $p \geq 1$, and suppose $\mathscr{A}$ is a p-weak differentiable atlas of X. For any $f \in N^{1,p}(X)$, the differential $\mathrm{d}f \in \Gamma_p(T_p^*X)$ is the element in the p-weak cotangent module defined by the collection $\{\mathrm{d}_{(U,\varphi)}f : U \to (\mathbb{R}^N)^*\}_{(U,\varphi)\in\mathscr{A}}$.

We record the following properties of the differential.

Proposition 5.7. *Let $A \subset X$ be a Borel set and $F : X \to Y$ a Lipschitz map to a metric measure space (Y, d, ν) admitting a p-weak differentiable structure, with $F_*\mu \leq C\nu$.*

(1) *If $f, g \in N^{1,p}(X)$ agree on $A \subset X$, then $\mathrm{d}f = \mathrm{d}g$ μ-a.e. on A.*

(2) *If $f, g \in N^{1,p}(X) \cap L^\infty(X)$, then $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$ μ-a.e.*

(3) *If E is the set of $y \in Y$ for which $T_{p,y}^*Y$ does not exist, then $\mu(F^{-1}(E)) = 0$ and, for μ-a.e. $x \in X \setminus F^{-1}(E)$ there exists a unique linear map $D_xF : T_{p,x}X \to T_{p,F(x)}Y$ such that*

$$\mathrm{d}_x(h \circ F) = \mathrm{d}_{F(x)}h \circ D_xF \quad \text{for } \mu\text{-a.e. } x,$$

for every $h \in N^{1,p}(Y)$.

(4) *If $h \in C^1(\mathbb{R})$ and if $h \circ f \in N^{1,p}(X)$, then $\mathrm{d}(h \circ f) = h'(f(x))\,\mathrm{d}f$.*

(5) *If $f_i \in N^{1,p}(X)$ and there is a function $f \in L^p$ and a $w \in \Gamma_p(T_p^*X)$ so that $\lim_{i\to\infty} f_i = f(x)$ converges in $L^p(X)$ and $\mathrm{d}f_i \to w$ converges in $\Gamma_p(T_p^*X)$, then, there is a function $\tilde{f} \in N^{1,p}(X)$ so that $\tilde{f} = f$ almost everywhere with $\mathrm{d}\tilde{f} = w$.*

Proof. The proofs of the first four claims follow directly from Proposition 4.10 together with the compatibility condition of sections. Indeed, one can verify the identities for each chart (U, φ), from which the identities follows for everywhere.

Consider now $f_i \in N^{1,p}(X)$ which converge in $L^p(X)$ to $f \in L^p(X)$ and so that $\mathrm{d}f_i$ converge in $\Gamma_p(T_p^*X)$ to $w \in \Gamma_p(T_p^*X)$. We have therefore that $g_i = |Df_i|_p = |\mathrm{d}f_i|$ converges in L^p to $g = |w|$. By Fuglede's theorem [1957, Theorem 3(f)] we can pass to a subsequence so that $f_i \to \tilde{f}$ converges pointwise and so that

$$\int_\gamma |f_i - \tilde{f}|\,\mathrm{d}s \to 0 \quad \text{and} \quad \int_\gamma |g_i - g|\,\mathrm{d}s \to 0$$

for p-a.e. absolutely continuous curves $\gamma : [0, 1] \to X$. Then, for all such curves, we have that $|f_i(\gamma(0)) - f_i(\gamma(1))| \leq \int_\gamma g_i\mathrm{d}s$, which converges to

$$|\tilde{f}(\gamma(0)) - \tilde{f}(\gamma(1))| \leq \int_\gamma g\,\mathrm{d}s.$$

Thus $\tilde{f} \in N^{1,p}$. Finally, one only needs to show that $w = \mathrm{d}\tilde{f}$. This follows by another diagonal argument and computing $\mathrm{d}\tilde{f}$ in charts using the argument from Lemma 4.7. $\qquad\square$

We finish the subsection with a proof of the density of Lipschitz functions in Newtonian spaces.

Proof of Theorem 1.9. Let $f \in N^{1,p}(X)$. By [Eriksson-Bique 2023], there exists a sequence $(f_n) \subset \text{LIP}_b(X)$ with $f_n \to f$ and $|Df_n|_p \to |Df|_p$ in $L^p(\mu)$. It follows that $(\mathrm{d}f_n) \subset \Gamma_p(T^*X)$ is equi-integrable, and Remark 4.8 and Lemma 4.7, together with a diagonalization argument over a union of charts covering X, show that $\mathrm{d}\tilde{f}_n \to \mathrm{d}f$ in $\Gamma_p(T^*X)$ for convex combinations $\tilde{f}_n \in \text{LIP}_b(X)$ of f_n's. Consequently $|D(\tilde{f}_n - f)|_p \to 0$ in $L^p(\mu)$. $\qquad\square$

5C. *Dependence of the p-weak differentiable structures on p.* Suppose $1 \leq p < q$. We have that $|Df|_p \leq |Df|_q$ μ-a.e. for every $f \in \text{LIP}_b(X)$, and the inequality may be strict; see [Di Marino and Speight 2015]. As a consequence, if $\varphi \in \text{LIP}_b(X; \mathbb{R}^N)$ is q-maximal in $U \subset X$, then it is p-maximal. It follows (using this dimension upper bound and [Keith 2004b, Proposition 3.1]) that if X admits a q-weak differentiable structure then X also admits a p-weak differentiable structure. We remark that the structures may be different.

