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ON BLOWUP FOR THE SUPERCRITICAL QUADRATIC WAVE EQUATION

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We study singularity formation for the quadratic wave equation in the energy supercritical case, i.e., for $d \geq 7$. We find in closed form a new, nontrivial, radial, self-similar blow-up solution u^* which exists for all $d \geq 7$. For $d = 9$, we study the stability of u^* without any symmetry assumptions on the initial data and show that there is a family of perturbations which lead to blowup via u^*. In similarity coordinates, this family represents a codimension-1 Lipschitz manifold modulo translation symmetries. The stability analysis relies on delicate spectral analysis for a non-self-adjoint operator. In addition, in $d = 7$ and $d = 9$, we prove nonradial stability of the well-known ODE blow-up solution. Also, for the first time we establish persistence of regularity for the wave equation in similarity coordinates.

1. Introduction

In this paper, we are concerned with the quadratic wave equation

$$(\partial_t^2 - \Delta_x)u(t,x) = u(t,x)^2, \tag{1-1}$$

where $(t,x) \in I \times \mathbb{R}^d$, for some interval $I \subset \mathbb{R}$ containing zero.

It is well known that in all space dimensions (1-1) admits solutions that blow up in finite time, starting from smooth and compactly supported initial data. This follows from a classical result by Levine [1974], which provides an open set of such initial data. However, Levine's argument is indirect, and therefore

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does not give insight into the blow-up profile. A more concrete example can be produced using the well-known *ODE solution*

$$u_T^{\mathrm{ODE}}(t,x) := \frac{6}{(T-t)^2}, \quad T > 0. \tag{1-2}$$

By truncating the initial data $(u_T^{\mathrm{ODE}}(0,\cdot), \partial_t u_T^{\mathrm{ODE}}(0,\cdot))$ outside a ball of radius larger than T and using finite speed of propagation, one constructs smooth and compactly supported initial data that lead to blowup at $t = T$. What is more, invariance of (1-1) under the rescaling

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{-2} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \quad \lambda > 0, \tag{1-3}$$

allows one to look for *self-similar* blow-up solutions of the form

$$u(t,x) = \frac{1}{(T-t)^2}\phi\left(\frac{x}{T-t}\right).$$

Note that (1-2) is a self-similar solution with trivial profile $\phi \equiv 6$. We note that the rescaling (1-3) leaves invariant the energy norm $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ of $(u(t,\cdot), \partial_t u(t,\cdot))$ precisely when $d = 6$, in which case (1-1) is referred to as *energy critical*. In this case, it can be easily shown that in addition to (1-2) no other radial and smooth self-similar solutions to (1-1) exist; see [Kavian and Weissler 1990]. However, in the *energy supercritical* case, i.e., for $d \geq 7$, numerics [Kycia 2011] indicate that in addition to (1-2) there are nontrivial, radial, globally defined, smooth, and decaying similarity profiles. In fact, for $d = 7$, there are infinitely many of them, all of which are positive, as proven by Dai and Duyckaerts [2021]. A similar result is expected to hold for all $7 \leq d \leq 15$; see [Kycia 2011].

From the point of view of the Cauchy problem for (1-1), the relevant similarity profiles appear to be the trivial one (1-2) and its first nontrivial "excitation". Namely, numerical work on supercritical power nonlinearity wave equations in the radial case [Bizoń et al. 2004; Glogić et al. 2020] yields evidence that generic blowup is described by the ODE profile, while the threshold separating generic blowup from global existence is given by the stable manifold of the first excited profile; see also [Bizoń 2001]. The first step in showing such genericity results would be to establish stability of the ODE profile and show that its first excitation is codimension-1 stable (which indicates that the stable manifold splits the phase space locally into two connected components). The only result so far for (1-1) in this direction is by Donninger and the third author [Donninger and Schörkhuber 2017], who proved radial stability of u_T for all odd $d \geq 7$. In this paper, we exhibit in closed form what appears to be the first excitation of (1-2) for every $d \geq 7$. Namely, we have the following self-similar solution to (1-1):

$$u^*(t,x) := \frac{1}{t^2} U\left(\frac{|x|}{t}\right), \tag{1-4}$$

where

$$U(\rho) = \frac{c_1 - c_2\rho^2}{(c_3 + \rho^2)^2}, \tag{1-5}$$

with

$$c_1 = \tfrac{4}{25}((3d-8)d_0 + 8d^2 - 56d + 48), \quad c_2 = \tfrac{4}{5}d_0, \quad c_3 = \tfrac{1}{15}(3d - 18 + d_0),$$

and $d_0 = \sqrt{6(d-1)(d-6)}$. We note that $c_3 > 0$ when $d \geq 7$, and thus $U \in C^\infty[0,\infty)$. To the best of our knowledge, this solution has not been known before, and with the intent of studying threshold behavior, the main object of this paper is to show a variant of codimension-1 stability of u^*.

Note that U has precisely one zero at $\rho^* = \rho^*(d) > 2$. In particular, this profile is not positive and therefore not a member of the family of self-similar profiles constructed in [Dai and Duyckaerts 2021]. However, it is strictly positive inside the backward light cone of the blow-up point $(0,0)$. Hence, in this local sense u^* provides a solution to the more frequently studied focusing equation

$$(\partial_t^2 - \Delta_x)u(t,x) = |u(t,x)|u(t,x). \tag{1-6}$$

What is more, as an outcome of our stability analysis we get that small perturbations of both the ODE profile and u^* stay positive under the evolution of (1-1) and therefore yield solutions to (1-6) as well.

1A. *Main results.*

Preliminaries. By action of symmetries, the solution (1-4) gives rise to a $(2d+1)$-parameter family of (in general nonradial) blow-up solutions. Namely, (1-1) is invariant under spacetime translations

$$S_{T,x_0}(t,x) := (t-T, x-x_0)$$

for $T > 0,\ x_0 \in \mathbb{R}^d$, time reflections

$$R(t,x) := (-t,x),$$

as well as Lorentz boosts, which we write in terms of hyperbolic rotations as

$$\Lambda(a) := \Lambda^d(a^d) \circ \Lambda^{d-1}(a^{d-1}) \circ \cdots \circ \Lambda^1(a^1),$$

where $a \in \mathbb{R}^d$ and $\Lambda^j(a^j)$ for $j = 1, \dots, d$ are given by

$$\begin{cases} t \mapsto t\cosh(a^j) + x^j \sinh(a^j), \\ x^j \mapsto t\sinh(a^j) + x^j \cosh(a^j), \\ x^k \mapsto x^k \quad (k \neq j). \end{cases}$$

We then let

$$\Lambda_{T,x_0}(a) := R \circ \Lambda(a) \circ S_{T,x_0}, \tag{1-7}$$

and thereby obtain the following $(2d+1)$-parameter family of solutions to (1-1):

$$u^*_{T,x_0,a}(t,x) := u^* \circ \Lambda_{T,x_0}(a)(t,x).$$

We note that, for

$$(t',x') := \Lambda_{T,x_0}(a)(t,x),$$

we have

$$|x'|^2 - t'^2 = |x-x_0|^2 - (T-t)^2. \tag{1-8}$$

Furthermore, for $\xi, a \in \mathbb{R}^d$, we set[1]

$$\gamma(\xi,a) := A_0(a) - A_j(a)\xi^j, \tag{1-9}$$

[1] For simplicity, we use Einstein's summation convention throughout the paper.

where

$$\begin{aligned} A_0(a) &:= \cosh(a^1)\cosh(a^2)\cdots\cosh(a^d),\\ A_1(a) &:= \sinh(a^1)\cosh(a^2)\cdots\cosh(a^d),\\ A_2(a) &:= \sinh(a^2)\cosh(a^3)\cdots\cosh(a^d),\\ &\;\vdots\\ A_d(a) &:= \sinh(a^d). \end{aligned}$$

Then, it is easy to check that

$$t' = (T-t)\gamma\left(\frac{x-x_0}{T-t}, a\right) \quad \text{and} \quad x'^j = (t-T)\partial_{a^j}\gamma\left(\frac{x-x_0}{T-t}, a\right)B^j(a) \tag{1-10}$$

for $j = 1, \dots, d$, where

$$B^j(a) = \prod_{i=j+1}^{d} \cosh(a^i)^{-1}.$$

Now, by using relations (1-8) and (1-10) we find more explicitly that

$$u^*_{T,x_0,a}(t,x) = \frac{1}{(T-t)^2}U_a\left(\frac{x-x_0}{T-t}\right), \tag{1-11}$$

with $U_a : \mathbb{R}^d \to \mathbb{R}$ given by

$$U_a(\xi) = \frac{(c_1-c_2)\gamma(\xi,a)^2 + c_2(1-|\xi|^2)}{((1+c_3)\gamma(\xi,a)^2 + |\xi|^2 - 1)^2}. \tag{1-12}$$

Note that for $a = 0$, we have $U_0(\xi) = U(|\xi|)$ with U being the radial profile in (1-5). Also, since $c_1 > c_2$ for all $d \geq 7$, there exists a positive constant $c_0 = c_0(d)$ such that

$$U_a \geq c_0 > 0 \quad \text{on } \mathbb{B}^d \tag{1-13}$$

for all $a \in \mathbb{R}^d$, where $\mathbb{B}^d$ denotes the open unit ball in $\mathbb{R}^d$. In summary, we have that, for $a \in \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and $T > 0$, (1-1) admits an explicit solution (1-11), which starts off smooth, blows up at $x = x_0$ as $t \to T^-$, and is strictly positive on the backward light cone

$$\mathcal{C}_{T,x_0} := \bigcup_{t\in[0,T)} \{t\} \times \overline{\mathbb{B}}^d_{T-t}(x_0)$$

of the blow-up point (T, x_0) — see Section 1C for the notation — which makes it a solution inside $\mathcal{C}_{T,x_0}$ to (1-6) as well. Furthermore, simply by scaling we have that, for $k \in \mathbb{N}_0$,

$$\left\| U_a\left(\frac{\cdot - x_0}{T-t}\right)\right\|_{\dot{H}^k(\mathbb{B}^d_{T-t}(x_0))} \simeq (T-t)^{\frac{d}{2}-k}, \tag{1-14}$$

and hence

$$\|u^*_{T,x_0,a}(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^d_{T-t}(x_0))} \simeq (T-t)^{\frac{d}{2}-2-k},$$

which implies that the solution blows up in local homogeneous Sobolev seminorms of order $k > s_c = \frac{1}{2}d - 2$. Here, s_c denotes the critical regularity, i.e., $\dot{H}^{s_c}(\mathbb{R}^d)$ is left-invariant under the rescaling (1-3).

Conditional stability of blowup via u^*. The main goal of this paper is to investigate stability of blowup governed by u^*. For $T=1$, $x_0=0$, and $a=0$, the blow-up initial data are given by

$$u^*_{1,0,0}(0,x)=U(|x|) \quad \text{and} \quad \partial_t u^*_{1,0,0}(0,x)=2U(|x|)+|x|U'(|x|).$$

We can now formulate the following stability result, where we restrict ourselves to the case $d=9$.

Theorem 1.1. *Let $d=9$. Define functions $h_j:\mathbb{R}^9\to\mathbb{R}$, $j=1,2$, by*

$$h_1(x)=\frac{1}{(7+5|x|^2)^3} \quad \text{and} \quad h_2(x)=\frac{35-5|x|^2}{(7+5|x|^2)^4}. \tag{1-15}$$

There exist constants $M>0$, $\delta>0$, and $\omega>0$ such that, for all real-valued $(f,g)\in C^\infty(\overline{\mathbb{B}}^9_2)\times C^\infty(\overline{\mathbb{B}}^9_2)$ satisfying

$$\|(f,g)\|_{H^6(\mathbb{B}^9_2)\times H^5(\mathbb{B}^9_2)}\le\frac{\delta}{M},$$

the following holds: There are parameters $a\in\overline{\mathbb{B}}^9_{M\delta/\omega}$, $x_0\in\overline{\mathbb{B}}^9_\delta$, $T\in[1-\delta,1+\delta]$, and $\alpha\in[-\delta,\delta]$ depending Lipschitz-continuously on (f,g) such that, for initial data

$$u(0,\cdot)=U(|\cdot|)+f+\alpha h_1 \quad \text{and} \quad \partial_t u(0,\cdot)=2U(|\cdot|)+|\cdot|U'(|\cdot|)+g+\alpha h_2, \tag{1-16}$$

there exists a unique solution $u\in C^\infty(\mathcal{C}_{T,x_0})$ to (1-1). *Furthermore, this solution blows up at (T,x_0) and can be written as*

$$u(t,x)=\frac{1}{(T-t)^2}\left[U_a\left(\frac{x-x_0}{T-t}\right)+\varphi(t,x)\right],$$

where $\|\varphi(t,\cdot)\|_{L^\infty(\mathbb{B}^9_{T-t}(x_0))}\lesssim(T-t)^\omega$ and

$$(T-t)^{k-\frac{9}{2}}\|\varphi(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^9_{T-t}(x_0))}\lesssim(T-t)^\omega$$

for $k=0,\dots,5$. In particular,

$$\begin{aligned}(T-t)^{k-\frac{5}{2}}\|u(t,\cdot)-u^*_{T,x_0,a}(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^9_{T-t}(x_0))}&\lesssim(T-t)^\omega,\\(T-t)^{k-\frac{5}{2}}\|\partial_t u(t,\cdot)-\partial_t u^*_{T,x_0,a}(t,\cdot)\|_{\dot{H}^{k-1}(\mathbb{B}^9_{T-t}(x_0))}&\lesssim(T-t)^\omega\end{aligned} \tag{1-17}$$

for $k=1,\dots,5$. Moreover, u is strictly positive on $\mathcal{C}_{T,x_0}$, and hence the statement above applies to (1-6) *as well.*

We note that the normalizing factor on the left-hand side of (1-17) appears naturally and corresponds to the behavior of the blow-up solution we perturbed around; see (1-14).

Some further remarks on the result are in order.

Remark 1.2. The proof of Theorem 1.1 relies on stability analysis in similarity coordinates, in which the above set of perturbations has a codimension-1 interpretation. More precisely, we construct a Lipschitz manifold which is of codimension 11, where ten codimensions are related to instabilities caused by translation symmetries of the equation and the remaining codimension is characterized by (h_1,h_2). This is elaborated on in Section 2; see in particular Propositions 2.1 and 2.4. We believe that this manifold gives rise to a proper codimension-1 manifold in a suitable physical data space. However, by the local nature of our approach and the presence of translation symmetries, this is not entirely clear.

Remark 1.3 (Regularity of the initial data). It is only the transformation from similarity coordinates to physical coordinates that induces the higher-regularity assumption on the data, from which we can easily deduce the Lipschitz-dependence on the blow-up parameters. We nonetheless believe that this can be optimized by a more refined analysis.

Remark 1.4 (Persistence of regularity). While persistence of regularity is standard for the wave equation in physical coordinates, it has not yet been considered for the local problem in similarity coordinates. In fact, all of the related works so far, such as [Chatzikaleas and Donninger 2019; Donninger and Schörkhuber 2016; Glogić and Schörkhuber 2021], are based on a notion of strong solutions in similarity coordinates. In this paper, we close this gap and rigorously prove regularity of solutions for smooth initial data. Our proof relies on estimates for the free wave evolution in similarity coordinates in arbitrarily high Sobolev spaces; see Proposition 3.1 on page 629.

Remark 1.5 (Generalization to other space dimensions). Large parts of the proof of Theorem 1.1 can be generalized to other odd space dimensions. However, the analysis of the underlying spectral problem is quite delicate and only for $d = 9$ we are able to solve it rigorously. Nevertheless, from numerical computations, we have strong evidence that the situation is analogous in other space dimensions in the sense that the linearization has exactly one *genuine* unstable eigenvalue.

Stable ODE blowup without symmetry. For both (1-1) and (1-6), stability of the ODE blow-up solution under small radial perturbations has been proven by Donninger and the third author [Donninger and Schörkhuber 2017] in all odd space dimensions $d \geq 7$. By exploiting the framework of the proof of Theorem 1.1, we generalize the result from that paper to nonradial perturbations in dimensions $d = 7$ and $d = 9$.

Before we state the result, we apply the symmetry transformations (1-7) to the ODE profile (1-2) to obtain the following family of blow-up solutions to both (1-1) and (1-6):

$$u^{\text{ODE}}_{T,x_0,a}(t,x) := \frac{1}{(T-t)^2}\kappa_a\left(\frac{x-x_0}{T-t}\right), \tag{1-18}$$

where

$$\kappa_a(\xi) = 6\gamma(\xi,a)^{-2}. \tag{1-19}$$

To shorten the notation, we write $\mathcal{C}_T := \mathcal{C}_{T,0}$ for the backward light cone with vertex $(T, 0)$.

Theorem 1.6. *Let $d \in \{7, 9\}$. There are constants $C > 0$, $\delta > 0$, and $\omega > 0$ such that, for any real-valued $(f, g) \in C^\infty(\overline{\mathbb{B}}^d_2) \times C^\infty(\overline{\mathbb{B}}^d_2)$ satisfying*

$$\|(f,g)\|_{H^{(d+3)/2}(\mathbb{B}^d_2)\times H^{(d+1)/2}(\mathbb{B}^d_2)} \leq \frac{\delta}{C}, \tag{1-20}$$

the following holds: There exist parameters $a \in \overline{\mathbb{B}}^d_{C\delta/\omega}$ and $T \in [1-\delta, 1+\delta]$ depending Lipschitz continuously on (f, g) such that, for initial data

$$u(0,\cdot) = 6 + f \quad \textit{and} \quad \partial_t u(0,\cdot) = 12 + g,$$

there exists a unique solution $u \in C^\infty(\mathcal{C}_T)$ *to* (1-1). *This solution blows up at* $(T, 0)$ *and can be written as*

$$u(t,x) = \frac{1}{(T-t)^2}\left[\kappa_a\left(\frac{x}{T-t}\right) + \varphi(t,x)\right],$$

where φ *satisfies* $\|\varphi(t,\cdot)\|_{L^\infty(\mathbb{B}^d_{T-t})} \lesssim (T-t)^\omega$ *and*

$$(T-t)^{k-\frac{d}{2}}\|\varphi(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^d_{T-t})} \lesssim (T-t)^\omega$$

for $k = 0, \dots, \frac{1}{2}(d+1)$. *In particular,*

$$\begin{aligned}(T-t)^{k-\frac{d}{2}+2}\|u(t,\cdot) - u^{\mathrm{ODE}}_{T,0,a}(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^d_{T-t})} &\lesssim (T-t)^\omega,\\ (T-t)^{k-\frac{d}{2}+2}\|\partial_t u(t,\cdot) - \partial_t u^{\mathrm{ODE}}_{T,0,a}(t,\cdot)\|_{\dot{H}^{k-1}(\mathbb{B}^d_{T-t})} &\lesssim (T-t)^\omega\end{aligned} \tag{1-21}$$

for $k = 1, \dots, \frac{1}{2}(d+1)$. *Furthermore, u is strictly positive and the statement above therefore applies to* (1-6) *as well.*

We note that due to the invariance of $u^{\mathrm{ODE}}_{1,0,0}$ under spatial translations the blow-up location $x_0 = 0$ does not change under small perturbations.

Remark 1.7. Stability of the ODE blow-up solution for energy supercritical wave equations outside radial symmetry was established in $d = 3$ by Donninger and the third author [Donninger and Schörkhuber 2016]. For the cubic wave equation, the corresponding result was obtained by Chatzikaleas and Donninger [2019] in $d = 5, 7$. Compared to these works, one important improvement in Theorem 1.6 is the regularity of the solution which allows for the classical interpretation. Furthermore, we prove Lipschitz dependence of the blow-up time and the blow-up point on the initial data. Finally, from a technical perspective, the adapted inner product defined in Section 3 is simpler than the corresponding expressions in [Chatzikaleas and Donninger 2019] and can easily be generalized.

1B. ***Related results.*** Wave equations with focusing power nonlinearities provide the simplest possible models for the study of nonlinear wave dynamics and have been investigated intensively in the past decades. Consequently, local well-posedness and the behavior of solutions for small initial data are by now well understood; see, e.g., [Lindblad and Sogge 1995]. Concerning global dynamics for large initial data, substantial progress has been made more recently for energy critical problems. This includes fundamental works on the characterization of the threshold between finite-time blowup and dispersion in terms of the well-known stationary ground state solution; see [Kenig and Merle 2008; Krieger et al. 2015].

In contrast, large data results for energy supercritical equations are rare. For various models, the ODE blowup is known to provide a stable blow-up mechanism and Theorem 1.6 further extends these results; see Remark 1.7. In [Bizoń et al. 2007], nontrivial self-similar solutions are constructed for odd supercritical nonlinearities in dimension 3, and [Dai and Duyckaerts 2021] provides a generalization to $d \geq 4$. Also, in the three-dimensional case, large global solutions were obtained for a supercritical nonlinearity in [Krieger and Schlag 2017]. Finally, for $d \geq 11$ and large enough nonlinearities, manifolds of codimension greater than or equal to two have been constructed in [Collot 2018] that lead to non-self-similar blowup in finite time.

In the description of threshold dynamics for energy supercritical wave equations, self-similar solutions appear to play the key role. This has been observed numerically for power-type nonlinearities [Bizoń et al. 2004; Glogić et al. 2020], but also for more physically relevant models such as wave maps [Biernat et al. 2017; Bizoń et al. 2000] or the Yang–Mills equation in equivariant symmetry [Bizoń and Tabor 2001; Bizoń 2002]. We note that the latter reduces essentially to a radial quadratic wave equation in $d \geq 7$, hence (1-1) provides a toy model. From an analytic point of view, threshold phenomena for energy supercritical wave equations are entirely unexplored. Moreover, results analogous to the energy critical case seem completely out of reach.

However, very recently, the first *explicit* candidate for a self-similar threshold solution has been found by the second and third authors in [Glogić and Schörkhuber 2021] for the focusing cubic wave equation in all supercritical space dimensions $d \geq 5$. In $d = 7$, by the conformal symmetry of the linearized equation, the genuine unstable direction could be given in closed form, see also [Glogić et al. 2020], which allowed for a rigorous stability analysis. Interestingly, the same effect occurs for the quadratic wave equation and the new self-similar solution (1-4) in $d = 9$, which explains the specific choice of the space dimension in Theorem 1.1. In view of our results, we conjecture that the self-similar profile U given in (1-5) plays an important role in the threshold dynamics for (1-1) and (1-6).

In the proofs of Theorems 1.1 and 1.6 we build on methods developed in earlier works, in particular, [Donninger and Schörkhuber 2016; Glogić and Schörkhuber 2021]. However, several aspects, in particular the spectral analysis, are specific to the problem and rather delicate. Furthermore, we add important generalizations such as the preservation of regularity, which improves the statements of these earlier works. The presentation of our results is completely self-contained and all necessary details are provided in the proofs.

1C. ***Notation.*** Throughout the whole paper the Einstein summation convention is in force, i.e., we sum over repeated upper and lower indices, where latin indices run from 1 to d. We write $\mathbb{N}$ for the natural numbers $\{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Furthermore, $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$. Also, $\overline{\mathbb{H}}$ stands for the closed complex right half-plane. By $\mathbb{B}^d_R(x_0)$ we denote the open ball of radius $R > 0$ in $\mathbb{R}^d$ centered at $x_0 \in \mathbb{R}^d$. The unit ball is abbreviated by $\mathbb{B}^d := \mathbb{B}^d_1(0)$, and $\mathbb{S}^{d-1} := \partial\mathbb{B}^d$. The notation $a \lesssim b$ means $a \leq Cb$ for an absolute constant $C > 0$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. If $a \leq C_\varepsilon b$ for a constant $C_\varepsilon > 0$ depending on some parameter ε, we write $a \lesssim_\varepsilon b$.

By $L^2(\mathbb{B}^d_R(x_0))$ and $H^k(\mathbb{B}^d_R(x_0))$, $k \in \mathbb{N}_0$, we denote the Lebesgue and Sobolev spaces, respectively, obtained from the completion of $C^\infty(\overline{\mathbb{B}^d_R(x_0)})$ with respect to the usual norm

$$\|u\|^2_{H^k(\mathbb{B}^d_R(x_0))} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|^2_{L^2(\mathbb{B}^d_R(x_0))},$$

with $\alpha \in \mathbb{N}^d_0$ denoting a multi-index and $\partial^\alpha u = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} u$, where $\partial_i u(x) = \partial_{x_i} u(x)$. For vector-valued functions, we use boldface letters, e.g., $\boldsymbol{f} = (f_1, f_2)$ and we sometime write $[\boldsymbol{f}]_1 := f_1$ to extract a single component. Throughout the paper, $W(f, g)$ denotes the Wronskian of two functions $f, g \in C^1(I)$, $I \subset \mathbb{R}$, where we use the convention $W(f, g) = fg' - f'g$, with f' denoting the first derivative. On a Hilbert space $\mathcal{H}$ we denote by $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators. For a closed linear operator $(L, \mathcal{D}(L))$

on $\mathcal{H}$, we define the resolvent set $\rho(L)$ as the set of all $\lambda \in \mathbb{C}$ such that $R_L(\lambda) := (\lambda - L)^{-1}$ exists as a bounded operator on the whole underlying space. Furthermore, the spectrum of L is defined as $\sigma(L) := \mathbb{C} \setminus \rho(L)$ and the point spectrum is denoted by $\sigma_p(L) \subset \sigma(L)$.

Spherical harmonics. Fix a dimension $d \geq 3$. For $\ell \in \mathbb{N}_0$, an eigenfunction for the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ with eigenvalue $\ell(\ell + d - 2)$ is called a spherical harmonic function of degree ℓ. For each $\ell \in \mathbb{N}$, we denote by $M_{d,\ell}$ the number of linearly independent spherical harmonics of degree ℓ, and for $\Omega_\ell := \{1, \dots, M_{d,\ell}\}$ we designate by $\{Y_{\ell,m} : m \in \Omega_\ell\}$ a set of orthonormal spherical harmonics, i.e.,

$$\int_{\mathbb{S}^{d-1}} Y_{\ell,m}(\omega)\overline{Y_{\ell,m'}(\omega)}\, d\sigma(\omega) = \delta_{mm'}.$$

Obviously, one has $\Omega_0 = \{1\}$ and $\Omega_1 = \{1, \dots, d\}$, and we can take $Y_{0,1}(\omega) = c_1$ and $Y_{1,m}(\omega) = \tilde{c}_m \omega_m$ for suitable normalization constants $c_1, \tilde{c}_m \in \mathbb{R}$. For $g \in C^\infty(\mathbb{S}^{d-1})$, we define $\mathcal{P}_\ell : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ by

$$\mathcal{P}_\ell g(\omega) := \sum_{m \in \Omega_\ell} (g \,|\, Y_{\ell,m})_{L^2(\mathbb{S}^{d-1})} Y_{\ell,m}(\omega).$$

It is well known, see, e.g., [Atkinson and Han 2012], that $\mathcal{P}_\ell$ defines a self-adjoint projection on $L^2(\mathbb{S}^{d-1})$ and that $\lim_{n\to\infty} \| g - \sum_{\ell=0}^n \mathcal{P}_\ell g \|_{L^2(\mathbb{S}^{d-1})} = 0$. This can be extended to Sobolev spaces, in particular, $\lim_{n\to\infty} \| g - \sum_{\ell=0}^n \mathcal{P}_\ell g \|_{H^k(\mathbb{S}^{d-1})} = 0$ for all $g \in C^\infty(\mathbb{S}^{d-1})$, see, e.g., [Donninger and Schörkhuber 2016], Lemma A.1. Furthermore, given $f \in C^\infty(\overline{\mathbb{B}}^d_R)$, by setting

$$[P_\ell f](x) := \sum_{m \in \Omega_\ell} (f(|x|\,\cdot) \,|\, Y_{\ell,m})_{L^2(\mathbb{S}^{d-1})} Y_{\ell,m}\left(\frac{x}{|x|}\right), \tag{1-22}$$

we have that (see for example Lemma A.2 in [Donninger and Schörkhuber 2016])

$$\lim_{n\to\infty} \left\| f - \sum_{\ell=0}^n P_\ell f \right\|_{H^k(\mathbb{B}^d_R)} = 0. \tag{1-23}$$

2. The stability problem in similarity coordinates

In this section we formulate (1-1) in similarity variables. The advantage of the new setting is the fact that self-similar solutions become time-independent and stability of finite-time blowup turns into asymptotic stability of static solutions. Then we state the main results in the new coordinate system.

Given $T > 0$ and $x_0 \in \mathbb{R}^d$, we define *similarity coordinates*

$$\tau := -\log(T - t) + \log T \quad \text{and} \quad \xi := \frac{x - x_0}{T - t}.$$

Note that in (τ, ξ), the backward light cone $\mathcal{C}_{T,x_0}$ corresponds to the infinite cylinder

$$\mathcal{Z} := \bigcup_{\tau \geq 0} \{\tau\} \times \overline{\mathbb{B}}^d.$$

Furthermore, by setting

$$\psi(\tau, \xi) := T^2 e^{-2\tau} u(T - Te^{-\tau}, Te^{-\tau}\xi + x_0),$$

(1-1) transforms into

$$(\partial_\tau^2 + 5\partial_\tau + 2\xi\cdot\nabla\partial_\tau + (\xi\cdot\nabla)^2 - \Delta + 5\xi\cdot\nabla + 6)\psi(\tau,\xi) = \psi(\tau,\xi)^2. \tag{2-1}$$

To get a first-order formulation we define

$$\psi_1(\tau,\xi) := \psi(\tau,\xi) \quad\text{and}\quad \psi_2(\tau,\xi) := \partial_\tau\psi(\tau,\xi) + \xi\cdot\nabla\psi(\tau,\xi) + 2\psi(\tau,\xi), \tag{2-2}$$

and let $\Psi(\tau) = (\psi_1(\tau,\cdot),\psi_2(\tau,\cdot))$, by means of which (2-1) can be written as

$$\partial_\tau\Psi(\tau) = \widetilde{\boldsymbol{L}}\Psi(\tau) + \boldsymbol{F}(\Psi(\tau)), \tag{2-3}$$

where

$$\widetilde{\boldsymbol{L}}\boldsymbol{u}(\xi) = \begin{pmatrix} -\xi\cdot\nabla u_1(\xi) - 2u_1(\xi) + u_2(\xi) \\ \Delta u_1(\xi) - \xi\cdot\nabla u_2(\xi) - 3u_2(\xi)\end{pmatrix} \quad\text{and}\quad \boldsymbol{F}(\boldsymbol{u}) = \begin{pmatrix} 0 \\ u_1^2\end{pmatrix}$$

for $\boldsymbol{u} = (u_1,u_2)$. Note that in the new variables, the solutions $u^*_{T,x_0,a}$ and $u^{\mathrm{ODE}}_{T,x_0,a}$ become static. Namely, every $a\in\mathbb{R}^d$ yields smooth, positive, and τ-independent solutions

$$\boldsymbol{U}_a = (U_{1,a}, U_{2,a}) \quad\text{and}\quad \boldsymbol{\kappa}_a = (\kappa_{1,a},\kappa_{2,a})$$

of (2-3) given by

$$\begin{aligned} U_{1,a}(\xi) &= U_a(\xi), \quad U_{2,a}(\xi) = \xi\cdot\nabla U_a(\xi) + 2U_a(\xi), \\ \kappa_{1,a}(\xi) &= \kappa_a(\xi), \quad \kappa_{2,a}(\xi) = \xi\cdot\nabla\kappa_a(\xi) + 2\kappa_a(\xi). \end{aligned}$$

We study (2-3) for small perturbations of $\boldsymbol{U}_a$ and $\boldsymbol{\kappa}_a$ in the Hilbert space

$$\mathcal{H} := H^{\frac{d+1}{2}}(\mathbb{B}^d)\times H^{\frac{d-1}{2}}(\mathbb{B}^d)$$

equipped with the standard norm

$$\|\boldsymbol{u}\|^2 := \|u_1\|^2_{H^{(d+1)/2}(\mathbb{B}^d)} + \|u_2\|^2_{H^{(d-1)/2}(\mathbb{B}^d)}.$$

Also, write $\mathcal{B}_R := \{\boldsymbol{u}\in\mathcal{H} : \|\boldsymbol{u}\|\le R\}$.

In Proposition 3.1 on page 629 we show that, for $d\in\{7,9\}$, the operator

$$\widetilde{\boldsymbol{L}} : C^\infty(\overline{\mathbb{B}}^d)\times C^\infty(\overline{B}^d)\subset\mathcal{H}\to\mathcal{H},$$

which describes the free wave evolution in similarity coordinates, is closable and its closure, which we denote by

$$\boldsymbol{L} : \mathcal{D}(\boldsymbol{L})\subset\mathcal{H}\to\mathcal{H},$$

generates a strongly continuous one-parameter semigroup $(\boldsymbol{S}(\tau))_{\tau\ge 0}\subset\mathcal{B}(\mathcal{H})$. By using the Sobolev embedding, it is easy to see that the nonlinearity satisfies

$$\|\boldsymbol{F}(\boldsymbol{u})\| = \|u_1^2\|_{H^{(d-1)/2}(\mathbb{B}^d)} \le \|u_1^2\|_{H^{(d+1)/2}(\mathbb{B}^d)} \lesssim \|u_1\|^2_{H^{(d+1)/2}(\mathbb{B}^d)} \lesssim \|\boldsymbol{u}\|^2$$

for all $\boldsymbol{u}\in\mathcal{H}$; hence $\boldsymbol{F}$ is well defined on $\mathcal{H}$.

2A. ***Stability of $\boldsymbol{U_a}$.*** The key to proving Theorem 1.1 is the following result, which establishes, for $d = 9$, conditional orbital asymptotic stability of the family of static solutions $\{\boldsymbol{U}_a : a \in \mathbb{R}^9\}$.

Proposition 2.1. *Let $d = 9$. There are constants $C > 0$ and $\omega > 0$ such that the following holds. For all sufficiently small $\delta > 0$ there exists a codimension-11 Lipschitz manifold $\mathcal{M} = \mathcal{M}_{\delta,C} \subset \mathcal{B}_{\delta/C}$ with $\mathbf{0} \in \mathcal{M}$ such that, for any $\Phi_0 \in \mathcal{M}$, there are $\Psi \in C([0,\infty),\mathcal{H})$ and $a \in \overline{\mathbb{B}}^9_{\delta/\omega}$ such that*

$$\Psi(\tau) = \boldsymbol{S}(\tau)(\boldsymbol{U}_0 + \Phi_0) + \int_0^\tau \boldsymbol{S}(\tau-\sigma)\boldsymbol{F}(\Psi(\sigma))\,d\sigma \tag{2-4}$$

and

$$\|\Psi(\tau) - \boldsymbol{U}_a\| \lesssim \delta e^{-\omega\tau}$$

for all $\tau \geq 0$.

The number of codimensions in Proposition 2.1 is related to the number of unstable eigenvalues of the linearization around $\boldsymbol{U}_a$ and the dimension of the corresponding eigenspaces; see Section 5. In fact, ten of these instabilities are caused by the translation symmetries of the problem, and can be controlled by choosing appropriately the blow-up parameters (T, x_0). There is, therefore, only one genuine unstable direction. Next, we state a persistence of regularity result for solutions to (2-4).

