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LEGENDRE SPECTRAL-COLLOCATION METHOD  
FOR VOLTERRA INTEGRAL DIFFERENTIAL  
EQUATIONS WITH NONVANISHING DELAY

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## LEGENDRE SPECTRAL-COLLOCATION METHOD FOR VOLTERRA INTEGRAL DIFFERENTIAL EQUATIONS WITH NONVANISHING DELAY

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The main purpose of this paper is to propose the Legendre spectral-collocation method to solve the Volterra integral differential equations with nonvanishing delay which arise in many problems, such as modeling in biosciences and population. In our method we divide the definition domain of the solution into several subintervals where the solution is sufficiently smooth. Then we can use the spectral-collocation method for these equations in each subinterval. We provide convergence analysis for this method, which shows that the numerical errors decay exponentially. Numerical examples are presented to confirm these theoretical results.

### 1. Introduction

VIDEs (Volterra integral differential equations) with delay arise in many problems, for example, ecological competition systems [54], modeling in biosciences and population [1; 9; 26] and models for transmission of disease with immigration of infectives [10]. Nonlinear Volterra integral and integrodifferential equations with nonvanishing delay have been used since the 1920s as mathematical models of population growth and related phenomena in biology. Volterra [46] refined his earlier predator-prey model to include situations where “historical actions cease after a certain interval of time” [47]. This leads to a system of nonlinear Volterra

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integrodifferential equations with constant delay  $T_0 > 0$  (using Volterra's notation):

$$\begin{aligned} N_1'(t) &= N_1(t) \left( \varepsilon_1 - \gamma_1 N_2(t) - \int_{t-T_0}^t F_1(t-\tau) N_1(\tau) d\tau \right), \\ N_2'(t) &= N_2(t) \left( -\varepsilon_2 + \gamma_2 N_1(t) + \int_{t-T_0}^t F_2(t-\tau) N_2(\tau) d\tau \right), \end{aligned}$$

with  $\varepsilon_i > 0$ ,  $\gamma_i \geq 0$  and continuous  $F_i(t) \geq 0$ , where  $N_i(t) = \phi_i(t)$ ,  $t \leq 0$ ,  $i = 1, 2$ .  $N_1(t)$  and  $N_2(t)$  represent the sizes of two populations (prey and predator) at time  $t \geq 0$ . These equations can be extended naturally to describe the dynamics of multispecies ecological systems. A further development of such population models based on VIDEs can be found in [20].

There exist many numerical methods for the VIDEs with delay, for example, general linear methods [53], linear multistep methods [5], block-by-block methods [32], Runge–Kutta methods [6; 7; 21; 29; 38], Petrov–Galerkin methods [31], piecewise polynomial collocation methods [13; 14; 36; 37]. Brunner investigated the numerical solution of nonlinear VIDEs with infinite delay in [11] and neutral VIDEs with constant delay in [12]. The superconvergence of the collocation method for VIDEs with nonvanishing delay is investigated in [13; 37; 39]. The monograph by Brunner [13] contains a wealth of material on the theory and numerical methods for VIDEs, with the focus being on the basic theory of Volterra equations with delay and the collocation methods and their convergence analysis.

Without the integral terms in VIDEs we obtain DDEs (delay differential equations). DDE models arise in many problems, such as the growth of tumors [45], population dynamics [28], hepatitis B virus infection [23], harmful algal blooms in the presence of toxic substances [16]. More applications of DDEs are described in [28]. Numerically solving DDEs has many of the same difficulties discussed for delay VIDEs. Many numerical methods are investigated for DDEs [6; 22; 34; 55]. The monograph by Bellen and Zennaro [8] gives a comprehensive account of numerical methods for DDEs, with the focus being on (classical and continuous) Runge–Kutta methods and their asymptotic stability properties which were also investigated by Baker and Tang [7]. There are some well-developed softwares for delay differential equations or systems. The popular solver developed by Shampine and Thompson [40; 44] for DDEs is well tested and user-friendly.

Spectral methods receive considerable attention mainly due to their high accuracy. Tang, Xu and Cheng [43] proposed a Legendre spectral-collocation method to solve VIEs (Volterra integral equations) of the second kind whose kernel and solutions are sufficiently smooth. Chen and Tang [17; 18; 19] proposed and analyzed a Jacobi spectral-collocation approximation for linear VIEs of the second kind with weakly singular kernels provided that the underlying solutions of the VIEs are sufficiently smooth. Then, in [30], the Jacobi spectral-collocation method was extended to

solve VIEs with Abel-type kernel. Recently, another spectral method, i.e., the Legendre spectral Galerkin method, was investigated in [48; 52] for VIEs. The spectral-collocation methods also attract the interest of those people who study the Volterra-type integral and related functional differential equations (see, e.g., [2; 3; 4; 27; 42; 49; 50; 51]).

However, there is very little literature about the spectral method to solve VIDEs with nonvanishing delay. The main difficulty in applying the spectral method to VIDEs with nonvanishing delay is that the solutions of these equations are not smooth enough at the primary discontinuous points associated with the delay function. In this paper, we overcome this difficulty and propose a Legendre spectral-collocation method to solve these equations. In our method we divide the definition domain into several subintervals according to the primary discontinuous points associated with the nonvanishing delay function. In each subinterval, where the solution is smooth enough, we can apply the Legendre spectral-collocation method to approximate the solution. We provide convergence analysis to show that the numerical errors decay exponentially. Numerical examples are presented to confirm this theoretical prediction.

The VIDEs with nonvanishing delay considered in this paper are as follows:

$$\begin{aligned}
 y'(t) &= a(t)y(t) + b(t)y(\theta(t)) + g(t) + \int_0^t K_1(t, s)y(s) ds + \int_0^{\theta(t)} K_2(t, s)y(s) ds, \\
 & \hspace{15em} t \in (0, T], \\
 y(t) &= \phi(t), \quad t \in [\theta(0), 0].
 \end{aligned} \tag{1}$$

In population models,  $y(t)$  means the population size at time  $t$ . The delay  $\theta(t)$  means that the growth of population size depends on the historical action. We assume that the functions describing the above equation all possess continuous derivatives of at least order  $m \geq 1$  on their respective domains; i.e.,

$$\begin{aligned}
 a(t), b(t), g(t) &\in C^m([0, T]), \quad \phi(t) \in C^m([\theta(0), 0]), \\
 K_1(t, s) &\in C^m(\Omega_1), \quad \Omega_1 := \{(t, s) : 0 \leq s \leq t \leq T\}, \\
 K_2(t, s) &\in C^m(\Omega_2), \quad \Omega_2 := \{(t, s) : \theta(0) \leq s \leq \theta(t), 0 \leq t \leq T\},
 \end{aligned} \tag{2}$$

and the delay function  $\theta$  will be subject to the following conditions:

$$\begin{aligned}
 \theta(t) &:= t - \tau(t), \quad \tau \in C^m([0, T]), \\
 \tau(t) &\geq \tau_0 > 0 \quad \text{for all } t \in [0, T], \\
 \theta &\text{ is strictly increasing on } [0, T].
 \end{aligned} \tag{3}$$

The nonvanishing delay  $\theta$  gives rise to the primary discontinuity points  $\{\xi_\mu\}$  for the solution of (1): they are determined by the recursion

$$\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu \geq 0 \quad (\xi_{-1} := \theta(0), \xi_0 = 0).$$

These points have the uniform separation property

$$\xi_\mu - \xi_{\mu-1} \geq \tau_0 > 0 \quad \text{for all } \mu \geq 0.$$

For ease of notation we will assume that

$$T = \xi_{M+1} \quad \text{for some } M \geq 1.$$

Theorem 4.1.9 in [13] states that the unique solution of (1) is in  $C^{m+1}(\xi_\mu, \xi_{\mu+1}]$  for each  $\mu = 0, 1, \dots, M$  and is bounded on  $Z_M := \{\xi_\mu : \mu = 0, 1, \dots, M\}$  and hence on  $[0, T]$ . At  $t = \xi_\mu$  ( $\mu = 1, \dots, \min\{m, M\}$ ),

$$\lim_{t \rightarrow \xi_\mu^-} y^{(\mu)}(t) = \lim_{t \rightarrow \xi_\mu^+} y^{(\mu)}(t),$$

while the  $(\mu + 1)$ -th derivative of  $y$  is in general not continuous at  $\xi_\mu$ . In addition, if  $\min\{m, M\} = m < M$ , the solution also lies in  $C^{m+1}[\xi_m, T]$ . This motivates us to apply the spectral-collocation method to approximate the solution on the subinterval  $(\xi_\mu, \xi_{\mu+1}]$ ,  $\mu = 0, 1, \dots, M$ .

This paper is organized as follows. In Section 2, we introduce the Legendre spectral-collocation method for VIDEs with nonvanishing delay. Some useful lemmas for the convergence analysis will be provided in Section 4, and the convergence analysis, in both  $L^\infty$  and  $L^2$ , will be given in Section 5. Numerical experiments are carried out in Section 6. Finally, in Section 7, we end with the conclusion and future work.

## 2. Legendre spectral-collocation method

For ease of analysis we change the interval  $[0, T]$  to the standard interval  $[-1, 1]$ . Precisely we use the variable transformation

$$t(x) = \frac{1}{2}T(x+1), \quad s(z) = \frac{1}{2}T(z+1). \quad (4)$$

Then (1) can be written as

$$\begin{aligned} u'(x) &= A(x)u(x) + B(x)u(\vartheta(x)) + f(x) \\ &\quad + \int_{-1}^x R_1(x, z)u(z) dz + \int_{-1}^{\vartheta(x)} R_2(x, z)u(z) dz, \quad x \in (-1, 1], \end{aligned} \quad (5)$$

$$u(x) = \psi(x), \quad x \in [\vartheta(-1), -1],$$

where

$$\begin{aligned} u(x) &:= y(t(x)), \quad A(x) := \frac{1}{2}Ta(t(x)), \quad B(x) := \frac{1}{2}Tb(t(x)), \\ f(x) &:= \frac{1}{2}Tg(t(x)), \quad \vartheta(x) := \frac{2}{T}\theta(t(x)) - 1, \quad \psi(x) := \phi(t(x)), \\ R_1(x, z) &:= \left(\frac{1}{2}T\right)^2 K_1(t(x), s(z)), \quad R_2(x, z) := \left(\frac{1}{2}T\right)^2 K_2(t(x), s(z)). \end{aligned} \quad (6)$$

The primary discontinuity point  $\xi_\mu$  becomes

$$\eta_\mu := (2\xi_\mu/T) - 1, \quad \mu = -1, 0, 1, \dots, M.$$

Define

$$\delta_\mu := (\eta_\mu, \eta_{\mu+1}], \quad \mu = -1, 0, \dots, M.$$

Set the collocation points as follows:

$$X_N := \bigcup_{\mu=0}^M X^\mu, \quad X^\mu := \{x_n^\mu : \eta_\mu = x_0^\mu < x_1^\mu < \dots < x_N^\mu = \eta_{\mu+1}\}, \quad (7)$$

where

$$x_i^\mu := \frac{\eta_{\mu+1} - \eta_\mu}{2} x_i + \frac{\eta_{\mu+1} + \eta_\mu}{2}; \quad (8)$$

here  $x_i$ ,  $i = 0, 1, \dots, N$ , are the  $N + 1$  Legendre Gauss–Lobatto points in the standard interval  $[-1, 1]$ . Then (5) holds at  $x_i^\mu$ ,  $i = 0, 1, \dots, N$ ,  $\mu = 0, 1, \dots, M$ :

$$\begin{aligned} u'(x_i^\mu) &= A(x_i^\mu)u(x_i^\mu) + B(x_i^\mu)u(\vartheta(x_i^\mu)) + f(x_i^\mu) \\ &\quad + \int_{-1}^{x_i^\mu} R_1(x_i^\mu, z)u(z) dz + \int_{-1}^{\vartheta(x_i^\mu)} R_2(x_i^\mu, z)u(z) dz. \end{aligned} \quad (9)$$