For the following statement we say that a *bundle map* $\pi : \mathcal{T} \to \mathcal{T}'$ between two measurable bundles $\mathcal{T} = (\{U_i, V_{i,x}\}, \{\phi_{i,l,x}\})_{i \in I}$ and $\mathcal{T}' = (\{U_j', V_{j,x}'\}, \{\psi_{j,k,x}\})_{j \in J}$ over X is a collection of linear maps $\{\pi_{i,j,x} : V_i \to V_j'\}$ for μ-a.e. $x \in U_i \cap U_j'$ such that

 (a) for each $i \in I$, $j \in J$ the map $x \mapsto \pi_{i,j,x}(v) : U_i \cap U_j' \to V_j'$ is Borel for any $v \in V_i$,

 (b) for each $i, l \in I$, $j, k \in J$ and μ-a.e. $x \in U_i \cap U_l \cap U_j' \cap U_k'$, we have the compatibility condition
   $$\psi_{j,k,x} \circ \pi_{i,j,x} = \phi_{l,j,x} \circ \pi_{i,l,x}.$$

When the underlying index sets agree and $U_i = V_i$ for all $i \in I$, it is sufficient to consider the family $\{\pi_{i,x} := \pi_{i,i,x}\}$, since these determine a unique bundle map.

Proposition 5.8. *Suppose $q > p \geq 1$ and X admits a q-weak differentiable structure. Then X admits p-weak differentiable structure and there is a bundle map $\pi_{p,q} : T_q^*X \to T_p^*X$ which is a linear 1-Lipschitz surjection μ-a.e. Moreover, this map satisfies $\pi_{p,q} = \pi_{p,s} \circ \pi_{s,q}$ for $q > s > p$, and $\pi_{p,q}(\mathrm{d}_q f) = \mathrm{d}_p f$ for any $f \in \text{LIP}_b(X)$, where $\mathrm{d}_q f, \mathrm{d}_p f$ are the p- and q-weak differentials respectively.*

Proof. Since X admits a q-differential structure, we can find q-charts $(U_i, \varphi_{q,i})$ so that $X = \bigcup_{i \in \mathbb{N}} U_i \cup N$, with $\mu(N) = 0$, and $\varphi_{q,i} \in N^{1,p}(X; \mathbb{R}^{m_i})$ is Lipschitz. Assume that U_i are chosen to be pairwise disjoint. As $|Df|_p \leq |Df|_q$ (a.e.) for any $f \in \text{LIP}_b(X)$, any p-independent map is also q-independent. Any map $\varphi \in N^{1,p}(X; \mathbb{R}^n)$ which is p-independent on some positive-measure subset of U_i must have $n \leq m_i$; see Proposition 4.14. By [Keith 2004b, Proposition 3.1] and this dimension bound we can cover X by maximal p-independent maps, i.e., charts, $(V_j, \varphi_{p,j})$. By considering the countable collection of sets $V_i \cap U_j$, and reindexing, we may assume that $(U_i, \varphi_{q,i})$ and $(U_i, \varphi_{p,i})$ are q- and p- charts, respectively.

We define the matrix A_x for $x \in U_i$ by taking as rows the vectors $\mathrm{d}_{i,p}\varphi_{q,i}^k$ for each component $k = 1, \ldots, m_i$. We define the bundle map $\pi_{p,q}$ by setting $\pi_{p,q}^x(\xi) = \xi \circ A_x$ for μ-a.e. $x \in U_i$. For each ξ we get $\mathrm{d}_p(\xi \circ \varphi_{q,i}) = \xi \circ A_x$. Thus, for p-a.e. curve $\gamma \in \text{AC}(I; X)$ and a.e. $t \in \gamma^{-1}(U)$ we have

$$\xi(\varphi_{q,i} \circ \gamma)_t' = (\xi \circ A_x)(\varphi_{p,i} \circ \gamma)_t'.$$

By the definition of the differential, we get immediately that $\pi_{p,q}(d_q f) = d_p f$ for every $f \in \mathrm{LIP}_b(X)$. Thus, the 1-Lipschitz property follows immediately from the definition of norms combined with $|Df|_p \leq |Df|_q$. The map is clearly a surjective bundle map as well, and by uniqueness of the p-differential, we automatically get $\pi_{p,s} \circ \pi_{s,q} = \pi_{p,q}$. $\qquad\square$

6. Relationship with Cheeger's and Gigli's differentiable structures

6A. *Gigli's cotangent module.* Fix $p \geq 1$. Gigli's cotangent module is the L^p-normed L^∞-module given by the following theorem.

Theorem 6.1. *There exists an L^p-normed L^∞-module $L^p(T^*X)$, with pointwise norm denoted by $|\cdot|_G$, and a bounded linear map $d_G : N^{1,p}(X) \to L^p(T^*X)$ satisfying*

$$|d_G f|_G = |Df|_p, \quad f \in N^{1,p}(X), \tag{6-1}$$

such that the subspace $\mathcal{V}$ defined by

$$\mathcal{V} := \left\{ \sum_j^M \chi_{A_j} d_G f_j : (A_j)_j \text{ Borel partition of } X, \ f_j \in N^{1,p}(X) \right\}$$

is dense in $L^p(T^*X)$. The module $L^p(T^*X)$ is uniquely determined up to isometric isomorphism of normed modules by these properties.

Following [Gigli 2018, Definition 1.4.1] we say that a collection $\{v_1, \ldots, v_N\} \subset L^p(T^*X)$ is *linearly independent* in a Borel set $U \subset X$ if, whenever $g_1, \ldots, g_N \in L^\infty(X)$ satisfy $\left|\sum_j^N g_j v_j\right|_G = 0$ μ-a.e. on U, we have $g_1 = \cdots = g_N = 0$ μ-a.e. in U. A linearly independent collection $\{v_1, \ldots, v_N\}$ in U is a *basis* of $L^p(T^*X)$ in U if, for any $v \in L^p(T^*X)$, there exists a Borel partition $\{U_i\}_{i \in \mathbb{N}}$ of U and $g_1^i, \ldots, g_N^i \in L^\infty(X)$ such that $\left|v - \sum_j^N g_j^i v_j\right|_G = 0$ μ-a.e. on U_i, for every $i \in \mathbb{N}$.