Proposition 2.2. *If the initial data Φ_0 from Proposition 2.1 is in $C^\infty(\overline{\mathbb{B}}^9) \times C^\infty(\overline{\mathbb{B}}^9)$ then the corresponding solution Ψ of (2-3) belongs to $C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z})$. In particular, Ψ satisfies (2-3) in the classical sense.*

Remark 2.3. That this proposition is not vacuous, i.e., that there exists $\Phi_0 \in \mathcal{M} \cap (C^\infty(\overline{\mathbb{B}}^9) \times C^\infty(\overline{\mathbb{B}}^9))$, follows from Proposition 2.4.

The proofs of Propositions 2.1 and 2.2 are provided in Section 7D.

In order to derive Theorem 1.1 from the above results we prescribe in physical variables initial data of the form

$$u(0,\cdot) = u^*_{1,0,0}(0,\cdot) + f \quad \text{and} \quad \partial_t u(0,\cdot) = \partial_t u^*_{1,0,0}(0,\cdot) + g \tag{2-5}$$

for free functions (f, g) defined on a suitably large ball centered at the origin. In similarity variables, this transforms into initial data $\Psi(0) = \boldsymbol{U}_0 + \Phi_0$ for (2-3), with

$$\Phi_0 = \Upsilon((f,g), T, x_0), \tag{2-6}$$

where

$$\Upsilon((f,g),T,x_0) := \mathcal{R}((f,g),T,x_0) + \mathcal{R}(\boldsymbol{U}_0,T,x_0) - \mathcal{R}(\boldsymbol{U}_0,1,0) \tag{2-7}$$

and

$$\mathcal{R}((f_1,f_2),T,x_0) = \begin{pmatrix} T^2 f_1(T\cdot+x_0) \\ T^3 f_2(T\cdot+x_0) \end{pmatrix}.$$

The next statement asserts that, for all small (f,g), there is a choice of parameters x_0, T, and α for which $\Upsilon((f+\alpha h_1, g+\alpha h_2), T, x_0)$ belongs to the manifold $\mathcal{M}$ from Proposition 2.1.

Proposition 2.4. *Let* (h_1, h_2) *be defined as in* (1-15). *There exists* $M > 0$ *such that*, *for all sufficiently small* $\delta > 0$, *the following holds. For any* $(f, g) \in H^6(\mathbb{B}^9_2) \times H^5(\mathbb{B}^9_2)$ *satisfying*

$$\|(f, g)\|_{H^6(\mathbb{B}^9_2)\times H^5(\mathbb{B}^9_2)} \leq \frac{\delta}{M^2},$$

there are $x_0 \in \overline{\mathbb{B}}^9_{\delta/M}$, $T \in [1-\delta/M, 1+\delta/M]$, *and* $\alpha \in [-\delta/M, \delta/M]$ *depending Lipschitz continuously on* (f, g) *such that*

$$\Upsilon((f+\alpha h_1, g+\alpha h_2), T, x_0) \in \mathcal{M}_{\delta,C},$$

where $\mathcal{M}_{\delta,C}$ *is the manifold from Proposition 2.1.*

Theorem 1.1 is then obtained by transforming the results of Propositions 2.1, 2.2, and 2.4 back to coordinates (t, x).

Remark 2.5. We note that when proving stability of the ODE blow-up solution for $d \in \{7, 9\}$ similar results are obtained. In fact, the proof implies the existence of a Lipschitz manifold $\mathcal{N}$ of codimension $d+1$ in the Hilbert space $\mathcal{H}$, according to $d+1$ directions of instability induced by translation invariance. A result similar to Proposition 2.4 guarantees that for *any* small enough data (f, g) one can suitably adjust the blow-up time T and the blow-up point x_0 such that $\Upsilon((f, g), T, x_0) \in \mathcal{N}$, which gives Theorem 1.6 on stable blowup. This point of view further justifies using codimension-1 terminology to describe the stability of u^*.

Time-evolution for small perturbations: *modulation ansatz.* In the following, we assume that $a = a(\tau)$, $a(0) = 0$, and $\lim_{\tau\to\infty} a(\tau) = a_\infty$. Inserting the ansatz

$$\Psi(\tau) = \boldsymbol{U}_{a(\tau)} + \Phi(\tau) \tag{2-8}$$

into (2-3) we obtain

$$\partial_\tau \Phi(\tau) = (\widetilde{\boldsymbol{L}} + \boldsymbol{L}'_{a(\tau)})\Phi(\tau) + \boldsymbol{F}(\Phi(\tau)) - \partial_\tau \boldsymbol{U}_{a(\tau)},$$

with

$$\boldsymbol{L}'_{a(\tau)}\boldsymbol{u} = \begin{pmatrix} 0 \\ V_{a(\tau)}u_1 \end{pmatrix} \quad \text{and} \quad V_a(\xi) = 2U_a(\xi).$$

In the following, we define

$$\boldsymbol{G}_{a(\tau)}(\Phi(\tau)) := [\boldsymbol{L}'_{a(\tau)} - \boldsymbol{L}'_{a_\infty}]\Phi(\tau) + \boldsymbol{F}(\Phi(\tau))$$

and study the evolution equation

$$\partial_\tau \Phi(\tau) = [\widetilde{\boldsymbol{L}} + \boldsymbol{L}'_{a_\infty}]\Phi(\tau) + \boldsymbol{G}_{a(\tau)}(\Phi(\tau)) - \partial_\tau \boldsymbol{U}_{a(\tau)}, \tag{2-9}$$

with initial data $\Phi(0) = \boldsymbol{u} \in \mathcal{H}$. This naturally splits into three parts: First, in Section 3, we study the time evolution governed by $\widetilde{\boldsymbol{L}}$ using semigroup theory. In Section 4, we analyze the linearized problem, where we consider $\widetilde{\boldsymbol{L}} + \boldsymbol{L}'_{a_\infty}$ as a (compact) perturbation of the free evolution and investigate the underlying spectral problem, restricting to $d = 9$. Resolvent bounds allow us to transfer the spectral information to suitable growth estimates for the linearized time evolution. The nonlinear problem will be analyzed in integral form in Section 7, using modulation theory and fixed-point arguments. Also, we prove Propositions 2.1–2.4 and, based on this, Theorem 1.1. In Section 8 we give the main arguments to prove Theorem 1.6.

3. The free wave evolution in similarity variables

In this section we prove well-posedness of the linear version of (2-3) in $\mathcal{H}$. In other words, we show that the (closure of the) operator $\widetilde{\boldsymbol{L}}$ generates a strongly continuous one-parameter semigroup of bounded operators on $\mathcal{H}$. What is more, in view of the regularity result Proposition 2.2, we consider the evolution in Sobolev spaces of arbitrarily high integer order. In Section 4 we then restrict the problem again to $\mathcal{H}$.

For $k \geq 1$, let

$$\mathcal{H}_k := H^k(\mathbb{B}^d) \times H^{k-1}(\mathbb{B}^d)$$

be equipped with the standard norm denoted by $\|\cdot\|_{H^k(\mathbb{B}^d)\times H^{k-1}(\mathbb{B}^d)}$. We set

$$\mathcal{D}(\widetilde{\boldsymbol{L}}) := C^\infty(\overline{\mathbb{B}}^d) \times C^\infty(\overline{\mathbb{B}}^d)$$

and consider the densely defined operator

$$\widetilde{\boldsymbol{L}} : \mathcal{D}(\widetilde{\boldsymbol{L}}) \subset \mathcal{H}_k \to \mathcal{H}_k.$$

We now state the central result of this section.

Proposition 3.1. *Let $d \in \{7, 9\}$ and $k \geq 3$. The operator $\widetilde{\boldsymbol{L}} : \mathcal{D}(\widetilde{\boldsymbol{L}}) \subset \mathcal{H}_k \to \mathcal{H}_k$ is closable and its closure $\boldsymbol{L}_k : \mathcal{D}(\boldsymbol{L}_k) \subset \mathcal{H}_k \to \mathcal{H}_k$ generates a strongly continuous semigroup $\boldsymbol{S}_k : [0, \infty) \to \mathcal{B}(\mathcal{H}_k)$ which satisfies*

$$\|\boldsymbol{S}_k(\tau)\boldsymbol{u}\|_{H^k(\mathbb{B}^d)\times H^{k-1}(\mathbb{B}^d)} \leq M_k e^{-\frac{1}{2}\tau} \|\boldsymbol{u}\|_{H^k(\mathbb{B}^d)\times H^{k-1}(\mathbb{B}^d)} \tag{3-1}$$

for all $\boldsymbol{u} \in \mathcal{H}_k$, all $\tau \geq 0$, and some $M_k > 1$. Furthermore, the following holds for the spectrum of $\boldsymbol{L}_k$:

$$\sigma(\boldsymbol{L}_k) \subset \left\{z \in \mathbb{C} : \operatorname{Re} z \leq -\tfrac{1}{2}\right\}, \tag{3-2}$$

and the resolvent has the bound

$$\|\boldsymbol{R}_{\boldsymbol{L}_k}(\lambda)\boldsymbol{f}\|_{H^k(\mathbb{B}^d)\times H^{k-1}(\mathbb{B}^d)} \leq \frac{M_k}{\operatorname{Re}\lambda + \frac{1}{2}} \|\boldsymbol{f}\|_{H^k(\mathbb{B}^d)\times H^{k-1}(\mathbb{B}^d)}$$

for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\frac{1}{2}$ and $\boldsymbol{f} \in \mathcal{H}_k$.

Remark 3.2. We prove Proposition 3.1 via the Lumer–Phillips Theorem. By using the standard inner product on $\mathcal{H}_k$, one can easily prove existence of the semigroup $(\boldsymbol{S}_k(\tau))_{\tau\geq 0}$, but in order to show that it decays exponentially and to prove the growth bound (3-1) in particular, we need to introduce an appropriate equivalent inner product. The necessity for such an approach will become apparent in the proof of Lemma 3.4 in the Appendix. We note that, for $d = 9$, the restriction on k is optimal within the class of integer Sobolev spaces. In particular, for scaling reasons exponential decay cannot be expected at lower integer regularities. For $d = 7$, a similar statement can be obtained for $k = 2$.

For $d \in \{7, 9\}$ and $k \geq 3$ we define the sesquilinear form

$$(\cdot\,|\,\cdot)_{\mathcal{H}_k} : (C^\infty(\overline{\mathbb{B}}^d) \times C^\infty(\overline{\mathbb{B}}^d))^2 \to \mathbb{C}, \quad (\boldsymbol{u}\,|\,\boldsymbol{v})_{\mathcal{H}_k} = \sum_{j=1}^{k} (\boldsymbol{u}\,|\,\boldsymbol{v})_j,$$

where

$$(\boldsymbol{u}\,|\,\boldsymbol{v})_1 = \int_{\mathbb{S}^{d-1}} \partial_i u_1(\omega)\overline{\partial^i v_1(\omega)}\, d\sigma(\omega) + \int_{\mathbb{S}^{d-1}} u_1(\omega)\overline{v_1(\omega)}\, d\sigma(\omega) + \int_{\mathbb{S}^{d-1}} u_2(\omega)\overline{v_2(\omega)}\, d\sigma(\omega),$$

$$(\boldsymbol{u}\,|\,\boldsymbol{v})_2 = \int_{\mathbb{B}^d} \partial_i \Delta u_1(\xi)\overline{\partial^i \Delta v_1(\xi)}\, d\xi + \int_{\mathbb{B}^d} \partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j v_2(\xi)}\, d\xi + \int_{\mathbb{S}^{d-1}} \partial_i u_2(\omega)\overline{\partial^i v_2(\omega)}\, d\sigma(\omega),$$

$$(\boldsymbol{u}\,|\,\boldsymbol{v})_3 = 4\int_{\mathbb{B}^d} \partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k v_1(\xi)}\, d\xi + 4\int_{\mathbb{B}^d} \partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j v_2(\xi)}\, d\xi$$
$$+ 4\int_{\mathbb{S}^{d-1}} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j v_1(\omega)}\, d\sigma(\omega),$$

and for $j \geq 4$ we use the standard $\dot{H}^j(\mathbb{B}^d) \times \dot{H}^{j-1}(\mathbb{B}^d)$ inner product

$$(\boldsymbol{u}\,|\,\boldsymbol{v})_j = (u_1\,|\,v_1)_{\dot{H}^j(\mathbb{B}^d)} + (u_2\,|\,v_2)_{\dot{H}^{j-1}(\mathbb{B}^d)}. \tag{3-3}$$

We then set

$$\|\boldsymbol{u}\|_{\mathcal{H}_k} := \sqrt{(\boldsymbol{u}\,|\,\boldsymbol{u})_{\mathcal{H}_k}}.$$

For brevity, we will use the notation $(\cdot\,|\,\cdot)_j = \|\cdot\|_j^2$, $j = 1, \dots, k$, for different parts of $(\cdot\,|\,\cdot)_{\mathcal{H}_k}$.

Lemma 3.3. *Let $d \in \{7, 9\}$ and $k \geq 3$. We have*

$$\|\boldsymbol{u}\|_{\mathcal{H}_k} \simeq \|\boldsymbol{u}\|_{H^k(\mathbb{B}^d)\times H^{k-1}(\mathbb{B}^d)}$$

for all $\boldsymbol{u} \in C^\infty(\overline{\mathbb{B}}^d) \times C^\infty(\overline{\mathbb{B}}^d)$. In particular, $\|\cdot\|_{\mathcal{H}_k}$ defines an equivalent norm on $\mathcal{H}_k$.

Proof. Note that it suffices to prove

$$\|\boldsymbol{u}\|^2_{H^3(\mathbb{B}^d)\times H^2(\mathbb{B}^d)} \lesssim \sum_{j=1}^{3} \|\boldsymbol{u}\|_j^2 \lesssim \|\boldsymbol{u}\|^2_{H^3(\mathbb{B}^d)\times H^2(\mathbb{B}^d)}. \tag{3-4}$$

The first estimate in (3-4) follows from the fact that

$$\|u\|^2_{L^2(\mathbb{B}^d)} \lesssim \|\nabla u\|^2_{L^2(\mathbb{B}^d)} + \|u\|^2_{L^2(\mathbb{S}^{d-1})}$$

for all $u \in C^\infty(\overline{\mathbb{B}}^d)$, which is a simple consequence of the identity

$$\int_{\mathbb{S}^{d-1}} |u(\omega)|^2\, d\sigma(\omega) = \int_{\mathbb{B}^d} \operatorname{div}(\xi|u(\xi)|^2)\, d\xi$$
$$= \int_{\mathbb{B}^d} (d|u(\xi)|^2 + \xi^i u(\xi)\overline{\partial_i u(\xi)} + \xi^i\overline{u(\xi)}\partial_i u(\xi))\, d\xi. \tag{3-5}$$

Using this, it is easy to see that

$$\|u\|_{H^2(\mathbb{B}^d)} \lesssim \int_{\mathbb{B}^d} \partial_i\partial_j u(\xi)\overline{\partial^i\partial^j u(\xi)}\, d\xi + \int_{\mathbb{S}^{d-1}} \partial_i u(\omega)\overline{\partial^i u(\omega)}\, d\sigma(\omega) + \int_{\mathbb{S}^{d-1}} |u(\omega)|^2\, d\sigma(\omega)$$

for all $u \in C^\infty(\overline{\mathbb{B}}^d)$. Similar bounds imply the first inequality in (3-4). Another consequence of (3-5) is the trace theorem, which asserts that

$$\int_{\mathbb{S}^{d-1}} |u(\omega)|^2 \, d\sigma(\omega) \lesssim \|u\|^2_{H^1(\mathbb{B}^d)}$$

for all $u \in C^\infty(\overline{\mathbb{B}}^d)$; using this, it is straightforward to obtain the second inequality in (3-4). Hence we obtain the claimed estimates in Lemma 3.3 for all $\boldsymbol{u} \in C^\infty(\overline{\mathbb{B}}^d) \times C^\infty(\overline{\mathbb{B}}^d)$ and, by density, we extend this to all of $\mathcal{H}_k$. □

Now we turn to proving Proposition 3.1. As the first auxiliary result, we have the following dissipation property of $\widetilde{\boldsymbol{L}}$.

Lemma 3.4. *Let $d \in \{7, 9\}$ and $k \geq 3$. Then*

$$\operatorname{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u} \,|\, \boldsymbol{u})_{\mathcal{H}_k} \leq -\tfrac{1}{2}\|\boldsymbol{u}\|^2_{\mathcal{H}_k}$$

for all $\boldsymbol{u} \in \mathcal{D}(\widetilde{\boldsymbol{L}})$.

The proof is provided in the Appendix. To apply the Lumer–Phillips theorem, we also need the following density property of $\widetilde{\boldsymbol{L}}$.

Lemma 3.5. *Let $d \in \{7, 9\}$ and $k \geq 3$. There exists $\lambda > -\frac{1}{2}$ such that* $\operatorname{ran}(\lambda - \widetilde{\boldsymbol{L}})$ *is dense in $\mathcal{H}_k$.*

Proof. Let $d \in \{7, 9\}$ and $k \geq 3$. We prove the statement by showing that there exists a λ such that, given $\boldsymbol{f} \in \mathcal{H}_k$ and $\varepsilon > 0$, there is some $\boldsymbol{f}_\varepsilon$ in the ε-neighborhood of $\boldsymbol{f}$ for which the equation $(\lambda - \widetilde{\boldsymbol{L}})\boldsymbol{u} = \boldsymbol{f}_\varepsilon$ admits a solution in $\mathcal{D}(\widetilde{\boldsymbol{L}})$. First, by density, there is $\tilde{\boldsymbol{f}} \in C^\infty(\overline{\mathbb{B}}^d) \times C^\infty(\overline{\mathbb{B}}^d)$ for which $\|\tilde{\boldsymbol{f}} - \boldsymbol{f}\|_{H^k(\mathbb{B}^d)\times H^{k-1}(\mathbb{B}^d)} < \frac{1}{2}\varepsilon$. Then, for $n \in \mathbb{N}$, we define $\boldsymbol{f}_n := (f_{1,n}, f_{2,n})$ with

$$f_{1,n} = \sum_{\ell=0}^{n} P_\ell \tilde{f}_1 \quad \text{and} \quad f_{2,n} = \sum_{\ell=0}^{n} P_\ell \tilde{f}_2,$$

where the P_ℓ are the projection operators defined in (1-22). Furthermore, according to (1-23) there exists an index $N \in \mathbb{N}$ for which $\|\boldsymbol{f}_N - \tilde{\boldsymbol{f}}\|_{H^k(\mathbb{B}^d)\times H^{k-1}(\mathbb{B}^d)} < \frac{1}{2}\varepsilon$. It is therefore sufficient to consider

$$(\lambda - \widetilde{\boldsymbol{L}})\boldsymbol{u} = \boldsymbol{f}_N \tag{3-6}$$

and produce a solution $\boldsymbol{u} \in \mathcal{D}(\widetilde{\boldsymbol{L}})$. First, we rewrite (3-6) as a system of equations in u_1 and u_2:

$$-(\delta^{ij} - \xi^i\xi^j)\partial_i\partial_j u_1(\xi) + 2(\lambda+3)\xi^i\partial_i u_1(\xi) + (\lambda+3)(\lambda+2)u_1(\xi) = g_N(\xi), \tag{3-7}$$

$$u_2(\xi) = \xi^i\partial_i u_1(\xi) + (\lambda+1)u_1(\xi) - f_{1,N}(\xi), \tag{3-8}$$

where

$$g_N(\xi) = \xi^i\partial_i f_{1,N}(\xi) + (\lambda+3)f_{1,N}(\xi) + f_{2,N}(\xi).$$

We now treat the case $d = 9$, for which we choose $\lambda = \frac{5}{2}$. With this choice, (3-7) reads as

$$-(\delta^{ij} - \xi^i\xi^j)\partial_i\partial_j u_1(\xi) + 11\xi^i\partial_i u_1(\xi) + \tfrac{99}{4}u_1(\xi) = g_N(\xi). \tag{3-9}$$

Note that g_N is a finite linear combination of spherical harmonics, and this allows us to decompose the PDE (3-9) which is posed on $\mathbb{B}^9$ into a finite number of ODEs posed on the interval $(0, 1)$. To this end, we switch to spherical coordinates $\rho = |\xi|$ and $\omega = \xi/|\xi|$. In particular, the relevant differential expressions transform in the following way:

$$\begin{aligned} \xi^i \partial_i u(\xi) &= \rho \partial_\rho u(\rho\omega), \\ \xi^i \xi^j \partial_i \partial_j u(\xi) &= \rho^2 \partial_\rho^2 u(\rho\omega), \\ \partial^i \partial_i u(\xi) &= \left(\partial_\rho^2 + \frac{8}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_\omega^{\mathbb{S}^8}\right) u(\rho\omega). \end{aligned}$$

Consequently, (3-9) becomes

$$\left(-(1-\rho^2)\partial_\rho^2 + \left(-\frac{8}{\rho} + 11\rho\right)\partial_\rho + \frac{99}{4} - \frac{1}{\rho^2}\Delta_\omega^{\mathbb{S}^8}\right) u(\rho\omega) = g_N(\rho\omega). \tag{3-10}$$

Now we take the decomposition of the right-hand side of (3-10) into spherical harmonics:

$$g_N(\rho\omega) = \sum_{\ell=0}^{N} \sum_{m\in\Omega_\ell} g_{\ell,m}(\rho) Y_{\ell,m}(\omega)$$

for some $g_{\ell,m} \in C^\infty[0, 1]$. Then by inserting the ansatz

$$u_1(\rho\omega) = \sum_{\ell=0}^{N} \sum_{m\in\Omega_\ell} u_{\ell,m}(\rho) Y_{\ell,m}(\omega) \tag{3-11}$$

into (3-10), we obtain the system of ODEs

$$\left(-(1-\rho^2)\partial_\rho^2 + \left(-\frac{8}{\rho} + 11\rho\right)\partial_\rho + \frac{\ell(\ell+7)}{\rho^2} + \frac{99}{4}\right) u_{\ell,m}(\rho) = g_{\ell,m}(\rho) \tag{3-12}$$

for $\ell = 0, \dots, N$ and $m \in \Omega_\ell$. For later convenience, we first set $v_{\ell,m}(\rho) = \rho^3 u_{\ell,m}(\rho)$ and thereby transform (3-12) into

$$\left(-(1-\rho^2)\partial_\rho^2 + \left(-\frac{2}{\rho} + 5\rho\right)\partial_\rho + \frac{(\ell+4)(\ell+3)}{\rho^2} + \frac{15}{4}\right) v_{\ell,m}(\rho) = \rho^3 g_{\ell,m}(\rho). \tag{3-13}$$

Then, by means of a further change of variables $v_{\ell,m}(\rho) = \rho^{\ell+3} w_{\ell,m}(\rho^2)$, we turn the homogeneous version of (3-13) into a hypergeometric equation in its canonical form:

$$z(1-z)w''_{\ell,m}(z) + (c - (a+b+1)z)w'_{\ell,m}(z) - ab w_{l,m}(z) = 0, \tag{3-14}$$

where

$$a = \tfrac{1}{4}(9+2\ell), \quad b = a + \tfrac{1}{2} = \tfrac{1}{4}(11+2\ell), \quad \text{and} \quad c = 2a = \tfrac{1}{2}(9+2\ell).$$

Equation (3-14) admits the two solutions

$$\phi_{0,\ell}(z) = {}_2F_1\big(a, a+\tfrac{1}{2}, 2a, z\big) \quad \text{and} \quad \phi_{1,\ell}(z) = {}_2F_1\big(a, a+\tfrac{1}{2}, \tfrac{3}{2}, 1-z\big),$$

which are analytic around $z = 0$ and $z = 1$, respectively; see [DLMF 2010]. In fact, the functions $\phi_{0,\ell}$ and $\phi_{1,\ell}$ can be expressed in closed form as

$$\phi_{0,\ell}(z) = \frac{1}{\sqrt{1-z}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{\frac{7}{2}+\ell},$$
$$\phi_{1,\ell}(z) = \sqrt{1-z}\left(\left(\frac{1}{1-\sqrt{1-z}}\right)^{\frac{7}{2}+\ell} - \left(\frac{1}{1+\sqrt{1-z}}\right)^{\frac{7}{2}+\ell}\right);$$

see [DLMF 2010, pp. 386-387]. Now by undoing the change of variables from above, we get solutions $\psi_{\ell,0} = \rho^{\ell+3}\phi_{0,\ell}(\rho^2)$ and $\psi_{\ell,1} = \rho^{\ell+3}\phi_{1,\ell}(\rho^2)$ to the homogeneous version of (3-13). Furthermore, the Wronskian is $W(\psi_{0,\ell}, \psi_{1,\ell})(\rho) = C_\ell(1-\rho^2)^{-3/2}\rho^{-2}$ for some nonzero constant C_ℓ. Then, by the variation of constants formula we obtain a solution to (3-13) on $(0, 1)$:

$$\begin{aligned} v_{\ell,m}(\rho) &= -\psi_{\ell,0}(\rho)\int_\rho^1 \frac{\psi_{\ell,1}(s)}{W(\psi_{\ell,0},\psi_{\ell,1})(s)}\frac{s^3 g_{\ell,m}(s)}{1-s^2}\,ds - \psi_{\ell,1}(\rho)\int_0^\rho \frac{\psi_{\ell,0}(s)}{W(\psi_{\ell,0},\psi_{\ell,1})(s)}\frac{s^3 g_{\ell,m}(s)}{1-s^2}\,ds \\ &= -\psi_{\ell,0}(\rho)\int_\rho^1 \psi_{\ell,1}(s)\sqrt{1-s}\,h_{\ell,m}(s)\,ds - \psi_{\ell,1}(\rho)\int_0^\rho \psi_{\ell,0}(s)\sqrt{1-s}\,h_{\ell,m}(s)\,ds, \end{aligned} \tag{3-15}$$

where $h_{\ell,m} \in C^\infty[0, 1]$. Obviously $v_{\ell,m} \in C^\infty(0, 1)$. We claim that $v_{\ell,m} \in C^\infty(0, 1]$. To see this, we note that, at $\rho = 1$, the set of Frobenius indices of (3-13) is $\{-\frac{1}{2}, 0\}$. Hence near $\rho = 1$, there is another solution, linearly independent of $\psi_{\ell,1}$, which has the form $(1-\rho)^{-1/2}\psi_{\ell,2}(\rho)$, where $\psi_{\ell,2}$ is analytic at $\rho = 1$. Hence

$$\psi_{\ell,0}(\rho) = c_{\ell,1}\psi_{\ell,1}(\rho) + c_{\ell,2}\frac{\psi_{\ell,2}(\rho)}{\sqrt{1-\rho}} \tag{3-16}$$

for some constants $c_{\ell,1}$ and $c_{\ell,2}$. Now, by letting

$$\alpha_{\ell,m} := \int_0^1 \psi_{\ell,0}(s)\sqrt{1-s}\,h_{\ell,m}(s)\,ds$$

and inserting (3-16) into (3-15), we get

$$\begin{aligned} v_{\ell,m}(\rho) = -c_{\ell,2}\frac{\psi_{\ell,2}(\rho)}{\sqrt{1-\rho}}&\int_\rho^1 \psi_{\ell,1}(s)\sqrt{1-s}\,h_{\ell,m}(s)\,ds \\ &- \alpha_{\ell,m}\psi_{\ell,1}(\rho) + c_{\ell,2}\psi_{\ell,1}(\rho)\int_\rho^1 \psi_{\ell,2}(s)h_{\ell,m}(s)\,ds. \end{aligned}$$

The second and the third term above are obviously smooth up to $\rho = 1$; for the first term, the square root factors in fact cancel out, as can easily be seen via the substitution $s = \rho + (1-\rho)t$, and smoothness of $v_{\ell,m}$ up to $\rho = 1$ follows. Consequently, the function u_1 defined in (3-11) belongs to $C^\infty(\overline{\mathbb{B}}^9 \setminus \{0\})$, and it solves (3-9) in the classical sense away from zero. Furthermore, from (3-15) one can check that $u_{\ell,m}$ and $u'_{\ell,m}$ are bounded near zero, and hence $u_1 \in H^1(\mathbb{B}^9)$. In particular, u_1 solves (3-9) in the weak sense on $\mathbb{B}^9$, and since the right-hand side is a smooth function, we conclude that $u_1 \in C^\infty(\mathbb{B}^9)$ by elliptic regularity. Consequently, $u_1 \in C^\infty(\overline{\mathbb{B}}^9)$, and therefore $u_2 \in C^\infty(\overline{\mathbb{B}}^9)$ according to (3-8). In conclusion, $\boldsymbol{u} := (u_1, u_2) \in \mathcal{D}(\widetilde{\boldsymbol{L}})$ solves (3-6).

For $d=7$, the same proof can be repeated by choosing $\lambda=\frac{3}{2}$. Namely, by taking the decomposition of the functions into spherical harmonics and by introducing the new variable

$$\tilde{v}_{\ell,m}(\rho)=\rho^2 u_{\ell,m}(\rho),$$

the problem is reduced to

$$\left(-(1-\rho^2)\partial_\rho^2+\left(-\frac{2}{\rho}+5\rho\right)\partial_\rho+\frac{(\ell+3)(\ell+2)}{\rho^2}+\frac{15}{4}\right)\tilde{v}_{\ell,m}(\rho)=\rho^2 g_{\ell,m}(\rho),$$

which is the same as (3-13) up to a shift in ℓ and the weight on the right-hand side. Hence the same reasoning applies. □

Proof of Proposition 3.1. Based on Lemmas 3.4 and 3.5, the Lumer–Phillips theorem (see [Engel and Nagel 2000, p. 83, Theorem 3.15]) together with Lemma 3.3 implies that $\widetilde{\boldsymbol{L}}$ is closable in $\mathcal{H}_k$, and that its closure $\boldsymbol{L}_k$ generates a semigroup $(\boldsymbol{S}_k(\tau))_{\tau\geq 0}$ for which (3-1) holds. The rest of the proposition follows from standard semigroup theory results; see, e.g., [Engel and Nagel 2000, p. 55, Theorem 1.10]. □

We conclude this section by proving certain restriction properties of the semigroups $(\boldsymbol{S}_k(\tau))_{\tau\geq 0}$. This will be crucial in showing persistence of regularity for the nonlinear equation.

Lemma 3.6. *Let $d\in\{7,9\}$ and $k\geq 3$. For any $j\in\mathbb{N}$, the semigroup $(\boldsymbol{S}_{k+j}(\tau))_{\tau\geq 0}$ is the restriction of $(\boldsymbol{S}_k(\tau))_{\tau\geq 0}$ to $\mathcal{H}_{k+j}$, i.e.,*

$$\boldsymbol{S}_{k+j}(\tau)=\boldsymbol{S}_k(\tau)|_{\mathcal{H}_{k+j}}$$

for all $\tau\geq 0$. In particular, we have the growth bound

$$\|\boldsymbol{S}_k(\tau)\boldsymbol{u}\|_{H^{k+j}(\mathbb{B}^d)\times H^{k+j-1}(\mathbb{B}^d)}\lesssim_j e^{-\frac{1}{2}\tau}\|\boldsymbol{u}\|_{H^{k+j}(\mathbb{B}^d)\times H^{k+j-1}(\mathbb{B}^d)}$$

for all $\boldsymbol{u}\in\mathcal{H}_{k+j}$ and all $\tau\geq 0$.

Proof. Let $d\in\{7,9\}$ and $k\geq 3$. We prove the claim only for $j=1$, as the general case follows from the arbitrariness of k. The crucial ingredients of the proof are continuity of the embedding $\mathcal{H}_{k+1}\hookrightarrow\mathcal{H}_k$ and the fact that $\mathcal{D}(\widetilde{\boldsymbol{L}})$ is a core for both $\boldsymbol{L}_k$ and $\boldsymbol{L}_{k+1}$. First, we prove that $\boldsymbol{L}_{k+1}$ is a restriction of $\boldsymbol{L}_k$; more precisely we show

$$\mathcal{D}(\boldsymbol{L}_{k+1})\subset\mathcal{D}(\boldsymbol{L}_k)\quad\text{and}\quad \boldsymbol{L}_{k+1}\boldsymbol{u}=\boldsymbol{L}_k\boldsymbol{u} \tag{3-17}$$

for all $\boldsymbol{u}\in\mathcal{D}(\boldsymbol{L}_{k+1})$. For $\boldsymbol{u}\in\mathcal{D}(\widetilde{\boldsymbol{L}})$, from the definition of $\boldsymbol{L}_{k+1}$ and $\boldsymbol{L}_k$ it follows that

$$\boldsymbol{u}\in\mathcal{D}(\boldsymbol{L}_{k+1})\cap\mathcal{D}(\boldsymbol{L}_k)\quad\text{and}\quad \boldsymbol{L}_{k+1}\boldsymbol{u}=\boldsymbol{L}_k\boldsymbol{u}=\widetilde{\boldsymbol{L}}\boldsymbol{u}.$$

Let now $\boldsymbol{u}\in\mathcal{D}(\boldsymbol{L}_{k+1})$. Since $(\boldsymbol{L}_{k+1},\mathcal{D}(\boldsymbol{L}_{k+1}))$ is closed, there exists a sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}\subset\mathcal{D}(\widetilde{\boldsymbol{L}})$ such that

$$\boldsymbol{u}_n\xrightarrow{\mathcal{H}_{k+1}}\boldsymbol{u}\quad\text{and}\quad \widetilde{\boldsymbol{L}}\boldsymbol{u}_n\xrightarrow{\mathcal{H}_{k+1}}\boldsymbol{L}_{k+1}\boldsymbol{u}.$$

From the embedding $\mathcal{H}_{k+1}\hookrightarrow\mathcal{H}_k$ we infer

$$\boldsymbol{u}_n\xrightarrow{\mathcal{H}_k}\boldsymbol{u}\quad\text{and}\quad \widetilde{\boldsymbol{L}}\boldsymbol{u}_n\xrightarrow{\mathcal{H}_k}\boldsymbol{L}_{k+1}\boldsymbol{u},$$

and by the closedness of $\boldsymbol{L}_k$ it follows that $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_k)$ and $\boldsymbol{L}_{k+1}\boldsymbol{u} = \boldsymbol{L}_k\boldsymbol{u}$. Now let $\lambda \in \rho(\boldsymbol{L}_{k+1}) \cap \rho(\boldsymbol{L}_k)$. From (3-17) we get that $\boldsymbol{R}_{\boldsymbol{L}_{k+1}}(\lambda) = \boldsymbol{R}_{\boldsymbol{L}_k}(\lambda)|_{\mathcal{H}_{k+1}}$. Now, given $\boldsymbol{u} \in \mathcal{H}_{k+1}$, we get by the Post–Widder inversion formula (see [Engel and Nagel 2000, p. 223, Corollary 5.5]) and the embedding $\mathcal{H}_{k+1} \hookrightarrow \mathcal{H}_k$ that, for every $\tau > 0$,

$$\boldsymbol{S}_{k+1}(\tau)\boldsymbol{u} = \lim_{n\to\infty}\left[\frac{n}{\tau}\boldsymbol{R}_{\boldsymbol{L}_{k+1}}\left(\frac{n}{\tau}\right)\right]^n \boldsymbol{u} = \lim_{n\to\infty}\left[\frac{n}{\tau}\boldsymbol{R}_{\boldsymbol{L}_{k}}\left(\frac{n}{\tau}\right)\right]^n \boldsymbol{u} = \boldsymbol{S}_k(\tau)\boldsymbol{u}.$$

This proves that $(\boldsymbol{S}_{k+1}(\tau))_{\tau\geq 0}$ is the restriction of $(\boldsymbol{S}_k(\tau))_{\tau\geq 0}$ to $\mathcal{H}_{k+1}$. As a result, from Proposition 3.1 we have

$$\|\boldsymbol{S}_k(\tau)\boldsymbol{u}\|_{H^{k+1}(\mathbb{B}^d)\times H^k(\mathbb{B}^d)} = \|\boldsymbol{S}_{k+1}(\tau)\boldsymbol{u}\|_{H^{k+1}(\mathbb{B}^d)\times H^k(\mathbb{B}^d)} \lesssim e^{-\frac{1}{2}\tau}\|\boldsymbol{u}\|_{H^{k+1}(\mathbb{B}^d)\times H^k(\mathbb{B}^d)}$$

for all $\boldsymbol{u} \in \mathcal{H}_{k+1}$ and all $\tau \geq 0$. □

4. Linearization around a self-similar solution: preliminaries on the structure of the spectrum

From now on, for fixed $d \in \{7, 9\}$, we work solely in the Sobolev space $H^{(d+1)/2}(\mathbb{B}^d) \times H^{(d-1)/2}(\mathbb{B}^d)$, which we earlier denoted by $\mathcal{H}_{(d+1)/2}$. To abbreviate the notation, we write

$$\mathcal{H} := \mathcal{H}_{(d+1)/2}.$$

We also denote by $(\boldsymbol{S}(\tau))_{\tau\geq 0}$ and $\boldsymbol{L} : \mathcal{D}(\boldsymbol{L}) \subset \mathcal{H} \to \mathcal{H}$ the corresponding semigroup $(\boldsymbol{S}_k(\tau))_{\tau\geq 0}$ and its generator $\boldsymbol{L}_k$, respectively, for $k = \frac{1}{2}(d+1)$.

