ON THE NUMBER OF IRREDUCIBLE COMPONENTS OF LOCAL DEFORMATION RINGS IN THE UNEQUAL CHARACTERISTIC CASE

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ABSTRACT. Fix two odd primes $l \neq p$ and let L be a finite extension of $\mathbb{Q}_l$ and K be a finite extension of $\mathbb{Q}_p$ with residue field k. Let $\bar{\rho}$ be a continuous representation of $\mathrm{Gal}(\bar{\mathbb{Q}}_l/L)$ on a two-dimensional k-vector space. Using results of M. Kisin ([4], [3]) and V. Pilloni ([5]), we determine the number of geometrically irreducible components of the K-scheme $X = \mathrm{Spec}\, R_{\bar{\rho}}^{\square,\psi}[\frac{1}{p}]$. We prove moreover that each component has a point rational over $K(\zeta_{p^{\delta+1}})$, where p^{δ} is the number of p-power roots of unity contained in the quadratic extension of the residue field of L. Furthermore, for each component $\mathcal{C}$ of X, we determine the set of $\bar{K}[G]$-isomorphism classes of $\mathcal{O}_{\bar{K}}$-lifts of $\bar{\rho}$ associated to the closed points of $\mathcal{C}$.

1. INTRODUCTION

In this paper we make use of results of M. Kisin ([4], [3]) and V. Pilloni ([5]) to determine the number and some rationality properties of the geometrically irreducible components of the deformation space associated to a local Galois representation of dimension two, in the $l \neq p$ case.

Fix two distinct odd primes l and p and let L be a finite extension of $\mathbb{Q}_l$ and K be a finite extension of $\mathbb{Q}_p$. Denote by k the residue field of K and fix a continuous representation $\bar{\rho}$ of $G := \mathrm{Gal}(\bar{\mathbb{Q}}_l/L)$ over a two dimensional k-vector space. Since p is odd, to study the deformations of $\bar{\rho}$ to complete local noetherian $\mathcal{O}_K$-algebras it is enough to study deformations with fixed determinant ψ. Without losing generality we can also change $\bar{\rho}$ by a twist and assume that it is either absolutely reducible or absolutely irreducible.

Denote by X the generic fiber of $\mathrm{Spec}\, R_{\bar{\rho}}^{\square,\psi}$. It is well known that X is a three dimensional reduced K-scheme and results of Kisin and Pilloni imply that any geometrically irreducible component $\mathcal{C}$ of X admits one of the following descriptions: (I) there exists a character $\gamma : G \to \bar{K}^{\times}$ such that for any $x \in \mathcal{C}(\bar{K})$, the corresponding $\bar{K}$-representation ρ_x is an extension of γ by $\gamma(1)$; (II) there exists a character $\gamma : G \to \bar{K}^{\times}$ such that for any $x \in \mathcal{C}(\bar{K})$, $\gamma^{-1} \otimes \rho_x$ is unramified; (III) there exists an irreducible representation $\sigma : G \to GL_2(\bar{K})$ such that for any $x \in \mathcal{C}(\bar{K})$, $\rho_x \simeq \sigma$; (IV) there are characters $\eta, \lambda : G \to \bar{K}^{\times}$ such that $\eta\lambda^{-1}$ is ramified and for any $x \in \mathcal{C}(\bar{K})$ we have $\rho_x \simeq \eta\phi \oplus \lambda\phi^{-1}$ where ϕ is an unramified lifting of the trivial character. We say that a component of $X_{\bar{K}}$ has type I, II, III or IV depending of which of the above descriptions it fits.

For any fixed pair $(\bar{\rho}, \psi)$ as above, we determine the number of components of $X_{\bar{K}}$ of each type. At this purpose, denote by l^f the cardinality of the residue field of L and set $\alpha = \mathrm{ord}_p(l^f - 1)$, $\beta = \mathrm{ord}_p(l^f + 1)$ so that $\alpha\beta = 0$. Let χ be the p-adic cyclotomic character of G and let ω be its reduction modulo p. The symbol $\sim$ denotes Teichmüller liftings of elements in the residue field of $\bar{\mathbb{Q}}_p$. For ? in the set $\{I, II, III, IV\}$, we let $c_?$ be the number of geometrically irreducible components of type ? contained in X and we set $\mathbf{c} = (c_I, c_{II}, c_{III}, c_{IV})$. The determination of

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the vector $\mathbf{c}$ follows three different paths, according to whether ω is (A) trivial, (B) quadratic or (C) neither trivial nor quadratic. We prove (Theorems 4.0.3, 4.0.4 and 4.0.5):

Theorem 1.0.1 (A)**.** *Assume $[L(\zeta_p):L]=1$ (i.e., $\alpha>0$ and $\beta=0$).*

A1. *Let $\bar\rho=\vartheta\oplus 1$ where $\vartheta:G\to k^\times$ is a character, and set $\psi=\tilde\vartheta\chi$.*
  A1.1. *If $\vartheta=1$, then $\mathbf{c}=\left(1,1,0,\frac{p^\alpha-1}{2}\right)$.*
  A1.2. *If $\vartheta\neq 1$ is unramified, then $\mathbf{c}=(0,1,0,p^\alpha-1)$.*
  A1.3. *If ϑ is ramified, then $\mathbf{c}=(0,0,0,p^\alpha)$.*

A2. *Let $\bar\rho$ be a non-trivial extension of the trivial character by itself, and set $\psi=\chi$.*
  A2.1. *If $\bar\rho$ is unramified, then $\mathbf{c}=\left(1,1,0,\frac{p^\alpha-1}{2}\right)$.*
  A2.2. *If $\bar\rho$ is ramified, then $\mathbf{c}=\left(1,0,0,\frac{p^\alpha-1}{2}\right)$.*

A3. *Let $\bar\rho$ be absolutely irreducible and set $\psi=\widetilde{\det\bar\rho}$. Then $\mathbf{c}=(0,0,1,0)$.*

Theorem 1.0.2 (B)**.** *Assume $[L(\zeta_p):L]=2$ (i.e., $\alpha=0$ and $\beta>0$).*

B1. *Let $\bar\rho=\vartheta\oplus 1$ where $\vartheta:G\to k^\times$ is a character.*
  B1.1. *If $\vartheta=\omega$ set $\psi=\chi$. Then $\mathbf{c}=\left(2,1,\frac{p^\beta-1}{2},0\right)$.*
  B1.2. *If $\vartheta\neq\omega$ is unramified set $\psi=\tilde\vartheta$. Then $\mathbf{c}=(0,1,0,0)$.*
  B1.3. *If $\vartheta\neq\omega$ is ramified set $\psi=\tilde\vartheta$. Then $\mathbf{c}=(0,0,0,1)$.*

B2. *Let $\bar\rho$ be a non-trivial extension of the trivial character by a character ϑ.*
  B2.1. *If $\vartheta=1$ set $\psi=1$. Then $\mathbf{c}=(0,1,0,0)$.*
  B2.2. *If $\vartheta=\omega$ set $\psi=\chi$. Then $\mathbf{c}=\left(1,0,\frac{p^\beta-1}{2},0\right)$.*

B3. *Let $\bar\rho$ be absolutely irreducible and set $\psi=\widetilde{\det\bar\rho}$.*
  B3.1. *If $\bar\rho$ cannot be induced from a character of $G_{L(\zeta_p)}$, then $\mathbf{c}=(0,0,1,0)$.*
  B3.2. *If $\bar\rho$ can be induced from a character of $G_{L(\zeta_p)}$, then $\mathbf{c}=\left(0,0,p^\beta,0\right)$.*

Theorem 1.0.3 (C)**.** *Assume $[L(\zeta_p):L]>2$ (i.e., $\alpha=\beta=0$).*

C1. *Let $\bar\rho=\vartheta\oplus 1$ where $\vartheta:G\to k^\times$ is a character.*
  C1.1. *If $\vartheta=\omega^{\pm1}$ set $\psi=\chi^{\pm1}$. Then $\mathbf{c}=(1,1,0,0)$.*
  C1.2. *If $\vartheta\neq\omega^{\pm1}$ is unramified set $\psi=\tilde\vartheta$. Then $\mathbf{c}=(0,1,0,0)$.*
  C1.3. *If $\vartheta\neq\omega^{\pm1}$ is ramified set $\psi=\tilde\vartheta$. Then $\mathbf{c}=(0,0,0,1)$.*

C2. *Let $\bar\rho$ be a non-trivial extension of the trivial character by a character ϑ.*
  C2.1. *If $\vartheta=1$ set $\psi=1$. Then $\mathbf{c}=(0,1,0,0)$.*
  C2.2. *If $\vartheta=\omega$ set $\psi=\chi$. Then $\mathbf{c}=(1,0,0,0)$.*

C3. *If $\bar\rho$ is absolutely irreducible, set $\psi=\widetilde{\det\bar\rho}$. Then $\mathbf{c}=(0,0,1,0)$.*

In particular, when p is big enough with respect to l and f (*e.g.*, $p>l^f$) the scheme X is geometrically irreducible, unless $\bar\rho$ is twist-equivalent to $\omega^{\pm1}\oplus 1$, in which case $X_{\bar K}$ has two geometrically irreducible components. On the other side, when $[L(\zeta_p):L]\leq 2$, the number of irreducible components of $X_{\bar K}$ can be large and it depends on α and β.

In sections 2 and 3, we determine many $\bar K$-rational points on each geometrically irreducible component of X. This is used to write the list $\mathrm{Iso}(\mathcal{C})$ of $\bar K[G]$-isomorphism classes of $\mathcal{O}_{\bar K}$-lifts of $\bar\rho$ associated to closed points on each component $\mathcal{C}$.

When dealing with components $\mathcal{C}$ of type I and II, the construction of $\mathrm{Iso}(\mathcal{C})$ is carried on in Props. 2.2.2, 2.2.3, 2.2.9, 2.2.10, 2.3.3, 2.3.5, 2.3.8, 2.3.9. Notice that the number of components of type I or II in the deformation space is always small (at most two, for each type) and it does not depend on the specific values of α and β. On the other side the number of components of type III can be large when ω is quadratic, and it depends on β. The determination of $\mathrm{Iso}(\mathcal{C})$ for components of type III is correspondingly more difficult: it is carried on in section 3 (cf. Props. 3.1.2, 3.2.2, 3.3.2, 3.4.1, 3.4.2), where results of S.H. Choi ([1]) are used. The number of components of type IV can be large only when ω is trivial, and in this case it depends on α. The set $\mathrm{Iso}(\mathcal{C})$ in these settings is computed in section 2 (cf. Props. 2.2.5, 2.2.6, 2.2.7, 2.3.10, 2.3.11, 2.3.12). In some cases, explicit computations with the Bruhat-Tits tree of $GL_2(K)$ were used to exclude the existence of liftings of $\bar{\rho}$ of type IV.

From the above constructions we deduce the following rationality result (Corollary 4.0.7):

Proposition 1.0.4. *Each absolutely irreducible component of X has a point rational over $K(\zeta_{p^{\alpha+\beta}})$, unless $\alpha > 0$, $\beta = 0$ and $\bar{\rho}$ is non-semisimple and twist-unramified, in which case every absolutely irreducible component of X has a point rational over $K(\zeta_{p^{\alpha+1}})$.*

It would be interesting to understand the meaning of the numbers appearing as components of the vector $\mathbf{c}$.

Notation 1.0.5. All the homomorphisms, cocycle, and coboundary maps in this paper are assumed to be continuous unless otherwise stated. If ρ_1 and ρ_2 are two representations of a group G with coefficients in a domain R contained in a field K, the notation $\rho_1 \simeq_K \rho_2$ means that ρ_1 and ρ_2 are K-linearly isomorphic as G-representations. If X is a scheme over a perfect field K, a geometrically irreducible component of X is an irreducible component of $X_{\bar{K}}$. The term *component* is sometimes used to mean irreducible component and, if no ambiguity arises, to mean absolutely irreducible component.

2. Reducible lifts

Let l and p be two distinct odd primes and fix a finite extensions L of $\mathbb{Q}_l$. Denote by G the absolute Galois group of L and let χ be the p-adic cyclotomic character of G and ω its reduction modulo p. Set $f = f(L/\mathbb{Q}_l)$. Let k be a finite extension of $\mathbb{F}_p$ and let $\bar{\rho} : G \to GL_2(k)$ be a representation; let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbb{Q}_p$ having residue field k. Denote by $CLN_{\mathcal{O}}$ the category whose objects are pairs (R, i) where R is a complete local noetherian $\mathcal{O}$-algebra, i is an isomorphism from the residue field of R onto k, and a morphism $(R, i) \to (R', i')$ is a local $\mathcal{O}$-algebra homomorphism compatible with i and i'. The framed deformation functor $D^{\square}_{\bar{\rho}}$ from the category $CLN_{\mathcal{O}}$ to the category of sets is represented by an object $R^{\square}_{\bar{\rho}}$ of $CLN_{\mathcal{O}}$. For any fixed lift $\psi : G \to \mathcal{O}^{\times}$ of $\det \bar{\rho}$, the functor $D^{\square,\psi}_{\bar{\rho}}$ cutting out those framed deformations whose determinant is ψ is represented by $R^{\square,\psi}_{\bar{\rho}}$.

Denote by 1 the trivial $\mathcal{O}$-valued character of G. There is a natural morphism of functors $D_1 \times D^{\square,\psi}_{\bar{\rho}} \longrightarrow D^{\square}_{\bar{\rho}}$ induced by sending a pair $(\alpha, \sigma) \in D_1(A) \times D^{\square,\psi}_{\bar{\rho}}(A)$ to $\alpha \otimes \sigma$, for any ring A in $CLN_{\mathcal{O}}$. Since $p \neq 2$, this induces an isomorphism $R^{\square}_{\bar{\rho}} \simeq R_1 \hat{\otimes}_{\mathcal{O}} R^{\square,\psi}_{\bar{\rho}} \simeq \mathcal{O}[[G^{ab,(l)}]] \hat{\otimes}_{\mathcal{O}} R^{\square,\psi}_{\bar{\rho}}$, so that:

$$R^{\square}_{\bar{\rho}} \simeq \frac{R^{\square,\psi}_{\bar{\rho}}[[T, X]]}{((1+X)^{p^{\alpha}} - 1)}$$

where $\alpha = \mathrm{ord}_p(l^f - 1)$. To determine the geometrically irreducible components of the generic fiber of $\mathrm{Spec}\, R^{\square}_{\bar{\rho}}$ it is therefore enough to study the irreducible components of $R^{\square,\psi}_{\bar{\rho}}[\frac{1}{p}]$ for any fixed choice of ψ.

If $\vartheta : G \to k^\times$ is a character, there is a canonical isomorphism $R^\square_{\bar\rho} \to R^\square_{\vartheta\otimes\bar\rho}$. After replacing k with a finite extension if necessary, we are then left to consider deformation spaces for representations $\bar\rho$ falling in one of the following cases:

- **(1):** $\bar\rho = \vartheta \oplus 1$, where $\vartheta : G \to k^\times$ is a character and 1 is the trivial k-valued character;
- **(2):** $\bar\rho$ is a non-split extension of 1 by 1 or a non split extension of 1 by ω;
- **(3):** $\bar\rho$ is absolutely irreducible.