We use  $u_i^\mu$  to approximate  $u(x_i^\mu)$ ,  $v_i^\mu$  to approximate  $u(\vartheta(x_i^\mu))$ ,  $\rho_i^\mu$  to approximate  $u'(x_i^\mu)$ . Then we can use

$$u_\mu(x) := \sum_{j=0}^N u_j^\mu F_j^\mu(x), \quad x \in [\eta_\mu, \eta_{\mu+1}]$$

to approximate  $u|_{\delta_\mu}(x)$ , i.e., the restriction of  $u(x)$  to the interval  $[\eta_\mu, \eta_{\mu+1}]$ .  $F_j^\mu(x)$ ,  $x \in [\eta_\mu, \eta_{\mu+1}]$ , is the  $j$ -th Lagrange interpolation basic function associated with the collocation points  $x_0^\mu, x_1^\mu, \dots, x_N^\mu$  in the interval  $[\eta_\mu, \eta_{\mu+1}]$ . Similarly, we use

$$\rho_\mu(x) := \sum_{j=0}^N \rho_j^\mu F_j^\mu(x), \quad x \in [\eta_\mu, \eta_{\mu+1}]$$

to approximate  $u'|_{\delta_\mu}(x)$ , i.e., the restriction of  $u'(x)$  to the subinterval  $[\eta_\mu, \eta_{\mu+1}]$ . Eventually  $u(x)$  can be approximated by

$$u^N(x) := u_\mu(x) \quad \text{if } x \in [\eta_\mu, \eta_{\mu+1}], \quad \mu = 0, 1, \dots, M,$$

and  $u'(x)$  can be approximated by

$$\rho^N(x) := \rho_\mu(x) \quad \text{if } x \in [\eta_\mu, \eta_{\mu+1}], \quad \mu = 0, 1, \dots, M.$$

Then (9) can be approximated by

$$\begin{aligned} \rho_i^\mu \approx & A(x_i^\mu)u_i^\mu + B(x_i^\mu)v_i^\mu + f(x_i^\mu) \\ & + \int_{-1}^{x_i^\mu} R_1(x_i^\mu, z)u^N(z) dz + \int_{-1}^{\vartheta(x_i^\mu)} R_2(x_i^\mu, z)u^N(z) dz, \quad (10) \end{aligned}$$

which can be written as

$$\begin{aligned} \rho_i^\mu \approx & A(x_i^\mu)u_i^\mu + B(x_i^\mu)v_i^\mu + f(x_i^\mu) \\ & + \sum_{r=0}^{\mu-1} \int_{\eta_r}^{\eta_{r+1}} R_1(x_i^\mu, z)u_r(z) dz + \int_{\eta_\mu}^{x_i^\mu} R_1(x_i^\mu, z)u_\mu(z) dz \\ & + \sum_{r=0}^{\mu-2} \int_{\eta_r}^{\eta_{r+1}} R_2(x_i^\mu, z)u_r(z) dz + \int_{\eta_{\mu-1}}^{\vartheta(x_i^\mu)} R_2(x_i^\mu, z)u_{\mu-1}(z) dz. \quad (11) \end{aligned}$$

In order to compute the integral term by the Gauss quadrature rule, we change the integration interval to the standard interval  $[-1, 1]$ . Note that the variable transformation

$$z(a, b, v) := \frac{b-a}{2}v + \frac{b+a}{2}, \quad v \in [-1, 1] \quad (12)$$

can change the interval  $[a, b]$  to  $[-1, 1]$ . For simplicity, we define

$$z_r(v) := z(\eta_r, \eta_{r+1}, v), \quad v \in [-1, 1], \quad r \geq 0. \quad (13)$$

Using the Gauss quadrature formula to approximate the integration term in (11) we obtain

$$\begin{aligned} \rho_i^\mu = & A(x_i^\mu)u_i^\mu + B(x_i^\mu)v_i^\mu + f(x_i^\mu) + \sum_{r=0}^{\mu-1} \frac{\eta_{r+1} - \eta_r}{2} \sum_{k=0}^N R_1(x_i^\mu, z_r(v_k))u_r(z_r(v_k))\omega_k \\ & + \frac{\eta_{\mu+1} - \eta_\mu}{2} \frac{x_i + 1}{2} \sum_{k=0}^N R_1(x_i^\mu, z_\mu(z(-1, x_i, v_k)))u_\mu(z_\mu(z(-1, x_i, v_k)))\omega_k \\ & + \sum_{r=0}^{\mu-2} \frac{\eta_{r+1} - \eta_r}{2} \sum_{k=0}^N R_2(x_i^\mu, z_r(v_k))u_r(z_r(v_k))\omega_k \\ & + \frac{\eta_\mu - \eta_{\mu-1}}{2} \frac{\tilde{\vartheta}(x_i^\mu) + 1}{2} \sum_{k=0}^N R_2(x_i^\mu, z_{\mu-1}(z(-1, \tilde{\vartheta}(x_i^\mu), v_k))) \\ & \quad \times u_{\mu-1}(z_{\mu-1}(z(-1, \tilde{\vartheta}(x_i^\mu), v_k)))\omega_k, \quad (14) \end{aligned}$$

where  $v_k, k = 0, 1, \dots, N$ , are the  $N + 1$  Legendre Gauss–Lobatto points in the standard interval  $[-1, 1]$ , corresponding to the weights  $\omega_k, k = 0, 1, \dots, N$ , and



$$\tilde{\vartheta}(x_i^\mu) := \frac{2}{\eta_\mu - \eta_{\mu-1}} \vartheta(x_i^\mu) - \frac{\eta_\mu + \eta_{\mu-1}}{\eta_\mu - \eta_{\mu-1}}, \quad \mu > 0. \quad (15)$$

Note that, for  $j, k = 0, 1, \dots, N$ ,  $r = 0, 1, \dots, M$ ,

$$\begin{aligned} F_j^r(z_r(v_k)) &= F_j(v_k) = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \\ F_j^r(z_r(z(-1, x, v))) &= F_j(z(-1, x, v)), \end{aligned} \quad (16)$$

where  $F_j(v)$  is the  $j$ -th Lagrange interpolation basic function associated with the  $N + 1$  Legendre Gauss–Lobatto points in the standard interval  $[-1, 1]$ . Then (14) can be simplified to

$$\begin{aligned} \rho_i^\mu &= A(x_i^\mu)u_i^\mu + B(x_i^\mu)v_i^\mu + f(x_i^\mu) + \beta(x_i^\mu) + \lambda(x_i^\mu), \\ &\mu = 0, 1, \dots, M, \quad i = 0, 1, \dots, N, \end{aligned} \quad (17)$$

where

$$\begin{aligned} \beta(x_i^\mu) &:= \sum_{r=0}^{\mu-1} \frac{\eta_{r+1} - \eta_r}{2} \beta_1^r(x_i^\mu) + \frac{\eta_{\mu+1} - \eta_\mu}{2} \frac{x_i + 1}{2} \beta_3(x_i^\mu), \\ \beta_1^r(x_i^\mu) &:= \sum_{k=0}^N R_1(x_i^\mu, z_r(v_k)) u_k^r \omega_k, \quad r = 0, 1, \dots, \mu - 1, \\ \beta_3(x_i^\mu) &:= \sum_{j=0}^N u_j^\mu \sum_{k=0}^N R_1(x_i^\mu, z_\mu(z(-1, x_i, v_k))) F_j(z(-1, x_i, v_k)) \omega_k, \end{aligned}$$

and

$$\begin{aligned} \lambda(x_i^\mu) &:= \begin{cases} \frac{\vartheta(x_i^0) + 1}{2} \lambda_2(x_i^0), & \mu = 0, \\ \sum_{r=0}^{\mu-2} \frac{\eta_{r+1} - \eta_r}{2} \lambda_1^r(x_i^\mu) + \frac{\eta_\mu - \eta_{\mu-1}}{2} \frac{\tilde{\vartheta}(x_i^\mu) + 1}{2} \lambda_3(x_i^\mu), & \mu > 0, \end{cases} \\ \lambda_1^r(x_i^\mu) &:= \sum_{j=0}^N u_j^r R_2(x_i^\mu, z_r(v_j)) \omega_j, \quad r = 0, 1, \dots, \mu - 2, \\ \lambda_2(x_i^0) &:= \sum_{k=0}^N R_2(x_i^0, z(-1, \vartheta(x_i^0), v_k)) \psi(z(-1, \vartheta(x_i^0), v_k)) \omega_k, \\ \lambda_3(x_i^\mu) &:= \sum_{j=0}^N u_j^{\mu-1} \sum_{k=0}^N R_2(x_i^\mu, z_{\mu-1}(z(-1, \tilde{\vartheta}(x_i^\mu), v_k))) F_j(z(-1, \tilde{\vartheta}(x_i^\mu), v_k)) \omega_k. \end{aligned}$$

However, the linear systems (17) alone are not enough to find out the unknown elements. We need two other linear systems associated with  $u_i^\mu$ ,  $v_i^\mu$ ,  $\rho_i^\mu$ ,

$i = 0, 1, \dots, N, \mu = 0, 1, \dots, M$ . Note that

$$\begin{aligned} u(x_i^\mu) &= u(-1) + \int_{-1}^{x_i^\mu} u'(z) dz \\ &= \psi(-1) + \sum_{r=0}^{\mu-1} \frac{\eta_{r+1} - \eta_r}{2} \int_{-1}^1 u'(z_r(v)) dv \\ &\quad + \frac{\eta_{\mu+1} - \eta_\mu}{2} \frac{x_i + 1}{2} \int_{-1}^1 u'(z_\mu(z(-1, x_i, v))) dv. \end{aligned} \quad (18)$$

Then we can approximate the above equation by

$$u_i^\mu = \psi(-1) + \alpha_1(x_i^\mu), \quad \mu = 0, 1, \dots, M, \quad i = 0, 1, \dots, N, \quad (19)$$

where

$$\begin{aligned} \alpha_1(x_i^\mu) &= \sum_{r=0}^{\mu-1} \frac{\eta_{r+1} - \eta_r}{2} \sum_{k=0}^N \rho_k^r \omega_k \\ &\quad + \frac{\eta_{\mu+1} - \eta_\mu}{2} \frac{x_i + 1}{2} \sum_{j=0}^N \rho_j^\mu \sum_{k=0}^N F_j(z(-1, x_i, v_k)) \omega_k. \end{aligned} \quad (20)$$

Similarly, the equation

$$\begin{aligned} u(\vartheta(x_i^\mu)) &= u(-1) + \int_{-1}^{\vartheta(x_i^\mu)} u'(z) dz \\ &= \psi(-1) + \sum_{r=0}^{\mu-2} \frac{\eta_{r+1} - \eta_r}{2} \int_{-1}^1 u'(z_r(v)) dv \\ &\quad + \frac{\eta_\mu - \eta_{\mu-1}}{2} \frac{\tilde{\vartheta}(x_i^\mu) + 1}{2} \int_{-1}^1 u'(z_{\mu-1}(z(-1, \tilde{\vartheta}(x_i^\mu), v))) dv \end{aligned} \quad (21)$$

can be approximated by

$$v_i^\mu = \psi(-1) + \alpha_2(x_i^\mu), \quad \mu = 0, 1, \dots, M, \quad i = 0, 1, \dots, N, \quad (22)$$

where

$$\alpha_2(x_i^\mu) = \begin{cases} \psi(\vartheta(x_i^0)) - \psi(-1), & \mu = 0, \\ \sum_{r=0}^{\mu-2} \frac{\eta_{r+1} - \eta_r}{2} \sum_{k=0}^N \rho_k^r \omega_k \\ \quad + \frac{\eta_\mu - \eta_{\mu-1}}{2} \frac{\tilde{\vartheta}(x_i^\mu) + 1}{2} \sum_{j=0}^N \rho_j^{\mu-1} \sum_{k=0}^N F_j(z(-1, \tilde{\vartheta}(x_i^\mu), v_k)) \omega_k, & \mu > 0. \end{cases}$$

Equations (19) and (22) are two linear systems we want to find.