With an eye towards studying the flow near the orbit $\{\boldsymbol{U}_a : a \in \mathbb{R}^d\}$ — see the section on page 628 — in this section we describe some general properties of the underlying linear operator

$$\widetilde{\boldsymbol{L}} + \boldsymbol{L}'_a, \quad \boldsymbol{L}'_a\boldsymbol{u} := \begin{pmatrix} 0 \\ V_a u_1 \end{pmatrix},$$

where

$$V_a(\xi) := 2U_a(\xi), \tag{4-1}$$

with U_a given in (1-12).

Remark 4.1. We emphasize that the results of this section apply to any smooth $V_a : \overline{\mathbb{B}}^d \to \mathbb{R}$ that depends smoothly on the parameter a. Obviously, such potentials arise in the linearization around smooth self-similar profiles.

Proposition 4.2. *Fix $d \in \{7, 9\}$. For every $a \in \mathbb{R}^d$, the operator $\boldsymbol{L}'_a : \mathcal{H} \to \mathcal{H}$ is compact, and the operator*

$$\boldsymbol{L}_a := \boldsymbol{L} + \boldsymbol{L}'_a, \quad \mathcal{D}(\boldsymbol{L}_a) := \mathcal{D}(\boldsymbol{L}) \subset \mathcal{H} \to \mathcal{H},$$

generates a strongly continuous semigroup $\boldsymbol{S}_a : [0, \infty) \to \mathcal{B}(\mathcal{H})$. Furthermore, given $\delta > 0$, there is $K > 0$ such that

$$\|\boldsymbol{L}_a - \boldsymbol{L}_b\| \leq K|a - b|$$

for all $a, b \in \overline{\mathbb{B}}^d_\delta$.

Proof. The compactness of $\boldsymbol{L}'_a$ follows from the smoothness of V_a and the compactness of the embedding $H^{(d+1)/2}(\mathbb{B}^d) \hookrightarrow H^{(d-1)/2}(\mathbb{B}^d)$. The fact that $\boldsymbol{L}_a$ generates a semigroup is a consequence of the bounded perturbation theorem; see, e.g., [Engel and Nagel 2000, p. 158]. For the Lipschitz dependence on the parameter a, we first note that by the fundamental theorem of calculus we have

$$V_a(\xi) - V_b(\xi) = (a^j - b^j) \int_0^1 \partial_{\alpha_j} V_{\alpha(s)}(\xi)\, ds \tag{4-2}$$

for $\alpha(s) = b + s(a-b)$. This implies that, given $\delta > 0$, we have

$$\|V_a - V_b\|_{\dot{H}^k(\mathbb{B}^d)} \lesssim_k |a-b| \tag{4-3}$$

for all $a, b \in \overline{\mathbb{B}}^d_\delta$. In particular,

$$\|V_a - V_b\|_{W^{(d-1)/2,\infty}(\mathbb{B}^d)} \lesssim |a-b|,$$

and we thus have

$$\|(V_a - V_b)u\|_{H^{(d-1)/2}(\mathbb{B}^d)} \lesssim |a-b| \|u\|_{H^{(d-1)/2}(\mathbb{B}^d)} \lesssim |a-b| \|u\|_{H^{(d+1)/2}(\mathbb{B}^d)}$$

for all $u \in C^\infty(\overline{\mathbb{B}}^d)$ and all $a, b \in \overline{\mathbb{B}}^d_\delta$, which implies the claim. □

Next, we show that the unstable spectrum of $\boldsymbol{L}_a : \mathcal{D}(\boldsymbol{L}_a) \subset \mathcal{H} \to \mathcal{H}$ consists of isolated eigenvalues and is confined to a compact region. This is achieved by proving bounds on the resolvent and using compactness of the perturbation.

Proposition 4.3. *Fix $d \in \{7, 9\}$. Let $\varepsilon > 0$ and $\delta > 0$. Then there are constants $\kappa > 0$ and $c > 0$ such that*

$$\|\boldsymbol{R}_{\boldsymbol{L}_a}(\lambda)\| \le c \tag{4-4}$$

for all $a \in \overline{\mathbb{B}}^d_\delta$ and for all $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re}\lambda \ge -\frac{1}{2} + \varepsilon$ and $|\lambda| \ge \kappa$. Furthermore, if $\lambda \in \sigma(\boldsymbol{L}_a)$ with $\operatorname{Re}\lambda > -\frac{1}{2}$, then λ is an isolated eigenvalue.

Proof. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\frac{1}{2}$. Then Proposition 3.1 implies that $\lambda \in \rho(\boldsymbol{L})$, and we therefore have the identity

$$\lambda - \boldsymbol{L}_a = [1 - \boldsymbol{L}'_a \boldsymbol{R}_{\boldsymbol{L}}(\lambda)](\lambda - \boldsymbol{L}). \tag{4-5}$$

In what follows we prove that, for suitably chosen λ, the Neumann series $\sum_{k=0}^\infty [\boldsymbol{L}'_a \boldsymbol{R}_{\boldsymbol{L}}(\lambda)]^k$ converges. According to (4-5), this yields

$$\boldsymbol{R}_{\boldsymbol{L}_a}(\lambda) = \boldsymbol{R}_{\boldsymbol{L}}(\lambda) \sum_{k=0}^\infty [\boldsymbol{L}'_a \boldsymbol{R}_{\boldsymbol{L}}(\lambda)]^k,$$

and then (4-4) follows from Proposition 3.1. First, observe that, given $\delta > 0$, we have

$$\|\boldsymbol{L}'_a \boldsymbol{R}_{\boldsymbol{L}}(\lambda) \boldsymbol{f}\| = \|V_a [\boldsymbol{R}_{\boldsymbol{L}}(\lambda)\boldsymbol{f}]_1\|_{H^{(d-1)/2}(\mathbb{B})^d} \lesssim \|[\boldsymbol{R}_{\boldsymbol{L}}(\lambda)\boldsymbol{f}]_1\|_{H^{(d-1)/2}(\mathbb{B}^d)} \tag{4-6}$$

for all $a \in \overline{\mathbb{B}}^d_\delta$ and all $\boldsymbol{f} \in \mathcal{H}$. Now, given $\boldsymbol{f} \in \mathcal{H}$, let $\boldsymbol{u} = \boldsymbol{R}_{\boldsymbol{L}}(\lambda)\boldsymbol{f}$. Since $(\lambda - \boldsymbol{L})\boldsymbol{u} = \boldsymbol{f}$, from the first component of this equation, we get

$$\xi^j \partial_j u_1(\xi) + (\lambda + 2) u_1(\xi) - u_2(\xi) = f_1(\xi)$$

in the weak sense on the ball $\mathbb{B}^d$. Consequently,

$$\|u_1\|_{H^{(d-1)/2}(\mathbb{B}^d)} \lesssim \frac{1}{|\lambda+2|}(\|u_1\|_{H^{(d+1)/2}(\mathbb{B}^d)} + \|u_2\|_{H^{(d-1)/2}(\mathbb{B}^d)} + \|f_1\|_{H^{(d-1)/2}(\mathbb{B}^d)}).$$

Then Proposition 3.1 implies that, given $\varepsilon > 0$,

$$\|[\boldsymbol{R_L}(\lambda)\boldsymbol{f}]_1\|_{H^{(d-1)/2}(\mathbb{B}^d)} \lesssim |\lambda|^{-1}(\|\boldsymbol{R_L}(\lambda)\boldsymbol{f}\| + \|\boldsymbol{f}\|) \lesssim |\lambda|^{-1}\|\boldsymbol{f}\|$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\frac{1}{2} + \varepsilon$ and all $\boldsymbol{f} \in \mathcal{H}$. Together with (4-6), this gives

$$\|\boldsymbol{L}'_a \boldsymbol{R_L}(\lambda)\boldsymbol{f}\| \lesssim |\lambda|^{-1}\|\boldsymbol{f}\|,$$

and the uniform bound (4-4) holds for some $c > 0$ when we restrict to $|\lambda| \geq \kappa$ for suitably large κ. The second statement follows from the compactness of $\boldsymbol{L}'_a$. Indeed, if $\operatorname{Re}\lambda > -\frac{1}{2}$ then $\lambda \in \rho(\boldsymbol{L})$, and according to (4-5) we have that $\lambda \in \sigma(\boldsymbol{L}_a)$ only if $1 - \boldsymbol{L}'_a\boldsymbol{R_L}(\lambda)$ is not a bounded invertible operator, which is equivalent to 1 being an eigenvalue of the compact operator $\boldsymbol{L}'_a\boldsymbol{R_L}(\lambda)$, which according to (4-5) implies that λ is an eigenvalue of $\boldsymbol{L}_a$. The fact that λ is isolated follows from the analytic Fredholm theorem (see [Simon 2015, Theorem 3.14.3, p. 194]) applied to the mapping $\lambda \mapsto \boldsymbol{L}'_a\boldsymbol{R_L}(\lambda)$ defined on $\mathbb{H}_{-1/2} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\frac{1}{2}\}$. ☐

Remark 4.4. The previous proposition implies that there are finitely many unstable spectral points of $\boldsymbol{L}_a$, i.e., the ones belonging to $\overline{\mathbb{H}} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$, all of which are eigenvalues. This can actually be abstractly shown just by using the compactness of $\boldsymbol{L}'_a$; see [Glogić 2022, Theorem B.1]. We nonetheless need Proposition 4.3 as it allows us later on to reduce the spectral analysis of $\boldsymbol{L}_a$ for all small a to the case $a = 0$; see Section 5C.

Note that the eventual presence of unstable spectral points of $\boldsymbol{L}_a$ prevents decay of the associated semigroup $(\boldsymbol{S}_a(\tau))_{\tau\geq 0}$ on the whole space $\mathcal{H}$. What is more, since $\boldsymbol{L}'_a$ is compact, a spectral mapping theorem for the unstable spectrum holds (see [Glogić 2022, Theorem B.1]), and hence eventual growing modes of $(\boldsymbol{S}_a(\tau))_{\tau\geq 0}$ are completely determined by the unstable spectrum of $\boldsymbol{L}_a$ and the associated eigenspaces. Therefore, in what follows we turn to spectral analysis of $\boldsymbol{L}_a$. First, we show an important result which relates solvability of the spectral equation $(\lambda - \boldsymbol{L}_a)\boldsymbol{u} = 0$ for $a = 0$, $\lambda \in \overline{\mathbb{H}}$, to the existence of *smooth* solutions to a certain ordinary differential equation. We note that, for $a = 0$, the potential V_a is radial; more precisely,

$$V_0(\xi) = 2U_0(\xi) = 2U(|\xi|) =: V(|\xi|),$$

with U given in (1-5).

Proposition 4.5. *Fix $d \in \{7, 9\}$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. Then $\lambda \in \sigma(\boldsymbol{L}_0)$ if and only if there are $\ell \in \mathbb{N}_0$ and $f \in C^\infty[0,1]$ such that*

$$\mathcal{T}_\ell^{(d)}(\lambda)f(\rho) := (1-\rho^2)f''(\rho) + \left(\frac{d-1}{\rho} - 2(\lambda+3)\rho\right)f'(\rho) - \left((\lambda+2)(\lambda+3) + \frac{\ell(\ell+d-2)}{\rho^2} - V(\rho)\right)f(\rho) = 0 \quad (4\text{-}7)$$

for all $\rho \in (0,1)$.

Proof. Let $\lambda \in \overline{\mathbb{H}} \cap \sigma(\boldsymbol{L}_0)$. By Proposition 4.3, λ is an eigenvalue, and hence there is a nontrivial $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$ satisfying $(\lambda - \boldsymbol{L}_0)\boldsymbol{u} = 0$. By a straightforward calculation, we get that the components u_1 and u_2 satisfy the equations

$$-(\delta^{ij} - \xi^i \xi^j)\partial_i \partial_j u_1(\xi) + 2(\lambda+3)\xi^j \partial_j u_1(\xi) + (\lambda+3)(\lambda+2)u_1(\xi) - V_0(\xi)u_1(\xi) = 0 \tag{4-8}$$

and

$$u_2(\xi) = \xi^j \partial_j u_1(\xi) + (\lambda+2)u_1(\xi) \tag{4-9}$$

weakly on $\mathbb{B}^d$. Since $u_1 \in H^{(d+1)/2}(\mathbb{B}^d)$, we get by elliptic regularity that $u_1 \in C^\infty(\mathbb{B}^d)$. Furthermore, we may take the decomposition of u_1 into spherical harmonics:

$$u_1(\xi) = \sum_{\ell=0}^{\infty} \sum_{m \in \Omega_\ell} (u_1(|\xi|\,\cdot) \,|\, Y_{\ell,m})_{L^2(\mathbb{S}^{d-1})} Y_{\ell,m}\left(\frac{\xi}{|\xi|}\right) = \sum_{\ell=0}^{\infty} \sum_{m \in \Omega_\ell} u_{\ell,m}(\rho) Y_{\ell,m}(\omega), \tag{4-10}$$

where $\rho = |\xi|$ and $\omega = \xi/|\xi|$. To be precise, the expansion above holds in $H^k(\mathbb{B}^d_{1-\epsilon})$ for any $k \in \mathbb{N}$ and $\epsilon > 0$; see (1-22) and (1-23). Since the potential V_0 is radially symmetric, (4-8) decouples by means of (4-10) into a system of infinitely many ODEs:

$$\mathcal{T}_\ell^{(d)}(\lambda) u_{\ell,m}(\rho) = 0, \tag{4-11}$$

posed on the interval $(0,1)$, where the operator $\mathcal{T}_\ell^{(d)}(\lambda)$ is given by (4-7). Since u_1 is nontrivial, there are indices $\ell \in \mathbb{N}_0$ and $m \in \Omega_\ell$ such that $u_{\ell,m}$ is nonzero and satisfies (4-11). Furthermore, since $u_1 \in C^\infty(\mathbb{B}^d) \cap H^{(d+1)/2}(\mathbb{B}^d)$, we have that $u_{\ell,m} \in C^\infty[0,1) \cap H^{(d+1)/2}\left(\frac{1}{2},1\right)$. Now we prove that $u_{\ell,m}$ is smooth up to $\rho = 1$. Note that $\rho = 1$ is a regular singular point of (4-11), and the corresponding set of Frobenius indices is $\{0, 2-\lambda\}$ when $d = 9$, and $\{0, 1-\lambda\}$ when $d = 7$. In the first case, if $\lambda \notin \{0,1,2\}$, then $u_{\ell,m}$ is either analytic or behaves like $(1-\rho)^{2-\lambda}$ near $\rho = 1$. If $\lambda \in \{0,1,2\}$, then the nonanalytic behavior can be described by $(1-\rho)^2 \log(1-\rho)$, $(1-\rho)\log(1-\rho)$, or $\log(1-\rho)$. In each case, singularity can be excluded by the requirement that $u_{\ell,m} \in H^5\left(\frac{1}{2},1\right)$. This implies that $u_{\ell,m}$ belongs to $C^\infty[0,1]$ and solves (4-7) on $(0,1)$. The same reasoning applies to the case $d = 7$. Implication in the other direction is now obvious. ☐

Remark 4.6. Note that Frobenius theory implies that smooth solutions f from Proposition 4.5 are in fact analytic on $[0,1]$, in the sense that they can be extended to an analytic function on an open interval that contains $[0,1]$. Consequently, determining the unstable spectrum of $\boldsymbol{L}_0$ amounts to solving the connection problem for a family of ODEs. We note that the connection problem is so far completely resolved only for hypergeometric equations, i.e., the ones with three regular singular points, while the ODE (4-7) has six of them. In fact, their number can, by a suitable change of variables, be reduced to four, but this nonetheless renders the standard ODE theory useless. Nevertheless, by building on the techniques developed recently to treat such problems (see [Costin et al. 2016; 2017; Glogić 2018; Glogić and Schörkhuber 2021]), for $d = 9$, we are able to solve the connection problem for (4-7) and we thereby provide in the following section a complete characterization of the unstable spectrum of $\boldsymbol{L}_0$.

5. Spectral analysis for perturbations around U_a: the case $d = 9$

From now on we restrict ourselves to $d = 9$.

5A. ***Analysis of the spectral ODE.*** In this section we investigate the ODE (4-7) for $d = 9$, and for convenience we shorten the notation by letting $\mathcal{T}_\ell(\lambda) := \mathcal{T}_\ell^{(9)}(\lambda)$, i.e., we have

$$\mathcal{T}_\ell(\lambda) f(\rho) := (1-\rho^2) f''(\rho) + \left(\frac{8}{\rho} - 2(\lambda+3)\rho\right) f'(\rho) - \left((\lambda+2)(\lambda+3) + \frac{\ell(\ell+7)}{\rho^2} - V(\rho)\right) f(\rho),$$

where the potential is given by

$$V(\rho) = \frac{480(7-\rho^2)}{(7+5\rho^2)^2}.$$

Now, in view of Proposition 4.5, given $\ell \in \mathbb{N}_0$, we define the set

$$\Sigma_\ell := \{\lambda \in \overline{\mathbb{H}} : \text{there exists } f_\ell(\,\cdot\,;\lambda) \in C^\infty[0,1] \text{ satisfying } \mathcal{T}_\ell(\lambda) f_\ell(\,\cdot\,;\lambda) = 0 \text{ on } (0,1)\}.$$

The central result of our spectral analysis is the following proposition.

Proposition 5.1. *The structure of* Σ_ℓ *is as follows*:

(1) *For* $\ell = 0$, *we have* $\Sigma_0 = \{1, 3\}$, *with corresponding solutions*

$$f_0(\rho; 1) = \frac{1-\rho^2}{(7+5\rho^2)^3} \quad \textit{and} \quad f_0(\rho; 3) = \frac{1}{(7+5\rho^2)^3},$$

which are unique up to a constant multiple.

(2) *For* $\ell = 1$, *we have* $\Sigma_1 = \{0, 1\}$, *and the corresponding solutions are*

$$f_1(\rho; 0) = \frac{\rho(7-3\rho^2)}{(7+5\rho^2)^3} \quad \textit{and} \quad f_1(\rho; 1) = \frac{\rho(77-5\rho^2)}{(7+5\rho^2)^3}.$$

(3) *For all* $\ell \geq 2$, *we have* $\Sigma_\ell = \varnothing$.

To prove this proposition, we use an adaptation of the ODE techniques devised in [Costin et al. 2016; 2017; Glogić 2018; Glogić and Schörkhuber 2021]. We will therefore occasionally refer to these works throughout the proof. Also, we found it convenient to split the proof into two cases: $\ell \in \{0, 1\}$ and $\ell \geq 2$.

Proof of Proposition 5.1 for $\ell \in \{0, 1\}$. For a detailed heuristic discussion of our approach we refer the reader to [Glogić and Schörkhuber 2021, Section 4.1]. Namely, the first step is to transform $\mathcal{T}_\ell(\lambda) f(\rho) = 0$ to an "isospectral" equation with four regular singular points. For this, we let $x = \rho^2$, and we define the new dependent variable y via

$$f(\rho) = \rho^\ell \big(\tfrac{7}{5} + \rho^2\big)^{-3} y(\rho^2).$$

This yields the following equation in its canonical Heun form (see [DLMF 2010]):

$$y''(x) + \left(\frac{\gamma(\ell)}{x} + \frac{\delta(\lambda)}{x-1} - \frac{6}{x-\mu}\right) y'(x) + \frac{\alpha(\ell,\lambda)\beta(\ell,\lambda)x - q(\ell,\lambda)}{x(x-1)(x-\mu)} y(x) = 0, \tag{5-1}$$

with singularities at $x \in \{0, 1, \mu, \infty\}$, where $\mu = -\frac{7}{5}$, $\gamma(\ell) = \frac{1}{2}(9 + 2\ell)$, $\delta(\lambda) = \lambda - 1$,

$$\alpha(\ell, \lambda) = \tfrac{1}{2}(\lambda - 3 + \ell), \quad \beta(\ell, \lambda) = \tfrac{1}{2}(\lambda - 4 + \ell),$$
$$q(\ell, \lambda) = -\tfrac{1}{20}(7(\lambda - 3)(\lambda + 8) + 7\ell^2 + (14\lambda + 95)\ell).$$

By Frobenius' theory, any $y \in C^\infty[0, 1]$ that solves (5-1) on $(0, 1)$ is in fact analytic on the closed interval $[0, 1]$. Furthermore, the Frobenius indices of (5-1) at $x = 0$ are $s_1 = 0$ and $s_2 = -\frac{1}{2}(7 + 2\ell)$. Therefore, for every $\lambda \in \mathbb{C}$ there is a unique solution (up to a constant multiple) to (5-1), which is analytic at $x = 0$. Furthermore, this solution has a power series expansion of the form

$$y_{\ell,\lambda}(x) = \sum_{n=0}^{\infty} a_n(\ell, \lambda) x^n, \quad a_0(\ell, \lambda) = 1. \tag{5-2}$$

To determine the coefficients a_n, we insert the ansatz (5-2) into (5-1) and obtain the recurrence relation

$$a_{n+2}(\ell, \lambda) = A_n(\ell, \lambda) a_{n+1}(\ell, \lambda) + B_n(\ell, \lambda) a_n(\ell, \lambda), \tag{5-3}$$

where

$$A_n(\ell, \lambda) = \frac{7\lambda(\lambda + 9) + 7\ell^2 + \ell(8n + 14\lambda + 103) + 8n^2 + 4(7\lambda + 34)n - 40}{14(n + 2)(2\ell + 2n + 11)} \tag{5-4}$$

and

$$B_n(\ell, \lambda) = \frac{5(\lambda + \ell + 2n - 4)(\lambda + \ell + 2n - 3)}{14(n + 2)(2\ell + 2n + 11)}, \tag{5-5}$$

with the initial condition

$$a_{-1}(\ell, \lambda) = 0 \quad \text{and} \quad a_0(\ell, \lambda) = 1. \tag{5-6}$$

Now, note that $\lambda \in \Sigma_\ell$ precisely when the radius of convergence of the series (5-2) is larger than 1. To analyze this radius, we resort to results from the theory of difference equations with variable coefficients. Namely, since

$$\lim_{n\to\infty} A_n(\ell, \lambda) = \tfrac{2}{7} \quad \text{and} \quad \lim_{n\to\infty} B_n(\ell, \lambda) = \tfrac{5}{7},$$

the so-called characteristic equation of (5-3) is

$$t^2 - \tfrac{2}{7}t - \tfrac{5}{7} = 0,$$

and according to Poincaré's theorem (see, for example, [Elaydi 2005, p. 343], or [Glogić and Schörkhuber 2021, Appendix A]) we have that either $a_n(\ell, \lambda) = 0$ eventually in n or

$$\lim_{n\to\infty} \frac{a_{n+1}(\ell, \lambda)}{a_n(\ell, \lambda)} = 1 \quad \text{or} \quad \lim_{n\to\infty} \frac{a_{n+1}(\ell, \lambda)}{a_n(\ell, \lambda)} = -\tfrac{5}{7}.$$

To explore this further, we treat cases $\ell = 0$ and $\ell = 1$ separately.

The case $\ell = 0$. First, we observe that in this case there are explicit polynomial solutions for $\lambda = 1$ and $\lambda = 3$, given by

$$y_{0,1}(x) = 1 - x \quad \text{and} \quad y_{0,3}(x) = 1, \tag{5-7}$$

respectively. These in turn correspond to $f_0(\cdot\,;1)$ and $f_0(\cdot\,;3)$, stated in Proposition 5.1. So we have that $\{1,3\} \subset \Sigma_0$. We now show the reverse inclusion. Let $\lambda \in \overline{\mathbb{H}} \setminus \{1,3\}$. Since $\ell = 0$, from (5-4) and (5-5) we have

$$A_n(0,\lambda) = \frac{7\lambda(\lambda+9) + 8n^2 + 4(7\lambda+34)n - 40}{14(n+2)(2n+11)} \quad \text{and} \quad B_n(0,\lambda) = \frac{5(\lambda+2n-4)(\lambda+2n-3)}{14(n+2)(2n+11)}.$$

Now, note that the assumption that $a_n(0,\lambda) = 0$ eventually in n contradicts the initial condition (5-6), as follows by backward substitution. Consequently, we have that either

$$\lim_{n\to\infty} \frac{a_{n+1}(0,\lambda)}{a_n(0,\lambda)} = 1, \tag{5-8}$$

or

$$\lim_{n\to\infty} \frac{a_{n+1}(0,\lambda)}{a_n(0,\lambda)} = -\tfrac{5}{7}. \tag{5-9}$$

We prove that (5-8) holds, from which it follows that the radius of convergence of the series (5-2) (that is when $\ell = 0$) is 1, and therefore $\lambda \notin \Sigma_0$. To that end, we first compute

$$a_2(0,\lambda) = \tfrac{1}{5544}(\lambda-3)(\lambda-1)(7\lambda^2 + 126\lambda + 680)$$

and

$$a_3(0,\lambda) = \tfrac{1}{3027024}(\lambda-3)(\lambda-1)(49\lambda^4 + 1519\lambda^3 + 18494\lambda^2 + 84224\lambda + 46080).$$

Then we define

$$r_2(0,\lambda) := \frac{a_3(0,\lambda)}{a_2(0,\lambda)},$$

where the common factor $(\lambda-3)(\lambda-1)$ (which is an artifact of the existence of the polynomial solutions (5-7)) is canceled, and consequently, according to (5-3), for $n \geq 2$, we let

$$r_{n+1}(0,\lambda) = A_n(0,\lambda) + \frac{B_n(0,\lambda)}{r_n(0,\lambda)}. \tag{5-10}$$

To show (5-8), our strategy is the following. For (5-10) we construct an approximate solution $\tilde{r}_n$ (which we also call a *quasi-solution*) for which $\lim_{n\to\infty} \tilde{r}_n(0,\lambda) = 1$ and which is provably close enough to r_n so as to rule out (5-9). The quasi-solution we use is

$$\tilde{r}_n(0,\lambda) := \frac{\lambda^2}{2(2n+9)(n+1)} + \frac{\lambda(4n+9)}{2(2n+9)(n+1)} + \frac{2n+2}{2n+9}. \tag{5-11}$$

We have elaborated on constructing such expressions in [Glogić and Schörkhuber 2021, Section 4.2.2] and in [Costin et al. 2016, Section 4.1]; one can also check [Glogić 2018, Sections 2.6.3 and 2.7.2]. Concerning (5-11), suffice it to say here that we chose a quadratic polynomial in λ with rational coefficients in n so as to emulate the behavior of $r_n(0,\lambda)$ for both large and small values of the participating parameters. To show that the quasi-solution indeed resembles $r_n(0,\lambda)$, we define the relative difference function

$$\delta_n(0,\lambda) := \frac{r_n(0,\lambda)}{\tilde{r}_n(0,\lambda)} - 1 \tag{5-12}$$

and show that it is small uniformly in λ and n. To this end we substitute (5-12) into (5-10) and thereby derive the recurrence relation for δ_n:

$$\delta_{n+1}(0,\lambda) = \varepsilon_n(0,\lambda) - C_n(0,\lambda)\frac{\delta_n(0,\lambda)}{1+\delta_n(0,\lambda)}, \tag{5-13}$$

where

$$\varepsilon_n(0,\lambda) = \frac{A_n(0,\lambda)\tilde{r}_n(0,\lambda) + B_n(0,\lambda)}{\tilde{r}_n(0,\lambda)\tilde{r}_{n+1}(0,\lambda)} - 1 \quad \text{and} \quad C_n(0,\lambda) = \frac{B_n(0,\lambda)}{\tilde{r}_n(0,\lambda)\tilde{r}_{n+1}(0,\lambda)}. \tag{5-14}$$

We have the following result.

Lemma 5.2. *For all $n \geq 6$ and $\lambda \in \overline{\mathbb{H}}$, the following estimates hold*:

$$|\delta_6(0,\lambda)| \leq \frac{1}{5}, \quad |\varepsilon_n(0,\lambda)| \leq \frac{3}{140} + \frac{23}{40n}, \quad \textit{and} \quad |C_n(0,\lambda)| \leq \frac{5}{7} - \frac{23}{10n}. \tag{5-15}$$

Note that from (5-15) and (5-13), by a simple induction we infer that $|\delta_n(0,\lambda)| \leq \frac{1}{5}$ for all $n \geq 6$. This then via (5-12) and the fact that $\lim_{n\to\infty} \tilde{r}_n(0,\lambda) = 1$ excludes (5-9), and we are done. It therefore remains to prove the preceding lemma.

Proof. First we show that for $n \geq 6$ the functions $\delta_6(0,\cdot)$, $\varepsilon_n(0,\cdot)$, and $C_n(0,\cdot)$ are analytic in $\overline{\mathbb{H}}$. This, based on (5-12) and (5-14), follows from the fact that the zeros of $\tilde{r}_n(0,\cdot)$ and the poles of $r_6(0,\cdot)$ are all contained in the (open) left half-plane. This is immediate for $\tilde{r}_n(0,\cdot)$ as it is a quadratic polynomial with two negative zeros. As for the zeros of the denominator of $r_6(0,\lambda)$, which is a polynomial of degree 10, this, although it can be proven by elementary means, can be straightforwardly checked by the Routh–Hurwitz stability criterion; see [Glogić and Schörkhuber 2021, Section A.2]. Furthermore, being rational functions, $\delta_6(0,\cdot)$, $\varepsilon_n(0,\cdot)$, and $C_n(0,\cdot)$ are all polynomially bounded in $\overline{\mathbb{H}}$. Therefore, to prove the lemma, it is enough to establish the estimates (5-15) on the imaginary axis only as they can be then extended to all of $\overline{\mathbb{H}}$ by the Phragmen–Lindelöf principle (in its sectorial form); see, e.g., [Titchmarsh 1939, p. 177].

In the following we prove only the third estimate in (5-15), as the first two are shown similarly. We proceed with writing $C_{n+6}(0,\lambda)$ (note the shift in the index) as the ratio of two polynomials $P_1(n,\lambda)$ and $P_2(n,\lambda)$, both of which belong to $\mathbb{Z}[n,\lambda]$. Then, for $t \in \mathbb{R}$, we have the following representation on the imaginary line:

$$|P_j(n,it)|^2 = Q_j(n,t^2)$$

for $j \in \{1,2\}$, where $Q_1(n,t^2) \in \mathbb{Z}[n,t^2]$ and $Q_2(n,t^2) \in \mathbb{N}_0[n,t^2]$. Now the desired estimate is equivalent to

$$\frac{Q_1(n,t^2)}{Q_2(n,t^2)} \leq \left(\frac{5}{7} - \frac{23}{10(n+6)}\right)^2,$$

which is in turn equivalent to

$$(50n+139)^2 Q_2(n,t^2) - (70(n+6))^2 Q_1(n,t^2) \geq 0.$$

Finally, the last inequality trivially holds as the polynomial on the left (when expanded) has manifestly positive coefficients. □

The case $\ell = 1$. We proceed similarly to the previous case, and we therefore only provide the relevant expressions. For $\lambda = 0$ and $\lambda = 1$, we have explicit polynomial solutions

$$y_{1,0}(x) = 1 - \tfrac{3}{7}x \quad \text{and} \quad y_{1,1}(x) = 1 - \tfrac{5}{77}x,$$

respectively, which correspond to $f_1(\,\cdot\,; 0)$ and $f_1(\,\cdot\,; 1)$ from the statement of the proposition. Therefore $\{0, 1\} \subset \Sigma_1$, and we proceed by showing that there are no additional elements in Σ_1. Let $\lambda \in \overline{\mathbb{H}} \setminus \{0, 1\}$. For $\ell = 1$, the series (5-2) yields a solution to (5-1), which is analytic at $x = 0$. According to (5-3), we have

$$a_{n+2}(1, \lambda) = A_n(1, \lambda)a_{n+1}(1, \lambda) + B_n(1, \lambda)a_n(1, \lambda), \tag{5-16}$$

where

$$A_n(1, \lambda) = \frac{7(\lambda + 1)(\lambda + 10) + 8n^2 + 4(7\lambda + 36)n}{14(n + 2)(2n + 13)}$$

and

$$B_n(1, \lambda) = \frac{5(\lambda + 2n - 3)(\lambda + 2n - 2)}{14(n + 2)(2n + 13)}.$$

Since

$$a_2(1, \lambda) = \tfrac{1}{8008}\lambda(\lambda - 1)(7\lambda^2 + 133\lambda + 786)$$

and

$$a_3(1, \lambda) = \tfrac{1}{720720}\lambda(\lambda - 1)(7\lambda^4 + 238\lambda^3 + 3263\lambda^2 + 17828\lambda + 22476),$$

we define

$$r_2(1, \lambda) := \frac{a_3(1, \lambda)}{a_2(1, \lambda)},$$

where the common linear factors are canceled, and according to (5-16) we define r_n for $n \geq 2$ by the recurrence

$$r_{n+1}(1, \lambda) = A_n(1, \lambda) + \frac{B_n(1, \lambda)}{r_n(1, \lambda)}.$$

As a quasi-solution, we let

$$\tilde{r}_n(1, \lambda) := \frac{\lambda^2}{2(2n + 11)(n + 1)} + \frac{(4n + 11)\lambda}{2(2n + 11)(n + 1)} + \frac{n + 1}{n + 4},$$

and analogously to the previous case we define $\delta_n(1, \lambda)$, $\varepsilon_n(1, \lambda)$, and $C_n(1, \lambda)$. Also, by the same method as above, we establish the following result.