Fix from now on a choice of arithmetic Frobenius Φ of G. We denote by $t_p : I_G \to \mathbb{Z}_p \subset \mathcal{O}_K$ the tame character of inertia arising from the canonical isomorphism $I^t_G \simeq \prod_{l'\neq l}\mathbb{Z}_{l'}$. We identify t_p with a generator $\tilde\xi$ of $H^1(G, K(1))$ via the isomorphism $H^1(G, K(1)) \to \mathrm{Hom}(I_G, K)$ induced by restriction. We can choose a representative $\tilde t_p \in Z^1(G, K(1))$ for the class $\tilde\xi$ having the following properties: $\tilde t_{p|I_G} = t_p$; $\tilde t_p$ takes values in $\mathbb{Z}_p(1)$; $\tilde t_p(\Phi) = 0$. We fix for the rest of the paper such a choice of representative $\tilde t_p$ and we also denote it by t_p if no confusion arises.

Assume that $\omega \neq 1$; after reducing $t_p : I_G \to \mathbb{Z}_p$ modulo p, we obtain a group homomorphism $\bar t_p : I_G \to \mathbb{F}_p \subset k$ that we identify with a generator ξ of $H^1(G, k(1))$ via the isomorphism $H^1(G, k(1)) \to \mathrm{Hom}(I_G, k)$. A representative for the class ξ is given by the reduction modulo p of the 1-cocycle $\tilde t_p$ that we choose above. We fix this choice of representative for ξ and we denote it by $\bar{\tilde t}_p$ or simply by $\bar t_p$ when no confusion arises.

2.1. **Classification theorems.** We keep the notation just introduced and we fix a lift ψ of the determinant of $\bar\rho$. The following results are proved in [4], [3], and [5].

Theorem 2.1.1. *The deformation space $X = \mathrm{Spec}\, R^{\square,\psi}_{\bar\rho}[\frac{1}{p}]$ is three dimensional, reduced, and scheme-theoretic union of formally smooth components. Any geometrically irreducible component $\mathcal{C}$ of X admits one of the following descriptions:*

- **(I):** *There exists a character $\gamma : G \to \bar K^\times$ such that for any geometric point $x : \mathrm{Spec}\,\bar K \to \mathcal{C}$ the corresponding $\bar K$-representation ρ_x is an extension of γ by $\gamma(1)$.*
- **(II):** *There exists a character $\gamma : G \to \bar K^\times$ such that for any geometric point $x : \mathrm{Spec}\,\bar K \to \mathcal{C}$ the $\bar K$-representation $\gamma^{-1} \otimes \rho_x$ is unramified.*
- **(III):** *There exists an irreducible representation $\sigma : G \to GL_2(\bar K)$ such that for any geometric point $x : \mathrm{Spec}\,\bar K \to \mathcal{C}$ the $\bar K$-representation ρ_x is isomorphic to σ.*
- **(IV):** *There are two characters $\eta, \lambda : G \to \bar K^\times$ such that $\eta\lambda^{-1}$ is ramified and such that for any geometric point $x : \mathrm{Spec}\,\bar K \to \mathcal{C}$ the $\bar K$-representation ρ_x is isomorphic to $\eta\phi \oplus \lambda\phi^{-1}$ where ϕ is an unramified lifting of the trivial character.*

It two distinct components intersect, they are respectively of type I and of type II for the same character γ; if x is a geometric point lying on the intersection the associated $\bar K$-representation is isomorphic to $\gamma \oplus \gamma(1)$.
If x and y are two geometric points of X with corresponding $\bar K$-representations isomorphic and reducible, then x and y belong to the same geometrically irreducible component.

We will use the above theorem without explicit mention. We will also refer to the component of X as of being of type I_γ, II_γ, III_σ and $\mathrm{IV}_{\{\eta_{|I_G},\lambda_{|I_G}\}}$ with the obvious meaning of this notation. We also have (cf. *loc.cit.*):

Theorem 2.1.2. *For any character $\gamma : G \to \bar K^\times$, there is at most one geometrically irreducible component of X of type I_γ or of type II_γ. There is at most one geometrically irreducible component of X of type $IV_{\{\eta_{|I_G},\lambda_{|I_G}\}}$, unless $\bar\rho \simeq \bar\eta \oplus \bar\lambda$ with $\bar\eta\bar\lambda\bar\lambda^{-1}$ non-trivial and unramified, in which case there are exactly two components of X of type $IV_{\{\eta_{|I_G},\lambda_{|I_G}\}}$.*

2.2. **Semisimple reducible $\bar{\rho}$.** Assume $\bar{\rho} = \vartheta \oplus 1$ for some character $\vartheta : G \to k^\times$. It is easy to see that if $\vartheta \neq \omega^{\pm 1}$ and $\omega \neq 1$ then $H^2(G, \mathrm{ad}^0 \bar{\rho}) \simeq H^0(G, \mathrm{ad}^0 \bar{\rho}(1))$ is trivial, so that $R_{\bar{\rho}}^{\square,\psi}$ is isomorphic to the formally smooth algebra $\mathcal{O}[[X_1, X_2, X_3]]$ and X has exactly one irreducible component, which is of type II_1 (resp. IV) if ϑ is unramified (resp. ramified).

We can therefore assume that either $\omega = 1$ or $\omega \neq 1$ and $\vartheta = \omega$; the case $\omega \neq 1$ and $\vartheta = \omega^{-1}$ gives rise to a deformation space which is isomorphic to the one of the case $\omega \neq 1$ and $\vartheta = \omega$.

2.2.1. *The case $\omega = 1$.* We assume $\zeta_p \in L$ (*i.e.*, $p | l^f - 1$) and we work with $\psi = \tilde{\vartheta}\chi$, where the symbol $\sim$ denotes the Teichmüller lifting.

Proposition 2.2.2. *If $\vartheta \neq 1$, X has no geometrically irreducible components of type I. Assume $\vartheta = 1$; then there is exactly one component $\mathcal{C}_I$ of type I, which is associated to the character $\gamma = 1$ and which contains a K-rational point x such that $\rho_x = \chi \oplus 1$ as $\mathcal{O}_K$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_I$ are all and only $\bar{K}$-isomorphic to either:*

$$\chi \oplus 1 \quad or \quad \begin{pmatrix} \chi & t_p \\ & 1 \end{pmatrix}.$$

Proof Let $\mathcal{C}$ be a component of $X_{\bar{K}}$ of type I_γ and let x be a point on $\mathcal{C}$, so that $\rho_x : G \to GL_2(\mathcal{O}_{K'})$ is isomorphic over K' to an extension of γ by $\gamma(1)$ and $\bar{\rho}_x = \bar{\rho} \otimes_k k'$, where K' is some finite extension of K with residue field k'. Since $\bar{\rho}$ is semisimple we obtain $\vartheta \oplus 1 \simeq \bar{\gamma} \oplus \bar{\gamma}$ so that $\vartheta = 1 = \bar{\gamma}$. If $\vartheta = 1$ the representation $\chi \oplus 1$ defines point on $\mathcal{C}$ by Proposition 4.6.4 of [5]. To show that $\mathcal{C}$ is the only component of type I, notice that a component of type $\mathrm{I}_{\gamma'}$ is such that $\bar{\gamma}' = 1$ and $(\gamma')^2 = 1$, because of our choice of ψ: this implies $\gamma = 1$ as p is odd. The final statement follows from the fact that $H^1(G, K(1))$ is one dimensional. ■

Proposition 2.2.3. *If ϑ is ramified, there are no components of type II; otherwise there is exactly one component $\mathcal{C}_{II}$ of type II which is associated to the character $\gamma = 1$ and contains a K-rational point x such that $\rho_x = \tilde{\vartheta}\chi \oplus 1$ as $\mathcal{O}_K$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{II}$ are all and only $\bar{K}$-isomorphic to either:*

$$\tilde{\vartheta}\chi\varphi^{-1} \oplus \varphi \quad or \quad \chi^{1/2} \otimes \begin{pmatrix} 1 & nr(1) \\ & 1 \end{pmatrix},$$

where φ is any unramified character $G \to \mathcal{O}_{\bar{K}}^\times$ such that $\bar{\varphi} = 1$ or $\bar{\varphi} = \vartheta$, and $nr(1)$ is the additive unramified character of G sending the arithmetic Frobenius to $1 \in \bar{K}$. The second representation only occurs if $\vartheta = 1$.

Proof It is obvious that a necessary condition for the existence of components of type II is the unramifiedness of ϑ. Assuming this, a component of type II_1 exists as $\rho = \tilde{\vartheta}\chi \oplus 1$ is unramified. Assume now that X contains a geometrically irreducible component of type II_γ for some character γ valued in the ring of integers of some finite extension K' of K. Notice that if γ' is another $\mathcal{O}_{K'}$-valued character of G which coincides on I_G with γ, then γ and γ' define the same component of $X_{\bar{K}}$, since two distinct type II components cannot intersect. We can then assume that γ, viewed as a character of G^{ab}, is trivial on $G^{nr} = Gal(L^{nr}/L)$, so that it induces by local class field theory a homomorphism $G^{ab} \to \mu_{l^f-1} \times (1 + \mathfrak{M}_{\mathcal{O}_L}) \to \mathcal{O}_{K'}^\times$. Since γ^2 is unramified, γ is trivial on $\mu_{l^f-1}^2 \times (1 + \mathfrak{M}_{\mathcal{O}_L})$ and we conclude that $\gamma = 1$ as $\bar{\gamma}_{|I_G} = 1$.

For the final statement, notice that any unramified two dimensional representation ρ of G is either split or it is an extension of an unramified character by itself. In the latter case, write $\rho \simeq \alpha \otimes \begin{pmatrix} 1 & r \\ & 1 \end{pmatrix}$ where r is a continuous unramified additive homomorphism $G \to \bar{K}$. By the lifting requirement we have $\bar{\alpha} = 1 = \vartheta$ and the determinant condition implies that $\alpha^2 = \chi$; notice that since $\omega = 1$, the cyclotomic character has a unique well defined square root. Finally, since we are working up to $\bar{K}$-isomorphisms, we can rescale r so that it sends an arithmetic Frobenius of L to one. ■

Corollary 2.2.4. *Assume $\vartheta = 1$. The unique component $\mathcal{C}_I$ of $X_{\bar{K}}$ of type I and the unique component $\mathcal{C}_{II}$ of $X_{\bar{K}}$ of type II intersect and if x is a geometric point in their intersection the $\bar{K}$-representation ρ_x is isomorphic to $\chi \oplus 1$.*

Components of type IV. Let Λ be the set of equivalence classes of characters $\eta : G \to \mathcal{O}_{\bar{K}}^{\times}$ such that $\bar{\eta} = 1$ and $\eta^2\tilde{\vartheta}^{-1}$ is ramified, where two such characters η and η_1 are considered equivalent if and only if $\eta_{|I_G} = \eta_{1|I_G}$ or $\eta\eta_{1|I_G} = \tilde{\vartheta}_{|I_G}$. Let r be equal to two if $\vartheta \neq 1$ is unramified, otherwise set $r = 1$.

Lemma 2.2.5. *The number of geometrically irreducible components of X of type IV equals $r \cdot \#\Lambda$.*

Proof Let K'/K be a finite extension and let η be a character representing a class in Λ. Denote by $\mathcal{L}^{\square,\eta^2\tilde{\vartheta}^{-1}\chi}[\frac{1}{p}]$ the smooth closed subscheme of Spec $R_{\bar{\rho}^{-1}}^{\square,\eta^2\tilde{\vartheta}^{-1}\chi}[\frac{1}{p}]$ parametrizing lifts of $\bar{\rho}^{-1}$ which are isomorphic over $\bar{K}$ to $\eta^2\tilde{\vartheta}^{-1}\chi\varphi \oplus \varphi^{-1}$ for some unramified character φ such that $\bar{\varphi} = 1$ ([5], 4.9). If $\vartheta \neq 1$ is unramified, Proposition 4.9.4 in *loc.cit.* implies that $\mathcal{L}^{\square,\eta^2\tilde{\vartheta}^{-1}\chi}[\frac{1}{p}]$ is the union of two geometrically irreducible components; in the other cases, $\mathcal{L}^{\square,\eta^2\tilde{\vartheta}^{-1}\chi}[\frac{1}{p}]$ is geometrically irreducible. After identifying the spectrum of $R_{\bar{\rho}^{-1}}^{\square,\eta^2\tilde{\vartheta}^{-1}\chi}[\frac{1}{p}]$ with X, we deduce that if $\vartheta \neq 1$ is unramified, the above subscheme cuts out two irreducible components of $X_{\bar{K}}$: one parametrizing lifts isomorphic over $\bar{K}$ to $\eta^{-1}\tilde{\vartheta}\varphi^{-1} \oplus \varphi\eta\chi$ for some non-trivial unramified character φ such that $\bar{\varphi} = 1$, and the other parametrizing lifts isomorphic to $\eta^{-1}\tilde{\vartheta} \oplus \eta\chi$. In the other cases, we obtain just one irreducible component of $X_{\bar{K}}$, parametrizing lifts that become isomorphic to $\eta^{-1}\tilde{\vartheta}\varphi^{-1} \oplus \varphi\eta\chi$ over $\bar{K}$ for some unramified character φ such that $\bar{\varphi} = 1$.

Assume now that η_1 is in the same equivalence class of η, say $\eta_{|I_G} = \eta_{1|I_G}$, and set $\eta = \beta\eta_1$. If ρ is a lift of $\bar{\rho}$ which is $\bar{K}$-isomorphic to $\eta^{-1}\tilde{\vartheta}\varphi^{-1} \oplus \varphi\eta\chi$ for some unramified character φ such that $\bar{\varphi} = 1$, then ρ is also isomorphic to $\eta_1^{-1}\tilde{\vartheta}\,(\beta\varphi)^{-1} \oplus (\beta\varphi)\eta_1\chi$ and $\beta\varphi$ is unramified with trivial reduction. Similar considerations can be made if $\eta\eta_{1|I_G} = \tilde{\vartheta}_{|I_G}$ so we conclude that if η and η_1 represent the same class in Λ, then they determine the same component of $X_{\bar{K}}$ (or the same pair of components if $\vartheta \neq 1$ is unramified). Thus we defined an injective map Θ from the set Λ (or $\Lambda \times \{0, 1\}$, if $\vartheta \neq 1$ is unramified) to the collection of geometrically irreducible components of X of type IV. We now check that Θ is surjective: if x is a geometric point of X that belongs to a component of type IV, then $\rho_x \simeq \tilde{\vartheta}\chi\beta^{-1} \oplus \beta$ for some character β as $\bar{K}$-representation. Notice that since x cannot belong to a component of type II, $\tilde{\vartheta}\beta^{-2}$ is ramified. We have $\bar{\beta} = 1$ or $\bar{\beta} = \vartheta$ and we deduce that x is a geometric point on $\Theta(\chi^{-1}\beta)$ in the first case, and on $\Theta(\tilde{\vartheta}\beta^{-1})$ in the second case. ■

Lemma 2.2.6. *Let $\alpha = \mathrm{ord}_p(l^f - 1)$. Then Λ contains $\frac{p^\alpha - 1}{2}$ element if ϑ is unramified, otherwise it contains p^α elements.*

Proof Fix a topological isomorphism $G^{ab} = \mathcal{O}_L^{\times} \times \hat{\mathbb{Z}}$ and set $K_\alpha = K(\zeta_{p^\alpha})$. Assume first that ϑ is unramified. Suppose $\alpha > 0$ and for any integer a such that $1 \leq a \leq p^\alpha - 1$

define an $\mathcal{O}_{K_\alpha}$-valued character η_a of G by composing the continuous homomorphism $G^{ab} \to \mathcal{O}_L^\times \to \mu_{p^\alpha} \subset 1 + \mathfrak{M}_{\mathcal{O}_{K_\alpha}}$ with the ath power map. It is clear that in this way we obtain $\frac{p^\alpha - 1}{2}$ distinct classes in Λ, as $\eta_a \sim \eta_b$ if and only if $a \equiv \pm b (\bmod p^\alpha)$. Viceversa, if η defines an element of Λ, then it factors via the maximal pro-p quotient of G^{ab}, hence through a map $\mu_{p^\alpha} \times \mathbb{Z}_p \to 1 + \mathfrak{M}_{\mathcal{O}_{\bar{K}}}$; after changing if necessary representative for η, we can assume that η is trivial on $\mathbb{Z}_p$, which implies that $\eta = \eta_a$ for some a as above. When $\alpha = 0$ and ϑ is unramified, we see that Λ is empty as η cannot be trivial by definition.

