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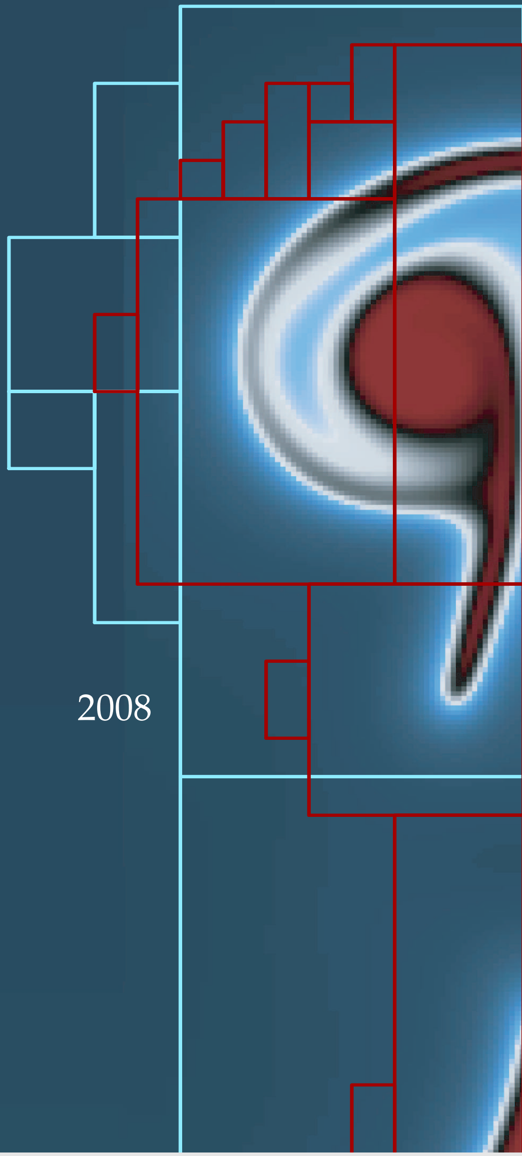

A BALANCING DOMAIN DECOMPOSITION METHOD BY CONSTRAINTS FOR ADVECTION-DIFFUSION PROBLEMS

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A BALANCING DOMAIN DECOMPOSITION METHOD BY CONSTRAINTS FOR ADVECTION-DIFFUSION PROBLEMS

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The balancing domain decomposition methods by constraints are extended to solving nonsymmetric, positive definite linear systems resulting from the finite element discretization of advection-diffusion equations. A preconditioned GMRES iteration is used to solve a Schur complement system of equations for the subdomain interface variables. In the preconditioning step of each iteration, a partially subassembled interface problem is solved. A convergence rate estimate for the GMRES iteration is established for the cases where the advection is not strong, under the condition that the mesh size is small enough. The estimate deteriorates with a decrease of the viscosity and for fixed viscosity it is independent of the number of subdomains and depends only slightly on the subdomain problem size. Numerical experiments for several two-dimensional advection-diffusion problems illustrate the fast convergence of the proposed algorithm for both diffusion-dominated and advection-dominated cases.

1. Introduction

Domain decomposition methods have been widely used and studied for solving large sparse linear systems arising from finite element discretization of partial differential equations. The balancing domain decomposition methods by constraints (BDDC) were introduced by Dohrmann [12] and they represent an interesting redesign of the balancing Neumann-Neumann algorithms; see also [17; 11] for related algorithms. Scalable convergence rates for the BDDC methods have been proved by Mandel and Dohrmann [27] for symmetric positive definite problems. Connections and spectral equivalence between the BDDC algorithms and the earlier dual-primal finite element tearing and interconnecting methods (FETI-DP) [15] have been established by Mandel et al. [28]; see also [25; 4]. The BDDC methods have also

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been extended to solving saddle point problems, for example, for Stokes equations by Li and Widlund [24], for nearly incompressible elasticity by Dohrmann [13], and for the flow in porous media by Tu [35; 37; 36].

The systems of linear equations arising from the finite element discretization of advection-diffusion equations are nonsymmetric, but usually positive definite. A number of domain decomposition methods have been proposed and analyzed for solving nonsymmetric and indefinite problems. Cai and Widlund [5; 6; 7] studied overlapping Schwarz methods for such problems, using a perturbation approach in their analysis, and established that the convergence rates of the two-level overlapping Schwarz methods are independent of the mesh size if the coarse mesh is fine enough. Motivated by the FETI-DPH method proposed by Farhat and Li [16] for solving symmetric indefinite problems, the authors [23] studied a BDDC algorithm for solving Helmholtz equations and estimated its convergence rate using a similar perturbation approach. For some other results using the perturbation approach for domain decomposition methods, see [40; 38; 19].

For advection-diffusion problems, standard iterative substructuring methods do not usually perform well when advection is strong. Dirichlet and Neumann boundary conditions used for the local subdomain problems in these algorithms are not appropriate because of loss of positive definiteness of the local bilinear forms. More general boundary conditions need to be considered. Therefore, a class of methods has been developed in [8; 9; 34; 18; 29], where additional adaptively chosen subdomain boundary conditions are used to stabilize the local subdomain problems; see also [30, Chapter 6] and the references therein for other similar approaches.

The Robin-Robin algorithm, a modification of the Neumann-Neumann approach for solving advection-diffusion problems, has been developed by Achdou et al. [3; 1; 2], where new local bilinear forms corresponding to Robin boundary conditions for the subdomains are used and a coarse level basis function, determined by the solution to an adjoint problem on each subdomain, is added to accelerate the convergence. Equipped with the same type local subdomain bilinear forms with Robin boundary conditions and a similar coarse level basis function, one-level and two-level FETI algorithms were proposed by Toselli [32] for solving advection-diffusion problems. Some additive and multiplicative BDDC algorithms with vertex constraints and edge average constraints have also been studied by Conceição [10]. All these algorithms, based on subdomain Robin boundary conditions, have been shown to be successful for solving advection-diffusion problems, including some advection-dominated cases, but a theoretical analysis is still missing.

In this paper, we develop BDDC algorithms for advection-diffusion problems. As in [2], local subdomain bilinear forms corresponding to Robin boundary conditions are used. The original system of linear equations is reduced to a Schur complement problem for the subdomain interface variables and a preconditioned GMRES iteration is then used. In the preconditioning step of each iteration, a partially subassembled finite element problem is solved, for which only the coarse level, primal interface degrees of freedom are shared by neighboring subdomains. The convergence analysis of our BDDC algorithms requires that the coarse level primal variable space contains certain flux average constraints, which depend on the coefficient of the first order term of the problem, across the subdomain interface, in addition to the standard subdomain vertex and edge/face average continuity constraints. A convergence rate estimate for the GMRES iterations is established for the cases where the advection is not strong, under the condition that the mesh size is small enough. The estimate deteriorates with a decrease of the viscosity and for fixed viscosity it is independent of the number of subdomains and depends only slightly on the subdomain problem size. A perturbation approach is used in our analysis to handle the nonsymmetry of the problem. A key point is to obtain an error bound for the partially subassembled finite element problem; we view this problem as a nonconforming finite element approximation.

For the cases where the advection is dominant, our BDDC algorithm also performs satisfactorily, even though where the convergence rate estimate does not apply.

The rest of this paper is organized as follows. The advection-diffusion equation and its adjoint form are described in Section 2. In Section 3, the finite element space and a stabilized finite element problem are introduced. The local subdomain bilinear forms and a partially subassembled finite element space are introduced in Section 4. In Section 5, an error estimate for the partially subassembled finite element problem is proved. The preconditioned interface problem for our BDDC algorithm is presented in Section 6 and its convergence analysis is given in Section 7. To conclude, numerical experiments in Section 8 demonstrate the effectiveness of our algorithm.

2. Problem setting

We consider the following second order scalar advection-diffusion problem in a bounded polyhedral domain $\Omega \in \mathbf{R}^d$, d = 2, 3,

$$\begin{cases} Lu := -\nu \triangle u + \boldsymbol{a} \cdot \nabla u + cu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here the viscosity ν is a positive constant. The velocity field $\boldsymbol{a}(x) \in (L^\infty(\Omega))^d$ and $\nabla \cdot \boldsymbol{a}(x) \in L^\infty(\Omega)$. The reaction coefficient $c(x) \in L^\infty(\Omega)$ and $f(x) \in L^2(\Omega)$. We define

$$\begin{aligned} \tilde{c}(x) &= c(x) - \tfrac{1}{2}\nabla \cdot \boldsymbol{a}(x), & \tilde{c}_s &= \|\tilde{c}(x)\|_{L^\infty(\Omega)}, \\ a_s &= \|\boldsymbol{a}(x)\|_{L^\infty(\Omega)}, & c_s &= \|c(x)\|_{L^\infty(\Omega)}. \end{aligned} \tag{2}$$

For simplicity we assume that $\tilde{c}_s$, a_s, c_s, and the diameter of Ω are of the order $O(1)$; we focus on studying the dependence on ν of the performance of our algorithms. We also assume that there exists a positive constant c_0 such that

$$\tilde{c}(x) \geq c_0 > 0, \qquad \text{for all } \ x \in \Omega. \tag{3}$$

The bilinear form associated with the operator L is defined, for functions in the space $H_0^1(\Omega)$, by

$$a_o(u, v) = \int_\Omega (\nu \nabla u \cdot \nabla v + \boldsymbol{a} \cdot \nabla u v + c u v)\, dx, \tag{4}$$

which is positive definite under assumption (3). The weak solution $u \in H_0^1(\Omega)$ of (1) satisfies

$$a_o(u, v) = \int_\Omega f v \, dx, \qquad \text{for all } \ v \in H_0^1(\Omega). \tag{5}$$

We assume that the weak solution u of the original problem (1), as well as the weak solution of the adjoint problem $L^* u = -\nu \triangle u - \nabla \cdot (\boldsymbol{a} u) + cu = f$, satisfies the regularity result,

$$\|u\|_{H^2(\Omega)} \leq \frac{C}{\nu} \|f\|_{L^2(\Omega)}, \tag{6}$$

where C is a positive constant independent of ν. Here we assume that $\|u\|_{H^2(\Omega)}$ grows proportionally with a decrease of the viscosity ν. In this paper, C always represents a generic positive constant independent of ν and the mesh size.

3. Finite element discretization and stabilization

Let $\widehat{W} \subset H_0^1(\Omega)$ be the standard continuous, piecewise linear finite element function space on a shape-regular triangulation of Ω. We denote elements of the triangulation by e, and their diameters by h_e. We set $h = \max_e h_e$.

It is well known that the original bilinear form $a_o(\cdot, \cdot)$ has to be stabilized to remove spurious oscillations in the finite element solution for advection-dominated problems. There are a large number of strategies for this purpose; see [20] and the references therein. Here, we follow [20; 32] and consider the Galerkin/least-squares method (GALS) of [20]. On each element e, we define the local Peclet number by

$$P_{e_e} = \frac{h_e \|\boldsymbol{a}\|_{L^\infty(e)}}{2\nu},$$

and we define a positive function $C(x)$ by

$$C(x) = \begin{cases} \frac{\tau h_e}{2\|\boldsymbol{a}\|_{L^\infty(e)}} & \text{if } P_{e_e} \geq 1, \\ \frac{\tau h_e^2}{4\nu} & \text{if } P_{e_e} < 1, \end{cases} \qquad \text{for all } \ x \in e, \tag{7}$$

where τ is a constant. We set $\tau = 0.7$ in our numerical experiments. We define $C_s = \max_{x\in\Omega} |C(x)|$ and $C_m = \min_{x\in\Omega} |C(x)|$. From the definition of C_s, we know that

$$C_s \le \max_{e\subset\Omega} \left\{ \min\left(\frac{\tau h_e}{2\|\boldsymbol{a}\|_{L^\infty(e)}}, \frac{\tau h_e^2}{4\nu} \right) \right\}. \tag{8}$$

The stabilized finite element problem for solving (5) is: find $u \in \widehat{W}$, such that

$$\begin{aligned} a(u, v) :&= a_o(u, v) + \int_\Omega C(x) Lu Lv \, dx \\ &= \int_\Omega f v \, dx + \int_\Omega C(x) f Lv \, dx, \quad \text{for all } v \in \widehat{W}. \end{aligned} \tag{9}$$

Here and from now on, the integration over Ω in the stabilization terms always represents a sum of integrals over all elements of Ω. We note that for all piecewise linear finite element functions u, $Lu = -\nu \triangle u + \boldsymbol{a} \cdot \nabla u + cu = \boldsymbol{a} \cdot \nabla u + cu$, on each element.

The symmetric and skew-symmetric parts of $a(u, v)$, respectively, are denoted by

$$b(u, v) = \int_\Omega \big(\nu \nabla u \cdot \nabla v + C(x) Lu Lv + \tilde{c} u v \big) \, dx, \tag{10}$$

$$z(u, v) = \frac{1}{2} \int_\Omega (\boldsymbol{a} \cdot \nabla u v - \boldsymbol{a} \cdot \nabla v u) \, dx. \tag{11}$$

The system of linear equations corresponding to the stabilized finite element problem (9) is denoted by

$$Au = f, \tag{12}$$

where the coefficient matrix A is nonsymmetric but positive definite. We denote the symmetric part of A by B and its skew-symmetric part by Z; they correspond to the bilinear forms $b(\cdot,\cdot)$ and $z(\cdot,\cdot)$ in (10) and (11), respectively. In this paper, we will use the same notation, for example, u, to denote both a finite element function and the vector of its coefficients with respect to the finite element basis; we will also use the same notation to denote the space of finite element functions and the space of their corresponding vectors, for example, $\widehat{W}$.

