Communications in Applied Mathematics and Computational Science

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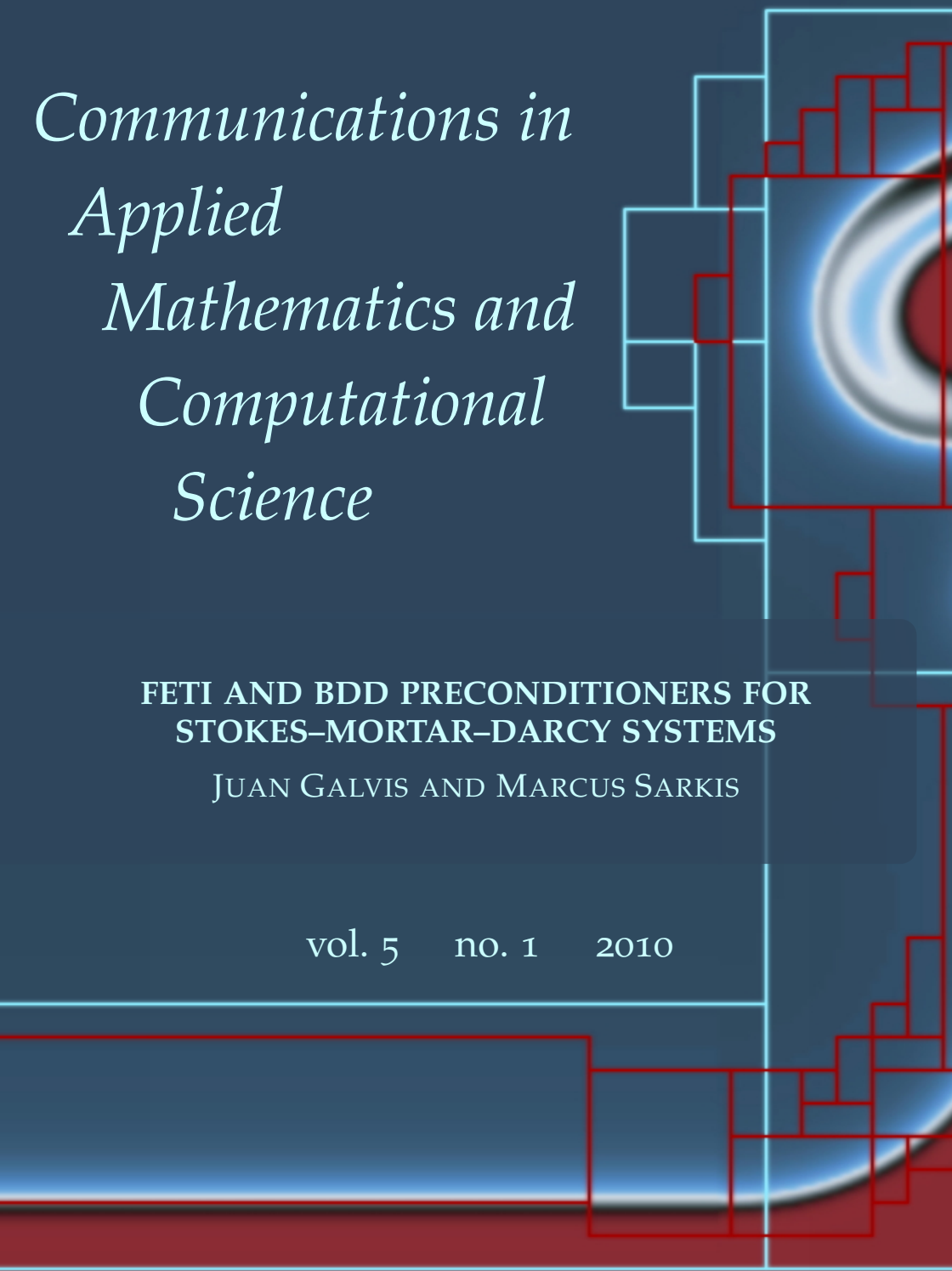
vol. 5 no. 1 2010



mathematical sciences publishers

COMM. APP. MATH. AND COMP. SCI.
Vol. 5, No. 1, 2010

FETI AND BDD PRECONDITIONERS FOR STOKES–MORTAR–DARCY SYSTEMS

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We consider the coupling across an interface of a fluid flow and a porous media flow. The differential equations involve Stokes equations in the fluid region, Darcy equations in the porous region, plus a coupling through an interface with Beaver–Joseph–Saffman transmission conditions. The discretization consists of P2/P1 triangular Taylor–Hood finite elements in the fluid region, the lowest order triangular Raviart–Thomas finite elements in the porous region, and the mortar piecewise constant Lagrange multipliers on the interface. We allow for nonmatching meshes across the interface. Due to the small values of the permeability parameter κ of the porous medium, the resulting discrete symmetric saddle point system is very ill conditioned. We design and analyze preconditioners based on the finite element by tearing and interconnecting (FETI) and balancing domain decomposition (BDD) methods and derive a condition number estimate of order $C_1(1 + (1/\kappa))$ for the preconditioned operator. In case the fluid discretization is finer than the porous side discretization, we derive a better estimate of order $C_2((\kappa + 1)/(\kappa + (h^p)^2))$ for the FETI preconditioner. Here h^p is the mesh size of the porous side triangulation. The constants C_1 and C_2 are independent of the permeability κ, the fluid viscosity ν, and the mesh ratio across the interface. Numerical experiments confirm the sharpness of the theoretical estimates.

1. Introduction

We consider the coupling across an interface of a fluid flow and a porous media flow. The model consists of Stokes equations in the fluid region, Darcy equations for the filtration velocity in the porous medium, and an adequate transmission condition for coupling of these equations through an interface. Such problems appear in several applications such as well-reservoir coupling in petroleum engineering, transport of substances across groundwater and surface water, and (bio)fluid-organ interactions. There are works that address numerical analysis issues of this model. For inf − sup conditions and approximation results associated to the continuous and

MSC2000: 35Q30, 65N22, 65N30, 65N55, 76D07.

Keywords: Stokes–Darcy coupling, mortar, balancing domain decomposition, FETI, saddle point problems, nonmatching grids, discontinuous coefficients, mortar elements.

discrete formulations for Stokes–Laplacian systems we refer [15; 12], for Stokes–Darcy systems we refer [31; 39; 2; 22], for Stokes–Mortar–Darcy systems, see [41; 26], and for DG discretizations [11; 41]. For studies on preconditioning analysis for Stokes-Laplacian systems, see [13; 14; 16; 17], and for Stokes-Darcy systems [3]. In this paper, we are interested in balancing domain decomposition (BDD) and finite element by tearing and interconnecting (FETI) preconditioned conjugate gradient methods for *Stokes–Mortar–Darcy* systems. For general references on BDD and FETI type methods, see [18; 19; 23; 24; 30; 33; 34; 35; 36; 40; 42; 43; 44].

In this paper we both extend some preliminary results contained in [25] and introduce and analyze new methods. We note that the BDD-I preconditioner introduced in [25] is not effective for small permeabilities (in real applications permeabilities are very small) while the preconditioner BDD-II in [25] requires constructing interface base functions which are orthogonal in the Stokes inner product (this construction is very expensive and impractical because it requires, as a precomputational step, solving many Stokes problems). Here in this paper we circumvent these issues by introducing a dual formulation and considering FETI-based methods. We propose and analyze FETI methods and present numerical experiments in order to verify the theory. We note that the analysis of the FETI algorithms for Stokes–Mortar–Darcy problems is very challenging due to the following issues:

(i) the mortar map from the Stokes to the Darcy side has a large kernel since the Stokes velocity space is in general richer than the Darcy velocity space on the interface;

(ii) the trace space of the Stokes velocity ($H^{1/2}$) is more regular than the trace space of the Darcy flux ($H^{-1/2}$), and due to a priori error estimates [31; 41; 26], the Stokes side must be chosen as the master side;

(iii) the energy associated to the Darcy region is much larger than the energy associated to the Stokes region due to the small value of the permeability.

Such issues imply that the master side must be chosen on the Stokes side and where the energy is smaller and velocity space is richer. The mathematical analysis under this choice is very hard to analyze even for simpler problems such as for transmission problems with discontinuous coefficients using Mortar or DG discretizations [19; 20; 21]. For problems where both the smallest coefficient and the finest mesh are placed on the master side, as far as we know, there are no optimal preconditioners developed in the literature for transmission problems, and typically there is a condition to rule out such a choice.

The rest of the paper is organized as follows: in Section 2 we present the Stokes–Darcy coupling model. In Section 3 we describe the weak formulation of this model. In Section 4 we introduce a finite element discretization. In Section 5 we

study the primal and dual formulation of the discrete problem. Section 6 presents a complete analysis of the BDD-I preconditioner introduced in [25]. In Section 7, we design and analyze the FETI preconditioner; see Lemma 3 and Theorem 4. In particular we obtain the condition number estimate of order $C_1(1+(1)/(\kappa))$ for this preconditioner and also prove Theorem 7, which gives a better estimate of order $C_2((\kappa+1)/(\kappa+(h^p)^2))$ for the FETI preconditioner in case the fluid discretization is finer than the porous side discretization; the case where the Stokes mesh is not a refinement of the Darcy mesh is also discussed (see Remark 8). In Section 7 we also consider more general fluid bilinear forms by allowing the presence of a tangential interface fluid velocity energy (Remark 10), and also translate the FETI results to analyze certain BDD methods (Remark 9). In Section 8 we present the numerical results, and in Section 9 we discuss the multisubdomain case.

Here h^p is the mesh size of the porous side triangulation. The constants C_1 and C_2 are independent of the permeability κ, the fluid viscosity ν, and the mesh ratio across the interface. In Section 8 we present numerical results that confirm the theoretical estimates concerning the BDD and the FETI preconditioners.

2. Problem setting

Let $\Omega^f, \Omega^p \subset \mathbb{R}^n$ be polyhedral subdomains, define $\Omega := \operatorname{int}(\overline{\Omega}^f \cup \overline{\Omega}^p)$ and $\Gamma := \partial\Omega^f \cap \partial\Omega^p$, with outward unit normal vectors η^i on $\partial\Omega^i$, $i = f, p$. The tangent vectors on Γ are denoted by τ_1 ($n = 2$), or τ_l, $l = 1, 2$ ($n = 3$). The exterior boundaries are $\Gamma^i := \partial\Omega^i \setminus \Gamma$, $i = f, p$. Fluid velocities are denoted by $\boldsymbol{u}^i : \Omega^i \to \mathbb{R}^n$, $i = f, p$, and pressures by $p^i : \Omega^i \to \mathbb{R}$, $i = f, p$.

We consider Stokes equations in the fluid region Ω^f and Darcy equations for the filtration velocity in the porous medium Ω^p. More precisely, we have the following systems of equations in each subdomain:

$$\begin{array}{cc} \text{Stokes equations} & \text{Darcy equations} \\ \left\{\begin{array}{rll} -\nabla\cdot T(\boldsymbol{u}^f, p^f) &= \boldsymbol{f}^f & \text{in } \Omega^f, \\ \nabla\cdot\boldsymbol{u}^f &= g^f & \text{in } \Omega^f, \\ \boldsymbol{u}^f &= \boldsymbol{h}^f & \text{on } \Gamma^f, \end{array}\right. & \left\{\begin{array}{rll} \boldsymbol{u}^p &= -\frac{\kappa}{\nu}\nabla p^p & \text{in } \Omega^p, \\ \nabla\cdot\boldsymbol{u}^p &= g^p & \text{in } \Omega^p, \\ \boldsymbol{u}^p\cdot\eta^p &= h^p & \text{on } \Gamma^p. \end{array}\right. \end{array} \tag{1}$$

Here $T(\boldsymbol{v}, p) := -pI + 2\nu\boldsymbol{D}\boldsymbol{v}$, where ν is the fluid viscosity, $\boldsymbol{D}\boldsymbol{v} := \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)$ is the linearized strain tensor and κ denotes the rock permeability. For simplicity on the analysis, we assume that κ is a real positive constant. We impose the following conditions:

(1) Interface matching conditions across Γ; see [15; 12; 16; 31] and references therein.
   (a) Conservation of mass across Γ: $\boldsymbol{u}^f\cdot\eta^f + \boldsymbol{u}^p\cdot\eta^p = 0$ on Γ.
   (b) Balance of normal forces across Γ: $p^f - 2\nu\eta^{fT}\boldsymbol{D}(\boldsymbol{u}^f)\eta^f = p^p$ on Γ.

