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We consider an urn model leading to a random walk that can be solved explicitly in terms of the well-known Jacobi polynomials.

1. Urn models and orthogonal polynomials

There are two simple and classical models in statistical mechanics which have recently been associated with very important classes of orthogonal polynomials. The oldest one of these models is due to D. Bernoulli (1770) and S. Laplace (1810), while the more recent model is due to Paul and Tatiana Ehrenfest (1907) [6]. While both of these models are featured in very classical texts in probability theory, such as [7], the connection with orthogonal polynomials is of much more recent vintage. In fact, the polynomials in question due to Krawtchouk and Hahn had not been recognized as basic objects with rich properties until around 1950. For a few pertinent and useful references, see [1; 2; 4; 5; 11; 14; 16; 18].

From these comments one could get the impression that the relations between orthogonal polynomials — especially some well-known classes of them — are only a matter of historical interest. Nothing could be further from the truth: there are several areas of probability and mathematical physics where recent important progress hinges on the connections with orthogonal polynomials.

The entire area of random matrix theory starts with the work of E. Wigner and F. Dyson and reaches a new stage in the hands of M. Mehta who brought the power of orthogonal polynomials into the picture.

In the area of random growth processes, the seminal work of K. Johansson depends heavily on orthogonal polynomials, specifically Laguerre and Meixner ones; see [15].

The connection between birth-and-death processes and orthogonal polynomials has many parents, but the people that made the most of it are S. Karlin and J. McGregor [17]. We will have a chance to go back to their work in connection with our model here. The ideas of using the spectral analysis of the corresponding

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one-step transition matrix have been pushed recently in the case of quantum random walks, an area where physics, computer science and mathematics could make important contributions. See [3; 12].

The study of the so-called ASEP (asymmetric simple exclusion processes), going back to F. Spitzer [21] and very much connected with the work of K. Johansson mentioned earlier, has recently profited from connections with the Askey–Wilson polynomials. All of this has important and deep connections with combinatorics and a host of other areas of mathematics; for example, the study of nonintersecting or noncolliding random processes, which goes back to F. Dyson.

There are many interrelations among these areas. For one example: the Hahn polynomials that were mentioned in connection with the Bernoulli–Laplace model were studied by Karlin and McGregor [18] in connection with a model in genetics due to Moran. They have also been found to be useful in discussing random processes with nonintersecting paths [8].

All of these areas are places where orthogonal polynomials have been put to very good use. For a review of several of these items, see [19]. Orthogonal polynomials of several variables, as well as matrix-valued orthogonal polynomials, have recently been connected to certain random walks. For three examples, see [9; 10; 13].

2. The Jacobi polynomials

The classical Jacobi polynomials are usually considered either over the interval $[-1, 1]$ or, as we will do, over $[0, 1]$.

These polynomials are orthogonal with respect to the weight function

$$W(x) = x^{\alpha}(1-x)^{\beta}.$$

Here we assume that $\alpha, \beta > -1$; in fact it will be assumed throughout that α, β are nonnegative integers.

These polynomials are eigenfunctions of the differential operator

$$x(1-x)\frac{d^2}{dx^2} + \bigl(\alpha + 1 + x(\alpha+\beta+2)\bigr)\frac{d}{dx},$$

a fact that will not play any role in our discussion but which is crucial in most physical and geometrical applications of Jacobi polynomials. These applications cover a vast spectrum including potential theory, electromagnetism and quantum mechanics.

Neither the orthogonality, nor the fact that our polynomials are eigenfunctions of this differential operator are enough to determine them uniquely. One can multiply each polynomial by a constant and preserve these properties. We chose to normalize our polynomials by the condition

$$Q_n(1) = 1.$$

For us it will be important that these polynomials satisfy (and in fact be defined by) the three-term recursion relation

$$x\,Q_n(x) = A_n\,Q_{n+1}(x) + B_n\,Q_n(x) + C_n\,Q_{n-1}(x),$$

with $Q_0 = 1$ and $C_0\,Q_{-1} = 0$.

The coefficients A_n, B_n, C_n are given by

$$A_n = \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \qquad n \geq 0,$$

$$B_n = 1 + \frac{n(n+\beta)}{2n+\alpha+\beta} - \frac{(n+1)(n+\beta+1)}{2n+\alpha+\beta+2}, \quad n \geq 0,$$

$$C_n = \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \qquad n \geq 1.$$

The coefficient B_n can be rewritten as

$$B_n = \frac{2n(n+\alpha+\beta+1) + (\alpha+1)\beta + \alpha(\alpha+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

which makes it clear that, along with the other coefficients, it is nonnegative.