The Legendre spectral-collocation method is to find  $\rho_i^\mu, u_i^\mu, i = 0, 1, \dots, N, \mu = 0, 1, 2, \dots, M$ , which satisfy (17), (19) and (22). The approximation to  $y(t)$

is  $u^N(2t/T - 1)$ ; the approximation to  $y'(t)$  is  $(2/T)\rho^N(2t/T - 1)$ . An efficient computation of  $F_j(s)$  can be found in [15] or [43].

### 3. The existence of the solution to the discrete system

In this section we will discuss the existence of the solution to the discrete system (17), (19) and (22). We write the linear system (17), (19) and (22) into matrix form:

$$\begin{cases} U^{(\mu)} = \Phi_1^{(\mu)} + A^{(\mu)}U^{(\mu)} + R_1^{(\mu)}U^{(\mu)} + B^{(\mu)}V^{(\mu)}, \\ U^{(\mu)} = \Phi_2^{(\mu)} + \frac{\eta_{\mu+1} - \eta_\mu}{2} R_3^{(\mu)}U^{(\mu)}, \\ V^{(\mu)} = \Phi_3^{(\mu)} + R_4^{(\mu)}U^{(\mu-1)}, \end{cases} \quad (23)$$

where

$$\begin{aligned} U^{(\mu)} &:= [\rho_0^\mu, \rho_1^\mu, \dots, \rho_N^\mu]', \\ U^{(\mu)} &:= [u_0^\mu, u_1^\mu, \dots, u_N^\mu]', \\ V^{(\mu)} &:= [v_0^\mu, v_1^\mu, \dots, v_N^\mu]', \quad \mu > 0, \\ V^{(0)} &:= [\psi(\vartheta(x_0^0)), \psi(\vartheta(x_1^0)), \dots, \psi(\vartheta(x_N^0))]', \quad \mu = 0, \\ \Phi_1^{(\mu)} &:= F^{(\mu)} + \sum_{r=0}^{\mu-1} R_1^{(r)}U^{(r)} + \sum_{r=0}^{\mu-2} R_2^{(r)}U^{(r)} + R_2^{(\mu)}U^{(\mu-1)}, \quad \mu > 0, \\ \Phi_1^{(0)}(i) &:= F^{(0)}(i) + \frac{\vartheta(x_i^0) + 1}{2} \sum_{k=0}^N R_2(x_i^0, z(-1, \vartheta(x_i^0), v_k)) \\ &\quad \times \psi(z(-1, \vartheta(x_i^0), v_k))\omega_k, \\ \Phi_2^{(\mu)} &:= \psi(-1)[1, 1, \dots, 1]' + \sum_{r=0}^{\mu-1} R_3^{(r)}U^{(r)}, \\ \Phi_3^{(\mu)} &:= \psi(-1)[1, 1, \dots, 1]' + \sum_{r=0}^{\mu-2} R_3^{(r)}U^{(r)}, \\ F^{(\mu)} &:= [f(x_0^\mu), f(x_1^\mu), \dots, f(x_N^\mu)]', \\ A^{(\mu)} &:= \text{diag}[A(x_0^\mu), A(x_1^\mu), \dots, A(x_N^\mu)], \\ B^{(\mu)} &:= \text{diag}[B(x_0^\mu), B(x_1^\mu), \dots, B(x_N^\mu)], \\ R_j^{(r)}(i, k) &:= \frac{\eta_{r+1} - \eta_r}{2} R_j(x_i^\mu, z_r(v_k))\omega_k, \quad j = 1, 2, r = 0, 1, \dots, \mu - 1, \\ R_3^{(r)}(i, k) &:= \frac{\eta_{r+1} - \eta_r}{2} \omega_k, \quad r = 0, 1, \dots, \mu - 1, \\ R_1^{(\mu)}(i, j) &:= \frac{\eta_{\mu+1} - \eta_\mu}{2} \frac{x_i + 1}{2} \sum_{k=0}^N R_1(x_i^\mu, z_\mu(z(-1, x_i, v_k)))F_j(z(-1, x_i, v_k))\omega_k, \end{aligned}$$



has a uniformly bounded inverse. This ensures the corresponding discrete system based on the interval  $[\tau_i^\mu, \tau_{i+1}^\mu]$  possesses a unique solution.

#### 4. Some useful lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of error estimates in Section 5. In order to give the subsequent lemmas conveniently, we first introduce some spaces. For simplicity, we denote by  $\partial_x^k u(x)$  the  $k$ -th derivative of  $u$ ; i.e.,  $\partial_x^k u(x) := (d^k u/dx^k)(x)$ .

Let  $(a, b)$  be a bounded interval of the real line. We denote by  $L^2(a, b)$  the space of measurable functions  $u : (a, b) \rightarrow \mathbb{R}$  such that  $\int_a^b |u(x)|^2 dx < +\infty$ . It is a Hilbert space for the inner product

$$(u, v) := \int_a^b u(x)v(x) dx,$$

which induces the norm

$$\|v\|_{L^2(a,b)} := \left( \int_a^b |v(x)|^2 dx \right)^{1/2}.$$

Let  $m \geq 1$  be an integer. We define  $H^m(a, b)$  to be the vector space of the functions  $v \in L^2(a, b)$  such that all the distributions of  $v$  of order up to  $m$  can be represented by functions in  $L^2(a, b)$ . In short,

$$H^m(a, b) := \{v \in L^2(a, b) : \text{for } 0 \leq k \leq m, \partial_x^k v(x) \in L^2(a, b)\}.$$

$H^m(a, b)$  is endowed with the inner product

$$(u, v)_m = \sum_{k=0}^m \int_a^b \partial_x^k u(x) \partial_x^k v(x) dx$$

for which  $H^m(a, b)$  is a Hilbert space. The associated norm is

$$\|v\|_{H^m(a,b)} := ((v, v)_m)^{1/2}.$$

In bounding the approximation error from above, only some of the  $L^2$  norms appearing on the right-hand side of the above norm enter into play. Thus, for a nonnegative integer  $N$ , it is convenient to introduce the seminorm

$$|v|_{H^{m;N}(a,b)} := \left( \sum_{k=\min(m,N+1)}^m \|\partial_x^k v(x)\|_{L^2(a,b)}^2 \right)^{1/2},$$

which implies that if  $N \geq m - 1$  then  $|v|_{H^{m;N}(a,b)} = \|\partial_x^m v\|_{L^2(a,b)}$ .

Let  $\Lambda_h$  denote the collection of subintervals  $\delta_\mu$ ,  $\mu = 0, 1, \dots, M$ . Referring to [25], we define the broken Sobolev space  $H^m(\Lambda_h)$  as

$$H^m(\Lambda_h) := \{u : u|_{\delta_\mu} \in H^m(\delta_\mu), \mu = 0, 1, \dots, M\}.$$

The associated norm is

$$\|u\|_{H^m(\Lambda_h)} := \left( \sum_{k=0}^m \|u^{(k)}\|_{L^2(\Lambda_h)}^2 \right)^{1/2},$$

where

$$\|u^{(k)}\|_{L^2(\Lambda_h)}^2 := \sum_{\mu=0}^M \|\partial_x^k(u|_{\sigma_\mu})\|_{L^2(\delta_\mu)}^2, \quad k = 0, 1, \dots, m.$$

For a nonnegative integer  $N$ , the associated seminorm is

$$|u|_{H^{m;N}(\Lambda_h)} := \left( \sum_{k=\min(m, N+1)}^m \|u^{(k)}\|_{L^2(\Lambda_h)}^2 \right)^{1/2}.$$

If  $N \geq m - 1$  then  $|u|_{H^{m;N}(\Lambda_h)} = \|u^{(m)}\|_{L^2(\Lambda_h)}$ .

The space  $L^\infty(a, b)$  is the Banach space of measurable functions  $u$  that are bounded outside a set of measure zero, equipped with the norm

$$\|u\|_{L^\infty(a, b)} := \operatorname{ess\,sup}_{x \in (a, b)} |u(x)|.$$

We denote by  $C([a, b])$  the space of continuous functions on the interval  $[a, b]$ .

We define an interpolation operator  $I_N$  associated with the collocation points  $X_N$  as follows: for any continuous function  $v \in C([-1, 1])$ ,

$$I_N v(x) := I_N^\mu(v|_{\delta_\mu})(x) \quad \text{if } x \in (\eta_\mu, \eta_{\mu+1}], \quad 0 \leq \mu \leq M, \quad (25)$$

where  $v|_{\delta_\mu}(x)$  is the restriction of  $v(x)$  to the subinterval  $[\eta_\mu, \eta_{\mu+1}]$ , and  $I_N^\mu$  is the interpolation operator associated with the collocation points  $X^\mu$  in the subinterval  $[\eta_\mu, \eta_{\mu+1}]$ ; i.e.,

$$I_N^\mu(v|_{\delta_\mu})(x) := \sum_{j=0}^N v|_{\delta_\mu}(x_j^\mu) F_j^\mu(x), \quad x \in [\eta_\mu, \eta_{\mu+1}].$$

Hereafter,  $C$  denotes a generic positive constant that is independent of  $N$ .

**Lemma 1.** *Assume that  $u \in H^m(-1, 1)$ ,  $m \geq 1$ ,  $v(x)$  is a bounded function. Then there exists a constant  $C$  independent of  $u$  and  $v$  such that, for  $N \geq m - 1$ ,*

$$\|u - J_N u\|_{L^2(-1, 1)} \leq C N^{-m} \|\partial_x^m u\|_{L^2(-1, 1)}, \quad (26)$$

$$\|u - J_N u\|_{L^\infty(-1, 1)} \leq C N^{1/2-m} \|\partial_x^m u\|_{L^2(-1, 1)}, \quad (27)$$

$$\sup_N \|J_N v\|_{L^2(-1, 1)} \leq C \|v\|_{L^\infty(-1, 1)}, \quad (28)$$

$$\|J_N\|_{L^\infty(-1, 1)} \leq (2/\pi) \log(N + 1) + 0.685, \quad (29)$$

where  $J_N$  is the interpolation operator associated with the  $N + 1$  Legendre Gauss-Lobatto points in the interval  $[-1, 1]$ .