Lemma 5.3. *For $n = 5$, we have $|\delta_5(1, \lambda)| \leq \frac{1}{5}$. Furthermore, for every $n \geq 5$,*

$$|\varepsilon_n(1, \lambda)| \leq \frac{3}{140} + \frac{5}{8(n+1)} \quad \textit{and} \quad |C_n(1, \lambda)| \leq \frac{5}{7} - \frac{5}{2(n+1)} \tag{5-17}$$

uniformly for all $\lambda \in \overline{\mathbb{H}}$. Consequently, $|\delta_n(1, \lambda)| \leq \frac{1}{5}$ for all $n \geq 5$ and $\lambda \in \overline{\mathbb{H}}$. This implies $\lim_{n\to\infty} r_n(1, \lambda) = 1$.

Proof of Proposition 5.1 for $\ell \geq 2$. Since the parameter ℓ is now free, the analysis is more complicated. Namely, in addition to having to emulate the global behavior in ℓ as well, a quasi-solution also has to approximate the actual solution well enough so as to, with an additional parameter ℓ, obey the estimates analogous to (5-17). We note that a similar problem was treated by the second and the third authors in [Glogić and Schörkhuber 2021, Sections 4.2.1 and 4.2.2], and we closely follow their approach. First, we introduce the change of variable $x = 12\rho^2/(5\rho^2+7)$, by means of which the singular points $\rho = 0$ and $\rho = 1$ remain fixed, while the remaining finite singularity (which corresponds to $\rho = \infty$) is now further away from the unit disk at $x = \frac{12}{5}$. Furthermore, by applying also the transformation

$$f(\rho) = x^{\frac{\ell}{2}}\left(\tfrac{12}{5} - x\right)^{\frac{\lambda+3}{2}} \tilde{y}(x)$$

to $\mathcal{T}_\ell(\lambda) f(\rho) = 0$, we arrive at a Heun equation for $\tilde{y}$:

$$\tilde{y}''(x) + \left(\frac{\tilde{\gamma}(\ell)}{x} + \frac{\tilde{\delta}(\lambda)}{x-1} + \frac{\epsilon}{x-\tilde{\mu}}\right)\tilde{y}'(x) + \frac{\tilde{\alpha}(\ell,\lambda)\tilde{\beta}(\ell,\lambda)x - \tilde{q}(\ell,\lambda)}{x(x-1)(x-\mu)}\tilde{y}(x) = 0, \tag{5-18}$$

where $\tilde{\mu} = \frac{12}{5}$, $\tilde{\gamma}(\ell) = \frac{1}{2}(9+2\ell)$, $\tilde{\delta}(\lambda) = \lambda - 1$, $\epsilon = \frac{3}{2}$, $\tilde{\alpha}(\lambda) = \frac{1}{2}(\lambda - 3 + \ell)$, $\tilde{\beta}(\lambda) = \frac{1}{2}(\lambda + 11 + \ell)$, and

$$\tilde{q}(\ell,\lambda) = \tfrac{1}{20}(17\ell^2 + 2\ell(55+12\lambda) - 7\lambda^2 + 80\lambda - 303).$$

The Frobenius indices of (5-18) at $x = 0$ are $s_1 = 0$ and $s_2 = -\frac{1}{2}(7+2\ell)$. Therefore, we consider the (normalized) analytic solution at $x = 0$:

$$\tilde{y}(x) = \sum_{n=0}^{\infty} \tilde{a}_n(\ell,\lambda)x^n, \quad \tilde{a}_0(\ell,\lambda) = 1. \tag{5-19}$$

Inserting (5-19) into (5-18) yields

$$\tilde{a}_{n+2}(\ell,\lambda) = \tilde{A}_n(\ell,\lambda)\tilde{a}_{n+1}(\ell,\lambda) + \widetilde{B}_n(\ell,\lambda)\tilde{a}_n(\ell,\lambda), \tag{5-20}$$

with

$$\tilde{A}_n(\ell,\lambda) = \frac{68n^2 + (48\lambda + 68\ell + 356)n + 7\lambda^2 + 17\ell^2 + 24\lambda\ell + 128\lambda + 178\ell - 15}{24(n+2)(2n+2\ell+11)}$$

and

$$\widetilde{B}_n(\ell,\lambda) = \frac{-5(2n+\lambda+\ell+11)(2n+\lambda+\ell-3)}{24(n+2)(2n+2\ell+11)},$$

supplied with the initial condition $\tilde{a}_{-1}(\ell,\lambda) = 0$ and $\tilde{a}_0(\ell,\lambda) = 1$. Now, $\lim_{n\to\infty}\tilde{A}_n(\ell,\lambda) = \frac{17}{12}$ and $\lim_{n\to\infty}\widetilde{B}_n(\ell,\lambda) = -\frac{5}{12}$, and consequently the characteristic equation of (5-3) is $t^2 - \frac{17}{12}t + \frac{5}{12} = 0$, with solutions $t_1 = \frac{5}{12}$ and $t_2 = 1$. Hence, for

$$\hat{r}_n(\ell,\lambda) := \frac{\tilde{a}_{n+1}(\ell,\lambda)}{\tilde{a}_n(\ell,\lambda)},$$

either $\tilde{a}_n(\ell,\lambda) = 0$ eventually in n or

$$\lim_{n\to\infty}\hat{r}_n(\ell,\lambda) = 1 \tag{5-21}$$

or

$$\lim_{n\to\infty} \hat{r}_n(\ell,\lambda) = \tfrac{5}{12}. \tag{5-22}$$

Now, for $\lambda \in \overline{\mathbb{H}}$, similarly to the previous cases, we exclude the first option by backward substitution. Then, from (5-20), we derive the recurrence relation for $\hat{r}_n$

$$\hat{r}_{n+1}(\ell,\lambda) = \tilde{A}_n(\ell,\lambda) + \frac{\widetilde{B}_n(\ell,\lambda)}{\hat{r}_n(\ell,\lambda)}, \tag{5-23}$$

along with the initial condition $r_0(\ell,\lambda) = A_{-1}(\ell,\lambda)$. For a quasi-solution to (5-23) we use

$$R_n(\ell,\lambda) := \frac{7\lambda^2}{24(n+1)(2n+2\ell+9)} + \frac{\lambda(6n+3\ell+10)}{3(n+1)(2n+2\ell+9)} + \frac{17\ell}{48(n+1)} + \frac{n-1}{n+1}.$$

Again, for the exact way of constructing such quasi-solutions we refer the reader to [Glogić and Schörkhuber 2021, Section 4.2.2] or [Glogić 2018, Section 2.7.2]. Thereupon we set

$$\tilde{\delta}_n(\ell,\lambda) := \frac{\hat{r}_n(\ell,\lambda)}{R_n(\ell,\lambda)} - 1 \tag{5-24}$$

to obtain

$$\tilde{\delta}_{n+1}(\ell,\lambda) = \tilde{\varepsilon}_n(\ell,\lambda) - \widetilde{C}_n(\ell,\lambda)\frac{\tilde{\delta}_n(\ell,\lambda)}{1+\tilde{\delta}_n(\ell,\lambda)},$$

where

$$\tilde{\varepsilon}_n(\ell,\lambda) = \frac{\tilde{A}_n(\ell,\lambda)R_n(\ell,\lambda) + \widetilde{B}_n(\ell,\lambda)}{R_n(\ell,\lambda)R_{n+1}(\ell,\lambda)} - 1 \quad \text{and} \quad \widetilde{C}_n(\ell,\lambda) = \frac{\widetilde{B}_n(\ell,\lambda)}{R_n(\ell,\lambda)R_{n+1}(\ell,\lambda)}.$$

Now, similarly to the previous cases, we establish the following lemma.

Lemma 5.4. *For all $\ell \geq 2$, $n \geq 3$, and $\lambda \in \overline{\mathbb{H}}$, the following estimates hold*:

$$|\tilde{\delta}_3(\ell,\lambda)| \leq \tfrac{1}{3}, \quad |\tilde{\varepsilon}_n(\ell,\lambda)| \leq \tfrac{1}{8}, \quad \textit{and} \quad |\widetilde{C}_n(\ell,\lambda)| \leq \tfrac{5}{12}.$$

As a consequence, $|\tilde{\delta}_n(\ell,\lambda)| \leq \frac{1}{3}$ *for all $n \geq 3$.*

From this lemma, (5-24), and the fact that $\lim_{n\to\infty} R_n(\ell,\lambda) = 1$, we exclude (5-22) and we therefore have $\lim_{n\to\infty} \tilde{r}_n(\ell,\lambda) = 1$. Hence, given $\lambda \in \overline{\mathbb{H}}$, there are no solutions to (5-18) which are analytic on $[0,1]$, and consequently $\Sigma_\ell = \varnothing$.

5B. ***The spectrum of L_0.*** With the results from above at hand, we can provide a complete description of the unstable spectrum of $\boldsymbol{L}_0$.

Proposition 5.5. *There exists $\omega_0 \in \left(0, \frac{1}{2}\right]$ such that*

$$\sigma(\boldsymbol{L}_0) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\omega_0\} = \{\lambda_0, \lambda_1, \lambda_2\}, \tag{5-25}$$

where $\lambda_0 = 0$, $\lambda_1 = 1$, and $\lambda_2 = 3$ are eigenvalues. The geometric eigenspace of λ_2 is spanned by $\boldsymbol{h}_0 = (h_{0,1}, h_{0,2})$, where

$$h_{0,1}(\xi) = \frac{1}{(7+5|\xi|^2)^3} \quad \textit{and} \quad h_{0,2}(\xi) = \xi^i \partial_i h_{0,1}(\xi) + 5h_{0,1}(\xi). \tag{5-26}$$

Moreover, the geometric eigenspaces of λ_1 *and* λ_0 *are spanned by* $\{\boldsymbol{g}_0^{(k)}\}_{k=0}^9 = \{(g_{0,1}^{(k)}, g_{0,2}^{(k)})\}_{k=0}^9$ *and* $\{\boldsymbol{q}_0^{(j)}\}_{j=1}^9 = \{(q_{0,1}^{(j)}, q_{0,2}^{(j)})\}_{j=1}^9$, *respectively, where we have in closed form*

$$\begin{aligned} g_{0,1}^{(0)}(\xi) &= \frac{1-|\xi|^2}{(7+5|\xi|^2)^3}, & g_{0,2}^{(0)}(\xi) &= \xi^i \partial_i g_{0,1}^{(0)}(\xi) + 3 g_{0,1}^{(0)}(\xi), \\ g_{0,1}^{(j)}(\xi) &= \frac{\xi^j(77-5|\xi|^2)}{(7+5|\xi|^2)^3}, & g_{0,2}^{(j)}(\xi) &= \xi^i \partial_i g_{0,1}^{(j)}(\xi) + 3 g_{0,1}^{(j)}(\xi) \end{aligned} \tag{5-27}$$

for $j = 1, \dots, 9$ *as well as*

$$q_{0,1}^{(j)}(\xi) = \frac{\xi^j(7-3|\xi|^2)}{(7+5|\xi|^2)^3} \quad \textit{and} \quad q_{0,2}^{(j)}(\xi) = \xi^i \partial_i q_{0,1}^{(j)}(\xi) + 2 q_{0,1}^{(j)}(\xi). \tag{5-28}$$

Remark 5.6. Recall that $\boldsymbol{U}_a$ solves the stationary equation $\boldsymbol{L}\boldsymbol{U}_a + \boldsymbol{F}(\boldsymbol{U}_a) = 0$. By the chain rule we get, for any $k = 1, \dots, d$,

$$(\boldsymbol{L} + \boldsymbol{F}'(\boldsymbol{U}_a))\partial_{a^k} \boldsymbol{U}_a = \boldsymbol{L}_a \partial_{a^k} \boldsymbol{U}_a = 0.$$

This implies that $\partial_{a^k} \boldsymbol{U}_a$ is an eigenvector of $\boldsymbol{L}_a$ with eigenvalue $\lambda = 0$. In particular, a direct calculation shows that $q_{0,1}^{(j)}(\xi) = c\,\partial_{a^j} U_a(\xi)|_{a=0}$.

Proof. From Propositions 4.3, 4.5, and 5.1 we deduce the existence of $\omega_0 \in \left(0, \frac{1}{2}\right]$ for which (5-25) holds. To determine the eigenspaces, we do the following. First, in view of Proposition 5.1, if $\lambda = 3$ then $\ell = 0$, and setting $u_{0,1}(\rho) = (7+5\rho^2)^{-3}$ in the expansion (4-10) yields (5-26). If $\lambda = 1$, then either $\ell = 0$ and $u_{0,m} = f_0(\cdot\,; 1)$, or $\ell = 1$ and $u_{1,m} = f_1(\cdot\,; 1)$, for $m = 1, \dots, 9$. Since we can choose $Y_{1,m}(\omega) = \tilde{c}_m \omega_m$ for $m = 1, \dots, 9$, these yield (5-27). For $\lambda = 0$, we have $\ell = 1$ with $u_{1,m} = f_1(\cdot\,, 0)$, which similarly leads to (5-28). □

In what follows, we prove that for each unstable eigenvalue the geometric and the algebraic eigenspaces are the same. To this end, we define the associated Riesz projections. Namely, we set

$$\boldsymbol{H}_0 := \frac{1}{2\pi i}\int_{\gamma_2} \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\,d\lambda, \quad \boldsymbol{P}_0 := \frac{1}{2\pi i}\int_{\gamma_1} \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\,d\lambda, \quad \text{and} \quad \boldsymbol{Q}_0 := \frac{1}{2\pi i}\int_{\gamma_0} \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\,d\lambda,$$

where $\gamma_j(s) = \lambda_j + \frac{1}{2}\omega_0 e^{2i\pi s}$ for $s \in [0,1]$ and $j = 0, 1, 2$.

Lemma 5.7. *We have*

$$\dim \operatorname{ran} \boldsymbol{H}_0 = 1, \quad \dim \operatorname{ran} \boldsymbol{P}_0 = 10, \quad \textit{and} \quad \dim \operatorname{ran} \boldsymbol{Q}_0 = 9.$$

Proof. We start with the observation that the ranges of the projections are finite-dimensional. Indeed, λ_j would otherwise belong to the essential spectrum of $\boldsymbol{L}_0$ (see [Kato 1976, Theorems 5.28 and 5.33]) which coincides with the essential spectrum of $\boldsymbol{L}$ (since $\boldsymbol{L}_0$ is a compact perturbation of $\boldsymbol{L}$), but this is in contradiction with (3-2). Now we show that $\dim \operatorname{ran} \boldsymbol{P}_0 = 10$. We know from properties of the Riesz integral that $\ker(\boldsymbol{L}_0 - \lambda_1) \subset \operatorname{ran} \boldsymbol{P}_0$. We therefore only need to prove the reverse inclusion. First, note that the space $\operatorname{ran} \boldsymbol{P}_0$ reduces the operator $\boldsymbol{L}_0$, and we have

$$\sigma(\boldsymbol{L}_0|_{\operatorname{ran} \boldsymbol{P}_0}) = \{1\};$$

see, e.g., [Hislop and Sigal 1996, Proposition 6.9]. Consequently, since $\boldsymbol{P}_0$ is finite-rank, the operator $1-\boldsymbol{L}_0|_{\operatorname{ran}\boldsymbol{P}_0}$ is nilpotent; i.e., there is $m\in\mathbb{N}$ such that $(1-\boldsymbol{L}_0|_{\operatorname{ran}\boldsymbol{P}_0})^m=0$. Note that it suffices to show that $m=1$. We argue by contradiction, and hence assume that $m\geq 2$. Then there is $\boldsymbol{u}\in\mathcal{D}(\boldsymbol{L}_0)$ such that

$$(1-\boldsymbol{L}_0)\boldsymbol{u}=\boldsymbol{v}$$

for a nontrivial $\boldsymbol{v}\in\ker(1-\boldsymbol{L}_0)$. This yields for u_1 the elliptic equation

$$-(\delta^{ij}-\xi^i\xi^j)\partial_i\partial_j u_1(\xi)+2(\lambda+3)\xi^j\partial_j u_1(\xi)+(\lambda+3)(\lambda+2)u_1(\xi)-V_0(\xi)u_1(\xi)=F(\xi), \quad (5\text{-}29)$$

where $\lambda=1$ and

$$F(\xi)=\xi^i\partial_i v_1(\xi)+(\lambda+3)v_1(\xi)+v_2(\xi).$$

Since $\boldsymbol{v}\in\ker(1-\boldsymbol{L}_0)=\operatorname{span}(\boldsymbol{g}_0^{(0)},\dots,\boldsymbol{g}_0^{(9)})$, we have that $\boldsymbol{v}=\sum_{k=0}^{9}\alpha_k\boldsymbol{g}_0^{(k)}$ for some $\alpha_0,\dots,\alpha_9\in\mathbb{C}$, not all of which are zero. To avoid cumbersome notation we let $g_k=g_{0,1}^{(k)}$. In the new notation, based on (5-27), we have

$$F(\xi)=\sum_{k=0}^{9}\alpha_k(2\xi^i\partial_{\xi^i}g_k+7g_k).$$

Furthermore, according to Proposition 5.1 we can rewrite F in polar coordinates as

$$F(\rho\omega)=\alpha_0(2\rho f_0'(\rho)+7f_0(\rho))Y_{0,1}(\omega)+\sum_{i=1}^{9}\alpha_i(2\rho f_1'(\rho)+7f_1(\rho))Y_{1,i}(\omega),$$

where we write $f_0=f_0(\,\cdot\,;1)$ and $f_1=f_1(\,\cdot\,;1)$. By taking the decomposition of u_1 into spherical harmonics as in (4-10), (5-29) can be written as a system of ODEs:

$$\mathcal{T}_0(1)u_{0,1}=-\alpha_0G_0,\qquad \mathcal{T}_1(1)u_{1,j}=-\alpha_jG_1,\quad j=1,\dots,9, \quad (5\text{-}30)$$

posed on the interval $(0,1)$, where $G_i(\rho)=2\rho f_i'(\rho)+7f_i(\rho)$ for $i=0,1$. Moreover, from the properties of u_1, we infer that $u_{\ell,m}\in C^\infty[0,1)\cap H^5\big(\frac{1}{2},1\big)$, and by the Sobolev embedding we have $u_{\ell,m}\in C^2[0,1]$. To obtain a contradiction, we show that if some α_k is nonzero then the corresponding ODE in (5-30) does not admit a $C^2[0,1]$ solution. To start, we assume that $\alpha_0\neq 0$. For convenience, we can without loss of generality assume that $\alpha_0=-1$. Then $u_{0,1}$ solves the ODE

$$(1-\rho^2)u''(\rho)+\Big(\frac{8}{\rho}-8\rho\Big)u'(\rho)-(12-V(\rho))u(\rho)=G_0(\rho), \quad (5\text{-}31)$$

where

$$G_0(\rho)=\frac{5\rho^4-102\rho^2+49}{(7+5\rho^2)^4}.$$

Note that

$$u_1(\rho)=f_0(\rho)=\frac{1-\rho^2}{(7+5\rho^2)^3}$$

is a solution to the homogeneous version of (5-31), and by reduction of order we find a second solution:

$$u_2(\rho)=u_1(\rho)\int_{1/2}^{\rho}\frac{ds}{s^8u_1(s)^2}=\frac{1-\rho^2}{(7+5\rho^2)^3}\int_{1/2}^{\rho}\frac{(7+5s^2)^6}{s^8(1-s^2)^2}\,ds.$$

Furthermore, a simple calculation yields

$$u_2(\rho) \simeq \rho^{-7} \quad \text{as } \rho \to 0^+$$

and

$$u_2(\rho) = 864 - 3456(1-\rho)\ln(1-\rho) + O(1-\rho) \quad \text{as } \rho \to 1^-. \tag{5-32}$$

With the fundamental system $\{u_1, u_2\}$ at hand, we can solve (5-31) by the variation of parameters formula. Namely, we have

$$u(\rho) = c_1 u_1(\rho) + c_2 u_2(\rho) - u_1(\rho)\int_0^\rho \frac{u_2(s)G_0(s)s^8}{1-s^2}\,ds + u_2(\rho)\int_0^\rho \frac{u_1(s)G_0(s)s^8}{1-s^2}\,ds$$

for some constants $c_1, c_2 \in \mathbb{C}$. If $u \in C^2[0,1]$, then c_2 must be equal to zero in the above expression, owing to the singular behavior of $u_2(\rho)$ near $\rho = 0$. Then by differentiation we obtain, for $\rho \in (0,1)$,

$$u'(\rho) = c_1 u_1'(\rho) - u_1'(\rho)\int_0^\rho \frac{u_2(s)G_0(s)s^8}{1-s^2}\,ds + u_2'(\rho)\int_0^\rho \frac{u_1(s)G_0(s)s^8}{1-s^2}\,ds.$$

Now we inspect the asymptotic behavior of u' as $\rho \to 1^-$. We first note that u_1' is bounded near $\rho = 1$. Furthermore, note that

$$\int_0^1 \frac{u_1(s)G_0(s)s^8}{1-s^2}\,ds = \int_0^1 \frac{s^2}{1-s^2}\frac{d}{ds}\left[\frac{s^7(1-s^2)^2}{(7+5s^2)^6}\right]ds = -2\int_0^1 \frac{s^8(1-s^2)}{(7+5s^2)^6} =: -C$$

for some $C > 0$, which can be calculated explicitly, and $C < 4 \times 10^{-8}$. Hence, based on (5-32), we have

$$u_2'(\rho)\int_0^\rho \frac{u_1(s)G_0(s)s^8}{1-s^2}\,ds \sim -3456\,C\ln(1-\rho) \quad \text{as } \rho \to 1^-.$$

Moreover,

$$-u_1'(\rho)\int_0^\rho \frac{u_2(s)G_0(s)s^8}{1-s^2}\,ds \sim \frac{1}{864}\ln(1-\rho) \quad \text{as } \rho \to 1^-.$$

Finally, we infer that the two integral terms cannot cancel, and thus

$$u'(\rho) \simeq \ln(1-\rho) \quad \text{as } \rho \to 1^-.$$

In conclusion, there is no choice of c_1 and c_2 for which u belongs to $C^2[0,1]$.

We similarly treat α_j for $j \in \{1, \dots, 9\}$. It is enough to consider just α_1, and without loss of generality assume that $\alpha_1 = -1$. Then (5-30) yields the ODE

$$(1-\rho^2)u''(\rho) + \left(\frac{8}{\rho} - 8\rho\right)u'(\rho) - \left(12 + \frac{8}{\rho^2} - V(\rho)\right)u(\rho) = G_1(\rho), \tag{5-33}$$

where

$$G_1(\rho) = \frac{\rho(4851 - 1610\rho^2 - 25\rho^4)}{(7+5\rho^2)^4}.$$

Note that

$$u_1(\rho) = f_1(\rho) = \frac{\rho(77-5\rho^2)}{(7+5\rho^2)^3}$$

is a solution for the homogeneous problem. Similarly as above, we obtain another solution by the reduction formula

$$u_2(\rho) = u_1(\rho)\int_1^\rho \frac{ds}{s^8 u_1(s)^2} = \frac{\rho(77-5\rho)}{(7+5\rho^2)^3}\int_1^\rho \frac{(7+5s^2)^6}{s^{10}(77-5s)^2}\,ds,$$

and by inspection of the integral we get $u_2(\rho) \simeq \rho^{-8}$ near the origin and $u_2(\rho) \simeq 1-\rho$ near $\rho = 1$. Now, the general solution of (5-33) on (0, 1) is given by

$$u(\rho) = c_1 u_1(\rho) + c_2 u_2(\rho) - u_1(\rho)\int_0^\rho \frac{u_2(s)G_1(s)s^8}{1-s^2}\,ds + u_2(\rho)\int_0^\rho \frac{u_1(s)G_1(s)s^8}{1-s^2}\,ds. \tag{5-34}$$

Assumption that u belongs to $C^2[0,1]$ forces $c_2 = 0$ above, due to the singular behavior of u_2 at $\rho = 0$. Furthermore, from the last term in (5-34) we see that $u'(\rho) \simeq \ln(1-\rho)$ as $\rho \to 1^-$. In conclusion, (5-33) admits no $C^2[0,1]$ solutions, and this finishes the proof for $\boldsymbol{P}_0$.

The remaining two projections are treated similarly, so we omit some details. For $\boldsymbol{H}_0$ we obtain the analogue of (5-29) with

$$F(\xi) = 2\xi^i \partial_i h_{0,1}(\xi) + 11 h_{0,1}(\xi).$$

This leads to the ODE

$$(1-\rho^2)u''(\rho) + \Big(\frac{8}{\rho} - 12\rho\Big)u'(\rho) - (30 - V(\rho))u(\rho) = H(\rho) \tag{5-35}$$

for

$$H(\rho) = \frac{77-5\rho^2}{(7+5\rho^2)^4}.$$

The argument, similarly as above, reduces to showing that (5-35) does not admit $C^2[0,1]$ solutions. By Proposition 5.1, we have that $u_1(\rho) = (7+5\rho^2)^{-3}$ solves the homogeneous variant of (5-35), with the reduction formula yielding another solution

$$u_2(\rho) = u_1(\rho)\int_{1/2}^\rho \frac{ds}{s^8(1-s^2)^2 u_1(s)^2} = \frac{1}{(7+5\rho^2)^3}\int_{1/2}^\rho \frac{(7+5s^2)^6}{s^8(1-s^2)^2}\,ds. \tag{5-36}$$

Note that u_2 is singular at both $\rho = 0$ and $\rho = 1$; more precisely

$$u_2(\rho) \simeq \rho^{-7} \quad \text{as } \rho \to 0^+ \qquad \text{and} \qquad u_2(\rho) \simeq (1-\rho)^{-1} \quad \text{as } \rho \to 1^-.$$

With u_1 and u_2 at hand, the general solution of (5-35) on the interval (0, 1) can be written as

$$u(\rho) = c_1 u_1(\rho) + c_2 u_2(\rho) - u_1(\rho)\int_0^\rho (1-s^2)s^8 H(s)u_2(s)\,ds + u_2(\rho)\int_0^\rho (1-s^2)s^8 H(s)u_1(s)\,ds,$$

where the parameters $c_1, c_2 \in \mathbb{C}$ are free. The assumption that u is bounded near $\rho = 0$ forces c_2 to equal 0. Note that the first and the third term in (5-36) are bounded near $\rho = 1$. However, due to the singular behavior of u_2, the last term is unbounded near $\rho = 1$, owing to the integrand being strictly positive on (0, 1). In conclusion, the general solution u in (5-36) is unbounded on (0, 1).

Finally, for $\boldsymbol{Q}_0$, we have

$$F(\xi) = \sum_{j=1}^{9} \alpha_j (2\xi^i \partial_{\xi^i} q_{0,1}^j(\xi) + 5 q_{0,1}^j(\xi)),$$

and the accompanying analogue of (5-31) is

$$(1-\rho^2)u''(\rho)+\left(\frac{8}{\rho}-6\rho\right)u'(\rho)-\left(6+\frac{8}{\rho^2}-V(\rho)\right)=Q(\rho),$$

where

$$Q(\rho)=\frac{15\rho^5-406\rho^3+343\rho}{(7+5\rho^2)^4}.$$

A fundamental solution set to the homogeneous version of the above ODE is given by

$$u_1(\rho)=\frac{\rho(7-3\rho^2)}{(7+5\rho^2)^3}\quad\text{and}\quad u_2(\rho)=u_1(\rho)\int_1^\rho\frac{1-s^2}{s^8u_1(s)^2}\,ds,$$

and therefore any solution to it on $(0,1)$ can be written as

$$u(\rho)=c_1u_1(\rho)+c_2u_2(\rho)-u_1(\rho)\int_0^\rho\frac{u_2(s)Q(s)s^8}{(1-s^2)^2}\,ds+u_2(\rho)\int_0^\rho\frac{u_1(s)Q(s)s^8}{(1-s^2)^2}\,ds$$

for a choice of $c_1,c_2\in\mathbb{C}$. Again, by similar asymptotic considerations as above, we infer that u'' is necessarily unbounded on $(0,1)$, and this concludes the proof. □

5C. ***The spectrum of $\boldsymbol{L_a}$ for $\boldsymbol{a\neq 0}$.*** We now investigate the spectrum of $\boldsymbol{L}_a$. In particular, by a perturbative argument we show that, for small a, an analogue of Proposition 5.5 holds for $\boldsymbol{L}_a$ as well.

Lemma 5.8. *There exists $\delta^*>0$ such that, for all $a\in\overline{\mathbb{B}}^9_{\delta^*}$, the following holds*:

$$\sigma(\boldsymbol{L}_a)\cap\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geq-\tfrac{1}{2}\omega_0\}=\{\lambda_0,\lambda_1,\lambda_2\},$$

where ω_0 is the constant from Proposition 5.5 and $\lambda_0=0$, $\lambda_1=1$, and $\lambda_2=3$ are eigenvalues. The geometric eigenspace of λ_2 is spanned by $\boldsymbol{h}_a=(h_{a,1},h_{a,2})$, where

$$h_{a,1}(\xi)=\frac{\gamma(\xi,a)}{(12\gamma(\xi,a)^2+5|\xi|^2-5)^3}\quad\text{and}\quad h_{a,2}(\xi)=\xi^j\partial_jh_{a,1}(\xi)+5h_{a,1}(\xi).$$

Moreover, the geometric eigenspaces of λ_0 and λ_1 are spanned by $\{\boldsymbol{g}_a^{(k)}\}_{k=0}^9=\{(g_{a,1}^{(k)},g_{a,2}^{(k)})\}_{k=0}^9$ and $\{\boldsymbol{q}_a^{(j)}\}_{j=1}^9=\{(q_{a,1}^{(j)},q_{a,2}^{(j)})\}_{j=1}^9$, respectively, where

$$g_{a,1}^{(0)}(\xi)=\frac{(|\xi|^2-1)\gamma(\xi,a)}{(12\gamma(\xi,a)^2+5|\xi|^2-5)^3},\qquad g_{a,2}^{(0)}(\xi)=\xi^j\partial_{\xi^j}g_{a,1}^{(0)}(\xi)+3g_{a,1}^{(0)}(\xi),$$

$$g_{a,1}^{(k)}(\xi)=\frac{(72\gamma(\xi,a)^2+5-5|\xi|^2)\partial_{a_j}\gamma(\xi,a)}{(12\gamma(\xi,a)^2+5|\xi|^2-5)^3},\quad g_{a,2}^{(k)}(\xi)=\xi^j\partial_{\xi^j}g_{a,1}^{(k)}(\xi)+3g_{a,1}^{(k)}(\xi),$$

and

$$q_{a,1}^{(j)}(\xi)=\partial_{a_j}U_a(\xi)\quad\text{and}\quad q_{a,2}^{(j)}(\xi)=\xi^j\partial_jq_{a,1}^{(j)}(\xi)+2q_{a,1}^{(j)}(\xi).$$

Additionally, the eigenfunctions depend Lipschitz continuously on the parameter a, i.e.,

$$\|\boldsymbol{h}_a-\boldsymbol{h}_b\|+\|\boldsymbol{g}_a^{(k)}-\boldsymbol{g}_b^{(k)}\|+\|\boldsymbol{q}_a^{(j)}-\boldsymbol{q}_b^{(j)}\|\lesssim|a-b|$$

for all $a,b\in\overline{\mathbb{B}}^9_{\delta^*}$.

Proof. Let $\varepsilon = -\frac{1}{2}\omega_0 + \frac{1}{2}$ and $\delta > 0$. Then take κ defined by Proposition 4.3, and introduce the sets

$$\Omega = \left\{z \in \mathbb{C} : \operatorname{Re} z \geq -\tfrac{1}{2}\omega_0 \text{ and } |z| \leq \kappa\right\}$$

and

$$\tilde{\Omega} = \left\{z \in \mathbb{C} : \operatorname{Re} z \geq -\tfrac{1}{2}\omega_0\right\} \setminus \Omega.$$

Note that Proposition 4.3 implies that $\tilde{\Omega} \subset \rho(\boldsymbol{L_a})$ for all $a \in \overline{\mathbb{B}}^9_\delta$. Hence we only need to investigate the spectrum in the compact set Ω. First, note that by Proposition 4.3, the set Ω contains a finite number of eigenvalues. By a direct calculation it can be checked that $\boldsymbol{q}_a^{(j)}$, $\boldsymbol{g}_a^{(k)}$, and $\boldsymbol{h_a}$ are eigenfunctions that correspond to $\lambda_0 = 0$, $\lambda_1 = 1$, and $\lambda_2 = 3$, respectively. Note that we get the explicit expression above simply by Lorentz transforming the corresponding eigenfunctions for $a = 0$. We now show that there are no other eigenvalues in Ω. For this, we utilize the Riesz projection onto the spectrum contained in Ω; see (5-39). This, however, necessitates that $\partial\Omega \subset \rho(\boldsymbol{L_a})$, and we now show that this holds for small enough a. First, note that for $\lambda \in \partial\Omega$ we have the identity

$$\lambda - \boldsymbol{L_a} = [1 - (\boldsymbol{L}'_a - \boldsymbol{L}'_0)\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)](\lambda - \boldsymbol{L}_0). \tag{5-37}$$

Then, from Proposition 4.2, we have

$$\|\boldsymbol{L}'_a - \boldsymbol{L}'_0\| \|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\| \lesssim |a| \max_{\lambda \in \partial\Omega} \|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\|$$

for all $a \in \overline{\mathbb{B}}^9_\delta$. Therefore, there is small enough $\delta^* > 0$ such that

$$\|\boldsymbol{L}'_a - \boldsymbol{L}'_0\| \|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\| < 1 \tag{5-38}$$

for all $\lambda \in \partial\Omega$ and all $a \in \overline{\mathbb{B}}^9_{\delta^*}$. Now from (5-38) and (5-37) we infer that $\partial\Omega \subset \rho(\boldsymbol{L_a})$ for all $a \in \overline{\mathbb{B}}^9_{\delta^*}$. Thereupon we define the projection

$$\tilde{\boldsymbol{T}}_a = \frac{1}{2\pi i} \int_{\partial\Omega} \boldsymbol{R}_{\boldsymbol{L}_a}(\lambda)\, d\lambda. \tag{5-39}$$

For $a = 0$, by Lemma 5.7 the rank of the operator $\tilde{\boldsymbol{T}}_a$ is 20. Furthermore, continuity of $a \mapsto \boldsymbol{R}_{\boldsymbol{L}_a}(\lambda)$ (which follows from (5-37)) implies continuity of $a \mapsto \tilde{\boldsymbol{T}}_a$ on $\overline{\mathbb{B}}^9_{\delta^*}$. Thus, we conclude that $\dim \operatorname{ran} \tilde{\boldsymbol{T}}_a = 20$ for all $a \in \overline{\mathbb{B}}^9_{\delta^*}$; see, e.g., [Kato 1976, p. 34, Lemma 4.10]. By this, we exclude any further eigenvalues. Lipschitz continuity for the eigenfunctions follows from the fact that they depend smoothly on a; see (4-2) and (4-3). □

6. Perturbations around U_a: bounds for the linearized time-evolution

We fix $\delta^* > 0$ as in Lemma 5.8 for the rest of this paper. In this section we propagate Lemma 5.7 to $\boldsymbol{L_a}$. For that, given $a \in \overline{\mathbb{B}}^9_{\delta^*}$, we define the Riesz projections

$$\boldsymbol{H}_a := \frac{1}{2\pi i} \int_{\gamma_2} \boldsymbol{R}_{\boldsymbol{L}_a}(\lambda)\, d\lambda, \quad \boldsymbol{P}_a := \frac{1}{2\pi i} \int_{\gamma_1} \boldsymbol{R}_{\boldsymbol{L}_a}(\lambda)\, d\lambda, \quad \text{and} \quad \boldsymbol{Q}_a := \frac{1}{2\pi i} \int_{\gamma_0} \boldsymbol{R}_{\boldsymbol{L}_a}(\lambda)\, d\lambda,$$

where $\gamma_j(s) = \lambda_j + \frac{1}{4}\omega_0 e^{2\pi i s}$ for $s \in [0, 1]$.