4. Domain decomposition and a partially subassembled finite element space

The original finite element triangulation of Ω is decomposed into N nonoverlapping polyhedral subdomains Ω_i; each subdomain is a union of shape regular elements. The typical diameter of the subdomains is denoted by H. The nodes on the boundaries of neighboring subdomains match across the subdomain interface $\Gamma = (\cup \partial\Omega_i) \backslash \partial\Omega$. The interface Γ is composed of subdomain faces $\mathcal{F}^l$ and/or edges $\mathcal{E}^k$, which are regarded as open subsets of Γ, and of the subdomain vertices, which are end points of edges. In three dimensions, the subdomain faces are shared by two subdomains, and the edges typically by more than two; in two dimensions, each edge is shared by two subdomains. The interface of subdomain Ω_i is defined by $\Gamma_i = \partial\Omega_i \cap \Gamma$. We denote the space of finite element functions on Ω_i, which vanish at the nodes of $\partial\Omega$, by $W^{(i)}$. The local bilinear and stabilized bilinear forms are defined on $W^{(i)}$ by

$$a_o^{(i)}(u^{(i)}, v^{(i)}) = \int_{\Omega_i} \left(\nu \nabla u^{(i)} \cdot \nabla v^{(i)} + \boldsymbol{a} \cdot \nabla u^{(i)} v^{(i)} + c u^{(i)} v^{(i)}\right) dx, \tag{13}$$

and

$$\begin{aligned}\overline{a}^{(i)}(u^{(i)}, v^{(i)}) &= \int_{\Omega_i} \left(\nu \nabla u^{(i)} \cdot \nabla v^{(i)} + \boldsymbol{a} \cdot \nabla u^{(i)} v^{(i)} + c u^{(i)} v^{(i)} + C(x) L u^{(i)} L v^{(i)}\right) dx \\ &= \int_{\Omega_i} \left(\nu \nabla u^{(i)} \cdot \nabla v^{(i)} + C(x) L u^{(i)} L v^{(i)} + \tilde{c} u^{(i)} v^{(i)}\right) dx \\ &\quad + \frac{1}{2} \int_{\Omega_i} \left(\boldsymbol{a} \cdot \nabla u^{(i)} v^{(i)} - \boldsymbol{a} \cdot \nabla v^{(i)} u^{(i)}\right) dx + \frac{1}{2} \int_{\Gamma_i} \boldsymbol{a} \cdot \boldsymbol{n} u^{(i)} v^{(i)}\, ds.\end{aligned}$$

We note that, in general, we cannot ensure that the stabilized bilinear form $\overline{a}^{(i)}(\cdot, \cdot)$ is positive definite on $W^{(i)}$ since the boundary integral on Γ_i does not vanish and the sign of $\boldsymbol{a} \cdot \boldsymbol{n}$ depends on the orientation of the flow $\boldsymbol{a}$ in relation to the external normal direction $\boldsymbol{n}$ on Γ_i. We therefore modify $\overline{a}^{(i)}(\cdot, \cdot)$ as in [2] and introduce

$$a^{(i)}(u^{(i)}, v^{(i)}) = \overline{a}^{(i)}(u^{(i)}, v^{(i)}) - \frac{1}{2} \int_{\Gamma_i} \boldsymbol{a} \cdot \boldsymbol{n} u^{(i)} v^{(i)}\, ds, \tag{14}$$

which corresponds to the Robin boundary condition on Γ_i. The assumption (3) now ensures that the modified local bilinear forms $a^{(i)}(\cdot, \cdot)$ are positive definite on $W^{(i)}$, $i = 1, 2. \dots, N$. The symmetric and skew-symmetric parts of $a^{(i)}(u^{(i)}, v^{(i)})$ are represented, respectively, by

$$b^{(i)}(u^{(i)}, v^{(i)}) = \int_{\Omega_i} \left(\nu \nabla u^{(i)} \cdot \nabla v^{(i)} + C(x) L u^{(i)} L v^{(i)} + \tilde{c} u^{(i)} v^{(i)}\right) dx, \tag{15}$$

$$z^{(i)}(u^{(i)}, v^{(i)}) = \frac{1}{2} \int_{\Omega_i} \left(\boldsymbol{a} \cdot \nabla u^{(i)} v^{(i)} - \boldsymbol{a} \cdot \nabla v^{(i)} u^{(i)}\right) dx. \tag{16}$$

We now introduce a partially subassembled finite element space, which was introduced by Klawonn et al. [22] in their analysis of FETI-DP algorithms for symmetric positive definite problems. The partially subassembled finite element space $\widetilde{W}$ is the direct sum of a coarse level primal subspace $\widehat{W}_\Pi$, which is a space of continuous coarse level finite element functions, and a dual subspace W_r, which is the product of local dual spaces $W_r^{(i)}$. The space $\widehat{W}_\Pi$ corresponds to a few selected subdomain interface degrees of freedom for each subdomain and is typically spanned by subdomain vertex nodal basis functions, and/or interface edge and/or face basis functions with weights at the nodes of the edge or face. These basis functions will correspond to the primal interface continuity constraints enforced in the BDDC algorithm. To simplify our analysis, we will always assume that the basis has been changed so that we have explicit primal unknowns corresponding to the primal continuity constraints of edges or faces; these coarse level primal degrees of freedom are shared by neighboring subdomains. For more details on the change of basis, see [21; 25]. Each subdomain dual space $W_r^{(i)}$ corresponds to the subdomain interior and dual interface degrees of freedom and it is spanned by all the basis functions of $W^{(i)}$ which vanish at the primal degrees of freedom. Thus, functions in the space $\widetilde{W}$ have a continuous coarse level, primal part and typically a discontinuous dual part across the subdomain interfaces. We have $\widehat{W} \subset \widetilde{W}$ and we denote the injection operator from $\widehat{W}$ to $\widetilde{W}$ by $\widetilde{R}$.

We define the bilinear form on the partially subassembled finite element space $\widetilde{W}$ by

$$\widetilde{a}_o(u, v) = \sum_{i=1}^{N} a_o^{(i)}(u^{(i)}, v^{(i)}), \qquad \text{for all } \ u, v \in \widetilde{W},$$

where $u^{(i)}$ and $v^{(i)}$ represent restrictions of u and v to subdomain Ω_i. Corresponding to the stabilized forms, we define, for all $u, v \in \widetilde{W}$,

$$\widetilde{a}(u, v) = \sum_{i=1}^{N} a^{(i)}(u^{(i)}, v^{(i)}), \qquad \widetilde{b}(u, v) = \sum_{i=1}^{N} b^{(i)}(u^{(i)}, v^{(i)}),$$
$$\widetilde{z}(u, v) = \sum_{i=1}^{N} z^{(i)}(u^{(i)}, v^{(i)}).$$

Denote the partially subassembled matrices corresponding to the bilinear forms $\widetilde{a}(\cdot,\cdot)$, $\widetilde{b}(\cdot,\cdot)$, and $\widetilde{z}(\cdot,\cdot)$ by $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{Z}$, respectively. We have $A = \widetilde{R}^T \widetilde{A} \widetilde{R}$, $B = \widetilde{R}^T \widetilde{B} \widetilde{R}$, and $Z = \widetilde{R}^T \widetilde{Z} \widetilde{R}$. We note that the use of the modified bilinear form $a^{(i)}(\cdot,\cdot)$, defined in Equation (14) corresponding to the Robin boundary condition, does not affect the matrix A of the original problem when it is assembled from $\widetilde{A}$, since the additional interface terms in (14) cancel.

We define broken norms on the space $\widetilde{W}$ by

$$\|w\|_{L^2(\Omega)}^2 = \sum_{i=1}^{N} \|w^{(i)}\|_{L^2(\Omega_i)}^2 \qquad |w|_{H^1(\Omega)}^2 = \sum_{i=1}^{N} |w^{(i)}|_{H^1(\Omega_i)}^2.$$

In this paper, $\|w\|_{L^2(\Omega)}$ and $|w|_{H^1(\Omega)}$, for functions $w \in \widetilde{W}$, always represent these broken norms. Since the subdomain bilinear forms $b^{(i)}(\cdot,\cdot)$, $i = 1, 2, ..., N$, are symmetric positive definite on $W^{(i)}$, we define

$$\|u^{(i)}\|^2_{B^{(i)}} = b^{(i)}(u^{(i)}, u^{(i)}) \quad \text{and} \quad \|u^{(i)}\|^2_{B_0^{(i)}} = b^{(i)}(u^{(i)}, u^{(i)}) - \int_{\Omega_i} \tilde{c} u^{(i)} u^{(i)}\, dx,$$

for any $u^{(i)} \in W^{(i)}$. We define

$$\|u\|^2_B = \sum_{i=1}^N \|u^{(i)}\|^2_{B^{(i)}}, \qquad \|u\|^2_{B_0} = \sum_{i=1}^N \|u^{(i)}\|^2_{B_0^{(i)}},$$

for any $u \in \widehat{W}$, and

$$\|w\|^2_{\widetilde{B}} = \sum_{i=1}^N \|w^{(i)}\|^2_{B^{(i)}}, \qquad \|w\|^2_{\widetilde{B}_0} = \sum_{i=1}^N \|w^{(i)}\|^2_{B_0^{(i)}},$$

for any $w \in \widetilde{W}$. All the B-, B_0-, $\widetilde{B}$-, and $\widetilde{B}_0$- norms are also well defined for functions in the space $H^2(\Omega)$.

Lemmas 4.1 and 4.2 are immediate consequences of the definitions of $B^{(i)}$-, $B_0^{(i)}$- and B_0- norms.

Lemma 4.1. There exists a positive constant C, such that for all $w^{(i)} \in W^{(i)}$, $i = 1, 2, ..., N$,

$$\begin{aligned}
\|w^{(i)}\|_{B^{(i)}} &\leq C \|w^{(i)}\|_{H^1(\Omega_i)}, \\
\sqrt{\nu}|w^{(i)}|_{H^1(\Omega_i)} &\leq \|w^{(i)}\|_{B_0^{(i)}}, \\
\|w^{(i)}\|_{B_0^{(i)}} &\leq C \max(\sqrt{\nu}, \sqrt{C_s}) \|w^{(i)}\|_{H^1(\Omega_i)}, \\
\max(\sqrt{\nu}, \sqrt{C_m}) |\boldsymbol{a} \cdot w^{(i)}|_{H^1(\Omega_i)} &\leq \|w^{(i)}\|_{B_0^{(i)}}.
\end{aligned}$$

Lemma 4.2. For all $u \in H^2(\Omega)$, $\|u\|_{B_0} \leq \max(\sqrt{\nu}, \sqrt{C_s}) \|u\|_{H^2(\Omega)}$.

Lemma 4.3. There exists a positive constant C, such that for all $u^{(i)}, v^{(i)} \in W^{(i)}$, $i = 1, 2, ..., N$,

$$|z^{(i)}(u^{(i)}, v^{(i)})| \leq C \frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})} \|u^{(i)}\|_{B^{(i)}} \|v^{(i)}\|_{B^{(i)}},$$

and

$$|a^{(i)}(u^{(i)}, v^{(i)})| \leq C \frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})} \|u^{(i)}\|_{B^{(i)}} \|v^{(i)}\|_{B^{(i)}}.$$

Proof. We only need prove the first inequality; the second one immediately follows. Since

$$|\boldsymbol{a} \cdot \nabla u^{(i)} v^{(i)}| \leq \frac{C}{\max(\sqrt{\nu}, \sqrt{C_m})} \sqrt{\nu |\nabla u|^2 + C(x) |\boldsymbol{a} \cdot \nabla u|^2} \cdot |v|,$$

we have

$$
\begin{aligned}
|z^{(i)}(u^{(i)}, v^{(i)})| &\le \frac{C}{\max(\sqrt{\nu}, \sqrt{C_m})} \cdot \Big(\int_\Omega \sqrt{\nu |\nabla u|^2 + C(x) |\boldsymbol{a} \cdot \nabla u|^2} \cdot |v| \, dx \\
&\qquad + \int_\Omega \sqrt{\nu |\nabla v|^2 + C(x) |\boldsymbol{a} \cdot \nabla v|^2} \cdot |u| \, dx \Big) \\
&\le \frac{C}{\max(\sqrt{\nu}, \sqrt{C_m})} \cdot \Big(\int_\Omega u^2 + \nu |\nabla u|^2 + C(x) |\boldsymbol{a} \cdot \nabla u|^2 \, dx \Big)^{1/2} \\
&\qquad \cdot \Big(\int_\Omega v^2 + \nu |\nabla v|^2 + C(x) |\boldsymbol{a} \cdot \nabla v|^2 \, dx \Big)^{1/2} \\
&\le \frac{C}{\max(\sqrt{\nu}, \sqrt{C_m})} \cdot \Big(\int_\Omega (1 + C(x) c^2) u^2 + \nu |\nabla u|^2 + C(x) |\boldsymbol{a} \cdot \nabla u + cu|^2 \, dx \Big)^{1/2} \\
&\qquad \cdot \Big(\int_\Omega (1 + C(x) c^2) v^2 + \nu |\nabla v|^2 + C(x) |\boldsymbol{a} \cdot \nabla v + cv|^2 \, dx \Big)^{1/2} \\
&\le C \frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})} \|u^{(i)}\|_{B^{(i)}} \|v\|_{B^{(i)}},
\end{aligned}
$$

where we use the Cauchy–Schwartz inequality in the middle. □

Lemma 4.4. There exists a positive constant C, such that, for all $u, v \in \widehat{W}$,

$$
|z(u, v)| \le C \, \frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})} \, \|u\|_B \, \|v\|_{L^2(\Omega)}.
$$

Proof. We find, by integration by parts and using Lemma 4.1, that

$$
\begin{aligned}
|z(u, v)| &\le \frac{1}{2} \int_\Omega |2\boldsymbol{a} \cdot \nabla u v + \nabla \cdot \boldsymbol{a} u v| \, dx \\
&\le C \left(a_s |u|_{H^1(\Omega)} \|v\|_{L^2(\Omega)} + \|\nabla \cdot \boldsymbol{a}\|_\infty \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \right) \\
&\le C \frac{1}{\sqrt{\nu}} \|u\|_B \|v\|_{L^2(\Omega)}.
\end{aligned}
$$

At the same time we have

$$
\begin{aligned}
|z(u, v)| &\le \frac{1}{2} \int_\Omega |2\boldsymbol{a} \cdot \nabla u v + \nabla \cdot \boldsymbol{a} u v| \, dx \\
&\le \int_\Omega \left(|(\boldsymbol{a} \cdot \nabla u + cu) v| + |cuv| \right) \, dx + \frac{1}{2} \int_\Omega |\nabla \cdot \boldsymbol{a} u v| \, dx \\
&\le C \frac{1}{\sqrt{C_m}} \|u\|_B \|v\|_{L^2(\Omega)}.
\end{aligned}
$$

□

Lemma 4.5. There exists a positive constant C, such that for all $u^{(i)}, v^{(i)} \in W^{(i)}$, if $v^{(i)}$ has zero values on $\partial\Omega_i$, then $|a^{(i)}(u^{(i)}, v^{(i)})| \le C \|u^{(i)}\|_{H^1(\Omega_i)} \|v^{(i)}\|_{B^{(i)}}$.