Definition 6.2. Let $p \geq 1$. The cotangent module $L^p(T^*X)$ is locally finitely generated if there exists a Borel partition such that $L^p(T^*X)$ has a finite basis in each set of the partition.

By [Gigli 2018, Proposition 1.4.5], there exists a Borel partition $\{A_N\}_{N \in \mathbb{N} \cup \{\infty\}}$ of X such that $L^p(T^*X)$ has a basis of N elements on A_N for each $N \in \mathbb{N} \cup \{\infty\}$. We call the partition $\{A_N\}$ the *dimensional decomposition* of X. Notice that $L^p(T^*X)$ is locally finitely generated if and only if $\mu(A_\infty) = 0$.

In the forthcoming discussion we identify vectors (and vector fields) $\xi \in \mathbb{R}^N$ with their dual element $v \mapsto v \cdot \xi$ where necessary.

Lemma 6.3. *Let $p \geq 1$, $N \geq 0$, $\varphi = (\varphi_1, \ldots, \varphi_N) \in N^{1,p}(X)^N$, and Φ be the canonical minimal gradient associated to φ. If $\boldsymbol{g} = (g_1, \ldots, g_N) \in L^\infty(X; (\mathbb{R}^N)^*)$, then*

$$\left|\sum_{k=1}^N g_k d_G \varphi_k\right|_{G,x} = \Phi^x(\boldsymbol{g}) \quad \text{for } \mu\text{-a.e. } x \in X.$$

In particular, φ is p-independent on $U \subset X$ if and only if $d_G \varphi_1, \ldots, d_G \varphi_N \in L^p(T^*X)$ are linearly independent on U.

Proof. If $g_1, \ldots, g_N$ are simple functions, then $\boldsymbol{g} = \sum_j^M \chi_{A_j} \xi_j$ for disjoint Borel A_j and some $\xi_j \in (\mathbb{R}^N)^*$. It follows that $\sum_{k=1}^N g_k \, \mathrm{d}_G \varphi_k = \sum_j^M \chi_{A_j} \, \mathrm{d}_G(\xi_j \circ \varphi)$ as elements of $L^p(T^*X)$. Thus

$$\left| \sum_{k=1}^N g_k \, \mathrm{d}_G \varphi_k \right|_x = \left| \sum_j^M \chi_{A_j} \, \mathrm{d}_G(\xi_j \circ \varphi) \right|_x = \sum_j^M \chi_{A_j} |D(\xi_j \circ \varphi)|_p = \Phi^x(\boldsymbol{g})$$

for μ-a.e. $x \in X$.

The estimate

$$\Phi^x(\boldsymbol{g}) \le \left(\sum_k^N |g_k|^q \right)^{1/q} \left(\sum_k^N |D\varphi_k|_p^p \right)^{1/p} \le C |\boldsymbol{g}| \sum_k^N |D\varphi_k|_p,$$

valid for all simple vector-valued $\boldsymbol{g}$, implies that the equality in the claim is stable under local L^∞-convergence of $\boldsymbol{g}$. Since simple functions are dense in L^∞, the claim follows. The remaining claim follows in a straightforward way from the equality. $\square$

Remark 6.4. If $\varphi \in \mathrm{LIP}(X; \mathbb{R}^N)$ is a chart in U, and $f \in N^{1,p}(X)$, then for the canonical minimal upper gradient $\Phi^x(a, \boldsymbol{\xi}))$ of $(f, \varphi) \in N_{\mathrm{loc}}^{1,p}(X; \mathbb{R}^{N+1})$ we have by Lemma 4.3(2) that $\Phi^x(1, -\mathrm{d}f) = 0$. Thus, by the previous lemma, we get $\mathrm{d}_G f - \sum_{k=1}^N g^k \, \mathrm{d}_G \varphi_k = 0$, where g^k are the components of $\mathrm{d}f$ and φ_k are the components of φ. Indeed, this follows by considering this first on the sets $U_M = \{x \in U : |g^k(x)| \le M, \ k = 1, \ldots, N\}$ and sending $M \to \infty$ combined with locality.

Lemma 6.5. *If (U, φ) is an N-dimensional p-weak chart in X, then the differentials of the component functions $\mathrm{d}_G \varphi_1, \ldots, \mathrm{d}_G \varphi_N$ form a basis of $L^p(T^*X)$ in U.*

Proof. By Lemma 6.3, $\mathrm{d}_G \varphi_1, \ldots, \mathrm{d}_G \varphi_N \in L^p(T^*X)$ are linearly independent on U. To see that they span $L^p(T^*X)$ in U, let $f \in N^{1,p}(X)$, and set $g_k := \mathrm{d}f(e_k)$ for each $k = 1, \ldots, N$, where e_k is the standard basis of $\mathbb{R}^N$. Then, since $\mathrm{d}\varphi_k = e^k$, where e^k is the dual basis of $(\mathbb{R}^N)^*$, we get $\mathrm{d}f = \sum_{k=1}^N g_k \, \mathrm{d}\varphi_k$. Thus, by Remark 6.4 we have $\mathrm{d}_G f = \sum_{k=1}^N g_k \, \mathrm{d}_G \varphi_k$. Since the abstract differentials $\mathrm{d}_G f$ span $L^p(T^*X)$, this completes the proof. $\square$

Lemma 6.6. *Suppose $p \ge 1$ and X admits a p-weak differentiable structure. There exists an isometric isomorphism $\iota : \Gamma_p(T_p^* X) \to L^p(T^*X)$ of normed modules satisfying*

$$\iota(\mathrm{d}f) = \mathrm{d}_G f, \quad f \in N^{1,p}(X). \tag{6-2}$$

The map ι is uniquely determined by (6-2).

Uniqueness here means that if $A : \Gamma_p(T_p^* X) \to L^p(T^*X)$ is L^∞-linear and satisfies (6-2) then $A = \iota$.