If ϑ is ramified, then Λ coincide with the set of equivalence classes of characters with trivial reduction, where two such characters are equivalent if and only if they have the same restriction to inertia. As above we see that representatives for the classes in Λ are given by the Galois characters induced by composing the map $G^{ab} \to \mathcal{O}_L^\times \to \mu_{p^\alpha} \subset 1 + \mathfrak{M}_{\mathcal{O}_{K_\alpha}}$ with the ath power map for $0 \leq a < p^\alpha - 1$. ■

As in the above proof, we fix a topological isomorphism $G^{ab} = \mathcal{O}_L^\times \times \hat{\mathbb{Z}}$ and for any integer a we denote by η_a the character of G defined by composing the map $G^{ab} \to \mu_{p^\alpha} \subset 1 + \mathfrak{M}_{K(\zeta_{p^\alpha})}$ with the ath power morphism.

Corollary 2.2.7. *Let $\alpha = \mathrm{ord}_p(l^f - 1) > 0$ and $K_\alpha = K(\zeta_{p^\alpha})$.*

- **(a):** *If $\vartheta = 1$, X has $\frac{p^\alpha - 1}{2}$ geometrically irreducible components of type IV, say $\mathcal{C}_{IV}^a$ $(1 \leq a \leq \frac{p^\alpha - 1}{2})$. For any unramified $\mathcal{O}_{K_\alpha}$-valued character φ of G with trivial reduction, there is a point $x \in \mathcal{C}_{IV}^a(K_\alpha)$ such that $\rho_x = \eta_a^{-1}\varphi^{-1} \oplus \varphi\eta_a\chi$ as $\mathcal{O}_{K_\alpha}$-representations.*
- **(b):** *If $\vartheta \neq 1$ is unramified, X has $p^\alpha - 1$ geometrically irreducible components of type IV, say $\mathcal{C}_{IV,0}^a$, $\mathcal{C}_{IV,1}^a$ $(1 \leq a \leq \frac{p^\alpha - 1}{2})$. There is a point $x \in \mathcal{C}_{IV,0}^a(K_\alpha)$ (resp. $x \in \mathcal{C}_{IV,1}^a(K_\alpha)$) such that $\rho_x = \tilde{\vartheta}\eta_a^{-1} \oplus \eta_a\chi$ (resp. $\rho_x = \tilde{\vartheta}\eta_a^{-1}\varphi^{-1} \oplus \varphi\eta_a\chi$ for any unramified $\mathcal{O}_{K_\alpha}$-valued character $\varphi \neq 1$ of G with trivial reduction) as $\mathcal{O}_{K_\alpha}$-representations.*
- **(c):** *If ϑ is ramified, X has p^α geometrically irreducible components of type IV, say $\mathcal{C}_{IV}^a$ $(1 \leq a \leq p^\alpha)$. For any unramified $\mathcal{O}_{K_\alpha}$-valued character φ of G with trivial reduction, there is a point $x \in \mathcal{C}_{IV}^a(K_\alpha)$ such that $\rho_x = \tilde{\vartheta}\eta_a^{-1}\varphi^{-1} \oplus \varphi\eta_a\chi$ as $\mathcal{O}_{K_\alpha}$-representations.*

In any case, the lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on components of type IV are all and only $\bar{K}$-isomorphic to representations of the form

$$\tilde{\vartheta}\eta_a^{-1}\varphi^{-1} \oplus \varphi\eta_a\chi$$

where φ is an unramified $\bar{K}$-valued character of G with trivial reduction and $\eta_a : G \to \mathcal{O}_{K(\zeta_{p^\alpha})}^\times$ is as above.

2.2.8. *The case $\vartheta = \omega \neq 1$.* We assume $\zeta_p \notin L$ (*i.e.*, p does not divide $l^f - 1$) and $\vartheta = \omega$; we work with $\psi = \chi$.

Proposition 2.2.9. *X always has a geometrically irreducible component $\mathcal{C}_{I,1}$ of type I with associate character 1; this component has a K-rational point x such that $\rho_x = \chi \oplus 1$ as $\mathcal{O}_K$-representation. If $[L(\zeta_p) : L] \neq 2$, there is no other components of type I, otherwise there is exactly another component $\mathcal{C}_{I,\tilde{\omega}}$ of type I having associated character $\gamma = \tilde{\omega}$. This component has a K-rational point y such that $\rho_y = \tilde{\omega} \oplus \tilde{\omega}\chi$ as $\mathcal{O}_K$-representations. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{I,1}$ resp. $\mathcal{C}_{I,\tilde{\omega}}$ are all and only $\bar{K}$-isomorphic to:*

$$\chi \oplus 1 \quad \textit{or} \quad \begin{pmatrix} \chi & t_p \\ & 1 \end{pmatrix};$$

resp.

$$\tilde{\omega} \oplus \tilde{\omega}\chi \quad or \quad \tilde{\omega} \otimes \begin{pmatrix} \chi & t_p \\ & 1 \end{pmatrix}.$$

Proof The first statement is obvious. If $\gamma : G \to \mathcal{O}_{\bar{K}}^{\times}$ is a non-trivial character such that $X_{\bar{K}}$ has a component of type I_{γ}, then $\gamma^2 = 1$ because of the determinant condition. By looking at the semisemplification of the reduction of any representation corresponding to a point on this component we also obtain $\bar{\gamma} = \omega$. This implies that $\omega^2 = 1$ and hence $[L(\zeta_p) : L] = 2$. We deduce that γ is the quadratic character associated to the extension $L(\zeta_p)/L$, or $\gamma = \tilde{\omega}$. A component of type I with such a character do occur as ρ_y defines a point on it. Finally, notice that x does not belong to this component. The only thing to check for the last statement is that when $[L(\zeta_p) : L] = 2$ the K-representation $\rho = \tilde{\omega} \otimes \left(\begin{smallmatrix} \chi & t_p \\ & 1 \end{smallmatrix}\right)$ has an integral model whose reduction is equal (and not just isomorphic) to $\bar{\rho}$. This is obvious after conjugating ρ first by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and then by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. ■

Proposition 2.2.10. *X contains exactly one geometrically irreducible component $\mathcal{C}_{II}$ of type II, with associate character 1; this component has a K-rational point x such that $\rho_x = \chi \oplus 1$ as $\mathcal{O}_K$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{II}$ are all and only $\bar{K}$-isomorphic to:*

$$\chi\varphi^{-1} \oplus \varphi,$$

where φ is any unramified character $G \to \mathcal{O}_{\bar{K}}^{\times}$ such that $\bar{\varphi} = 1$ or $\bar{\varphi} = \omega$.

Proof It is clear that X has a component of type II_1 containing x as in the statement of the proposition. If $\mathcal{C}$ is a component of type II with associated character $\gamma \neq 1$, then γ^2 and $\bar{\gamma}$ are both unramified, so that γ is unramified as p does not divide $l^f - 1$. This implies that x also belongs to $\mathcal{C}$. Since two components of type II do not intersect, we deduce that $\mathcal{C} = \mathcal{C}_{II}$. The last statement follows from the fact that if an unramified representation $\alpha \otimes \left(\begin{smallmatrix} 1 & r \\ & 1 \end{smallmatrix}\right)$ is $\bar{K}$-isomorphic to a lift of $\bar{\rho}$ we have $\bar{\alpha} = 1$ or $\bar{\alpha} = \omega$ and the determinant condition implies $\alpha^2 = \chi$ contradicting the fact that $\omega \neq 1$. ■

Corollary 2.2.11. *The components $\mathcal{C}_{I,1}$ and $\mathcal{C}_{II}$ of $X_{\bar{K}}$ intersect and if x is a geometric point in their intersection the $\bar{K}$-representation ρ_x is isomorphic to $\chi \oplus 1$. When $[L(\zeta_p) : L] = 2$ also $\mathcal{C}_{I,\tilde{\omega}}$ and $\mathcal{C}_{II}$ intersect and if y is a geometric point in their intersection then ρ_y is isomorphic to $\chi \oplus 1$ over $\bar{K}$. There is no intersection between $\mathcal{C}_{I,1}$ and $\mathcal{C}_{I,\tilde{\omega}}$.*

Proposition 2.2.12. *X has no components of type IV.*

Proof Assume that x is a geometric point of a component of type IV, so that ρ_x is $\bar{K}$-isomorphic to $\chi\beta^{-1} \oplus \beta$ for some character β such that β^2 is ramified. The lifting property of ρ_x implies that the reduction of β is either trivial or equal to ω. In either case, the restriction of β to I_G must factors through the maximal pro-p quotient of $Gal(L^{ab}/L^{ur})$, which is trivial as $\zeta_p \notin L$. This contradicts the fact that β is ramified. ■

2.3. **Non-semisimple $\bar{\rho}$.** Assume $\bar{\rho}$ is a non-split extension of 1 by ϑ, so that either $\vartheta = 1$ or $\vartheta = \omega$. We consider the following three cases:

(1) $\omega \neq 1$ and $\vartheta = 1$;
(2) $\omega \neq 1$ and $\vartheta = \omega$;
(3) $\omega = \vartheta = 1$.

We fix a topological isomorphism $G^{ab} \simeq \mathcal{O}_L^\times \times \hat{\mathbb{Z}}$ identifying the arithmetic Frobenius Φ that we fixed earlier with $1 \in \hat{\mathbb{Z}}$. Denote by $\tilde{\varphi} : G \to \mathbb{Z}_p$ the unique group homomorphism factoring through $G^{nr,(p)}$ and obtained by identifying the latter with $\mathbb{Z}_p$; let φ be the reduction of $\tilde{\varphi}$ modulo p and view it as a homomorphism with values in k via the embedding $\mathbb{F}_p \subset k$.

2.3.1. *The case $\omega \neq 1$ and $\vartheta = 1$.* Assume $\omega \neq 1$, $\vartheta = 1$ and choose the determinant lift $\psi = 1$. Since $H^1(G, k) = \mathrm{Hom}(G^{ab,(p)}, k)$ is one dimensional, we can assume that:

$$\bar{\rho} = \begin{pmatrix} 1 & \varphi \\ & 1 \end{pmatrix}$$

where $\varphi : G \to k$ is as above.

Proposition 2.3.2. *The deformation space X has no geometrically irreducible components of type I or IV.*

Proof If ρ is a lift of $\bar{\rho}$ corresponding to a point on a component of type I_γ we have $\bar{\gamma}\omega \oplus \bar{\gamma} = 1 \oplus 1$, so that $\omega = 1$. Assume now that ρ corresponds to a point on a component of type IV and that $\rho \simeq \lambda \oplus \lambda^{-1}$ as K-representations, so that λ^2 is a ramified character of G. We deduce that $\bar{\lambda} = 1$ and hence λ is unramified. ■

Proposition 2.3.3. *X contains exactly one geometrically irreducible component $\mathcal{C}_{II}$ of type II, and its associate character is 1; this component has a K-rational point x such that $\rho_x = \left(\begin{smallmatrix} 1 & \tilde{\varphi} \\ & 1 \end{smallmatrix}\right)$ as $\mathcal{O}_K$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{II}$ are all and only $\bar{K}$-isomorphic to:*

$$\begin{pmatrix} 1 & \tilde{\varphi} \\ & 1 \end{pmatrix}.$$

Proof It is clear that X contains a geometrically irreducible component $\mathcal{C}_{II}$ of type II with associate character 1. If y is any geometric point on this component ρ_y is unramified, so that $\rho_y \simeq \left(\begin{smallmatrix} \alpha & b \\ & \alpha \end{smallmatrix}\right)$ as $\bar{K}$-representations. Since $\alpha^2 = 1$ and $\bar{\alpha} = 1$, we deduce $\alpha = 1$ so that $b \in H^1(G, K')$ for some finite extension K'/K. This implies that either $b = 0$ or, without loss of generality, $b = \tilde{\varphi}$ (after seeing $\mathbb{Z}_p \subset K'$). The first possibility does not occur as the trivial representation does not have an integral model lifting $\bar{\rho}$.

We now see that there is only one component of type II. Let $\gamma : G \to \bar{K}^\times$ be a character such that $X_{\bar{K}}$ has a component $\mathcal{C}$ of type II_γ; let $y \in \mathcal{C}(\bar{K})$ so that $\gamma^{-1} \otimes \rho_y$ is unramified. By the determinant condition we deduce that γ^2 is unramified; notice that $\overline{\gamma^{-1} \otimes \rho_y}$ is unramified, so that $\bar{\gamma}$ is unramified. We conclude that γ is unramified and therefore $y \in \mathcal{C}_{II}(\bar{K})$. Since two distinct components of type II do not intersect, we have $\mathcal{C}_{II} = \mathcal{C}$. ■

2.3.4. *The case $\omega \neq 1$ and $\vartheta = \omega$.* Assume $\vartheta = \omega$, $\omega \neq 1$ and set $\psi = \chi$. Since $H^1(G, k(1))$ is one dimensional, we can assume:

$$\bar{\rho} = \begin{pmatrix} \omega & \bar{t}_p \\ & 1 \end{pmatrix},$$

where the 1-cocycle $\bar{t}_p$ is defined at the beginning of section 2.

Proposition 2.3.5. *X contains exactly one geometrically irreducible component $\mathcal{C}_{I,1}$ of type I; its associate character is $\gamma = 1$. This component has a K-rational point x such that $\rho_x = \begin{pmatrix} \chi & t_p \\ & 1 \end{pmatrix}$ as $\mathcal{O}_K$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{I,1}$ are all and only $\bar{K}$-isomorphic to:*

$$\begin{pmatrix} \chi & t_p \\ & 1 \end{pmatrix}.$$

Proof By definition of the cocycles t_p and $\bar{t}_p$, ρ_x is a lift of $\bar{\rho}$ so that the generic fiber of our deformation space contains at least one component of type I_1. Notice that any lift of $\bar{\rho}$ associated to a $\bar{K}$-rational point on this component is $\bar{K}$-isomorphic to either ρ_x or to $\chi \oplus 1$; the latter case cannot happen as $\bar{\rho}$ is ramified.