Proof. As in the proof of Lemma 4.4, we have, by integration by parts,

$$
\begin{aligned}
|z^{(i)}(u^{(i)}, v^{(i)})| &\leq \frac{1}{2}\int_{\Omega} |2\boldsymbol{a}\cdot\nabla u^{(i)} v^{(i)} + \nabla\cdot\boldsymbol{a} u^{(i)} v^{(i)}|\, dx \\
&\leq C\Big(a_s|u^{(i)}|_{H^1(\Omega_i)}\|v^{(i)}\|_{L^2(\Omega_i)} \\
&\qquad\qquad + \|\nabla\cdot\boldsymbol{a}\|_{L^\infty(\Omega)}\|u^{(i)}\|_{L^2(\Omega_i)}\|v^{(i)}\|_{L^2(\Omega_i)}\Big) \\
&\leq C\|u^{(i)}\|_{H^1(\Omega_i)}\|v^{(i)}\|_{B^{(i)}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|a^{(i)}(u^{(i)}, v^{(i)})| \leq |b^{(i)}(u^{(i)}, v^{(i)})| + |z^{(i)}(u^{(i)}, v^{(i)})| \\
&\quad \leq C\|u^{(i)}\|_{B^{(i)}}\|v^{(i)}\|_{B^{(i)}} + \|u^{(i)}\|_{H^1(\Omega_i)}\|v^{(i)}\|_{B^{(i)}} \leq C\|u^{(i)}\|_{H^1(\Omega_i)}\|v^{(i)}\|_{B^{(i)}},
\end{aligned}
$$

where we use Lemma 4.1 at the last step. □

We also have the following approximation property in B-norm for the finite element space $\widehat{W}$.

Lemma 4.6. There exists a positive constant C, such that for all $u \in H^2(\Omega)$,

$$
\inf_{w\in\widehat{W}} \|u-w\|_B \leq C\max(\sqrt{\nu}, \sqrt{C_s}, h)h|u|_{H^2(\Omega)}.
$$

Proof. We have, for any $u \in H^2(\Omega)$ and $w \in \widehat{W}$, that

$$
\begin{aligned}
\|u-w\|_B^2 &= b(u-w, u-w) \leq \nu|u-w|^2_{H^1(\Omega)} \\
&\qquad + C_s\|L(u-w)\|^2_{L^2(\Omega)} + \tilde{c}_s\|u-w\|^2_{L^2(\Omega)} \\
&= \nu|u-w|^2_{H^1(\Omega)} + C_s\|\nu\Delta u + \boldsymbol{a}\cdot\nabla(u-w) + c(u-w)\|^2_{L^2(\Omega)} \\
&\qquad + \tilde{c}_s\|u-w\|^2_{L^2(\Omega)} \\
&\leq \nu^2 C_s|u|^2_{H^2(\Omega)} + (\nu + C_s a_s^2)|u-w|^2_{H^1(\Omega)} \\
&\qquad + (C_s c_s^2 + \tilde{c}_s^2)\|u-w\|^2_{L^2(\Omega)}.
\end{aligned}
$$

We complete the proof by using (7) and the following standard finite element approximation results [33, Lemma B.6],

$$
\inf_{w\in\widehat{W}} \left\{\|u-w\|^2_{L^2(\Omega)} + h^2|u-w|^2_{H^1(\Omega)}\right\} \leq Ch^4|u|^2_{H^2(\Omega)}.
$$

□

For each subdomain interface edge $\mathscr{E}^k$, let $\vartheta_{\mathscr{E}^k}$ be the standard finite element edge cut-off function which vanishes at all interface nodes except those of the edge $\mathscr{E}^k$ where it takes the value 1. For three-dimensional problems, we denote the finite element face cut-off functions by $\vartheta_{\mathscr{F}^l}$, which vanishes at all interface nodes except those of $\mathscr{F}^l$ where it takes the value 1. Let I_h be the interpolation operator into the finite element space. In the convergence analysis of our BDDC algorithm for advection-diffusion problems, we require that the coarse level primal subspace $\widehat{W}_\Pi$ satisfies the following assumption.

Assumption 4.7. For two-dimensional problems, the coarse level primal subspace $\widehat{W}_\Pi$ contains all subdomain corner degrees of freedom, and for each edge $\mathscr{E}^k$, one edge average degree of freedom and two edge flux average degrees of freedom such that for any $w \in \widetilde{W}$,

$$\int_{\mathscr{E}^k} w^{(i)}\, ds, \quad \int_{\mathscr{E}^k} \boldsymbol{a} \cdot \boldsymbol{n} w^{(i)}\, ds, \quad \int_{\mathscr{E}^k} \boldsymbol{a} \cdot \boldsymbol{n} w^{(i)} s\, ds,$$

respectively, are the same (with a difference of factor -1 corresponding to opposite normal directions) for the two subdomains Ω_i that share $\mathscr{E}^k$.

For three dimensional problems, $\widehat{W}_\Pi$ contains all subdomain corner degrees of freedom, and for each face $\mathscr{F}^l$, one face average degree of freedom and two face flux average degrees of freedom, and for each edge $\mathscr{E}^k$, one edge average degree of freedom, such that for any $w \in \widetilde{W}$,

$$\int_{\mathscr{F}^l} I_h\left(\vartheta_{\mathscr{F}^l} w^{(i)}\right)\, ds, \quad \int_{\mathscr{F}^l} \boldsymbol{a} \cdot \boldsymbol{n} I_h\left(\vartheta_{\mathscr{F}^l} w^{(i)}\right)\, ds, \quad \int_{\mathscr{F}^l} \boldsymbol{a} \cdot \boldsymbol{n} I_h\left(\vartheta_{\mathscr{F}^l} w^{(i)}\right) s\, ds,$$

respectively, are the same (with a difference of factor -1 corresponding to opposite normal directions) for the two subdomains Ω_i that share the face $\mathscr{F}^l$, and

$$\int_{\mathscr{E}^k} I_h\left(\vartheta_{\mathscr{E}^k} w^{(i)}\right)\, ds$$

are the same for all subdomains Ω_i that share the edge $\mathscr{E}^k$.

We will need an error bound for the approximation of partially subassembled finite element problems in the analysis of our BDDC algorithm. For this purpose, we make an assumption for our decomposition of the global domain Ω.

Assumption 4.8. Each subdomain Ω_i is triangular or quadrilateral in two dimensions, and tetrahedral or hexahedral in three dimensions. The subdomains form a shape regular coarse mesh of Ω.

Under Assumption 4.8, we can denote by $\widehat{W}_H$ the continuous linear, bilinear, or trilinear finite element space on the coarse subdomain mesh, and denote by I_H the finite element interpolation operator into $\widehat{W}_H$. We have the following Bramble–Hilbert lemma [39, Theorem 2.3]:

Lemma 4.9. There exists a constant C, such that for all $u \in H^2(\Omega)$, $i = 1, 2, ..., N$,

$$\|u - I_H u\|_{H^t(\Omega_i)} \leq C H^{2-t} |u|_{H^2(\Omega_i)},$$

where $t = 0, 1, 2$.

5. Error estimate for a partially subassembled finite element problem

In this section, we prove an error bound for the solution of a partially subassembled finite element problem.

Given $g \in L^2(\Omega)$, we define

$$\varphi_g \in H_0^1(\Omega) \quad \text{and} \quad \widetilde{\varphi}_g \in \widetilde{W}$$

as the solutions to the following problems, respectively,

$$a_o(u, \varphi_g) = (u, g), \qquad \text{for all } u \in H_0^1(\Omega), \tag{17}$$

$$\widetilde{a}_o(w, \widetilde{\varphi}_g) + \int_\Omega C(x) L^* w L^* \widetilde{\varphi}_g \, dx = (w, g) + \int_\Omega C(x) L^* w g \, dx, \text{ for all } w \in \widetilde{W}. \tag{18}$$

We know from Equation (17) that φ_g is the weak solution to the adjoint problem $L^* \varphi_g = g$, and $\varphi_g \in H^2(\Omega)$ under the regularity assumption Equation (6). We have the following result.

Lemma 5.1. Let Assumption 4.7 hold. For any $g \in L^2(\Omega)$, let φ_g be the solution to Equation (17) and let $L_h(q, \varphi_g) = \widetilde{a}_o(q, \varphi_g) - (q, g)$, for $q \in \widetilde{W}$. There exists a constant C, such that for all $q \in \widetilde{W}$,

$$|L_h(q, \varphi_g)| \le C C_L(\nu, H, h) \|\varphi_g\|_{H^2} \|q\|_{\widetilde{B}},$$

where

$$C_L(\nu, H, h) = \begin{cases} \frac{1}{\sqrt{\nu}} \max\left(H\nu, H^2\right), & d = 2, \\ \frac{1}{\sqrt{\nu}} \max\left(H\nu, H^2, \sqrt{Hh}\right)\left(1 + \log\left(\frac{H}{h}\right)\right), & d = 3. \end{cases}$$

Proof. We give the proof only for the three-dimensional case; the two-dimensional case can be proved in a similar manner. For any $q \in \widetilde{W}$, we have

$$\begin{aligned} L_h(q, \varphi_g) &= \widetilde{a}_o(q, \varphi_g) - (q, g) \\ &= \sum_{i=1}^N \int_{\Omega_i} \left(\nu \nabla q^{(i)} \nabla \varphi_g + \boldsymbol{a} \cdot \nabla q^{(i)} \varphi_g + c q^{(i)} \varphi_g - q^{(i)} g\right) dx \\ &= \sum_{i=1}^N \left\{ \int_{\partial\Omega_i} \left(\nu \partial_n \varphi_g q^{(i)} + \boldsymbol{a} \cdot \boldsymbol{n} \varphi_g q^{(i)}\right) ds \right. \\ &\qquad \left. - \int_{\Omega_i} \left(\nu \Delta \varphi_g q^{(i)} + \nabla \cdot (\boldsymbol{a} \varphi_g) q^{(i)} - c \varphi_g q^{(i)} + g q^{(i)}\right) dx \right\} \\ &= \sum_{i=1}^N \int_{\partial\Omega_i} \left(\nu \partial_n \varphi_g q^{(i)} + \boldsymbol{a} \cdot \boldsymbol{n} \varphi_g q^{(i)}\right) ds \end{aligned}$$

$$= \sum_{i=1}^{N} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \int_{\Gamma_{ij}} \left(\nu \partial_n \varphi_g q^{(i)} + \boldsymbol{a} \cdot \boldsymbol{n} \varphi_g q^{(i)} \right) ds,$$

where we use the fact that $L^* \varphi_g = g$ holds in the weak sense. Here Γ_{ij} represents the boundary faces of Ω_i.

Denote the common average of q on the face $\mathscr{F}^l$ of Γ_{ij} by $\overline{q}_{\mathscr{F}^l}$ and its common averages on the edges $\mathscr{E}^{lk}$ by $\overline{q}_{\mathscr{E}^{lk}}$. Since the finite element cut-off functions $\vartheta_{\mathscr{F}^l}$ and $\vartheta_{\mathscr{E}^{lk}}$ provide a partition of unity [33, Section 4.6], we have

$$\begin{aligned} L_h(q, \varphi_g) = &\\ \sum_{i=1}^{N} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \Bigg\{ &\int_{\mathscr{F}^l} \Big(\nu \partial_n \varphi_g I_h(\vartheta_{\mathscr{F}^l}(q^{(i)} - \overline{q}_{\mathscr{F}^l})) + \boldsymbol{a} \cdot \boldsymbol{n} \varphi_g I_h(\vartheta_{\mathscr{F}^l}(q^{(i)} - \overline{q}_{\mathscr{F}^l})) \Big) ds \\ &+ \sum_{\mathscr{E}^{lk} \subset \Gamma_{ij}} \int_{\mathscr{F}^l} \Big(\nu \partial_n \varphi_g I_h(\vartheta_{\mathscr{E}^{lk}}(q^{(i)} - \overline{q}_{\mathscr{E}^{lk}})) + \boldsymbol{a} \cdot \boldsymbol{n} \varphi_g I_h(\vartheta_{\mathscr{E}^{lk}}(q^{(i)} - \overline{q}_{\mathscr{E}^{lk}})) \Big) ds \Bigg\}, \end{aligned} \tag{19}$$

where we have also subtracted the constant average values $\overline{q}_{\mathscr{F}^l}$ and $\overline{q}_{\mathscr{E}^{lk}}$ from $q^{(i)}$, which does not change the sum.

Then, from Assumption 4.7, we know that

$$\begin{aligned} L_h(q, \varphi_g) = &\sum_{i=1}^{N} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \int_{\mathscr{F}^l} \left(\nu \partial_n (\varphi_g - I_H^{(i)} \varphi_g) \left(I_h \left(\vartheta_{\mathscr{F}^l}(q^{(i)} - \overline{q}_{\mathscr{F}^l}) \right) \right) \right) ds \\ &+ \sum_{i=1}^{N} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \int_{\mathscr{F}^l} \left(\boldsymbol{a} \cdot \boldsymbol{n} (\varphi_g - I_H^{(i)} \varphi_g) \left(I_h \left(\vartheta_{\mathscr{F}^l}(q^{(i)} - \overline{q}_{\mathscr{F}^l}) \right) \right) \right) ds \\ &+ \sum_{i=1}^{N} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \sum_{\mathscr{E}^{lk} \subset \Gamma_{ij}} \int_{\mathscr{F}^l} \left(\nu \partial_n \varphi_g \left(I_h \left(\vartheta_{\mathscr{E}^{lk}}(q^{(i)} - \overline{q}_{\mathscr{E}^{lk}}) \right) \right) \right) ds \\ &+ \sum_{i=1}^{N} \sum_{\Gamma_{ij} \subset \partial\Omega_i} \sum_{\mathscr{E}^{lk} \subset \Gamma_{ij}} \int_{\mathscr{F}^l} \left(\boldsymbol{a} \cdot \boldsymbol{n} \varphi_g \left(I_h \left(\vartheta_{\mathscr{E}^{lk}}(q^{(i)} - \overline{q}_{\mathscr{E}^{lk}}) \right) \right) \right) ds \\ =: &\; I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{20}$$

where $I_H \varphi_g$ represents the interpolation of φ_g into the space $\widehat{W}_H$ on the coarse subdomain mesh. Each of the four terms in Equation (20) is bounded as follows.