Since we insist on the condition $Q_n(1) = 1$, we can see, for instance by induction and using the recursion relation, that

$$A_n + B_n + C_n = 1, \quad n \geq 1.$$

We also have

$$A_0 + B_0 = 1.$$

There are, of course, several explicit expressions for the different variants of the Jacobi polynomials, and they can be used, for instance, in computing the integrals that appear in the last section of this paper.

In terms of hypergeometric functions, the usual Jacobi polynomials are given by

$$\big((\alpha+1)_n/n!\big)\,{}_2F_1\big(-n, n+\alpha+\beta+1;\,(1-x)/2\big),$$

while our polynomials $Q_n(x)$ are obtained by multiplying these standard Jacobi polynomials by $(-1)^n n!/(\beta+1)_n$ and replacing x by $1-2x$.

The normalization chosen above is natural when one thinks of these polynomials (at least for some values of α, β) as the spherical functions for some appropriate symmetric space, and insists that these functions take the value 1 at the north pole of the corresponding sphere. The simplest of all cases is the one where $\alpha = \beta = 0$; one gets the Legendre polynomials and the usual two-dimensional sphere sitting in $\mathbb{R}^3$ (see [23]).

The fact that the coefficients are nonnegative and add up to one cries out for a probabilistic interpretation of these quantities. This is the purpose of this paper. We have not seen in the literature concrete models of random walks where the Jacobi polynomials play this role. The urn model we give is admittedly a bit contrived, but it is quite concrete. Hopefully it will motivate other people to find a more natural and simpler model that goes along with this recursion relation.

3. The model

Here we consider a discrete-time random walk on the nonnegative integers whose one-step transition probability matrix coincides with the one that gives the three-term recursion relation satisfied by the Jacobi polynomials.

At times $t = 0, 1, 2, \ldots,$ an urn contains n blue balls and this determines the state of our random walk on $\mathbb{Z} \geq 0$.

The urn sits in a "bath" consisting of an infinite number of red balls. The transition mechanism is made up of a few steps which are described now, leaving some of the details for later.

In the first step a certain number of red balls from the surrounding bath are mixed with the n blue balls in the urn.

In the second step a ball is selected (with uniform distribution) from among the balls in the urn. This "chosen ball" can be blue or red. In either case an experiment is performed in a parallel world, using an appropriate "auxiliary urn", to determine if this chosen ball will retain its color or have it changed (from red to blue or vice versa).

Once this is settled, and the possible change of color has taken place, the main urn contains the initial n balls plus a certain number of balls taken form the bath in the first step, and we are ready for the third and last step. This final step consists of having all red balls in the main urn removed and dropped back into the bath.

The state of the system at time $t+1$ is given by the number of blue balls in the urn after these three steps are completed. Clearly, the new state can take any of the values $n-1$, n, $n+1$.

A more detailed description of the three steps above is given in the next section.

4. The details of the model

If at time t the urn contains n blue balls, with $n = 0, 1, 2, \ldots,$ we pick

$$n + \alpha + \beta + 1$$

red balls from the bath to get a total of $2n+\alpha+\beta+1$ balls in the urn at the end of step one.

We now perform step 2: this gives us a blue ball with probability

$$\frac{n}{2n+\alpha+\beta+1}$$

and a red ball with probability

$$\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}.$$

If the chosen ball is blue, then we throw α blue balls and $n+\beta$ red balls into an "auxiliary urn" with n blue balls, mix all these balls, and pick one with uniform distribution. We imagine the auxiliary urn surrounded by a bath of an infinite number of blue and red balls which are used to augment the n blue balls in this auxiliary urn.

The probability of getting a blue ball in the auxiliary urn is

$$\frac{n+\alpha}{2n+\alpha+\beta},$$

and if this is the outcome, the chosen ball in the main urn has its color changed from blue to red. If we get a red ball in this auxiliary urn, then the chosen ball retains its blue color.

On the other hand, if in step 2 we had chosen a red ball, then we throw $\alpha+1$ blue balls and $n+\beta+1$ red balls into a different auxiliary urn with n blue balls. This auxiliary urn is also surrounded by a bath of an infinite number of blue and red balls.

These balls are mixed and one is chosen with the uniform distribution. The probability that this ball in the auxiliary urn is red is given by

$$\frac{n+\beta+1}{2n+\alpha+\beta+2},$$

and if this is the case, the chosen ball in the main urn has its color changed from red to blue. Otherwise the chosen ball retains its red color.