Proof. The set

$$\mathcal{W} = \left\{ \sum_j^M \chi_{A_j} \mathrm{d}f_j : (A_j)_j \text{ Borel partition of } X, \ f_j \in N^{1,p}(X) \right\}$$

is dense in $\Gamma_p(T_p^* X)$, since it contains all the simple Borel sections of $T_p^* X$. We set

$$\iota(v) := \sum_j^M \chi_{A_j} \mathrm{d}_G f_j, \quad v = \sum_j^M \chi_{A_j} \mathrm{d}f_j \in \mathcal{W}.$$

We have that

$$|\iota(v)|_G = \sum_j^M \chi_{A_j} |Df_j|_p = \sum_j^M \chi_{A_j} |\mathrm{d}f_j| = |v| \quad \mu\text{-a.e.}$$

for $v \in \mathcal{W}$. This implies that ι is well-defined and preserves the pointwise norm on the dense set $\mathcal{W}$. By Remark 6.4 we have that ι is linear. Since $\iota(\mathcal{W}) = \mathcal{V}$, it follows that ι extends to an isometric isomorphism $\iota : \Gamma_p(T_p^*X) \to L^p(T^*X)$. Note that $\iota(\mathrm{d}f) = \mathrm{d}_G f$ for every $f \in N^{1,p}(X)$, establishing (6-2).

To prove uniqueness, note that if $A : \Gamma_p(T_p^*X) \to L_p(T^*X)$ is linear and satisfies (6-2), then $A(v) = \iota(v)$ for all $v \in \mathcal{W}$ which implies that $A = \iota$ by the density of $\mathcal{W}$.   □

Proof of Theorem 1.11. If X admits a p-weak differentiable structure, Lemma 6.5 implies that $L^p(T^*X)$ is locally finitely generated. To prove the converse implication, suppose $\{A_N\}_{N \in \mathbb{N} \cup \{\infty\}}$ is the dimensional decomposition of X and $\mu(A_\infty) = 0$.

Let $N \in \mathbb{N}$ be such that $\mu(A_N) \geq \mu(V) > 0$ for some Borel set V, and $v_1, \ldots, v_N \in L^p(T^*X)$ is a basis of $L^p(T^*X)$ on V. By possibly passing to a smaller subset of V, we may assume that there exists $C > 0$ for which

$$\int_V \left| \sum_k^N g_k v_k \right|_G^p \, \mathrm{d}\mu \geq \frac{1}{C} \int_V |\boldsymbol{g}|^p \, \mathrm{d}\mu \quad \text{for all } \boldsymbol{g} = (g_1, \ldots, g_N) \in L^\infty. \tag{6-3}$$

For each $k = 1, \ldots, N$ there are sequences

$$v_k^n = \sum_j^{M_k^n} \chi_{A_{j,k}^n} \mathrm{d}_G f_{j,k}^n,$$

with $\{A_{j,k}^n\}_j$ a Borel partition of X and $(f_j^n) \subset N^{1,p}(X)$ such that $v_k^n \to v_k$ in $L^p(T^*X)$ as $n \to \infty$, by the definition of $L^p(T^*X)$. We set $J^n = \{1, \ldots, M_1^n\} \times \cdots \times \{1, \ldots, M_N^n\}$ and define new partitions $A_{\bar{j}}^n := A_{j_1,1}^n \cap \cdots \cap A_{j_N,N}^n$ indexed by $\bar{j} = (j_1, \ldots, j_N) \in J^n$. Then

$$v_k^n = \sum_{\bar{j} \in J^n} \chi_{A_{\bar{j}}^n} \mathrm{d}_G(f_{j_k,k}^n), \quad \mu(V) = \sum_{\bar{j} \in J^n} \mu(A_{\bar{j}}^n \cap V),$$

$$\lim_{n \to \infty} \int_X |v_k^n - v_k|_G^p \, \mathrm{d}\mu = \lim_{n \to \infty} \sum_{\bar{j} \in J^n} \int_{A_{\bar{j}}^n} |\mathrm{d}_G \varphi_k^{n,\bar{j}} - v_k|_G^p \, \mathrm{d}\mu = 0 \tag{6-4}$$

for all n and $k = 1, \ldots, N$. We claim that there exists n so that $\varphi^{n,\bar{j}} := (f_{j_1,1}^n, \ldots, f_{j_N,N}^n) \in N^{1,p}(X; \mathbb{R}^N)$ is p-independent on a positive measure subset of $A_{\bar{j}}^n \cap V$ for some $\bar{j} \in J^n$.

By (6-3) we have the inequality

$$\frac{1}{C} \int_{A_{\bar{j}}^n \cap V} |\boldsymbol{g}|^p \, \mathrm{d}\mu \leq \int_{A_{\bar{j}}^n \cap V} \left| \sum_k^N g_k v_k \right|^p \, \mathrm{d}\mu$$

$$\leq C' \int_{A_{\bar{j}}^n \cap V} \left| \sum_k^N g_k \mathrm{d}_G \varphi_k^{n,\bar{j}} \right|_G^p \, \mathrm{d}\mu + C' \int_{A_{\bar{j}}^n \cap V} \left| \sum_k^N g_k (\mathrm{d}_G \varphi_k^{n,\bar{j}} - v_k) \right|_G^p \, \mathrm{d}\mu$$

$$\leq C' \int_{A_{\bar{j}}^n \cap V} \Phi_{n,\bar{j}}(\boldsymbol{g}(x), x) \, \mathrm{d}\mu + C'' \int_{A_{\bar{j}}^n \cap V} |\boldsymbol{g}|^p \left(\sum_k^N |\mathrm{d}_G \varphi_k^{n,\bar{j}} - v_k|_G^p \right) \, \mathrm{d}\mu$$

for all $\boldsymbol{g} = (g_1, \ldots, g_N) \in L^\infty$, where $\Phi_{n,\bar{j}}$ is the canonical minimal gradient of $\varphi^{n,\bar{j}}$ (see Lemma 6.3). By (6-4) there exists $n \in \mathbb{N}$, $\bar{j} \in J^n$ and a Borel set $U \subset A_{\bar{j}}^n \cap V$ with $0 < \mu(U) \leq \mu(A_{\bar{j}}^n \cap V)$ such that