For the first term I_1, from the Cauchy–Schwarz inequality, we have

$$|I_1| \le \nu \sum_{i=1}^{N} \sum_{\Gamma_{ij} \subset \partial\Omega} \left(\int_{\mathscr{F}^l} |\nabla(\varphi_g - I_H^{(i)} \varphi_g)|^2 ds \int_{\mathscr{F}^l} |I_h(\vartheta_{\mathscr{F}^l}(q^{(i)} - \overline{q}_{\mathscr{F}^l}))|^2 ds \right)^{1/2} \tag{21}$$

Using a trace theorem and Lemma 4.9, we have for the first factor

$$\begin{aligned}\int_{\mathcal{F}^l} |\nabla(\varphi_g - I_H^{(i)}\varphi_g)|^2\, ds &\le CH\|\nabla(\varphi_g - I_H^{(i)}\varphi_g)\|_{H^1(\Omega_i)}^2 \\ &\le CH\|\varphi_g - I_H^{(i)}\varphi_g\|_{H^2(\Omega_i)}^2 \le CH|\varphi_g|_{H^2(\Omega_i)}^2.\end{aligned} \tag{22}$$

For the second factor, we have

$$\begin{aligned}\int_{\mathcal{F}^l} |I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \overline{q}_{\mathcal{F}^l}))|^2\, ds &\le CH\|I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \overline{q}_{\mathcal{F}^l}))\|_{H^1(\Omega_i)}^2 \\ \le CH(1+\log\frac{H}{h})^2\|q^{(i)} - \overline{q}_{\mathcal{F}^l}\|_{H^1(\Omega_i)}^2 &\le CH\big(1+\log\frac{H}{h}\big)^2|q^{(i)}|_{H^1}^2,\end{aligned} \tag{23}$$

where we have used a trace theorem for the first step, a Poincaré–Friedrichs inequality and Lemma 4.24 of [33] for the second step, and a Poincaré–Friedrichs inequality in the last step. Combining Equation (21), (22), and (23), we have the following bound for I_1,

$$\begin{aligned}|I_1| &\le C\nu H(1+\log\frac{H}{h})\sum_{i=1}^{N}|\varphi_g|_{H^2(\Omega_i)}|q|_{H^1(\Omega_i)} \\ &\le C\frac{\nu H(1+\log\frac{H}{h})}{\sqrt{\nu}}|\varphi_g|_{H^2(\Omega)}\|q\|_{\widetilde{B}},\end{aligned}$$

where we use the Cauchy–Schwarz inequality and Lemma 4.1 in the last step.

To derive a bound for I_2, we find from the Cauchy–Schwarz inequality that

$$|I_2| \le \sum_{i=1}^{N}\sum_{\Gamma_{ij}\subset\partial\Omega}\Big(\int_{\mathcal{F}^l}|\varphi_g - I_H^{(i)}\varphi_g|^2 ds\int_{\mathcal{F}^l}|I_h(\vartheta_{\mathcal{F}^l}(q^{(i)}-\overline{q}_{\mathcal{F}^l}))|^2 ds\Big)^{1/2}. \tag{24}$$

Using a trace theorem and Lemma 4.9, we have, for the first factor on the right hand side of Equation (24),

$$\int_{\mathcal{F}^l} |\varphi_g - I_H^{(i)}\varphi_g|^2\, ds \le CH\|\varphi_g - I_H^{(i)}\varphi_g\|_{H^1(\Omega_i)}^2 \le CH^3|\varphi_g|_{H^2(\Omega_i)}^2. \tag{25}$$

Combining Equation (24), (25), and (23), and using Lemma 4.1, we have

$$\begin{aligned}|I_2| &\le CH^2(1+\log\frac{H}{h})\sum_{i=1}^{N}|\varphi_g|_{H^2(\Omega_i)}|q|_{H^1(\Omega_i)} \\ &\le C\frac{H^2(1+\log\frac{H}{h})}{\sqrt{\nu}}|\varphi_g|_{H^2(\Omega)}\|q\|_{\widetilde{B}}.\end{aligned}$$

The estimate for I_3 is similar to the estimate for I_1. Instead of using Equation (22) and (23), we have, by using a trace theorem,

$$\int_{\mathscr{F}^l} |\nabla \varphi_g|^2 \, ds \le CH \|\nabla \varphi_g\|^2_{H^1(\Omega_i)} \le CH \|\varphi_g\|^2_{H^2(\Omega_i)}, \tag{26}$$

and

$$\begin{aligned} \int_{\mathscr{F}^l} |I_h \vartheta_{\mathscr{E}^{lk}} (q^{(i)} - \overline{q}_{\mathscr{E}^{lk}})|^2 \, ds &\le Ch \|I_h \vartheta_{\mathscr{E}^{lk}} (q^{(i)} - \overline{q}_{\mathscr{E}^{lk}})\|^2_{L^2(\mathscr{E}^{lk})} \\ &\le Ch(1 + \log \frac{H}{h}) \|I_h \vartheta_{\mathscr{E}^{lk}} (q^{(i)} - \overline{q}_{\mathscr{E}^{lk}})\|^2_{H^1(\Omega_i)} \\ &\le Ch(1 + \log \frac{H}{h})^2 |q^{(i)}|^2_{H^1(\Omega_i)}. \end{aligned} \tag{27}$$

In the first step of Equation (27), we use the fact that $I_h \vartheta_{\mathscr{E}^{lk}} (q^{(i)} - \overline{q}_{\mathscr{E}^{lk}})$ is different from zero only in the strip of elements next to the edge $\mathscr{E}^{lk}$; in the second and the last steps, we use [33, Lemma 4.16], [33, Corollary 4.20], and a Poincaré–Friedrichs inequality. Combining Equation (26) and (27), we have

$$\begin{aligned} |I_3| &\le C\nu \sqrt{Hh} (1 + \log \frac{H}{h}) \sum_{i=1}^{N} \|\varphi_g\|_{H^2(\Omega_i)} |q|_{H^1(\Omega_i)} \\ &\le C \frac{\nu \sqrt{Hh} (1 + \log \frac{H}{h})}{\sqrt{\nu}} \|\varphi_g\|_{H^2(\Omega)} \|q\|_{\widetilde{B}}. \end{aligned}$$

Similarly, for I_4, we have

$$|I_4| \le C \frac{\sqrt{Hh} (1 + \log \frac{H}{h})}{\sqrt{\nu}} \|\varphi_g\|_{H^2(\Omega)} \|q\|_{\widetilde{B}}.$$

□

Remark 5.2. In the case of two-dimensional problems, Equation (19) becomes

$$L_h(q, \varphi_g) = \sum_{i=1}^{N} \sum_{\mathscr{E}^{ij} \subset \partial\Omega_i} \int_{\mathscr{E}^{ij}} \big(\nu \partial_n \varphi_g (q^{(i)} - \overline{q}_{\mathscr{E}^{ij}}) + \boldsymbol{a} \cdot \boldsymbol{n} \, \varphi_g (q^{(i)} - \overline{q}_{\mathscr{E}^{ij}})\big) \, ds,$$

where the finite element cut-off functions are no longer used. The bound for $L_h(q, \varphi_g)$ then follows from the similar steps as for the bounds of I_1 and I_2 in the proof applied on the edges, where the logarithmic factor related to the use of finite element cut-off functions disappears.

Remark 5.3. The factor in the bound of I_2 is proportional to $H^2/\sqrt{\nu}$, where H compensates for the effect of small ν in the advection-dominated case. Without using the two face flux average continuity constraints as in Assumption 4.7, this factor would become proportional to $H/\sqrt{\nu}$ instead. For two-dimensional problems, the same benefits can be obtained by enforcing the two edge flux average continuity constraints as in Assumption 4.7. Our numerical experiments in Section 8 show the effectiveness of using the two edge flux constraints for two-dimensional examples. The factor in the bound of I_4 (only appearing for three-dimensional problems) is proportional to $\sqrt{Hh/\nu}$ where $\sqrt{Hh}$ can be used to compensate for the effect of small ν; in fact this factor can be improved to $\sqrt{H^3h/\nu}$ by introducing a few extra edge normal flux average constraints [24, Equation (35)].

Lemma 5.4. There exists a positive constant C, such that for all $q \in \widetilde{W}$,

$$\int_\Omega C(x)\left(LqL(\widetilde{\varphi}_g - \widetilde{R}\varphi_g) - L^*qL^*(\widetilde{\varphi}_g - \widetilde{R}\varphi_g)\right)dx \\ \le C\sqrt{C_s}\|q\|_{\widetilde{B}}\left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right).$$

Proof. We have

$$\begin{aligned}
&\int_\Omega C(x)\left(LqL(\widetilde{\varphi}_g - \widetilde{R}\varphi_g) - L^*qL^*(\widetilde{\varphi}_g - \widetilde{R}\varphi_g)\right)dx \\
&\quad= \int_\Omega C(x)\left((L+L^*)qL(\widetilde{\varphi}_g - \widetilde{R}\varphi_g) - L^*q(L+L^*)(\widetilde{\varphi}_g - \widetilde{R}\varphi_g)\right)dx \\
&\quad\le \left(\int_\Omega C(x)((L+L^*)q)^2\right)^{1/2}\left(\int_\Omega C(x)\big(L(\widetilde{\varphi}_g - \widetilde{R}\varphi_g)\big)^2dx\right)^{1/2} \\
&\qquad\qquad + \left(\int_\Omega C(x)(L^*q)^2\right)^{1/2}\left(\int_\Omega C(x)\big((L+L^*)(\widetilde{\varphi}_g - \widetilde{R}\varphi_g)\big)^2dx\right)^{1/2} \\
&\quad\le 2\sqrt{C_s}\|q\|_{\widetilde{B}}\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}_0} + 2\sqrt{C_s}\left(\int_\Omega C(x)(-\boldsymbol{a}\cdot\nabla q + (c - \nabla\cdot\boldsymbol{a})q)^2\right)^{1/2} \\
&\qquad\qquad \times\left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right) \\
&\quad\le C\sqrt{C_s}\|q\|_{\widetilde{B}}\left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right).
\end{aligned}$$
□

Lemma 5.5. There exists a positive constant C, such that for all $q \in \widetilde{W}$ and $u \in \widetilde{W} \cup H^2(\Omega)$,

$$\int_\Omega C(x)LqLudx \le C\frac{\sqrt{C_s}}{\max(\sqrt{\nu}, \sqrt{C_m})}\|q\|_{\widetilde{B}}\|u\|_{\widetilde{B}_0}.$$

Proof. We have

$$\begin{aligned}
\int_\Omega C(x) Lq Lu dx &= \int_\Omega C(x)(\boldsymbol{a}\cdot\nabla q + cq) Lu dx \\
&\le \sqrt{C_s}\Big(\int_\Omega (\boldsymbol{a}\cdot\nabla q + cq)^2\Big)^{1/2}\Big(\int_\Omega C(x)(Lu)^2 dx\Big)^{1/2} \\
&\le \sqrt{C_s}\Big(\int_\Omega (\boldsymbol{a}\cdot\nabla q + cq)^2\Big)^{1/2}\Big(\int_\Omega C(x)(Lu)^2 dx\Big)^{1/2} \\
&\le C\frac{\sqrt{C_s}}{\max(\sqrt{\nu},\sqrt{C_m})}\|q\|_{\widetilde{B}}\|u\|_{\widetilde{B}_0},
\end{aligned}$$

where we use Lemma 4.1 for the last inequality. □

Lemma 5.6. Let Assumption 4.7 hold. φ_g and $\widetilde{\varphi}_g$ are solutions of Equation (17) and Equation (18), respectively, for $g \in L^2(\Omega)$. If h is sufficiently small, then there exists a positive constant C, such that

$$\|\varphi_g - \widetilde{\varphi}_g\|_{\widetilde{B}} \le C C_L(\nu, H, h)\|\varphi_g\|_{H^2},$$

where $C_L(\nu, H, h)$ is given in Lemma 5.1.