(c) Beavers–Joseph–Saffman condition: this condition is a kind of empirical law that gives an expression for the component of the Cauchy stress tensor in the tangential direction of Γ; see [4] and [29]. It is expressed by

$$\boldsymbol{u}^f \cdot \tau_l = -\frac{\sqrt{\kappa}}{\alpha^f} 2\eta^{fT} \boldsymbol{D}(\boldsymbol{u}^f)\tau_l, \quad l = 1, n-1, \qquad \text{on } \Gamma.$$

(2) Compatibility condition: the divergence and boundary data satisfy (see [26])

$$\langle g^f, 1\rangle_{\Omega^f} + \langle g^p, 1\rangle_{\Omega^p} - \langle \boldsymbol{h}^f \cdot \eta^f, 1\rangle_{\Gamma^f} - \langle h^p, 1\rangle_{\Gamma^p} = 0.$$

3. Weak formulation

In this section we present the weak version of the coupled system of partial differential equations introduced above. Without loss of generality, we consider $\boldsymbol{h}^f = \boldsymbol{\Gamma}$, $g^f = 0$, $h^p = 0$ and $g^p = 0$ in (1); see [26].

The problem can be formulated as: *Find* $(\boldsymbol{u}, p, \lambda) \in \boldsymbol{X} \times M_0 \times \Lambda$ *such that for all* $(\boldsymbol{v}, q, \mu) \in \boldsymbol{X} \times M_0 \times \Lambda$

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b_\Gamma(\boldsymbol{v}, \lambda) &= f(\boldsymbol{v}), \\ b(\boldsymbol{u}, q) &= 0, \\ b_\Gamma(\boldsymbol{u}, \mu) &= 0, \end{cases} \tag{2}$$

where

$$\boldsymbol{X} = \boldsymbol{X}^f \times \boldsymbol{X}^p := H_0^1(\Omega^f, \Gamma^f)^n \times \boldsymbol{H}_0(\mathrm{div}, \Omega^p, \Gamma^p)$$

and M_0 is the subset of $M := L^2(\Omega^f) \times L^2(\Omega^p) \equiv L^2(\Omega)$ of pressures with a zero average value in Ω. Here $H_0^1(\Omega^f, \Gamma^f)$ denotes the subspace of $H^1(\Omega^f)$ of functions that vanish on Γ^f. The space $\boldsymbol{H}_0(\mathrm{div}, \Omega^p, \Gamma^p)$ consists of functions in $\boldsymbol{H}(\mathrm{div}, \Omega^p)$ with zero normal trace on Γ^p, where

$$\boldsymbol{H}(\mathrm{div}, \Omega^p) := \left\{ \boldsymbol{v} \in L^2(\Omega^p)^n : \mathrm{div}\ \boldsymbol{v} \in L^2(\Omega^p) \right\}.$$

For the Lagrange multiplier space we consider $\Lambda := H^{1/2}(\Gamma)$. See [26] for a discussion on the choice of the Lagrange multipliers space Λ and how to derive the weak formulation (2) and other equivalent weak formulations; see also [31].

The global bilinear forms are

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &:= a^f_{\alpha^f}(\boldsymbol{u}^f, \boldsymbol{v}^f) + a^p(\boldsymbol{u}^p, \boldsymbol{v}^p), \\ b(\boldsymbol{v}, p) &:= b^f(\boldsymbol{v}^f, p^f) + b^p(\boldsymbol{v}^p, p^p), \end{aligned}$$

with local bilinear forms $a^f_{\alpha^f}$, b^f and b^p defined by

$$a^f_{\alpha^f}(\boldsymbol{u}^f, \boldsymbol{v}^f) := 2\nu(\boldsymbol{D}\boldsymbol{u}^f, \boldsymbol{D}\boldsymbol{v}^f)_{\Omega^f} + \sum_{\ell=1}^{n-1} \frac{\nu\alpha^f}{\sqrt{\kappa}} \langle \boldsymbol{u}^f \cdot \tau_\ell, \boldsymbol{v}^f \cdot \tau_\ell\rangle_\Gamma, \ \boldsymbol{u}^f, \boldsymbol{v}^f \in \boldsymbol{X}^f, \tag{3}$$

$$a^p(\boldsymbol{u}^p, \boldsymbol{v}^p) := ((\nu/\kappa)\boldsymbol{u}^p, \boldsymbol{v}^p)_{\Omega^p}, \quad \boldsymbol{u}^p, \boldsymbol{v}^p \in \boldsymbol{X}^p, \tag{4}$$

$$b^f(\boldsymbol{v}^f, q^f) := -(q^f, \nabla \cdot \boldsymbol{v}^f)_{\Omega^f}, \quad \boldsymbol{v}^f \in \boldsymbol{X}^f, \ q^f \in M^f, \tag{5}$$

$$b^p(\boldsymbol{v}^p, p^p) := -(p^p, \nabla \cdot \boldsymbol{v}^p)_{\Omega^p}, \quad \boldsymbol{v}^p \in \boldsymbol{X}^p, \ p^p \in M^p, \tag{6}$$

and with weak conservation of mass bilinear form defined by

$$b_\Gamma(\boldsymbol{v}, \mu) := \langle \boldsymbol{v}^f \cdot \eta^f, \mu \rangle_\Gamma + \langle \boldsymbol{v}^p \cdot \eta^p, \mu \rangle_\Gamma, \ \boldsymbol{v} = (\boldsymbol{v}^f, \boldsymbol{v}^p) \in \boldsymbol{X}, \mu \in \Lambda. \tag{7}$$

The second duality pairing of (7) is interpreted as $\langle \boldsymbol{v}^p \cdot \eta^p, E_{\eta^p}(\mu) \rangle_{\partial\Omega^p}$. Here E_{η^p} is any continuous lift-in operator from $H^{1/2}(\Gamma)$ to $H^{1/2}(\partial\Omega^p)$; recall that $\Gamma \subset \partial\Omega^p$ and $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \Omega^p, \Gamma^p)$. It easy to see that this duality pairing is independent of the lift-in operator E_{η^p}. In particular, one example of such a lift-in operator can be constructed by taking the trace on $\partial\Omega^p$ of the harmonic extension with Dirichlet data μ on Γ and homogeneous Neumann data on Γ^p; see [26].

The functional f in the right side of (2) is defined by

$$f(\boldsymbol{v}) := f^f(\boldsymbol{v}^f) + f^p(\boldsymbol{v}^p), \quad \text{for all } \boldsymbol{v} = (\boldsymbol{v}^f, \boldsymbol{v}^p) \in \boldsymbol{X},$$

where $f^i(\boldsymbol{v}^i) := (\boldsymbol{f}^i, \boldsymbol{v}^i)_{L^2(\Omega^i)}$ for all $\boldsymbol{v}^i \in \boldsymbol{X}^i$, $i = f, p$.

The bilinear forms $a^f_{\alpha^f}, b^f$ are associated to Stokes equations, and the bilinear forms a^p, b^p to Darcy law. The bilinear form $a^f_{\alpha^f}$ includes interface matching conditions 1.*b* and 1.*c* above. The bilinear form b_Γ is used to impose the weak version of the interface matching condition 1.*a* above. We have the following lemma that addresses the well-posedness of the problem.

Lemma 1 (See [26; 31]). *There exists $\beta > 0$ such that*

$$\inf_{\substack{(q,\mu)\in M_0\times\Lambda \\ (q,\mu)\neq 0}} \sup_{\substack{\boldsymbol{v}\in\boldsymbol{X} \\ \boldsymbol{v}\neq 0}} \frac{b(\boldsymbol{v}, q) + b_\Gamma(\boldsymbol{v}, \mu)}{\|\boldsymbol{v}\|_{\boldsymbol{X}} \left(\|p\|_M + \|\mu\|_\Lambda\right)} \geq \beta > 0. \tag{8}$$

where

$$\|\boldsymbol{v}\|^2_{\boldsymbol{X}} := \|\boldsymbol{v}^f\|^2_{H^1_0(\Omega_f)^2} + \|\boldsymbol{v}^p\|^2_{\boldsymbol{H}(\mathrm{div},\Omega_p)}.$$

This inf-sup condition, together with the fact that $a^f_{\alpha^f}$ is $\boldsymbol{X}^f \times \boldsymbol{H}(\mathrm{div}^0, \Omega^p)$-elliptic and $a^f_{\alpha^f}$, b and b_Γ are bounded, guarantees the well-posedness of the problem (2).

4. Discretization

From now on we consider only the two-dimensional case. We note that the ideas developed in the following can be easily extended to case of three-dimensional subdomains.

We assume that Ω^i, $i = f, p$, are *two-dimensional* polygonal subdomains. Let $\mathcal{T}^i_{h^i}(\Omega^i)$ be a geometrically conforming shape regular and quasiuniform triangulation of Ω^i with mesh size parameter h^i, $i = f, p$. We do not assume that these two

triangulations match at the interface Γ. For the fluid region, let $\boldsymbol{X}^f_{h^f}$ and $M^f_{h^f}$ be P2/P1 triangular Taylor–Hood finite elements; see [7; 8; 10]. More precisely,

$$\boldsymbol{X}^f_{h^f} := \left\{ \boldsymbol{u} \in \boldsymbol{X}^f \ : \ \begin{array}{l} \boldsymbol{u}_K = \hat{\boldsymbol{u}}_K \circ F_K^{-1} \text{ and } \hat{\boldsymbol{u}}_K \in P_2(\hat{K})^2 \\ \text{for all } K \in \mathcal{T}^f_{h^f}(\Omega_f) \end{array} \right\} \cap C^0(\overline{\Omega}^f)^2, \tag{9}$$

where $\boldsymbol{u}_K := \boldsymbol{u}|_K$ and

$$M^f_{h^f} := \left\{ p \in L^2(\Omega^f) \ : \ \begin{array}{l} p_K = \hat{p}_K \circ F_K^{-1} \text{ and } \hat{p}_K \in P_1(\hat{K}) \\ \text{for all } K \in \mathcal{T}^f_{h^f}(\Omega_f), \end{array} \right\} \cap C^0(\overline{\Omega}^f).$$

Denote by $\mathring{M}^f_{h^f} \subset M^f_{h^f}$ the discrete fluid pressures with zero average value in Ω^f. For the porous region, let $\boldsymbol{X}^p_{h^p} \subset \boldsymbol{X}^p$ and $M^p_{h^p} \subset L^2(\Omega^p)$ be the lowest order Raviart–Thomas finite elements based on triangles; see [7; 10]. Let $\mathring{M}^p_{h^p} \subset M^p_{h^p}$ be the subset of pressures in $M^p_{h^p}$ with zero average value in Ω^p.

Define $\boldsymbol{X}_h := \boldsymbol{X}^f_{h^f} \times \boldsymbol{X}^p_{h^p} \subset \boldsymbol{X}$ and $M_h := M^f_{h^f} \times M^p_{h^p} \subset L^2(\Omega^f) \times L^2(\Omega^p)$. Note that in the definition of the discrete velocities we assume that the boundary conditions are included, that is, for $\boldsymbol{v}^f_{h^f} \in \boldsymbol{X}^f_{h^f}$, we have $\boldsymbol{v}^f_{h^f} = \boldsymbol{\Gamma}$ on Γ^f and for $\boldsymbol{v}^p_{h^p} \in \boldsymbol{X}^p_{h^p}$ we have that $\boldsymbol{v}^p_h \cdot \eta^p = 0$ on Γ^p.

Let $\mathcal{T}^p_{h^p}(\Gamma)$ be the restriction to Γ of the porous side triangulation $\mathcal{T}^p_{h^p}(\Omega^p)$. For the Lagrange multipliers space we choose piecewise constant functions on Γ with respect to the triangulation $\mathcal{T}^p_{h^p}(\Gamma)$:

$$\Lambda_{h^p} := \left\{ \lambda \ : \ \lambda|_{e^p_j} = \lambda_{e^p_j} \text{ is constant in each edge } e^p_j \text{ of } \mathcal{T}^p_{h^p}(\Gamma) \right\}, \tag{10}$$

that is, the *master* is on the fluid region side and the *slave* is on the porous region side; see [5; 6; 19; 45]. The choice of piecewise constant Lagrange multipliers leads to a nonconforming approximation on Λ_{h^p} since piecewise constant functions do not belong to $H^{1/2}(\Gamma)$. For the analysis of this nonconforming discretization and a priori error estimates we refer to [26].