$\sum_k^N |d_G \varphi_k^{n,\bar{j}} - v_k|_G^p < \varepsilon$ on U, where $C'' \varepsilon < 1/(2C)$. Thus

$$\frac{1}{C} \int_U |\boldsymbol{g}|^p \, d\mu \leq C' \int_U \Phi_{n,\bar{j}}(\boldsymbol{g}(x), x) \, d\mu + \frac{1}{2C} \int_U |\boldsymbol{g}|^p \, d\mu$$

for all $\boldsymbol{g} = (g_1, \ldots, g_N) \in L^\infty(U; \mathbb{R}^N)$ by extending g by zero to $V \setminus U$. This readily implies that $I(\varphi^{n,\bar{j}}) > 0$ a.e. in U, proving the p-independence of $\varphi^{n,\bar{j}}$ in U. Note that $\varphi^{n,\bar{j}}$ is also maximal, since the existence of a Lipschitz map on a positive measure subset of U with a higher-dimensional target would imply that the local dimension of $L^p(T^*X)$ in V would be $> N$; see Lemma 6.3. By Proposition 4.14, U contains an N-dimensional p-weak chart, and [Keith 2004b, Proposition 3.1] implies that X admits a differentiable structure.

The argument above shows that each A_N with $\mu(A_N) > 0$ can be covered up to a null-set by N-dimensional p-weak charts, proving (b), while (a) follows directly from Lemma 6.6. Finally, (c) is implied by Proposition 4.13. □

Theorem 1.11 and [Gigli 2018, Chapter 2] immediately yield the following corollary.

Corollary 6.7. *Let $p \geq 1$ and suppose X admits a p-weak differentiable structure.*

(i) *If $p > 1$, then $N^{1,p}(X)$ is reflexive.*

(ii) *If $p = 2$, then $N^{1,2}(X)$ is infinitesimally Hilbertian if and only if, for μ-a.e. $x \in X$, the pointwise norm $|\cdot|_x$ (see Theorem 1.7) is induced by an inner product.* □

6B. *Lipschitz differentiability spaces.* A space X is said to be a Lipschitz differentiability space if it admits a Cheeger structure. Recall that a Cheeger structure is a countable collection of Cheeger charts (U_i, φ_i), see Section 4G, so that $\mu(X \setminus \bigcup_i U_i) = 0$. Following [Cheeger 1999, Section 4, p. 458], we note that the differentials $d_{C,i} f$ of a Lipschitz function f with respect to overlapping charts satisfy a cocycle condition almost everywhere and the transition maps preserve the pointwise norm. Thus, they define a measurable L^∞-bundle T_C^*X called the measurable cotangent bundle.

Suppose now that X admits a Cheeger structure. Denote by T_C^*X the associated measurable cotangent bundle, and by

$$|\xi|_{C,x} := \text{Lip}(\xi \circ \varphi)(x), \quad \xi \in (\mathbb{R}^N)^*,$$

the pointwise norm for μ-a.e. $x \in U$, where (U, φ) is an N-dimensional Cheeger chart of X.

Fix $p \geq 1$. Any Lipschitz differentiability space X admits a p-weak differentiable structure. Indeed, the asymptotic doubling property of the measure (see [Bate and Speight 2013]) implies, by [Bate 2015, Lemma 8.3], that X decomposes into finite-dimensional pieces. The existence of the p-weak differentiable structure now follows from Proposition 5.4, and the associated measurable cotangent bundle is denoted by T_p^*X. We have the following result from [Ikonen et al. 2022, Theorem 3.4]:

Theorem 6.8. *Let $p \geq 1$. There exists a morphism $P : \Gamma_p(T_C^*X) \to L^p(T^*X)$ of normed modules such that*

(a) *$P(d_C f) = d_G f$ for every $f \in \text{LIP}(X)$,*

(b) *$|P(\omega)|_G \leq |\omega|_C$ for every $\omega \in \Gamma_p(T_C^*X)$, and*

(c) *for every $w \in L^p(T^*X)$ there exists $\omega \in P^{-1}(w)$ with $|w|_G = |\omega|_c$.*

Remark 6.9. The proof of [Ikonen et al. 2022, Theorem 3.4] can be modified to cover the case $p = 1$: the energy density of Lipschitz functions holds for $p = 1$ by [Eriksson-Bique 2023], and equicontinuity can be used instead of L^p-boundedness to obtain the weakly convergent subsequence in the proof.

Proof of Theorem 1.10. Arguing as in the proof of Proposition 5.8 we may assume that X has a Borel partition $\{U_i\}$ and Lipschitz maps $\varphi_p^i = (\varphi_{p,1}^i, \ldots, \varphi_{p,N_i}^i)$, $\varphi_C^i = (\varphi_{C,1}^i, \ldots, \varphi_{C,M_i}^i)$ such that (U_i, φ_p^i) is a p-weak chart and (U_i, φ_C^i) is a Cheeger chart on X (of possibly different dimensions N_i and M_i) for each $i \in \mathbb{N}$. For each i and μ-a.e. $x \in U_i$ define

$$\sigma_{i,x} = (\mathrm{d}_{p,x}\varphi_{C,1}^i, \ldots, \mathrm{d}_{p,x}\varphi_{C,M_i}^i) : \mathbb{R}^{N_i} \to \mathbb{R}^{M_i}.$$