Lemma 6.1. *We have*

$$\operatorname{ran} \boldsymbol{H}_a = \operatorname{span}(\boldsymbol{h}_a), \quad \operatorname{ran} \boldsymbol{P}_a = \operatorname{span}(\boldsymbol{g}_a^{(0)}, \dots, \boldsymbol{g}_a^{(9)}), \quad \textit{and} \quad \operatorname{ran} \boldsymbol{Q}_a = \operatorname{span}(\boldsymbol{q}_a^{(1)}, \dots, \boldsymbol{q}_a^{(9)})$$

for all $a \in \overline{\mathbb{B}}^9_{\delta^*}$. *Moreover, the projections are mutually transversal,*

$$\boldsymbol{H}_a \boldsymbol{P}_a = \boldsymbol{P}_a \boldsymbol{H}_a = \boldsymbol{H}_a \boldsymbol{Q}_a = \boldsymbol{Q}_a \boldsymbol{H}_a = \boldsymbol{Q}_a \boldsymbol{P}_a = \boldsymbol{P}_a \boldsymbol{Q}_a = 0,$$

and depend Lipschitz continuously on the parameter a, *i.e.*,

$$\|\boldsymbol{H}_a - \boldsymbol{H}_b\| + \|\boldsymbol{P}_a - \boldsymbol{P}_b\| + \|\boldsymbol{Q}_a - \boldsymbol{Q}_b\| \lesssim |a - b|$$

for all $a, b \in \overline{\mathbb{B}}^9_{\delta^*}$.

Proof. The Riesz projections depend continuously on a, hence the dimensions of the ranges remain the same. Transversality follows from the definition of Riesz projections. The Lipschitz bounds follow from the second resolvent identity and Proposition 4.2. □

Since $\boldsymbol{P}_a$ and $\boldsymbol{Q}_a$ are finite-rank, for every $\boldsymbol{f} \in \mathcal{H}$, there are $\alpha^k \in \mathbb{C}$ and $\beta^j \in \mathbb{C}$ such that

$$\boldsymbol{P}_a \boldsymbol{f} = \sum_{k=0}^{9} \alpha^k \boldsymbol{g}_a^{(k)} \quad \text{and} \quad \boldsymbol{Q}_a \boldsymbol{f} = \sum_{j=1}^{9} \beta^j \boldsymbol{q}_a^{(j)}.$$

We thereby define the projections

$$\boldsymbol{P}_a^{(k)} \boldsymbol{f} := \alpha^k \boldsymbol{g}_a^{(k)} \quad \text{and} \quad \boldsymbol{Q}_a^{(j)} \boldsymbol{f} := \beta^j \boldsymbol{q}_a^{(j)}.$$

Clearly, the projections satisfy the identities

$$\boldsymbol{P}_a = \sum_{k=0}^{9} \boldsymbol{P}_a^{(k)}, \quad \boldsymbol{Q}_a = \sum_{j=1}^{9} \boldsymbol{Q}_a^{(j)} \qquad \text{and} \qquad \boldsymbol{P}_a^{(i)} \boldsymbol{P}_a^{(j)} = \delta^{ij} \boldsymbol{P}_a^{(i)}, \quad \boldsymbol{Q}_a^{(k)} \boldsymbol{Q}_a^{(l)} = \delta^{kl} \boldsymbol{Q}_a^{(k)}.$$

We also define

$$\boldsymbol{T}_a := \boldsymbol{I} - \boldsymbol{H}_a - \boldsymbol{P}_a - \boldsymbol{Q}_a.$$

By Lemma 6.1, we have that $\boldsymbol{T}_a$ is Lipschitz continuous with respect to a and the projections $\boldsymbol{T}_a$, $\boldsymbol{H}_a$, $\boldsymbol{P}_a^{(k)}$, and $\boldsymbol{Q}_a^{(j)}$ are mutually transversal. Moreover, the Lipschitz continuity of $\boldsymbol{Q}_a$ and $\boldsymbol{P}_a$ with respect to a implies

$$\|\boldsymbol{Q}_a^{(j)} - \boldsymbol{Q}_b^{(j)}\| \lesssim |a - b|, \quad j = 1, \dots, 9, \quad \text{and} \quad \|\boldsymbol{P}_a^{(k)} - \boldsymbol{P}_b^{(k)}\| \lesssim |a - b|, \quad k = 0, \dots, 9,$$

for all $a, b \in \overline{\mathbb{B}}^9_{\delta^*}$. We now describe the interaction of the semigroup $(\boldsymbol{S}_a(\tau))_{\tau \geq 0}$ with these projections.

Proposition 6.2. *The projection operators* $\boldsymbol{H}_a$, $\boldsymbol{P}_a^{(k)}$, *and* $\boldsymbol{Q}_a^{(j)}$ *commute with the semigroup* $\boldsymbol{S}_a(\tau)$, *i.e.*,

$$[\boldsymbol{S}_a(\tau), \boldsymbol{H}_a] = [\boldsymbol{S}_a(\tau), \boldsymbol{P}_a^{(k)}] = [\boldsymbol{S}_a(\tau), \boldsymbol{Q}_a^{(j)}] = 0 \tag{6-1}$$

for $j = 1, \dots, 9$, $k = 0, \dots, 9$, *and* $\tau \geq 0$. *Furthermore,*

$$\boldsymbol{S}_a(\tau) \boldsymbol{H}_a = e^{3\tau} \boldsymbol{H}_a, \quad \boldsymbol{S}_a(\tau) \boldsymbol{P}_a^{(k)} = e^{\tau} \boldsymbol{P}_a^{(k)}, \quad \textit{and} \quad \boldsymbol{S}_a(\tau) \boldsymbol{Q}_a^{(j)} = \boldsymbol{Q}_a^{(j)}, \tag{6-2}$$

and there exists $\omega > 0$ *such that*

$$\|\boldsymbol{S}_a(\tau)\boldsymbol{T}_a\boldsymbol{u}\| \lesssim e^{-\omega\tau}\|\boldsymbol{T}_a\boldsymbol{u}\| \tag{6-3}$$

for all $\boldsymbol{u} \in \mathcal{H}$, $a \in \overline{\mathbb{B}}^9_{\delta^*}$, *and* $\tau \geq 0$. *Moreover, we have*

$$\|\boldsymbol{S}_a(\tau)\boldsymbol{T}_a - \boldsymbol{S}_b(\tau)\boldsymbol{T}_b\| \lesssim e^{-\omega\tau}|a-b| \tag{6-4}$$

for all $a, b \in \overline{\mathbb{B}}^9_{\delta^*}$ *and* $\tau \geq 0$.

Proof. Equation (6-1) follows from the properties of the Riesz projections $\boldsymbol{H}_a$, $\boldsymbol{P}_a$, and $\boldsymbol{Q}_a$. In particular, they commute with $\boldsymbol{S}_a(\tau)$, and this yields, for example, that

$$\boldsymbol{P}^{(k)}_a\boldsymbol{S}_a(\tau)\boldsymbol{u} = \boldsymbol{P}_a\boldsymbol{P}^{(k)}_a\boldsymbol{S}_a(\tau)\boldsymbol{u} = \boldsymbol{P}^{(k)}_a\boldsymbol{S}_a\boldsymbol{P}_a(\tau)\boldsymbol{u} = e^{\tau}\boldsymbol{P}^{(k)}_a\boldsymbol{P}_a\boldsymbol{u} = \boldsymbol{S}_a(\tau)\boldsymbol{P}^{(k)}_a\boldsymbol{u}.$$

Equation (6-2) follows from the correspondence between point spectra of a semigroup and its generator. Equation (6-3) follows from Gearhart–Prüss theorem. More precisely, we have that $\operatorname{ran}\boldsymbol{T}_a$ reduces both $\boldsymbol{L}_a$ and $\boldsymbol{S}_a(\tau)$, and furthermore

$$\boldsymbol{R}_{\boldsymbol{L}_a|_{\operatorname{ran}\boldsymbol{T}_a}}(\lambda) \text{ exists in } \left\{z \in \mathbb{C} : \operatorname{Re} z \geq -\tfrac{1}{2}\omega_0\right\}$$

and is uniformly bounded there, i.e., according to Proposition 4.3 there exists $c > 0$ such that

$$\|\boldsymbol{R}_{\boldsymbol{L}_a|_{\operatorname{ran}\boldsymbol{T}_a}}(\lambda)\| \leq c$$

for all $\operatorname{Re}\lambda \geq -\frac{1}{2}\omega_0$ and all $a \in \overline{\mathbb{B}}^9_{\delta^*}$. Hence, by the Gearhart–Prüss theorem (see [Engel and Nagel 2000, p. 302, Theorem 1.11]), for every $\varepsilon > 0$, we have

$$\|\boldsymbol{S}_a(\tau)|_{\operatorname{ran}\boldsymbol{T}_a}\| \lesssim_\varepsilon e^{-(\frac{\omega_0}{2}-\varepsilon)\tau} \tag{6-5}$$

for all $a \in \overline{\mathbb{B}}^9_{\delta^*}$ and $\tau \geq 0$. From here (6-3) holds for any $\omega < \frac{1}{2}\omega_0$. We remark in passing that (6-3) also follows from purely abstract considerations; see [Glogić 2022, Theorem B.1]. Finally, to obtain (6-4) we do the following. First, for $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_a)$ we define the function

$$\Phi_{a,b}(\tau) = \frac{\boldsymbol{S}_a(\tau)\boldsymbol{T}_a\boldsymbol{u} - \boldsymbol{S}_b(\tau)\boldsymbol{T}_b\boldsymbol{u}}{|a-b|}.$$

Note that this function satisfies the evolution equation

$$\partial_\tau\Phi_{a,b}(\tau) = \boldsymbol{L}_a\boldsymbol{T}_a\Phi_{a,b}(\tau) + \frac{\boldsymbol{L}_a\boldsymbol{T}_a - \boldsymbol{L}_b\boldsymbol{T}_b}{|a-b|}\boldsymbol{S}_b(\tau)\boldsymbol{T}_b\boldsymbol{u}$$

with the initial condition

$$\Phi_{a,b}(0) = \frac{\boldsymbol{T}_a\boldsymbol{u} - \boldsymbol{T}_b\boldsymbol{u}}{|a-b|},$$

and therefore by Duhamel's principle we have

$$\Phi_{a,b}(\tau) = \boldsymbol{S}_a(\tau)\boldsymbol{T}_a\frac{\boldsymbol{T}_a\boldsymbol{u} - \boldsymbol{T}_b\boldsymbol{u}}{|a-b|} + \int_0^\tau \boldsymbol{S}_a(\tau-\tau')\boldsymbol{T}_a\frac{\boldsymbol{L}_a\boldsymbol{T}_a - \boldsymbol{L}_b\boldsymbol{T}_b}{|a-b|}\boldsymbol{S}_b(\tau')\boldsymbol{T}_b\boldsymbol{u}\,d\tau'.$$

Now, from Proposition 4.2 and Lemma 6.1, we get

$$\|\boldsymbol{L}_a\boldsymbol{T}_a - \boldsymbol{L}_b\boldsymbol{T}_b\| \lesssim |a-b|,$$

and from this and (6-5) we obtain

$$\|\Phi_{a,b}(\tau)\| \lesssim e^{-(\frac{\omega_0}{2}-\varepsilon)\tau}(1+\tau)\|\boldsymbol{u}\| \lesssim e^{-(\frac{\omega_0}{2}-2\varepsilon)}\|\boldsymbol{u}\|.$$

By choosing $\varepsilon > 0$ such that $\omega = \frac{1}{2}\omega_0 - 2\varepsilon > 0$, we conclude the proof. □

7. Nonlinear theory

With the linear theory at hand, we turn to studying the Cauchy problem for the nonlinear equation (2-9). Following the usual approach of first constructing strong solutions, we recast (2-9) in an integral form à la Duhamel,

$$\Phi(\tau) = \boldsymbol{S}_{a_\infty}(\tau)\Phi(0) + \int_0^\tau \boldsymbol{S}_{a_\infty}(\tau-\sigma)(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\,d\sigma \tag{7-1}$$

(where $(\boldsymbol{S}_{a_\infty}(\tau))_{\tau\geq 0}$ is the semigroup generated by $\boldsymbol{L}_{a_\infty}$), and resort to fixed-point arguments. Our aim is to construct global and decaying solutions to (7-1). An obvious obstruction to that is the presence of growing modes of $\boldsymbol{S}_{a_\infty}(\tau)$, see (6-2), and we deal with them in the following way. First, we note that the instabilities coming from $\boldsymbol{Q}_{a_\infty}$ and $\boldsymbol{P}_{a_\infty}$ are not genuine, as they arise from the Lorentz and space-time translation symmetries of (1-1).

We take care of the Lorentz instability by modulation. Namely, the presence of the unstable space ran $\boldsymbol{Q}_{a_\infty}$ is related to the freedom of choice of the function $a : [0,\infty) \mapsto \mathbb{R}^9$ in the ansatz (2-8), and, roughly speaking, we prove that given small enough initial data $\Phi(0)$, there is a way to choose a such that it leads to a solution Φ of (7-1) which eventually (in τ) gets stripped of any remnant of the unstable space ran $\boldsymbol{Q}_{a_\infty}$ brought about by initial data.

With the rest of the instabilities, which cause exponential growth, we deal differently. Namely, we introduce to the initial data suitable correction terms which serve to suppress the growth. Also, as mentioned, the unstable space ran $\boldsymbol{P}_{a_\infty}$ is another apparent instability as it is an artifact of the spacetime translation symmetries, and we use it to prove that the corrections corresponding to $\boldsymbol{P}_{a_\infty}$ can be annihilated by a proper choice of the parameters x_0 and T, which appear in the initial data $\Phi(0)$; see (2-6). The remaining instability, coming from $\boldsymbol{H}_{a_\infty}$, is the only genuine one, and the correction corresponding to it is reflected in the modification of the initial data in the main result; see (1-16).

To formalize the process described above, we first make some technical preparations. For the rest of this paper, we fix $\omega > 0$ from Proposition 6.2. Then, we introduce the function spaces

$$\mathcal{X} := \{\Phi \in C([0,\infty),\mathcal{H}) : \|\Phi\|_{\mathcal{X}} < \infty\}, \quad \text{where } \|\Phi\|_{\mathcal{X}} := \sup_{\tau>0} e^{\omega\tau}\|\Phi(\tau)\|,$$

and

$$X := \{a \in C^1([0,\infty),\mathbb{R}^9) : a(0) = 0,\ \|a\|_X < \infty\}, \quad \text{where } \|a\|_X := \sup_{\tau>0}[e^{\omega\tau}|\dot{a}(\tau)| + |a(\tau)|].$$

For $a \in X$, we define

$$a_\infty := \lim_{\tau\to\infty} a(\tau).$$

Furthermore, for $\delta > 0$, we set

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \le \delta\} \quad \text{and} \quad X_\delta := \Big\{a \in X : \sup_{\tau>0}[e^{\omega\tau}|\dot a(\tau)|] \le \delta\Big\}.$$

To ensure that all terms in (7-1) are defined, we must impose some size restriction on the function a. Note that it is enough to consider $a \in X_\delta$ for $\delta < \delta^*\omega$, as then $|a(\tau)| \le \delta/\omega < \delta^*$ for all $\tau \ge 0$. We will also frequently make use of the inequality

$$|a_\infty - a(\tau)| \le \int_\tau^\infty |\dot a(\sigma)|\, d\sigma \le \frac{\delta}{\omega} e^{-\omega\tau}. \tag{7-2}$$

Furthermore, note that, for $a, b \in X_\delta$ and $\tau \ge 0$, we have $|a(\tau) - b(\tau)| \le \|a - b\|_X$; in particular, we have $|a_\infty - b_\infty| \le \|a - b\|_X$.

7A. *Estimates of the nonlinear terms.* With an eye toward setting up a fixed-point scheme for (7-1), we now establish necessary bounds for the nonlinear terms. Namely, we treat

$$\boldsymbol{G}_{a(\tau)}(\Phi(\tau)) = [\boldsymbol{L}'_{a(\tau)} - \boldsymbol{L}'_{a_\infty}]\Phi(\tau) + \boldsymbol{F}(\Phi(\tau)).$$

Lemma 7.1. *Given $\delta \in (0, \delta^*\omega)$, we have*

$$\begin{gathered} \|\boldsymbol{G}_{a(\tau)}(\Phi(\tau))\| \lesssim \delta^2 e^{-2\omega\tau}, \\ \|\boldsymbol{G}_{a(\tau)}(\Phi(\tau)) - \boldsymbol{G}_{b(\tau)}(\Psi(\tau))\| \lesssim \delta e^{-2\omega\tau}(\|\Phi - \Psi\|_{\mathcal{X}} + \|a - b\|_X) \end{gathered} \tag{7-3}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$, $a, b \in X_\delta$, and $\tau \ge 0$, where the implicit constants in the above estimates are absolute.

Proof. First, since $H^5(\mathbb{B}^9)$ is a Banach algebra, we have that

$$\|u_1^2 - v_1^2\|_{H^4(\mathbb{B}^9)} \lesssim \|u_1 + v_1\|_{H^5}\|u_1 - v_1\|_{H^5},$$

and hence

$$\|\boldsymbol{F}(\boldsymbol{u}) - \boldsymbol{F}(\boldsymbol{v})\| \lesssim (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)\|\boldsymbol{u} - \boldsymbol{v}\| \tag{7-4}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}$. Next, we prove the second estimate in Lemma 7.1, as the first one follows from it. From (7-4), Proposition 4.2, and inequality (7-2), we obtain

$$\begin{gathered} \|\boldsymbol{F}(\Phi(\tau)) - \boldsymbol{F}(\Psi(\tau))\| \lesssim \delta e^{-2\omega\tau}\|\Phi - \Psi\|_{\mathcal{X}}, \\ \|[\boldsymbol{L}'_{a(\tau)} - \boldsymbol{L}'_{a_\infty}](\Phi(\tau) - \Psi(\tau))\| \lesssim \delta e^{-2\omega\tau}\|\Phi - \Psi\|_{\mathcal{X}} \end{gathered} \tag{7-5}$$

for $\Phi, \Psi \in \mathcal{X}_\delta$ and $a \in X_\delta$. Furthermore, using the fact that

$$V_{a_\infty}(\xi) - V_{a(\tau)}(\xi) = \int_\tau^\infty \partial_s V_{a(s)}(\xi)\, ds = \int_\tau^\infty \dot a^k(s)\varphi_{a(s),k}(\xi)\, ds, \tag{7-6}$$

with $\varphi_{a,k}(\xi) = \partial_{a^k} V_a(\xi)$, together with the smoothness of $\varphi_{a,k}$, we infer

$$\begin{aligned}\|([\boldsymbol{L}'_{a(\tau)}-\boldsymbol{L}'_{a_\infty}]-[\boldsymbol{L}'_{b(\tau)}-\boldsymbol{L}'_{b_\infty}])\boldsymbol{u}\| &\lesssim \|u_1\|_{H^4(\mathbb{B}^9)}\int_\tau^\infty \|\dot{a}^k(s)\varphi_{a(s),k}(\xi)-\dot{b}^k(s)\varphi_{b(s),k}(\xi)\|_{W^{4,\infty}(\mathbb{B}^9)}\,ds \\ &\lesssim \|\boldsymbol{u}\|\int_\tau^\infty |\dot{a}(s)-\dot{b}(s)|\,ds+\|\boldsymbol{u}\|\int_\tau^\infty |\dot{a}(s)||a(s)-b(s)|\,ds \\ &\lesssim \|\boldsymbol{u}\|\int_\tau^\infty e^{-\omega s}\|a-b\|_X\,ds.\end{aligned}$$

Hence

$$\|([\boldsymbol{L}'_{a(\tau)}-\boldsymbol{L}'_{a_\infty}]-[\boldsymbol{L}'_{b(\tau)}-\boldsymbol{L}'_{b_\infty}])\Psi(\tau)\| \lesssim \delta e^{-2\omega\tau}\|a-b\|_X$$

for $a, b \in X_\delta$ and $\Psi \in \mathcal{X}_\delta$, and this together with (7-5) concludes the proof. ☐

7B. ***Suppressing the instabilities.*** In this section we formalize the process of taming the instabilities. In particular, by introducing correction terms to the initial data we arrive at a modified equation, to which we prove existence of global and decaying solutions.

We first derive the so-called modulation equation for the parameter a. Recall that

$$\partial_\tau \boldsymbol{U}_{a(\tau)} = \dot{a}_j(\tau)\boldsymbol{q}^{(j)}_{a(\tau)} = \sum_{j=1}^{9} \dot{a}^j(\tau)\boldsymbol{q}^{(j)}_{a(\tau)};$$

see Remark 5.6. We introduce a smooth cut-off function $\chi : [0,\infty) \to [0,1]$ satisfying $\chi(\tau) = 1$ for $\tau \in [0,1]$, $\chi(\tau) = 0$ for $\tau \geq 4$, and $|\chi'(\tau)| \leq 1$ for all $\tau \in (0,\infty)$. The aim is to construct a function $a : [0,\infty) \mapsto \mathbb{R}^9$ such that it yields a solution Φ to (7-1) for which

$$\boldsymbol{Q}^{(j)}_{a_\infty}\Phi(\tau) = \chi(\tau)\boldsymbol{Q}^{(j)}_{a_\infty}\Phi(0) \tag{7-7}$$

for all $\tau \geq 0$. In that case, although $\boldsymbol{Q}^{(j)}_{a_\infty}\Phi(0) \neq 0$ in general, we have that $\boldsymbol{Q}^{(j)}_{a_\infty}\Phi(\tau) = 0$ eventually in τ. According to (7-1) and Proposition 6.2, (7-7) adopts the form

$$(1-\chi(\tau))\boldsymbol{Q}^{(j)}_{a_\infty}\boldsymbol{u} + \int_0^\tau (\boldsymbol{Q}^{(j)}_{a_\infty}\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \boldsymbol{Q}^{(j)}_{a_\infty}\dot{a}_i(\sigma)\boldsymbol{q}^i_{a(\sigma)})\,d\sigma = 0,$$

where for convenience we write $\boldsymbol{u}$ instead of $\Phi(0)$. Using $\boldsymbol{Q}^{(j)}_{a_\infty}\boldsymbol{q}^{(i)}_{a_\infty} = \delta^{ij}\boldsymbol{q}^{(j)}_{a_\infty}$, we get the modulation equation

$$a^j(\tau)\boldsymbol{q}^{(j)}_{a_\infty} = -\int_0^\tau \chi'(\sigma)\boldsymbol{Q}^{(j)}_{a_\infty}\boldsymbol{u}\,d\sigma + \int_0^\tau (\boldsymbol{Q}^{(j)}_{a_\infty}\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \boldsymbol{Q}^{(j)}_{a_\infty}\dot{a}_i(\sigma)(\boldsymbol{q}^{(i)}_{a(\sigma)} - \boldsymbol{q}^{(i)}_{a_\infty}))\,d\sigma$$

for $j = 1,\ldots,9$. By introducing the notation

$$\boldsymbol{A}_j(a,\Phi,\boldsymbol{u})(\sigma) := \chi'(\sigma)\boldsymbol{Q}^{(j)}_{a_\infty}\boldsymbol{u} + (\boldsymbol{Q}^{(j)}_{a_\infty}\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \boldsymbol{Q}^{(j)}_{a_\infty}\dot{a}_i(\sigma)(\boldsymbol{q}^{(i)}_{a(\sigma)} - \boldsymbol{q}^{(i)}_{a_\infty})),$$

the modulation equation can be written succinctly as

$$a_j(\tau) = A_j(\,\cdot\,,\Phi,\boldsymbol{u}) := \|\boldsymbol{q}^{(j)}_{a_\infty}\|^{-2}\int_0^\tau (\boldsymbol{A}_j(a,\Phi,\boldsymbol{u})(\sigma)\,|\,\boldsymbol{q}^{(j)}_{a_\infty})\,d\sigma, \quad j = 1,\ldots,9. \tag{7-8}$$

In the following we prove that, for small enough Φ and $\boldsymbol{u}$, the system (7-8) admits a global (in τ) solution.

Lemma 7.2. *For all sufficiently small $\delta > 0$ and all sufficiently large $C > 0$, the following holds: For every $\boldsymbol{u} \in \mathcal{H}$ satisfying $\|\boldsymbol{u}\| \le \delta/C$ and every $\Phi \in \mathcal{X}_\delta$, there exists a unique $a = a(\Phi, \boldsymbol{u}) \in X_\delta$ such that* (7-8) *holds for $\tau \ge 0$. Moreover,*

$$\|a(\Phi, \boldsymbol{u}) - a(\Psi, \boldsymbol{v})\|_X \lesssim \|\Phi - \Psi\|_{\mathcal{X}} + \|\boldsymbol{u} - \boldsymbol{v}\| \tag{7-9}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{B}_{\delta/C}$.

Proof. We use a fixed-point argument. Using the bounds from Lemma 7.1, one can show that, given $\boldsymbol{u}$ and Φ that satisfy the above assumptions, the following estimates hold:

$$\|\boldsymbol{A}_j(a, \Phi, \boldsymbol{u})(\tau)\| \lesssim \Big(\frac{\delta}{C} + \delta^2\Big) e^{-2\omega\tau} \quad \text{and} \quad \|\boldsymbol{A}_j(a, \Phi, \boldsymbol{u})(\tau) - \boldsymbol{A}_j(b, \Phi, \boldsymbol{u})(\tau)\| \lesssim \delta e^{-\omega\tau} \|a - b\|_X$$

for all $a, b \in X_\delta$. From here, according to the definition in (7-8), we have that, for all small enough $\delta > 0$ and all large enough $C > 0$, given $\Phi \in \mathcal{X}_\delta$ and $\boldsymbol{u} \in \mathcal{B}_{\delta/C}$, the ball X_δ is invariant under the action of the operator $A(\,\cdot\,, \Phi, \boldsymbol{u})$, which is furthermore a contraction on X_δ. Hence (7-8) has a unique solution in X_δ. The Lipschitz continuity of the solution map follows from the estimate

$$\begin{aligned}\|a - b\|_X &\le \|A(a, \Phi, \boldsymbol{u}) - A(b, \Phi, \boldsymbol{u})\|_X + \|A(b, \Phi, \boldsymbol{u}) - A(b, \Phi, \boldsymbol{v})\|_X + \|A(b, \Phi, \boldsymbol{v}) - A(b, \Psi, \boldsymbol{v})\|_X \\ &\lesssim \delta \|a - b\|_X + \|\boldsymbol{u} - \boldsymbol{v}\| + \|\Phi - \Psi\|_{\mathcal{X}}\end{aligned}$$

by taking small enough $\delta > 0$. ☐

For the remaining instabilities, we introduce the correction terms

$$\begin{aligned}\boldsymbol{C}_1(\Phi, a, \boldsymbol{u}) &:= \boldsymbol{P}_{a_\infty}\bigg(\boldsymbol{u} + \int_0^\infty e^{-\sigma}(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\, d\sigma\bigg), \\ \boldsymbol{C}_2(\Phi, a, \boldsymbol{u}) &:= \boldsymbol{H}_{a_\infty}\bigg(\boldsymbol{u} + \int_0^\infty e^{-3\sigma}(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\, d\sigma\bigg),\end{aligned}$$

and set $\boldsymbol{C} := \boldsymbol{C}_1 + \boldsymbol{C}_2$. Consequently, we investigate the modified integral equation

$$\begin{aligned}\Phi(\tau) &= \boldsymbol{S}_{a_\infty}(\tau)(\boldsymbol{u} - \boldsymbol{C}(\Phi, a, \boldsymbol{u})) + \int_0^\tau \boldsymbol{S}_{a_\infty}(\tau - \sigma)(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\, d\sigma \\ &=: \boldsymbol{K}(\Phi, a, \boldsymbol{u})(\tau).\end{aligned} \tag{7-10}$$

Proposition 7.3. *For all sufficiently small $\delta > 0$ and all sufficiently large $C > 0$, the following holds: For every $\boldsymbol{u} \in \mathcal{H}$ with $\|\boldsymbol{u}\| \le \delta/C$ there exist functions $\Phi \in \mathcal{X}_\delta$ and $a \in X_\delta$ such that* (7-10) *holds for $\tau \ge 0$. Furthermore, the solution map $\boldsymbol{u} \mapsto (\Phi(\boldsymbol{u}), a(\boldsymbol{u}))$ is Lipschitz continuous, i.e.,*

$$\|\Phi(\boldsymbol{u}) - \Phi(\boldsymbol{v})\|_{\mathcal{X}} + \|a(\boldsymbol{u}) - a(\boldsymbol{v})\|_X \lesssim \|\boldsymbol{u} - \boldsymbol{v}\| \tag{7-11}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{B}_{\delta/C}$.

Proof. We choose $C > 0$ and $\delta > 0$ such that Lemma 7.2 holds. Then, for fixed $\boldsymbol{u} \in \mathcal{B}_{\delta/C}$, there is a unique $a = a(\Phi, \boldsymbol{u}) \in X_\delta$ associated to every $\Phi \in \mathcal{X}_\delta$ such that the modulation equation (7-8) is satisfied. Hence we can define $\boldsymbol{K}_{\boldsymbol{u}}(\Phi) := \boldsymbol{K}(\Phi, a, \boldsymbol{u})$. We intend to show that for small enough $\delta > 0$ the operator $\boldsymbol{K}_{\boldsymbol{u}}$ is a contraction on $\mathcal{X}_\delta$. To show the necessary bounds, we first split $\boldsymbol{K}_{\boldsymbol{u}}(\Phi)$ according to projections $\boldsymbol{P}_{a_\infty}$, $\boldsymbol{Q}_{a_\infty}$, $\boldsymbol{H}_{a_\infty}$, and $\boldsymbol{T}_{a_\infty}$, and then estimate each part separately.

First, note that the transversality of the projections implies

$$\boldsymbol{P}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) = -\int_\tau^\infty e^{\tau-\sigma}\boldsymbol{P}_{a_\infty}(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\,d\sigma$$

and

$$\boldsymbol{H}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) = -\int_\tau^\infty e^{3(\tau-\sigma)}\boldsymbol{H}_{a_\infty}(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\,d\sigma.$$

Now, since

$$\partial_\tau \boldsymbol{U}_{a(\tau)} = \dot{a}_j(\tau)\boldsymbol{q}^{(j)}_{a_\infty} + \dot{a}_j(\tau)[\boldsymbol{q}^{(j)}_{a(\tau)} - \boldsymbol{q}^{(j)}_{a_\infty}]$$

and

$$\|\boldsymbol{q}^{(j)}_{a(\tau)} - \boldsymbol{q}^{(j)}_{a_\infty}\| \lesssim \delta e^{-\omega\tau},$$

we have

$$\|\boldsymbol{H}_{a_\infty}\partial_\tau \boldsymbol{U}_{a(\tau)}\| + \|\boldsymbol{P}_{a_\infty}\partial_\tau \boldsymbol{U}_{a(\tau)}\| + \|(1-\boldsymbol{Q}_{a_\infty})\partial_\tau \boldsymbol{U}_{a(\tau)}\| \lesssim \delta^2 e^{-2\omega\tau} \tag{7-12}$$

for all $a \in X_\delta$. This, together with Lemma 7.1 and the fact that

$$\boldsymbol{Q}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) = \chi(\tau)\boldsymbol{Q}_{a_\infty}\boldsymbol{u} \tag{7-13}$$

(see (7-7)), yields the bounds

$$\begin{aligned} \|\boldsymbol{H}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau)\| + \|\boldsymbol{P}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau)\| &\lesssim \delta^2 e^{-2\omega\tau}, \\ \|\boldsymbol{Q}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau)\| &\lesssim \frac{\delta}{C}e^{-2\omega\tau} \end{aligned} \tag{7-14}$$

for all $\Phi \in \mathcal{X}_\delta$. On the other hand, for the stable subspace we have

$$\boldsymbol{T}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) = \boldsymbol{S}_{a_\infty}(\tau)\boldsymbol{T}_{a_\infty}\boldsymbol{u} + \int_0^\tau \boldsymbol{S}_{a_\infty}(\tau-\sigma)\boldsymbol{T}_{a_\infty}(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\,d\sigma,$$

and by Lemma 7.1, Proposition 6.2, and (7-12), we get

$$\|\boldsymbol{T}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau)\| \lesssim \Big(\frac{\delta}{C} + \delta^2\Big)e^{-\omega\tau} \tag{7-15}$$

for all $\Phi \in \mathcal{X}_\delta$. Now, from (7-14) and (7-15) we see that $\boldsymbol{K}_{\boldsymbol{u}}$ maps $\mathcal{X}_\delta$ into itself for all $\delta > 0$ sufficiently small and all $C > 0$ sufficiently large. The contraction property of $\boldsymbol{K}_{\boldsymbol{u}}$ is established similarly. Namely, there is the analogue of (7-12):

$$\begin{aligned} &\|\boldsymbol{H}_{a_\infty}\partial_\tau \boldsymbol{U}_{a(\tau)} - \boldsymbol{H}_{b_\infty}\partial_\tau \boldsymbol{U}_{b(\tau)}\| + \|\boldsymbol{P}_{a_\infty}\partial_\tau \boldsymbol{U}_{a(\tau)} - \boldsymbol{P}_{b_\infty}\partial_\tau \boldsymbol{U}_{b(\tau)}\| \\ &\qquad + \|(1-\boldsymbol{Q}_{a_\infty})\partial_\tau \boldsymbol{U}_{a(\tau)} - (1-\boldsymbol{Q}_{b_\infty})\partial_\tau \boldsymbol{U}_{b(\tau)}\| \lesssim \delta^2 e^{-2\omega\tau} \end{aligned}$$

for all $a, b \in X_\delta$. Furthermore, by Lemma 7.1, (7-13), and Lemma 7.2, we get the analogous estimates to (7-14); namely, we have

$$\begin{aligned} &\|\boldsymbol{H}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) - \boldsymbol{H}_{b_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Psi)(\tau)\| + \|\boldsymbol{P}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) - \boldsymbol{P}_{b_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Psi)(\tau)\| \\ &\qquad + \|\boldsymbol{Q}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) - \boldsymbol{Q}_{b_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Psi)(\tau)\| \lesssim \delta e^{-2\omega\tau}\|\Phi - \Psi\|_{\mathcal{X}} \end{aligned}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$, where $a = a(\Phi, \boldsymbol{u})$ and $b = a(\Psi, \boldsymbol{u})$. Also, in line with (7-15), we have

$$\|\boldsymbol{T}_{a_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) - \boldsymbol{T}_{b_\infty}\boldsymbol{K}_{\boldsymbol{u}}(\Psi)(\tau)\| \lesssim \delta e^{-\omega\tau}\|\Phi - \Psi\|_{\mathcal{X}}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$. By combining these estimates we get

$$\|\boldsymbol{K}_{\boldsymbol{u}}(\Phi) - \boldsymbol{K}_{\boldsymbol{u}}(\Psi)\|_{\mathcal{X}} \lesssim \delta\|\Phi - \Psi\|_{\mathcal{X}} \tag{7-16}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$, and contractivity follows by taking small enough $\delta > 0$.

