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Seifert forms and concordance

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Abstract

If a knot K has Seifert matrix V_K and has a prime power cyclic branched cover that is not a homology sphere, then there is an infinite family of non–concordant knots having Seifert matrix V_K.

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1 Introduction

Levine's homomorphism $\psi\colon \mathcal{C} \to \mathcal{G}$ from the concordance group of knots in S^3 to the algebraic concordance group of Seifert matrices (defined in [12]) has an infinitely generated kernel, as proved by Jiang [8]. It follows that every algebraic concordance class can be represented by an infinite family of non–concordant knots. However, it is also the case that every class in $\mathcal{G}$ can be represented by an infinite number of distinct Seifert matrices, so Jiang's result alone tells us nothing about whether a given Seifert matrix can arise from non–concordant knots. In fact, all the knots in the kernel of ϕ identified by Jiang have distinct Seifert forms.

Examples of non–slice, algebraically slice, knots quickly yield pairs of non–concordant knots with the same Seifert matrix. Beyond this nothing has been known regarding the extent to which the Seifert matrix of a knot might determine its concordance class. We prove the following.

Theorem 1.1 *If a knot K has Seifert matrix V_K and its Alexander polynomial $\Delta_K(t)$ has an irreducible factor that is not a cyclotomic polynomial ϕ_n with n divisible by three distinct primes, then there is an infinite family $\{K_i\}$ of non–concordant knots such that each K_i has Seifert matrix V_K.*

The condition on the Alexander polynomial seems somewhat technical; we note three relevant facts. First, if the Alexander polynomial of the knot is trivial, $\Delta_K(t) = 1$, then K is topologically slice [4, 5]. Second we have:

Theorem 1.2 *All prime power cyclic branched covers of a knot K are homology spheres if and only if all nontrivial irreducible factors of $\Delta_K(t)$ are cyclotomic polynomials $\phi_n(t)$ with n divisible by three distinct primes. All branched covers of K are homology spheres if and only if $\Delta_K(t) = 1$.*

Finally, we note that Taehee Kim [9] has applied the recent advances in concordance theory of [2] to prove that for each n divisible by three distinct primes there is a knot with $\Delta_K(t) = (\phi_n(t))^2$ for which there is an infinite family of non–concordant knots having the same Seifert matrix.

A good reference for the basic knot theory in this paper is [17], for the algebraic concordance group [12, 13] are the main references, and for Casson–Gordon invariants references are [1, 7].

Remark We have chosen to use Seifert matrices instead of Seifert forms to be consistent with references [12, 13]. A basis free approach using Seifert forms could be carried out identically.

2 Proof of Theorem 1.1

Unless indicated, all homology groups are taken with integer coefficients.

Let F be a Seifert surface for K with associated Seifert matrix V_K. View F as a disk with $2g$ bands added and let $\{S_m\}_{m=1,\dots,2g}$, be a collection of unknotted circles, one linking each of the bands. Let K_i be the knot formed by replacing a tubular neighborhood of each S_m with a copy of the complement of a knot J_i, identifying the meridian and longitude of J_i with the longitude and meridian of the S_m, respectively. The correct choice of the J_i will be identified in the proof. Replacing the S_m with the knot complements has the effect of adding a local knot to each band of F. The Seifert form of K_i is independent of the choice of J_i. Applying Theorem 1.2, proved in the next section, we assume the p^k–fold cyclic branched cover of S^3 branched over K has nontrivial homology. Denote this cover by $M(K)$ and let q be a maximal prime power divisor of $|H_1(M(K))|$.

According to Casson and Gordon [1], if $K_i\# - K_j$ is slice (that is, if K_i and K_j are concordant) then for some nontrivial $\mathbf{Z}_q$–valued character χ on $H_1(M(K_i\# - K_j))$ the Casson–Gordon invariant $\sigma_1(\tau(K_i\# - K_j,\chi)) = 0$. Using the additivity of Casson–Gordon invariants (proved by Gilmer [6]), this equality can be rewritten as $\sigma_1(\tau(K_i,\chi_i)) = \sigma_1(\tau(K_j,\chi_j))$ where χ_i and χ_j are the restrictions of χ to $H_1(M(K_i))$ and $H_1(M(K_j))$, respectively. Notice that at least one of χ_i and χ_j is nontrivial. Furthermore, since according to [1] (see also [6]) the set of characters for which the Casson–Gordon invariants must vanish is a metabolizer for the linking form on $H^1(M(K_i\# - K_j),\mathbf{Q}/\mathbf{Z})$, there are such characters for which χ_j must be nontrivial. (If the metabolizer was contained in $H^1(M(K_i),\mathbf{Q}/\mathbf{Z})$ then order considerations would show that it equalled this summand, contradicting nonsingularity.)

Litherland's analysis [14] of companionship and Casson–Gordon invariants applies directly to the case of knotting the bands in the Seifert surface (see also [7])). Roughly stated, there is a correspondence between characters on $H_1(M(K))$ and on $H_1(M(K_i))$; it then follows that the difference of the corresponding Casson–Gordon invariants is determined by q–signatures of J_i: $\sigma_{a/q}(J_i) = \operatorname{sign}\left((1-\omega)V_{J_i} + (1-\overline{\omega})V_{J_i}^t\right)$ where $\omega = e^{2\pi ai/q}$. More precisely, it follows readily from the results of [14] and iteration that the equality of Casson–Gordon invariants for K_i and K_j is given by

$$(*) \qquad \sigma_1(\tau(K,\chi_i)) + \sum_l \sigma_{a_l/q}(J_i) = \sigma_1(\tau(K,\chi_j)) + \sum_l \sigma_{b_l/q}(J_j).$$

The two summations that appear have $2gp^k$ terms in them. The values of the a_l are given by the values of χ_i on the $2gp^k$ lifts of the circles S_m to $M(K)$. Similar statements hold for the b_l and χ_j. Observe also that since the lifts of the S_m generate $H_1(M(K))$ (see for instance [17]) and at least one of χ_i or χ_j is nontrivial, at least one of the a_l or b_l is nontrivial.