It is easy to see that the collection $\{\pi_{i,x} = \sigma_{i,x}^*\}$ defines a bundle map $T_C^*X \to T_p^*X$ satisfying

$$\mathrm{d}_{p,x}f = \mathrm{d}_{C,x}f \circ \sigma_x \quad \text{for } \mu\text{-a.e. } x \in X,$$

for every $f \in N^{1,p}(X)$. This proves (1-7). In particular, for each $i \in \mathbb{N}$ and $\xi \in (\mathbb{R}^{M_i})^*$ we have $\pi_{i,x}(\xi) = \xi \circ \sigma_{i,x} = \mathrm{d}_{p,x}(\xi \circ \varphi_C^i)$, and consequently

$$|\pi_{i,x}(\xi)|_x = |D(\xi \circ \varphi_C^i)|_p(x) \leq \mathrm{Lip}(\xi \circ \varphi_C^i)(x) = |\xi|_{C,x}$$

for μ-a.e. $x \in U_i$. Moreover, for any $\zeta \in (\mathbb{R}^{N_i})^*$, setting $\xi := \mathrm{d}_{C,x}(\zeta \circ \varphi_p^i)$, we have

$$\pi_{i,x}(\xi) = \mathrm{d}_{C,x}(\zeta \circ \varphi_p^i) \circ \sigma_x = \mathrm{d}_{p,x}(\zeta \circ \varphi_p^i) = \zeta,$$

proving that $\pi_{i,x}$ is surjective for μ-a.e. $x \in U_i$.

   To prove that $\pi_{i,x}$ is a submetry for μ-a.e. $x \in U_i$, suppose to the contrary that there exists a Borel set $B \subset U_i$, with $0 < \mu(B) < \infty$, such that $\pi_{i,x}$ is not a submetry for $x \in B$. Then there exists a Borel map $\zeta : B \to (\mathbb{R}^{N_i})^*$ with $|\zeta_x|_x = 1$ and

$$|\zeta_x|_x = 1 \quad \text{and} \quad \inf_{\xi \in \pi_{i,x}^{-1}(\zeta_x)} |\xi|_{C,x} > 1 \quad \text{for } \mu\text{-a.e. } x \in B. \tag{6-5}$$

We derive a contradiction using Theorem 6.8 and the isometric isomorphism $\iota : \Gamma_p(T_p^*X) \to L^p(T^*X)$ from Theorem 1.11(a). We may view ζ as an element of $\Gamma_p(T_p^*X)$ by extending it by zero outside B. Set $w := \iota(\zeta) \in L^p(T^*X)$. Then $|w|_G = \chi_B$. By Theorem 6.8(c) there exists $\omega \in \Gamma_p(T_C^*X)$ with $P(\omega) = w$ and $|\omega|_C = |w|_G = \chi_B$ μ-a.e. However, since $\omega_x \in \pi_{i,x}^{-1}(\zeta_x)$ for μ-a.e. $x \in B$, we have $|\omega|_{C,x} \geq \inf_{\xi \in \pi_{i,x}^{-1}(\zeta_x)} |\xi|_{C,x} > 1$ for μ-a.e. $x \in B$ by (6-5), which is a contradiction. This completes the proof that $\pi_{i,x}$ is a submetry for μ-a.e. $x \in U_i$.

   If $\mathrm{Lip}\, f \leq \omega(|Df|_p)$ holds for every $f \in \mathrm{LIP}_b(X)$, then by [Ikonen et al. 2022, Theorem 1.1] we have $|Df|_p = \mathrm{Lip}\, f$ μ-a.e. for every $f \in \mathrm{LIP}_b(X)$. It follows that p-weak charts are Cheeger charts (see Theorem 1.8 and Remark 4.15) and that the pointwise norms agree μ-almost everywhere. This implies that the maps $\pi_{i,x}$ are isometric bijections for μ-a.e. x. $\qquad\square$

Appendix: General measure theory

A1. *Measurability questions.* Here we record a host of measurability statements that are needed throughout the paper. See [Gigli and Pasqualetto 2020; Ambrosio et al. 2008; Bogachev 2007] for more details.

Given $f \in N^{1,p}(X)$ and a Borel representative g of p-weak upper gradient of f, we define

$$\Gamma(f) := \{\gamma \in \mathrm{AC}(I; X) : f \circ \gamma \in \mathrm{AC}(I; \mathbb{R})\},$$

$$\Gamma(f, g) := \{\gamma \in \mathrm{AC}(I; X) : g \text{ upper gradient of } f \text{ along } \gamma\} \subset \Gamma(f)$$

and

$$MD = \{(\gamma, t) \in \mathrm{AC}(I; X) \times I : |\gamma'_t| \text{ exists}\},$$

$$\mathrm{Diff}(f) = \{(\gamma, t) \in \mathrm{AC}(I; X) \times I : \gamma \in \Gamma(f), \ (f \circ \gamma)'_t \text{ and } |\gamma'_t| > 0 \text{ exist}\},$$

$$\mathrm{Diff}(f, g) = \{(\gamma, t) \in \mathrm{Diff}(f) : \gamma \in \Gamma(f, g), \ |(f \circ \gamma)'_t| \leq g_f(\gamma_t)|\gamma'_t|\}.$$

Also, let $\mathrm{Len}(\gamma)$ be the length of a curve γ, if the curve is rectifiable, and otherwise infinity. The function der is defined by $\mathrm{der}(\gamma, t) := |\gamma'_t| = \lim_{h \to 0} d(\gamma_{t+h}, \gamma_t)/|h|$, when the limit exists, and otherwise is infinity.