5. Primal and dual formulations

In order to simplify the notation and since there is no danger of confusion, we will denote the finite element functions and the corresponding vector representation by the same symbol, that is, when writing finite element functions we will drop the indices h^i. Recall that we have the pair of spaces $(\boldsymbol{X}_h, M_h)$ associated to the coupled problem, and spaces associated to each subproblem: $(\boldsymbol{X}^f_{h^f}, M^f_{h^f})$ and $(\boldsymbol{X}^p_{h^p}, M^p_{h^p})$. We will keep the subscript h^i, $i = f, p$, in the notation for local subspaces $\boldsymbol{X}^f_{h^f}$, $M^f_{h^f}$, $\boldsymbol{X}^p_{h^p}$ and $M^p_{h^p}$.

Since we are interested in preconditioning issues we assume $\alpha^f = 0$ in the definition of the fluid side local bilinear form $a^f_{\alpha^f}$ in (3). We denote $a^f = a^f_0$. See Remark 10 for the case $\alpha^f > 0$.

With the discretization chosen in Section 4 we obtain the following symmetric saddle point linear system

$$\left[\begin{array}{cc|cc|c} A^f & B^{fT} & 0 & 0 & C^{fT} \\ B^f & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & A^p & B^{pT} & -C^{pT} \\ 0 & 0 & B^p & 0 & 0 \\ \hline C^f & 0 & -C^p & 0 & 0 \end{array}\right] \left[\begin{array}{c} \boldsymbol{u}^f \\ p^f \\ \hline \boldsymbol{u}^p \\ p^p \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} \boldsymbol{f}^f \\ g^f \\ \hline \boldsymbol{f}^p \\ g^p \\ \hline 0 \end{array}\right], \tag{11}$$

with matrices A^i, B^i, C^i and columns vectors $\boldsymbol{f}^i$, g^i, $i = f, p$, defined by

$$\begin{aligned} a^i(\boldsymbol{u}^i, \boldsymbol{v}^i) &= \boldsymbol{v}^{iT} A^i \boldsymbol{u}^i, \\ b^i(\boldsymbol{u}^i, q^i) &= q^{iT} B^i \boldsymbol{u}^i, \\ (\boldsymbol{u}^i \cdot \eta^f, \mu)_\Gamma &= \mu^T C^i \boldsymbol{u}^i, \\ f^i(\boldsymbol{v}^i) &= \boldsymbol{v}^{iT} \boldsymbol{f}^i, \\ g^i(q^i) &= q^{iT} g^i. \end{aligned} \tag{12}$$

The matrix A^f corresponds to ν times the discrete version of the linearized stress tensor on Ω^f. Note that in the case $\alpha^f > 0$, the bilinear form $a^f_{\alpha^f}$ in (3) includes a boundary term; see Remark 10. The matrix A^p corresponds to ν/κ times a discrete L^2-norm on Ω^p. Matrix $-B^i$ is the discrete divergence in Ω^i, $i = f, p$, and matrices C^f and C^p correspond to the matrix form of the discrete conservation of mass on Γ. Note that ν can be viewed as a scaling factor since it appears in both matrices A^f and A^p. Therefore, it is not relevant for preconditioning issues.

Consider the following partition of the degrees of freedom: for $i = f, p$, let

$$\left[\begin{array}{l} \boldsymbol{u}^i_I \\ p^i_I \\ u^i_\Gamma \\ \bar{p}^i \end{array}\right] \begin{array}{l} \text{interior displacements + tangential velocities on } \Gamma, \\ \text{interior pressures with zero average in } \Omega^i, \\ \text{interface outward } \textit{normal velocities} \text{ on } \Gamma, \\ \text{constant pressure in } \Omega^i. \end{array}$$

For $i = f, p$, we have the block structure

$$A^i = \left[\begin{array}{cc} A^i_{II} & A^{iT}_{\Gamma I} \\ A^i_{\Gamma I} & A^i_{\Gamma\Gamma} \end{array}\right], \quad B^i = \left[\begin{array}{cc} B^i_{II} & B^{iT}_{\Gamma I} \\ 0 & \bar{B}^{iT} \end{array}\right] \quad \text{and } C^i = \begin{bmatrix} 0 & 0 & \tilde{C}^i & 0 \end{bmatrix}.$$

Note that the $(2, 1)$ entry of B^i corresponds to integrating an *interior* velocity against a constant pressure, then it vanishes due to the divergence theorem. We

have the following matrix representation of the coupled problem in (11):

$$\left[\begin{array}{cccc|cccc|c} A_{II}^f & B_{II}^{fT} & A_{\Gamma I}^{fT} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{II}^f & 0 & B_{\Gamma I}^{fT} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\Gamma I}^f & B_{I\Gamma}^{fT} & A_{\Gamma\Gamma}^f & \bar{B}^{fT} & 0 & 0 & 0 & 0 & \tilde{C}^{fT} \\ 0 & 0 & \bar{B}^f & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & A_{II}^p & B_{II}^{pT} & A_{\Gamma I}^{pT} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{II}^p & 0 & B_{I\Gamma}^p & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{\Gamma I}^p & B_{I\Gamma}^{pT} & A_{\Gamma\Gamma}^p & \bar{B}^{pT} & -\tilde{C}^{pT} \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{B}^p & 0 & 0 \\ \hline 0 & 0 & \tilde{C}^f & 0 & 0 & 0 & -\tilde{C}^p & 0 & 0 \end{array}\right] \left[\begin{array}{c} \boldsymbol{u}_I^f \\ p_I^f \\ u_\Gamma^f \\ \bar{p}^f \\ \hline \boldsymbol{u}_I^p \\ p_I^p \\ u_\Gamma^p \\ \bar{p}^p \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} \boldsymbol{f}_I^f \\ g_I^f \\ f_\Gamma^f \\ \bar{g}^f \\ \hline \boldsymbol{f}_I^p \\ g_I^p \\ f_\Gamma^p \\ \bar{g}^p \\ \hline 0 \end{array}\right]. \tag{13}$$

Following [19; 40], we choose the following matrix representation in each subdomain Ω^i, $i = f, p$:

$$\left[\begin{array}{cc|cc} A_{II}^i & B_{II}^{iT} & A_{\Gamma I}^{iT} & 0 \\ B_{II}^i & 0 & B_{I\Gamma}^i & 0 \\ \hline A_{\Gamma I}^i & B_{I\Gamma}^{iT} & A_{\Gamma\Gamma}^i & \bar{B}^{iT} \\ 0 & 0 & \bar{B}^i & 0 \end{array}\right] = \left[\begin{array}{c|c} K_{II}^i & K_{\Gamma I}^{iT} \\ \hline K_{\Gamma I}^i & K_{\Gamma\Gamma}^i \end{array}\right]. \tag{14}$$

5.1. ***The primal formulation.*** From the last equation in (13) we see that the mortar condition on Γ (using the Darcy side as the slave side) can be imposed as $u_\Gamma^p = (\tilde{C}^p)^{-1}\tilde{C}^f u_\Gamma^f = \Pi u_\Gamma^f$, where Π is the $L^2(\Gamma)$ projection on the space of piecewise constant functions on each subinterval $e^p \in \mathcal{T}_{h^p}^p(\Gamma)$. We note that $\tilde{C}^p$ is a diagonal matrix for the lowest order Raviart–Thomas elements.

Now we eliminate $\boldsymbol{u}_I^i$, p_I^i, $i = f, p$, and λ, to obtain the following (saddle point) Schur complement:

$$S \left[\begin{array}{c} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{array}\right] = \left[\begin{array}{c} b_\Gamma \\ \bar{b}^f \\ \bar{b}^p \end{array}\right]. \tag{15}$$

Here S is given by

$$S := \left[\begin{array}{ccc} S_\Gamma^f & \bar{B}^{fT} & 0 \\ \bar{B}^f & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] + \tilde{\Pi}^T \left[\begin{array}{ccc} S_\Gamma^p & 0 & \bar{B}^{pT} \\ 0 & 0 & 0 \\ \bar{B}^p & 0 & 0 \end{array}\right] \tilde{\Pi} = \tilde{S}^f + \tilde{S}^p$$

$$= \left[\begin{array}{c|cc} S_\Gamma^f + \Pi^T S_\Gamma^p \Pi & \bar{B}^{fT} & \Pi^T \bar{B}^{pT} \\ \hline \bar{B}^f & 0 & 0 \\ \bar{B}^p \Pi & 0 & 0 \end{array}\right] = \left[\begin{array}{cc} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{array}\right],$$

where

$$\tilde{\Pi} := \begin{bmatrix} \Pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and } \bar{B}^T := [\bar{B}^{fT} \ \Pi^T \bar{B}^{pT}]. \tag{16}$$

Here, we have introduced

$$\tilde{S}^f := \begin{bmatrix} S_\Gamma^f & \bar{B}^{fT} & 0 \\ \bar{B}^f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{S}^p := \tilde{\Pi}^T \begin{bmatrix} S_\Gamma^p & 0 & \bar{B}^{pT} \\ 0 & 0 & 0 \\ \bar{B}^p & 0 & 0 \end{bmatrix} \tilde{\Pi} \tag{17}$$

and

$$S_\Gamma := S_\Gamma^f + \Pi^T S_\Gamma^p \Pi. \tag{18}$$

The local matrices S_Γ^i and $\bar{B}^i$ and the local Schur complement S^i are given by

$$S^i = \begin{bmatrix} S_\Gamma^i & \bar{B}^{iT} \\ \bar{B}^i & 0 \end{bmatrix} := K_{\Gamma\Gamma}^i - K_{\Gamma I}^i \left(K_{II}^i\right)^{-1} K_{\Gamma I}^{iT}, \ i = p, f. \tag{19}$$

The right side of (15) is given by

$$\begin{bmatrix} b_\Gamma \\ \bar{b}^f \\ \bar{b}^p \end{bmatrix} = \left\{ \begin{bmatrix} f_\Gamma^f \\ \bar{g}^f \\ 0 \end{bmatrix} - \begin{bmatrix} K_{\Gamma I}^f \left(K_{II}^f\right)^{-1} \begin{bmatrix} \boldsymbol{f}_I^f \\ g_I^f \end{bmatrix} \\ 0 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} \Pi^T f_\Gamma^p \\ 0 \\ \bar{g}^p \end{bmatrix} - \tilde{\Pi}^T \begin{bmatrix} K_{\Gamma I}^p \left(K_{II}^p\right)^{-1} \begin{bmatrix} \boldsymbol{f}_I^p \\ g_I^p \end{bmatrix} \\ 0 \end{bmatrix} \right\}.$$

We note that the reduced system (15), as well as the original system (13), is solvable when $\bar{b}^f + \bar{b}^p = 0$, and the solution is unique when we restrict to pressures with zero average value on Ω.

From now on we only work with functions defined on Γ and extended inside the subdomain using the discrete Stokes and Darcy problems. It is convenient to define the space

$$V_\Gamma := \left\{ v_\Gamma = (v^f{}_\Gamma, v_\Gamma^p) : v^f{}_\Gamma = \mathcal{SH}(\boldsymbol{v}^f \cdot \eta^f|_\Gamma) \text{ and } v_\Gamma^p = \mathcal{DH}(\boldsymbol{v}^p \cdot \eta^p|_\Gamma) \right\} \tag{20}$$

and

$$M_0^h := \Big\{ q \in M^h : q^i = \text{piecewise constant in } \Omega^i \text{ for } i = f, p, \text{ and } \int_{\Omega^f} q^f + \int_{\Omega^p} q^p = 0 \Big\}. \tag{21}$$

Here $\mathcal{SH}$ ($\mathcal{DH}$) is the velocity component of the discrete Stokes (Darcy) harmonic extension operator that maps discrete interface normal velocity $u_\Gamma^f \in H_{00}^{1/2}(\Gamma)$ (respectively $u_\Gamma^p \in (H^{1/2}(\Gamma))'$) to the solution of following problem: Find $\boldsymbol{u}^i \in \boldsymbol{X}_{h^i}^i$ and $p^i \in \mathring{M}_{h^i}^i$ such that for all $\boldsymbol{v}^i \in \boldsymbol{X}_{h^i}^i$ and $q^i \in \mathring{M}_{h^i}^i$, $i = f, p$, we have

$$\begin{cases} a^f(\mathcal{SH}u^f, \boldsymbol{v}^f) + b^f(\boldsymbol{v}^f, p^f) &= 0, \\ b^f(\mathcal{SH}u^f, q^f) &= 0, \\ \mathcal{SH}u^f \cdot \eta^f &= u_\Gamma^f \quad \text{on } \Gamma, \\ \mathcal{SH}u^f &= \mathbf{\Gamma} \quad \text{on } \Gamma^f, \end{cases} \tag{22}$$

and

$$\begin{cases} a^p(\mathcal{DH}u^p, \boldsymbol{v}^p) + b^p(\boldsymbol{v}^p, p^p) &= 0, \\ b^p(\mathcal{DH}u^p, q^p) &= 0, \\ \mathcal{DH}u^p \cdot \eta^p &= u_\Gamma^p \quad \text{on } \Gamma, \\ \mathcal{DH}u^p \cdot \eta^p &= 0 \quad \text{on } \Gamma^p. \end{cases} \tag{23}$$

The degrees of freedom associated with $\mathcal{SH}u^f \cdot \tau^f$ on Γ are free. This corresponds to imposing the natural boundary condition $\tau^T \boldsymbol{D}(\mathcal{SH}u^f)\eta_f = 0$ on Γ which is the expression for interface condition of Beavers–Joseph–Saffman with $\alpha^f = 0$.