Let now $1 \neq \gamma : G \to \bar{K}^\times$ be a character such that X has a geometrically irreducible component $\mathcal{C}_{I,\gamma}$ of type I_γ. Fix a K'-rational point on this component for some finite extension K' of K and let ρ be the associated $\mathcal{O}_{K'}$-representation, which is K'-isomorphic to $\begin{pmatrix} \gamma\chi & * \\ & \gamma \end{pmatrix}$. Set $\rho_1 = \gamma$ and $\rho_2 = \gamma\chi$. The K'-representations $\rho_1 \oplus \rho_2$ and ρ have the same characteristic polynomial. Since $\omega \neq 1$, $\bar{\rho}_1$ is not isomorphic to $\bar{\rho}_2$. Also notice that $\bar{\rho}$, which is indecomposable by assumption, admits $\bar{\rho}_1$ as subrepresentation: this follows from the conditions $\bar{\rho}^{ss} \simeq \bar{\rho}_1 \oplus \bar{\rho}_2$, $\gamma^2 = 1$ and $\gamma \neq 1$ (recall that p is odd). We conclude that by the main theorem of [7] there is a matrix $\mathcal{L} \in GL_2(\mathcal{O}_{K'})$ such that:

$$\rho = \mathcal{L} \begin{pmatrix} \rho_1 & * \\ & \rho_2 \end{pmatrix} \mathcal{L}^{-1}$$

as $\mathcal{O}_{K'}$-linear representations. Since $\begin{pmatrix} \rho_1 & * \\ & \rho_2 \end{pmatrix} = \gamma\chi \otimes \begin{pmatrix} \chi^{-1} & * \\ & 1 \end{pmatrix}$ and $H^1(G, \mathcal{O}_{K'}(-1)) = 0$ we deduce that for some $c \in \mathcal{O}_{K'}$ we have:

$$\rho = \gamma\chi \otimes \left[\mathcal{L} \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \chi^{-1} & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} \mathcal{L}^{-1} \right]$$

so that $\bar{\rho}$ is semisimple, contradicting our assumption. ■

Proposition 2.3.6. *X has no geometrically irreducible components of type II or IV.*

Proof If ρ is a lift of $\bar{\rho}$ corresponding to a point on a component of type II_γ, ρ is isomorphic over $\bar{K}$ to the representation $\begin{pmatrix} \chi\gamma^{-1}\alpha & * \\ & \gamma\alpha^{-1} \end{pmatrix}$ for some unramified character α. After reduction and semisemplification we deduce that $\bar{\gamma} = \bar{\alpha}$ or $\bar{\gamma} = \bar{\alpha}\omega$. In any case, $\bar{\gamma}$ is unramified. Since $\omega \neq 1$, we conclude that γ - and hence ρ - is unramified, contradicting the fact that $\bar{\rho}(I_G) \neq 1$.

A lift of $\bar{\rho}$ corresponding to a point on a component of type IV is isomorphic over $\bar{K}$ to $\begin{pmatrix} \chi\alpha & \\ & \alpha^{-1} \end{pmatrix}$ for some character α such that α^2 is ramified. After reduction we see that $\bar{\alpha} = 1$ or $\bar{\alpha} = \omega^{-1}$, so that $\bar{\alpha}$ is unramified. This implies that α is unramified, and hence a contradiction. ■

2.3.7. *The case $\omega = \vartheta = 1$.* Assume $\omega = \vartheta = 1$ and $\psi = \chi$. Let $\alpha = \mathrm{ord}_p(l^f - 1) > 0$ and recall that we have fixed an isomorphism $G^{ab} = \mathcal{O}_L^\times \times \hat{\mathbb{Z}}$. The projection $I_G \to \mu_{p^\alpha} \to \mu_p$ induced by $\mathcal{O}_L^\times \to \mu_{p^\alpha}$ factors through $I_G^{(p)}$, which is identified with $\mathbb{Z}_p$ via t_p: this induces a surjection $\mathbb{Z}_p \to \mu_p$ and then an isomorphism $\mu_p \simeq \mathbb{F}_p$ that we consider fixed.

Let φ be the character defined at the beginning of 2.3. Denote by $\delta : G \to k$ the additive character obtained by composing the projection $G \to G^{ab} \to \mu_{p^\alpha}$ of G into the p-power roots of unity of $\mathcal{O}_L^\times$ with the surjection $\mu_{p^\alpha} \to \mu_p$ and then with the inclusion $\mu_p \subset k$ induced by $\mu_p \simeq \mathbb{F}_p$ and by the embedding of $\mathbb{F}_p$ in k. We have:

$$H^1(G,k) = \mathrm{Hom}(G^{ab,(p)},k) = k\varphi \oplus k\delta.$$

We can then assume that $\bar{\rho}$ has one of the following shapes:

$$\bar{\rho} = \begin{pmatrix} 1 & \delta + b\varphi \\ & 1 \end{pmatrix} \text{ for some } b \in k, \quad \text{or } \bar{\rho} = \begin{pmatrix} 1 & \varphi \\ & 1 \end{pmatrix}.$$

We assume for this paragraph that we choose an element ζ_{p^α} generating μ_{p^α} such that $\delta\left(\zeta_{p^\alpha}\right) = 1$.

Proposition 2.3.8. *X contains exactly one irreducible component $\mathcal{C}_I$ of type I, which has associated character $\gamma = 1$. Furthermore:*

(1) *If $\bar{\rho} = \left(\begin{smallmatrix} 1 & \delta \\ & 1 \end{smallmatrix}\right)$, $\mathcal{C}_I$ has a K-rational point x such that $\rho_x = \left(\begin{smallmatrix} \chi & t_p \\ & 1 \end{smallmatrix}\right)$ as $\mathcal{O}_K$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_I$ are all and only $\bar{K}$-isomorphic to:*

$$\begin{pmatrix} \chi & t_p \\ & 1 \end{pmatrix}.$$

(2) *If $\bar{\rho} = \left(\begin{smallmatrix} 1 & b\delta+\varphi \\ & 1 \end{smallmatrix}\right)$ for some $b \in k$, write $l^f - 1 = p^\alpha u$ and let a be an integer such that $au \equiv 1 (\mathrm{mod}\, p)$. Then $\mathcal{C}_I$ has a K-rational point x such that*

$$\rho_x = \begin{pmatrix} \chi & \tilde{b}t_p + a\frac{\chi-1}{p^\alpha} \\ & 1 \end{pmatrix}$$

as $\mathcal{O}_K$-representation. If $b = 0$ (resp. $b \neq 0$), the lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_I$ are all and only $\bar{K}$-isomorphic to:

$$\chi \oplus 1 \textit{ or } \begin{pmatrix} \chi & t_p \\ & 1 \end{pmatrix} \quad \textit{(resp. } \begin{pmatrix} \chi & t_p \\ & 1 \end{pmatrix}\textit{).}$$

Proof If X has a geometrically irreducible component of type I_γ then $\gamma^2 = 1$ and $\bar{\gamma} = 1$, which implies $\gamma = 1$.

Since $\omega = 1$, the reduction modulo p of the cocycle $t_p : G \to \mathbb{Z}_p(1)$ fixed in 2 is a homomorphism $\bar{t}_p : G \to \mathbb{F}_p \subset k$. Since $t_p(\Phi) = 0$ and $t_{p|I_G}$ is the tame pro-p character of inertia, $\bar{t}_p = \delta$. In particular, when $\bar{\rho} = \left(\begin{smallmatrix} 1 & \delta \\ & 1 \end{smallmatrix}\right)$, the $\mathcal{O}_K$-representation $\rho = \left(\begin{smallmatrix} \chi & t_p \\ & 1 \end{smallmatrix}\right)$ defines a point on a component of X having type I_γ. Since $\bar{\rho}$ is ramified, it cannot have any lift which is $\bar{K}$-isomorphic to $\chi \oplus 1$.

Assume now $\bar{\rho} = \left(\begin{smallmatrix} 1 & b\delta+\varphi \\ & 1 \end{smallmatrix}\right)$. The continuous function $\zeta := \tilde{b}t_p + a\frac{\chi-1}{p^\alpha}$ is a 1-cocycle of G with values in $K(1)$. Observe that $\tilde{b}t_p$ has values in $W(k) \subset \mathcal{O}_K$ and the image of $\chi - 1$ is contained in $p^\alpha\mathbb{Z}_p$, so that ζ assumes integral values. Since $a\frac{\chi-1}{p^\alpha}(\Phi) \equiv 1 (\mathrm{mod}\, p)$ by construction, the reduction of $\zeta : G \to W(k)$ modulo p equals $b\delta + \varphi$ and $\overline{\rho_x} = \bar{\rho}$, so that there is a component of type I_1. To prove the final statement, notice that if $b \neq 0$ we have $\rho_x \simeq \left(\begin{smallmatrix} \chi & t_p \\ & 1 \end{smallmatrix}\right)$ as K-representations, and $\bar{\rho}$ cannot have an unramified lift. When $b = 0$, $\rho_x \simeq \left(\begin{smallmatrix} \chi & \\ & 1 \end{smallmatrix}\right)$ over K and

$$\rho = \begin{pmatrix} \chi & pt_p + a\frac{\chi-1}{p^\alpha} \\ & 1 \end{pmatrix}$$

is an $\mathcal{O}_K$-linear representation which reduces to $\bar{\rho}$ and which is K-isomorphic to $\left(\begin{smallmatrix} \chi & t_p \\ & 1 \end{smallmatrix}\right)$. ■

Proposition 2.3.9. *If $\bar{\rho} = \left(\begin{smallmatrix} 1 & \delta+b\varphi \\ & 1 \end{smallmatrix}\right)$ for some $b \in k$, $X_{\bar{K}}$ has no components of type II. If $\bar{\rho} = \left(\begin{smallmatrix} 1 & \varphi \\ & 1 \end{smallmatrix}\right)$, $X_{\bar{K}}$ has exactly one component $\mathcal{C}_{II}$ of type II, corresponding to the character $\gamma = 1$. $\mathcal{C}_{II}$ contains a K-rational point x such that*

$$\rho_x = \chi^{1/2} \otimes \begin{pmatrix} 1 & \tilde{\varphi} \\ & 1 \end{pmatrix}$$

as $\mathcal{O}_K$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{II}$ are all and only $\bar{K}$-isomorphic to either:

$$\chi\beta \oplus \beta^{-1} \text{ or } \chi^{1/2} \otimes \begin{pmatrix} 1 & \tilde{\varphi} \\ & 1 \end{pmatrix},$$

where $\beta : G \to \bar{K}^{\times}$ is any unramified character such that $\bar{\beta} = 1$ and $\beta^2(\Phi) \neq l^{-f}$.

Proof If $X_{\bar{K}}$ contains a geometric point x lying on a component of type II_γ, γ^2 and $\bar{\gamma}$ are unramified, so that γ and ρ_x are unramified and $\bar{\rho} = \left(\begin{smallmatrix} 1 & \varphi \\ & 1 \end{smallmatrix}\right)$. In this case, a component of type II_1 exists as the above described representation ρ_x shows (we denoted by $\chi^{1/2}$ the composition of χ with the inverse of the automorphism $1 + p\mathbb{Z}_p \to 1 + p\mathbb{Z}_p$ given by squaring). Notice that this is the only component of type II as $\gamma = 1$ and two type II components do not intersect.

For the last statement, notice that if ρ is an unramified non split representation $\bar{K}$-isomorphic to a lift of $\bar{\rho}$, then $\rho \simeq_{\bar{K}} \alpha \otimes \left(\begin{smallmatrix} 1 & r \\ & 1 \end{smallmatrix}\right)$ where $\alpha^2 = \chi$ and $\bar{\alpha} = 1$. This implies $\alpha = \chi^{1/2}$ and since r is unramified, we can assume up to $\bar{K}$-isomorphism that $r = \tilde{\varphi}$.
If ρ is as above but split, we can find a finite extension K' of K and an unramified character $\beta : G \to \mathcal{O}_{K'}^{\times}$ such that $\bar{\beta} = 1$ and $\rho \simeq \chi\beta \oplus \beta^{-1}$. Notice that since $\bar{\rho}(\Phi)$ is not diagonalizable over $\bar{k}$, we need to have $\chi\beta(\Phi) \neq \beta^{-1}(\Phi)$, so that $l^f\beta^2(\Phi) - 1 \neq 0$. To show that these representations occur, up to $\bar{K}$-isomorphism, as liftings of $\bar{\rho}$, denote by ϖ a uniformizer of K' and let

$$c = \frac{\varpi^{\mathrm{ord}_\varpi(l^f\beta^2(\Phi)-1)}}{l^f\beta^2(\Phi) - 1} \in \mathcal{O}_{K'}^{\times}.$$

There is a K'-isomorphism:

$$\rho \simeq \beta^{-1} \otimes \begin{pmatrix} \chi\beta^2 & c \cdot \dfrac{\chi\beta^2 - 1}{\varpi^{\mathrm{ord}_\varpi(l^f\beta^2(\Phi)-1)}} \\ & 1 \end{pmatrix}.$$

By construction, the right hand side above is an $\mathcal{O}_{K'}$-linear representation which lifts $\bar{\rho}$. ■

Components of type IV. Fix a finite extension E of K containing μ_{p^α} and denote by ϖ a uniformizer of $\mathcal{O}_E$. Let e be the ramification degree of E over $\mathbb{Q}_p$.

Fix an integer r such that $1 \leq r \leq p^\alpha - 1$, an element t in $\mathbb{Z}_{>0} \cup \{\infty\}$ and a unit $u \in \mathcal{O}_E^{\times}$. Let

$$\varepsilon_{r,t,u} : G \longrightarrow 1 + \mathfrak{M}_E$$

be the character defined by composing the surjection $G \to G^{ab,(p)} \simeq \mu_{p^\alpha} \times \Phi^{\mathbb{Z}_p}$ with the homomorphism $\mu_{p^\alpha} \times \Phi^{\mathbb{Z}_p} \to \mathcal{O}_E^{\times}$ defined by:

$$\zeta_{p^\alpha} \longmapsto \zeta_{p^\alpha}^r, \qquad \Phi \longmapsto 1 + \varpi^t u.$$

(We adopt the convention $\varpi^\infty = 0$). Write $l^f = 1 + \varpi^{e\alpha} w$ with $w \in \mathcal{O}_E^{\times}$ and let $\gamma = \mathrm{ord}_p(r)$, so that $\zeta_{p^\alpha}^r = \zeta_{p^{\alpha-\gamma}}$ is a primitive $p^{\alpha-\gamma}$th root of one.

Lemma 2.3.10. *Set:*

$$c = \min\left\{\frac{e}{p^{\alpha-\gamma-1}(p-1)}, \mathrm{ord}_\varpi\left(\varpi^t(2u+\varpi^t u^2) - \varpi^{e\alpha}w\right)\right\}.$$

Then $c \geq 1$ and the continuous function $\varepsilon_{r,t,u} - \varepsilon_{r,t,u}^{-1}\chi : G \to \mathcal{O}_E$ takes values in $\varpi^c\mathcal{O}_E$.

Proof We have $\left(\varepsilon_{r,t,u} - \varepsilon_{r,t,u}^{-1}\chi\right)(\zeta_{p^\alpha}) = \left(\zeta_{p^{\alpha-\gamma}}^2 - 1\right)\zeta_{p^{\alpha-\gamma}}^{-1} \sim \zeta_{p^{\alpha-\gamma}} - 1$, where we write $A \sim B$ in $\mathcal{O}_E - \{0\}$ if and only if AB^{-1} is a unit. Since $\mathbb{Q}_p(\zeta_{p^{\alpha-\gamma}})$ is a subfield of E, we deduce:

$$\mathrm{ord}_\varpi\left((\varepsilon_{r,t,u} - \varepsilon_{r,t,u}^{-1}\chi)(\zeta_{p^\alpha})\right) = \frac{e}{p^{\alpha-\gamma-1}(p-1)} \geq 1.$$

We also see that

$$(\varepsilon_{r,t,u} - \varepsilon_{r,t,u}^{-1}\chi)(\Phi) \sim \varpi^t(2u+\varpi^t u^2) - \varpi^{e\alpha}w$$

and the ϖ-adic order of this element is positive.