Proof. For any $\widetilde{\psi} \in \widetilde{W}$, we have

$$\begin{aligned}
\|\widetilde{\varphi}_g - \widetilde{\psi}\|^2_{\widetilde{B}} &= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \widetilde{\varphi}_g - \widetilde{\psi}) \\
&= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g - \widetilde{\psi}) + (\widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \widetilde{\varphi}_g) - \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g)) \\
&= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g - \widetilde{\psi}) + (\widetilde{a}_o(\widetilde{\varphi}_g - \widetilde{\psi}, \widetilde{\varphi}_g) - \widetilde{a}_o(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g)) \\
&\quad + \int_\Omega C(x) L(\widetilde{\varphi}_g - \widetilde{\psi}) L(\widetilde{\varphi}_g - \varphi_g)\, dx \\
&= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g - \widetilde{\psi}) + ((\widetilde{\varphi}_g - \widetilde{\psi}, g) - \widetilde{a}_o(\widetilde{\varphi}_g - \widetilde{\psi}, \varphi_g)) \\
&\quad + \int_\Omega C(x) L(\widetilde{\varphi}_g - \widetilde{\psi}) L(\widetilde{\varphi}_g - \varphi_g) dx - \int_\Omega C(x) L^*(\widetilde{\varphi}_g - \widetilde{\psi}) L^*(\widetilde{\varphi}_g - \varphi_g) dx,
\end{aligned}$$

where in the last step we use Equation (18) and that $L^*\varphi_g = g$ holds in the weak sense. Dividing by $\|\widetilde{\varphi}_g - \widetilde{\psi}\|_{\widetilde{B}}$ on both sides and denoting $\widetilde{\varphi}_g - \widetilde{\psi}$ by q, we have, from Lemmas 4.3, 5.4, and 5.1, that

$$\begin{aligned}
\|\widetilde{\varphi}_g - \widetilde{\psi}\|_{\widetilde{B}} &\le C\frac{1}{\max(\sqrt{\nu},\sqrt{C_m})}\|\varphi_g - \widetilde{\psi}\|_{\widetilde{B}} + \frac{\big|(q, g) - \widetilde{a}_o(q, \varphi_g)\big|}{\|q\|_{\widetilde{B}}} \\
&\quad + C\sqrt{C_s}\left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right) \\
&\le C\frac{1}{\max(\sqrt{\nu},\sqrt{C_m})}\|\varphi_g - \widetilde{\psi}\|_{\widetilde{B}} + C C_L(\nu, H, h)\|\varphi_g\|_{H^2} \\
&\quad + C\sqrt{C_s}\left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right).
\end{aligned}$$

Then, using a triangle inequality, we have

$$\begin{aligned}\|\varphi_g - \widetilde{\varphi}_g\|_{\widetilde{B}} &\le \inf_{\widetilde{\psi}\in\widetilde{W}} \left\{\|\varphi_g - \widetilde{\psi}\|_{\widetilde{B}} + \|\widetilde{\varphi}_g - \widetilde{\psi}\|_{\widetilde{B}}\right\} \\ &\le C\frac{1}{\max(\sqrt{\nu},\sqrt{C_m})} \inf_{\widetilde{\psi}\in\widetilde{W}} \|\varphi_g - \widetilde{\psi}\|_{\widetilde{B}} + CC_L(\nu,H,h)\|\varphi_g\|_{H^2} \\ &\qquad + C\sqrt{C_s}\left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right) \\ &\le CC_L(\nu,H,h)\|\varphi_g\|_{H^2} + C\sqrt{C_s}\|\widetilde{\varphi}_g - \varphi_g\|_{\widetilde{B}},\end{aligned}$$

where we use Lemma 4.6 in the last step. From Equation (8), we know that if h is small enough such that $C\sqrt{C_s} < 1$, then the second term on the right hand side can be combined with the left hand side and our result is proved. □

6. The BDDC preconditioner

The BDDC algorithms and closely related primal versions of the FETI algorithms were proposed by Dohrmann [12], Fragakis and Papadrakakis [17], and Cros [11], for solving symmetric, positive definite problems. The formulation of BDDC preconditioners can be applied equally well to nonsymmetric problems. In our BDDC algorithm for solving the advection-diffusion problems, the global system of linear equations (12) is reduced to a Schur complement problem for the subdomain interface variables and then a preconditioned GMRES iteration is used to solve the interface problem.

We decompose the space $\widehat{W}$ into $W_I \oplus \widehat{W}_\Gamma$, where W_I is the product of local subdomain spaces $W_I^{(i)}$, $i = 1, 2, ..., N$, corresponding to the subdomain interior variables. $\widehat{W}_\Gamma$ is the subspace corresponding to the variables on the interface. The original discrete problem (12) can be written as: find $u_I \in W_I$ and $u_\Gamma \in \widehat{W}_\Gamma$, such that

$$\begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma}\end{bmatrix}\begin{bmatrix} u_I \\ u_\Gamma\end{bmatrix} = \begin{bmatrix} f_I \\ f_\Gamma\end{bmatrix}, \tag{28}$$

where A_{II} is block diagonal with one block for each subdomain, and $A_{\Gamma\Gamma}$ corresponds to the subdomain interface variables and is assembled from subdomain matrices across the subdomain interfaces.

Eliminating the subdomain interior variables u_I from (28), we have the Schur complement problem

$$S_\Gamma u_\Gamma = g_\Gamma,$$

where $S_\Gamma = A_{\Gamma\Gamma} - A_{\Gamma I}A_{II}^{-1}A_{I\Gamma}$, and $g_\Gamma = f_\Gamma - A_{\Gamma I}A_{II}^{-1}f_I$.

Correspondingly, we define a partially subassembled Schur complement operator $\widetilde{S}_\Gamma$ as follows. We decompose the space $\widetilde{W}$ into $W_I \oplus \widetilde{W}_\Gamma$. Here $\widetilde{W}_\Gamma$ contains the coarse level, continuous primal interface degrees of freedom, in the subspace $\widehat{W}_\Pi$, which are shared by neighboring subdomains, and the remaining dual subdomain interface degrees of freedom which are in general discontinuous across the subdomain interfaces. Then the partially subassembled problem matrix $\widetilde{A}$ can be written in a two by two block form

$$\begin{bmatrix} A_{II} & \widetilde{A}_{I\Gamma} \\ \widetilde{A}_{\Gamma I} & \widetilde{A}_{\Gamma\Gamma} \end{bmatrix}, \tag{29}$$

where $\widetilde{A}_{\Gamma\Gamma}$ is assembled only with respect to the coarse level primal degrees of freedom across the interface. The partially subassembled Schur complement operator $\widetilde{S}_\Gamma$ is defined by $\widetilde{S}_\Gamma = \widetilde{A}_{\Gamma\Gamma} - \widetilde{A}_{\Gamma I} A_{II}^{-1} \widetilde{A}_{I\Gamma}$. From the definition of S_Γ and $\widetilde{S}_\Gamma$, we see that S_Γ can be obtained from $\widetilde{S}_\Gamma$ by assembling with respect to the dual interface variables, i.e.,

$$S_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma,$$

where $\widetilde{R}_\Gamma$ is the injection operator from the space $\widehat{W}_\Gamma$ into $\widetilde{W}_\Gamma$. We also define $\widetilde{R}_{D,\Gamma} = D\widetilde{R}_\Gamma$, where D is a diagonal scaling matrix. The diagonal elements of D equal 1, for the rows of the primal interface variables, and equal $\delta_i^\dagger(x)$ for the others. Here, for a subdomain interface node x, the inverse counting function $\delta_i^\dagger(x)$ is defined by $\delta_i^\dagger(x) = 1/card(\mathcal{N}_x)$, where $\mathcal{N}_x$ is the set of indices of the subdomains which have x on their boundaries and $card(\mathcal{N}_x)$ is the number of the subdomains in the set $\mathcal{N}_x$.

The preconditioned interface problem in our BDDC algorithm is

$$\widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} g_\Gamma. \tag{30}$$

A GMRES iteration is used to solve Equation (30). In each iteration, to multiply S_Γ by a vector, subdomain Dirichlet boundary problems need to be solved; to multiply $\widetilde{S}_\Gamma^{-1}$ by a vector, a partially subassembled finite element problem with the coefficient matrix $\widetilde{A}$ needs to be solved, which requires solving subdomain Robin boundary problems and a coarse level problem; cf. [25]. After obtaining the interface solution u_Γ, we find u_I by solving subdomain Dirichlet problems.

7. Convergence rate of the GMRES iteration

In this section, we give a convergence analysis of the GMRES iteration for solving the preconditioned interface problem (30) for advection-diffusion problems.

For any $u_\Gamma \in \widetilde{W}_\Gamma$, we denote its standard discrete harmonic extension to the interior of subdomains by $u_{\mathcal{H},\Gamma} \in \widetilde{W}$; see [33, Section 4.4] for a definition of the discrete harmonic extension. We have the following result on the equivalence of the norms of local discrete harmonic extensions and traces on subdomain boundaries, cf. [33, Lemma 4.10].

Lemma 7.1. There exist positive constants c and C, which are independent of ν, H and h, such that for all $u_\Gamma \in \widetilde{W}_\Gamma$, and $i = 1, 2, ..., N$,

$$c|u^{(i)}_{\mathcal{H},\Gamma}|_{H^1(\Omega_i)} \leq |u^{(i)}_\Gamma|_{H^{1/2}(\partial\Omega_i)} \leq C|u^{(i)}_{\mathcal{H},\Gamma}|_{H^1(\Omega_i)}.$$

We define another discrete extension of $u_\Gamma \in \widetilde{W}_\Gamma$ to the interior of subdomains by

$$u_{\mathcal{A},\Gamma} = \begin{bmatrix} -A_{II}^{-1}\widetilde{A}_{I\Gamma}u_\Gamma \\ u_\Gamma \end{bmatrix} \in \widetilde{W}. \tag{31}$$

The discrete harmonic extension $u_{\mathcal{H},\Gamma}$ can be obtained from u_Γ by solving subdomain Dirichlet problems corresponding to discrete Laplacian and it minimizes the energy norms of all finite element functions which have the trace u_Γ on the interface. $u_{\mathcal{A},\Gamma}$ does not have this energy minimization property and it is obtained from u_Γ by solving subdomain advection-diffusion problems with Dirichlet boundary conditions as shown in Equation (31). We note that both $u_{\mathcal{H},\Gamma}$ and $u_{\mathcal{A},\Gamma}$ are also well defined for $u_\Gamma \in \widehat{W}_\Gamma$, and as a result $u_{\mathcal{H},\Gamma} \in \widehat{W}$ and $u_{\mathcal{A},\Gamma} \in \widehat{W}$.

We define two bilinear forms for vectors in $\widehat{W}_\Gamma$ and $\widetilde{W}_\Gamma$ respectively by

$$\langle u_\Gamma, v_\Gamma\rangle_{B_\Gamma} = v^T_{\mathcal{A},\Gamma}Bu_{\mathcal{A},\Gamma},\ \langle u_\Gamma, v_\Gamma\rangle_{Z_\Gamma} = v^T_{\mathcal{A},\Gamma}Zu_{\mathcal{A},\Gamma},\ \text{ for all } u_\Gamma, v_\Gamma \in \widehat{W}_\Gamma, \tag{32}$$

$$\langle u_\Gamma, v_\Gamma\rangle_{\widetilde{B}_\Gamma} = v^T_{\mathcal{A},\Gamma}\widetilde{B}u_{\mathcal{A},\Gamma},\ \langle u_\Gamma, v_\Gamma\rangle_{\widetilde{Z}_\Gamma} = v^T_{\mathcal{A},\Gamma}\widetilde{Z}u_{\mathcal{A},\Gamma},\ \text{ for all } u_\Gamma, v_\Gamma \in \widetilde{W}_\Gamma. \tag{33}$$

In general, we use the notation $\langle p, q\rangle_M$ to represent the product $q^T Mp$, for any given matrix M and vectors p and q.

From the definitions (31), (32), and (33), follows

Lemma 7.2. For any $v \in \widetilde{W}$, denote its restriction to Γ by $v_\Gamma \in \widetilde{W}_\Gamma$. Then for any $u_\Gamma \in \widetilde{W}_\Gamma$ and $v \in \widetilde{W}$,

$$\langle u_\Gamma, v_\Gamma\rangle_{\widetilde{S}_\Gamma} = \left\langle u_{\mathcal{A},\Gamma}, v\right\rangle_{\widetilde{A}} \ \text{ and } \ \langle u_\Gamma, v_\Gamma\rangle_{\widetilde{S}_\Gamma} = \langle u_\Gamma, v_\Gamma\rangle_{\widetilde{B}_\Gamma} + \langle u_\Gamma, v_\Gamma\rangle_{\widetilde{Z}_\Gamma}.$$

For any $u_\Gamma \in \widetilde{W}_\Gamma$, $\langle u_\Gamma, u_\Gamma\rangle_{\widetilde{S}_\Gamma} = \left\langle u_{\mathcal{A},\Gamma}, u_{\mathcal{A},\Gamma}\right\rangle_{\widetilde{A}} = \left\langle u_{\mathcal{A},\Gamma}, u_{\mathcal{A},\Gamma}\right\rangle_{\widetilde{B}} = \langle u_\Gamma, u_\Gamma\rangle_{\widetilde{B}_\Gamma} \geq 0$, and $\langle u_\Gamma, u_\Gamma\rangle_{\widetilde{Z}_\Gamma} = 0$. The same results also hold for functions and the corresponding bilinear forms in the space $\widehat{W}_\Gamma$.

From Lemma 7.2, we define B_Γ- and $\widetilde{B}_\Gamma$- norms for elements in the spaces $\widehat{W}_\Gamma$ and $\widetilde{W}_\Gamma$ respectively by:

$$\|u_\Gamma\|^2_{B_\Gamma} = \langle u_\Gamma, u_\Gamma\rangle_{B_\Gamma}, \qquad \text{for any } u_\Gamma \in \widehat{W}_\Gamma,$$

and

$$\|w_\Gamma\|^2_{\widetilde{B}_\Gamma} = \langle w_\Gamma, w_\Gamma\rangle_{\widetilde{B}_\Gamma}, \qquad \text{for any } w_\Gamma \in \widetilde{W}_\Gamma.$$

The following two lemmas can be obtained from definitions (32) and (33), and Lemmas 7.2, 4.3, and 4.4.

Lemma 7.3. There exist positive constants C_1 and C_2, which are independent of ν, H and h, such that for all $u_\Gamma, v_\Gamma \in \widetilde{W}_\Gamma$,

$$| \langle u_\Gamma, v_\Gamma \rangle_{\widetilde{Z}_\Gamma} | \leq C_1 \frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})} \| u_\Gamma \|_{\widetilde{B}_\Gamma} \| v_\Gamma \|_{\widetilde{B}_\Gamma},$$

and

$$| \langle u_\Gamma, v_\Gamma \rangle_{\widetilde{S}_\Gamma} | \leq C_2 \frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})} \| u_\Gamma \|_{\widetilde{B}_\Gamma} \| v_\Gamma \|_{\widetilde{B}_\Gamma}.$$

The same results hold for functions and the corresponding bilinear forms in the space $\widehat{W}_\Gamma$ as well.

Lemma 7.4. There exists a positive constant C, such that for all $u_\Gamma, v_\Gamma \in \widehat{W}_\Gamma$,

$$| \langle u_\Gamma, v_\Gamma \rangle_{Z_\Gamma} | \leq C \frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})} \| u_\Gamma \|_{B_\Gamma} \| v_{\mathscr{A},\Gamma} \|_{L^2(\Omega)}.$$

We denote the preconditioned operator

$$\widetilde{R}^T_{D,\Gamma} \widetilde{S}^{-1}_\Gamma \widetilde{R}_{D,\Gamma} S_\Gamma$$

in (30) by T. The convergence rate of the GMRES iteration can be estimated by using the following result due to Eisenstat et al. [14].