For $i = f, p$, define the normal trace component of $\boldsymbol{X}_{h^i}^i$ by

$$Z_{h^i}^i = \left\{ \boldsymbol{v}^i \cdot \eta^i|_\Gamma \ : \ \boldsymbol{v}^i \in \boldsymbol{X}_{h^i}^i \right\}. \tag{24}$$

Associated with the coupled problem (13) we introduce the *balanced subspace*:

$$V_{\Gamma,\bar{B}} := \left\{ v^f{}_\Gamma \in Z_{h^f}^f \ : \ (v^f{}_\Gamma, \Pi v_\Gamma^f) \in V_\Gamma \text{ and } \int_\Gamma v^f{}_\Gamma \cdot \eta_f = 0 \right\}, \tag{25}$$

with V_Γ defined in (20); see [40]. Observe that $V_{\Gamma,\bar{B}} = \mathrm{Ker}\bar{B}$, where $\bar{B}$ is defined in (16) and (19). Then for $v_\Gamma^f \in V_{\Gamma,\bar{B}}$ we have $\bar{B}v_\Gamma^f = 0$. We will refer to functions $v_\Gamma^f \in V_{\Gamma,\bar{B}}$ as *balanced functions*. If $v_\Gamma^p = \Pi v_\Gamma^f$ and v_Γ^f is a balanced function, then we also say that v_Γ^p is a balanced function or the pair $(v_\Gamma^f, \Pi v^f{}_\Gamma)$ is balanced.

5.2. ***Dual formulation.*** In the system (13), we first eliminate the unknowns $\boldsymbol{u}_I^f$, p_I^f and $\boldsymbol{u}_I^p$, p_I^p. We obtain

$$\left[\begin{array}{cc|cc|c} S_\Gamma^f & \bar{B}^{fT} & 0 & 0 & \tilde{C}^{fT} \\ \bar{B}^f & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & S_\Gamma^p & \bar{B}^{pT} & -\tilde{C}^{pT} \\ 0 & 0 & \bar{B}^p & 0 & 0 \\ \hline \tilde{C}^f & 0 & -\tilde{C}^p & 0 & 0 \end{array}\right] \left[\begin{array}{c} u_\Gamma^f \\ \bar{p}^f \\ \hline u_\Gamma^p \\ \bar{p}^p \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} \tilde{b}^f \\ \hline \tilde{b}^p \\ \hline 0 \end{array}\right], \tag{26}$$

where the right side of (26) is given by

$$\left[\begin{array}{c}\tilde{b}^f\\ \hline \tilde{b}^p\\ \hline 0\end{array}\right] = \left[\begin{array}{c}\left[\begin{array}{c}f_\Gamma^f\\ \bar{g}^f\end{array}\right] - K_{\Gamma I}^f \left(K_{II}^f\right)^{-1} \left[\begin{array}{c}\boldsymbol{f}_I^f\\ g_I^f\end{array}\right]\\ \hline \left[\begin{array}{c}f_\Gamma^p\\ \bar{g}^p\end{array}\right] - K_{\Gamma I}^p \left(K_{II}^p\right)^{-1} \left[\begin{array}{c}\boldsymbol{f}_I^p\\ g_I^p\end{array}\right]\\ \hline 0\end{array}\right].$$

Here S_Γ^i, K_{II}^i and $K_{I\Gamma}^i$, $i = f, p$, are defined in (19) and (14).

Let $N_i := \left[\tilde{C}^i \;\; 0\right]$ and consider S^i, $i = f, p$, defined in (19). Then the matrix in the left side of (26) can be rewritten as

$$\left[\begin{array}{c|c|c} S^f & 0 & N^{fT}\\ \hline 0 & S^p & -N^{pT}\\ \hline N^f & -N^p & 0\end{array}\right].$$

Now we eliminate the unknowns u_Γ^f, $\bar{p}^f$ and u_Γ^p, $\bar{p}^p$. We end up with the reduced system

$$F\lambda = c, \tag{27}$$

where the operator F is defined by

$$F := N^f (S^f)^{-1} N^{fT} + N^p (S^p)^{-1} N^{pT}, \tag{28}$$

and the right side c is given by

$$c = N^f (S^f)^{-1} \left\{ \left[\begin{array}{c}f_\Gamma^f\\ \bar{g}^f\end{array}\right] - K_{\Gamma I}^f \left(K_{II}^f\right)^{-1} \left[\begin{array}{c}\boldsymbol{f}_I^f\\ g_I^f\end{array}\right] \right\} - N^p (S^p)^{-1} \left\{ \left[\begin{array}{c}f_\Gamma^p\\ \bar{g}^p\end{array}\right] - K_{\Gamma I}^p \left(K_{II}^p\right)^{-1} \left[\begin{array}{c}\boldsymbol{f}_I^f\\ g_I^p\end{array}\right] \right\}.$$

Note that F is positive semidefinite and since a discrete Lagrange multiplier in Λ_{h^p} does not have necessarily zero mean average value on Γ, the operator F has one simple zero eigenvalue corresponding to a constant Lagrange multiplier. The linear system above, as well as the original linear system (13), is solvable for zero mean right side, that is, for $c^T \cdot (1, \ldots, 1) = 0$.

6. BDD preconditioner

In this section we design and analyze a BDD type preconditioner for the Schur complement system (15); see [9; 19; 42] and also [1; 21; 35; 40; 43]. For the sake

of simplicity on the analysis we assume that $\Gamma = \{1\} \times (0,1)$, $\Omega^f = (1,2) \times (0,1)$ and $\Omega^p = (0,1) \times (0,1)$. We introduce the velocity coarse space on Γ as the span of the normal velocity $v_0 = y(1-y)$ (with v_0 also denoting its vector representation). Define

$$R_0 := \begin{bmatrix} v_0^T & 0 \\ 0 & I_{2\times 2} \end{bmatrix}, \quad S_0 := R_0 S R_0^T \quad \text{and} \quad Q_0 := R_0^T S_0^\dagger R_0. \tag{29}$$

The system (15) is solvable when the right side satisfies $\bar{b}^f + \bar{b}^p = 0$ with uniqueness of the solution in the space of vectors with pressure component having zero average value on Ω. Then S_0 is invertible restricted to vectors with pressure component in M_0^h defined in (21). The low dimensionality of the coarse space (which is spanned by v_0 and a constant pressure per subdomain Ω^i, $i = f, p$) and the fact that the function v_0 is independent of the triangulation parameters imply stable discrete inf-sup condition for the coarse problem.

Denote $\tilde{S}_0 := v_0^T S_\Gamma v_0$ and $\tilde{S} := \bar{B} v_0 \tilde{S}_0^{-1} v_0^T \bar{B}^T$. We can write, see (18) and (29),

$$S_0 = \begin{bmatrix} \tilde{S}_0 & (\bar{B}v_0)^T \\ \bar{B}v_0 & 0 \end{bmatrix}.$$

A simple calculation using the formula for the inverse of a saddle point matrix gives

$$Q_0 = \begin{bmatrix} v_0\tilde{S}_0^{-1}v_0^T - v_0\tilde{S}_0^{-1}v_0^T\bar{B}^T\tilde{S}^{-1}\bar{B}v_0\tilde{S}_0^{-1}v_0^T & v_0\tilde{S}_0^{-1}v_0^T\bar{B}^T\tilde{S}^{-1} \\ \tilde{S}^{-1}\bar{B}v_0\tilde{S}_0^{-1}v_0^T & \tilde{S}^{-1} \end{bmatrix},$$

and using (18) we obtain

$$Q_0 S = \begin{bmatrix} v_0\tilde{S}_0^{-1}v_0^T S_\Gamma - v_0\tilde{S}_0^{-1}v_0^T\bar{B}^T\tilde{S}^{-1}\bar{B}v_0\tilde{S}_0^{-1}v_0^T S_\Gamma + v_0\tilde{S}_0^{-1}v_0^T\bar{B}^T\tilde{S}^{-1}\bar{B} & 0 \\ \tilde{S}^{-1}\bar{B}v_0\tilde{S}_0^{-1}v_0^T S_\Gamma - \tilde{S}^{-1}\bar{B} & I \end{bmatrix},$$

or

$$Q_0 S = \begin{bmatrix} \mathcal{P} & 0 \\ \mathcal{G} & I \end{bmatrix},$$

where we have defined

$$\mathcal{P} := \left(v_0\tilde{S}_0^{-1}v_0^T S_\Gamma - v_0\tilde{S}_0^{-1}v_0^T\bar{B}^T\tilde{S}^{-1}\bar{B}v_0\tilde{S}_0^{-1}v_0^T S_\Gamma\right) + v_0\tilde{S}_0^{-1}v_0^T\bar{B}^T\tilde{S}^{-1}\bar{B},$$
$$\mathcal{G} := \tilde{S}^{-1}\bar{B} - \tilde{S}^{-1}\bar{B}v_0\tilde{S}_0^{-1}v_0^T S_\Gamma.$$

With this notation we have that

$$I - Q_0 S = \begin{bmatrix} I - \mathcal{P} & 0 \\ \mathcal{G} & 0 \end{bmatrix}.$$

Elementary calculations show that $\mathcal{P}^2 = \mathcal{P}$ and $\bar{B}(I-\mathcal{P}) = 0$, hence $I-\mathcal{P}$ is a projection and its image is contained on the balanced subspace defined in (25); see also [40].

Given a residual $r = \left[f_\Gamma^T \;\; \bar{g}^T\right]^T$, the coarse problem $Q_0 r$, with Q_0 defined in (29), is the solution of the coupled problem (13) with one velocity degree of freedom (v_0), and a constant pressure per subdomain Ω^i, $i = f, p$, with mean zero in $\Omega = \text{int}(\overline{\Omega^f} \cup \overline{\Omega^p})$. Note that the matrix S_0 defined in (29) can be computed easily, and in order to ensure zero mean pressure on Ω we can use a Lagrange multiplier.