Notice that the chosen ball in the main urn has a change of color only when we get a match of colors for the balls drawn in the main and an auxiliary urn: blue followed by blue or red followed by red.

In either case once the possible change of color of the chosen ball in the main urn has been decided upon, we remove all the red balls from the main urn.

We see that the state of the system goes from n to $n-1$ when the chosen ball is blue and then its color gets changed into red. This event has probability

$$\frac{n}{2n+\alpha+\beta+1} \times \frac{n+\alpha}{2n+\alpha+\beta}.$$

Observe that this coincides with the value of C_n in the recursion relation satisfied by our version of the Jacobi polynomials.

The state increases from n to $n+1$ if the chosen ball is red and its color gets changed into blue. This event has probability

$$\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} \times \frac{n+\beta+1}{2n+\alpha+\beta+2}.$$

This coincides with the values of A_n given earlier.

As we noticed earlier, when the chosen ball is blue and the ball in the corresponding auxiliary urn is red then the chosen ball retains its color. Likewise if the chosen ball is red and the ball in the corresponding auxiliary urn is blue then the chosen one retains its color. In either case, the total number of blue balls in the main urn remains unchanged and the state goes from n to n.

Recall the basic property of the coefficients A_n, B_n, C_n, namely

$$A_n + B_n + C_n = 1.$$

This shows that the probability of going from state n to state n is given by B_n.

In summary, we have built a random walk whose one-step transition probability is the one given by the three-term relation satisfied by our version of the Jacobi polynomials.

5. Birth-and-death processes and orthogonal polynomials

A Markov chain with state space given by the nonnegative integers and a tridiagonal one-step transition probability matrix $\mathbb{P}$ is called a birth-and-death process. Our model given above clearly fits in this framework.

One of the most important connections between orthogonal polynomials and birth-and-death processes, such as the one considered here, is given by the Karlin–McGregor formula [17].

If the polynomials satisfy

$$\pi_j \int_0^1 Q_i(x) Q_j(x) W(x) dx = \delta_{ij},$$

one gets the following representation formula for the entries of the powers of the one-step transition probability matrix

$$(\mathbb{P}^n)_{ij} = \pi_j \int_0^1 x^n Q_i(x) Q_j(x) W(x) dx.$$

This compact expression gives the solution to the dynamics of our random walk and allows for the study of many of its properties.

In the case of our version of the Jacobi polynomials, the squares of the norms of the polynomials Q_i are given by

$$\frac{\Gamma(i+1)\,\Gamma(i+\alpha+1)\,\Gamma(\beta+1)^2}{\Gamma(i+\beta+1)\,\Gamma(i+\alpha+\beta+1)\,(2i+\alpha+\beta+1)}.$$

In our case, when α, β are assumed to be nonnegative integers, this expression can of course be written without any reference to the Gamma function.

We recall how one can compute in the case of our transition matrix P its invariant (stationary) distribution, that is, the (unique up to scalars) row vector

$$\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \dots)$$

such that

$$\boldsymbol{\pi} P = \boldsymbol{\pi}.$$

It is a simple matter of using the recursion relation for the polynomials Q_i to show that the components π_i are given, up to a common multiplicative constant, by the inverses of the integrals

$$\int_0^1 Q_i^2(x) W(x) dx$$

mentioned above. This justifies the notation π_i for these two apparently unrelated quantities, and in our case furnishes an explicit expression for an invariant distribution.

We close this paper with a note of historical interest. One of the referees suggested that I contact Dick Askey, who reportedly had pointed out that the Legendre polynomials had surfaced for the first time in connection with a problem in probability theory.

Askey recalls that Arthur Erdélyi told him once that this occurred in a work by J. L. Lagrange. Indeed in [20] , Lagrange considers such a problem. In the process of solving it he needs to find the power series expansion of the expression

$$\frac{1}{\sqrt{1-2az+(a^2-4b^2)z^2}}$$

in powers of z.

The three-term recursion for the coefficients in this expansion is explicitly written down and considered well-known by Lagrange. The account of Lagrange's work given in the very complete and scholarly book [22] has a derivation of this recursion. The case $a^2 - 4b^2 = 1$ gives the Legendre polynomials in the variable a. The work of Lagrange took place in the period 1770–1773, and predates the work of Legendre and Laplace. I am thankful to the anonymous referee and to Askey for pointing me in the correct direction.

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