For the Lipschitz continuity, similarly to proving (7-9), we use the integral equation (7-10) to show that, given sufficiently small $\delta > 0$,

$$\|\Phi(\boldsymbol{u}) - \Phi(\boldsymbol{v})\|_{\mathcal{X}} \lesssim \|\boldsymbol{u} - \boldsymbol{v}\|$$

for all $\boldsymbol{u} \in \mathcal{B}_{\delta/C}$, and then (7-9) implies (7-11). □

7C. *Conditional stability in similarity variables.* According to Proposition 7.3 and (2-8) there exists a family of initial data close to $\boldsymbol{U}_0$ which lead to global (strong) solutions to (2-3), which furthermore converge to $\boldsymbol{U}_{a_\infty}$ for some a_∞ close to $a = 0$; with minimal modifications, the same argument can be carried out for $\boldsymbol{U}_a$ for any $a \neq 0$. In conclusion, we have conditional asymptotic orbital stability of the family $\{\boldsymbol{U}_a : a \in \mathbb{R}^9\}$, the condition being that the initial data belong to the set which ensures global existence and convergence. In this section we show that this set represents a Lipschitz manifold of codimension 11.

Let $\delta > 0$ and $C > 0$ be as in Proposition 7.3, and let $\boldsymbol{u} \in \mathcal{B}_{\delta/C}$. Also, let us write

$$\boldsymbol{C}(\boldsymbol{u}) := \boldsymbol{C}(\Phi(\boldsymbol{u}), a(\boldsymbol{u}), \boldsymbol{u}),$$

where the mapping $\boldsymbol{u} \mapsto (\Phi(\boldsymbol{u}), a(\boldsymbol{u}))$ is defined in Proposition 7.3. Moreover, we denote the projection corresponding to all unstable directions by

$$\boldsymbol{J}_a := \boldsymbol{P}_a + \boldsymbol{H}_a.$$

Note that by definition $\boldsymbol{J}_{a_\infty}\boldsymbol{C}(\boldsymbol{u}) = \boldsymbol{C}(\boldsymbol{u})$, and we have the Lipschitz estimate

$$\|\boldsymbol{J}_a - \boldsymbol{J}_b\| \lesssim |a - b|$$

for all $a, b \in X_\delta$.

Proposition 7.4. *There exists C > 0 such that, for all sufficiently small $\delta > 0$, there exists a codimension-11 Lipschitz manifold $\mathcal{M} = \mathcal{M}_{\delta,C} \subset \mathcal{H}$ with $\boldsymbol{0} \in \mathcal{M}$, defined as the graph of a Lipschitz continuous function $\boldsymbol{M} : \ker \boldsymbol{J}_0 \cap \mathcal{B}_{\delta/2C} \to \operatorname{ran} \boldsymbol{J}_0$,*

$$\mathcal{M} := \left\{\boldsymbol{v} + \boldsymbol{M}(\boldsymbol{v}) : \boldsymbol{v} \in \ker \boldsymbol{J}_0, \|\boldsymbol{v}\| \leq \frac{\delta}{2C}\right\} \subset \{\boldsymbol{u} \in \mathcal{B}_{\delta/C} : \boldsymbol{C}(\boldsymbol{u}) = 0\}.$$

Furthermore, for every $\boldsymbol{u} \in \mathcal{M}$, there exists $(\Phi, a) = (\Phi_{\boldsymbol{u}}, a_{\boldsymbol{u}}) \in \mathcal{X}_\delta \times X_\delta$ satisfying

$$\Phi(\tau) = \boldsymbol{S}_{a_\infty}(\tau)\boldsymbol{u} + \int_0^\tau \boldsymbol{S}_{a_\infty}(\tau - \sigma)(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\, d\sigma \tag{7-17}$$

for all $\tau \geq 0$. Moreover, there exists $K > C$ such that $\{\boldsymbol{u} \in \mathcal{B}_{\delta/K} : \boldsymbol{C}(\boldsymbol{u}) = 0\} \subset \mathcal{M}_{\delta,C}$.

Proof. First, we show that, for small enough $\delta > 0$, we have $\boldsymbol{C}(\boldsymbol{u}) = 0$ if and only if $\boldsymbol{J}_0\boldsymbol{C}(\boldsymbol{u}) = 0$. Assume that $\boldsymbol{J}_0\boldsymbol{C}(\boldsymbol{u}) = 0$. Then we obtain the estimate

$$\|\boldsymbol{C}(\boldsymbol{u})\| \le \|\boldsymbol{J}_0\boldsymbol{C}(\boldsymbol{u}) + (\boldsymbol{J}_{a_{\boldsymbol{u},\infty}} - \boldsymbol{J}_0)\boldsymbol{C}(\boldsymbol{u})\| \lesssim |a_{\boldsymbol{u},\infty}|\|\boldsymbol{C}(\boldsymbol{u})\|.$$

Since $a_{\boldsymbol{u},\infty} = O(\delta)$, we get $\boldsymbol{C}(\boldsymbol{u}) = 0$. The other direction is obvious. Now we construct the mapping $\boldsymbol{M}$. Let $\boldsymbol{u} \in \mathcal{H}$ and take the decomposition $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w} \in \ker \boldsymbol{J}_0 \oplus \operatorname{ran} \boldsymbol{J}_0$. Fix $\boldsymbol{v} \in \ker \boldsymbol{J}_0$ and define

$$\widetilde{\boldsymbol{C}}_{\boldsymbol{v}} : \operatorname{ran} \boldsymbol{J}_0 \to \operatorname{ran} \boldsymbol{J}_0, \quad \widetilde{\boldsymbol{C}}_{\boldsymbol{v}}(\boldsymbol{w}) = \boldsymbol{J}_0\boldsymbol{C}(\boldsymbol{v} + \boldsymbol{w}).$$

We establish that this mapping is invertible at zero, provided $\boldsymbol{v}$ is small enough, and we obtain $\boldsymbol{w} = \widetilde{\boldsymbol{C}}_{\boldsymbol{v}}^{-1}(\boldsymbol{0})$. This defines a mapping

$$\boldsymbol{M} : \ker \boldsymbol{J}_0 \to \operatorname{ran} \boldsymbol{J}_0, \quad \boldsymbol{M}(\boldsymbol{v}) := \widetilde{\boldsymbol{C}}_{\boldsymbol{v}}^{-1}(\boldsymbol{0}).$$

To show this, we use a fixed-point argument. Recall the definition of the correction terms $\boldsymbol{C} = \boldsymbol{C}_1 + \boldsymbol{C}_2$, $\boldsymbol{C}_1 = \sum_{k=0}^{9} \boldsymbol{C}_1^k$ with

$$\boldsymbol{C}_1^k(\Phi, a, \boldsymbol{u}) = \boldsymbol{P}_{a\infty}^{(k)}\boldsymbol{u} + \boldsymbol{P}_{a\infty}^{(k)}\boldsymbol{I}_1(\Phi, a)$$

and

$$\boldsymbol{C}_2(\Phi, a, \boldsymbol{u}) = \boldsymbol{H}_{a_\infty}\boldsymbol{u} + \boldsymbol{H}_{a_\infty}\boldsymbol{I}_2(\Phi, a),$$

where

$$\boldsymbol{I}_1(\Phi, a) := \int_0^\infty e^{-\sigma}[\boldsymbol{G}_{a(\sigma)}(\Psi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)}]\, d\sigma$$

and

$$\boldsymbol{I}_2(\Phi, a) := \int_0^\infty e^{-3\sigma}[\boldsymbol{G}_{a(\sigma)}(\Psi(\sigma) - \partial_\sigma \boldsymbol{U}_{s(\sigma)})]\, d\sigma.$$

We write

$$\boldsymbol{F}_1(\boldsymbol{u}) := \sum_{k=0}^{9} \boldsymbol{F}_1^k(\boldsymbol{u}) = \sum_{k=0}^{9} \boldsymbol{P}_{a\infty}^{(k)}\boldsymbol{I}_1(\Phi_{\boldsymbol{u}}, a_{\boldsymbol{u}})$$

and

$$\boldsymbol{F}_2(\boldsymbol{u}) := \boldsymbol{H}_{a_\infty}\boldsymbol{I}_2(\Phi_{\boldsymbol{u}}, a_{\boldsymbol{u}}).$$

By Lemma 7.1 and (7-12) we infer

$$\|\boldsymbol{F}_1^k(\boldsymbol{u})\| \lesssim \delta^2 \quad \text{and} \quad \|\boldsymbol{F}_2(\boldsymbol{u})\| \lesssim \delta^2. \tag{7-18}$$

Now, for $\boldsymbol{v} \in \ker \boldsymbol{J}_0$, we get

$$\begin{aligned}
\widetilde{\boldsymbol{C}}_{\boldsymbol{v}}(\boldsymbol{w}) = \boldsymbol{J}_0\boldsymbol{C}(\boldsymbol{v} + \boldsymbol{w}) &= \boldsymbol{J}_0\boldsymbol{J}_{a_\infty}(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{J}_0(\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{F}_2(\boldsymbol{v} + \boldsymbol{w})) \\
&= \boldsymbol{J}_0^2\boldsymbol{w} + \boldsymbol{J}_0(\boldsymbol{J}_{a_\infty} - \boldsymbol{J}_0)\boldsymbol{w} + \boldsymbol{J}_0\boldsymbol{J}_{a_\infty}\boldsymbol{v} + \boldsymbol{J}_0(\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{F}_2(\boldsymbol{v} + \boldsymbol{w})) \\
&= \boldsymbol{w} + \boldsymbol{J}_0(\boldsymbol{J}_{a_\infty} - \boldsymbol{J}_0)(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{J}_0(\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{F}_2(\boldsymbol{v} + \boldsymbol{w})).
\end{aligned}$$

Introducing the notation

$$\Omega_{\boldsymbol{v}}(\boldsymbol{w}) := \boldsymbol{J}_0(\boldsymbol{J}_0 - \boldsymbol{J}_{a_\infty})(\boldsymbol{v} + \boldsymbol{w}) - \boldsymbol{J}_0(\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{F}_2(\boldsymbol{v} + \boldsymbol{w})),$$

we rewrite equation $\widetilde{\boldsymbol{C}}_{\boldsymbol{v}}(\boldsymbol{w}) = 0$ as

$$\boldsymbol{w} = \Omega_{\boldsymbol{v}}(\boldsymbol{w}). \tag{7-19}$$

Now, for $\delta > 0$ and $C > 0$ from Proposition 7.3, we set

$$\widetilde{B}_{\delta/C}(\boldsymbol{v}) := \left\{ \boldsymbol{w} \in \operatorname{ran} \boldsymbol{J}_0 : \|\boldsymbol{v} + \boldsymbol{w}\| \leq \frac{\delta}{C} \right\}.$$

We show that $\Omega_{\boldsymbol{v}} : \widetilde{B}_{\delta/C}(\boldsymbol{v}) \to \widetilde{B}_{\delta/C}(\boldsymbol{v})$ is a contraction mapping for sufficiently small $\boldsymbol{v}$. Let $\boldsymbol{v} \in \mathcal{H}$ with $\|\boldsymbol{v}\| \leq \delta/(2C)$, and let $\boldsymbol{w} \in \widetilde{B}_{\delta/C}(\boldsymbol{v})$. Using (7-18), we estimate

$$\|\Omega_{\boldsymbol{v}}(\boldsymbol{w})\| \leq \|\boldsymbol{J}_0 - \boldsymbol{J}_{a_\infty}\| \|\boldsymbol{v} + \boldsymbol{w}\| + \|\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w})\| + \|\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w})\| \lesssim \frac{\delta^2}{C} + \delta^2.$$

Hence, by fixing $C > 0$, we have $\|\boldsymbol{v} + \Omega_{\boldsymbol{v}}(\boldsymbol{w})\| \leq \delta/C$ for all small enough $\delta > 0$. So the ball $\widetilde{B}_{\delta/C}(\boldsymbol{v})$ is invariant under the action of $\Omega_{\boldsymbol{v}}$. To prove contractivity, first, for $\boldsymbol{w}, \tilde{\boldsymbol{w}} \in \widetilde{B}_{\delta/C}(\boldsymbol{v})$, we associate to $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} + \tilde{\boldsymbol{w}}$ the functions (Φ, a) and (Ψ, b) in $\mathcal{X}_\delta \times X_\delta$ by Proposition 7.3, respectively. Then we obtain

$$\begin{aligned}\|\Omega_{\boldsymbol{v}}(\boldsymbol{w}) - \Omega_{\boldsymbol{v}}(\tilde{\boldsymbol{w}})\| \leq \|&\boldsymbol{J}_0(\boldsymbol{J}_0 - \boldsymbol{J}_{a_\infty})(\boldsymbol{v} + \boldsymbol{w}) - \boldsymbol{J}_0(\boldsymbol{J}_0 - \boldsymbol{J}_{b_\infty})(\boldsymbol{v} + \tilde{\boldsymbol{w}})\| \\ &+ \|\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w}) - \boldsymbol{F}_1(\boldsymbol{v} + \tilde{\boldsymbol{w}})\| + \|\boldsymbol{F}_2(\boldsymbol{v} + \boldsymbol{w}) - \boldsymbol{F}_2(\boldsymbol{v} + \tilde{\boldsymbol{w}})\|,\end{aligned}$$

and writing

$$\boldsymbol{J}_0(\boldsymbol{J}_0 - \boldsymbol{J}_{a_\infty})(\boldsymbol{v} + \boldsymbol{w}) - \boldsymbol{J}_0(\boldsymbol{J}_0 - \boldsymbol{J}_{b_\infty})(\boldsymbol{v} + \tilde{\boldsymbol{w}}) = \boldsymbol{J}_0(\boldsymbol{J}_0 - \boldsymbol{J}_{a_\infty})(\boldsymbol{w} - \tilde{\boldsymbol{w}}) - \boldsymbol{J}_0(\boldsymbol{J}_{a_\infty} - \boldsymbol{J}_{b_\infty})(\boldsymbol{v} + \tilde{\boldsymbol{w}})$$

we get by Proposition 7.3 the estimate

$$\begin{aligned}\|\boldsymbol{J}_0(\boldsymbol{J}_0 - \boldsymbol{J}_{a_\infty})(\boldsymbol{w} - \tilde{\boldsymbol{w}})\| + \|\boldsymbol{J}_0(\boldsymbol{J}_{a_\infty} - \boldsymbol{J}_{b_\infty})(\boldsymbol{v} + \tilde{\boldsymbol{w}})\| &\lesssim |a_\infty| \|\boldsymbol{w} - \tilde{\boldsymbol{w}}\| + |a_\infty - b_\infty| \|\boldsymbol{v} + \boldsymbol{w}\| \\ &\lesssim \delta \|\boldsymbol{w} - \tilde{\boldsymbol{w}}\| + \frac{\delta}{C} \|a - b\|_X \\ &\lesssim \delta \|\boldsymbol{w} - \tilde{\boldsymbol{w}}\|.\end{aligned}$$

On the other hand, by Lemma 7.1 and (7-12), we obtain, for $k = 0, \dots, 9$, that

$$\|\boldsymbol{P}^{(k)}_{a_\infty} \boldsymbol{I}_1(\Phi, a) - \boldsymbol{P}^{(k)}_{b_\infty} \boldsymbol{I}_2(\Psi, b)\| \lesssim \delta(\|\Phi - \Psi\|_{\mathcal{X}} + \|a - b\|_X)$$

and

$$\|\boldsymbol{H}_{a_\infty} \boldsymbol{I}_2(\Phi, a) - \boldsymbol{H}_{b_\infty} \boldsymbol{I}_2(\Psi, b)\| \lesssim \delta(\|\Phi - \Psi\|_{\mathcal{X}} + \|a - b\|_X).$$

Thus we get the Lipschitz estimate

$$\|\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w}) - \boldsymbol{F}_1(\boldsymbol{v} + \tilde{\boldsymbol{w}})\| + \|\boldsymbol{F}_2(\boldsymbol{v} + \boldsymbol{w}) - \boldsymbol{F}_2(\boldsymbol{v} + \tilde{\boldsymbol{w}})\| \lesssim \delta \|\boldsymbol{w} - \tilde{\boldsymbol{w}}\|,$$

and we conclude that, for all small enough $\delta > 0$, the operator $\Omega_{\boldsymbol{v}} : \widetilde{B}_{\delta/C}(\boldsymbol{v}) \to \widetilde{B}_{\delta/C}(\boldsymbol{v})$ is contractive, with the contraction constant $\frac{1}{2}$. Consequently, by the contraction map principle we get that, for every $\boldsymbol{v} \in \ker \boldsymbol{J}_0 \cap \mathcal{B}_{\delta/2C}$, there exists a unique $\boldsymbol{w} \in \widetilde{B}_{\delta/C}(\boldsymbol{v})$ that solves (7-19); hence $\boldsymbol{C}(\boldsymbol{v} + \boldsymbol{w}) = \widetilde{\boldsymbol{C}}_{\boldsymbol{v}}(\boldsymbol{w}) = 0$.

Next, we establish the Lipschitz-continuity of the mapping $\boldsymbol{v} \mapsto \boldsymbol{M}(\boldsymbol{v})$. Let $\boldsymbol{v}, \tilde{\boldsymbol{v}} \in \ker \boldsymbol{J}_0 \cap \mathcal{B}_{\delta/2C}$ and $\boldsymbol{w}, \tilde{\boldsymbol{w}} \in \widetilde{B}_{\delta/C}$ be the corresponding solutions to (7-19). We get

$$\begin{aligned}\|\boldsymbol{M}(\boldsymbol{v}) - \boldsymbol{M}(\tilde{\boldsymbol{v}})\| = \|\boldsymbol{w} - \tilde{\boldsymbol{w}}\| &\leq \|\Omega_{\boldsymbol{v}}(\boldsymbol{w}) - \Omega_{\boldsymbol{v}}(\tilde{\boldsymbol{w}})\| + \|\Omega_{\boldsymbol{v}}(\tilde{\boldsymbol{w}}) - \Omega_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{w}})\| \\ &\leq \tfrac{1}{2} \|\boldsymbol{w} - \tilde{\boldsymbol{w}}\| + \|\Omega_{\boldsymbol{v}}(\tilde{\boldsymbol{w}}) - \Omega_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{w}})\|.\end{aligned}$$

The second term we estimate with

$$\begin{aligned}\|\Omega_{\boldsymbol{v}}(\tilde{\boldsymbol{w}})-\Omega_{\tilde{\boldsymbol{v}}}(\tilde{\boldsymbol{w}})\| &= \big\|\boldsymbol{J}_0(\boldsymbol{J}_0-\boldsymbol{J}_{a_{\boldsymbol{v}+\tilde{\boldsymbol{w}},\infty}})(\boldsymbol{v}+\tilde{\boldsymbol{w}})-\boldsymbol{J}_0(\boldsymbol{F}_1(\boldsymbol{v}+\tilde{\boldsymbol{w}})+\boldsymbol{F}_2(\boldsymbol{v}+\tilde{\boldsymbol{w}}))\\ &\qquad -\boldsymbol{J}_0(\boldsymbol{J}_0-\boldsymbol{J}_{a_{\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}},\infty}})+\boldsymbol{J}_0(\boldsymbol{F}_1(\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}})+\boldsymbol{F}_2(\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}}))\big\|\\ &\lesssim \|\boldsymbol{J}_0(\boldsymbol{J}_{a_{\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}}},\infty}-\boldsymbol{J}_{a_{\boldsymbol{v}+\tilde{\boldsymbol{w}},\infty}})\tilde{\boldsymbol{w}}\|+\|\boldsymbol{J}_0(\boldsymbol{J}_{a_{\boldsymbol{v}+\tilde{\boldsymbol{w}},\infty}}\boldsymbol{v}-\boldsymbol{J}_{a_{\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}},\infty}}\boldsymbol{w})\|\\ &\qquad +\|\boldsymbol{J}_0(\boldsymbol{F}_1(\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}})+\boldsymbol{F}_2(\tilde{\boldsymbol{v}})+\tilde{\boldsymbol{w}})\|\\ &\lesssim |a_{\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}},\infty}-a_{\boldsymbol{v}+\tilde{\boldsymbol{w}}}|\|\tilde{\boldsymbol{w}}\|+\|\tilde{\boldsymbol{v}}-\boldsymbol{v}\|+|a_{\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}},\infty}|\|\tilde{\boldsymbol{v}}\|+\delta\|\tilde{\boldsymbol{v}}-\boldsymbol{v}\|\\ &\lesssim \frac{\delta}{C}\|\tilde{\boldsymbol{v}}-\boldsymbol{v}\|+\frac{\delta}{2C}\|\tilde{\boldsymbol{v}}-\boldsymbol{v}\|+\delta\|\tilde{\boldsymbol{v}}-\boldsymbol{v}\|\\ &\lesssim \|\tilde{\boldsymbol{v}}-\boldsymbol{v}\|.\end{aligned}$$

Thereby we obtain the claimed Lipschitz estimate

$$\|\boldsymbol{M}(\boldsymbol{v})-\boldsymbol{M}(\tilde{\boldsymbol{v}})\|\le 2\|\Omega_{\boldsymbol{v}}(\tilde{\boldsymbol{w}})-\Omega(\tilde{\boldsymbol{w}})\|\lesssim\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|.$$

We note that, for $\boldsymbol{u}=0$, the associated (Φ,a) is trivial, i.e., $\Phi=0$ and $a=0$. Thus, we have $\boldsymbol{C}(\boldsymbol{0})=\boldsymbol{F}_1(\boldsymbol{0})+\boldsymbol{F}_2(\boldsymbol{0})=0$. Moreover, $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{w}=\boldsymbol{0}$ if and only if $\boldsymbol{v}=\boldsymbol{w}=\boldsymbol{0}$. Since in this case $\boldsymbol{v}$ satisfies the smallness condition, $\boldsymbol{w}$ solving $\boldsymbol{C}(\boldsymbol{0}+\boldsymbol{w})=\boldsymbol{0}$ is unique; hence $\boldsymbol{M}(\boldsymbol{0})=\boldsymbol{0}$.

Finally, let $\boldsymbol{u}\in\mathcal{H}$ satisfy $\boldsymbol{C}(\boldsymbol{u})=\boldsymbol{0}$. Then, since $1-\boldsymbol{J}_0$ is a bounded operator on $\mathcal{H}$,

$$\|(1-\boldsymbol{J}_0)\boldsymbol{u}\|\lesssim\|\boldsymbol{u}\|.$$

We obtain $\boldsymbol{v}_{\boldsymbol{u}}:=(1-\boldsymbol{J}_0)\boldsymbol{u}\in\ker\boldsymbol{J}_0$ and $\|\boldsymbol{v}_{\boldsymbol{u}}\|\le\delta/(2C)$ for $\|\boldsymbol{u}\|\le\delta/K$ for $K>C$ large enough. Uniqueness yields $\boldsymbol{w}_{\boldsymbol{u}}:=\boldsymbol{J}_0\boldsymbol{u}=\boldsymbol{M}(\boldsymbol{v}_{\boldsymbol{u}})$, and hence $\boldsymbol{u}\in\mathcal{M}_{\delta,C}$. □

Remark 7.5. For each correction term, the same argument yields the existence of Lipschitz manifolds $\mathcal{M}_1,\mathcal{M}_2\subset\mathcal{H}$ of codimensions 10 and 1, respectively, characterized by the vanishing of $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$. In particular, $\mathcal{M}$ can be characterized as a subset of the intersection $\mathcal{M}_1\cap\mathcal{M}_2$ in a small neighborhood around zero.

7D. *Proofs of Propositions 2.1 and 2.2.*

Proof of Proposition 2.1. Let $\Phi_0\in\mathcal{M}_{\delta,C}$, where $\mathcal{M}_{\delta,C}$ is the manifold defined in Proposition 7.4. In particular, $\|\Phi_0\|\le\delta/C$ and $\boldsymbol{C}(\Phi_0)=0$. By Proposition 7.4 there is a pair $(\Phi,a)\in\mathcal{X}_\delta\times X_\delta$ which solves (7-17) with initial data $\boldsymbol{u}=\Phi_0$. Furthermore, after substituting the variation of constants formula

$$\boldsymbol{S}_{a_\infty}(\tau)=\boldsymbol{S}(\tau)+\int_0^\tau\boldsymbol{S}(\tau-\sigma)\boldsymbol{L}'_{a_\infty}\boldsymbol{S}_{a_\infty}(\sigma)\,d\sigma$$

into (7-17), a straightforward calculation yields that $\Psi(\tau):=\boldsymbol{U}_{a(\tau)}+\Phi(\tau)$ satisfies

$$\Psi(\tau)=\boldsymbol{S}(\tau)(\boldsymbol{U}_0+\Phi_0)+\int_0^\tau\boldsymbol{S}(\tau-\sigma)\boldsymbol{F}(\Psi(\sigma))\,d\sigma \tag{7-20}$$

for all $\tau\ge0$. Then, based on (4-3) and (7-2) we infer that

$$\|\Psi(\tau)-\boldsymbol{U}_{a_\infty}\|\le\|\Phi(\tau)\|+\|\boldsymbol{U}_{a(\tau)}-\boldsymbol{U}_{a_\infty}\|\lesssim\delta e^{-\omega\tau}$$

for all $\tau\ge0$, as claimed. □

Proof of Proposition 2.2. Let $\Phi_0 \in \mathcal{M} \cap (C^\infty(\overline{\mathbb{B}}^9) \times C^\infty(\overline{\mathbb{B}}^9))$, and let $\Psi \in C([0,\infty), \mathcal{H})$ be the solution of (7-20) associated to Φ_0 via Proposition 2.1. To prove smoothness of $\Psi(\tau)$ (for fixed τ) we use the representation (7-20). Recall that we defined $\boldsymbol{S}(\tau) := \boldsymbol{S}_k(\tau)$ for $k = \frac{1}{2}(d+1) = 5$ with $(\boldsymbol{S}_k(\tau))_{\tau \geq 0}$ denoting the free wave evolution of Proposition 3.1. Now, using Lemma 3.6, we infer from (7-20) that

$$\begin{aligned}\|\Psi(\tau)\|_{H^6(\mathbb{B}^9)\times H^5(\mathbb{B}^9)} &\lesssim e^{-\frac{\tau}{2}}\|\boldsymbol{U}_0+\Phi_0\|_{H^6(\mathbb{B}^9)\times H^5(\mathbb{B}^9)}+\int_0^\tau e^{-\frac{\tau-\sigma}{2}}\|\boldsymbol{F}(\Psi(\sigma))\|_{H^6(\mathbb{B}^9)\times H^5(\mathbb{B}^9)}\,d\sigma \\ &\lesssim e^{-\frac{\tau}{2}}\|\boldsymbol{U}_0+\Phi_0\|_{H^6(\mathbb{B}^9)\times H^5(\mathbb{B}^9)}+\int_0^\tau e^{-\frac{\tau-\sigma}{2}}\|\Psi(\sigma)\|^2_{H^5(\mathbb{B}^9)\times H^4(\mathbb{B}^9)}\,d\sigma \lesssim 1\end{aligned}$$

for all $\tau \geq 0$. Then inductively, for $k \geq 5$, we get

$$\|\Psi(\tau)\|_{H^k(\mathbb{B}^9)\times H^{k-1}(\mathbb{B}^9)} \lesssim 1$$

for all $\tau \geq 0$. Consequently, by the Sobolev embedding we have $\Psi(\tau) \in C^\infty(\overline{\mathbb{B}}^9) \times C^\infty(\overline{\mathbb{B}}^9)$ for all $\tau \geq 0$.

To get regularity in τ we do the following. First, by local Lipschitz continuity of $\boldsymbol{F} : \mathcal{H}_k \mapsto \mathcal{H}_k$ for every $k \geq 5$ and Gronwall's lemma we get from (7-20) that $\Psi : [0, \mathcal{T}] \mapsto \mathcal{H}_k$ is Lipschitz continuous for every $\mathcal{T} > 0$ and $k \geq 5$. Consequently, $\boldsymbol{F}(\Psi(\cdot)), \boldsymbol{L}\boldsymbol{F}(\Psi(\cdot)) : [0, \mathcal{T}] \mapsto \mathcal{H}_k$ are Lipschitz continuous. The latter is immediate from interpreting $\boldsymbol{L}$ as a map from $\mathcal{H}_k$ to $\mathcal{H}_{k+2}$ and using the Lipschitz continuity of Ψ. Therefore, $\Psi \in C^1([0,\infty), \mathcal{H}_k)$, with

$$\begin{aligned}\partial_\tau \Psi(\tau) &= \boldsymbol{L}\Psi(\tau) + \boldsymbol{F}(\Psi(\tau)) \\ &= \boldsymbol{S}(\tau)\boldsymbol{L}(\boldsymbol{U}_0 + \Phi_0) + \int_0^\tau \boldsymbol{S}(\tau-\sigma)\boldsymbol{L}\boldsymbol{F}(\Psi(\sigma))\,d\sigma + \boldsymbol{F}(\Psi(\tau)) \qquad (7\text{-}21)\end{aligned}$$

for every $\tau \geq 0$; see, e.g., [Pazy 1983, p. 108, Corollary 2.6]. Consequently, by regularity of $\boldsymbol{F}$, $\boldsymbol{F}(\Psi(\cdot)), \boldsymbol{L}^m\boldsymbol{F}(\Psi(\cdot)) \in C^1([0,\infty), \mathcal{H}_k)$ for all $m \geq 0$ and $k \geq 5$. Therefore, from the second equality of (7-21), we get that $\partial_\tau \Psi \in C^1([0,\infty), \mathcal{H}_k)$, with

$$\partial_\tau^2 \Psi(\tau) = \boldsymbol{S}(\tau)\boldsymbol{L}^2(\boldsymbol{U}_0 + \Phi_0) + \int_0^\tau \boldsymbol{S}(\tau-\sigma)\boldsymbol{L}^2\boldsymbol{F}(\Psi(\sigma))\,d\sigma + \boldsymbol{L}\boldsymbol{F}(\Psi(\tau)) + \partial_\tau \boldsymbol{F}(\Psi(\tau)) \qquad (7\text{-}22)$$

for all $\tau \geq 0$. Inductively, we get that $\Psi \in C^m([0,\infty), \mathcal{H}_k)$ for all $m \geq 0$ and $k \geq 5$. In particular, by the Sobolev embedding, $\partial_\tau^m \Psi(\tau) \in C^\infty(\overline{\mathbb{B}}^9) \times C^\infty(\overline{\mathbb{B}}^9)$. Additionally, by the Sobolev embedding $H^k(\mathbb{B}^9) \hookrightarrow L^\infty(\mathbb{B}^9)$ for $k \geq 5$, we get that the derivatives in τ hold pointwise. As a consequence, by (a strong version of) the Schwarz theorem (see, e.g., [Rudin 1976, p. 235, Theorem 9.41]), we get that mixed derivatives of all orders in τ and ξ exist, so $\Psi \in C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z})$, and the first equality of (7-21) holds classically. □

7E. ***Variation of blow-up parameters and proof of Proposition 2.4.*** In this section we prove boundedness and continuity properties of the initial data operator Υ (see (2-6)) which are necessary to establish Proposition 2.4. We assume that $x_0 \in \overline{\mathbb{B}}^9_{1/2}$ and $T \in \left[\frac{1}{2}, \frac{3}{2}\right] =: I$. We also introduce the notation

$$\mathcal{Y} := H^6(\mathbb{B}^9_2) \times H^5(\mathbb{B}^9_2),$$

and denote by $\mathcal{B}_{\mathcal{Y}}$ the unit ball in $\mathcal{Y}$.

Lemma 7.6. *The initial data operator* $\Upsilon : \mathcal{B}_{\mathcal{Y}} \times I \times \overline{\mathbb{B}}^9_{1/2} \to \mathcal{H}$ *is Lipschitz continuous, i.e.,*

$$\|\Upsilon(\boldsymbol{v}, T_1, x_0) - \Upsilon(\boldsymbol{w}, T_2, y_0)\| \lesssim \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathcal{Y}} + |T_1 - T_2| + |x_0 - y_0|$$

for all $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{B}_{\mathcal{Y}}$, *all* $T_1, T_2 \in I$, *and all* $x_0, y_0 \in \overline{\mathbb{B}}^9_{1/2}$. *Furthermore, for* $\delta > 0$ *sufficiently small, we have*

$$\|\Upsilon(\boldsymbol{v}, T, x_0)\| \lesssim \delta$$

for all $\boldsymbol{v} \in \mathcal{Y}$ *with* $\|\boldsymbol{v}\|_{\mathcal{Y}} \leq \delta$, *all* $T \in [1-\delta, 1+\delta] \subset I$, *and all* $x_0 \in \overline{\mathbb{B}}^9_{\delta}$.