For balanced functions v_Γ^f and u_Γ^f, the S_Γ-inner product (see (18)) is defined by

$$\langle u_\Gamma^f, v_\Gamma^f \rangle_{S_\Gamma} := \langle S_\Gamma u_\Gamma^f, v_\Gamma^f \rangle = u_\Gamma^{fT} S_\Gamma v_\Gamma^f .$$

Recall that $\bar{B} u_\Gamma^f = 0$ when u_Γ^f is balanced. Then, on this subspace of balanced functions, the S_Γ inner product coincides with the S-inner product defined by

$$\left\langle \begin{bmatrix} v_\Gamma^f \\ \bar{q}^f \\ \bar{q}^p \end{bmatrix}, \begin{bmatrix} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{bmatrix} \right\rangle_S := \begin{bmatrix} v_\Gamma^f \\ \bar{q}^f \\ \bar{q}^p \end{bmatrix}^T S \begin{bmatrix} u_\Gamma^f \\ \bar{p}^f \\ \bar{p}^p \end{bmatrix} = \begin{bmatrix} v_\Gamma^f \\ \bar{q} \end{bmatrix}^T \begin{bmatrix} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{bmatrix} \begin{bmatrix} u_\Gamma^f \\ \bar{p} \end{bmatrix},$$

where $\bar{p}^T = \left[\bar{p}^p \;\; \bar{p}^p\right]^T$. Consider the BDD preconditioner operator given by

$$S_N^{-1} := Q_0 + (I - Q_0 S)\,(\tilde{S}^f)^\dagger\,(I - S Q_0)\,, \tag{30}$$

where $\tilde{S}^f$ is defined in (17); see [19; 40]. The notation $(\tilde{S}^f)^\dagger$ stands for the pseudo-inverse of $\tilde{S}^f$, that is,

$$(\tilde{S}^f)^\dagger = \begin{bmatrix} (S^f)^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

with S^f defined in (19). The preconditioned operator is given by

$$\begin{aligned} S_N^{-1} S &= Q_0 S + (I - Q_0 S)\,(\tilde{S}^f)^\dagger S\,(I - Q_0 S) \\ &= \begin{bmatrix} \mathcal{P} & 0 \\ \mathcal{G} & I \end{bmatrix} + \begin{bmatrix} I-\mathcal{P} & 0 \\ \mathcal{G} & 0 \end{bmatrix} (\tilde{S}^f)^\dagger \begin{bmatrix} S_\Gamma & \bar{B}^T \\ \bar{B} & 0 \end{bmatrix} \begin{bmatrix} I-\mathcal{P} & 0 \\ \mathcal{G} & 0 \end{bmatrix}. \end{aligned} \tag{31}$$

Note that applying $(S^f)^{-1}$ to a vector

$$\begin{bmatrix} u_\Gamma^f \\ \bar{p} \end{bmatrix}$$

is equivalent to solving the linear system

$$\begin{bmatrix} A_{II}^f & B_{II}^{fT} & A_{\Gamma I}^{fT} & 0 \\ B_{II}^f & 0 & B_{I\Gamma}^f & 0 \\ A_{\Gamma I}^f & B_{I\Gamma}^{fT} & A_{\Gamma\Gamma}^f & \bar{B}^{fT} \\ 0 & 0 & \bar{B}^f & 0 \end{bmatrix} \begin{bmatrix} w_I^f \\ s_I^f \\ w_\Gamma^f \\ \bar{s}^f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_\Gamma^f \\ \bar{p}^f \end{bmatrix}.$$

If u_Γ^f is balanced, so is the velocity component of

$$(S^f)^{-1} \begin{bmatrix} u_\Gamma^f \\ \bar{p}^f \end{bmatrix}.$$

Using elementary calculations with the matrices in (31) we obtain

$$\left\langle S_N^{-1} S \begin{bmatrix} u_\Gamma \\ \bar{p} \end{bmatrix}, \begin{bmatrix} v_\Gamma \\ \bar{q} \end{bmatrix} \right\rangle_S = \langle (S_\Gamma^f)^{-1} S_\Gamma u_\Gamma, v_\Gamma \rangle_{S_\Gamma},$$

for $u_\Gamma, v_\Gamma \in \text{Range}(I - \mathcal{P})$. In order to bound the condition number of the preconditioned operator $S_N^{-1} S$, we need only analyze the condition of the operator $(S_\Gamma^f)^{-1} S_\Gamma$. Note that

$$c\langle u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma} \leq \left\langle \left(S^f\right)^{-1} S_\Gamma u_\Gamma^f, u_\Gamma^f \right\rangle_{S_\Gamma} \leq C \langle u_\Gamma^f, u_\Gamma^f \rangle_{S_\Gamma}$$

is equivalent to

$$c\langle S^f u_\Gamma^f, u_\Gamma^f \rangle \leq \langle S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle \leq C \langle S^f u_\Gamma^f, u_\Gamma^f \rangle. \tag{32}$$

The next theorem shows that the condition number estimate for the BDD method introduced in (30) is of order $O(1 + (1/\kappa))$, where κ is the permeability of the porous medium; see (1).

Theorem 2. *If u_Γ^f is a balanced function then*

$$\langle S_\Gamma^f u_\Gamma^f, u_\Gamma^f \rangle \leq \langle S_\Gamma u_\Gamma^f, u_\Gamma^f \rangle \prec \left(1 + \frac{1}{\kappa}\right) \langle S_\Gamma^f u_\Gamma^f, u_\Gamma^f \rangle.$$

Proof. The lower bound follows trivially from $\tilde{S}_\Gamma^f$ and $\tilde{S}_\Gamma^p$ being positive on the subspace of balanced functions. Next we concentrate on the upper bound.

Let v_Γ^f be a balanced function and $v_\Gamma^p = \Pi v_\Gamma^f$. Define $\boldsymbol{v}^p = \mathcal{DH} v_\Gamma^p$; see (23). Using properties of the discrete operator $\mathcal{DH}$ [38] we obtain

$$\langle S_\Gamma^p v_\Gamma^p, v_\Gamma^p \rangle = a^p(\boldsymbol{v}^p, \boldsymbol{v}^p) \asymp \frac{\nu}{\kappa} \|v_\Gamma^p\|^2_{(H^{1/2})'(\Gamma)}.$$

Using the L_2-stability property of mortar projection Π, we have

$$\|v_\Gamma^p\|^2_{(H^{1/2})'(\Gamma)} \prec \|v_\Gamma^p\|^2_{L^2(\Gamma)} = \|v_\Gamma^f\|^2_{L^2(\Gamma)} \prec \|v_\Gamma^f\|^2_{H_{00}^{1/2}(\Gamma)}.$$

With $\mathcal{SH}$ defined in (22), define $\boldsymbol{v}^f = \mathcal{SH} v_\Gamma^f$. Using properties of $\mathcal{SH}$ [40], we have

$$\nu \|v_\Gamma^f\|^2_{H_{00}^{1/2}(\Gamma)} \asymp a^f(\boldsymbol{v}^f, \boldsymbol{v}^f)$$

and then

$$\langle S_\Gamma^p v_\Gamma^p, v_\Gamma^p \rangle \prec \frac{1}{\kappa} \langle S^f u_\Gamma^f, u_\Gamma^f \rangle. \tag{33}$$

This gives the upper bound and finishes the proof. □

Recall that we consider the preconditioned projected conjugate gradient method applied to the Schur complement problem (15). Here is the algorithm:

(1) Initialize

$$x^{(0)} = Q_0 b + w$$
$$d^{(0)} = b - S x^{(0)}$$

with $w \in \text{Range}(I - Q_0 S)$. Recall that all vectors have three components, for instance,

$$x = \begin{bmatrix} x_\Gamma \\ \bar{x}^f \\ \bar{x}^p \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_\Gamma \\ \bar{b}^f \\ \bar{b}^p \end{bmatrix}.$$

(2) Iterate $k = 1, 2, \ldots$ until convergence

$$\begin{aligned}
\text{Precondition: } z^{(k-1)} &= (\tilde{S}^f)^\dagger d^{(k-1)}, \\
\text{Project: } y^{(k-1)} &= (I - Q_0 S) z^{(k-1)} \\
\beta^k &= \langle y^{(k-1)}, d^{(k-1)} \rangle / \langle y^{(k-2)}, d^{(k-1)} \rangle \quad [\beta^{(1)} = 0], \\
r^{(k)} &= y^{(k-1)} + \beta^{(k)} r^{(k)} \quad [r^{(1)} = y^{(0)}], \\
\alpha^{(k)} &= \langle y^{(k-1)}, d^{(k-1)} \rangle / \langle d^{(k)}, S r^{(k)} \rangle, \\
x^{(k)} &= x^{(k-1)} + \alpha^{(k)} r^{(k)}, \\
d^{(k)} &= d^{(k-1)} - \alpha^{(k)} S r^{(k)}.
\end{aligned}$$

Implementation of the projected preconditioned conjugate gradient algorithm for the system (15) involving the BDD preconditioner (30).

7. FETI preconditioner

In this section we analyze a FETI preconditioner for the reduced linear system (27); see [9; 19; 42; 24; 30; 37]. Recall the definition of F in (28). We propose the following preconditioner

$$(N^p)^\dagger (S^p) (N^p)^{\dagger T}, \tag{34}$$

where $(N^p)^\dagger$ is the pseudo-inverse $(N^p)^\dagger = [(\tilde{C}^p)^{-1} \; 0]$.

Note that after computing the action of $(S^f)^{-1}$ and $(S^p)^{-1}$ in the application of F to a zero average Lagrange multiplier, we end up with balanced functions. Therefore, to apply the preconditioned operator $(N^p)^\dagger(S^p)(N^p)^{\dagger T}F$ to a zero mean Lagrange multiplier, we do not need to solve a coarse problem at the beginning of the CG, nor inside of the CG iteration.

The FETI preconditioner in (34) can be considered as the dual preconditioner of the BDD preconditioner defined in (30); see the proof of Lemma 3 below.

Recall the definition of S^i, $i = f, p$, in (19) and the definition of space of balanced functions $V_\Gamma = V_\Gamma^f \times V_\Gamma^p$ in (25) and (24). We prove the following result.

Lemma 3. *Let $\lambda \in \Lambda_{h^p} \cap L_0^2(\Gamma)$ be a zero mean Lagrange multiplier. Then*

$$\langle N^f (S^f)^{-1} N^{fT}\lambda, \lambda\rangle \prec \frac{1}{\kappa}\langle N^p (S^p)^{-1} N^{pT}\lambda, \lambda\rangle.$$

Proof. Consider a zero mean Lagrange multiplier λ. Define $t = (S_\Gamma^p)^{-1/2}\tilde{C}^{pT}\lambda$ and $w^f = \tilde{C}^{fT}\lambda$. Then it is enough to prove that

$$\|(S_\Gamma^f)^{-1/2} w^f\|^2 \prec \|t\|^2.$$

Since w^f is balanced, that is, $w^f \in V_\Gamma^f$, we have that

$$\begin{aligned}
\|(S_\Gamma^f)^{-1/2} w^f\|^2 &= \sup_{z^f \in Z_{h_f}^f} \frac{\langle (S_\Gamma^f)^{-1/2} w^f, z^f\rangle^2}{\|z^f\|^2} = \sup_{v^f \text{ balanced}} \frac{\langle w^f, v^f\rangle^2}{\|(S_\Gamma^f)^{1/2} v^f\|^2} \\
&= \sup_{v^f \text{ balanced}} \frac{\langle \lambda, N^f v^f\rangle^2}{\|(S_\Gamma^f)^{1/2} v^f\|^2} \\
&= \sup_{v^f \text{ balanced}} \frac{\langle (S_\Gamma^p)^{-1/2}\tilde{C}^p\lambda, (S_\Gamma^p)^{1/2}(\tilde{C}^p)^{-1}\tilde{C}^f v^f\rangle^2}{\|(S_\Gamma^f)^{1/2} v^f\|^2}.
\end{aligned}$$

Then using the Cauchy–Schwarz inequality and (33) in the proof of Theorem 2, we have

$$\begin{aligned}
\|(S_\Gamma^f)^{-1/2} w^f\|^2 &= \sup_{v^f \text{balanced}} \frac{\langle t, (S_\Gamma^p)^{1/2}(\tilde{C}^p)^{-1}\tilde{C}^f v^f\rangle^2}{\|(S_\Gamma^f)^{1/2} v^f\|^2} \\
&\le \|t\|^2 \sup_{v^f \text{balanced}} \frac{\|(S_\Gamma^p)^{1/2}(\tilde{C}^p)^{-1}\tilde{C}^f v^f\|^2}{\|(S_\Gamma^f)^{1/2} v^f\|^2} \prec \frac{1}{\kappa}\|t\|^2. \qquad \square
\end{aligned}$$

Using Lemma 3 we can derive the following estimate for the condition number of the FETI preconditioner defined in (34).

Theorem 4. *Let λ be a zero mean Lagrange multiplier. Then*

$$\langle N^p (S^p)^{-1} N_p^T \lambda, \lambda\rangle \prec \langle F\lambda, \lambda\rangle \prec \left(1 + \frac{1}{\kappa}\right)\langle N^p (S^p)^{-1} N^{pT}\lambda, \lambda\rangle.$$

The condition number estimate $O((\kappa+1)/\kappa)$ can be improved in the case where the fluid side triangulation is finer than the porous side triangulation. This case has some advantages when κ is small. In order to fix ideas and simplify notation we analyze in detail the case where the triangulation of the fluid side is a *refinement* of the porous side triangulation. In particular, in Theorem 7, we will prove that the condition of the FETI preconditioned operator is of order $O((\kappa+1)/(\kappa+(h^p)^2))$ in this simpler situation. The analysis that we will present to prove Theorem 7 can be extended easily for the case where the fluid side triangulation is finer than (and not necessarily a refinement of) the porous side triangulation; see Remark 8.