Theorem 7.5. Let c and C be two positive parameters such that

$$c \langle u, u \rangle_{B_\Gamma} \leq \langle u, Tu \rangle_{B_\Gamma}, \tag{34}$$

$$\langle Tu, Tu \rangle_{B_\Gamma} \leq C \langle u, u \rangle_{B_\Gamma}. \tag{35}$$

Then

$$\frac{\| r_m \|_{B_\Gamma}}{\| r_0 \|_{B_\Gamma}} \leq \left(1 - \frac{c^2}{C} \right)^{m/2},$$

where r_m is the residual of the GMRES iteration at iteration m.

Remark 7.6. In our convergence analysis of the GMRES iteration, we use the B_Γ-norm; the analysis in the L^2-norm is not available yet. In our numerical experiments, we have found that the convergence rates in both the B_Γ- and L^2- norms are quite similar. For a study of the convergence rates of the GMRES iteration combined with an additive Schwarz method in the Euclidean and energy norms, see Sarkis and Szyld [31].

We define an interface average operator $E_{D,\Gamma}$ for functions in the space $\widetilde{W}_\Gamma$ by

$$E_{D,\Gamma} w_\Gamma = \widetilde{R}_\Gamma \widetilde{R}^T_{D,\Gamma} w_\Gamma,$$

for any $w_\Gamma \in \widetilde{W}_\Gamma$. This operator computes an average of w_Γ across Γ. The following result on the stability of $E_{D,\Gamma}$ can be found in [22; 21; 26].

Lemma 7.7. Let Assumption 4.7 hold. There exists a positive constant C, such that for all $w_\Gamma \in \widetilde{W}_\Gamma$, and $i = 1, 2, ..., N$,

$$|\left(E_{D,\Gamma} w_\Gamma\right)^{(i)}|_{H^{1/2}(\partial\Omega_i)} \le \Phi(H,h)|w_\Gamma^{(i)}|_{H^{1/2}(\partial\Omega_i)},$$

where $\Phi(H,h) = C(1+\log(H/h))$.

Lemma 7.8. Let Assumption 4.7 hold. There exists a positive constant C, such that for all $w_\Gamma \in \widetilde{W}_\Gamma$,

$$\|E_{D,\Gamma} w_\Gamma\|^2_{\widetilde{B}_\Gamma} \le \frac{C}{\nu}\Phi^2(H,h)\|w_\Gamma\|^2_{\widetilde{B}_\Gamma},$$

where $\Phi(H,h)$ is given in Lemma 7.7.

Proof. It is sufficient to show that $\|E_{D,\Gamma}w_\Gamma - w_\Gamma\|^2_{\widetilde{B}_\Gamma} \le \frac{C}{\nu}\Phi^2(H,h)\|w_\Gamma\|^2_{\widetilde{B}_\Gamma}$. Denote $E_{D,\Gamma}w_\Gamma - w_\Gamma$ by v_Γ. Let $v_{\mathcal{A},\Gamma}$ and $v_{\mathcal{H},\Gamma}$ be the extensions defined by Equation (31) and the standard discrete harmonic extension of v_Γ, respectively. From the definition of $v_{\mathcal{A},\Gamma}$, we know that $a^{(i)}(v^{(i)}_{\mathcal{A},\Gamma}, q^{(i)}) = 0$, for any $q^{(i)} \in W^{(i)}$, which vanishes at the nodes of $\partial\Omega_i$. Take

$$q^{(i)} = v^{(i)}_{\mathcal{A},\Gamma} - v^{(i)}_{\mathcal{H},\Gamma},$$

and we find $a^{(i)}(v^{(i)}_{\mathcal{A},\Gamma}, v^{(i)}_{\mathcal{A},\Gamma} - v^{(i)}_{\mathcal{H},\Gamma}) = 0$. Therefore, we have

$$\begin{aligned}
\|v^{(i)}_{\mathcal{A},\Gamma} - v^{(i)}_{\mathcal{H},\Gamma}\|^2_{B^{(i)}} &= |a^{(i)}(v^{(i)}_{\mathcal{A},\Gamma} - v^{(i)}_{\mathcal{H},\Gamma}, v^{(i)}_{\mathcal{A},\Gamma} - v^{(i)}_{\mathcal{H},\Gamma})| \\
&= |a^{(i)}(v^{(i)}_{\mathcal{H},\Gamma}, v^{(i)}_{\mathcal{A},\Gamma} - v^{(i)}_{\mathcal{H},\Gamma})| \\
&\le C\|v^{(i)}_{\mathcal{H},\Gamma}\|_{H^1(\Omega_i)}\|v^{(i)}_{\mathcal{A},\Gamma} - v^{(i)}_{\mathcal{H},\Gamma}\|_{B^{(i)}},
\end{aligned}$$

where we use Lemma 4.5 in the last step. Canceling the common factor, we have

$$\|v^{(i)}_{\mathcal{A},\Gamma} - v^{(i)}_{\mathcal{H},\Gamma}\|_{B^{(i)}} \le C\|v^{(i)}_{\mathcal{H},\Gamma}\|_{H^1(\Omega_i)}.$$

Therefore, by Lemma 4.1,

$$\|v^{(i)}_{\mathcal{A},\Gamma}\|_{B^{(i)}} \le C\|v^{(i)}_{\mathcal{H},\Gamma}\|_{H^1(\Omega_i)}.$$

From this and using Equation (33) and Lemmas 7.1, 7.7, and 4.1, noting that $v^{(i)}_{\mathcal{H},\Gamma}$ vanishes at the coarse level primal degrees of freedom, we have

$$\|E_{D,\Gamma}w_\Gamma - w_\Gamma\|^2_{\widetilde{B}_\Gamma} = \|v_{\mathcal{A},\Gamma}\|^2_{\widetilde{B}}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \| v_{\mathcal{A},\Gamma}^{(i)} \|_{B^{(i)}}^2 \\
&\leq C \sum_{i=1}^{N} \| v_{\mathcal{H},\Gamma}^{(i)} \|_{H^1(\Omega_i)}^2 \leq C \sum_{i=1}^{N} | v_{\mathcal{H},\Gamma}^{(i)} |_{H^1(\Omega_i)}^2 \leq C \sum_{i=1}^{N} | v_{\Gamma}^{(i)} |_{H^{1/2}(\partial\Omega_i)}^2 \\
&\leq C \Phi^2(H,h) \sum_{i=1}^{N} | w_{\Gamma}^{(i)} |_{H^{1/2}(\partial\Omega_i)}^2 \leq C \Phi^2(H,h) \sum_{i=1}^{N} | w_{\mathcal{H},\Gamma}^{(i)} |_{H^1(\Omega_i)}^2 \\
&\leq C \Phi^2(H,h) \sum_{i=1}^{N} | w_{\mathcal{A},\Gamma}^{(i)} |_{H^1(\Omega_i)}^2 \leq \frac{C}{\nu} \Phi^2(H,h) \sum_{i=1}^{N} \| w_{\mathcal{A},\Gamma}^{(i)} \|_{B_0^{(i)}}^2 \\
&= \frac{C}{\nu} \Phi^2(H,h) \| w_{\mathcal{A},\Gamma} \|_{\widetilde{B}_0}^2 \leq \frac{C}{\nu} \Phi^2(H,h) \| w_{\Gamma} \|_{\widetilde{B}_\Gamma}^2. \qquad \square
\end{aligned}
$$

Lemma 7.9. Let $w_\Gamma = \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma$, for $u_\Gamma \in \widehat{W}_\Gamma$. Then

$$
\| w_\Gamma \|_{\widetilde{B}_\Gamma}^2 = \langle u_\Gamma, T u_\Gamma \rangle_{S_\Gamma} .
$$

Proof. Since $\widetilde{R}_{D,\Gamma}^T w_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u = T u_\Gamma$, we have, using Lemma 7.2,

$$
\begin{aligned}
\| w_\Gamma \|_{\widetilde{B}_\Gamma}^2 &= \langle w_\Gamma, w_\Gamma \rangle_{\widetilde{S}_\Gamma} = {w_\Gamma}^T \widetilde{S}_\Gamma w_\Gamma \\
&= {w_\Gamma}^T \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma \\
&= {w_\Gamma}^T \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma = \left\langle u_\Gamma, \widetilde{R}_{D,\Gamma}^T w_\Gamma \right\rangle_{S_\Gamma} = \langle u_\Gamma, T u_\Gamma \rangle_{S_\Gamma} . \qquad \square
\end{aligned}
$$

Lemma 7.10. Let Assumption 4.7 hold. Let $w_\Gamma = \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma$, for $u_\Gamma \in \widehat{W}_\Gamma$. There then exists a positive constant C, such that for all $u_\Gamma \in \widehat{W}_\Gamma$,

$$
\| w_\Gamma \|_{\widetilde{B}_\Gamma}^2 \leq C \frac{\Phi^2(H,h)}{\nu \max(\nu, C_m)} \| u_\Gamma \|_{B_\Gamma}^2 ,
$$

where $\Phi(H,h)$ is given in Lemma 7.7.

Proof. We have, from Lemma 7.2,

$$
\begin{aligned}
\langle T u_\Gamma, T u_\Gamma \rangle_{B_\Gamma} &= \langle T u_\Gamma, T u_\Gamma \rangle_{S_\Gamma} \\
&= \left\langle \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma, \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma \right\rangle_{S_\Gamma} \\
&= \left\langle \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T w_\Gamma, \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T w_\Gamma \right\rangle_{\widetilde{S}_\Gamma} \\
&= \langle E_D w_\Gamma, E_D w_\Gamma \rangle_{\widetilde{S}_\Gamma} = \| E_D w_\Gamma \|_{\widetilde{B}_\Gamma}^2 .
\end{aligned}
$$

Then, from Lemmas 7.8, 7.9, and 7.3, we have

$$\begin{aligned}\langle Tu_\Gamma, Tu_\Gamma\rangle_{B_\Gamma} = \|E_D w_\Gamma\|^2_{\widetilde{B}_\Gamma} &\le \frac{C}{\nu}\Phi^2(H,h)\|w_\Gamma\|^2_{\widetilde{B}_\Gamma}\\ &= \frac{C}{\nu}\Phi^2(H,h)\,\langle u_\Gamma, Tu_\Gamma\rangle_{S_\Gamma}\\ &\le C\frac{\Phi^2(H,h)}{\nu\max(\sqrt{\nu},\sqrt{C_m})}\|Tu_\Gamma\|_{B_\Gamma}\|u_\Gamma\|_{B_\Gamma}.\end{aligned}$$

Therefore, we have

$$\langle Tu_\Gamma, Tu_\Gamma\rangle_{B_\Gamma} \le C\frac{\Phi^4(H,h)}{\nu^2\max(\nu, C_m)}\,\langle u_\Gamma, u_\Gamma\rangle_{B_\Gamma}\,. \tag{36}$$

Then, using Lemmas 7.9 and 7.3, and Equation (36), we have,

$$\|w_\Gamma\|^2_{\widetilde{B}_\Gamma} = \langle u_\Gamma, Tu_\Gamma\rangle_{S_\Gamma} \le C\frac{\|u_\Gamma\|_{B_\Gamma}\ \|Tu_\Gamma\|_{B_\Gamma}}{\max(\sqrt{\nu},\sqrt{C_m})} \le C\frac{\Phi^2(H,h)}{\nu\max(\nu, C_m)}\|u_\Gamma\|^2_{B_\Gamma}. \qquad \square$$

Lemma 7.11. Let $w_\Gamma = \widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}S_\Gamma u_\Gamma$, for $u_\Gamma \in \widehat{W}_\Gamma$. Then for all $v \in \widetilde{R}(\widehat{W})$,

$$\left\langle w_{\mathcal{A},\Gamma}, v\right\rangle_{\widetilde{A}} = \left\langle \widetilde{R}u_{\mathcal{A},\Gamma}, v\right\rangle_{\widetilde{A}},$$

that is, $\left\langle w_{\mathcal{A},\Gamma} - \widetilde{R}u_{\mathcal{A},\Gamma}, v\right\rangle_{\widetilde{A}} = 0$.

Proof. For any $v \in \widetilde{R}(\widehat{W})$, denote its continuous interface part by

$$v_\Gamma \in \widetilde{R}_\Gamma(\widehat{W}_\Gamma).$$

Given $u_\Gamma \in \widehat{W}_\Gamma$, from Lemma 7.2 and the fact that $\widetilde{R}_\Gamma\widetilde{R}^T_{D,\Gamma}v_\Gamma = v_\Gamma$, we have

$$\begin{aligned}\left\langle w_{\mathcal{A},\Gamma}, v\right\rangle_{\widetilde{A}} = \langle w_\Gamma, v_\Gamma\rangle_{\widetilde{S}_\Gamma} &= v_\Gamma^T\widetilde{S}_\Gamma w_\Gamma\\ &= v_\Gamma^T\widetilde{S}_\Gamma\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}S_\Gamma u_\Gamma\\ &= v_\Gamma^T\widetilde{R}_{D,\Gamma}\widetilde{R}_\Gamma^T\widetilde{S}_\Gamma\widetilde{R}_\Gamma u_\Gamma\\ &= \left\langle \widetilde{R}_\Gamma u_\Gamma, \widetilde{R}_\Gamma\widetilde{R}^T_{D,\Gamma}v_\Gamma\right\rangle_{\widetilde{S}_\Gamma} = \left\langle \widetilde{R}_\Gamma u_\Gamma, v_\Gamma\right\rangle_{\widetilde{S}_\Gamma}\\ &= v_\Gamma^T\widetilde{S}_\Gamma\widetilde{R}_\Gamma u_\Gamma = \left\langle \widetilde{R}u_{\mathcal{A},\Gamma}, v\right\rangle_{\widetilde{A}}.\end{aligned} \qquad \square$$

Lemma 7.12. Let Assumption 4.7 hold. Let $w_\Gamma = \widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}S_\Gamma u_\Gamma$, for $u_\Gamma \in \widehat{W}_\Gamma$. If h is sufficiently small, there then exists a positive constant C, such that for all $u_\Gamma \in \widehat{W}_\Gamma$,

$$\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{L^2(\Omega)} \le C\frac{\Phi(H,h)\ C_L(\nu,H,h)}{\sqrt{\nu}\max(\nu^2, C_m\nu)}\|u_\Gamma\|_{B_\Gamma},$$

where $C_L(\nu, H, h)$ is given in Lemma 5.1, and $\Phi(H,h)$ is given in Lemma 7.7.