Since $\omega = 1$, we can view $\varepsilon_{r,t,u} - \varepsilon_{r,t,u}^{-1}\chi$ as a continuous function $\mu_{p^\alpha} \times \Phi^{\mathbb{Z}_p} \to \mathcal{O}_E$. To show that it takes values in $\varpi^c\mathcal{O}_E$ it is enough to show that $\left(\varepsilon_{r,t,u} - \varepsilon_{r,t,u}^{-1}\chi\right)(\zeta_{p^\alpha}^j\Phi^i)$ has ϖ-adic order larger than or equal to c for any $1 \leq j \leq p^\alpha$ and any integer $i > 0$. For simplicity set $d = e/p^{\alpha-\gamma-1}(p-1)$. Assume $t \neq \infty$, so that $(\varepsilon_{r,t,u} - \varepsilon_{r,t,u}^{-1}\chi)(\zeta_{p^\alpha}^j\Phi^i) \sim \zeta_{p^{\alpha-\gamma}}^{2j}(1+\varpi^t u)^{2i} - (1+\varpi^{e\alpha}w)^i$. The ϖ-adic order of the latter is larger than or equal to

$$\min\left\{\mathrm{ord}_\varpi\left(\zeta_{p^{\alpha-\gamma}}^{2j}(1+\varpi^t u)^{2i} - 1\right), \mathrm{ord}_\varpi\left((1+\varpi^{e\alpha}w)^i - 1\right)\right\}.$$

We have $\mathrm{ord}_\varpi\left((1+\varpi^{e\alpha}w)^i - 1\right) \geq e\alpha$ and a simple computation shows that $\mathrm{ord}_\varpi\left(\zeta_{p^{\alpha-\gamma}}^{2j}(1+\varpi^t u)^{2i} - 1\right) \geq \min\{t,d\}$. We conclude:

$$\begin{aligned}\mathrm{ord}_\varpi(\varepsilon_{r,t,u} - \varepsilon_{r,t,u}^{-1}\chi)(\zeta_{p^\alpha}^j\Phi^i) &\geq \min\{t,d,e\alpha\}\\ &= \min\{t,d\}.\end{aligned}$$

By definition of c we see that $c = \min\{t,d\}$ if $t < e\alpha$ and $c = d$ otherwise, so that $\min\{t,d\} \geq c$ as desired. If $t = \infty$, then $c = d$ and we obtain an analogous inequality. ■

Proposition 2.3.11. *If $\bar{\rho}$ is ramified, X has $\frac{p^\alpha-1}{2}$ geometrically irreducible components of type IV, say $\mathcal{C}_{IV}^r$ $(1 \leq r \leq \frac{p^\alpha-1}{2})$. Let $K_\alpha = K(\zeta_{p^\alpha})$ and denote by ϖ a uniformizer of K_α. Fix an integer $r \in [1, \frac{p^\alpha-1}{2}]$ and set $d = e(K_\alpha/\mathbb{Q}_p)/p^{\alpha-\mathrm{ord}_p(r)-1}(p-1)$. The characters $\varepsilon_{r,*,*} : G \to 1+\mathfrak{M}_{K_\alpha}$ appearing below are defined as at the beginning of the section:*

(a): *If $\bar{\rho} = \left(\begin{smallmatrix}1 & \delta\\ & 1\end{smallmatrix}\right)$, there is a point $x \in \mathcal{C}_{IV}^r(K_\alpha)$ such that*

$$\rho_x = \begin{pmatrix}\varepsilon_{r,\infty,1} & t_1 \cdot \dfrac{\varepsilon_{r,\infty,1} - \varepsilon_{r,\infty,1}^{-1}\chi}{\varpi^d}\\ & \varepsilon_{r,\infty,1}^{-1}\chi\end{pmatrix}$$

as $\mathcal{O}_{K_\alpha}$-representation. Here t_1 equals $\varpi^d/(\zeta_{p^\alpha}^r - \zeta_{p^\alpha}^{-r}) \in \mathcal{O}_{K_\alpha}^\times$. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{IV}^r$ are all and only $\bar{K}$-isomorphic to representations of the form $\varepsilon_{r,\infty,1}\beta \oplus \beta^{-1}\varepsilon_{r,\infty,1}^{-1}\chi$, where β is an unramified $\bar{K}$-valued character of G with trivial reduction.

(b): *If $\bar\rho = \begin{pmatrix} 1 & b\delta+\varphi \\ & 1 \end{pmatrix}$ for some $b \in k^\times$, there is a point $y \in \mathcal{C}^r_{IV}(K_\alpha)$ such that*

$$\rho_y = \begin{pmatrix} \varepsilon_{r,d,v} & t_2 \cdot \dfrac{\varepsilon_{r,d,v} - \varepsilon^{-1}_{r,d,v}\chi}{\varpi^d} \\ & \varepsilon^{-1}_{r,d,v}\chi \end{pmatrix}$$

as $\mathcal{O}_{K_\alpha}$-representation. Here $t_2 = \varpi^d\tilde b/(\zeta^r_{p^\alpha} - \zeta^{-r}_{p^\alpha}) \in \mathcal{O}^\times_{K_\alpha}$ and $v = \frac{1}{2t_2}$. The lifts of $\bar\rho$ associated to $\bar K$-rational points on $\mathcal{C}^r_{IV}$ are all and only $\bar K$-isomorphic to representations of the form $\varepsilon_{r,d,v}\beta \oplus \beta^{-1}\varepsilon^{-1}_{r,d,v}\chi$, where β is an unramified $\bar K$-valued character of G with trivial reduction.

Proof Let $\mathcal{C} \subset X_{\bar K}$ be a component of type IV and select a point $x \in \mathcal{C}(\bar K)$, so that $\rho_x \simeq_{\bar K} \begin{pmatrix} \lambda & \\ & \lambda^{-1}\chi \end{pmatrix}$ for some character $\lambda : G \to \mathcal{O}^\times_{\bar K}$ such that λ^2 is ramified and $\bar\lambda = 1$. As in the proof of Lemma 2.2.5, there is a three-dimensional formally smooth closed subscheme $\mathcal{L}[\frac{1}{p}] \simeq \mathcal{L}^{\square,\lambda^{-2}\chi}[\frac{1}{p}]$ of X parametrizing lifts of $\bar\rho$ which are $\bar K$-isomorphic to $\lambda\beta \oplus \beta^{-1}\lambda^{-1}\chi$ for some unramified character β of G such that $\bar\beta = 1$. Since $\bar\rho$ is not semisimple $\mathcal{L}[\frac{1}{p}]$ is geometrically irreducible ([5], 4.9.4) and $\mathcal{C} = \mathcal{L}[\frac{1}{p}]$. By associating to $\mathcal{C}$ the set of characters of the form $\lambda\beta$ with β as above, we obtain therefore a well defined map:

$$\Theta : \left\{ \begin{smallmatrix} \text{components of } X_{\bar K} \\ \text{of type IV} \end{smallmatrix} \right\} \longrightarrow \left\{ \begin{smallmatrix} \text{characters } \lambda : G \to \mathcal{O}^\times_{\bar K} \\ \text{with } \lambda^2 \text{ ramified and } \bar\lambda = 1 \end{smallmatrix} \right\} / \approx$$

where $\lambda \approx \lambda_1$ if and only if $\lambda\lambda^{\pm 1}_{1|I_G} = 1$. Since two distinct component of type IV do not intersect Θ is injective and Lemma 2.2.6 implies that $X_{\bar K}$ has at most $\frac{p^\alpha - 1}{2}$ components of type IV. Notice that under our assumptions this number is not zero.

Fix for the rest of this proof an integer r so that $1 \le r \le \frac{p^\alpha-1}{2}$.

(a) Assume $\bar\rho = \begin{pmatrix} 1 & \delta \\ & 1 \end{pmatrix}$ and set

$$\xi = \frac{\varepsilon_{r,\infty,1} - \varepsilon^{-1}_{r,\infty,1}\chi}{\varpi^d}.$$

Applying Lemma 2.3.10 with $E = K_\alpha$, $t = \infty$ and $u = 1$ we obtain $c = d$ so that the image of ξ is contained in $\mathcal{O}_{K_\alpha}$. We also have $\xi(\Phi) = (1 - l^f)/\varpi^d \equiv 0 (\operatorname{mod} \varpi)$ since $e\alpha > d$. As $p \neq 2$ we have $\left(\varepsilon_{r,\infty,1} - \varepsilon^{-1}_{r,\infty,1}\chi\right)(\zeta_{p^\alpha}) = \zeta^r_{p^\alpha} - \zeta^{-r}_{p^\alpha} \neq 0$ so that t_1 is well defined, and by construction $t_1 \cdot \xi(\zeta_{p^\alpha}) = 1$. Since $\varpi^d/(\zeta^r_{p^\alpha} - \zeta^{-r}_{p^\alpha}) \in \mathcal{O}^\times_{K_\alpha}$ we conclude that ρ_x has integral values and $\bar\rho_x = \bar\rho$ as desired.

(b) Assume $\bar\rho = \begin{pmatrix} 1 & b\delta+\varphi \\ & 1 \end{pmatrix}$ for some $b \in k^\times$ and set

$$\xi'_v = \frac{\varepsilon_{r,d,v} - \varepsilon^{-1}_{r,d,v}\chi}{\varpi^d}$$

where v is a unit in $\mathcal{O}_{K_\alpha}$ that we will determine. Using Lemma 2.3.10 applied with $E = K_\alpha$, $t = d$ and $u = v$ we see that ξ'_v is an $\mathcal{O}_{K_\alpha}$-valued function on $\mu_{p^\alpha} \times \Phi^{\mathbb{Z}_p}$. We have that

$$h = \varpi^d / \left(\varepsilon_{r,d,v} - \varepsilon^{-1}_{r,d,v}\chi\right)(\zeta_{p^\alpha}) = \varpi^d/(\zeta^r_{p^\alpha} - \zeta^{-r}_{p^\alpha})$$

is a unit in $\mathcal{O}_{K_\alpha}$ which does not depend on the choice of v. We deduce that the continuous function $\xi''_v := h\tilde b \cdot \xi'_v$ has values in $\mathcal{O}_{K_\alpha}$ and ρ_y is an $\mathcal{O}_{K_\alpha}$-linear representation. Observe that:

$$\xi_v''(\zeta_{p^\alpha})(\mathrm{mod}\, \varpi) = b$$
$$\xi_v''(\Phi) = h\tilde{b} \cdot \frac{2v + \varpi^d v^2 - \varpi^{e\alpha - d} w}{1 + \varpi^d v},$$

where we wrote $l^f = 1 + \varpi^{e\alpha} w$. In particular $\xi_v''(\Phi)(\mathrm{mod}\, \varpi) = 2b\bar{h}\bar{v}$, where the bar denotes reduction modulo ϖ. Since $b \neq 0$ we can choose v so that $2b\bar{h}\bar{v} = 1$ in the residue field of K_α. With this choice of v, $\bar{\rho}_y = \bar{\rho}$ as desired.

Finally, since $r \in [1, \frac{p^\alpha - 1}{2}]$ the definition of the equivalence relation $\approx$ implies that the components defined above are pairwise distinct. ■

Proposition 2.3.12. *If $\bar{\rho} = \left(\begin{smallmatrix} 1 & \varphi \\ & 1 \end{smallmatrix}\right)$, X has $\frac{p^\alpha - 1}{2}$ geometrically irreducible components of type IV, say $\mathcal{C}_{IV}^r$ ($1 \leq r \leq \frac{p^\alpha - 1}{2}$). Let $K_{\alpha+1} = K(\zeta_{p^{\alpha+1}})$ and denote by π a uniformizer of $K_{\alpha+1}$. Fix an integer $r \in [1, \frac{p^\alpha - 1}{2}]$ and set $d = e(K_{\alpha+1}/\mathbb{Q}_p)/p^{\alpha - \mathrm{ord}_p(r) - 1}(p-1)$. There is a point $z \in \mathcal{C}_{IV}^r(K_{\alpha+1})$ such that*

$$\rho_z = \begin{pmatrix} \varepsilon_{r,d-1,1} & t_3 \cdot \dfrac{\varepsilon_{r,d-1,1} - \varepsilon_{r,d-1,1}^{-1}\chi}{\pi^{d-1}} \\ & \varepsilon_{r,d-1,1}^{-1}\chi \end{pmatrix}$$

as $\mathcal{O}_{K_{\alpha+1}}$-representation. Here t_3 equals $\pi^{1-d}\left(\varepsilon_{r,d-1,1}(\Phi) - \varepsilon_{r,d-1,1}^{-1}\chi(\Phi)\right)^{-1} \in \mathcal{O}_{K_{\alpha+1}}^\times$. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{IV}^r$ are all and only $\bar{K}$-isomorphic to representations of the form $\varepsilon_{r,d-1,1}\beta \oplus \beta^{-1}\varepsilon_{r,d-1,1}^{-1}\chi$, where β is an unramified $\bar{K}$-valued character of G with trivial reduction.

Proof As in the proof of Proposition 2.3.11, $X_{\bar{K}}$ has at most $\frac{p^\alpha - 1}{2} > 0$ distinct components of type IV. Fix an integer r so that $1 \leq r \leq \frac{p^\alpha - 1}{2}$. Notice that $e(K_{\alpha+1}/\mathbb{Q}_p) \geq p^\alpha(p-1)$ so that $d - 1 > 0$ and the character $\varepsilon_{r,d-1,1} : G \to 1 + \mathfrak{M}_{K_{\alpha+1}}$ is well defined.

Set:

$$\xi = \frac{\varepsilon_{r,d-1,1} - \varepsilon_{r,d-1,1}^{-1}\chi}{\pi^{d-1}}$$

Using Lemma 2.3.10 applied with $E = K_{\alpha+1}$, $t = d-1$ and $u = 1$ we see that ξ is an $\mathcal{O}_{K_{\alpha+1}}$-valued function on $\mu_{p^\alpha} \times \Phi^{\mathbb{Z}_p}$. We see that $\mathrm{ord}_\pi(\xi(\zeta_{p^\alpha})) > 0$, while:

$$\xi(\Phi) \sim \frac{\pi^{d-1}(2 + \pi^{d-1}) - \pi^{e\alpha} w}{\pi^{d-1}}$$

for some $w \in \mathcal{O}_{K_{\alpha+1}}^\times$. This implies that $\xi(\Phi)$ is a unit in $\mathcal{O}_{K_{\alpha+1}}$ and $t_3 = \xi(\Phi)^{-1}$ is an element of $\mathcal{O}_{K_{\alpha+1}}^\times$. We deduce that the continuous function $t_3 \cdot \xi$ has values in $\mathcal{O}_{K_{\alpha+1}}$ and ρ_z is an $\mathcal{O}_{K_{\alpha+1}}$-linear representation such that $\bar{\rho}_z = \bar{\rho}$ as desired.

Finally, since $r \in [1, \frac{p^\alpha - 1}{2}]$ the definition of the equivalence relation $\approx$ implies that the components defined above are pairwise distinct. ■

Remark 2.3.13. One way to determine the possible shapes of the lifts of $\bar{\rho}$ in the above settings is to notice that if $\rho = \varepsilon_1 \oplus \varepsilon_2$ is a degree two K-linear representation $G \to \mathrm{Aut}_K(V)$, and if

$\tilde{\rho}: G \to \mathrm{Aut}_{\mathcal{O}_K}(L)$ is an $\mathcal{O}_K$-integral model of ρ, there is an $\mathcal{O}_K$-basis of the $\mathcal{O}_K$-lattice $L \subset V$ such that the matricial form of $\tilde{\rho}$ in this basis is:

$$\begin{pmatrix} \varepsilon_1 & r(\varepsilon_1 - \varepsilon_2) \\ & \varepsilon_2 \end{pmatrix}$$

for some $r \in K$ which makes the function $r(\varepsilon_1 - \varepsilon_2)$ to be $\mathcal{O}_K$-valued.

3. Irreducible lifts

We maintain the notation introduced at the beginning of 2. We furthermore set $\alpha = \mathrm{ord}_p(l^f - 1)$ and $\beta = \mathrm{ord}_p(l^f + 1)$. We now look at the existence of geometrically irreducible components of type III in the generic fiber X of the framed deformation space of $\bar{\rho}$.