We assume that the fluid side discretization on Γ, $\mathcal{T}^f_{h^f}(\Omega^f)|_\Gamma$, is a refinement of the corresponding porous side discretization, $\mathcal{T}^p_{h^p}(\Omega^p)|_\Gamma$. That is, assume that $h^p = rh^f$ for some positive integer r. We will refer to this assumption as the *nested refinement assumption*. For $j = 1, \ldots, m^p$, we introduce the normal fluid velocity ϕ^f_j as the P2 bubble function defined on $\mathcal{T}^p_{h^p}(\Omega^p)|_\Gamma$ and with support on the interval $e^p_j = \{0\} \times [(j-1)h^p, jh^p]$. Recall that we are using P2/P1 Taylor–Hood discretization on the fluid side. Under the nested refinement assumption we have $\phi^f_j \in Z^f_{h^f}$ with $Z^f_{h^f}$ defined in (24). Denote by $Z^f_{h^f,b}$ the subspace of $Z^f_{h^f}$ spanned by all ϕ^f_j, $j = 1, \ldots, m^p$, and by $Z^f_{h^f,0}$ the subspace of $Z^f_{h^f}$ spanned by functions with zero average on all edges e^p_j, $j = 1, \ldots, m^p$. Note that $Z^f_{h^f,b}$ and $Z^f_{h^f,0}$ form a direct sum for $Z^f_{h^f}$ and the image $\Pi Z^f_{h^f,0}$ is the zero vector.

Before deriving the condition number estimate of the FETI preconditioner under the nested refinement assumption we first prove a preliminary lemma.

Lemma 5. *Assume that $h^p = rh^f$, where r is a positive integer. If $v^f_{\Gamma,b} \in Z^f_{h^f,b}$ is a balanced function, then*

$$\langle S^f_\Gamma v^f_{\Gamma,b}, v^f_{\Gamma,b}\rangle \asymp \frac{\kappa}{(h^p)^2}\langle S^p_\Gamma \Pi v^f_{\Gamma,b}, \Pi v^f_{\Gamma,b}\rangle.$$

Proof. Let

$$v^f_{\Gamma,b} = \sum_{j=1}^{m^p} \beta_j \phi^f_j \in Z^f_{h^f,b} \subset Z^f_{h^f},$$

and note that since the basis functions ϕ^f_j, $j = 1, \ldots, m^p$, do not overlap each other on Γ, they are orthogonal in $L^2(\Gamma)$ and also in $H^1_0(\Gamma)$. Then

$$\|v^f_{\Gamma,b}\|^2_{L^2(\Gamma)} = \sum_{j=1}^{m^p} \beta_j^2 \|\phi^f_j\|^2_{L^2(\Gamma)} \asymp h^p \sum_{j=1}^{m^p} \beta_j^2, \tag{35}$$

$$|v^f_{\Gamma,b}|^2_{H^1(\Gamma)} = \sum_{j=1}^{m^p} \beta_j^2 |\phi^f_j|^2_{H^1_0(e^p_j)} \asymp \frac{1}{h^p} \sum_{j=1}^{m^p} \beta_j^2. \tag{36}$$

Using (35), (36) and a interpolation estimate we see that

$$\|v^f_{\Gamma,b}\|^2_{H^{1/2}_{00}(\Gamma)} \asymp \sum_{j=1}^{m^p} \beta_j^2 \asymp \frac{1}{h^p}\|v^f_{\Gamma,b}\|^2_{L^2(\Gamma)}.$$

Note also that $\langle S^f v^f_{\Gamma,b}, v^f_{\Gamma,b}\rangle \leq a^f(\mathcal{SH}v^f_{\Gamma,b}, \mathcal{SH}v^f_{\Gamma,b}) \asymp \nu\|v^f_{\Gamma,b}\|^2_{H^{1/2}_{00}(\Gamma)}$.

Denote by

$$z^p_{\Gamma,b} = \sum_{j=1}^{m^p} \rho_j \chi_{e^p_j}$$

the unique piecewise constant function such that $\Pi v^f_{\Gamma,b} = z^p_{\Gamma,b}$. Note that $|\rho_j| \asymp |\beta_j|$, $j = 1, \ldots, m^p$. We obtain

$$\langle S^f_\Gamma v^f_{\Gamma,b}, v^f_{\Gamma,b}\rangle \prec \frac{\nu}{h^p}\|v^f_{\Gamma,b}\|^2_{L^2(\Gamma)} \asymp \frac{\nu}{h^p}\|z^p_{\Gamma,b}\|^2_{L^2(\Gamma)} \tag{37}$$

$$\prec \frac{\nu}{(h^p)^2}\|z^p_{\Gamma,b}\|^2_{(H^{1/2})'(\Gamma)} \asymp \frac{\kappa}{(h^p)^2}\langle S^p_\Gamma z^p_{\Gamma,b}, z^p_{\Gamma,b}\rangle, \tag{38}$$

where we have used an inverse inequality for piecewise constant functions. □

We now translate Lemma 5 in a result concerning the dual preconditioner.

Lemma 6. *Assume that $h^p = rh^f$, where r is a positive integer and let λ be a zero mean Lagrange multiplier. Then*

$$\frac{(h^p)^2}{\kappa}\langle N^p(S^p)^{-1}N^{pT}\lambda, \lambda\rangle \prec \langle N^f(S^f)^{-1}N^{fT}\lambda, \lambda\rangle.$$

Proof. We proceed as before. Let $t = (S^f_\Gamma)^{-\frac{1}{2}}\tilde{C}^{fT}\lambda$ and $w = \tilde{C}^p\lambda$. Then

$$\|(S^p_\Gamma)^{-\frac{1}{2}}w\|^2 = \sup_{z^p\in Z^p_{h^p}} \frac{\langle (S^p_\Gamma)^{-\frac{1}{2}}w, z^p\rangle^2}{\|z^p\|^2} = \sup_{v^p \text{balanced}} \frac{\langle w, v^p\rangle^2}{\|(S^p_\Gamma)^{\frac{1}{2}}v^p\|^2}$$

$$= \sup_{v^p \text{balanced}} \frac{\langle \lambda, N^p v^p\rangle^2}{\|(S^p_\Gamma)^{\frac{1}{2}}v^p\|^2} = \sup_{v^f_b \text{balanced}} \frac{\langle \lambda, \tilde{C}^f v^f_b\rangle^2}{\|(S^p_\Gamma)^{\frac{1}{2}}(\tilde{C}^p)^{-1}N^f v^f_b\|^2}$$

$$= \sup_{v^f_b \text{balanced}} \frac{\langle (S^f_\Gamma)^{-\frac{1}{2}}\tilde{C}^{fT}\lambda, (S^f_\Gamma)^{\frac{1}{2}}v^f_b\rangle^2}{\|(S^p_\Gamma)^{\frac{1}{2}}(\tilde{C}^p)^{-1}\tilde{C}^f v^f_b\|^2}$$

$$\leq \|t\|^2 \sup_{v^f_b \text{balanced}} \frac{\|(S^f_\Gamma)^{\frac{1}{2}}v^f_b\|^2}{\|(S^p_\Gamma)^{\frac{1}{2}}(\tilde{C}^p)^{-1}\tilde{C}^f v^f_b\|^2} \prec \frac{\kappa}{(h^p)^2}\|t\|^2,$$

where the last step follows from Lemma 5. □

From Lemmas 3 and 6, the next theorem follows.

Theorem 7. *Assume that $h^p = rh^f$, where r is a positive integer. Let λ be a zero mean Lagrange multiplier, then*

$$\left(1+\frac{(h^p)^2}{\kappa}\right)\langle N^p(S^p)^{-1}N^{pT}\lambda,\lambda\rangle \prec \langle F\lambda,\lambda\rangle \prec \left(1+\frac{1}{\kappa}\right)\langle N^p(S^p)^{-1}N^{pT}\lambda,\lambda\rangle.$$

We solve the system (27) using preconditioned conjugate gradient. Here is the algorithm:

(1) Initialize:

$$x^{(0)} = 0 \ \text{(no coarse problem)}$$
$$\lambda^{(0)} = c$$

(2) Iterate $k = 1, 2, \ldots$ until convergence:

$$\begin{aligned}
\text{Precondition: } y^{(k-1)} &= (N^p)^\dagger (S^p)(N^{pT})^\dagger d^{(k-1)},\\
\beta^k &= \langle y^{(k-1)}, d^{(k-1)}\rangle / \langle y^{(k-2)}, d^{(k-1)}\rangle \ \ [\beta^{(1)} = 0]\,,\\
r^{(k)} &= y^{(k-1)} + \beta^{(k)} r^{(k)} \ [r^{(1)} = y^{(0)}]\,,\\
\alpha^{(k)} &= \langle y^{(k-1)}, d^{(k-1)}\rangle / \langle d^{(k)}, Fr^{(k)}\rangle,\\
x^{(k)} &= x^{(k-1)} + \alpha^{(k)} r^{(k)},\\
d^{(k)} &= d^{(k-1)} - \alpha^{(k)} Fr^{(k)}.
\end{aligned}$$

Implementation of the preconditioned conjugate gradient algorithm for the system (27) involving the FETI preconditioner (34).

Remark 8. Theorem 7 can be extended for the case where $h^f \le 2h^p$. We only need to extend the argument given in the proof of Lemma 5. The basic idea in the proof of Lemma 5 is to associate a bubble function $\phi_j^f \in Z_{h^f}^f$ to each porous side element e_j^p, $j = 1, \ldots, m^p$, in such a way that we can construct a one to one and continuous map $v_{\Gamma,b}^f \mapsto z_{\Gamma,b}^p$. The bubble functions ϕ_j^f, $j = 1, \ldots, m^p$, can be chosen orthogonal in $L^2(\Gamma)$ and in $H_0^1(\Gamma)$. This can also be done when $h^f \le h^p$. The smaller the h^f, the closer is the size of the support of the bubble ϕ_j^f to the size of the element e_j^p since more and more elements e^f can be associated to only one element e^p. This construction can also be carried out in the case $h^p < h^f \le 2h^p$ where nonorthogonal Taylor–Hood basis functions must be used. This last situation leads to the appearance of an additional constant that depends on the nonorthogonality; see Section 8.

Remark 9. We note that Lemma 5 can be used directly to obtain a bound for the balancing domain decomposition preconditioner similar to the one presented in Section 6 but with $\tilde{S}^p$ instead of $\tilde{S}^f$ in (30); see Proposition 2 of [25]. In this case an additional variable elimination is needed. We have to eliminate the component

of the normal fluid velocity in the space $Z^f_{h^f,0}$ and work with the Schur complement with respect to the space $Z^f_{h^f,b}$. This is rather difficult to implement (we can use Lagrange multipliers in this case). Then passing to the dual preconditioner permits us to take advantage of the case where the fluid side discretization on Γ is a refinement of the corresponding porous side discretization.

Remark 10. Theorems 2, 4 and 7 are also valid for the case $\alpha^f > 0$ in (3). To see this we need to compare, for different values of α^f, the energy of discrete extensions for a given normal velocity defined on Γ. Given the outward normal velocity v^f_Γ on Γ, let $\mathcal{SH}_{\alpha^f} v^f_\Gamma$ denote the discrete harmonic extension in the sense of $(a^f_{\alpha^f}, b^f)$, that is, the solution of problem (22) with a^f replaced by $a^f_{\alpha^f}$. Recall that $a^f = a^f_0$, where $a^f_0 = a_{\alpha^f}$ when $\alpha^f = 0$, and therefore, $\mathcal{SH} v^f_\Gamma = \mathcal{SH}_0 v^f_\Gamma$. Note that in (22) we have imposed the *natural* boundary condition $\tau^T \boldsymbol{D}(\mathcal{SH} u^f)\eta_f = 0$ on Γ. Now we define another extension denoted by $\widehat{\mathcal{SH}} v^f_\Gamma$. Given the outward normal velocity v^f_Γ on Γ, let $\widehat{\mathcal{SH}} v^f_\Gamma$ be the (a^f, b^f)-discrete harmonic extension given by the solution of (22) with the boundary condition $\widehat{\mathcal{SH}} v^f_\Gamma \cdot \tau = 0$. For both $\mathcal{SH}$ and $\widehat{\mathcal{SH}}$ are imposed essential boundary condition v^f_Γ for the normal component on Γ. The difference between them is in how the boundary condition is imposed for the tangential component on Γ: For the $\mathcal{SH}$, is imposed homogeneous *natural* boundary condition, while for $\widehat{\mathcal{SH}}$, is imposed homogeneous *essential* boundary condition.