*Proof.* Inequalities (26) and (28) can be found in [15; 33; 43], and (29) can be found in [24]. We only prove (27). Using the Sobolev inequality [15, p. 490], we have

$$\|u - J_N u\|_{L^\infty(-1,1)} \leq C \|u - J_N u\|_{L^2(-1,1)}^{1/2} \|u - J_N u\|_{H^1(-1,1)}^{1/2}.$$

Applying the result (26) to  $\|u - J_N u\|_{L^2(-1,1)}^{1/2}$  makes the above inequality become

$$\|u - J_N u\|_{L^\infty(-1,1)} \leq C N^{-m/2} \|\partial_x^m u\|_{L^2(-1,1)}^{1/2} \|u - J_N u\|_{H^1(-1,1)}^{1/2}, \quad (30)$$

which leads to (27) because  $\|u - J_N u\|_{H^1(-1,1)}^{1/2}$  can be estimated as follows [15, p. 289]:

$$\|u - J_N u\|_{H^1(-1,1)}^{1/2} \leq C N^{(1-m)/2} \|\partial_x^m u\|_{L^2(-1,1)}^{1/2}. \quad \square$$

**Lemma 2.** *Assume that  $u \in C([-1, 1]) \cap H^m(\Lambda_h)$ . Let  $I_N u$  be the interpolation function defined in (25) where  $N + 1$  is the number of collocation points in the intervals  $[\eta_\mu, \eta_{\mu+1}]$ ,  $\mu = 0, 1, \dots, M$ . Then the following estimates hold for  $N \geq m - 1$ :*

$$\|u - I_N u\|_{L^2(-1,1)} \leq C N^{-m} \|u^{(m)}\|_{L^2(\Lambda_h)}, \quad (31)$$

$$\|u - I_N u\|_{L^\infty(-1,1)} \leq C N^{1/2-m} \|u^{(m)}\|_{L^2(\Lambda_h)}, \quad (32)$$

$$\|I_N\|_{L^\infty(-1,1)} \leq C \log(N + 1), \quad (33)$$

$$\sup_N \|I_N u\|_{L^2(-1,1)} \leq C \|u\|_{L^\infty(-1,1)}. \quad (34)$$

*Proof.* By the definition of  $I_N^\mu$  we know that the  $(I_N^\mu(u|_{\delta_\mu}))(z)$  is a function defined on the subinterval  $[\eta_\mu, \eta_{\mu+1}]$ . The variable transformation  $z = z_\mu(v)$  changes it to be a function valued on the standard interval  $[-1, 1]$ ; i.e., for  $v \in [-1, 1]$ ,

$$(I_N^\mu(u|_{\delta_\mu}))(z_\mu(v)) = \sum_{j=0}^N u|_{\delta_\mu}(x_j^\mu) F_j^\mu(z_\mu(v)) = \sum_{j=0}^N u|_{\delta_\mu}(x_j^\mu) F_j(v). \quad (35)$$

The result (16) is used in the derivation of the second equality above. On the other hand, we note that  $u|_{\delta_\mu}(z_\mu(v))$  is a function defined on the interval  $[-1, 1]$ . Its interpolation polynomial associated with the Legendre Gauss–Lobatto points  $v_j$ ,  $j = 0, 1, \dots, N$ , in the interval  $[-1, 1]$  is

$$J_N(u|_{\delta_\mu}(z_\mu(v))) = \sum_{j=0}^N u|_{\delta_\mu}(z_\mu(v_j)) F_j(v), \quad v \in [-1, 1]. \quad (36)$$

Note that  $v_j = x_j$ ; then

$$z_\mu(v_j) = x_j^\mu, \quad j = 0, 1, \dots, N.$$

Plugging this into the right-hand side of (36) yields

$$J_N(u|_{\delta_\mu}(z_\mu(v))) = \sum_{j=0}^N u|_{\delta_\mu}(x_j^\mu) F_j(v), \quad v \in [-1, 1]. \quad (37)$$

Combining (35) with (37) yields

$$(I_N^\mu(u|_{\delta_\mu}))(z_\mu(v)) = J_N(u|_{\delta_\mu}(z_\mu(v))), \quad v \in [-1, 1]. \quad (38)$$

By (26), we have

$$\begin{aligned} \int_{\eta_\mu}^{\eta_{\mu+1}} (u|_{\delta_\mu}(z) - I_N^\mu(u|_{\delta_\mu})(z))^2 dz &= \frac{\eta_{\mu+1} - \eta_\mu}{2} \int_{-1}^1 (u|_{\delta_\mu}(z_\mu(v)) - J_N(u|_{\delta_\mu}(z_\mu(v))))^2 dv \\ &\leq CN^{-2m} \left( \frac{\eta_{\mu+1} - \eta_\mu}{2} \right)^{2m+1} \|\partial_v^m(u|_{\delta_\mu}(z_\mu(\cdot)))\|_{L^2(-1,1)}^2 \\ &\leq CN^{-2m} \|\partial_z^m(u|_{\delta_\mu}(\cdot))\|_{L^2(\delta_\mu)}^2. \end{aligned} \quad (39)$$

This helps to deduce that

$$\begin{aligned} \|u - I_N u\|_{L^2(-1,1)}^2 &= \sum_{\mu=0}^M \int_{\eta_\mu}^{\eta_{\mu+1}} (u|_{\delta_\mu}(z) - I_N^\mu(u|_{\delta_\mu})(z))^2 dz \\ &\leq CN^{-2m} \sum_{\mu=0}^M \|\partial_z^m(u|_{\delta_\mu}(\cdot))\|_{L^2(\delta_\mu)}^2 = CN^{-2m} \|u^{(m)}\|_{L^2(\Lambda_h)}^2, \end{aligned} \quad (40)$$

which leads to (31).

Using (27), we have

$$\begin{aligned} \|u - I_N u\|_{L^\infty(-1,1)} &= \max_{0 \leq \mu \leq M} \left\{ \|u|_{\delta_\mu}(z_\mu(\cdot)) - J_N(u|_{\delta_\mu}(z_\mu(\cdot)))\|_{L^\infty(-1,1)} \right\} \\ &\leq CN^{1/2-m} \max_{0 \leq \mu \leq M} \left\{ \|\partial_v^m(u|_{\delta_\mu}(z_\mu(\cdot)))\|_{L^2(-1,1)} \right\} \\ &\leq CN^{1/2-m} \|u^{(m)}\|_{L^2(\Lambda_h)}. \end{aligned} \quad (41)$$

This is (32).

Now we begin to prove (33). It is evident that

$$\|I_N u\|_{L^\infty(-1,1)} = \max_{0 \leq \mu \leq M} \|I_N^\mu(u|_{\delta_\mu})\|_{L^\infty(\sigma_\mu)}. \quad (42)$$

We use (29) to estimate  $\|I_N^\mu(u|_{\delta_\mu})\|_{L^\infty(\delta_\mu)}$ :

$$\begin{aligned} \|I_N^\mu(u|_{\delta_\mu})\|_{L^\infty(\delta_\mu)} &= \|(I_N^\mu(u|_{\delta_\mu}))(z_\mu(\cdot))\|_{L^\infty(-1,1)} = \|J_N(u|_{\delta_\mu}(z_\mu(\cdot)))\|_{L^\infty(-1,1)} \\ &\leq C \log(N+1) \|u|_{\delta_\mu}(z_\mu(\cdot))\|_{L^\infty(-1,1)} = C \log(N+1) \|u|_{\delta_\mu}\|_{L^\infty(\delta_\mu)} \\ &\leq C \log(N+1) \|u\|_{L^\infty(-1,1)}, \end{aligned} \quad (43)$$



which together with (42) give that

$$\|I_N u\|_{L^\infty(-1,1)} \leq C \log(N+1) \|u\|_{L^\infty(-1,1)}.$$

This leads to the desired result (33).

Now we begin to prove (34). The result (28) is useful in the following derivation:

$$\begin{aligned} \|I_N u\|_{L^2(-1,1)}^2 &= \sum_{\mu=0}^M \|I_N^\mu(u|_{\delta_\mu})\|_{L^2(\delta_\mu)}^2 = \sum_{\mu=0}^M \frac{\eta_{\mu+1} - \eta_\mu}{2} \|(I_N^\mu(u|_{\delta_\mu}))(z_\mu(\cdot))\|_{L^2(-1,1)}^2 \\ &= \sum_{\mu=0}^M \frac{\eta_{\mu+1} - \eta_\mu}{2} \|J_N(u|_{\delta_\mu}(z_\mu(\cdot)))\|_{L^2(-1,1)}^2 \\ &\leq C \sum_{\mu=0}^M \frac{\eta_{\mu+1} - \eta_\mu}{2} \|u|_{\delta_\mu}(z_\mu(\cdot))\|_{L^\infty(-1,1)}^2 \\ &\leq C \sum_{\mu=0}^M \frac{\eta_{\mu+1} - \eta_\mu}{2} \|u\|_{L^\infty(-1,1)}^2 \leq C \|u\|_{L^\infty(-1,1)}^2, \end{aligned} \quad (44)$$

which leads to the desired result (34). Now we have completed the whole proof for this lemma.  $\square$

**Lemma 3** [15; 41]. *Assume that  $u \in H^m(-1, 1)$  for some  $m \geq 1$  and  $\varphi \in \mathcal{P}_N$ , which denotes the space of all polynomials of degree not exceeding  $N$ . Then there exists a constant  $C$  independent of  $N \geq m - 1$  such that*

$$\left| \int_{-1}^1 u(x)\varphi(x) dx - \sum_{j=0}^N u(x_j)\varphi(x_j)\omega_j \right| \leq CN^{-m} \|\partial_x^m u\|_{L^2(-1,1)} \|\varphi\|_{L^2(-1,1)},$$

where  $x_j$  are the  $N+1$  Legendre Gauss–Lobatto points, with corresponding weights  $\omega_j$ ,  $j = 0, 1, \dots, N$ .

**Lemma 4** [43]. *Suppose  $0 \leq M < +\infty$ . If a nonnegative integrable function  $e(x)$  satisfies*

$$e(x) \leq v(x) + M \int_{-1}^x e(z) dz,$$

where  $v(x)$  is also a nonnegative integrable function, then

$$\|e(x)\|_{L^p(-1,1)} \leq C \|v(x)\|_{L^p(-1,1)}, \quad p = 2, +\infty.$$

## 5. Convergence analysis

This section is devoted to providing a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential; i.e., the spectral

accuracy can be obtained for the proposed approximations. Firstly, we will carry out convergence analysis in the  $L^\infty(-1, 1)$  space.

**Theorem 1.** *Let  $u(x)$  be the exact solution to (5),  $u^N(x)$  be the approximate solution, and  $\rho^N(x)$  be the approximate derivative obtained by using the spectral-collocation schemes (17), (19) and (22). Then, for  $N \geq m - 1$  sufficiently large,*

$$\|e_i(x)\|_{L^\infty(-1,1)} \leq CN^{1/2-m} (R\|u\|_{L^\infty(-1,1)} + \|u^{(m+1)}\|_{L^2(\Lambda_h)}), \quad i = 0, 1, \quad (45)$$

where

$$e_0(x) := \begin{cases} 0, & x \in [\vartheta(-1), -1], \\ u(x) - u^N(x), & x \in (-1, 1], \end{cases}$$

$$e_1(x) := \begin{cases} 0, & x \in [\vartheta(-1), -1], \\ u'(x) - \rho^N(x), & x \in (-1, 1], \end{cases}$$

and  $R$  is a constant dependent on the  $m$ -order derivatives of  $R_j(x, z)$ ,  $\psi(z)$ ,  $j = 1, 2$ .