Proof. Let $v \in C^\infty(\overline{\mathbb{B}}^9_2)$. Let $T \in \left[\frac{1}{2}, \frac{3}{3}\right]$ and $x_0, y_0 \in \overline{\mathbb{B}}^9_{1/2}$. Then we get by the fundamental theorem of calculus that

$$v(T\xi + x_0) - v(T\xi + y_0) = (x_0^i - y_0^i) \int_0^1 \partial_i v(T\xi + y_0 + s(x_0 - y_0)) \, ds.$$

This implies that $\|v(T \cdot + x_0) - v(T \cdot + y_0)\|_{L^2(\mathbb{B}^9)} \lesssim \|v\|_{H^1(\mathbb{B}^9_2)} |x_0 - y_0|$. The same argument yields, for all $k \in \mathbb{N}$, that

$$\|v(T \cdot + x_0) - v(T \cdot + y_0)\|_{H^k(\mathbb{B}^9)} \lesssim \|v\|_{H^{k+1}(\mathbb{B}^9_2)} |x_0 - y_0|. \tag{7-23}$$

Similarly, we get, for all $T_1, T_2 \in \left[\frac{1}{2}, \frac{3}{2}\right]$ and all $x_0 \in \overline{\mathbb{B}}^9_{1/2}$, that

$$\|v(T_1 \cdot + x_0) - v(T_2 \cdot + x_0)\|_{H^k(\mathbb{B}^9)} \lesssim \|v\|_{H^{k+1}(\mathbb{B}^9_2)} |T_1 - T_2|, \tag{7-24}$$

where $k \in \mathbb{N}$. The estimates (7-23) and (7-24) can be extended to $v \in H^{k+1}(\mathbb{B}^9_2)$ by density. Now let $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{Y}$, $T_1, T_2 \in \left[\frac{1}{2}, \frac{3}{2}\right]$, and $x_0, y_0 \in \overline{\mathbb{B}}^9_{1/2}$. Inequalities (7-23) and (7-24) imply

$$\|\mathcal{R}(\boldsymbol{v}, T_1, x_0) - \mathcal{R}(\boldsymbol{w}, T_2, y_0)\| \lesssim \|v\|_{\mathcal{Y}}(|T_1 - T_2| + |x_0 - y_0|) + \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathcal{Y}}. \tag{7-25}$$

Moreover, since $\boldsymbol{U}_0$ is smooth, we have

$$\|\mathcal{R}(\boldsymbol{U}_0, T_1, x_0) - \mathcal{R}(\boldsymbol{U}_0, T_2, y_0)\| \lesssim |T_1 - T_2| + |x_0 - y_0| \tag{7-26}$$

for all $T_1, T_2 \in \left[\frac{1}{2}, \frac{3}{2}\right]$ and $x_0, y_0 \in \overline{\mathbb{B}}^9_{1/2}$. Now the inequalities (7-25) and (7-26) imply the first part of the statement. The same inequalities imply

$$\|\Upsilon(\boldsymbol{v}, T, x_0)\| \lesssim \|\boldsymbol{v}\|_{\mathcal{Y}} + |T - 1| + |x_0|,$$

which proves the second part of the statement. ☐

We have the following result, which has Proposition 2.4 as a direct consequence. To shorten the notation, we write $\boldsymbol{h} := \boldsymbol{h}_0$.

Lemma 7.7. *There exists* $M > 0$ *such that, for all sufficiently small* $\delta > 0$, *the following holds*: *For every real-valued* $\boldsymbol{v} \in \mathcal{Y}$ *that satisfies* $\|\boldsymbol{v}\|_{\mathcal{Y}} \leq \delta/M^2$, *there exist* $\Phi \in \mathcal{X}_\delta$, $a \in X_\delta$, *and parameters* $\alpha \in [-\delta/M, \delta/M]$, $T \in [1 - \delta/M, 1 + \delta/M] \subset \left[\frac{1}{2}, \frac{3}{2}\right]$, *and* $x_0 \in \overline{\mathbb{B}}^9_{\delta/M} \subset \overline{\mathbb{B}}^9_{1/2}$ *such that*

$$\boldsymbol{C}(\Phi, a, \Upsilon(\boldsymbol{v} + \alpha \boldsymbol{h}_0, T, x_0)) = 0. \tag{7-27}$$

Moreover, the parameters depend Lipschitz continuously on the data, i.e.,

$$|\alpha(\boldsymbol{v})-\alpha(\boldsymbol{w})|+|T(\boldsymbol{v})-T(\boldsymbol{w})|+|x_0(\boldsymbol{v})-x_0(\boldsymbol{w})| \lesssim \|\boldsymbol{v}-\boldsymbol{w}\|_{\mathcal{Y}}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{Y}$ satisfying the above assumptions. In particular, $\Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h}, T, x_0) \in \mathcal{M}_{\delta,\boldsymbol{C}}$.

Proof. Fix constants $C > 0$ and $K > 0$ from Proposition 7.4. By Lemma 7.6, we have that, for all $M > 0$ large enough and all $\delta > 0$ small enough, the inequality

$$\|\Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h}, T, x_0)\| \le \frac{\delta}{K} \tag{7-28}$$

holds for every $\|\boldsymbol{v}\|_{\mathcal{Y}} \le \delta/M$, $\alpha \in [-\delta/M, \delta/M]$, $T \in [1-\delta/M, 1+\delta/M]$, and $x_0 \in \overline{\mathbb{B}}^9_{\delta/M}$. Furthermore, in view of (7-28) and Proposition 7.3, we get that, given $\|\boldsymbol{v}\|_{\mathcal{Y}} \le \delta/M^2$, for every $\alpha \in [-\delta/M, \delta/M]$, $T \in [1-\delta/M, 1+\delta/M]$, and $x_0 \in \overline{\mathbb{B}}^9_{\delta/M}$, there are functions

$$\Phi = \Phi(\boldsymbol{v}+\alpha\boldsymbol{h}, T, x_0) \quad \text{and} \quad a = a(\boldsymbol{v}+\alpha\boldsymbol{h}, T, x_0)$$

which solve the modified integral equation

$$\Phi(\tau) = \boldsymbol{S}_{a_\infty}(\tau)(\Upsilon(\boldsymbol{v}, T, x_0) - \boldsymbol{C}(\Phi, a, \Upsilon(\boldsymbol{v}, T, x_0))) + \int_0^\tau \boldsymbol{S}_{a_\infty}(\tau-\sigma)(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\, d\sigma \tag{7-29}$$

for all $\tau \ge 0$. For such Φ and a, we show that one can associate to any $\|\boldsymbol{v}\|_{\mathcal{Y}} \le \delta/M^2$ suitable parameters T, x_0, and α such that (7-27) holds. From this, via Proposition 7.4, we conclude that $\Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h}, T, x_0) \in \mathcal{M}_{\delta,\boldsymbol{C}}$. Recall that the correction terms can be written as $\boldsymbol{C} = \boldsymbol{C}_1 + \boldsymbol{C}_2 = \sum_{k=0}^{9} \boldsymbol{C}_1^k + \boldsymbol{C}_2$, where

$$\boldsymbol{C}_1^k(\Phi, a, \boldsymbol{u}) = \boldsymbol{P}^{(k)}_{a_\infty}\boldsymbol{u} + \boldsymbol{P}^{(k)}_{a_\infty}\boldsymbol{I}_1(\Phi, a) \quad \text{and} \quad \boldsymbol{C}_2(\Phi, a, \boldsymbol{u}) = \boldsymbol{H}_{a_\infty}\boldsymbol{u} + \boldsymbol{H}_{a_\infty}\boldsymbol{I}_2(\Phi, a),$$

and where the integrals are denoted by

$$\boldsymbol{I}_1(\Phi, a) = \int_0^\infty e^{-\sigma}(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\, d\sigma,$$

$$\boldsymbol{I}_2(\Phi, a) = \int_0^\infty e^{-3\sigma}(\boldsymbol{G}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{U}_{a(\sigma)})\, d\sigma,$$

and we have

$$\|\boldsymbol{P}^{(k)}_{a_\infty}\boldsymbol{I}_1(\Phi, a)\| \lesssim \delta^2 \quad \text{and} \quad \|\boldsymbol{H}_{a_\infty}\boldsymbol{I}_2(\Phi, a)\| \lesssim \delta^2; \tag{7-30}$$

see (7-18). We will show that there are parameters T, α, and x_0 such that, for $k = 0, \dots, 9$,

$$(\boldsymbol{C}_1^k(\Phi, a, \Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h}, T, x_0)) \,|\, \boldsymbol{g}^{(k)}_{a_\infty}) = 0 \quad \text{and} \quad (\boldsymbol{C}_2(\Phi, a, \Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h}, T, x_0)) \,|\, \boldsymbol{h}_{a_\infty}) = 0, \tag{7-31}$$

which implies (7-27). To this end we expand the initial data operator. First, by Taylor expansion we get, for $T \in [1-\delta/M, 1+\delta/M]$ and $x_0 \in \overline{\mathbb{B}}^9_{\delta/M}$,

$$\mathcal{R}(\boldsymbol{U}_0, T, x_0) - \mathcal{R}(\boldsymbol{U}_0, 1, 0) = c_0(T-1)\boldsymbol{g}_0^{(0)} + \sum_{j=1}^{9} c_j x_0^j \boldsymbol{g}_0^{(j)} + r(T, x_0),$$

where the remainder satisfies

$$\|r(T, x_0) - r(\widetilde{T}, \tilde{x}_0)\| \lesssim \delta(|T-\widetilde{T}| + |x_0 - \tilde{x}_0|).$$

Hence we obtain

$$\Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h},T,x_0)=\mathcal{R}(\boldsymbol{v}+\alpha\boldsymbol{h},T,x_0)+c_0(T-1)\boldsymbol{g}^{(0)}_{a_\infty}+\sum_{j=1}^{9}c_jx_0^j\boldsymbol{g}^{(j)}_{a_\infty}+r_{a_\infty}(T,x_0),$$

where

$$r_{a_\infty}(T,x_0)=c_0(T-1)(\boldsymbol{g}^{(0)}_0-\boldsymbol{g}^{(0)}_{a_\infty})+\sum_{j=1}^{9}c_jx_0^j(\boldsymbol{g}^{(j)}_0-\boldsymbol{g}^{(j)}_{a_\infty})+r(T,x_0).$$

It is straightforward to check that

$$\|r_a(T,x_0)-r_b(\widetilde{T},\tilde{x}_0)\|\lesssim\delta(|a-b|+|T-\widetilde{T}|+|x_0-\tilde{x}_0|) \tag{7-32}$$

for all $a,b\in\mathbb{B}^9_\delta$, $T,\widetilde{T}\in[1-\delta/M,1+\delta/M]$, and $x_0,\tilde{x}_0\in\overline{\mathbb{B}}^9_{\delta/M}$. We now write

$$\mathcal{R}(\boldsymbol{v}+\alpha\boldsymbol{h},T,x_0)=\mathcal{R}(\boldsymbol{v},T,x_0)+\alpha\mathcal{R}(\boldsymbol{h}_{a_\infty},T,x_0)+\alpha\mathcal{R}(\boldsymbol{h}-\boldsymbol{h}_{a_\infty},T,x_0).$$

The last term can be estimated by

$$\|\mathcal{R}(\boldsymbol{h}-\boldsymbol{h}_{a_\infty},T,x_0)\|\lesssim|a_\infty|.$$

By taking the Taylor expansion of $\mathcal{R}(\boldsymbol{h}_{a_\infty},T,x_0)$ at $(T,x_0)=(1,0)$, we obtain

$$\mathcal{R}(\boldsymbol{v}+\alpha\boldsymbol{h},T,x_0)=\mathcal{R}(\boldsymbol{v},T,x_0)+\alpha\boldsymbol{h}_{a_\infty}+\alpha\tilde{r}_a(T,x_0),$$

where the remainder satisfies

$$\|\tilde{r}_a(T,x_0)-\tilde{r}_b(\widetilde{T},\tilde{x}_0)\|\lesssim|a-b|+|T-\widetilde{T}|+|x_0-\tilde{x}_0|. \tag{7-33}$$

Hence we obtain the expansion

$$\Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h},T,x_0)=\mathcal{R}(\boldsymbol{v},T,x_0)+\alpha\boldsymbol{h}_{a_\infty}+c_0(T-1)\boldsymbol{g}^{(0)}_{a_\infty}+\sum_{j=1}^{9}c_jx_0^j\boldsymbol{g}^{(j)}_{a_\infty}+r_{a_\infty}(T,x_0)+\alpha\tilde{r}_{a_\infty}(T,x_0).$$

By applying the projections to the initial data operator we get

$$\begin{aligned}
\boldsymbol{P}^{(0)}_{a_\infty}\Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h},T,x_0)&=\boldsymbol{P}^{(0)}_{a_\infty}\mathcal{R}(\boldsymbol{v},T,x_0)+c_0(T-1)\boldsymbol{g}^{(0)}_{a_\infty}+\boldsymbol{P}^{(0)}_{a_\infty}r_{a_\infty}(T,x_0)+\alpha\boldsymbol{P}^{(0)}_{a_\infty}\tilde{r}_{a_\infty}(T,x_0),\\
\boldsymbol{P}^{(j)}_{a_\infty}\Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h},T,x_0)&=\boldsymbol{P}^{(j)}_{a_\infty}\mathcal{R}(\boldsymbol{v},T,x_0)+c_jx_0^j\boldsymbol{g}^{(j)}_{a_\infty}+\boldsymbol{P}^{(j)}_{a_\infty}r_{a_\infty}(T,x_0)+\alpha\boldsymbol{P}^{(j)}_{a_\infty}\tilde{r}_{a_\infty}(T,x_0),\\
\boldsymbol{H}_{a_\infty}\Upsilon(\boldsymbol{v}+\alpha\boldsymbol{h},T,x_0)&=\boldsymbol{H}_{a_\infty}\mathcal{R}(\boldsymbol{v},T,x_0)+\alpha\boldsymbol{h}_{a_\infty}+\boldsymbol{H}_{a_\infty}r_{a_\infty}(T,x_0)+\alpha\boldsymbol{H}_{a_\infty}\tilde{r}_{a_\infty}(T,x_0).
\end{aligned}$$

Hence, by introducing the notation $\beta=T-1$, we define, for $k=0,\dots,9$,

$$\begin{aligned}
\boldsymbol{\Gamma}^{(k)}_{\boldsymbol{v}}(\alpha,\beta,x_0)&=\boldsymbol{P}^{(k)}_{a_\infty}\mathcal{R}(\boldsymbol{v},\beta+1,x_0)+\boldsymbol{P}^{(k)}_{a_\infty}r_{a_\infty}(\beta,x_0)+\alpha\boldsymbol{P}^{(k)}_{a_\infty}\tilde{r}_{a_\infty}(\beta,x_0)+\boldsymbol{P}^{(k)}_{a_\infty}\boldsymbol{I}_1(\alpha,\beta,x_0),\\
\boldsymbol{\Gamma}^{(10)}_{\boldsymbol{v}}(\alpha,\beta,x_0)&=\boldsymbol{H}_{a_\infty}\mathcal{R}(\boldsymbol{v},\beta+1,x_0)+\boldsymbol{H}_{a_\infty}r_{a_\infty}(\beta,x_0)+\alpha\boldsymbol{H}_{a_\infty}\tilde{r}_{a_\infty}(\beta,x_0)+\boldsymbol{H}_{a_\infty}\boldsymbol{I}_2(\alpha,\beta,x_0).
\end{aligned}$$

Using this notation we can rewrite (7-31) as

$$\begin{aligned}
\beta&=\Gamma^{(0)}_{\boldsymbol{v}}(\alpha,\beta,x_0):=\tilde{c}_0(\boldsymbol{\Gamma}^{(0)}_{\boldsymbol{v}}(\alpha,\beta,x_0)\,|\,\boldsymbol{g}^{(0)}_{a_\infty}),\\
x_0^j&=\Gamma^{(j)}_{\boldsymbol{v}}(\alpha,\beta,x_0):=\tilde{c}_j(\boldsymbol{\Gamma}^{(j)}_{\boldsymbol{v}}(\alpha,\beta,x_0)\,|\,\boldsymbol{g}^{(j)}_{a_\infty}),\\
\alpha&=\Gamma^{(10)}_{\boldsymbol{v}}(\alpha,\beta,x_0):=\tilde{c}_{10}(\boldsymbol{\Gamma}^{(10)}_{\boldsymbol{v}}(\alpha,\beta,x_0)\,|\,\boldsymbol{h}_{a_\infty})
\end{aligned} \tag{7-34}$$

for $j = 1, \dots, 9$ and some constants $\tilde{c}_0, \tilde{c}_j, \tilde{c}_{10} \in \mathbb{R}$. We will show that $\Gamma_{\boldsymbol{v}} = (\Gamma_{\boldsymbol{v}}^{(0)}, \dots, \Gamma_{\boldsymbol{v}}^{(10)})$ is a contraction on $\overline{\mathbb{B}}^{11}_{\delta/M}$ for sufficiently small $\delta > 0$ and for sufficiently large $M > 0$. Thereby the first part of the statement follows by Banach's fixed-point theorem.

First we observe that $\Gamma_{\boldsymbol{v}}$ maps $\overline{\mathbb{B}}^{11}_{\delta/M}$ into itself. Indeed, by the proof of Lemma 7.6, we know that $\|\mathcal{R}(\boldsymbol{v}, 1+\beta, x_0)\| \lesssim \|\boldsymbol{v}\|_{\mathcal{Y}}$. Now estimates (7-32)–(7-33), and the integral estimates (7-30) imply

$$\Gamma_{\boldsymbol{v}}^{(j)}(\alpha, \beta, x_0) = O\left(\frac{\delta}{M^2}\right) + O(\delta^2)$$

for all $j = 0, \dots, 10$. Thus, there is a choice of large enough $M > 0$ such that, for all sufficiently small $\delta > 0$, the inequality

$$|\Gamma_{\boldsymbol{v}}(\alpha, \beta, x_0)| \le \frac{\delta}{M}$$

holds for all $(\alpha, \beta, x_0) \in \overline{\mathbb{B}}^{11}_{\delta/M}$. Next we show that by restricting, if necessary, to even smaller $\delta > 0$, the operator $\Gamma_{\boldsymbol{v}}$ is a contraction on $\overline{\mathbb{B}}^{11}_{\delta/M}$. Let $(\Phi, a) \in \mathcal{X}_\delta \times X_\delta$ be the functions solving (7-29) corresponding to parameters $\boldsymbol{v} + \alpha \boldsymbol{h}$, $T = 1+\beta$, and x_0. Furthermore, let $(\Psi, b) \in \mathcal{X}_\delta \times X_\delta$ be the functions corresponding to $\boldsymbol{v} + \tilde{\alpha}\boldsymbol{h}$, $\widetilde{T} = 1 + \tilde{\beta}$, and $\tilde{x}_0$. Then, we obtain

$$\|\Phi - \Psi\|_{\mathcal{X}} + \|a - b\|_X \lesssim \|\Upsilon(\boldsymbol{v} + \alpha \boldsymbol{h}, T, x_0) - \Upsilon(\boldsymbol{v} + \tilde{\alpha}\boldsymbol{h}, \widetilde{T}, \tilde{x}_0)\| \lesssim |\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|.$$

Hence, by Lemma 7.1, we get, for $k = 0, \dots, 9$, that

$$\|\boldsymbol{P}^{(k)}_{a_\infty} \boldsymbol{I}_1(\Phi, a) - \boldsymbol{P}^{(k)}_{b_\infty} \boldsymbol{I}_1(\Psi, b)\| \lesssim \delta(\|\Phi - \Psi\|_{\mathcal{X}} + \|a - b\|_X) \lesssim \delta(|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|)$$

and

$$\|\boldsymbol{H}_{a_\infty} \boldsymbol{I}_2(\Phi, a) - \boldsymbol{H}_{b_\infty} \boldsymbol{I}_2(\Psi, b)\| \lesssim \delta(|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|).$$

Furthermore, by (7-25) and the Lipschitz continuity of the Riesz projections $\boldsymbol{P}^{(k)}_a$ and $\boldsymbol{H}_a$, we obtain

$$\begin{aligned}\|\boldsymbol{P}^{(k)}_{a_\infty} \mathcal{R}(\boldsymbol{v}, T, x_0) - \boldsymbol{P}^{(k)}_{b_\infty} \mathcal{R}(\boldsymbol{v}, \widetilde{T}, \tilde{x}_0)\| + \|\boldsymbol{H}_{a_\infty} \mathcal{R}(\boldsymbol{v}, T, x_0) - \boldsymbol{H}_{b_\infty} \mathcal{R}(\boldsymbol{v}, \widetilde{T}, \tilde{x}_0)\| \\ \lesssim \|\boldsymbol{v}\|_{\mathcal{Y}}(\|a - b\|_X + |T - \widetilde{T}| + |x_0 - \tilde{x}_0|) \lesssim \delta(|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|).\end{aligned}$$

Moreover, for $k = 0, \dots, 9$, we have

$$\begin{aligned}\|\boldsymbol{P}^{(k)}_{a_\infty} r_{a_\infty}(T, x_0) - \boldsymbol{P}^{(k)}_{b_\infty} r_{b_\infty}(\widetilde{T}, \tilde{x}_0)\| + \|\alpha \boldsymbol{P}^{(k)}_{a_\infty} \tilde{r}_{a_\infty}(T, x_0) - \tilde{\alpha} \boldsymbol{P}^{(k)}_{b_\infty} \tilde{r}_{b_\infty}(\widetilde{T}, \tilde{x}_0)\| \\ \lesssim \delta(|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|)\end{aligned}$$

and

$$\begin{aligned}\|\boldsymbol{H}_{a_\infty} r_{a_\infty}(T, x_0) - \boldsymbol{H}_{b_\infty} r_{b_\infty}(\widetilde{T}, \tilde{x}_0)\| + \|\alpha \boldsymbol{H}_{a_\infty} \tilde{r}_{a_\infty}(T, x_0) - \tilde{\alpha} \boldsymbol{H}_{b_\infty} \tilde{r}_{b_\infty}(\widetilde{T}, \tilde{x}_0)\| \\ \lesssim \delta(|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|).\end{aligned}$$

From these estimates we infer that

$$\|\Gamma_{\boldsymbol{v}}^{(j)}(\alpha, \beta, x_0) - \Gamma_{\boldsymbol{v}}^{(j)}(\tilde{\alpha}, \tilde{\beta}, \tilde{x}_0)\| \lesssim \delta(|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|) \tag{7-35}$$

for $j = 0, \dots, 10$. Therefore, $\Gamma_{\boldsymbol{v}}$ is a contraction for all small enough $\delta > 0$, and this concludes the proof of the first part of the statement.

It remains to establish the Lipschitz continuity of the parameters with respect to the initial data. Let $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{Y}$ satisfy the smallness condition and let (α, β, x_0) and $(\tilde{\alpha}, \tilde{\beta}, \tilde{x}_0)$ be the corresponding set of parameters. The first line in (7-34) implies

$$\begin{aligned}|\beta - \tilde{\beta}| &= |\Gamma_{\boldsymbol{v}}^{(0)}(\alpha, \beta, x_0) - \Gamma_{\boldsymbol{w}}^{(0)}(\tilde{\alpha}, \tilde{\beta}, \tilde{x}_0)| \\ &\lesssim |\Gamma_{\boldsymbol{v}}^{(0)}(\alpha, \beta, x_0) - \Gamma_{\boldsymbol{w}}^{(0)}(\alpha, \beta, x_0)| + |\Gamma_{\boldsymbol{w}}^{(0)}(\alpha, \beta, x_0) - \Gamma_{\boldsymbol{w}}^{(0)}(\tilde{\alpha}, \tilde{\beta}, \tilde{x}_0)|.\end{aligned}$$

The second term can be estimated with (7-35). To estimate the first term, we use the Lipschitz continuity of the Riesz projections to get

$$\begin{aligned}&\|\boldsymbol{P}^{(0)}_{a_\infty(\boldsymbol{v},\beta,x_0)}\mathcal{R}(\boldsymbol{v}, 1+\beta, x_0) - \boldsymbol{P}^{(0)}_{a_\infty(\boldsymbol{w},\beta,x_0)}\mathcal{R}(\boldsymbol{w}, 1+\beta, x_0)\| \\ &\qquad\lesssim \|\boldsymbol{v}\|_{\mathcal{Y}} \|a_\infty(\boldsymbol{v}, \beta, x_0) - a_\infty(\boldsymbol{w}, \beta, x_0)\|_X + \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathcal{Y}} \lesssim \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathcal{Y}}.\end{aligned}$$

Similar estimates using (7-32)–(7-33) and Lemma 7.1 yield

$$|\Gamma_{\boldsymbol{v}}^{(0)}(\alpha, \beta, x_0) - \Gamma_{\boldsymbol{w}}^{(0)}(\alpha, \beta, x_0)| \lesssim \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathcal{Y}}.$$

In summary, we obtain

$$|\beta - \tilde{\beta}| \lesssim \delta(|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|) + \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathcal{Y}},$$

and similar estimates for the remaining components yield

$$|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0| \lesssim \delta(|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |x_0 - \tilde{x}_0|) + \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathcal{Y}},$$

which concludes the proof. □

7F. ***Proof of Theorem 1.1.*** Let $M > 0$ be from Proposition 2.4. For $\delta > 0$ define $\delta' := \delta/M$. Then consider $(f, g) \in C^\infty(\overline{\mathbb{B}}^9_2) \times C^\infty(\overline{\mathbb{B}}^9_2)$ satisfying

$$\|(f, g)\|_{H^6(\mathbb{B}^9_2) \times H^5(\mathbb{B}^9_2)} \leq \frac{\delta'}{M} = \frac{\delta}{M^2}.$$

By Propositions 2.4 and 2.1, we have that, for all $\delta > 0$ sufficiently small, there exist $a \in \overline{\mathbb{B}}^9_{M\delta'/\omega}$, $T \in [1-\delta', 1+\delta']$, $x_0 \in \overline{\mathbb{B}}^9_{\delta'}$, and $\alpha \in [-\delta', \delta']$, depending Lipschitz continuously on (f, g) with respect to the norm on $\mathcal{Y}$, as well a function $\Psi \in C([0, \infty), \mathcal{H})$ that solves

$$\Psi(\tau) = \boldsymbol{S}(\tau)[\boldsymbol{U}_0 + \Upsilon((f, g) + \alpha \boldsymbol{h}, T, x_0)] + \int_0^\tau \boldsymbol{S}(\tau - \sigma)\boldsymbol{F}(\Psi(\sigma))\, d\sigma, \tag{7-36}$$

and obeys the estimate

$$\|\Psi(\tau) - \boldsymbol{U}_a\| \lesssim \delta e^{-\omega\tau} \tag{7-37}$$

for all $\tau \geq 0$. By standard arguments, Ψ is the unique solution to (7-36) in $C([0, \infty), \mathcal{H})$. Now, from the smoothness of f and g, we have that the initial data $\Psi(0) = \boldsymbol{U}_0 + \Upsilon((f, g) + \alpha \boldsymbol{h}, T, x_0)$ belongs to $C^\infty(\overline{\mathbb{B}}^9_2) \times C^\infty(\overline{\mathbb{B}}^9_2)$, and therefore from Proposition 2.2 we infer that Ψ is smooth and solves (2-3) classically. More precisely, by writing $\Psi(\tau) = (\psi_1(\tau,\cdot), \psi_2(\tau,\cdot))$, we have that $\psi_j \in C^\infty(\mathcal{Z})$ for $j = 1, 2$ and

$$\begin{aligned}\partial_\tau \psi_1(\tau,\xi) &= -\xi\cdot\nabla\psi_1(\tau,\xi) - 2\psi_1(\tau,\xi) + \psi_2(\tau,\xi),\\ \partial_\tau \psi_2(\tau,\xi) &= \Delta\psi_1(\tau,\xi) - \xi\cdot\nabla\psi_2(\tau,\xi) - 3\psi_2(\tau,\xi), + \psi_1(\tau,\xi)^2\end{aligned}$$

for $(\tau,\xi) \in \mathcal{Z}$, with

$$\begin{aligned}\psi_1(0,\cdot) &= T^2[\boldsymbol{U}_0]_1(T\cdot + x_0) + T^2 f(T\cdot + x_0) + \alpha T^2 h_1(T\cdot + x_0),\\ \psi_2(0,\cdot) &= T^3[\boldsymbol{U}_0]_2(T\cdot + x_0) + T^3 g(T\cdot + x_0) + \alpha T^3 h_2(T\cdot + x_0).\end{aligned}$$

Furthermore, by writing $\Phi(\tau) = \Psi(\tau) - \boldsymbol{U}_a$, where $\Phi(\tau) = (\varphi_1(\tau,\cdot), \varphi_2(\tau,\cdot))$, from (7-37) we have

$$\|\varphi_1(\tau,\cdot)\|_{H^5(\mathbb{B}^9)} \lesssim \delta e^{-\omega\tau} \quad \text{and} \quad \|\varphi_2(\tau,\cdot)\|_{H^4(\mathbb{B}^9)} \lesssim \delta e^{-\omega\tau} \tag{7-38}$$

for all $\tau \geq 0$. Furthermore, by the Sobolev embedding we have, for the first component, that

$$\|\varphi_1(\tau,\cdot)\|_{L^\infty(\mathbb{B}^9)} \lesssim \delta e^{-\omega\tau} \tag{7-39}$$

for all $\tau \geq 0$. Now, we translate these results back to physical coordinates and let

$$u(t,x) = \frac{1}{(T-t)^2}\psi_1\left(-\log(T-t) + \log T, \frac{x-x_0}{T-t}\right).$$

Based on the smoothness properties of ψ_1, we conclude that $u \in C^\infty(\mathcal{C}_{T,x_0})$. Furthermore, u solves

$$(\partial_t^2 - \Delta_x)u(t,x) = u(t,x)^2$$

on $\mathcal{C}_{T,x_0}$ and satisfies

$$u(0,\cdot) = U(|\cdot|) + f + \alpha h_1, \quad \partial_t u(0,\cdot) = 2U(|\cdot|) + |\cdot|U'(|\cdot|) + g + \alpha h_2$$

on $\overline{\mathbb{B}}^9_T(x_0)$. Uniqueness of u follows from uniqueness of Ψ, though it also follows by standard results concerning wave equations in physical coordinates. Furthermore,

$$u(t,x) = \frac{1}{(T-t)^2}\left[U_a\left(\frac{x-x_0}{T-t}\right) + \varphi(t,x)\right],$$

with $\varphi(t,x) := \varphi_1(-\log(T-t) + \log T, (x-x_0)/(T-t))$. The bound (7-39) yields

$$\|\varphi(t,\cdot)\|_{L^\infty(\mathbb{B}^9_{T-t}(x_0))} = \|\varphi_1(-\log(T-t) + \log T,\cdot)\|_{L^\infty(\mathbb{B}^9)} \lesssim \delta(T-t)^\omega \tag{7-40}$$

for all $t \in [0,T)$. Furthermore, by (7-38),

$$(T-t)^{k-\frac{9}{2}}\|\varphi(t,\cdot)\|_{\dot H^k(\mathbb{B}^9_{T-t}(x_0))} = \|\varphi_1(-\log(T-t) + \log T,\cdot)\|_{\dot H^k(\mathbb{B}^9)} \lesssim (T-t)^\omega$$

for $k = 0,\dots,5$, which implies the first line in (1-17). The second line follows also from (7-38) and the fact that

$$\partial_t u(t,x) = \frac{1}{(T-t)^3}\psi_2\left(-\log(T-t) + \log T, \frac{x-x_0}{T-t}\right).$$

Relabelling δ' with δ concludes the proof of Theorem 1.1 for (1-1). Now, let c_0 be the constant from (1-13). Recall that the above conclusions hold for all sufficiently small $\delta > 0$. Therefore, from (7-40) we see that we can choose small enough $\delta > 0$ so as to ensure

$$\|\varphi(t,\cdot)\|_{L^\infty(\mathbb{B}^9_{T-t}(x_0))} \leq \tfrac{1}{2}c_0$$

for all $t \in [0, T)$. As a consequence, u is strictly positive on $\mathcal{C}_{T,x_0}$ and therefore provides a solution to (1-6) as well. ☐

8. Proof of Theorem 1.6: stable ODE blowup

The proof of Theorem 1.6 follows mutatis mutandis the proof of Theorem 1.1. However, for the convenience of the reader we outline the most important steps and stick to the notation introduced above. Starting with (2-3), we consider solutions of the form $\Psi(\tau) = \boldsymbol{\kappa}_{a(\tau)} + \Phi(\tau)$, which yields

$$\partial_\tau \Phi(\tau) = [\boldsymbol{L} + \boldsymbol{L}'_{\kappa_{a_\infty}}]\Phi(\tau) + \widetilde{\boldsymbol{G}}_{a(\tau)}(\Phi(\tau)) - \partial_\tau \boldsymbol{\kappa}_{a(\tau)}, \tag{8-1}$$

where

$$\boldsymbol{L}'_{\kappa_a}\boldsymbol{u}(\xi) = \begin{pmatrix} 0 \\ 2\kappa_a(\xi)u_1(\xi) \end{pmatrix}$$

and

$$\widetilde{\boldsymbol{G}}_{a(\tau)}(\Phi(\tau)) = [\boldsymbol{L}'_{\kappa_{a(\tau)}} - \boldsymbol{L}'_{\kappa_{a_\infty}}]\Phi(\tau) + \boldsymbol{F}(\Phi(\tau)).$$

In this equation, $\boldsymbol{L} : \mathcal{D}(\boldsymbol{L}) \subset \mathcal{H} \to \mathcal{H}$ denotes, as usual, the operator describing the free wave evolution. This is fully characterized for both $d = 7$ and $d = 9$ in Section 3; recall that $\mathcal{H} := \mathcal{H}_k$ for $k = \frac{1}{2}(d+1)$. For the perturbation theory, the spectral analysis is crucial. Once this is obtained, most results are purely abstract and the proofs can be adapted from previous sections.

Since $\boldsymbol{L}'_{\kappa_a}$ is compact and depends Lipschitz continuously on a, the results of Section 4 apply. In particular, for small enough a, the spectrum of $\boldsymbol{L} + \boldsymbol{L}'_{\kappa_a}$ in the right half-plane consists of isolated eigenvalues confined to a compact region. Furthermore, an analogous result to Proposition 4.5 holds with V replaced by a constant. This substantially simplifies the spectral analysis and with the above prerequisites it is easy to derive the following statement. For all of the ensuing statements, $d \in \{7, 9\}$.

Proposition 8.1. *There are constants $\delta^* > 0$ and $\omega > 0$ such that the following holds: For any $a \in \overline{\mathbb{B}}^d_{\delta^*}$, the operator $\boldsymbol{L} + \boldsymbol{L}'_{\kappa_a} : \mathcal{D}(\boldsymbol{L}) \subset \mathcal{H} \to \mathcal{H}$ generates a strongly continuous semigroup $(\boldsymbol{S}_{\kappa_a}(\tau))_{\tau\geq 0}$ on $\mathcal{H}$. Furthermore, there exist projections $\widetilde{\boldsymbol{P}}_a, \widetilde{\boldsymbol{Q}}^{(k)}_a \in \mathcal{B}(\mathcal{H})$, $k = 1, \dots, d$, of rank 1 that are mutually transversal and depend Lipschitz continuously on a. Furthermore, they commute with $\boldsymbol{S}_{\kappa_a}(\tau)$ and, for all $\boldsymbol{u} \in \mathcal{H}$ and $\tau \geq 0$,*

$$\boldsymbol{S}_{\kappa_a}(\tau)\widetilde{\boldsymbol{P}}_a\boldsymbol{u} = e^\tau \boldsymbol{u} \quad \text{and} \quad \boldsymbol{S}_{\kappa_a}(\tau)\widetilde{\boldsymbol{Q}}^{(k)}_a\boldsymbol{u} = \boldsymbol{u},$$

as well as

$$\|\boldsymbol{S}_{\kappa_a}(\tau)[1 - \widetilde{\boldsymbol{P}}_a - \widetilde{\boldsymbol{Q}}_a]\boldsymbol{u}\| \lesssim e^{-\omega\tau}\|[1 - \widetilde{\boldsymbol{P}}_a - \widetilde{\boldsymbol{Q}}_a]\boldsymbol{u}\|$$

with $\widetilde{\boldsymbol{Q}}_a = \sum_{k=1}^d \widetilde{\boldsymbol{Q}}^{(k)}_a$. Moreover,

$$\|\boldsymbol{S}_{\kappa_a}(\tau)[1 - \widetilde{\boldsymbol{P}}_a - \widetilde{\boldsymbol{Q}}_a] - \boldsymbol{S}_{\kappa_b}(\tau)[1 - \widetilde{\boldsymbol{P}}_b - \widetilde{\boldsymbol{Q}}_b]\| \lesssim e^{-\omega\tau}|a - b| \tag{8-2}$$

for all $a, b \in \overline{\mathbb{B}}^9_{\delta^*}$ *and* $\tau \geq 0$. *Also,*

$$\operatorname{ran} \widetilde{\boldsymbol{P}}_a = \operatorname{span}(\tilde{\boldsymbol{g}}_a) \quad \textit{and} \quad \operatorname{ran} \widetilde{\boldsymbol{Q}}_a^{(k)} = \operatorname{span}(\tilde{\boldsymbol{q}}_a^{(k)}), \tag{8-3}$$

where

$$\tilde{\boldsymbol{g}}_a(\xi) = \begin{pmatrix} A_0(a)[A_0(a) - A_j \xi^j]^{-3} \\ 3A_0(a)^2 [A_0(a) - A_j \xi^j]^{-4} \end{pmatrix} \quad \textit{and} \quad \tilde{\boldsymbol{q}}_a^{(k)} = \partial_{a_k} \boldsymbol{\kappa}_a \tag{8-4}$$

for $k = 1, \dots, d$.