Both extensions $\mathcal{SH}_{\alpha^f}$ and $\widehat{\mathcal{SH}}$ satisfy the zero discrete divergence and boundary conditions in (22). Using this fact and the minimization property of the $(a^f_{\alpha^f}, b^f)$-discrete harmonic extension $\mathcal{SH}_{\alpha^f}$ and the (a^f, b^f)-discrete harmonic extension $\widehat{\mathcal{SH}}$, we get

$$
\begin{aligned}
a^f(\mathcal{SH} v^f_\Gamma, \mathcal{SH} v^f_\Gamma) & \\
&= a^f_0(\mathcal{SH} v^f_\Gamma, \mathcal{SH} v^f_\Gamma) \ \text{(by definition)} \\
&\le a^f_0(\mathcal{SH}_{\alpha^f} v^f_\Gamma, \mathcal{SH}_{\alpha^f} v^f_\Gamma) \ \text{(by the minimization property of } \mathcal{SH}) \\
&\le a^f_{\alpha^f}(\mathcal{SH}_{\alpha^f} v^f_\Gamma, \mathcal{SH}_{\alpha^f} v^f_\Gamma) \ \ (\alpha^f > 0) \\
&\le a^f_{\alpha^f}(\widehat{\mathcal{SH}} v^f_\Gamma, \widehat{\mathcal{SH}} v^f_\Gamma) \ \text{(by the minimization property of } \mathcal{SH}_{\alpha^f}) \\
&= a^f_0(\widehat{\mathcal{SH}}_0 v^f_\Gamma, \widehat{\mathcal{SH}}_0 v^f_\Gamma) \ \ \text{(because } \widehat{\mathcal{SH}} u^f \cdot \tau^f = 0 \text{ on } \Gamma) \\
&\asymp \nu \| v^f_\Gamma \|^2_{H^{1/2}_{00}(\Gamma)} \\
&\asymp a^f(\mathcal{SH}_0 v^f_\Gamma, \mathcal{SH} v^f_\Gamma).
\end{aligned}
$$

The last two equivalences follow from properties of the (a^f, b)-discrete harmonic extensions $\mathcal{SH}$ and $\widehat{\mathcal{SH}}$ (which coincides with the discrete Stokes harmonic extension) [28; 40]. The two equivalences appearing above are independent of the

permeability, fluid viscosity and mesh sizes. Then, the energy of the $(a^f_{\alpha^f}, b)$-discrete harmonic extensions is equivalent to the energy of the (a^f, b)-discrete harmonic extension, that is, the discrete Stokes harmonic extension. This equivalence guarantees the extensions of Theorems 2, 4 and 7 to the case $\alpha^f > 0$.

8. Numerical results

In this section we present numerical tests in order to verify the estimates in Theorems 2, 4 and 7. We consider $\Omega^f = (1, 2) \times (0, 1)$ and $\Omega^p = (0, 1) \times (0, 1)$. See [11] and [26] for examples of exact solutions and compatible divergence and boundary data. Note that the reduced systems (15) and (27) involve only degrees of freedom on the interface Γ. To solve both reduced systems (15) and (27) we can use the PCG algorithms described on pages 15 and 19. Recall that the original system (11) is a "three times" saddle point problem. Note that since the finite element basis of $M^f_{h^f} \times M^p_{h^p}$ and Λ^{h^p} have no zero mean, the finite element matrix in (13) has the kernel composed by constant pressures in $\Omega = \text{int}(\overline{\Omega^f} \cup \overline{\Omega^p})$ and constant Lagrange multipliers on Γ. The corresponding system is solved up to a constant pressure and a constant Lagrange multiplier. These constants can be recovered when imposing the zero average pressure constraint [26].

In our test problems we compute the eigenvalues of the preconditioned operators. We also run PCG until the initial residual is reduced by a factor of 10^{-6}.

8.1. ***BDD preconditioner.*** In the case of the BDD preconditioner (30) for (15), we solve a coarse problem before reducing the system to ensure balanced velocities at the beginning of the CG iterations.

We consider $\alpha^f = 0$ and $\nu = 1$, and different values of h^f and h^p with non-matching grids across the interface Γ. Table 1 shows results for $\kappa = 1$, Table 2 for $\kappa = 10^{-3}$ and Table 3 for $\kappa = 10^{-5}$. These three tables reveal growth of order $O(1 + (1/\kappa))$ in κ and hence, verify the sharpness of the estimate in Theorem 2.

$h^f \downarrow$ $h^p \rightarrow$	$3^{-1} * 2^{-0}$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1, 1.0189(3)	1, 1.0198(3)	1, 1.0194(3)	1, 1.0193(3)	1, 1.0193(3)
$2^{-1} * 2^{-1}$	1, 1.0209(3)	1, 1.0200(3)	1, 1.0197(3)	1, 1.0196(3)	1, 1.0196(3)
$2^{-1} * 2^{-2}$	1, 1.0217(3)	1, 1.0205(3)	1, 1.0202(3)	1, 1.0201(3)	1, 1.0201(3)
$2^{-1} * 2^{-3}$	1, 1.0220(3)	1, 1.0208(3)	1, 1.0204(3)	1, 1.0203(3)	1, 1.0203(3)
$2^{-1} * 2^{-4}$	1, 1.0221(3)	1, 1.0209(3)	1, 1.0205(3)	1, 1.0204(3)	1, 1.0204(3)

Table 1. Minimum and maximum eigenvalues (and number of PCG iterations) for the BDD preconditioned operator. Here $\kappa = 1$ and $\alpha^f = 0$.

$h^f\downarrow \quad h^p\rightarrow$	$3^{-1}*2^{-1}$	$3^{-1}*2^{-2}$	$3^{-1}*2^{-3}$	$3^{-1}*2^{-4}$
$2^{-1}*2^{-0}$	1, 21.0147(3)	1, 20.6035(3)	1, 20.3686(3)	1, 20.2893(3)
$2^{-1}*2^{-1}$	1, 21.3303(6)	1, 20.8549(7)	1, 20.6550(7)	1, 20.5836(7)
$2^{-1}*2^{-2}$	1, 22.0017(6)	1, 21.3392(9)	1, 21.1424(10)	1, 21.0735(10)
$2^{-1}*2^{-3}$	1, 22.2367(6)	1, 21.6045(10)	1, 21.3626(9)	1, 21.2955(10)
$2^{-1}*2^{-4}$	1, 22.3479(6)	1, 21.7006(10)	1, 21.4666(11)	1, 21.3929(9)

Table 2. Minimum and maximum eigenvalues (and number of PCG iterations) for the BDD preconditioned operator. Here $\kappa = 10^{-3}$ and $\alpha^f = 0$.

$h^f\downarrow \quad h^p\rightarrow$	$3^{-1}*2^{-1}$	$3^{-1}*2^{-2}$	$3^{-1}*2^{-3}$	$3^{-1}*2^{-4}$
$2^{-1}*2^{-0}$	1, 1977.08(3)	1, 1945.05(3)	1, 1932.10(3)	1, 1928.32(3)
$2^{-1}*2^{-1}$	1, 1997.27(6)	1, 1972.77(7)	1, 1961.34(7)	1, 1957.88(7)
$2^{-1}*2^{-2}$	1, 2053.57(6)	1, 2021.03(13)	1, 2010.27(17)	1, 2006.90(17)
$2^{-1}*2^{-3}$	1, 2079.68(6)	1, 2044.05(13)	1, 2032.42(21)	1, 2029.13(31)
$2^{-1}*2^{-4}$	1, 2090.10(6)	1, 2054.33(13)	1, 2042.26(22)	1, 2038.90(28)

Table 3. Minimum and maximum eigenvalues (and number of PCG iterations) for the BDD preconditioned operator. Here $\kappa = 10^{-5}$ and $\alpha^f = 0$.

8.2. ***FETI preconditioner.*** In the case of the FETI preconditioner (34), we solve the reduced system (27) up to a constant Lagrange multiplier and a constant pressure. These constants are recovered after enforcing zero mean pressure on $\Omega = \text{int}\,(\overline{\Omega}^f \cup \overline{\Omega}^p)$ [26]. We recall that the FETI method can be viewed as the dual preconditioner counterpart of the BDD preconditioner. We repeat the same experiments mentioned above for the latter preconditioner.

$h^f\downarrow \quad h^p\rightarrow$	$3^{-1}*2^{-1}$	$3^{-1}*2^{-2}$	$3^{-1}*2^{-3}$	$3^{-1}*2^{-4}$
$2^{-1}*2^{-0}$	1.0000, 1.0208(3)	1.0000, 1.0194(3)	1.0000, 1.0193(3)	1.0000, 1.0193(3)
$2^{-1}*2^{-1}$	1.0017, 1.0200(3)	1.0000, 1.0197(3)	1.0000, 1.0196(3)	1.0000, 1.0196(3)
$2^{-1}*2^{-2}$	1.0026, 1.0205(3)	1.0004, 1.0202(3)	1.0000, 1.0200(3)	1.0000, 1.0201(3)
$2^{-1}*2^{-3}$	1.0027, 1.0208(3)	1.0007, 1.0204(3)	1.0001, 1.0203(3)	1.0000, 1.0203(3)
$2^{-1}*2^{-4}$	1.0028, 1.0209(2)	1.0007, 1.0205(3)	1.0002, 1.0204(3)	1.0000, 1.0204(3)
$2^{-1}*2^{-5}$	1.0028, 1.0209(2)	1.0007, 1.0206(3)	1.0002, 1.0205(3)	1.0000, 1.0204(3)

Table 4. Minimum and maximum eigenvalues (and number of PCG iterations) of the FETI preconditioned operator. Here $\kappa = 1$ and $\alpha^f = 0$.

$h_f \downarrow \quad h_p \rightarrow$	$3^{-1} * 2^{-1}$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.000, 20.7608(3)	1.000, 20.4405(3)	1.000, 20.3110(3)	1.000, 20.2732(3)
$2^{-1} * 2^{-1}$	2.707, 20.9627(5)	1.000, 20.7177(7)	1.000, 20.6034(7)	1.000, 20.5688(7)
$2^{-1} * 2^{-2}$	3.634, 21.5257(5)	1.425, 21.2003(10)	1.000, 21.0927(12)	1.000, 21.0590(12)
$2^{-1} * 2^{-3}$	3.714, 21.7868(5)	1.651, 21.4305(9)	1.106, 21.3142(11)	1.000, 21.2813(12)
$2^{-1} * 2^{-4}$	3.760, 21.891 (5)	1.663, 21.5333(9)	1.162, 21.4126(11)	1.026, 21.3790(12)
$2^{-1} * 2^{-5}$	3.771, 21.937 (5)	1.673, 21.5768(9)	1.164, 21.4561(11)	1.040, 21.4220(12)

Table 5. Minimum and maximum eigenvalues (and number of PCG iterations) for the FETI preconditioned operator. Here $\kappa = 10^{-3}$ and $\alpha^f = 0$.

$h^f \downarrow \quad h^p \rightarrow$	$3^{-1} * 2^{-2}$	$3^{-1} * 2^{-3}$	$3^{-1} * 2^{-4}$
$2^{-1} * 2^{-0}$	1.00, 1945.05(3)	1.00, 1932.10(3)	1.00, 1928.32(3)
$2^{-1} * 2^{-1}$	1.00, 1972.77(7)	1.00, 1961.34(7)	1.00, 1957.88(7)
$2^{-1} * 2^{-2}$	43.45, 2021.03(11)	1.00, 2010.27(17)	1.00, 2006.90(17)
$2^{-1} * 2^{-3}$	66.10, 2044.05(11)	11.58, 2032.42(20)	1.00, 2029.13(37)
$2^{-1} * 2^{-4}$	67.29, 2054.33(10)	17.20, 2042.26(19)	3.64, 2038.90(35)
$2^{-1} * 2^{-5}$	68.32, 2058.68(10)	17.42, 2046.61(10)	5.04, 2043.20(36)

Table 6. Minimum and maximum eigenvalues (and number of PCG iterations) for the FETI preconditioned operator. Here $\kappa = 10^{-5}$ and $\alpha^f = 0$.