*Proof.* In each subinterval  $(\eta_\mu, \eta_{\mu+1}]$ ,  $\mu = 0, 1, \dots, M$ , the degree of the polynomial  $\rho^N(s)$  does not exceed  $N$ . Then

$$\alpha_1(x_i^\mu) = \int_{-1}^{x_i^\mu} \rho^N(z) dz \quad \text{and} \quad \alpha_2(x_i^\mu) = \int_{-1}^{\vartheta(x_i^\mu)} \rho^N(z) dz, \quad (46)$$

which implies that

$$u(x_i^\mu) - u_i^\mu = \int_{-1}^{x_i^\mu} e_1(z) dz, \quad (47)$$

$$u(\vartheta(x_i^\mu)) - v_i^\mu = \int_{-1}^{\vartheta(x_i^\mu)} e_1(z) dz.$$

Subtracting (17) from (9) yields

$$u'(x_i^\mu) - \rho_i^\mu = A(x_i^\mu) \int_{-1}^{x_i^\mu} e_1(z) dz + B(x_i^\mu) \int_{-1}^{\vartheta(x_i^\mu)} e_1(z) dz + \int_{-1}^{x_i^\mu} R_1(x_i^\mu, z) e_0(z) dz$$

$$+ \int_{-1}^{\vartheta(x_i^\mu)} R_2(x_i^\mu, z) e_0(z) dz + \sum_{j=0}^1 E_j(x_i^\mu), \quad (48)$$

where, for  $x \in [-1, 1]$ ,

$$E_1(x) := \int_{-1}^x R_1(x, z) u^N(z) dz - \beta(x), \quad E_0(x) := \int_{-1}^{\vartheta(x)} R_2(x, z) u^N(z) dz - \lambda(x),$$

Multiplying by  $F_i^\mu(x)$  on both sides of (48) and summing from  $i = 0$  to  $N$ , we obtain that

$$\begin{aligned}
 & \sum_{i=0}^N u'(x_i^\mu) F_i^\mu(x) - \sum_{i=0}^N \rho_i^\mu F_i^\mu(x) \\
 &= \sum_{i=0}^N \left( A(x_i^\mu) \int_{-1}^{x_i^\mu} e_1(z) dz \right) F_i^\mu(x) + \sum_{i=0}^N \left( B(x_i^\mu) \int_{-1}^{\vartheta(x_i^\mu)} e_1(z) dz \right) F_i^\mu(x) \\
 &+ \sum_{i=0}^N \left( \int_{-1}^{x_i^\mu} R_1(x_i^\mu, z) e_0(z) dz \right) F_i^\mu(x) + \sum_{i=0}^N \left( \int_{-1}^{\vartheta(x_i^\mu)} R_2(x_i^\mu, z) e_0(z) dz \right) F_i^\mu(x) \\
 &+ \sum_{j=0}^1 \sum_{i=0}^N E_j(x_i^\mu) F_i^\mu(x), \quad x \in [\eta_\mu, \eta_{\mu+1}]. \tag{49}
 \end{aligned}$$

By the definitions of  $I_N$  and  $\rho^N(x)$ , we have

$$\begin{aligned}
 I_N u'(x) - \rho^N(x) &= I_N \left( A(x) \int_{-1}^x e_1(z) dz \right) + I_N \left( B(x) \int_{-1}^{\vartheta(x)} e_1(z) dz \right) \\
 &+ I_N \left( \int_{-1}^x R_1(x, z) e_0(z) dz \right) + I_N \left( \int_{-1}^{\vartheta(x)} R_2(x, z) e_0(z) dz \right) \\
 &+ \sum_{j=0}^1 I_N E_j(x), \quad x \in [-1, 1]. \tag{50}
 \end{aligned}$$

This leads to

$$\begin{aligned}
 e_1(x) &= \sum_{j=0}^1 I_N E_j(x) + \sum_{j=2}^6 E_j(x) + A(x) \int_{-1}^x e_1(z) dz + B(x) \int_{-1}^{\vartheta(x)} e_1(z) dz \\
 &+ \int_{-1}^x R_1(x, z) e_0(z) dz + \int_{-1}^{\vartheta(x)} R_2(x, z) e_0(z) dz, \tag{51}
 \end{aligned}$$

where

$$\begin{aligned}
 E_2(x) &:= (I - I_N) u'(x), \\
 E_3(x) &:= (I_N - I) \int_{-1}^x R_1(x, z) e_0(z) dz, \\
 E_4(x) &:= (I_N - I) \int_{-1}^{\vartheta(x)} R_2(x, z) e_0(z) dz, \\
 E_5(x) &:= (I_N - I) \left( A(x) \int_{-1}^x e_1(z) dz \right), \\
 E_6(x) &:= (I_N - I) \left( B(x) \int_{-1}^{\vartheta(x)} e_1(z) dz \right). \tag{52}
 \end{aligned}$$

Applying the Dirichlet formula to the last two terms in the right-hand side of (51) yields

$$\int_{-1}^x R_1(x, z)e_0(z) dz = \int_{-1}^x \left[ \int_s^x R_1(x, z) dz \right] e_1(s) ds, \quad (53)$$

$$\int_{-1}^{\vartheta(x)} R_2(x, z)e_0(z) dz = \int_{-1}^{\vartheta(x)} \left[ \int_s^{\vartheta(x)} R_2(x, z) dz \right] e_1(s) ds, \quad (54)$$

which help to deduce that there exist constants  $C_1, C_2, C > 0$  such that

$$\begin{aligned} & \left| A(x) \int_{-1}^x e_1(z) dz + B(x) \int_{-1}^{\vartheta(x)} e_1(z) dz + \int_{-1}^x R_1(x, z)e_0(z) dz + \int_{-1}^{\vartheta(x)} R_2(x, z)e_0(z) dz \right| \\ & \leq C_1 \int_{-1}^x e_1(z) dz + C_2 \int_{-1}^{\vartheta(x)} e_1(z) dz \leq C \int_{-1}^x e_1(z) dz. \end{aligned} \quad (55)$$

Then, by Lemma 4,  $e_1(x)$  in (51) can be estimated as follows:

$$\|e_1(x)\|_{L^\infty(-1,1)} \leq C \left( \sum_{j=0}^1 \|I_N E_j(x)\|_{L^\infty(-1,1)} + \sum_{j=2}^6 \|E_j(x)\|_{L^\infty(-1,1)} \right). \quad (56)$$

We estimate each term of the right-hand side of the above inequality one by one.

First we estimate  $\|I_N E_0(x)\|_{L^\infty(-1,1)}$ . By (33) we have

$$\|I_N E_0(x)\|_{L^\infty(-1,1)} \leq C \log(N+1) \|E_0(x)\|_{L^\infty(-1,1)}. \quad (57)$$

We estimate  $\|E_0(x)\|_{L^\infty(-1,1)}$ . Note that  $E_0(x)$  can be written as

$$E_0(x) = \begin{cases} \int_{-1}^{\vartheta(x)} R_2(x, z)\psi(z) dz - \frac{\vartheta(x)+1}{2} \lambda_2(x), & x \in \delta_0, \\ \sum_{r=0}^{\mu-2} \left( \int_{\eta_r}^{\eta_{r+1}} R_2(x, z)u_r(z) dz - \frac{\eta_{r+1}-\eta_r}{2} \lambda_1^r(x) \right) \\ \quad + \int_{\eta_{\mu-1}}^{\vartheta(x)} R_2(x, z)u_{\mu-1}(z) dz - \frac{\eta_\mu - \eta_{\mu-1}}{2} \frac{\tilde{\vartheta}(x)+1}{2} \lambda_3(x), & x \in \delta_\mu, \mu > 0. \end{cases} \quad (58)$$

Lemma 3 helps to deduce that

$$\begin{aligned} & \left| \int_{-1}^{\vartheta(x)} R_2(x, z)\psi(z) dz - \frac{\vartheta(x)+1}{2} \lambda_2(x) \right| \\ & \leq CN^{-m} \left\| \partial_v^m (R_2(x, z(-1, \vartheta(x), \cdot))\psi(z(-1, \vartheta(x), \cdot))) \right\|_{L^2(-1,1)} \\ & \leq CN^{-m} \left| \frac{\vartheta(x)+1}{2} \right|^m \left\| \partial_z^m (R_2(x, z)\psi(z)) \Big|_{z=z(-1, \vartheta(x), \cdot)} \right\|_{L^2(-1,1)} \\ & \leq CN^{-m} \left\| \partial_z^m (R_2(x, \cdot)\psi(\cdot)) \right\|_{L^2(\vartheta(x), -1)}, \quad x \in \delta_0, \end{aligned} \quad (59)$$

and, for  $x \in \delta_\mu$ ,  $\mu > 0$ ,

$$\begin{aligned}
 & \left| \int_{\eta_r}^{\eta_{r+1}} R_2(x, z) u_r(z) dz - \frac{\eta_{r+1} - \eta_r}{2} \lambda_1^r(x) \right| \\
 & \leq CN^{-m} \left| \frac{\eta_{r+1} - \eta_r}{2} \right| \left\| \partial_v^m (R_2(x, z_r(x, \cdot))) \right\|_{L^2(-1,1)} \|u_r(z_r(\cdot))\|_{L^2(-1,1)} \\
 & \leq CN^{-m} \left| \frac{\eta_{r+1} - \eta_r}{2} \right|^{m+1/2} \left\| \partial_z^m (R_2(x, z)) \Big|_{z=z_r(\cdot)} \right\|_{L^2(-1,1)} \|u_r\|_{L^2(\delta_r)} \\
 & \leq CN^{-m} \left\| \partial_z^m (R_2(x, \cdot)) \right\|_{L^2(\delta_r)} \|u_r\|_{L^2(\delta_r)}.
 \end{aligned} \tag{60}$$

Similarly,

$$\begin{aligned}
 & \left| \int_{\eta_{\mu-1}}^{\vartheta(x)} R_2(x, z) u_{\mu-1}(z) dz - \frac{\eta_\mu - \eta_{\mu-1}}{2} \frac{\tilde{\vartheta}_{\mu-1}(x) + 1}{2} \lambda_3(x) \right| \\
 & \leq CN^{-m} \left\| \partial_z^m (R_2(x, \cdot)) \right\|_{L^2(\eta_{\mu-1}, \vartheta(x))} \|u_{\mu-1}\|_{L^2(\delta_{\mu-1})}.
 \end{aligned} \tag{61}$$