Lemma A.1. (1) *The functions* $\mathrm{Len} : C(I; X) \to [0, \infty]$ *and* $\mathrm{der} : \mathrm{AC}(I; X) \times I \to [0, \infty]$ *are Borel measurable.*

(2) *If* $g : X \to [0, \infty]$ *is a Borel function, then* $I : \mathrm{AC}(I; X) \to \mathbb{R}$, *given by* $\gamma \mapsto \int_\gamma g \, \mathrm{d}s$ *or* ∞ *if the curve is not rectifiable, is Borel.*

(3) *If* $H : \mathrm{AC}(I; X) \times I \to [0, \infty]$ *is Borel, then* $I_H(\gamma) := \int_0^1 H(\gamma, s) \, \mathrm{d}s : \mathrm{AC}(I; X) \to [0, \infty]$ *is Borel.*

(4) *The set* MD *is Borel, and the map* $MD \to \mathbb{R}$ *defined by* $(\gamma, t) \to |\gamma'_t|$ *is Borel.*

Proof. (1) The length function is a lower semicontinuous function with respect to uniform convergence, and thus is Borel. Fix $r, p \in \mathbb{Q}$ positive. Then define

$$A_{p,r} = \bigcup_{n \in \mathbb{N}} \bigcap_{q \in \mathbb{Q} \cap (-1/n, 1/n)} \{(\gamma, t) : |d(\gamma_{t+q}, \gamma_t) - qp| < r|q|\},$$

which is Borel. The set M where the metric derivative exists is of the form $\bigcap_{r \in \mathbb{Q} \cap (0, \infty)} \bigcup_{p \in \mathbb{Q} \cap (0, \infty)} A_{p,r}$. On this set we have $M \cap A_{p,r} = \mathrm{der}^{-1}(B(p, r))$ and thus $\mathrm{der}(\gamma, t)$ is Borel.

(2) The claims for the integral function being Borel follow from a monotone family argument, and considering g first a characteristic function of an open set and using lower semicontinuity of the integral in that case.

(3) If H is a characteristic function of a product set $A \times B$, where A and B are open sets such that $A \subset C(I; X)$, $B \subset I$, then the claim follows just as in statement (2). Again, by a monotone family argument, we obtain the claim for all Borel measurable functions.

(4) Define for every $q \in \mathbb{Q}$ and $\varepsilon, h > 0$ the sets $A(\varepsilon, q, h)$ and $B(\varepsilon, q)$ by

$$A(\varepsilon, q, h) := \left\{ (\gamma, t) \in C(I; X) \times I : \left| \frac{d(\gamma_{t+h}, \gamma_t)}{|h|} - q \right| < \varepsilon \right\},$$

$$B(\varepsilon, q) := \bigcup_{\delta \in \mathbb{Q}_+} \bigcap_{h \in (0, \delta) \cap \mathbb{Q}} A(\varepsilon, q, h).$$

We note that $|\gamma'_t|$ exists if and only if $(\gamma, t) \in \bigcap_{j \in \mathbb{N}} \bigcup_{q \in \mathbb{Q}} B(2^{-j}, q) = MD$. On the set MD, where the limit exists, we can write $|\gamma'_t| = \lim_{n \to \infty} n(d(\gamma_{t+n^{-1}}, \gamma_t))$, which shows measurability. $\square$

Lemma A.2. *Let g be a Borel p-weak upper gradient of $f \in N^{1,p}(X)$. There exists a Borel set $\Gamma_0 \subset$ $\mathrm{AC}(I; X)$ with $\mathrm{Mod}_p(\Gamma_0) = 0$ such that $AC \setminus \Gamma_0 \subset \Gamma(f, g)$.*

Suppose moreover that f is Borel. Then the set $A := \Gamma_0^c \times I \cap \mathrm{Diff}(f, g)$ is Borel, and $\boldsymbol{\pi}(A^c) = 0$ whenever $\boldsymbol{\pi} = \mathcal{L}^1 \times \boldsymbol{\eta}$ and $\boldsymbol{\eta}$ is a q-test plan.

If f is Lipschitz, and $g = \mathrm{Lip}[f]$, then we can choose $\Gamma_0 = \varnothing$, and $\mathrm{Diff}(f, g) = \mathrm{Diff}(f)$ is Borel.

Note that we make no claims about the Borel measurability of the set $\Gamma(f, g)$.

Proof. We model the argument after [Pasqualetto 2022, Lemma 1.9]. Since $\mathrm{Mod}_p(\Gamma(f, g)^c) = 0$, there exists an L^p-integrable Borel function $\rho : X \to [0, \infty]$ with $\int_\gamma \rho \, ds = \infty$ for every $\gamma \notin \Gamma_{f,g}$. Then $\Gamma_0 := \left\{ \gamma \in \mathrm{AC}(I; X) : \int_\gamma \rho \, ds = \infty \right\} \supset \Gamma_{f,g}^c$ is a Borel set, by Lemma A.1 and $\boldsymbol{\eta}(\Gamma_0) = 0$ for every q-plan $\boldsymbol{\eta}$ (see Remark 2.2). If f is Lipschitz, then $\Gamma(f, g) = \mathrm{AC}(I; X)$. Thus, we can choose $\Gamma_0 = \varnothing$.

For the second part assume $f \in N^{1,p}(X)$ is Borel, and set

$$A(\varepsilon, q, h) = \left\{ (\gamma, t) \in \Gamma_0^c \times I : \left| \frac{f(\gamma_{t+h}) - f(\gamma_t)}{h} - q \right| < \varepsilon \right\},$$

$$B(\varepsilon, q) = \bigcup_{\delta \in \mathbb{Q}_+} \bigcap_{h \in (0, \delta) \cap \mathbb{Q}} A(\varepsilon, q, h)$$

for each $q \in \mathbb{Q}$ and $\varepsilon, h > 0$. It is easy to see that for each $\gamma \notin \Gamma_0$, $(f \circ \gamma)'_t$ exists if and only if

$$(\gamma, t) \in \bigcap_{j \in \mathbb{N}} \bigcup_{q \in \mathbb{Q}} B(2^{-j}, q) =: A.$$

Note that A is a Borel set with $A \cap MD \subset \mathrm{Diff}(f)$. Moreover, $(\gamma, t) \mapsto (f \circ \gamma)'_t$ is Borel when restricted to $A \cap MD$.

Define the Borel function $H(\gamma, t) = (f \circ \gamma)'_t$ if $(\gamma, t) \in A \cap MD$ and $H = +\infty$ otherwise, and $G(\gamma, t) = |H| - g(\gamma_t)|\gamma'_t|$ (here we use the convention $\infty - \infty = \infty$). Then the set

$$\{G \leq 0\} = \Gamma_0^c \times I \cap \mathrm{Diff}(f, g)$$

is Borel.