We consider $\alpha^f = 0$, $\nu = 1$ and different values of h^f and h^p with nonmatching grids across the interface Γ; see Table 4 on the previous page for the results when $\kappa = 1$, Table 5 for $\kappa = 10^{-3}$ and Table 6 for the case $\kappa = 10^{-5}$. Note that in Tables 4–6 the minimum eigenvalues are strictly greater than one when $h^f \leq 2h^p$, and the value of the minimum eigenvalues seem to stabilize very quickly for smaller h^f with fixed h^p. This confirms the extension of Theorem 7 for the case where $h^f \leq 2h^p$ (Remark 8). In Table 7 we present the numerical results where one of the meshes on the interface is a refinement of the other side triangulation on the interface. We observe a behavior similar to the behavior of Table 6 with a bigger value for the minimum eigenvalue when $h_f \leq h_p$. This verifies the estimates of Theorem 7. This shows that the FETI preconditioner is scalable for the parameters faced in practice, that is, when the fluid side mesh is finer than the porous side mesh, and the permeability κ is very small. We conclude that the numerical experiments concerning the FETI preconditioner reveal the sharpness of the results obtained in Theorems 4 and 7 and Remark 8.

Recall that we have assumed $\alpha^f = 0$. Now consider $\alpha^f > 0$. Numerical experiment were performed with $\alpha^f > 0$ revealing results similar to the ones presented

$h^f\downarrow \quad h^p\rightarrow$	$2^{-1}*2^{-2}$	$2^{-1}*2^{-3}$	$2^{-1}*2^{-4}$
$2^{-1}*2^{-0}$	1.00, 1961.35(3)	1.00, 1937.86(3)	1.00, 1929.93(3)
$2^{-1}*2^{-1}$	1.00, 1986.49(7)	1.00, 1966.50(7)	1.00, 1959.36(7)
$2^{-1}*2^{-2}$	176.56, 2034.92(7)	1.00, 2015.24(18)	1.00, 2008.35(17)
$2^{-1}*2^{-3}$	151.62, 2061.45(7)	44.91, 2037.26(13)	1.00, 2030.55(45)
$2^{-1}*2^{-4}$	154.45, 2071.06(7)	38.04, 2047.66(13)	11.98, 2040.29(21)
$2^{-1}*2^{-5}$	154.86, 2075.43(7)	38.73, 2051.91(13)	10.20, 2044.66(24)

Table 7. Minimum and maximum eigenvalues (and number of PCG iterations) for the FETI preconditioned operator. Here $\kappa = 10^{-5}$ and $\alpha^f = 0$. The refinement condition of Theorem 7 is satisfied under the diagonal.

$h^f\downarrow \quad h^p\rightarrow$	$3^{-1}\ 2^{-2}$	$3^{-1}\ 2^{-3}$	$3^{-1}\ 2^{-4}$
$2^{-1}\ 2^{-0}$	1.00, 1678.07(3)	1.00, 1666.84(3)	1.00, 1663.55(3)
$2^{-1}\ 2^{-1}$	1.00, 1787.53(7)	1.00, 1776.50(7)	1.00, 1773.22(7)
$2^{-1}\ 2^{-2}$	41.65, 1812.69(17)	1.00, 1801.61(17)	1.00, 1798.29(17)
$2^{-1}\ 2^{-3}$	63.63, 1816.43(17)	11.24, 1804.66(13)	1.00, 1801.34(43)
$2^{-1}\ 2^{-4}$	66.82, 1817.38(17)	16.75, 1805.30(13)	3.58, 1801.91(23)
$2^{-1}\ 2^{-5}$	67.99, 1817.68(17)	17.37, 1805.57(13)	4.97, 1802.14(24)

Table 8. Minimum and maximum eigenvalues (and number of PCG iterations) for the FETI preconditioned operator. Here $\kappa = 10^{-5}$ and $\alpha^f = 1$.

above for the case $\alpha^f = 0$. We only include Table 8 which shows the extreme eigenvalues of the FETI preconditioned operator for the case $\alpha^f = 1$, $\nu = 1$ and $\kappa = 10^{-5}$. This table presents a similar behavior to the one with $\alpha^f = 0$ in Table 6 and hence confirms Remark 10, which says that the parameter α^f does not play much of a role for preconditioning.

9. The multisubdomain case

The methods introduced in the previous sections considered only the two-subdomain cases where discrete Stokes and Darcy indefinite subproblems are solved exactly in each subdomain and in each CG iteration. These methods might be very costly for large subproblems since direct or accurate iterative local solvers for the indefinite systems have to be used. In this section we show that the methodology developed for the two-subdomain cases can be developed also for the multisubdomain case. The analysis (using tools developed in Section 7) and numerical experiments for the multisubdomain case will be presented elsewhere.

We now extend the FETI method of Section 7 for many subdomains when the triangulations $\mathcal{T}^f_{h^f}$ and $\mathcal{T}^p_{h^p}$ coincide on the interface Γ. Let $\{\Omega^i_j\}_{j=1}^{n^i}$ be a geometrically conforming substructures of Ω^i, $i = f, p$. We also assume that $\{\Omega^f_j\}_{j=1}^{n^f} \cup \{\Omega^p_j\}_{j=1}^{n^p}$ form a geometrically conforming decomposition of Ω; hence, the two decompositions are aligned on the interface Γ. We define the local inner interfaces as $\Gamma^i_j = \partial\Omega^i_j \setminus \partial\Omega^i$, $j = 1, \ldots, n^i$, $i = f, p$. We also define

$$\widetilde{\Gamma} = \bigcup_{j=1}^{n^f} \Gamma^f_j \cup \bigcup_{j=1}^{n^p} \Gamma^p_j \cup \Gamma.$$

See Figure 1. In order to simplify the presentation, we assume that for the fluid region, the spaces $\boldsymbol{X}^f_{h^f}$ and $M^f_{h^f}$ are the P2/P0 triangular finite elements, while for the porous region, the spaces $\boldsymbol{X}^p_{h^p} \subset \boldsymbol{X}^p$ and $M^p_{h^p} \subset L^2(\Omega^p)$ are the lowest order Raviart–Thomas finite elements based on triangles. Similar as in the previous sections, and using the FETI-DP framework [42], we decompose the velocity and pressure spaces as follows:

$\boldsymbol{X}^f_I$: interior velocities in the subdomains $\{\Omega^f_j\}_{j=1}^{n^f}$

$\boldsymbol{X}^f_{\widetilde{\Gamma}}$: interface velocities on $\widetilde{\Gamma} \cap \overline{\Omega}^f$

$\boldsymbol{X}^p_I$: interior velocities in the subdomains $\{\Omega^p_j\}_{j=1}^{n^p}$

$\boldsymbol{X}^p_{\widetilde{\Gamma}}$: interface velocities on $\widetilde{\Gamma} \cap \overline{\Omega}^p$

M^i_I, $(i = p, f)$: interior zero mean pressure in each subdomain $\{\Omega^i_j\}_{j=1}^{n^i}$, $i = f, p$

M^i_0, $(i = p, f)$: constant pressure in each subdomain $\{\Omega^i_j\}_{j=1}^{n^i}$, $i = f, p$

$M_I = M^f_I \times M^p_I$

$\boldsymbol{X}_I = \boldsymbol{X}^f_I \times \boldsymbol{X}^p_I$, $\boldsymbol{X}_{\widetilde{\Gamma}} = \boldsymbol{X}^f_{\widetilde{\Gamma}} \times \boldsymbol{X}^p_{\widetilde{\Gamma}}$, $M_I = M^f_I \times M^p_I$ and $M_0 = M^f_0 \times M^p_0$

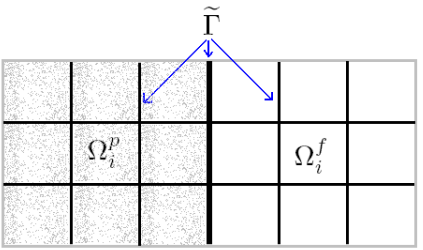

Figure 1. Global interface $\widetilde{\Gamma}$ that includes all local interfaces and the Stokes–Darcy interface Γ.

After imposing the mortar condition as in Section 4 we can reduce (11) to a Schur complement system on the interface $\widetilde{\Gamma}$,

$$\widetilde{S}_{\widetilde{\Gamma}}\boldsymbol{u}_{\widetilde{\Gamma}} = \widetilde{b}_{\widetilde{\Gamma}} \tag{39}$$

which is the multisubdomain generalization of the reduced system (15).

The $\widetilde{\Gamma}$-interface velocity space $\boldsymbol{X}_{\widetilde{\Gamma}}$ can be decomposed in primal and dual degrees of freedom, that is, $\boldsymbol{X}_{\widetilde{\Gamma}} = \boldsymbol{X}_C \oplus \boldsymbol{X}_\Delta$ where $\boldsymbol{X}_C$ consists of functions which are continuous with respect to the primal degrees of freedom. The primal variables for the fluid velocity field satisfy the continuity of the fluid velocities at the substructure corners and the continuity of the mean normal and mean tangential component on each face of the substructures $\{\Omega_j^f\}_{j=1}^{n^f}$. For the porous side, the primal variables satisfy the continuity of the mean normal flux on the each face of the subsubstructures $\{\Omega_j^p\}_{j=1}^{n^p}$ [27; 32; 33; 34; 43]. For faces of the subdomains on Γ, only the continuity of the mean fluxes is satisfied. The space $\boldsymbol{X}_\Delta$ includes the remaining fluid side velocity degrees of freedom and the remaining porous media velocity degrees of freedom.

Functions in $\boldsymbol{X}_\Delta$ do not satisfy the continuity requirements on $\widetilde{\Gamma}$. The continuity requirement can be enforced using Lagrange multipliers $\widetilde{\lambda}$ on $\widetilde{\Gamma}$ and represented by the equation

$$B_\Delta \boldsymbol{v}_\Delta = 0.$$

We ensure that this condition coincides with the last equation of (13) that corresponds to the flux continuity across the Stokes–Darcy interface Γ. On that interface we use the same Lagrange multipliers of the dual formulation (27). Proceeding as in [32] we can obtain a reduced system of the form

$$\widetilde{F}\widetilde{\lambda} = \widetilde{b},$$

which corresponds to the multisubdomain version of (27). The preconditioner operator is of the form

$$B_\Delta \widetilde{S}_{\widetilde{\Gamma}} B_\Delta^T,$$

where $\widetilde{S}_{\widetilde{\Gamma}}$ was introduced in (39). See [27] for a more detailed discussion and numerical experiments for the FETI method in the multisubdomain case.

10. Conclusions and final comments

We consider the problem of coupling fluid flows with porous media flows with Beavers–Joseph–Saffman condition on the interface. We choose a discretization consisting of Taylor–Hood finite elements of order two on the free fluid side and the lowest order Raviart–Thomas finite element on the porous fluid side. The meshes are allowed to be nonmatching across the interface.

We design and analyze two preconditioners for the resulting symmetric linear system. We note that the original linear system is symmetric indefinite and involves three Lagrange multipliers: one for each subdomain pressure and a third one to impose the weak conservation of mass across the interface Γ; see Section 1.

One preconditioner is based on BDD methods and the other one is based on FETI methods. In the case of the BDD preconditioner, the energy is controlled by the Stokes side, while in the FETI preconditioner, the energy is controlled by the Darcy system; see Theorems 2 and 4. In both cases a bound $C_1((\kappa+1)/\kappa)$ is derived. Furthermore, under the assumption that the fluid side mesh on the interface is finer than the corresponding porous side mesh, we derive the better bound $C_2((\kappa+1)/(\kappa+(h^p)^2))$ for the FETI preconditioner; see Theorem 7 and Remark 8. This better bound also shows that the FETI preconditioner is more scalable for parameters faced in practice, for example, problems with small permeability κ and where the fluid side mesh is finer than the porous side mesh. The constants C_1 and C_2 above are independent of the fluid viscosity ν, the mesh ratio across the interface and the permeability κ.

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Received November 22, 2008. Revised November 17, 2009.

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Communications in Applied Mathematics and Computational Science, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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http://www.mathscipub.org
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Typeset in LaTeX

Communications in Applied Mathematics and Computational Science

vol. 5 no. 1 2010

FETI and BDD preconditioners for Stokes–Mortar–Darcy Systems 1
Juan Galvis and Marcus Sarkis

A cut-cell method for simulating spatial models of biochemical reaction networks in arbitrary geometries 31
Wanda Strychalski, David Adalsteinsson and Timothy Elston

An urn model associated with Jacobi polynomials 55
F. Alberto Grünbaum

Ensemble samplers with affine invariance 65
Jonathan Goodman and Jonathan Weare

A second-order accurate method for solving the signed distance function equation 81
Peter Schwartz and Phillip Colella

On the second-order accuracy of volume-of-fluid interface reconstruction algorithms: convergence in the max norm 99
Elbridge Gerry Puckett