A prime power branched cover of a knot is a rational homology sphere and hence $H_1(M(K))$ is finite. A short proof of this is given in the next section. Hence, there is only a finite set of characters to consider and $\sigma_1(\tau(K,\chi_1))$ lies in a bounded range, say $[-N_0, N_0]$. If we can choose J_1 so that $\sum_l \sigma_{a_l/q}(J_1)$ lies in a range $[2N_0+1, N_1]$ (for some N_1 and for all possible sums with some $a_l \neq 0 \in \mathbf{Z}_q$) then it would follow that K and K_1 are not concordant. Similarly, by selecting each J_{i+1} so that the sum lies in the range $[2N_0+N_i+1, N_{i+1}]$ we will have that the equality $(*)$ cannot hold for any pair i and j and the theorem is proved.

The desired J_i are constructed by taking ever larger multiples of a knot T for which $\sigma_{a/q}(T) \geq 2$ for all $a \neq 0 \in \mathbf{Z}_q$. Such a knot is given in the following lemma, which completes the proof of Theorem 1.1.

Lemma 2.1 *The $(2,q)$–torus knot $T_{2,q}$ has $\sigma_{a/q}(T) \geq 2$ for all $a \neq 0 \in \mathbf{Z}_q$.*

Proof The signature function of a knot K, $\operatorname{sign}((1-\omega)V_K + (1-\overline{\omega})V_K)$, has jumps only at roots of the Alexander polynomial, and if these roots are simple the jump is either ± 2 [15]. The $(2,q)$–torus knot has cyclotomic Alexander polynomial ϕ_{2q} with $(q-1)/2$ simple roots on the upper unit circle in the complex plane. Hence the signature $\sigma_{-1}(T_{2,q}) \leq q-1$. On the other hand, this -1 signature is easily computed from the standard rank $q-1$ Seifert form for $T_{2,q}$ to be exactly $q-1$, and so all the jumps must be positive 2. The first of these jumps occurs at a primitive $2q$–root of unity, so all q–signatures must be positive as desired. □

3 Proof of Theorem 1.2

We have the following result of Fox [3] and include as a corollary a result used above.

Theorem 3.1 *If $M(K)$ is the r–fold cyclic branched cover of S^3 branched over K, then*

$$|H_1(M(K))| = \prod_{i=0}^{r-1} \Delta_K(\zeta_r^i)$$

where ζ_r is a primitive r–root of unity. If the product is 0 then $H_1(M(K))$ is infinite.

Corollary 3.2 *If r is a prime power, then $M(K)$ is a rational homology sphere: $H_1(M(K),\mathbf{Q}) = 0$.*

Proof Suppose that $r = p^k$ and $\Delta_K(\zeta_r^i) = 0$. Then the r–cyclotomic polynomial, $\phi_r(t) = (t^{p^k} - 1)/(t^{p^{k-1}} - 1)$ would divide $\Delta_K(t)$. But $\phi_r(1) = p$ while $\Delta_K(1) = \pm 1$. □

We now proceed with the proof of Theorem 1.2.

Proof of Theorem 1.2 According to Riley [16] the order of the homology of the k–fold cyclic branched cover of a knot K grows exponentially as a function of k if the Alexander polynomial has a root that is not a root of unity. Hence, we only need to consider the case that all irreducible factors of the Alexander polynomial are cyclotomic polynomials, $\phi_n(t)$. Using Theorem 3.1, the result is reduced to the case that that $\Delta_K(t) = \phi_n(t)$. As in the proof of Corollary 3.2, n cannot be a prime power.

An elementary argument using the resultant of polynomials (see for instance [10]) gives

$$\prod_{i=0}^{p^k-1} \phi_n(\zeta_{p^k}^i) = \prod((\omega_n)^{p^k} - 1)$$

where the second product is taken over all primitive n–roots of unity. Let $g = \gcd(n, p^k)$ and let $m = n/g$. One has that $\omega_n^{p^k} = \omega_m$ for some primitive m–root of unity and with a bit of care one sees that the product can be rewritten as

$$\prod(\omega_m - 1)^b$$

where now the product is over all primitive m–roots of unity and $b \geq 1$. (Though we don't need it, a close examination shows that if k is greater than or equal to the maximal power of p in n then $b = p^k - p^{k-1}$, otherwise $b = p^k$.)

If n has three distinct prime factors then m has at least two distinct prime factors and this product is 1 (see for instance [11, page 73]). On the other hand, if n has two distinct prime factors, then by letting p be one of those factors and letting k be large, it is arranged that m is a prime power and the product yields that prime and in particular is greater than 1. This concludes the proof of the first statement of Theorem 1.2.

Finally, suppose that all cyclic branched covers of K are homology spheres. By the above discussion we just need to show that no factor of the Alexander polynomial is $\phi_n(t)$ for any n. But from Theorem 3.1 we see that if $\phi_n(t)$ divides the Alexander polynomial then the n–fold cyclic branched cover would have infinite homology. This concludes the proof. □

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