Set $N := \{G > 0\}$, suppose $\boldsymbol{\eta}$ is a q-test plan and $\boldsymbol{\pi} := \mathcal{L}^1 \times \boldsymbol{\eta}$. Note that

$$N \subset \Gamma_0 \times I \cup \{(\gamma, t) \in \Gamma_0^c \times I : G(\gamma, t) > 0\}.$$

But, for all $\gamma \notin \Gamma_0$, we have $G(\gamma, t) \leq 0$ for $\mathcal{L}^1$-a.e. $t \in I$. Thus

$$\boldsymbol{\pi}(N) \leq \boldsymbol{\eta}(\Gamma_0) + \int_{\Gamma_0^c} \int_0^1 \chi_{\{G(\gamma, \cdot) > 0\}}(t) \, dt \, d\boldsymbol{\eta}(\gamma) = 0,$$

finishing the proof of the second part. $\square$

Corollary A.3. *Every pointwise defined function $f \in N^{1,p}(X)$ has a Borel representative $\bar{f} \in N^{1,p}(X)$. Moreover, if $f \in N^{1,p}(X)$ and g is a Borel p-weak upper gradient of f, there exists a Borel set $N \subset$ $C(I; X) \times I$, with $N^c \subset \mathrm{Diff}(f, g)$ and $\boldsymbol{\pi}(N) = 0$ whenever $\boldsymbol{\pi} = \mathcal{L}^1 \times \boldsymbol{\eta}$, $\boldsymbol{\eta}$ a q-test plan. The map $(\gamma, t) \mapsto (f \circ \gamma)'_t$ if $(\gamma, t) \notin N$ and $+\infty$ otherwise is Borel. If f is Lipschitz the representative can be chosen as the same function.*

Proof. The first claim follows directly from [Eriksson-Bique 2023, Theorem 1.1]. To see the second, let $\bar{f} \in N^{1,p}(X)$ be a Borel representative of f. The set $E := \{f \neq \bar{f}\}$ is p-exceptional, i.e., $\Gamma_E := \{\gamma : \gamma^{-1}(E) \neq \varnothing\}$ has zero p-modulus. Note that, if f is Lipschitz, then f is automatically Borel and we do not need to change representatives, and we can set $\Gamma_E = \varnothing$.

If $\bar{A}$ is the set in Lemma A.2 for $\bar{f}$, g, then $A := \bar{A} \setminus (\Gamma_E \times I) \subset \text{Diff}(f, g)$ and $N := A^c$ satisfies the claim since it is Borel and $N \subset \Gamma_E \times I \cup \bar{A}^c$.

The last claim follows since N^c is Borel and, if $(\gamma, t) \notin N$, we have

$$(f \circ \gamma)_t' = \lim_{n \to \infty} n(f(\gamma_{t+1/n}) - f(\gamma_t)). \qquad \square$$

A2. *Essential supremum.*

Definition A.4. Let X be a σ-finite measure space and $\mathcal{F}$ a collection of measurable functions on X, then there exists a function $g : X \to \mathbb{R} \cup \{\infty, -\infty\}$ which is measurable, and:

(A) For each $f \in \mathcal{F}$,

$$f \leq g$$

almost everywhere.

(B) For each g' that satisfies (A), will satisfy $g \leq g'$ almost everywhere.

We call $g = \text{ess sup}_{f \in \mathcal{F}} f$. Similarly, we define $g = \text{ess inf}_{f \in \mathcal{F}} f$, by switching the directions of the inequalities and assuming $g : X \to \mathbb{R} \cup \{\infty, -\infty\}$.

We will need the following standard lemma. While its proof is standard, we provide it for the sake of completeness.

Lemma A.5. *If X is any σ-finite measure space and $\mathcal{F}$ is any collection of measurable functions, then* $\text{ess sup}_{f \in \mathcal{F}} f$ *and* $\text{ess inf}_{f \in \mathcal{F}} f$ *exists and is unique, and further, there are sequences $f_n, g_n \in \mathcal{F}$ so that* $\text{ess sup}_{f \in \mathcal{F}} f = \sup_n f_n$ *and* $\text{ess inf}_{f \in \mathcal{F}} f = \inf_n g_n$ *almost everywhere.*

Proof. The uniqueness follows from (B) in Definition A.4. Indeed, if g and g' are essential suprema, they both satisfy A, and thus $g \leq g'$ and $g' \leq g$.

By considering $\{\arctan(f) : f \in \mathcal{F}\}$, we can assume that the collection is bounded. Further, by σ-finiteness, and after exhausting the space by finite measure sets, it suffices to consider a bounded measure. Define $\mathcal{G}$ to be the collection of all functions of the form $\max(f_1, \ldots, f_k)$ for some $f_i \in \mathcal{F}$. By construction, if $g, g' \in G$, then $\max(g, g') \in G'$.

Consider $U = \sup_{g \in \mathcal{G}} \int g \, d\mu$. There is a sequence g_n so that $\lim_{n \to \infty} \int g_n \, d\mu = U$. By modifying the sequence if necessary, we may take it increasing in n, and define $g = \lim_{n \to \infty} g_n$.

We claim that g is an essential supremum for $\mathcal{F}$. First, if $f \in \mathcal{F}$, and $f > g$ on a positive measure set, then $\lim_{n \to \infty} \int \max(f, g_n) \, d\mu > U$, contradicting the definition of U. Thus the condition A in the definition is satisfied.

Now, if h is any other function satisfying A, then $h \geq g_n$, and thus $h \geq g$ almost everywhere, by construction. Thus B is also satisfied. Finally, the construction gives a countable collection g_n formed each from finitely many $f_i \in \mathcal{F}$, and thus gives the final claim in the statement. $\qquad \square$

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