Proof. We have, from Equation (18), that

$$
\begin{aligned}
&(w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}, g)\\
&\quad = \widetilde{a}_o(w_{\mathscr{A},\Gamma}, \widetilde{\varphi}_g) - a_o(u_{\mathscr{A},\Gamma}, \varphi_g) - \int_\Omega C(x) L^* w_{\mathscr{A},\Gamma} L^*(\varphi_g - \widetilde{\varphi}_g)\, dx\\
&\quad = \widetilde{a}(w_{\mathscr{A},\Gamma}, \widetilde{\varphi}_g) - \widetilde{a}(u_{\mathscr{A},\Gamma}, \varphi_g)\\
&\qquad\quad + \int_\Omega C(x)\big(Lu_{\mathscr{A},\Gamma} L\varphi_g - Lw_{\mathscr{A},\Gamma} L\widetilde{\varphi}_g - L^* w_{\mathscr{A},\Gamma} L^*(\varphi_g - \widetilde{\varphi}_g)\big)\, dx\\
&\quad = \widetilde{a}(w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}, \widetilde{\varphi}_g) - \widetilde{a}(u_{\mathscr{A},\Gamma}, \varphi_g - \widetilde{\varphi}_g) + \int_\Omega C(x)\big(L(u_{\mathscr{A},\Gamma} - w_{\mathscr{A},\Gamma}) L\varphi_g\\
&\qquad\quad + Lw_{\mathscr{A},\Gamma} L(\varphi_g - \widetilde{\varphi}_g) - L^* w_{\mathscr{A},\Gamma} L^*(\varphi_g - \widetilde{\varphi}_g)\big)\, dx\\
&\quad \le \widetilde{a}(w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}, \widetilde{\varphi}_g) - \widetilde{a}(u_{\mathscr{A},\Gamma}, \varphi_g - \widetilde{\varphi}_g)\\
&\qquad\quad + C\frac{\sqrt{C_s}}{\max(\sqrt{\nu}, \sqrt{C_m})} \|w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}\|_{\widetilde{B}} \|\varphi_g\|_{\widetilde{B}_0}\\
&\qquad\qquad + C\sqrt{C_s}\|w_{\mathscr{A},\Gamma}\|_{\widetilde{B}} \left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right),
\end{aligned}
$$

where we use Lemmas 5.5 and 5.4 in the last step.

Let ψ be any finite element function in the space $\widehat{W}$. Then from Lemma 7.11, we know that $\widetilde{a}(w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}, \psi) = 0$. Therefore,

$$
\widetilde{a}(w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}, \widetilde{\varphi}_g) - \widetilde{a}(u_{\mathscr{A},\Gamma}, \varphi_g - \widetilde{\varphi}_g) = \widetilde{a}(w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}, \widetilde{\varphi}_g - \psi) - \widetilde{a}(u_{\mathscr{A},\Gamma}, \varphi_g - \widetilde{\varphi}_g).
$$

Then, using Lemmas 4.3, 4.2, 4.6, and 5.6, and that $\|\varphi_g\|_{H^2(\Omega)} \le C\frac{1}{\nu}\|g\|_{L^2(\Omega)}$, we have

$$
\begin{aligned}
&\|(w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}, g)\|\\
&\quad \le C\frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})}(\|w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}\|_{\widetilde{B}} + \|u_{\mathscr{A},\Gamma}\|_{\widetilde{B}})(\|\widetilde{\varphi}_g - \psi\|_{\widetilde{B}} + \|\varphi_g - \widetilde{\varphi}_g\|_{\widetilde{B}})\\
&\qquad + C\frac{\sqrt{C_s}}{\max(\sqrt{\nu}, \sqrt{C_m})}\|w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}\|_{\widetilde{B}}\|\varphi_g\|_{\widetilde{B}_0}\\
&\qquad\quad + C\sqrt{C_s}\|w_{\mathscr{A},\Gamma}\|_{\widetilde{B}}\left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right)\\
&\quad \le C\frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})}(\|w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}\|_{\widetilde{B}} + \|u_{\mathscr{A},\Gamma}\|_{\widetilde{B}})(\|\varphi_g - \psi\|_{\widetilde{B}} + 2\|\varphi_g - \widetilde{\varphi}_g\|_{\widetilde{B}})\\
&\qquad + C\frac{\sqrt{C_s}}{\max(\sqrt{\nu}, \sqrt{C_m})}\max(\sqrt{\nu}, \sqrt{C_s})\|w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}\|_{\widetilde{B}}\|\varphi_g\|_{H^2(\Omega)}\\
&\qquad\quad + C\sqrt{C_s}\|w_{\mathscr{A},\Gamma}\|_{\widetilde{B}}\left(\|\widetilde{\varphi}_g - \widetilde{R}\varphi_g\|_{\widetilde{B}} + \sqrt{\nu}\|\varphi_g\|_{H^2(\Omega)}\right)\\
&\quad \le C\frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})}\Big(C_L(\nu, H, h) + \sqrt{C_s}\max(\sqrt{\nu}, \sqrt{C_s})\Big)\big(\|w_{\mathscr{A},\Gamma}\|_{\widetilde{B}}\\
&\qquad + \|u_{\mathscr{A},\Gamma}\|_{\widetilde{B}}\big)\|\varphi_g\|_{H^2(\Omega)}
\end{aligned}
$$

$$\leq C \frac{C_L(\nu, H, h)}{\nu \max(\sqrt{\nu}, \sqrt{C_m})} \left(\|w_{\mathcal{A},\Gamma}\|_{\widetilde{B}} + \|u_{\mathcal{A},\Gamma}\|_{\widetilde{B}} \right) \|g\|_{L^2(\Omega)}.$$

Therefore, using Lemmas Lemma 7.2 and Lemma 7.10, we have

$$\begin{aligned}
\|w_{\mathcal{A},\Gamma} - u_{\mathcal{A},\Gamma}\|_{L^2(\Omega)} &= \sup_{g \in L^2(\Omega)} \frac{\left|(w - u_{\mathcal{A},\Gamma}, g)\right|}{\|g\|_{L^2(\Omega)}} \\
&\leq C \frac{C_L(\nu, H, h)}{\nu \max(\sqrt{\nu}, \sqrt{C_m})} \left(\|u_{\mathcal{A},\Gamma}\|_{\widetilde{B}} + \|w_{\mathcal{A},\Gamma}\|_{\widetilde{B}} \right) \\
&= C \frac{C_L(\nu, H, h)}{\nu \max(\sqrt{\nu}, \sqrt{C_m})} \left(\|u_\Gamma\|_{\widetilde{B}_\Gamma} + \|w_\Gamma\|_{\widetilde{B}_\Gamma} \right) \\
&\leq C \frac{\Phi(H, h)\, C_L(\nu, H, h)}{\sqrt{\nu} \max(\nu^2, C_m \nu)} \|u_\Gamma\|_{B_\Gamma}. \qquad \square
\end{aligned}$$

Lemma 7.13. Let Assumption 4.7 hold and let $v_\Gamma = \widetilde{R}_{D,\Gamma}^T w_\Gamma$, for $w_\Gamma \in \widetilde{W}_\Gamma$. There exists a positive constant C, such that

$$\|v_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2 \leq C \frac{\max(\nu, C_s)\ \Phi^2(H, h)}{\nu^2} \frac{H^2}{h^2} \|w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2,$$

for all $w_\Gamma \in \widetilde{W}_\Gamma$, where $\Phi(H, h)$ is given in Lemma 7.7.

Proof. We only need to show that

$$\|v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2 \leq C \frac{\max(\nu, C_s)\ \Phi^2(H, h)}{\nu^2} \frac{H^2}{h^2} \|w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2.$$

From Assumption 4.7, we know for any $w_\Gamma \in \widetilde{W}_\Gamma$, $v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}$ has zero averages over the subdomain interfaces. Using a Poincaré–Friedrichs inequality, Lemmas 4.1, 7.2, the proof in Lemma 7.8, and an inverse inequality, we have

$$\begin{aligned}
\|v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2 &\leq C H^2 |v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}|_{H^1(\Omega)}^2 \\
&\leq C \frac{H^2}{\nu} \|v_{\mathcal{A},\Gamma} - w_{\mathcal{A},\Gamma}\|_{\widetilde{B}}^2 \\
&= C \frac{H^2}{\nu} \|v_\Gamma - w_\Gamma\|_{\widetilde{B}_\Gamma}^2 \\
&\leq C \frac{H^2\ \Phi^2(H, h)}{\nu^2} \|w_{\mathcal{A},\Gamma}\|_{\widetilde{B}_0}^2 \\
&\leq C \frac{\max(\nu, C_s) H^2\ \Phi^2(H, h)}{\nu^2} \|w_{\mathcal{A},\Gamma}\|_{H^1(\Omega)}^2 \\
&\leq C \frac{\max(\nu, C_s)\ \Phi^2(H, h)}{\nu^2} \frac{H^2}{h^2} \|w_{\mathcal{A},\Gamma}\|_{L^2(\Omega)}^2. \qquad \square
\end{aligned}$$

Lemma 7.14. Let Assumption 4.7 hold and let $v_\Gamma = Tu_\Gamma - u_\Gamma$, for $u_\Gamma \in \widehat{W}_\Gamma$. If h is sufficiently small, there then exists a positive constant C, such that for all $u_\Gamma \in \widehat{W}_\Gamma$,

$$\|v_{\mathscr{A},\Gamma}\|_{L^2(\Omega)} \leq C \frac{\max(\sqrt{\nu}, \sqrt{C_s})\Phi^2(H,h)\ C_L(\nu, H, h)}{\nu^{5/2}\max(\nu, C_m)} \frac{H}{h} \|u_\Gamma\|_{B_\Gamma},$$

where $C_L(\nu, H, h)$ is given in Lemma 5.1, and $\Phi(H, h)$ is given in Lemma 7.7.

Proof. Since $Tu_\Gamma = \widetilde{R}^T_{D,\Gamma} w_\Gamma$ and $\widetilde{R}^T_{D,\Gamma}\widetilde{R}_\Gamma = I$, we have

$$v_\Gamma = Tu_\Gamma - u_\Gamma = \widetilde{R}^T_{D,\Gamma} w_\Gamma - \widetilde{R}^T_{D,\Gamma}\widetilde{R}_\Gamma u_\Gamma = \widetilde{R}^T_{D,\Gamma}(w_\Gamma - \widetilde{R}_\Gamma u_\Gamma).$$

Using Lemmas 7.13 and 7.12, we have

$$\begin{aligned}
\|v_{\mathscr{A},\Gamma}\|^2_{L^2(\Omega)} &\leq C \frac{\max(\nu, C_s)\ \Phi^2(H,h)}{\nu^2} \frac{H^2}{h^2} \|w_{\mathscr{A},\Gamma} - u_{\mathscr{A},\Gamma}\|^2_{L^2(\Omega)} \\
&\leq C \frac{\max(\nu, C_s)\ \Phi^2(H,h)}{\nu^2} \frac{H^2}{h^2} \frac{\Phi^2(H,h)\ C_L^2(\nu, H, h)}{\nu^3\ (\max(\nu, C_m))^2} \|u_\Gamma\|^2_{B_\Gamma} \\
&= C \frac{\max(\nu, C_s)\Phi^4(H,h)\ C_L^2(\nu, H, h)}{\nu^5 \max(\nu^2, C_m^2)} \frac{H^2}{h^2} \|u_\Gamma\|^2_{B_\Gamma}. \qquad \square
\end{aligned}$$

Theorem 7.15. Let Assumption 4.7 hold. If h is sufficiently small, there then exist positive constants C_1, C_2, and C_3, which are independent of H, h, and ν, such that for all $u_\Gamma \in \widehat{W}_\Gamma$,

$$\langle Tu_\Gamma, Tu_\Gamma \rangle_{B_\Gamma} \leq C_1 \frac{\Phi^4(H,h)}{\nu^2 \max(\nu, C_m)} \langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma}, \tag{37}$$

and

$$c_0 \langle u_\Gamma, u_\Gamma \rangle_{B_\Gamma} \leq \frac{C_2}{\max(\nu, C_m)} \langle u_\Gamma, Tu_\Gamma \rangle_{B_\Gamma}, \tag{38}$$

where $\Phi(H, h)$ is given in Lemma 7.7. For two dimensions,

$$c_0 = 1 - C_3 \frac{\max(\sqrt{\nu}, \sqrt{C_s}) \max(H\nu, H^2)}{\nu^3 \max(\nu^2\sqrt{\nu}, C_m^2\sqrt{C_m})} \frac{H}{h} \Phi^2(H,h),$$

and for three dimensions

$$c_0 = 1 - C_3 \frac{\max(\sqrt{\nu}, \sqrt{C_s}) \max(H\nu, H^2, \sqrt{Hh})}{\nu^3 \max(\nu^2\sqrt{\nu}, C_m^2\sqrt{C_m})} \frac{H}{h} \Phi^2(H,h)\big(1 + \log(H/h)\big).$$

Proof. The upper bound Equation (37) is an immediate result of (36).