By the Cauchy inequality, which states that

$$\sum_{r=0}^{\mu-1} a_r b_r \leq \left( \sum_{r=0}^{\mu-1} a_r^2 \right)^{1/2} \left( \sum_{r=0}^{\mu-1} b_r^2 \right)^{1/2},$$

in which we let

$$\begin{aligned}
 a_r &= \left\| \partial_z^m (R_2(x, \cdot)) \right\|_{L^2(\delta_r)}, \quad b_r = \|u_r\|_{L^2(\delta_r)}, \quad r = 0, 1, \dots, \mu - 2, \\
 a_{\mu-1} &= \left\| \partial_z^m (R_2(x, \cdot)) \right\|_{L^2(\eta_{\mu-1}, \vartheta(x))}, \quad b_{\mu-1} = \|u_{\mu-1}\|_{L^2(\delta_{\mu-1})},
 \end{aligned}$$

we have, for  $x \in \delta_\mu$ ,  $\mu > 0$ ,

$$\begin{aligned}
 |E_0(x)| &\leq CN^{-m} \left\| \partial_z^m (R_2(x, \cdot)) \right\|_{L^2(-1, \vartheta(x))} \|u^N\|_{L^2(-1,1)} \\
 &\leq CN^{-m} \left\| \partial_z^m (R_2(x, \cdot)) \right\|_{L^2(-1, \vartheta(x))} (\|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}).
 \end{aligned} \tag{62}$$

Then

$$\|E_0(x)\|_{L^\infty(-1,1)} \leq CN^{-m} \tilde{R}_2 (\|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}), \tag{63}$$

where

$$\tilde{R}_2 := \max \left\{ \max_{x \in \delta_0} \left\| \partial_z^m (R_2(x, \cdot)) \psi(\cdot) \right\|_{L^2(\vartheta(x), -1)}, \max_{x \in [\eta_1, 1]} \left\| \partial_z^m (R_2(x, \cdot)) \right\|_{L^2(-1, \vartheta(x))} \right\}.$$

Therefore, combining (63) with (57) gives

$$\|I_N E_0(x)\|_{L^\infty(-1,1)} \leq CN^{-m} \log(N+1) \tilde{R}_2 (\|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}). \tag{64}$$

Using the same analysis as for  $\|I_N E_0(x)\|_{L^\infty(-1,1)}$ , we can obtain the estimate

$$\|I_N E_1(x)\|_{L^\infty(-1,1)} \leq CN^{-m} \log(N+1) \tilde{R}_1 (\|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}), \tag{65}$$

where

$$\tilde{R}_1 := \max_{x \in [-1, 1]} \|\partial_z^m (R_1(x, \cdot))\|_{L^2(-1, x)}.$$

Now we begin to estimate  $\|E_j(x)\|_{L^\infty(-1, 1)}$ ,  $j = 2, 3, 4, 5, 6$ . Note that, in each subinterval  $\delta_\mu$ ,  $\mu = 0, 1, \dots, M$ ,  $u|_{\delta_\mu}(x) \in H^m(\delta_\mu)$ . Applying (32) to  $u'(x)$ , we have

$$\|E_2(x)\|_{L^\infty(-1, 1)} \leq CN^{1/2-m} \|u^{(m+1)}\|_{L^2(\Lambda_h)}. \quad (66)$$

Now we begin to estimate  $\|E_4(x)\|_{L^\infty(-1, 1)}$ . For simplicity of notation, we set

$$b(x) := \int_{-1}^{\vartheta(x)} R_2(x, z) e_0(z) dz.$$

Applying (32) with  $m = 1$  to  $b(x)$  yields

$$\|(I_N - I)b(x)\|_{L^\infty(-1, 1)} \leq CN^{-1/2} \|\partial_x^1 b\|_{L^2(-1, 1)}. \quad (67)$$

Note that

$$\begin{aligned} |\partial_x^1 b(x)| &= \left| R_2(x, \vartheta(x)) e_0(\vartheta(x)) \vartheta'(x) + \int_{-1}^{\vartheta(x)} \frac{\partial R_2}{\partial x}(x, z) e_0(z) dz \right| \\ &\leq \|e_0\|_{L^\infty(-1, 1)} \left| \left( R_2(x, \vartheta(x)) \vartheta'(x) + \int_{-1}^{\vartheta(x)} \frac{\partial R_2}{\partial x}(x, z) dz \right) \right| \\ &\leq C \|e_0\|_{L^\infty(-1, 1)}, \end{aligned} \quad (68)$$

which, together with (67), yields

$$\|E_4(x)\|_{L^\infty(-1, 1)} = \|(I_N - I)b(x)\|_{L^\infty(-1, 1)} \leq CN^{-1/2} \|e_0\|_{L^\infty(-1, 1)}. \quad (69)$$

Similarly,

$$\begin{aligned} \|E_3(x)\|_{L^\infty(-1, 1)} &\leq CN^{-1/2} \|e_0\|_{L^\infty(-1, 1)}, \\ \|E_5(x)\|_{L^\infty(-1, 1)} &\leq CN^{-1/2} \|e_1\|_{L^\infty(-1, 1)}, \\ \|E_6(x)\|_{L^\infty(-1, 1)} &\leq CN^{-1/2} \|e_1\|_{L^\infty(-1, 1)}. \end{aligned} \quad (70)$$

Combining (56) with (64), (65), (66), (69) and (70) yields that

$$\begin{aligned} \|e_1(x)\|_{L^\infty(-1, 1)} &\leq CN^{-m} (\log(N+1) R \|u\|_{L^\infty(-1, 1)} + N^{1/2} \|u^{(m+1)}\|_{L^2(\Lambda_h)}) \\ &\quad + CN^{-1/2} \|e_0\|_{L^\infty(-1, 1)}, \end{aligned} \quad (71)$$

where

$$R := \max\{\tilde{R}_1, \tilde{R}_2\}.$$

Now we need another relation between  $\|e_1(x)\|_{L^\infty(-1, 1)}$  and  $\|e_0\|_{L^\infty(-1, 1)}$ . Multiplying by  $F_i^\mu(x)$  on both sides of (47) and summing from  $i = 0$  to  $N$  for

$\mu = 0, 1, \dots, M$ , we obtain that

$$e_0(x) = E_7(x) + (I_N - I) \left( \int_{-1}^x e_1(s) ds \right) + \int_{-1}^x e_1(s) ds, \quad (72)$$

where

$$E_7(x) := (I - I_N)u(x).$$

Then

$$\begin{aligned} & \|e_0\|_{L^\infty(-1,1)} \\ & \leq C \left( \|E_7(x)\|_{L^\infty(-1,1)} + \left\| (I_N - I) \left( \int_{-1}^x e_1(s) ds \right) \right\|_{L^\infty(-1,1)} + \|e_1\|_{L^\infty(-1,1)} \right). \end{aligned}$$

Using (32) for  $E_7(x)$ , and applying (32) with  $m = 1$  to the middle term of the right-hand side of the above inequality, we have

$$\|e_0\|_{L^\infty(-1,1)} \leq CN^{-m-1/2} \|u^{(m+1)}\|_{L^2(\Lambda_h)} + C \|e_1\|_{L^\infty(-1,1)}. \quad (73)$$

Plugging the above result into the last term of (71) yields the desired estimate (45) for  $e_1$ , which, in turn, substituted into the last term of (73), yields the estimate (45) for  $e_0$ .  $\square$

Next, we will give the error estimate in the  $L^2(-1, 1)$  space.

**Theorem 2.** *Let  $u(x)$  be the exact solution to (5). Let  $u^N(x)$  be the approximate solution, and  $\rho^N(x)$  be the approximate derivative obtained by using the spectral-collocation schemes (17), (19) and (22). Then, for  $N \geq m - 1$  sufficiently large,*

$$\|e_i\|_{L^2(-1,1)} \leq CN^{-m} R(R+1) (\|u\|_{L^\infty(-1,1)} + \|u^{(m+1)}\|_{L^2(\Lambda_h)}), \quad i = 0, 1. \quad (74)$$

*Proof.* By Lemma 4, it follows from (51) and (55) that

$$\|e_1(x)\|_{L^2(-1,1)} \leq C \left( \sum_{j=0}^1 \|I_N E_j(x)\|_{L^2(-1,1)} + \sum_{j=2}^6 \|E_j(x)\|_{L^2(-1,1)} \right). \quad (75)$$

We estimate each term on the right of the above inequality one by one. Applying (34) to  $E_0(x)$  yields

$$\|I_N E_0(x)\|_{L^2(-1,1)} \leq C \|E_0(x)\|_{L^\infty(-1,1)}. \quad (76)$$

Recalling the result (63) and using the result of Theorem 1, we obtain that

$$\|I_N E_0(x)\|_{L^2(-1,1)} \leq CN^{-m} R(R+1) (\|u\|_{L^\infty(-1,1)} + \|u^{(m+1)}\|_{L^2(\Lambda_h)}). \quad (77)$$

Similarly,

$$\|I_N E_1(x)\|_{L^2(-1,1)} \leq CN^{-m} R(R+1) (\|u\|_{L^\infty(-1,1)} + \|u^{(m+1)}\|_{L^2(\Lambda_h)}). \quad (78)$$

Note that, in each subinterval  $\delta_\mu$ ,  $\mu = 0, 1, \dots, M$ ,  $u'|_{\delta_\mu}(x) \in H^m(\delta_\mu)$ . Applying (31) to  $u'(x)$ , we have

$$\|E_2(x)\|_{L^2(-1,1)} \leq CN^{-m} \|u^{(m+1)}\|_{L^2(\Lambda_h)}. \quad (79)$$

Applying the analysis from (67)–(69), using (31) in Lemma 2 with  $m = 1$  for  $b(x)$  yields

$$\|E_4\|_{L^2(-1,1)} = \|(I - I_N)b(x)\|_{L^2(-1,1)} \leq CN^{-1} \|e_0\|_{L^\infty(-1,1)}. \quad (80)$$

Using the estimate for  $e_0$  in Theorem 1 makes the above inequality become

$$\|E_4\|_{L^2(-1,1)} \leq CN^{-m-1/2} (R\|u\|_\infty + \|u^{(m+1)}\|_{L^2(\Lambda_h)}). \quad (81)$$

Similarly,

$$\|E_3\|_{L^2(-1,1)} \leq CN^{-m-1/2} (R\|u\|_\infty + \|u^{(m+1)}\|_{L^2(\Lambda_h)}). \quad (82)$$

Using the same analysis from (67)–(69), using (31) in Lemma 2 with  $m = 1$  we obtain

$$\|E_i\|_{L^2(-1,1)} \leq CN^{-1} \|e_1\|_{L^\infty(-1,1)}, \quad i = 5, 6. \quad (83)$$

Combining (75) with (77), (78), (79), (81), (82) and (83) we obtain the estimate (74) for  $e_1$ .

Now we begin to estimate  $\|e_0\|_{L^2(-1,1)}$ . From (72) we have

$$\|e_0\|_{L^2(-1,1)} \leq C \left( \|E_7(x)\|_{L^2(-1,1)} + \left\| (I_N - I) \int_{-1}^x e_1(s) ds \right\|_{L^2(-1,1)} + \|e_1\|_{L^2(-1,1)} \right).$$

Using (31) for  $E_7(x)$ , and applying (31) with  $m = 1$  to the middle term of the right-hand side of the above inequality, we have

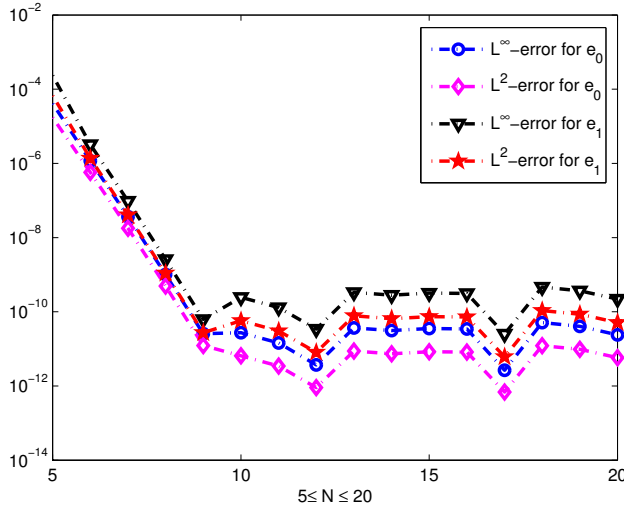
$$\|e_0\|_{L^2(-1,1)} \leq CN^{-m-1} \|u^{(m+1)}\|_{L^2(\Lambda_h)} + C \|e_1\|_{L^2(-1,1)}, \quad (84)$$

which leads to the estimate (74) for  $e_0$  by plugging the result (74) for  $e_1$  into the last term of (84).  $\square$

## 6. Numerical examples

In this section, we give four numerical examples. The first one is the linear case with smooth solution. The second one is the linear case with solution unsmooth at the primary discontinuous points. The third one is the nonlinear case. The fourth one is the case in which the delay is a function of the solution to the equations. These examples confirm the theoretical results obtained in the previous section.