3.1. **Preliminary results.**

Proposition 3.1.1. *Let $\mathcal{C}$ be a geometrically irreducible component of X and suppose that there is a $\bar{K}$-rational point on $\mathcal{C}$ whose associated $\bar{K}$-linear representation σ is absolutely irreducible. Then:*

- **(a):** *If x is any geometric point lying on $\mathcal{C}$, we have $\rho_x \simeq \sigma$ as representations over $\bar{K}$.*
- **(b):** *$\mathcal{C}$ is the only geometrically irreducible component of X containing a geometric point whose associated representation is $\bar{K}$-isomorphic to σ.*

Proof Part (a) follows from the proof of Proposition 4.10.3 in [5]. Part (b) is a consequence of the results of Chapter 4 of [1] (notice that in *loc. cit.* the role of p and l is switched with respect to our conventions). When $\bar{\rho}$ is reducible, (b) also follows from the proof of Proposition 4.10.8 of [5]. ■

We adopt the following terminology. Let F be an algebraically closed field of characteristic different from two. Fix an index-two closed subgroup H of a topological group P and let γ be a homomorphism from H to $F^\times$. For a choice of element s in $P - H$ we denote by γ^c the character of H defined by $\gamma^c(h) = \gamma(s^{-1}hs)$ for $h \in H$; notice that the choice of s is irrelevant and that γ coincides with its P/H-conjugate γ^c if and only if it can be continuously extended to a character of P. If that is the case, γ has exactly two extensions to P: if we denote one of them by $\tilde{\gamma}$, the other is $\tilde{\gamma}\chi_{P/H}$ where $\chi_{P/H}$ is the only character of P over F whose kernel equals H.

We denote by $\mathrm{Ind}_H^P(\gamma)$ the group homomorphism $\rho : P \to GL_2(F)$ defined by the following matrices:

$$\rho(g) = \begin{cases} \begin{pmatrix} \gamma(g) & \\ & \gamma^c(g) \end{pmatrix} & \text{if } g \in H \\ \\ \begin{pmatrix} & \gamma(gs) \\ \gamma(s^{-1}g) & \end{pmatrix} & \text{if } g \in P - H. \end{cases}$$

We have $\mathrm{Ind}_H^P(\gamma) \simeq \mathrm{Ind}_H^P(\gamma^c)$. The F-representation $\mathrm{Ind}_H^P(\gamma)$ is irreducible if and only if $\gamma \neq \gamma^c$. When $\gamma = \gamma^c$, $\mathrm{Ind}_H^P(\gamma)$ is isomorphic to $\tilde{\gamma}\chi_{P/H} \oplus \tilde{\gamma}$, with $\tilde{\gamma}$ an extension of γ to P.

Lemma 3.1.2. *Let M/L be a quadratic extension and let E/K be a finite extension. Fix a character $\gamma : G_M \to \mathcal{O}_E^\times$ with trivial reduction. Then:*

- **(a):** *If M/L is ramified, then γ extends to a character of G.*
- **(b):** *If $\alpha = \beta = 0$ or $\alpha > 0$, and M/L is unramified, then γ extends to G.*
- **(c):** *If $\alpha = 0$, $\beta > 0$ and M/L is unramified, then γ extends to G if and only if γ is unramified.*

Proof Let $\alpha' = \mathrm{ord}_p(l^{f(M/\mathbb{Q}_l)} - 1)$. Choose a uniformizer ς of M; if M/L is ramified (resp. unramified) we can assume that $\epsilon = \varsigma^2$ (resp. $\epsilon = \varsigma$) is a uniformizer for L.

We normalize the reciprocity law of local class field theory so that uniformizers corresponds to arithmetic Frobenii. There is a commutative diagram:

$$\begin{array}{ccccccc} G_M & \longrightarrow & G_M^{ab,(p)} & \overset{\sim}{\longrightarrow} & \mu_{p^{\alpha'}} \times \varsigma^{\mathbb{Z}_p} & \overset{\gamma}{\longrightarrow} & 1 + \mathfrak{M}_E \\ \downarrow & & \downarrow & & \downarrow \mathfrak{N} & & \\ G & \longrightarrow & G^{ab,(p)} & \overset{\sim}{\longrightarrow} & \mu_{p^\alpha} \times \epsilon^{\mathbb{Z}_p} & & \end{array}$$

in which the central horizontal isomorphisms are induced by the reciprocity maps, $\mathfrak{N}$ is induced by the norm map of M over L, the first two vertical maps are induced by the inclusion of L in M, and γ denotes the morphism induced by the given character $G_M \to \mathcal{O}_E^\times$ (also denoted γ) on $\mu_{p^{\alpha'}} \times \varsigma^{\mathbb{Z}_p}$. Assume M/L is ramified, so that $\alpha = \alpha'$. As the image under $G^{ab} \overset{\sim}{\to} \mathcal{O}_L^\times \times \epsilon^{\hat{\mathbb{Z}}}$ of $\ker(G^{ab} \to G^{ab,(p)})$ contains -1, we see that $\mathfrak{N}(\zeta_{p^\alpha}^i \varsigma^j) = \zeta_{p^\alpha}^{2i} \epsilon^j$ for all integers i and j. In particular $\mathfrak{N}$ is an isomorphism and γ extends to a character of G with values in $1 + \mathfrak{M}_E$. This proves (a).

Assume that M/L is unramified, so that $\mathfrak{N}(\zeta_{p^{\alpha'}}^i \varsigma^j) = \zeta_{p^\alpha}^{2i} \epsilon^{2j}$. In this case $\alpha' = \alpha + \beta$, so that if $\alpha > 0$ we have $\alpha = \alpha'$ and $\mathfrak{N}$ is an isomorphism. Similarly if $\alpha = \beta = 0$ we have $\alpha' = 0$ and $\mathfrak{N}$ is bijective. This proves (b).

To prove (c), assume M/L is unramified. Suppose that $\alpha = 0$ and $\alpha' = \beta > 0$. Define a continuous character $\gamma' : \epsilon^{\mathbb{Z}_p} \to 1 + \mathfrak{M}_E$ by setting $\gamma'(\epsilon) = \gamma(\varsigma)^{1/2}$. If γ is unramified, $\gamma' \circ \mathfrak{N} = \gamma$ so that γ' defined an extension of γ to G. Viceversa, assume that γ extends to a character $\tilde{\gamma} : G \to \mathcal{O}_E^\times$. By contradiction suppose that there is $g \in G_M$ such that $\gamma(g) \neq 1$ and g is sent by the fixed epimorphism $G_M \to \mu_{p^{\alpha'}} \times \varsigma^{\mathbb{Z}_p}$ to $\zeta_{p^{\alpha'}}$. Then

$$1 \neq \tilde{\gamma}(g) = \gamma''(\mathfrak{N}(\zeta_{p^{\alpha'}})) = \gamma''(1) = 1$$

for some character $\gamma'' : \mathcal{O}_L^\times \times \epsilon^{\hat{\mathbb{Z}}} \to \mathcal{O}_E^\times$ induced by $\tilde{\gamma}$. This is impossible. ■

We introduce another piece of notation. Recall that we have chosen an arithmetic Frobenius Φ of G. Assume for this definition that $L(\zeta_p)$ is a quadratic extension of L so that Φ^2 is an arithmetic Frobenius for $G_{L(\zeta_p)}$ and $\beta > 0$. By local class field theory we obtain an identification $G_{L(\zeta_p)}^{ab,(p)} \simeq \mu_{p^\beta} \times \varsigma^{\mathbb{Z}_p}$ sending Φ^2 to a uniformizer ς of $L(\zeta_p)$. For any integer r and for any extension E of K containing ζ_{p^β} we denote by γ_r (resp. $\gamma_{r,1}$) the homomorphism

$$G_{L(\zeta_p)} \longrightarrow 1 + \mathfrak{M}_E$$

obtained by sending ζ_{p^β} to $\zeta_{p^\beta}^r$ and ς to $-l^f$ (resp. ς to 1). If an extension E is not specified, we see γ_r (resp. $\gamma_{r,1}$) as a character with $K(\zeta_{p^\beta})$-values.

3.2. **Semisimple reducible $\bar{\rho}$.** Assume $\bar{\rho} = \vartheta \oplus 1$ for some character $\vartheta : G \to k^\times$. We saw that if $\vartheta \neq \omega^{\pm 1}$ and $\omega \neq 1$ then $R_{\bar{\rho}}^{\square,\psi}$ is an integral domain and X has exactly one irreducible component of type II_1 (resp. IV) if ϑ is unramified (resp. ramified). We therefore assume that either $\omega = 1$ or $\omega \neq 1$ and $\vartheta = \omega$.

Proposition 3.2.1. *If $\omega = 1$ and $\psi = \tilde{\vartheta}\chi$, or if $\vartheta = \omega \neq 1$, $[L(\zeta_p) : L] \neq 2$ and $\psi = \chi$, then X has no geometrically irreducible components of type III.*

Proof Let $\rho : G \to GL_2(\bar{K})$ be an absolutely irreducible representation lifting $\bar{\rho}$, say $\rho \simeq_{\bar{K}} \mathrm{Ind}_M^L(\gamma)$ for a quadratic extension M/L and a character $\gamma : G_M \to \mathcal{O}_{\bar{K}}^{\times}$ such that γ is different from its G/G_M-conjugate γ^c.

Assume $\omega = 1$ and $\psi = \tilde{\vartheta}\chi$. Since $\bar{\rho}$ is reducible, we have $\bar{\gamma} = \bar{\gamma}^c$ and $\bar{\gamma}$ extends to G; denoting by $\tilde{\gamma}$ any such extension, we obtain $\bar{\rho} \simeq \tilde{\gamma}\bar{\chi}_{M/L} \oplus \tilde{\gamma}$ where $\chi_{M/L}$ is the quadratic $\bmod p$ character associated to the extension M/L. If $\tilde{\gamma} = 1$, parts (a) and (b) of Lemma 3.1.2 imply that $\gamma = \gamma^c$. Then $\tilde{\gamma}\bar{\chi}_{M/L} = 1$ which again implies $\bar{\gamma} = 1$ and $\gamma = \gamma^c$, so that there are no components of type III in this case.

Assume now $\vartheta = \omega \neq 1$ and $\psi = \chi$. As before we find $\bar{\rho} \simeq \tilde{\gamma}\bar{\chi}_{M/L} \oplus \tilde{\gamma}$, which implies $\omega = \bar{\chi}_{M/L}$ and therefore $[L(\zeta_p) : L] = 2$. ■

Proposition 3.2.2. *Assume $\vartheta = \omega \neq 1$, $[L(\zeta_p) : L] = 2$ and $\psi = \chi$. Then $\beta > 0$ and $X_{\bar{K}}$ has $\frac{p^{\beta}-1}{2}$ geometrically irreducible components of type III, say $\mathcal{C}_{III}^r$ ($1 \le r \le \frac{p^{\beta}-1}{2}$). Let $K_{\beta} = K(\zeta_{p^{\beta}})$: there is a point $x \in \mathcal{C}_{III}^r(K_{\beta})$ such that*

$$\rho_x = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \mathrm{Ind}_{L(\zeta_p)}^L(\gamma_r) \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

as $\mathcal{O}_{K_{\beta}}$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{III}^r$ are all and only $\bar{K}$-isomorphic to the above ρ_x.

Proof By way of Proposition 3.1.1 we need to classify the $\bar{K}$-linear isomorphism classes of absolutely irreducible lifts of $\bar{\rho}$. Fix a finite extension K' of K and let $\rho : G \to GL_2(K')$ be an absolutely irreducible representation having determinant χ; assume that ρ has an integral model whose reduction is equal to $\bar{\rho}$. We can assume $\rho = \mathrm{Ind}_M^L(\gamma)$ with M a quadratic extension of L and $\gamma : G_M \to \mathcal{O}_{K'}^{\times}$ a character which is different from its G/G_M-conjugate. Since any integral model of ρ has reducible reduction we deduce that $\bar{\gamma} = \bar{\gamma}^c$ and $\vartheta \oplus 1 \simeq \tilde{\gamma}\bar{\chi}_{M/L} \oplus \tilde{\gamma}$, where $\tilde{\gamma}$ denotes any fixed extension of $\bar{\gamma}$ to G. This implies that $\vartheta = \omega = \bar{\chi}_{M/L}$ so that $M = L(\zeta_p)$. Since $\bar{\gamma} = 1$, part (c) of Lemma 3.1.2 implies that γ is ramified. Assuming without loss of generality that $\zeta_{p^{\beta}} \in K'$ we see that γ factors as:

$$\gamma : G_{L(\zeta_p)} \longrightarrow \mu_{p^{\beta}} \times \varsigma^{\mathbb{Z}_p} \longrightarrow 1 + \mathfrak{M}_{K'}$$

and satisfies $\gamma(\zeta_{p^{\beta}}) = \zeta_{p^{\beta}}^r$ for some integer $r \in [1, p^{\beta} - 1]$.

Since $\Phi \notin G$ and Φ^2 is an arithmetic Frobenius in $G_{L(\zeta_p)}$ we obtain using the determinant condition:

$$\gamma\gamma^c(\Phi^2) = \gamma(\Phi^2)^2 = l^{2f}.$$

Assume that in the projection $G_{L(\zeta_p)} \to \mu_{p^{\beta}} \times \varsigma^{\mathbb{Z}_p}$, Φ^2 corresponds to the uniformizer ς, so that $\gamma(\varsigma) = \pm l^f$. Since the image of γ is contained in $1 + \mathfrak{M}_{K'}$, we have that $\gamma(\varsigma) - 1$ is an integer divisible by p. Since $\alpha = 0$ this means that $\gamma(\varsigma) = -l^f$.

We conclude that our original representation ρ is $\bar{K}$-isomorphic to $\rho_r := \mathrm{Ind}_{L(\zeta_p)}^L(\gamma_r)$ for some integer $r \in [1, p^{\beta} - 1]$. We claim that if $r, s \in [1, p^{\beta} - 1]$ then:

$$\rho_r \simeq_{\bar{K}} \rho_s \text{ if and only if } r \equiv \pm s (\bmod p^{\beta}).$$

Indeed, if $\rho_r \simeq_{\bar{K}} \rho_s$ pick an element $i \in I_G = I_{G_{L(\zeta_p)}}$ whose image under the surjection $G_{L(\zeta_p)} \to \mu_{p^{\beta}} \times \varsigma^{\mathbb{Z}_p}$ equals $\zeta_{p^{\beta}}$. By Kummer theory together with the fact that $\bar{\gamma}_r = 1$, we see that

$\gamma_r^c(i) = \gamma_r(i)^{l^{-f}}$. The equality of traces of $\rho_r(i)$ and $\rho_s(i)$ implies then that:

$$\zeta_{p^\beta}^r + \zeta_{p^\beta}^{rl^{-f}} = \zeta_{p^\beta}^s + \zeta_{p^\beta}^{sl^{-f}}.$$

Since $l^{-f} = -1$ in $\mathbb{Z}/p^\beta\mathbb{Z}$ we deduce that $\zeta_{p^\beta}^r + \zeta_{p^\beta}^{-r} = \zeta_{p^\beta}^s + \zeta_{p^\beta}^{-s}$ so that $r \equiv \pm s (\operatorname{mod} p^\beta)$. On the other side, by similar computations one sees that $\gamma_r^c = \gamma_{-r}$, so that the claim is proved.