To prove the lower bound Equation (38), we have, from $\widetilde{R}_\Gamma^T \widetilde{R}_{D,\Gamma} = I$ and Lemmas 7.2 and 7.3:

$$\begin{aligned}\langle u_\Gamma, u_\Gamma\rangle_{B_\Gamma} &= \langle u_\Gamma, u_\Gamma\rangle_{S_\Gamma}\\ &= {u_\Gamma}^T \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} S_\Gamma u_\Gamma = \left\langle w_\Gamma, \widetilde{R}_\Gamma u_\Gamma\right\rangle_{\widetilde{S}_\Gamma}\\ &\le C\frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})}\|w_\Gamma\|_{\widetilde{B}_\Gamma}\|u_\Gamma\|_{B_\Gamma}\\ &= C\frac{1}{\max(\sqrt{\nu}, \sqrt{C_m})}\langle u_\Gamma, Tu_\Gamma\rangle_{S_\Gamma}^{1/2}\|u_\Gamma\|_{B_\Gamma}.\end{aligned}$$

Here we use Lemma 7.9 in the last step. Canceling the common factor, we have

$$\|u_\Gamma\|_{B_\Gamma}^2 \le C\frac{1}{\max(\nu, C_m)}\langle u_\Gamma, Tu_\Gamma\rangle_{S_\Gamma}. \tag{39}$$

Let $v_\Gamma = Tu_\Gamma - u_\Gamma$. We have, from Equation (39), Lemmas 7.2, 7.4 and 7.14,

$$\begin{aligned}\langle u_\Gamma, u_\Gamma\rangle_{B_\Gamma} &\le C\frac{1}{\max(\nu, C_m)}\left(\langle u_\Gamma, Tu_\Gamma\rangle_{B_\Gamma} + \langle u_\Gamma, Tu_\Gamma - u_\Gamma\rangle_{Z_\Gamma}\right)\\ &\le \frac{C\,\langle u_\Gamma, Tu_\Gamma\rangle_{B_\Gamma}}{\max(\nu, C_m)} + C\frac{1}{\max(\nu\sqrt{\nu}, C_m\sqrt{C_m})}\|u_\Gamma\|_{B_\Gamma}\|v_{\mathscr{A},\Gamma}\|_{L^2(\Omega)}\\ &\le \frac{C\,\langle u_\Gamma, Tu_\Gamma\rangle_{B_\Gamma}}{\max(\nu, C_m)}\\ &\quad + \frac{C\max(\sqrt{\nu}, \sqrt{C_s})}{\nu^2\max(\nu^3, C_m^2\sqrt{\nu C_m})}\frac{H}{h}\Phi^2(H,h)C_L(\nu, H, h)\|u_\Gamma\|_{B_\Gamma}^2.\end{aligned}$$

The second term on the right hand side can be combined with the left hand side and Equation (38) is proved. □

Remark 7.16. For the case $\nu = O(1)$, in Theorem 7.15,

$$c_0 = 1 - CH\frac{H}{h}\Phi^2(H,h)$$

for two dimensions and

$$c_0 = 1 - CH\frac{H}{h}\Phi^2(H,h)\big(1 + \log(H/h)\big),$$

for three dimensions, which will be positive for appropriate choice of H and h. In this case Theorem 7.5 applies and we can see that the convergence rate of our BDDC algorithm deteriorates with a decrease of ν and for fixed ν the convergence rate is independent of the number of subdomains and depends on H/h slightly. For the advection-dominated cases, it is hard to choose practical H and h to make c_0 in Theorem 7.15 positive, and Theorem 7.5 does not apply for those cases. However, the numerical experiments in the following section show that our BDDC algorithm in fact performs satisfactorily even for the advection-dominated examples.

8. Numerical experiments

We test our BDDC algorithm by solving three advection-diffusion examples in the square domain $\Omega = [-1, 1]^2$. These examples were used by Toselli [32] for testing his FETI algorithms.

The domain Ω is decomposed into square subdomains and each subdomain into uniform triangles. Piecewise linear finite elements are used in our experiments. The stabilization function $C(x)$ in Equation (9) is defined in (7). We also take $f = 0$ and $c = 10^{-4}$ in (1) in our examples.

A GMRES iteration with the L^2-norm is used without restart to solve the preconditioned interface problem (30). The iteration is stopped when the L^2-norm of the residual has been reduced by a factor of 10^{-6}; we have found consistently that the convergence rate using the B_Γ-norm is quite similar to that using the L^2-norm.

We test two different sets of coarse level primal continuity constraints in our BDDC algorithms. In BDDC-1, only subdomain vertex and edge average continuity constraints are included in the coarse level primal subspace; in BDDC-2, as in Assumption 4.7, two additional edge flux average constraints for each edge are also included in the coarse level variable space. We also compare the performance of our BDDC algorithms with that of the one-level and two-level Robin-Robin algorithms which were developed in [3; 1; 2]. They are denoted by RR-1 and RR-2 in our tables. We do not present numerical results for the one-level and two-level FETI algorithms here; their performances are in fact similar to the Robin-Robin algorithms, cf. [32].

8.1. ***Thermal boundary layer (test problem I).*** First we consider a thermal boundary layer problem. The velocity field $\boldsymbol{a}$ in Equation (1) is defined by

$$\boldsymbol{a} = \left(\frac{1+y}{2}, 0\right).$$

The boundary condition is given by:

$$u = 1, \quad \begin{cases} x = -1 & -1 < y \leq 1, \\ y = 1, & -1 \leq x \leq 1, \end{cases}$$

$$\begin{aligned} u &= 0, & y &= -1, & -1 &\leq x \leq 1, \\ u &= \frac{1+y}{2}, & x &= 1, & -1 &\leq y \leq 1. \end{aligned}$$

In our first set of experiments, reported in Table 1, we fix the subdomain problem size and change the number of subdomains. We see that, for viscosity values ν larger than 10^{-4} , the iteration counts of BDDC-2 do not change with an increase of the number of subdomains and that it converges faster than BDDC-1. We believe that for that range of viscosity values, the subdomain diameters in our experiments satisfy that c_0 is positive in Theorem 7.15; cf. Remark 7.16. For smaller viscosity, the improvement in the convergence rate of BDDC-2 over BDDC-1 is no longer clear in this example. In fact c_0 may no longer be positive. We also see from Table 1 that RR-1 and RR-2 converge slower than the BDDC algorithms, and that their iteration counts are more sensitive to an increase of the number of subdomains.

In our second set of experiments, in Table 2, we fix the number of subdomains and change the local subdomain problem size. We see that for all four algorithms, the iteration counts are not sensitive to an increase of the subdomain problem size especially with small viscosity, and that BDDC-2 converges the fastest.

We can also see from Table 1 and Table 2 that the iteration counts of all the algorithms are bounded when ν goes to zero.

8.2. ***Variable flow field (test problem II).*** We next consider a more complicated flow. The velocity field is

$$\boldsymbol{a} = \tfrac{1}{2}\big((1-x^2)(1+y), -(4-(1+y)^2)\big).$$

The boundary condition is given by $u = 1$, for $y = -1$ and $-1 < x < 0$, with $u = 0$, elsewhere on the boundary of Ω.

Table 3 gives the iteration counts of the four algorithms, for different number of subdomains with a fixed subdomain problem size. We have similar findings as for the first example in Table 1. We see that BDDC-2 scales well with respect to an increase of the number of subdomains for viscosity values larger than 10^{-4}, and that it converges the fastest among the four algorithms. The improvement in the convergence rate of BDDC-2 over BDDC-1 is obvious, especially when $\nu > 10^{-5}$.

In Table 4, we can see that the iteration counts of each algorithm are insensitive to an increase of the subdomain problem size, and they are bounded when ν goes to zero.

8.3. ***Rotating flow field (test problem III).*** This example is the most difficult one of the three examples, cf. [32]. Here the velocity field is $\boldsymbol{a} = (y, -x)$. The boundary condition is given by:

$$u = 1, \text{ for } \begin{cases} y = -1, & 0 < x \le 1, \\ y = 1, & 0 < x \le 1, \\ x = 1, & -1 \le y \le 1, \end{cases}$$

with $u = 0$, elsewhere on $\partial\Omega$.

# of Sub.	4^2	8^2	16^2	32^2	4^2	8^2	16^2	32^2
ν	BDDC-1				BDDC-2			
$1e0$	3	3	3	3	3	3	3	3
$1e-1$	5	4	4	4	4	4	4	4
$1e-2$	6	7	8	7	4	5	5	5
$1e-3$	5	8	12	18	5	6	7	6
$1e-4$	5	8	12	20	5	7	11	17
$1e-5$	5	8	12	21	5	8	12	20
$1e-6$	5	8	13	21	5	8	12	21
ν	RR-1				RR-2			
$1e0$	13	45	176	>500	6	7	7	6
$1e-1$	13	36	115	343	9	11	12	12
$1e-2$	10	18	45	156	8	13	18	23
$1e-3$	10	14	24	51	9	13	20	30
$1e-4$	11	16	25	40	10	16	25	41
$1e-5$	11	17	26	45	11	17	27	46
$1e-6$	11	17	27	45	11	17	28	46

Table 1. Iteration counts for changing number of subdomains and $H/h = 6$ for test problem I.

H/h	6	12	24	48	6	12	24	48
ν	BDDC-1				BDDC-2			
$1e0$	3	4	5	5	3	4	5	5
$1e-1$	5	6	6	7	4	5	5	6
$1e-2$	6	7	8	9	4	5	5	6
$1e-3$	5	6	7	7	5	5	6	6
$1e-4$	5	5	6	7	5	5	5	6
$1e-5$	5	5	5	5	5	4	4	4
$1e-6$	5	4	4	4	5	4	4	4
ν	RR-1				RR-2			
$1e0$	13	14	16	16	6	8	10	12
$1e-1$	13	15	16	18	9	11	13	16
$1e-2$	10	11	13	15	8	10	11	14
$1e-3$	10	9	9	10	9	9	9	10
$1e-4$	11	10	10	10	10	10	10	10
$1e-5$	11	11	11	11	11	10	11	11
$1e-6$	11	11	11	11	11	10	11	11

Table 2. Iteration counts for 4×4 subdomains and changing subdomain problem size for test problem I.

# of Sub.	4^2	8^2	16^2	32^2	4^2	8^2	16^2	32^2
ν	BDDC-1				BDDC-2			
$1e0$	4	4	4	3	2	2	1	1
$1e-1$	5	5	5	4	2	2	2	2
$1e-2$	6	9	9	8	4	3	3	3
$1e-3$	8	12	18	23	6	8	8	7
$1e-4$	9	14	25	42	7	11	19	23
$1e-5$	9	15	27	50	7	11	22	42
$1e-6$	9	15	27	51	7	11	22	45
ν	RR-1				RR-2			
$1e0$	13	45	152	390	6	7	7	7
$1e-1$	15	32	81	216	9	12	14	14
$1e-2$	10	19	41	106	9	14	19	29
$1e-3$	13	21	34	64	12	19	29	43
$1e-4$	14	27	52	84	14	26	50	76
$1e-5$	14	29	63	135	14	28	60	128
$1e-6$	14	29	67	156	14	28	64	151

Table 3. Iteration counts for changing number of subdomains and $H/h = 6$ for test problem II.

H/h	6	12	24	48	6	12	24	48
ν	BDDC-1				BDDC-2			
$1e0$	4	4	5	6	2	1	1	1
$1e-1$	5	6	7	8	2	2	2	2
$1e-2$	6	7	8	9	4	4	4	4
$1e-3$	8	8	8	8	6	6	6	6
$1e-4$	9	9	9	9	7	7	7	7
$1e-5$	9	9	9	9	7	8	7	7
$1e-6$	9	9	9	9	7	8	7	7
ν	RR-1				RR-2			
$1e0$	13	15	16	17	6	9	10	11
$1e-1$	15	17	18	19	9	11	14	16
$1e-2$	10	11	13	14	9	10	11	13
$1e-3$	13	12	11	10	12	12	11	10
$1e-4$	14	14	13	13	14	14	13	13
$1e-5$	14	14	14	14	14	14	14	14
$1e-6$	14	14	14	14	14	14	14	14

Table 4. Iteration counts for 4×4 subdomains and changing subdomain problem size for test problem II.

# of Sub.	4^2	8^2	16^2	32^2	4^2	8^2	16^2	32^2
ν	BDDC-1				BDDC-2			
$1e0$	4	3	3	3	2	2	1	1
$1e-1$	5	5	4	4	2	2	2	2
$1e-2$	9	9	7	6	4	3	3	3
$1e-3$	25	33	30	22	8	7	6	5
$1e-4$	38	67	111	112	11	12	14	14
$1e-5$	41	84	183	284	12	14	17	24
$1e-6$	41	86	199	434	12	14	18	26
ν	RR-1				RR-2			
$1e0$	13	45	148	379	6	7	7	7
$1e-1$	17	37	92	233	9	11	12	12
$1e-2$	28	50	95	218	15	22	31	49
$1e-3$	52	94	170	319	34	59	87	96
$1e-4$	68	171	360	>500	49	114	251	475
$1e-5$	71	201	>500	>500	52	141	359	>500
$1e-6$	71	205	>500	>500	53	145	389	>500

Table 5. Iteration counts for changing number of subdomains and $H/h = 6$ for test problem III.

H/h	6	12	24	48	6	12	24	48
ν	BDDC-1				BDDC-2			
$1e0$	4	4	5	6	2	2	1	1
$1e-1$	5	6	7	7	2	2	2	2
$1e-2$	9	11	12	13	4	4	4	4
$1e-3$	25	35	39	41	8	12	14	14
$1e-4$	38	72	104	122	11	26	39	45
$1e-5$	41	85	156	245	12	33	74	96
$1e-6$	41	87	165	290	12	34	88	142
ν	RR-1				RR-2			
$1e0$	13	14	15	16	6	8	10	12
$1e-1$	17	17	19	21	9	12	14	16
$1e-2$	28	30	31	30	15	17	18	20
$1e-3$	52	62	68	70	34	43	47	49
$1e-4$	68	104	141	166	49	83	110	128
$1e-5$	71	119	189	279	52	98	171	250
$1e-6$	71	121	197	318	53	100	180	296

Table 6. Iteration counts for 4×4 subdomains and changing subdomain problem size for test problem III.

Table 5 gives the iteration counts of the four algorithms for different number of subdomains with a fixed subdomain problem size. We see that BDDC-2 converges much faster than BDDC-1 and the two Robin-Robin algorithms. For the cases where $\nu > 10^{-5}$, the iteration counts are almost independent of the number of subdomains. Even when the viscosity ν goes to zero, the convergence of BDDC-2 is still very fast, while the convergence rates of BDDC-1 and the two Robin-Robin algorithms are not satisfactory at all.

From Table 6, we see that the iteration counts of all the algorithms increase with an increase of the subdomain problem size; the increase for BDDC-2 is the smallest.

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