**Figure 1.** Example 1: Errors versus  $N$  in  $L^\infty$  and  $L^2$  norms.

**Example 1.** Consider (1) with

$$T = 4, \quad a(t) = t, \quad b(t) = t^2, \quad K_1(t, s) = \sin(t + s), \quad (85)$$

$$K_2(t, s) = \cos(t + s), \quad \theta(t) = t - 1, \quad \phi(t) = e^t,$$

$$g(t) = e^t - te^t - t^2e^{t-1} + \sin t - \frac{1}{2}[e^t(\sin 2t - \cos 2t) + e^{t-1}(\cos(2t-1) + \sin(2t-1))].$$

The corresponding exact solution is  $y(t) = e^t, t \in (0, T)$ .

Figure 1 plots the errors for  $5 \leq N \leq 20$  in both  $L^\infty$  and  $L^2$  norms. Moreover, the corresponding errors versus several values of  $N$  are displayed in Table 1. As expected, the errors decay exponentially, which confirms our theoretical predictions. This example shows that our method is also valid for the nonvanishing delay VIDES with smooth solution.

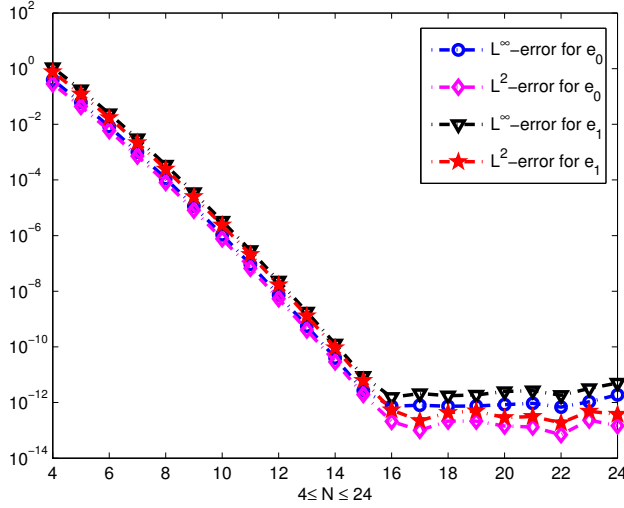
**Example 2.** Consider (1) with

$$T = 3, \quad a(t) = 0, \quad b(t) = g(t) = e^t, \quad K_1(t, s) = 0,$$

$$K_2(t, s) = e^{t+s}, \quad \theta(t) = t - \left(\frac{1}{2} + \frac{1}{2}t\right), \quad \phi(t) = 1.$$

$N$	5	8	11	14	17	20
$L^\infty$ -error for $e_0$	$4.04 \cdot 10^{-05}$	$1.01 \cdot 10^{-09}$	$1.45 \cdot 10^{-11}$	$3.11 \cdot 10^{-11}$	$2.66 \cdot 10^{-12}$	$2.38 \cdot 10^{-11}$
$L^2$ -error for $e_0$	$1.71 \cdot 10^{-05}$	$4.92 \cdot 10^{-10}$	$3.44 \cdot 10^{-12}$	$7.37 \cdot 10^{-12}$	$6.87 \cdot 10^{-13}$	$5.81 \cdot 10^{-12}$
$L^\infty$ -error for $e_1$	$2.32 \cdot 10^{-04}$	$2.58 \cdot 10^{-09}$	$1.30 \cdot 10^{-10}$	$2.81 \cdot 10^{-10}$	$2.53 \cdot 10^{-11}$	$2.20 \cdot 10^{-10}$
$L^2$ -error for $e_1$	$6.65 \cdot 10^{-05}$	$1.10 \cdot 10^{-09}$	$3.03 \cdot 10^{-11}$	$6.53 \cdot 10^{-11}$	$6.09 \cdot 10^{-12}$	$5.12 \cdot 10^{-11}$

**Table 1.** Example 1: Errors versus  $N$  in  $L^\infty$  and  $L^2$  norms.



**Figure 2.** Example 2: Errors versus  $N$  in  $L^\infty$  and  $L^2$  norms.

The corresponding exact solution is

$$y(t) = \begin{cases} e^t + \frac{2}{3}e^{-1/2}(e^{3t/2} - 1), & t \in (0, 1], \\ y_1(t), & t \in (1, 3], \end{cases}$$

where

$$y_1(t) := \frac{2}{3}e^{(3t-1)/2} + \frac{8}{21}e^{(7t-5)/4} + \frac{1}{2}e^t + \frac{1}{4}e^{2t-1} + \frac{16}{135}e^{(9t-7)/4} - \frac{4}{15}e^{t-1/2} - \frac{4}{9}e^{(3t-2)/2} + \frac{1}{4}e - \frac{40}{189}e^{1/2} - \frac{2}{3}e^{-1/2}. \quad (86)$$

It is worth noting that the solution of this example possesses primary discontinuities at  $t = 0, 1$ , where

$$0 = y^{(k)}(0-) \neq y^{(k)}(0+) \quad \text{and} \quad y^{(k)}(1-) \neq y^{(k)}(1+), \quad k \geq 1.$$

Figure 2 plots the errors for  $4 \leq N \leq 24$  in both  $L^\infty$  and  $L^2$  norms. The corresponding errors versus several values of  $N$  are displayed in Table 2. The spectral accuracy is obtained although the solution for the equation is unsmooth at the primary discontinuous points.

$N$	4	8	12	16	20	24
$L^\infty$ -error for $e_0$	$3.99 \cdot 10^{-01}$	$1.12 \cdot 10^{-04}$	$7.43 \cdot 10^{-09}$	$7.39 \cdot 10^{-13}$	$8.38 \cdot 10^{-13}$	$1.89 \cdot 10^{-12}$
$L^2$ -error for $e_0$	$2.86 \cdot 10^{-01}$	$8.06 \cdot 10^{-05}$	$5.37 \cdot 10^{-09}$	$2.10 \cdot 10^{-13}$	$1.42 \cdot 10^{-13}$	$1.46 \cdot 10^{-13}$
$L^\infty$ -error for $e_1$	$1.10 \cdot 10^{-00}$	$3.42 \cdot 10^{-04}$	$2.37 \cdot 10^{-08}$	$1.53 \cdot 10^{-12}$	$2.50 \cdot 10^{-12}$	$5.06 \cdot 10^{-12}$
$L^2$ -error for $e_1$	$7.65 \cdot 10^{-01}$	$2.44 \cdot 10^{-04}$	$1.69 \cdot 10^{-08}$	$5.39 \cdot 10^{-13}$	$2.86 \cdot 10^{-13}$	$3.75 \cdot 10^{-13}$

**Table 2.** Example 2: Errors versus  $N$  in  $L^\infty$  and  $L^2$  norms.

For the nonlinear VIDes with nonvanishing delay of the form

$$\begin{aligned}
 y'(t) &= d(t, y(t), y(\theta(t))) + \int_0^t K_1(t, s, y(s)) ds + \int_0^{\theta(t)} K_2(t, s, y(s)) ds, \\
 & \qquad \qquad \qquad t \in (0, T], \\
 y(t) &= \phi(t), \quad t \in [\theta(0), 0],
 \end{aligned} \tag{87}$$

we can design a spectral-collocation method similar to the linear case. Equation (87) can be written as

$$\begin{aligned}
 u'(x) &= h(x, u(x), u(\vartheta(x))) + \int_{-1}^x R_1(x, z, u(z)) dz + \int_{-1}^{\vartheta(x)} R_2(x, z, u(z)) dz, \\
 & \qquad \qquad \qquad z \in (-1, 1], \\
 u(x) &= \psi(x), \quad x \in [\vartheta(-1), -1],
 \end{aligned} \tag{88}$$

where

$$\begin{aligned}
 u(x) &:= y(t(x)), \quad \vartheta(x) := \frac{2}{T}\theta(t(x)) - 1, \quad \psi(x) := \phi(t(x)), \\
 h(x, u(x), u(\vartheta(x))) &= \frac{T}{2}d(t(x), y(t(x)), y(\theta(t(x)))), \\
 R_1(x, z, u(z)) &:= \left(\frac{T}{2}\right)^2 K_1(t(x), s(z), y(s(z))), \\
 R_2(x, z, u(z)) &:= \left(\frac{T}{2}\right)^2 K_2(t(x), s(z), y(s(z))).
 \end{aligned} \tag{89}$$

We assume that (88) holds at the collocation points  $x_i^\mu$ , where  $i = 0, 1, \dots, n$  and  $\mu = 0, 1, \dots, M$ :

$$\begin{aligned}
 u'(x_i^\mu) &= h(x_i^\mu, u(x_i^\mu), u(\vartheta(x_i^\mu))) + \int_{-1}^{x_i^\mu} R_1(x_i^\mu, z, u(z)) dz \\
 & \quad + \int_{-1}^{\vartheta(x_i^\mu)} R_2(x_i^\mu, z, u(z)) dz \\
 &= h(x_i^\mu, u(x_i^\mu), u(\vartheta(x_i^\mu))) + \sum_{r=0}^{\mu-1} \int_{\eta_r}^{\eta_{r+1}} R_1(x_i^\mu, z, u(z)) dz \\
 & \quad + \int_{\eta_\mu}^{x_i^\mu} R_1(x_i^\mu, z, u(z)) dz \\
 & \quad + \sum_{r=0}^{\mu-2} \int_{\eta_r}^{\eta_{r+1}} R_2(x_i^\mu, z, u(z)) dz + \int_{\eta_{\mu-1}}^{\vartheta(x_i^\mu)} R_2(x_i, z)u(z) dz.
 \end{aligned} \tag{90}$$

We use  $u_i^\mu$  to approximate  $u(x_i^\mu)$ ,  $v_i^\mu$  to approximate  $u(\vartheta(x_i^\mu))$ ,  $\rho_i^\mu$  to approximate  $u'(x_i^\mu)$ ,  $i = 0, 1, \dots, N$ ,  $\mu = 0, 1, \dots, M$ , and use  $u^N(x)$  to approximate  $u(x)$ ,