Fix now an integer $r \in [1, \frac{p^\beta-1}{2}]$ and set $\tilde{\rho}_r = \left(\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right)^{-1} \rho_r \left(\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, viewed as a representation with coefficients in K_β. To conclude the proof of the proposition, we are left with checking that $\tilde{\rho}_r$ is an integral representation with determinant χ and that it reduces to $\bar{\rho}$.

Since $\gamma_r^c = \gamma_{-r}$, the determinant of $\tilde{\rho}_r$ is unramified. As $\Phi \notin G_{L(\zeta_p)}$ we have $\det \tilde{\rho}_r(\Phi) = -\gamma_r(\Phi^2) = l^f$ and we conclude $\det \tilde{\rho}_r = \chi$. An explicit matrix computation shows that:

$$\tilde{\rho}_r(g) = \begin{cases} \dfrac{1}{2}\begin{pmatrix} \gamma_r(g) + \gamma_r^c(g) & \gamma_r^c(g) - \gamma_r(g) \\ \gamma_r^c(g) - \gamma_r(g) & \gamma_r(g) + \gamma_r^c(g) \end{pmatrix} & \text{if } g \in G_{L(\zeta_p)} \\ \\ \dfrac{1}{2}\begin{pmatrix} -\gamma_r(g\Phi) - \gamma_r(\Phi^{-1}g) & -\gamma_r(g\Phi) + \gamma_r(\Phi^{-1}g) \\ \gamma_r(g\Phi) - \gamma_r(\Phi^{-1}g) & \gamma_r(g\Phi) + \gamma_r(\Phi^{-1}g) \end{pmatrix} & \text{if } g \in G - G_{L(\zeta_p)}. \end{cases}$$

This implies that $\tilde{\rho}_r$ is $\mathcal{O}_{K_\beta}$-integral and its reduction modulo $\mathfrak{M}_{K_\beta}$ equals $\bar{\rho}$, as $\bar{\gamma}_r = \bar{\gamma}_r^c = 1$ and $\vartheta = \omega = \bar{\chi}_{L(\zeta_p)/L}$. ■

3.3. **Non-semisimple $\bar{\rho}$.** Assume $\bar{\rho}$ is a non-split extension of 1 by ϑ, so that either $\vartheta = 1$ or $\vartheta = \omega$.

Proposition 3.3.1. *If $\bar{\rho}$ is a non-split extension of 1 by 1, then X does not have any geometrically irreducible component of type III.*

Proof If x is a $\bar{K}$-rational point on a component of type III, $\rho_x \simeq_{\bar{K}} \operatorname{Ind}_M^L(\gamma)$ where M is a quadratic extension of L and γ is a character of G_M which satisfies $\gamma \neq \gamma^c$. Since $\bar{\rho}$ is reducible, we have $\bar{\gamma} = \bar{\gamma}^c$; since $\bar{\rho}^{ss} = 1 \oplus 1$, if we denote by $\tilde{\gamma}$ any extension of $\bar{\gamma}$ to G we have $\tilde{\gamma} = 1 = \tilde{\gamma}\bar{\chi}_{M/L}$, so that $\bar{\chi}_{M/L} = 1$, which is impossible. ■

Assume $\vartheta = \omega$, $\omega \neq 1$ and set $\psi = \chi$. We can assume:

$$\bar{\rho} = \begin{pmatrix} \omega & \bar{t}_p \\ & 1 \end{pmatrix},$$

where the 1-cocycle $\bar{t}_p$ is defined at the beginning of section 2.

Proposition 3.3.2. *Assume $\vartheta = \omega \neq 1$ and $\psi = \chi$.*

- **(a):** *If $[L(\zeta_p) : L] \neq 2$, X does not have any geometrically irreducible component of type III.*
- **(b):** *If $[L(\zeta_p) : L] = 2$, X has $\frac{p^\beta-1}{2}$ geometrically irreducible components of type III, say $\mathcal{C}_{III}^r$ $(1 \leq r \leq \frac{p^\beta-1}{2})$. Let $K_\beta = K(\zeta_{p^\beta})$: there is a point $x \in \mathcal{C}_{III}^r(K_\beta)$ such that for some $X \in GL_2(K_\beta)$*

$$\rho_x = X^{-1} \cdot \operatorname{Ind}_{L(\zeta_p)}^L(\gamma_r) \cdot X$$

as $\mathcal{O}_{K_\beta}$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{III}^r$ are all and only $\bar{K}$-isomorphic to the above ρ_x.

Proof Let K' be a finite extension of K. If x is a K'-rational point on a component of type III, $\rho_x \simeq_{K'} \mathrm{Ind}_M^L(\gamma)$ where M is a quadratic extension of L and γ is a character of G_M which satisfies $\gamma \neq \gamma^c$. We have $\bar{\gamma} = \bar{\gamma}^c$ and from $\bar{\rho}^{ss} = \omega \oplus 1$ we deduce $\omega = \bar{\chi}_{M/L}$ and hence $M = L(\zeta_p)$. Let us now assume $[L(\zeta_p) : L] = 2$ and $M = L(\zeta_p)$.

As in the proof of Proposition 3.2.2, we see that we can assume $K' = K_\beta$ and we have $\rho_x \simeq_{K_\beta} \rho_r$ where $\rho_r = \mathrm{Ind}_{L(\zeta_p)}^L(\gamma_r)$ for a uniquely determined integer $r \in [1, \frac{p^\beta - 1}{2}]$. Since the semisimple reduction of ρ_r is isomorphic to $\omega \oplus 1$, by Ribet's lemma ([6], Proposition 2.1) we can find a matrix $Y \in GL_2(K_\beta)$ such that $Y^{-1}\rho_r Y$ is $\mathcal{O}_{K_\beta}$-integral and has reduction equal to $\begin{pmatrix} \omega & c\bar{t}_p \\ & 1 \end{pmatrix}$ for some $c \neq 0$ in the residue field of K_β. Then we can also find $X \in GL_2(K_\beta)$ such that $X^{-1}\rho_r X$ is $\mathcal{O}_{K_\beta}$-integral and has reduction equal to $\bar{\rho}$. Since $\det(X^{-1}\rho_r X) = \chi$, we are done. ■

3.4. **Absolutely irreducible $\bar{\rho}$.** Suppose that $\bar{\rho}$ is absolutely irreducible. We can assume that ([2] 2.3, 2.4):

$$\bar{\rho} = \mathrm{Ind}_M^L(\theta)$$

where M is a quadratic extension of L and $\theta : G_M \to k^\times$ is a character which is different from its G/G_M conjugate θ^c. We set ψ to be the Teichmüller lift of $\det \bar{\rho}$ and we consider as usual framed deformations of $\bar{\rho}$ having determinant ψ. The generic fiber $X_{\bar{K}}$ of the corresponding deformation space has only components of type III.

A straightforward computation shows that $H^2(G, \mathrm{ad}^0 \bar{\rho})$ is non-trivial if and only if one of the following mutually exclusive conditions is satisfied:

(IRR1) $M = L(\zeta_p)$, or
(IRR2) $L(\zeta_p)/L$ is quadratic and $\theta^c = \theta \otimes \omega_{|G_M}$.

If (IRR1) and (IRR2) are both false, X is an integral scheme and up to $\bar{K}$-isomorphism there is only one integral lift of $\bar{\rho}$ defining a point on X, namely $\mathrm{Ind}_M^L(\tilde{\theta})$.

Lemma 3.4.1. *Assume that* (IRR1) *does not hold and* (IRR2) *holds. Then $\bar{\rho}$ is isomorphic to $\mathrm{Ind}_{L(\zeta_p)}^L(\eta)$ for some character $\eta : G_{L(\zeta_p)} \to k^\times$ which satisfies $\eta \neq \eta^c$.*

Proof Let δ denote the restriction of θ to $G_{M(\zeta_p)}$. Fix an element $s \in G_{L(\zeta_p)} - G_{M(\zeta_p)}$; the value at $g \in G_{M(\zeta_p)}$ of the conjugate δ^c of δ with respect to the quadratic extension $M(\zeta_p)/L(\zeta_p)$ can be computed as:

$$\delta^c(g) = \delta(s^{-1}gs) = \theta^c(g),$$

where θ^c is the conjugate of θ with respect to the extension M/L. By (IRR2), $\delta^c(g)$ equals $(\theta \otimes \omega)(g) = \theta(g)\bar{\chi}_{L(\zeta_p)/L}(g) = \delta(g)$, so that δ extends to a character η of $G_{L(\zeta_p)}$. Since $\theta_{|M(\zeta_p)} = \eta_{|M(\zeta_p)}$ we have:

$$\mathrm{Ind}_M^L \mathrm{Ind}_{M(\zeta_p)}^M(\theta_{|M(\zeta_p)}) \simeq \mathrm{Ind}_{L(\zeta_p)}^L \mathrm{Ind}_{M(\zeta_p)}^{L(\zeta_p)}(\eta_{|M(\zeta_p)}). \quad (1)$$

As $\theta_{|M(\zeta_p)}$ extends to M we have

$$\mathrm{Ind}_{M(\zeta_p)}^M(\theta_{|M(\zeta_p)}) \simeq \theta \oplus \theta\bar{\chi}_{|M(\zeta_p)/M}.$$

Similarly:

$$\mathrm{Ind}_{M(\zeta_p)}^{L(\zeta_p)}(\eta_{|M(\zeta_p)}) \simeq \eta \oplus \eta\bar{\chi}_{L(\zeta_p)/M(\zeta_p)}.$$

We conclude from (1) that $\mathrm{Ind}_M^L(\theta) \oplus \mathrm{Ind}_M^L\left(\theta\bar{\chi}_{|M(\zeta_p)/M}\right) \simeq \mathrm{Ind}_{L(\zeta_p)}^L(\eta) \oplus \mathrm{Ind}_{L(\zeta_p)}^L(\eta\bar{\chi}_{L(\zeta_p)/M(\zeta_p)})$. Since $\bar{\chi}_{|M(\zeta_p)/M} = \omega_{|G_M}$, (IRR2) implies that the representation $\mathrm{Ind}_M^L\left(\theta\bar{\chi}_{|M(\zeta_p)/M}\right)$ is isomorphic to $\mathrm{Ind}_M^L(\theta)$ and:

$$\mathrm{Ind}_M^L(\theta) \oplus \mathrm{Ind}_M^L(\theta) \simeq \mathrm{Ind}_{L(\zeta_p)}^L(\eta) \oplus \mathrm{Ind}_{L(\zeta_p)}^L(\eta\bar{\chi}_{L(\zeta_p)/M(\zeta_p)}).$$

As the left hand side above has length two, it follows that $\bar{\rho}$ is isomorphic to $\mathrm{Ind}_{L(\zeta_p)}^L(\eta)$. ■

By the above lemma, we can assume from now on that $M = L(\zeta_p)$, which implies $\omega \neq 1$ (*i.e.*, $\alpha = 0$) and $\beta > 0$. Recall the characters $\gamma_{r,1}$ introduced in 3.1 and recall that the symbol $\sim$ denotes Teichmüller lifting for some finite extension of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$.

Proposition 3.4.2. *Assume $M = L(\zeta_p)$ and set $\psi = \widetilde{\det\bar{\rho}}$. Then $X_{\bar{K}}$ has p^β geometrically irreducible components of type III, say $\mathcal{C}_{III}^r$ ($0 \leq r \leq p^\beta - 1$). Let $K_\beta = K(\zeta_{p^\beta})$: there is a point $x \in \mathcal{C}_{III}^r(K_\beta)$ such that*

$$\rho_x = \mathrm{Ind}_{L(\zeta_p)}^L(\tilde{\theta} \otimes \gamma_{r,1})$$

as $\mathcal{O}_{K_\beta}$-representation. The lifts of $\bar{\rho}$ associated to $\bar{K}$-rational points on $\mathcal{C}_{III}^r$ are all and only $\bar{K}$-isomorphic to ρ_x.

Proof Let M_1 be a quadratic extension of L inside $\bar{\mathbb{Q}}_l$ and let $\rho = \mathrm{Ind}_{M_1}^L(\delta)$ be a degree two absolutely irreducible representation of G defined over some finite extension K' of K. Assume that ρ has determinant ψ and has an integral model whose reduction equals $\bar{\rho}$. This implies $\mathrm{Ind}_{M_1}^L(\bar{\delta}) \simeq \mathrm{Ind}_{L(\zeta_p)}^L(\theta)$, so that $\bar{\delta}$ is different from its G/G_{M_1}-conjugate $\bar{\delta}^c$.

We claim that we can suppose $M_1 = L(\zeta_p)$. Assume $M_1 \neq L(\zeta_p)$ and set $F = M_1(\zeta_p)$. Since $\mathrm{Ind}_{M_1}^L(\bar{\delta}) \simeq \mathrm{Ind}_{L(\zeta_p)}^L(\theta)$ we can assume $\bar{\delta}_{|G_F} = \theta_{|G_F}$, which implies

$$\delta_{|G_F} = \tilde{\theta}_{|G_F} \otimes \eta$$

for some character $\eta : G_F \to 1 + \mathfrak{M}_{K'}$. Denote by ς a uniformizer of $L(\zeta_p)$, so that $\varsigma = \nu^2$ for a choice of uniformizer ν of F. The norm map $F^\times \to L(\zeta_p)^\times$ induces an isomorphism $\mu_{p^\beta} \times \nu^{\mathbb{Z}_p} \to \mu_{p^\beta} \times (-\varsigma)^{\mathbb{Z}_p}$, so that the character η can be extended to a character of $G_{L(\zeta_p)}$. Since the same is true by assumption for $\tilde{\theta}_{|G_F}$, we conclude that $\delta_{|G_F}$ extends to a character δ_1 of $G_{L(\zeta_p)}$. It is easy to check that δ_1 is different from its $G/G_{L(\zeta_p)}$-conjugate. We conclude that $\rho = \mathrm{Ind}_{M_1}^L(\delta) \simeq \mathrm{Ind}_{L(\zeta_p)}^L(\delta_1)$.

Assume now $M_1 = L(\zeta_p)$ and $\rho = \mathrm{Ind}_{L(\zeta_p)}^L(\delta)$ as above; also suppose $\zeta_{p^\beta} \in K'$. The isomorphism $\mathrm{Ind}_{L(\zeta_p)}^L(\bar{\delta}) \simeq \mathrm{Ind}_{L(\zeta_p)}^L(\theta)$ implies $\bar{\delta} = \theta$ or $\bar{\delta} = \theta^c$. We assume $\bar{\delta} = \theta$ as the other case can be treated similarly and does not produce new irreducible components. We write $\delta = \tilde{\theta} \otimes \gamma$ where

$$\gamma : G_{L(\zeta_p)} \to G_{L(\zeta_p)}^{ab,(p)} \simeq \mu_{p^\beta} \times \varsigma^{\mathbb{Z}_p} \to 1 + \mathfrak{M}_{K'}$$

(ς is a uniformizer of $L(\zeta_p)$) sends ζ_{p^β} to $\zeta_{p^\beta}^r$ for some integer $r \in [0, p^\beta - 1]$ and sends ς to some $m \in 1 + \mathfrak{M}_{K'}$. By the proof of Proposition 3.2.2 and by Lemma 3.1.2 (c), it follows that the $G/G_{L(\zeta_p)}$-conjugate γ^c of γ sends ζ_{p^β} to $\zeta_{p^\beta}^{-r}$ and ς to m.