Proof. We only sketch the main steps of the proof, since many parts are abstract operator theory and can be copied verbatim from previous sections.

The results of Section 3 together with the bounded perturbation theorem immediately imply that $\boldsymbol{L} + \boldsymbol{L}'_{\kappa_a}$ generates a strongly continuous semigroup, which we denote by $(\boldsymbol{S}_{\kappa_a}(\tau))_{\tau \geq 0}$. Furthermore, the results of Propositions 4.3 and 4.5 hold in particular for our case at hand, and we infer that, for $\operatorname{Re} \lambda > -\frac{1}{2}$, the spectrum of $\boldsymbol{L} + \boldsymbol{L}'_{\kappa_a}$ consists of isolated eigenvalues confined to a compact region. For $a = 0$, Proposition 4.5 holds mutatis mutandis with V replaced by the constant potential $2\kappa_0 = 12$. In this case, in the spectral ODE the number of regular singular points can be reduced to three, and we can therefore resolve the connection problem by using the standard theory of hypergeometric equations. This is outlined in the following, where we show that there exists $0 < \mu_0 \leq \frac{1}{2}$ such that

$$\sigma(\boldsymbol{L} + \boldsymbol{L}'_{\kappa_0}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\mu_0\} \cup \{0, 1\}.$$

In fact, we convince ourselves that

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\} \setminus \{0, 1\} \subset \rho(\boldsymbol{L} + \boldsymbol{L}'_{\kappa_0}). \tag{8-5}$$

We argue by contradiction. Let us assume that $\lambda \in \sigma(\boldsymbol{L} + \boldsymbol{L}'_{\kappa_0}) \setminus \{0, 1\}$ and $\operatorname{Re} \lambda \geq 0$. Then, for some $\ell \in \mathbb{N}_0$, (4-7) with potential $2\kappa_0$ must have an analytic solution on $[0, 1]$. We show that this cannot be the case. By changing variables and setting $f(\rho) = \rho^\ell v(\rho^2)$, (4-7) transforms into the standard hypergeometric form

$$z(1-z)v''(z) + (c - (a+b+1)z)v'(z) + ab\,v(z) = 0,$$

with

$$a = \tfrac{1}{2}(\lambda + \ell - 1), \quad b = \tfrac{1}{2}(\lambda + \ell + 6), \quad \text{and} \quad c = \tfrac{1}{2}d + \ell.$$

Fundamental systems around the regular singular points $\rho = 0$ and $\rho = 1$ are given by $\{v_0, \tilde{v}_0\}$ and $\{v_1, \tilde{v}_1\}$, respectively, where

$$\begin{aligned}
v_0(z) &= {}_2F_1(a, b; c; z), \\
\tilde{v}_0(z) &= z^{1-c}\,{}_2F_1(a+1-c, b+1-c; 2-c; z), \\
v_1(z) &= {}_2F_1(a, b; a+b+1-c; 1-z), \\
\tilde{v}_1(z) &= (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; 1+c-a-b; 1-z).
\end{aligned}$$

In fact, this holds if $a + b - c \neq 0$, and, for $a + b - c = 0$, the function $\tilde{v}_1$ behaves logarithmically. In either case, the solutions $\tilde{v}_1$ and $\tilde{v}_0$ are not admissible since they are not analytic for $\operatorname{Re} \lambda \geq 0$. We therefore look for λ which connect v_0 and v_1, i.e., for which v_0 and v_1 are constant multiples of each other. For the hypergeometric equation, the connection coefficients are known explicitly and we have

$$v_0(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} v_1(z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \tilde{v}_1(z);$$

see [DLMF 2010]. So the condition that quantifies our eigenvalues is

$$\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} = 0.$$

This can only be the case if a or b is a pole of the gamma function, i.e., $-a \in \mathbb{N}_0$ or $-b \in \mathbb{N}_0$. In particular, this implies that $\lambda \in \mathbb{R}$. Since $\operatorname{Re}\lambda \geq 0$, we can exclude $-b \in \mathbb{N}_0$. On the other hand, $-a \in \mathbb{N}_0$ is possible only if $\lambda \in \{0, 1\}$, which contradicts our assumption and proves (8-5). For $\lambda = 1$ and $\lambda = 0$, one can easily check that explicit solutions to the eigenvalue equation are given by $\tilde{\boldsymbol{g}}_0$ and $\tilde{\boldsymbol{q}}_0^{(k)}$, respectively, where

$$\tilde{\boldsymbol{g}}_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{q}}_0^{(k)} = \partial_{a_k} \boldsymbol{\kappa}_a|_{a=0} = 6 \begin{pmatrix} \xi_k \\ 3\xi_k \end{pmatrix}$$

for $k = 1, \ldots, d$. Similar to the above reasoning one shows that these functions indeed span the eigenspaces for the corresponding eigenvalues, i.e., the geometric multiplicities of $\lambda = 1$ and $\lambda = 0$ are 1 and d, respectively. The algebraic multiplicities are determined by the dimension of the ranges of the corresponding Riesz projections

$$\widetilde{\boldsymbol{P}}_0 = \frac{1}{2\pi i} \int_{\gamma_1} \boldsymbol{R}_{\boldsymbol{L}+\boldsymbol{L}'_{\kappa_0}}(\lambda)\, d\lambda \quad \text{and} \quad \widetilde{\boldsymbol{Q}}_0 = \frac{1}{2\pi i} \int_{\gamma_0} \boldsymbol{R}_{\boldsymbol{L}+\boldsymbol{L}'_{\kappa_0}}(\lambda)\, d\lambda,$$

where, for $j \in \{0, 1\}$, we have $\gamma_j(s) = \lambda_j + \frac{1}{4}\omega_0 e^{2\pi i s}$ for $s \in [0, 1]$. An ODE argument analogous to the proof of Lemma 5.7 yields

$$\operatorname{ran} \widetilde{\boldsymbol{P}}_0 = \operatorname{span}(\tilde{\boldsymbol{g}}_0) \quad \text{and} \quad \operatorname{ran} \widetilde{\boldsymbol{Q}}_0 = \operatorname{span}(\tilde{\boldsymbol{q}}_0^{(1)}, \ldots, \tilde{\boldsymbol{q}}_0^{(d)}).$$

The perturbative characterization of the spectrum of $\boldsymbol{L} + \boldsymbol{L}'_{\kappa_a}$ for $a \in \overline{\mathbb{B}}^d_{\delta^*}$ is purely abstract. Along the lines of the proof of Lemma 5.8, one shows that

$$\sigma(\boldsymbol{L} + \boldsymbol{L}'_{\kappa_a}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\tfrac{1}{2}\omega_0\} \cup \{0, 1\},$$

where for $\lambda = 0$ and $\lambda = 1$, the eigenfunctions are Lorentz boosted versions of $\tilde{\boldsymbol{g}}_0$ and $\tilde{\boldsymbol{q}}_0^{(k)}$. In fact, one can check by direct calculations that the functions $\tilde{\boldsymbol{g}}_a$ and $\tilde{\boldsymbol{q}}_a^{(k)}$ stated in (8-4) solve the corresponding eigenvalue equations. Equation (8-3) for the spectral projections

$$\widetilde{\boldsymbol{P}}_a = \frac{1}{2\pi i} \int_{\gamma_1} \boldsymbol{R}_{\boldsymbol{L}+\boldsymbol{L}'_{\kappa_a}}(\lambda)\, d\lambda \quad \text{and} \quad \widetilde{\boldsymbol{Q}}_a = \frac{1}{2\pi i} \int_{\gamma_0} \boldsymbol{R}_{\boldsymbol{L}+\boldsymbol{L}'_{\kappa_a}}(\lambda)\, d\lambda$$

follows from the abstract arguments provided in the proof of Lemma 6.1. The same holds for the Lipschitz dependence of the projections on the parameter a. The growth bounds for the semigroup follow from the structure of the spectrum, resolvent bounds and the Gearhart–Prüss theorem analogous to the proof of Proposition 6.2. Finally, the proof for the Lipschitz continuity (8-2) can be copied verbatim. ☐

The analysis of the integral equation

$$\Phi(\tau) = \boldsymbol{S}_{\kappa_{a_\infty}}(\tau)\boldsymbol{u} + \int_0^\tau \boldsymbol{S}_{\kappa_{a_\infty}}(\tau-\sigma)(\widetilde{\boldsymbol{G}}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{\kappa}_{a(\sigma)})\, d\sigma$$

is completely analogous to Section 7. In particular, to derive the modulation equation for a, one uses the fact that $\partial_\tau \boldsymbol{\kappa}_{a(\tau)} = \dot{a}_k(\tau)\tilde{\boldsymbol{q}}_0^{(k)}$. By introducing the correction

$$\widetilde{\boldsymbol{C}}(\Phi, a, \boldsymbol{u}) = \widetilde{\boldsymbol{P}}_{a_\infty}\boldsymbol{u} + \widetilde{\boldsymbol{P}}_{a_\infty}\int_0^\infty e^{-\sigma}(\widetilde{\boldsymbol{G}}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \boldsymbol{\kappa}_{a(\sigma)})\, d\sigma,$$

it is straightforward to prove the following result.

Proposition 8.2. *There exists $\omega > 0$ such that, for all sufficiently large $C > 0$ and all sufficiently small $\delta > 0$, the following holds: For every $\|\boldsymbol{v}\|_{\mathcal{Y}} \le \delta/C$, every $T \in [1-\delta/C, 1+\delta/C]$, and every $x_0 \in \overline{\mathbb{B}}^d_{\delta/C}$, there exist functions $\Phi \in \mathcal{X}_\delta$ and $a \in X_\delta$ such that the integral equation*

$$\Phi(\tau) = \boldsymbol{S}_{\kappa_{a_\infty}}(\tau)(\Upsilon(\boldsymbol{v}, T, x_0) - \widetilde{\boldsymbol{C}}(\Phi, a, \Upsilon(\boldsymbol{v}, T, x_0))) + \int_0^\tau \boldsymbol{S}_{\kappa_{a_\infty}}(\tau-\sigma)(\widetilde{\boldsymbol{G}}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \beta \kappa_{a(\sigma)})\, d\sigma$$

holds for $\tau \ge 0$, and also $\|\Phi(\tau)\| \lesssim \delta e^{-\omega\tau}$ for all $\tau \ge 0$. Moreover, the solution map is Lipschitz continuous, i.e.,

$$\|\Phi(\boldsymbol{v}, T_1, x_0) - \Phi(\boldsymbol{w}, T_2, y_0)\|_{\mathcal{X}} + \|a(\boldsymbol{v}, T_1, x_0) - a(\boldsymbol{w}, T_2, y_0)\|_X \lesssim \|\boldsymbol{v}-\boldsymbol{w}\|_{\mathcal{Y}} + |T_1 - T_2| + |x_0 - y_0|$$

for all $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{Y}$ satisfying the smallness assumption, all $T_1, T_2 \in [1-\delta/C, 1+\delta/C]$, and all $x_0, y_0 \in \overline{\mathbb{B}}^d_{\delta/C}$.

We note that similarly to the manifold $\mathcal{M}$ one can construct a manifold $\mathcal{N} \subset \ker \widetilde{\boldsymbol{P}}_0 \oplus \operatorname{ran} \widetilde{\boldsymbol{P}}_0$ of codimension $(1+d)$ characterized by the vanishing of the correction term $\widetilde{\boldsymbol{C}}$. However, in the context of stable blowup this is not of much interest, since the existence of this manifold is solely caused by the translation instability. In particular, no correction of the physical initial data is necessary, if blow-up time and point are chosen appropriately, i.e., for suitably small (f, g), there are T and x_0 such that $\Upsilon(\boldsymbol{v}, T, x_0) \in \mathcal{N}$. This is contained in the following result, where $\mathcal{Y} = H^{(d+3)/2}(\mathbb{B}^d) \times H^{(d+1)/2}(\mathbb{B}^d)$.

Lemma 8.3. *There exists $C > 0$ such that, for all sufficiently small $\delta > 0$, the following holds: For every $\boldsymbol{v} \in \mathcal{Y}$ satisfying $\|\boldsymbol{v}\|_{\mathcal{Y}} \le \delta/C^2$, there is a choice of parameters $T \in [1-\delta/C, 1+\delta/C]$ and $x_0 \in \overline{\mathbb{B}}^d_{\delta/C}$ in Proposition 8.2 such that*

$$\widetilde{\boldsymbol{C}}(\Phi, a, \Upsilon(\boldsymbol{v}, T, x_0)) = 0.$$

Moreover, the parameters depend Lipschitz continuously on the data, i.e.,

$$|T(\boldsymbol{v}) - T(\boldsymbol{w})| + |x_0(\boldsymbol{v}) - x_0(\boldsymbol{w})| \lesssim \|\boldsymbol{v}-\boldsymbol{w}\|_{\mathcal{Y}}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{Y}$ satisfying the above smallness assumption.

The proof is along the lines of the proof of Lemma 7.7 on page 668 with obvious simplifications. With these results, in combination with persistence of regularity that is completely analogous to Proposition 2.2, Theorem 1.6 is obtained by the same arguments as in Section 7F.

Appendix: Proof of Lemma 3.4

We will frequently use the identities

$$2\,\mathrm{Re}[\xi^j \partial_j f(\xi) \bar{f}(\xi)] = \partial_j [\xi^j |f(\xi)|^2] - d|f(\xi)|^2 \tag{A-1}$$

and

$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}[\xi^j \partial_j f] = k\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} f(\xi) + \xi^j \partial_j \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} f(\xi), \tag{A-2}$$

which hold for all $k \in \mathbb{N}$ and $f \in C^\infty(\overline{\mathbb{B}}^d)$. Furthermore, by the divergence theorem, we have

$$\int_{\mathbb{B}^d} \partial_i \Delta u(\xi) \overline{\partial^i v(\xi)}\, d\xi = -\int_{\mathbb{B}^d} \Delta u(\xi) \overline{\Delta v(\xi)}\, d\xi + \int_{\mathbb{S}^{d-1}} \Delta u(\omega) \overline{\omega_i \partial^i v(\omega)}\, d\sigma(\omega) \tag{A-3}$$

for smooth u and v, and similarly

$$\int_{\mathbb{B}^d} \partial_i \Delta u(\xi) \overline{\partial^i v(\xi)}\, d\xi = -\int_{\mathbb{B}^d} \partial_i \partial_j u(\xi) \overline{\partial^j \partial^i v(\xi)}\, d\xi + \int_{\mathbb{S}^{d-1}} \omega_j \partial^j \partial_i u(\omega) \overline{\partial^i v(\omega)}\, d\sigma(\omega). \tag{A-4}$$

We first prove the result for $d = 9$, starting with those parts of $(\cdot\,|\,\cdot)_{\mathcal{H}_k}$ that correspond to the standard $\dot{H}^k \times \dot{H}^{k-1}$ inner product.

For the sake of concreteness, we consider the case $k = 5$, which corresponds to the space we are going to use later on. Using the above identities, we infer

$$\begin{aligned}
&\mathrm{Re} \int_{\mathbb{B}^9} \partial_i \partial_j \partial_k \partial_l \partial_m [\widetilde{\boldsymbol{L}}\boldsymbol{u}]_1(\xi) \overline{\partial^i \partial^j \partial^k \partial^l \partial^m u_1(\xi)}\, d\xi \\
&\quad = \mathrm{Re} \int_{\mathbb{B}^9} \partial_i \partial_j \partial_k \partial_l \partial_m u_2(\xi) \overline{\partial^i \partial^j \partial^k \partial^l \partial^m u_1(\xi)}\, d\xi - \tfrac{5}{2} \int_{\mathbb{B}^9} \partial_i \partial_j \partial_k \partial_l \partial_m u_1(\xi) \overline{\partial^i \partial^j \partial^k \partial^l \partial^m u_1(\xi)}\, d\xi \\
&\qquad - \tfrac{1}{2} \int_{\mathbb{S}^8} \partial_i \partial_j \partial_k \partial_l \partial_m u_1(\omega) \overline{\partial^i \partial^j \partial^k \partial^l \partial^m u_1(\omega)}\, d\sigma(\omega).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathrm{Re} \int_{\mathbb{B}^9} \partial_i \partial_j \partial_k \partial_l [\widetilde{\boldsymbol{L}}\boldsymbol{u}]_2(\xi) \overline{\partial^i \partial^j \partial^k \partial^l u_2(\xi)}\, d\xi \\
&\quad = -\tfrac{5}{2} \int_{\mathbb{B}^9} \partial_i \partial_j \partial_k \partial_l u_2(\xi) \overline{\partial^i \partial^j \partial^k \partial^l u_2(\xi)}\, d\xi - \mathrm{Re} \int_{\mathbb{B}^9} \partial_i \partial_j \partial_k \partial_l \partial_m u_1(\xi) \overline{\partial^i \partial^j \partial^k \partial^l \partial^m u_2(\xi)}\, d\xi \\
&\qquad - \tfrac{1}{2} \int_{\mathbb{S}^8} \partial_i \partial_j \partial_k \partial_l u_2(\omega) \overline{\partial^i \partial^j \partial^k \partial^l u_2(\omega)}\, d\sigma(\omega) \\
&\qquad + \mathrm{Re} \int_{\mathbb{S}^8} \omega^m \partial_i \partial_j \partial_k \partial_l \partial_m u_1(\omega) \overline{\partial^i \partial^j \partial^k \partial^l u_2(\omega)}\, d\sigma(\omega).
\end{aligned} \tag{A-5}$$

Hence

$$\begin{aligned}
\mathrm{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_5 &\le -\tfrac{5}{2}\|\boldsymbol{u}\|_5^2 - \tfrac{1}{2} \int_{\mathbb{S}^8} \partial_i \partial_j \partial_k \partial_l \partial_m u_1(\omega) \overline{\partial^i \partial^j \partial^k \partial^l \partial^m u_1(\omega)}\, d\sigma(\omega) \\
&\quad - \tfrac{1}{2} \int_{\mathbb{S}^8} \partial_i \partial_j \partial_k \partial_l u_2(\omega) \overline{\partial^i \partial^j \partial^k \partial^l u_2(\omega)}\, d\sigma(\omega) \\
&\quad + \mathrm{Re} \int_{\mathbb{S}^8} \omega^m \partial_i \partial_j \partial_k \partial_l \partial_m u_1(\omega) \overline{\partial^i \partial^j \partial^k \partial^l u_2(\omega)}\, d\sigma(\omega) \le -\tfrac{5}{2}\|\boldsymbol{u}\|_5^2,
\end{aligned} \tag{A-6}$$

where we used $\mathrm{Re}(a\bar{b}) \le \frac{1}{2}(|a|^2 + |b|^2)$ as well as the bound

$$\Big|\sum_k \omega_k \partial_k u(\omega)\Big|^2 \le \sum_k (\omega_k)^2 \sum_k |\partial_k u(\omega)|^2 = \sum_k |\partial_k u(\omega)|^2.$$

A similar calculation yields

$$\mathrm{Re}(\widetilde{\boldsymbol{L}}\,|\,\boldsymbol{u})_4 \le -\tfrac{3}{2}\|\boldsymbol{u}\|_4^2.$$

In view of the logic of these estimates, it is clear that we cannot use the standard homogeneous inner products for integer regularities lower than $j = 3$, since the bound shifts to the right and will be positive eventually. For this reason, we use tailor-made expressions for the remaining $H^3(\mathbb{B}^9) \times H^2(\mathbb{B}^9)$ part. In the following, we prove that

$$\sum_{j=1}^{3} \mathrm{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_j \le -\tfrac{1}{2}\sum_{j=1}^{3} \|\boldsymbol{u}\|_j^2, \tag{A-7}$$

which in combination with the above bounds implies the first claim in Lemma 3.4 for $d = 9$ and $k = 5$ (and in fact for any $3 \le k \le 5$.) For higher regularities, we add again the corresponding standard homogeneous parts. Analogous to the above calculations, one shows that

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{B}^9} \partial_{i_1\cdots i_k}[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_1(\xi)\overline{\partial^{i_1\cdots i_k}u_1(\xi)}\,d\xi \\
&\quad = \big(\tfrac{5}{2}-k\big)\int_{\mathbb{B}^9} \partial_{i_1\cdots i_k}u_1(\xi)\overline{\partial^{i_1\cdots i_k}u_1(\xi)}\,d\xi - \tfrac{1}{2}\int_{\mathbb{S}^8} \partial_{i_1\cdots i_k}u_1(\omega)\overline{\partial^{i_1\cdots i_k}u_1(\omega)}\,d\sigma(\omega) \\
&\qquad + \mathrm{Re}\int_{\mathbb{B}^9} \partial_{i_1\cdots i_k}u_1(\xi)\overline{\partial^{i_1\cdots i_k}u_2(\xi)}\,d\xi
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{B}^9} \partial_{i_1\cdots i_{k-1}}[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_2(\xi)\overline{\partial^{i_1\cdots i_{k-1}}u_2(\xi)}\,d\xi \\
&\quad = \big(\tfrac{5}{2}-k\big)\int_{\mathbb{B}^9} \partial_{i_1\cdots i_{k-1}}u_2(\xi)\overline{\partial^{i_1\cdots i_{k-1}}u_2(\xi)}\,d\xi - \mathrm{Re}\int_{\mathbb{B}^9} \partial_{i_1\cdots i_k}u_1(\xi)\overline{\partial^{i_1\cdots i_k}u_2(\xi)}\,d\xi \\
&\qquad + \mathrm{Re}\int_{\mathbb{S}^8} \omega^{i_k}\partial_{i_1\cdots i_k}u_1(\omega)\overline{\partial^{i_1\cdots i_{k-1}}u_2(\omega)}\,d\sigma(\omega) - \tfrac{1}{2}\int_{\mathbb{S}^8} \partial_{i_1\cdots i_{k-1}}u_2(\omega)\overline{\partial^{i_1\cdots i_{k-1}}u_2(\omega)}\,d\sigma(\omega).
\end{aligned}$$

As in (A-6), we thus obtain for $j \ge 6$ the bound

$$\mathrm{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_j \le \big(\tfrac{5}{2}-j\big)\|\boldsymbol{u}\|_j^2.$$

It is left to prove (A-7). We first consider $\mathrm{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_3$. Using (A-2), (A-1), and the divergence theorem, we calculate

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{B}^9} \partial_i\partial_j\partial_k[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_1(\xi)\overline{\partial^i\partial^j\partial^k u_1(\xi)}\,d\xi \\
&\quad = -\tfrac{1}{2}\int_{\mathbb{B}^9} \partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k u_1(\xi)}\,d\xi - \tfrac{1}{2}\int_{\mathbb{S}^8} \partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j\partial^k u_1(\omega)}\,d\sigma(\omega) \\
&\qquad + \mathrm{Re}\int_{\mathbb{B}^9} \partial_i\partial_j\partial_k u_2(\xi)\overline{\partial^i\partial^j\partial^k u_1(\xi)}\,d\xi.
\end{aligned}$$

An application of (A-4) shows

$$\begin{aligned}
\operatorname{Re}\int_{\mathbb{B}^9} &\partial_i\partial_j[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_2(\xi)\overline{\partial^i\partial^j u_2(\xi)}\,d\xi \\
&= \operatorname{Re}\int_{\mathbb{B}^9}\partial_i\partial_j\partial^k\partial_k u_1(\xi)\overline{\partial^i\partial^j u_2(\xi)}\,d\xi - \tfrac{1}{2}\int_{\mathbb{B}^9}\partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j u_2(\xi)}\,d\xi \\
&\qquad - \tfrac{1}{2}\int_{\mathbb{S}^8}\partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j u_2(\omega)}\,d\sigma(\omega) \\
&= \operatorname{Re}\int_{\mathbb{S}^8}\omega^k\partial_k\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_2(\omega)}\,d\sigma(\omega) - \operatorname{Re}\int_{\mathbb{B}^9}\partial_i\partial_j\partial_k u_1(\xi)\overline{\partial^i\partial^j\partial^k u_2(\xi)}\,d\xi \\
&\qquad - \tfrac{1}{2}\int_{\mathbb{B}^9}\partial_i\partial_j u_2(\xi)\overline{\partial^i\partial^j u_2(\xi)}\,d\xi - \tfrac{1}{2}\int_{\mathbb{S}^8}\partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j u_2(\omega)}\,d\sigma(\omega),
\end{aligned}$$

and finally

$$\begin{aligned}
\int_{\mathbb{S}^8}&\partial_i\partial_j[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega) \\
&= -\operatorname{Re}\int_{\mathbb{S}^8}\omega^k\partial_k\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega) \\
&\qquad - 4\int_{\mathbb{S}^8}\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega) + \operatorname{Re}\int_{\mathbb{S}^8}\partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega).
\end{aligned}$$

In summary, we infer that

$$\operatorname{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_3 = -\tfrac{1}{2}\|\boldsymbol{u}\|_3^2 - 12\int_{\mathbb{S}^8}\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega) + 4\int_{\mathbb{S}^8}A(\omega)\,d\sigma(\omega),$$

where

$$\begin{aligned}
A(\omega) = &-\tfrac{1}{2}\partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j\partial^k u_1(\omega)} - \tfrac{1}{2}\partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j u_2(\omega)} \\
&- \tfrac{1}{2}\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)} + \operatorname{Re}(\omega^k\partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j u_2(\omega)}) \\
&- \operatorname{Re}(\omega^k\partial_i\partial_j\partial_k u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}) + \operatorname{Re}(\partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j u_1(\omega)}).
\end{aligned}$$

By using the inequality

$$\operatorname{Re}(a\bar{b}) + \operatorname{Re}(a\bar{c}) - \operatorname{Re}(b\bar{c}) \le \tfrac{1}{2}(|a|^2+|b|^2+|c|^2), \quad a,b,c\in\mathbb{C},$$

we get $A(\omega)\le 0$. Analogously, to estimate $\operatorname{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_2$, we get

$$\begin{aligned}
\operatorname{Re}\int_{\mathbb{B}^9}&\partial_i\partial^j\partial_j[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_1(\xi)\overline{\partial^i\partial_l\partial^l u_1(\xi)}\,d\xi \\
&= -\tfrac{1}{2}\operatorname{Re}\int_{\mathbb{B}^9}\partial_i\partial^j\partial_j u_1(\xi)\overline{\partial^i\partial_l\partial^l u_1(\xi)}\,d\xi - \tfrac{1}{2}\int_{\mathbb{S}^8}\partial_i\partial^j\partial_j u_1(\omega)\overline{\partial^i\partial_l\partial^l u_1(\omega)}\,d\sigma(\omega) \\
&\qquad + \operatorname{Re}\int_{\mathbb{B}^9}\partial_i\partial^j\partial_j u_1(\xi)\overline{\partial^i\partial_l\partial^l u_2(\xi)}\,d\xi
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re}\int_{\mathbb{S}^8}&\partial_i[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_2(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega) \\
&= \operatorname{Re}\int_{\mathbb{S}^8}\partial_i\partial^j\partial_j u_1(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega) \\
&\qquad - \operatorname{Re}\int_{\mathbb{S}^8}\omega^j\partial_i\partial_j u_2(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega) - 4\operatorname{Re}\int_{\mathbb{S}^8}\partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega).
\end{aligned}$$

For the remaining term, we do a similar calculation as in (A-5), but we use instead (A-3) in order to cancel the mixed term. In summary, we obtain

$$\operatorname{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_2 = -\tfrac{1}{2}\|\boldsymbol{u}\|_2^2 - 3\int_{\mathbb{S}^8} \partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8} B(\omega)\,d\sigma(\omega),$$

where

$$\begin{aligned}B(\omega) = &-\tfrac{1}{2}\partial_i\partial^j\partial_j u_1(\omega)\overline{\partial^i\partial_l\partial^l u_1(\omega)} - \tfrac{1}{2}\partial_i\partial_j u_2(\omega)\overline{\partial^i\partial^j u_2(\omega)} - \tfrac{1}{2}\partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}\\ &+\operatorname{Re}(\omega^k\partial_i\partial^j\partial_j u_1(\omega)\overline{\partial^i\partial_k u_2(\omega)}) + \operatorname{Re}(\partial_i\partial^j\partial_j u_1(\omega)\overline{\partial^i u_2(\omega)}) - \operatorname{Re}(\omega^j\partial_i\partial_j u_2(\omega)\overline{\partial^i u_2(\omega)}),\end{aligned}$$

and we observe that $B(\omega)\le 0$. Now, we consider $\operatorname{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_1$, which consists only of boundary integrals. For the first term, we get

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{S}^8} &\partial_i[\boldsymbol{L}\boldsymbol{u}]_1(\omega)\overline{\partial^i u_1(\omega)}\,d\sigma(\omega)\\ &= -3\operatorname{Re}\int_{\mathbb{S}^8}\partial_i u_1(\omega)\overline{\partial^i u_1(\omega)}\,d\sigma(\omega) - \operatorname{Re}\int_{\mathbb{S}^8}\omega^j\partial_i\partial_j u_1(\omega)\overline{\partial^i u_1(\omega)}\,d\sigma(\omega)\\ &\qquad + \operatorname{Re}\int_{\mathbb{S}^8}\partial_i u_2(\omega)\overline{\partial^i u_1(\omega)}\,d\sigma(\omega).\end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{S}^8}&(\partial_i u_2(\omega) - \omega^j\partial_i\partial_j u_1(\omega))\overline{\partial^i u_1(\omega)}\,d\sigma(\omega)\\ &\le \tfrac{1}{2}\int_{\mathbb{S}^8}\partial_i u_1(\omega)\overline{\partial^i u_1(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8}\partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8}\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega),\end{aligned}$$

which implies

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{S}^8}&\partial_i[\boldsymbol{L}\boldsymbol{u}]_1(\omega)\overline{\partial^i u_1(\omega)}\,d\sigma(\omega)\\ &\le -\tfrac{5}{2}\operatorname{Re}\int_{\mathbb{S}^8}\partial_i u_1(\omega)\overline{\partial^i u_1(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8}\partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8}\partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega).\end{aligned}$$

Analogously,

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{S}^8}&[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_2(\omega)\overline{u_2(\omega)}\,d\sigma(\omega)\\ &= -3\int_{\mathbb{S}^8}|u_2(\omega)|^2\,d\sigma(\omega) + \operatorname{Re}\int_{\mathbb{S}^8}\partial^i\partial_i u_1(\omega)\overline{u_2(\omega)}\,d\sigma(\omega) - \operatorname{Re}\int_{\mathbb{S}^8}\omega^i\partial_i u_2(\omega)\overline{u_2(\omega)}\,d\sigma(\omega)\\ &\le -\tfrac{5}{2}\int_{\mathbb{S}^8}|u_2(\omega)|^2\,d\sigma(\omega) + \int_{\mathbb{S}^8}|\Delta u_1(\omega)|^2\,d\sigma(\omega) + \int_{\mathbb{S}^8}\partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega),\end{aligned}$$

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{S}^8}&[\widetilde{\boldsymbol{L}}\boldsymbol{u}]_1(\omega)\overline{u_1(\omega)}\,d\sigma(\omega)\\ &= -2\int_{\mathbb{S}^8}|u_1(\omega)|^2\,d\sigma(\omega) - \operatorname{Re}\int_{\mathbb{S}^8}\omega^i\partial_i u_1(\omega)\overline{u_1(\omega)}\,d\sigma(\omega) + \operatorname{Re}\int_{\mathbb{S}^8}u_2(\omega)\overline{u_1(\omega)}\,d\sigma(\omega)\\ &\le -\tfrac{3}{2}\int_{\mathbb{S}^8}|u_1(\omega)|^2\,d\sigma(\omega) + \int_{\mathbb{S}^8}\partial_i u_1(\omega)\overline{\partial^i u_1(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8}|u_2(\omega)|^2\,d\sigma(\omega),\end{aligned}$$

and hence

$$\operatorname{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_1 \le -\tfrac{3}{2}\|\boldsymbol{u}\|_1^2 + 2\int_{\mathbb{S}^8} \partial_i u_2(\omega)\overline{\partial^i u_2(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8} |\Delta u_1(\omega)|^2\,d\sigma(\omega).$$

In conclusion,

$$\sum_{j=1}^{3}\operatorname{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_j \le -\tfrac{1}{2}\sum_{j=1}^{3}\|\boldsymbol{u}\|_j^2 - 11\int_{\mathbb{S}^8} \partial_i\partial_j u_1(\omega)\overline{\partial^i\partial^j u_1(\omega)}\,d\sigma(\omega) + \int_{\mathbb{S}^8} |\Delta u_1(\omega)|^2\,d\sigma(\omega).$$

By the Cauchy–Schwarz inequality,

$$|\Delta u(\omega)|^2 \le \left|\sum_{i=1}^{9}\partial_i^2 u(\omega)\right|^2 \le 9\sum_{i=1}^{9}|\partial_i^2 u(\omega)|^2 \le 9\sum_{i,j=1}^{9}|\partial_i\partial_j u(\omega)|^2,$$

which proves (A-7).

Analogous calculations for $d = 7$ and $k \ge 3$ yield an even better bound, namely,

$$\operatorname{Re}(\widetilde{\boldsymbol{L}}\boldsymbol{u}\,|\,\boldsymbol{u})_{\mathcal{H}_k} \le -\tfrac{3}{2}\|\boldsymbol{u}\|_{\mathcal{H}_k}^2 \tag{A-8}$$

for all $\boldsymbol{u} \in \mathcal{D}(\widetilde{\boldsymbol{L}})$, from which we obtain in particular the claimed estimate. Another way to see that (A-8) holds is by Lemma 3.2 of [Glogić and Schörkhuber 2021], which is formulated in terms of the above inner product for the specific case $d = 7$ and $k = 3$. The operator considered there corresponds to $\widetilde{\boldsymbol{L}}$ shifted by a constant, which immediately gives the inequality (A-8). □

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