$\rho^N(x)$  to approximate  $u'(x)$ . Similarly to (17), the numerical scheme for (88) is

$$\rho_i^\mu = h(x_i^\mu, u_i^\mu, v_i^\mu) + \varpi(x_i^\mu) + \gamma(x_i^\mu), \quad (91)$$

where

$$\begin{aligned} \varpi(x_i^\mu) &= \sum_{r=0}^{\mu-1} \frac{\eta_{r+1} - \eta_r}{2} \varpi_1^r(x_i^\mu) + \frac{\eta_{\mu+1} - \eta_\mu}{2} \frac{x_i + 1}{2} \varpi_3(x_i^\mu), \quad \mu \geq 0, \\ \varpi_1^r(x_i^\mu) &:= \sum_{k=0}^N R_1(x_i^\mu, z_r(v_k), u_k^r) \omega_k, \quad r = 0, 1, \dots, \mu - 1, \\ \varpi_3(x_i^\mu) &:= \sum_{k=0}^N R_1\left(x_i^\mu, z_\mu(z(-1, x_i, v_k)), \sum_{j=0}^N u_j^\mu F_j(z(-1, x_i, v_k))\right) \omega_k, \end{aligned}$$

and

$$\begin{aligned} \gamma(x_i^\mu) &= \begin{cases} \frac{\vartheta(x_i^0) + 1}{2} \gamma_2(x_i^0), & \mu = 0, \\ \sum_{r=0}^{\mu-2} \frac{\eta_{r+1} - \eta_r}{2} \gamma_1^r(x_i^\mu) + \frac{\eta_\mu - \eta_{\mu-1}}{2} \frac{\tilde{\vartheta}(x_i^\mu) + 1}{2} \gamma_3(x_i^\mu), & \mu > 0, \end{cases} \\ \gamma_1^r(x_i^\mu) &:= \sum_{k=0}^N R_2(x_i^\mu, z_r(v_k), u_k^r) \omega_k, \quad r = 0, 1, \dots, \mu - 2, \\ \gamma_2(x_i^0) &:= \sum_{k=0}^N R_2(x_i^0, z(-1, \vartheta(x_i^0), v_k), \psi(z(-1, \vartheta(x_i^0), v_k))) \omega_k, \\ \gamma_3(x_i^\mu) &:= \sum_{k=0}^N R_2\left(x_i^\mu, z_{\mu-1}(z(-1, \tilde{\vartheta}(x_i^\mu), v_k)), \sum_{j=0}^N u_j^{\mu-1} F_j(z(-1, \tilde{\vartheta}(x_i^\mu), v_k))\right) \omega_k. \end{aligned}$$

Combining (19) with (22) and (91), we obtain the numerical scheme for the nonlinear VIDEs (87).

**Example 3.** Consider (87) with

$$\begin{aligned} T = 2, \quad \phi(t) = 1, \quad \theta(t) = t - 1, \quad d(t, y(t), y(\theta(t))) &= y^2(\theta(t)) - 1 - e^{2t-1} + 2e^t, \\ K_1(t, s, y(s)) = 0, \quad K_2(t, s, y(s)) &= e^{t+s} y^2(s). \end{aligned} \quad (92)$$

The corresponding exact solution is

$$y(t) = \begin{cases} e^t, & t \in (0, 1], \\ \frac{1}{2}e^{2t-2} - \frac{1}{2}e^{2t-1} + \frac{5}{3}e^t + \frac{1}{12}e^{4t-3} - t + \frac{1}{4}e + \frac{1}{2}, & t \in (1, 2]. \end{cases}$$

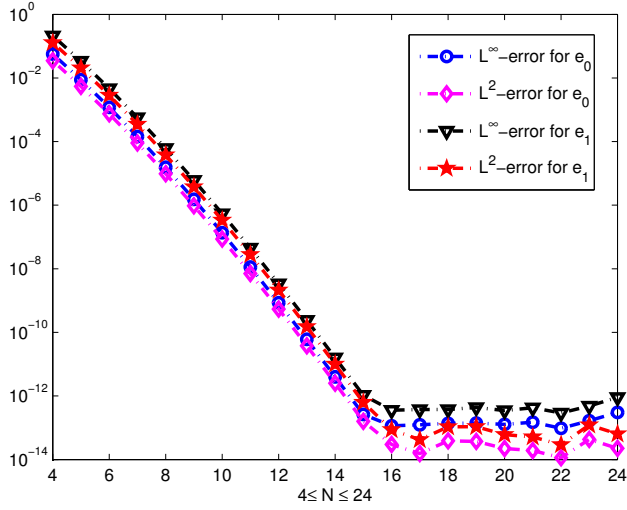
In this example, the primary discontinuous points are  $t = 0, 1$ , where

$$0 = y^{(k)}(0-) \neq y^{(k)}(0+) \quad \text{and} \quad y^{(k)}(1-) \neq y^{(k)}(1+), \quad k \geq 1.$$

Numerical errors versus several values of  $N$  are displayed in Table 3 and Figure 3. These results indicate that the desired spectral accuracy is obtained.

$N$	4	8	12	16	20	24
$L^\infty$ -error for $e_0$	$5.67 \cdot 10^{-02}$	$1.56 \cdot 10^{-05}$	$8.55 \cdot 10^{-10}$	$1.15 \cdot 10^{-13}$	$1.28 \cdot 10^{-13}$	$3.06 \cdot 10^{-13}$
$L^2$ -error for $e_0$	$3.51 \cdot 10^{-02}$	$9.75 \cdot 10^{-06}$	$5.36 \cdot 10^{-10}$	$3.00 \cdot 10^{-14}$	$2.23 \cdot 10^{-14}$	$2.24 \cdot 10^{-14}$
$L^\infty$ -error for $e_1$	$2.12 \cdot 10^{-01}$	$6.13 \cdot 10^{-05}$	$3.40 \cdot 10^{-09}$	$3.55 \cdot 10^{-13}$	$3.48 \cdot 10^{-13}$	$9.02 \cdot 10^{-13}$
$L^2$ -error for $e_1$	$1.28 \cdot 10^{-01}$	$3.80 \cdot 10^{-05}$	$2.11 \cdot 10^{-09}$	$8.71 \cdot 10^{-14}$	$6.06 \cdot 10^{-14}$	$6.39 \cdot 10^{-14}$

**Table 3.** Example 3: Errors versus  $N$  in  $L^\infty$  and  $L^2$  norms.



**Figure 3.** Example 3: Errors versus  $N$  in  $L^\infty$  and  $L^2$  norms.

**Example 4.** Now we consider the case where the delay is a function of the solution; i.e.,

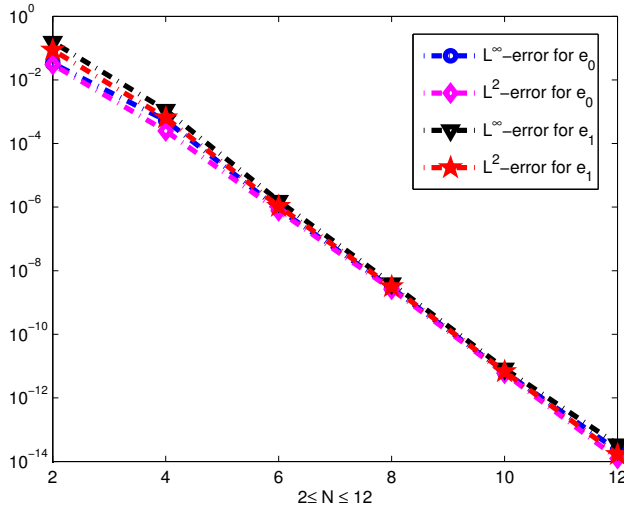
$$\begin{aligned}
 y'(t) &= d(t, y(t), y(\theta(t, y(t)))) + \int_0^t K_1(t, s, y(s)) ds + \int_0^{\theta(t, y(t))} K_2(t, s, y(s)) ds, \\
 & \qquad \qquad \qquad t \in [0, T], \\
 y(t) &= \phi(t), \quad t \in [\theta(0, y(0)), 0].
 \end{aligned} \tag{93}$$

If we take

$$\begin{aligned}
 T &= 2, \quad \phi(t) = 2.5, \quad \theta(t, y(t)) = t - y(t), \\
 d(t, y(t), y(\theta(t, y(t)))) &= y^2(t) + y^2(\theta(t, y(t))) + g(t), \\
 g(t) &= -\frac{1}{2}e^{-t} - \frac{1}{4}(e^{-t} + 4)^2 \\
 &\quad - \frac{1}{4}e^t \left( -\frac{1}{3}e^{-3t} - 4e^{-2t} - 16e^{-t} + \frac{1}{3} + 20 \right) - (2.5)^2 \left( t - \frac{1}{2}e^{-t} - 1 \right), \\
 K_1(t, s, y(s)) &= e^{t-s} y^2(s), \quad K_2(t, s, y(s)) = y^2(s),
 \end{aligned} \tag{94}$$

$N$	2	4	6	8	10	12
$L^\infty$ -error for $e_0$	$3.32 \cdot 10^{-02}$	$5.33 \cdot 10^{-04}$	$8.78 \cdot 10^{-07}$	$2.87 \cdot 10^{-09}$	$6.54 \cdot 10^{-12}$	$2.22 \cdot 10^{-14}$
$L^2$ -error for $e_0$	$3.05 \cdot 10^{-02}$	$2.50 \cdot 10^{-04}$	$7.65 \cdot 10^{-07}$	$2.62 \cdot 10^{-09}$	$5.91 \cdot 10^{-12}$	$1.24 \cdot 10^{-14}$
$L^\infty$ -error for $e_1$	$1.57 \cdot 10^{-01}$	$1.14 \cdot 10^{-03}$	$1.56 \cdot 10^{-06}$	$3.99 \cdot 10^{-09}$	$8.29 \cdot 10^{-12}$	$3.42 \cdot 10^{-14}$
$L^2$ -error for $e_1$	$8.73 \cdot 10^{-02}$	$6.41 \cdot 10^{-04}$	$1.06 \cdot 10^{-06}$	$3.16 \cdot 10^{-09}$	$6.75 \cdot 10^{-12}$	$1.66 \cdot 10^{-14}$

**Table 4.** Example 4: Errors versus  $N$  in  $L^\infty$  and  $L^2$  norms.



**Figure 4.** Example 4: Errors versus  $N$  in  $L^\infty$  and  $L^2$  norms.

then the corresponding exact solution is

$$y(t) = \begin{cases} 2.5, & t \in [-1, 0], \\ \frac{1}{2}(e^{-t} + 4), & t \in (0, 2]. \end{cases}$$

This solution possesses a primary discontinuous point  $t = 0$  where  $y^{(k)}(0-) = 0$  while  $y^{(k)}(0+) = (-1)^k \frac{1}{2}$ ,  $k \geq 1$ .

We use the Newton iterative method to solve the nonlinear discrete system corresponding to this example. Errors versus  $N$  are listed in Table 4 and plotted in Figure 4 from which we can see that the spectral accuracy is obtained. This example shows that our method can also handle the case where the delay is a function of the solution.

## 7. Conclusion and future work

We propose the Legendre spectral-collocation method to solve VIDEs with nonvanishing delay, and provide convergence analysis for the proposed method. Numerical

examples are provided to confirm the theoretical results that the numerical errors decay exponentially. The main difficulty in applying the spectral method to VIDES with nonvanishing delay is the solution of this equation possesses primary discontinuous points associated with the nonvanishing delay function. We overcome this difficulty by dividing the global definition domain of the solution into several subintervals where the solution is smooth enough. Then spectral method can be used to approximate the solution in each subinterval.

Our future work will focus on the spectral method for the Volterra functional integral and differential integral equation with nonvanishing delay.

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
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