Since the determinant of the representation $\mathrm{Ind}_{L(\zeta_p)}^L(\delta)$ is assumed to be equal to ψ, we obtain that for $g \in G_{L(\zeta_p)}$ which is sent to ς by the reciprocity map we have

$$\delta\delta^c(g) = \tilde{\theta}\gamma\tilde{\theta}^c\gamma^c(g) = \tilde{\theta}\tilde{\theta}^c(g).$$

This implies $m^2 = 1$ and hence $m = 1$ since $1+\mathfrak{M}_{K'}$ is a pro-p group. We conclude that $\gamma = \gamma_{r,1}$ and:

$$\rho = \rho_r := \mathrm{Ind}_{L(\zeta_p)}^{L}(\tilde{\theta} \otimes \gamma_{r,1}).$$

Observe that ρ_r is an $\mathcal{O}_{K_\beta}$-integral representation whose reduction modulo $\mathfrak{M}_{K_\beta}$ is irreducible and equals $\mathrm{Ind}_{L(\zeta_p)}^{L}(\theta)$, as $\bar{\gamma}_{r,1} = 1$. Furthermore, if $\rho_r \simeq_{\bar{K}} \rho_s$ we must have $\tilde{\theta}\gamma_{r,1} = \tilde{\theta}\gamma_{s,1}$ or $\tilde{\theta}\gamma_{r,1} = \tilde{\theta}^c\gamma_{-s,1}$: the first option implies $r \equiv s (\mathrm{mod}\, p^\beta)$, while the second option cannot occur as reducing modulo $\mathfrak{M}_{K_\beta}$ it would imply $\theta = \theta^c$.

To conclude the proof, in view of Proposition 3.1.1, it is then enough that we show $\det \rho_r = \psi$. If $g \in G_{L(\zeta_p)}$ we have $\det \rho_r(g) = \tilde{\theta}\gamma_{r,1}\tilde{\theta}^c\gamma_{-r,1}(g) = \tilde{\theta}\tilde{\theta}^c(g) = \psi(g)$. Assume $g \in G - G_{L(\zeta_p)}$: we need to check that $\gamma_{r,1}(g^2) = 1$ (notice that g^2 is an element of $G_{L(\zeta_p)}$).
Assume $g = \iota\Phi^{-n}$ with $\iota \in I_G$ and $n \in \hat{\mathbb{Z}} - 2\hat{\mathbb{Z}}$, so that

$$\gamma_{r,1}(g^2) = \gamma_{r,1}(\iota)\gamma_{r,1}(\Phi^{-n}\iota\Phi^n)\gamma_{r,1}(\Phi^{-2n}) = \gamma_{r,1}(\iota)\gamma_{r,1}(\Phi^{-n}\iota\Phi^n).$$

Let $\zeta_{p^\beta}^a$ be the image of ι under the projection $G_{L(\zeta_p)}^{ab,(p)} \to \mu_{p^\beta}$. By Kummer theory and the definition of $\gamma_{r,1}$ we obtain:

$$\gamma_{r,1}(\Phi^{-n}\iota\Phi^n) = \zeta_{p^\beta}^{l^{-nf}ar}$$

so that $\gamma_{r,1}(\iota)\gamma_{r,1}(\Phi^{-n}\iota\Phi^n) = \zeta_{p^\beta}^{(1+l^{-nf})ar} = 1$, as $l^f \equiv -1 (\mathrm{mod}\, p^\beta)$ and $n \notin 2\hat{\mathbb{Z}}$. ■

4. Counting the irreducible components

For convenience of the reader we recall our notation. Let l be an odd prime and fix a finite extension L of $\mathbb{Q}_l$ whose residue field has cardinality l^f. Let G be the absolute Galois group of L. Fix an odd prime $p \neq l$ and let K be a finite extension of $\mathbb{Q}_p$ having residue field k. Denote by χ be the p-adic cyclotomic character of G and by ω its reduction modulo p. Set

$$\alpha = \mathrm{ord}_p(l^f - 1), \;\; \beta = \mathrm{ord}_p(l^f + 1).$$

We denote by $\sim$ the operation of taking Teichmüller liftings of elements in the residue field of $\bar{\mathbb{Q}}_p$. For any non-negative integer m we denote by K_m the extension of K in $\bar{\mathbb{Q}}_p$ generated by the p^mth roots of unity.

Fix a continuous representation $\bar{\rho} : G \to GL_2(k)$ and let $\psi : G \to \mathcal{O}_K^\times$ be a lifting of the determinant character of $\bar{\rho}$. Denote by $R_{\bar{\rho}}^{\square,\psi}$ the complete local noetherian $\mathcal{O}_K$-algebra representing the functor $D_{\bar{\rho}}^{\square,\psi}$ and let X denotes the generic fiber $\mathrm{Spec}\, R_{\bar{\rho}}^{\square,\psi}\left[\frac{1}{p}\right]$ of the corresponding deformation K-scheme. In view of what remarked at the beginning of 2, in what follows we can replace k with a finite extension, we can change $\bar{\rho}$ by a twist, and we are allowed to fix a specific value for ψ.

If $\mathcal{C}$ is a geometrically irreducible component of X, we say that it is of type I, II, III or IV according to the classification given in Theorem 2.1.1. For ? in the set $\{I, II, III, IV\}$, we let $c_?$ be the number of geometrically irreducible components of type ? contained in X. We will denote by $K_?$ a finite extension of K inside $\bar{\mathbb{Q}}_p$ over which *every* component of type ? has a rational point. We set:

$$\mathbf{c} = (c_I, c_{II}, c_{III}, c_{IV})$$
$$\mathbf{K} = (K_I, K_{II}, K_{III}, K_{IV})$$

and let $c = c_I + c_{II} + c_{III} + c_{IV}$ and $K_c = K_I K_{II} K_{III} K_{IV}$.

We now summarize some of the results of the previous sections. We distinguish three cases depending on whether (A) $[L(\zeta_p) : L] = 1$, (B) $[L(\zeta_p) : L] = 2$ or (C) $[L(\zeta_p) : L] > 2$. These three cases also correspond to, respectively: $\alpha > 0$ and $\beta = 0$; $\alpha = 0$ and $\beta > 0$; $\alpha = \beta = 0$. Notice that since p is odd it cannot happen that $\alpha > 0$ and $\beta > 0$.

Theorem 4.0.3 (A)**.** *Assume $[L(\zeta_p) : L] = 1$ (i.e., $\omega = 1$).*

- A1. *Let $\bar{\rho} = \vartheta \oplus 1$ where $\vartheta : G \to k^\times$ is a character, and set $\psi = \tilde{\vartheta}\chi$.*
  - A1.1. *If $\vartheta = 1$, then $\mathbf{c} = \left(1, 1, 0, \frac{p^\alpha - 1}{2}\right)$ and $\mathbf{K} = (K, K, -, K_\alpha)$. In particular $c = \frac{p^\alpha+3}{2}$ and $K_c = K_\alpha$.*
  - A1.2. *If $\vartheta \neq 1$ is unramified, then $\mathbf{c} = (0, 1, 0, p^\alpha - 1)$ and $\mathbf{K} = (-, K, -, K_\alpha)$. In particular $c = p^\alpha$ and $K_c = K_\alpha$.*
  - A1.3. *If ϑ is ramified, then $\mathbf{c} = (0, 0, 0, p^\alpha)$ and $\mathbf{K} = (-, -, -, K_\alpha)$. In particular $c = p^\alpha$ and $K_c = K_\alpha$.*
- A2. *Let $\bar{\rho}$ be a non-trivial extension of the trivial character by itself, and set $\psi = \chi$.*
  - A2.1. *If $\bar{\rho}$ is unramified, then $\mathbf{c} = \left(1, 1, 0, \frac{p^\alpha - 1}{2}\right)$ and $\mathbf{K} = (K, K, -, K_{\alpha+1})$. In particular $c = \frac{p^\alpha+3}{2}$ and $K_c = K_{\alpha+1}$.*
  - A2.2. *If $\bar{\rho}$ is ramified, then $\mathbf{c} = \left(1, 0, 0, \frac{p^\alpha - 1}{2}\right)$ and $\mathbf{K} = (K, -, -, K_\alpha)$. In particular $c = \frac{p^\alpha+1}{2}$ and $K_c = K_\alpha$.*
- A3. *Let $\bar{\rho}$ be absolutely irreducible and set $\psi = \widetilde{\det \bar{\rho}}$. Then $\mathbf{c} = (0, 0, 1, 0)$ and $\mathbf{K} = (-, -, K, -)$. In particular $c = 1$ and $K_c = K$.*

Proof The result follows from the propositions proved in sections 2.2.1, 2.3.7, 3.2, 3.3 and 3.4. ■

Theorem 4.0.4 (B)**.** *Assume $[L(\zeta_p) : L] = 2$ (i.e., $\omega = \omega^{-1} \neq 1$).*

- B1. *Let $\bar{\rho} = \vartheta \oplus 1$ where $\vartheta : G \to k^\times$ is a character.*
  - B1.1. *If $\vartheta = \omega$ set $\psi = \chi$. Then $\mathbf{c} = \left(2, 1, \frac{p^\beta - 1}{2}, 0\right)$ and $\mathbf{K} = (K, K, K_\beta, -)$. In particular $c = \frac{p^\beta+5}{2}$ and $K_c = K_\beta$.*
  - B1.2. *If $\vartheta \neq \omega$ is unramified set $\psi = \tilde{\vartheta}$. Then $\mathbf{c} = (0, 1, 0, 0)$ and $\mathbf{K} = (-, K, -, -)$. In particular $c = 1$ and $K_c = K$.*
  - B1.3. *If $\vartheta \neq \omega$ is ramified set $\psi = \tilde{\vartheta}$. Then $\mathbf{c} = (0, 0, 0, 1)$ and $\mathbf{K} = (-, -, -, K)$. In particular $c = 1$ and $K_c = K$.*
- B2. *Let $\bar{\rho}$ be a non-trivial extension of the trivial character by a character ϑ.*
  - B2.1. *If $\vartheta = 1$ set $\psi = 1$. Then $\mathbf{c} = (0, 1, 0, 0)$ and $\mathbf{K} = (-, K, -, -)$. In particular $c = 1$ and $K_c = K$.*
  - B2.2. *If $\vartheta = \omega$ set $\psi = \chi$. Then $\mathbf{c} = \left(1, 0, \frac{p^\beta - 1}{2}, 0\right)$ and $\mathbf{K} = (K, -, K_\beta, -)$. In particular $c = \frac{p^\beta+1}{2}$ and $K_c = K_\beta$.*
- B3. *Let $\bar{\rho}$ be absolutely irreducible and set $\psi = \widetilde{\det \bar{\rho}}$.*
  - B3.1. *If $\bar{\rho}$ cannot be induced from a character of $G_{L(\zeta_p)}$, then $\mathbf{c} = (0, 0, 1, 0)$ and $\mathbf{K} = (-, -, K, -)$. In particular $c = 1$ and $K_c = K$.*
  - B3.2. *If $\bar{\rho}$ can be induced from a character of $G_{L(\zeta_p)}$, then $\mathbf{c} = \left(0, 0, p^\beta, 0\right)$ and $\mathbf{K} = (-, -, K_\beta, -)$. In particular $c = p^\beta$ and $K_c = K_\beta$.*

Proof The result follows from the propositions proved in sections 2.2.8, 2.3.1, 2.3.4, 3.2, 3.3 and 3.4. ■

Theorem 4.0.5 (C). *Assume $[L(\zeta_p) : L] > 2$ (i.e., $\omega \neq \omega^{-1}$).*

- C1. *Let $\bar{\rho} = \vartheta \oplus 1$ where $\vartheta : G \to k^\times$ is a character.*
  - C1.1. *If $\vartheta = \omega^{\pm 1}$ set $\psi = \chi^{\pm 1}$. Then $\mathbf{c} = (1, 1, 0, 0)$ and $\mathbf{K} = (K, K, -, -)$. In particular $c = 2$ and $K_c = K$.*
  - C1.2. *If $\vartheta \neq \omega^{\pm 1}$ is unramified set $\psi = \tilde{\vartheta}$. Then $\mathbf{c} = (0, 1, 0, 0)$ and $\mathbf{K} = (-, K, -, -)$. In particular $c = 1$ and $K_c = K$.*
  - C1.3. *If $\vartheta \neq \omega^{\pm 1}$ is ramified set $\psi = \tilde{\vartheta}$. Then $\mathbf{c} = (0, 0, 0, 1)$ and $\mathbf{K} = (-, -, -, K)$. In particular $c = 1$ and $K_c = K$.*
- C2. *Let $\bar{\rho}$ be a non-trivial extension of the trivial character by a character ϑ.*
  - C2.1. *If $\vartheta = 1$ set $\psi = 1$. Then $\mathbf{c} = (0, 1, 0, 0)$ and $\mathbf{K} = (-, K, -, -)$. In particular $c = 1$ and $K_c = K$.*
  - C2.2. *If $\vartheta = \omega$ set $\psi = \chi$. Then $\mathbf{c} = (1, 0, 0, 0)$ and $\mathbf{K} = (K, -, -, -)$. In particular $c = 1$ and $K_c = K$.*
- C3. *If $\bar{\rho}$ is absolutely irreducible, set $\psi = \widetilde{\det \bar{\rho}}$. Then $\mathbf{c} = (0, 0, 1, 0)$ and $\mathbf{K} = (-, -, K, -)$. In particular $c = 1$ and $K_c = K$.*

Proof The result follows from the propositions proved in sections 2.2.8, 2.3.1, 2.3.4, 3.2, 3.3 and 3.4. ■

If $l \neq 2$ and f are fixed and we let p grow, we eventually end up in the case $\alpha = \beta = 0$, so that:

Corollary 4.0.6. *Fix an odd prime l and assume that p is a prime big enough with respect to l and f (for example, $p > l^f$). Then X is geometrically irreducible, unless $\bar{\rho}$ is twist-equivalent to $\omega^{\pm 1} \oplus 1$, in which case $X_{\bar{K}}$ has two geometrically irreducible components.*

Notice that the above corollary can be proved easily without making use the Theorems 4.0.3, 4.0.4 or 4.0.5.

Corollary 4.0.7. *Each absolutely irreducible component of X has a point rational over $K_{\alpha+\beta}$, unless $\alpha > 0$, $\beta = 0$ and $\bar{\rho}$ is non-semisimple and twist-unramified, in which case every absolutely irreducible component of X has a point rational over $K_{\alpha+1}$.*

Proof This follows from the determination of K_c in Theorems 4.0.3, 4.0.4 and 4.0.5. The special case corresponds to case A2.1. of Theorem 4.0.3. ■

Remark 4.0.8. The results proved in sections 2 and 3 give additional information on the components: in the mentioned sections, for any geometrically irreducible component $\mathcal{C}$ of X, the list of $\bar{K}[G]$-isomorphism classes of lifts of $\bar{\rho}$ corresponding to $\bar{K}$-points of $\mathcal{C}$ is determined. Furthermore, some rational points on $\mathcal{C}$ are explicitly computed. We did not summarize in this section these results.

References

1. S. H. Choi, *Local deformation lifting spaces of mod l Galois representations*, Ph.D. thesis, Harvard University, 2009.
2. F. Diamond, *An extensions of Wiles' results*, Modular Forms and Fermat's last theorem, Springer, 1997, pp. 475–489.
3. M. Kisin, *Modularity of 2-adic Barsotti-Tate representations*, Invent. Math. **178** (2009), 587–634.
4. ———, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. **170** (2009), 1085–1180.
5. V. Pilloni, *The study of 2-dimensional p-adic Galois representations in the $\ell \neq p$ case*, Preprint, 2008.
6. K. A. Ribet, *A modular construction of unramified p-extensions of* $\mathbf{Q}(\mu_p)$, Invent. Math. **34** (1976), 151–162.
7. E. Urban, *On residually reducible representations on local rings*, J. Algebra **212** (1999), 738–742.

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