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The index of projective families of elliptic operators

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Abstract

An index theory for projective families of elliptic pseudodifferential operators is developed. The topological and the analytic index of such a family both take values in twisted K–theory of the parametrizing space, X. The main result is the equality of these two notions of index when the twisting class is in the torsion subgroup of $H^3(X;\mathbb{Z})$. The Chern character of the index class is then computed.

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Introduction

In this paper we develop an index theory for projective families of elliptic pseudodifferential operators. Such a family, $\{D_b,\ b \in X\}$, on the fibers of a fibration

$$\phi\colon\ M \longrightarrow X \tag{1}$$

with base X, and typical fibre F, is a collection of local elliptic families, for an open covering of the base, acting on finite-dimensional vector bundles of fixed rank where the usual compatibility condition on triple overlaps, to give a global family, may fail by a scalar factor. These factors define an integral 3–cohomology class on the base, the Dixmier–Douady class, $\Theta \in H^3(X,\mathbb{Z})$. We show that both the analytic and topological index of D_b may be defined as elements of twisted K–theory, with twisting class Θ, and that they are equal. In this setting of finite-dimensional bundles the twisting class is necessarily a *torsion* class. We also compute the Chern character of the index in terms of characteristic classes. When the twisting class Θ is trivial, these results reduce to the Atiyah–Singer index theorem for families of elliptic operators, [2].

The vector bundles on which D_b acts, with this weakened compatibility condition, form a *projective vector bundle*[1]. In the torsion case, elements of twisted K–theory may be represented by differences of such projective vector bundles and, after stabilization, the local index bundles of the family give such a difference and so define the analytic index. Nistor and Troitsky have recently given a similar definition in [13]. The topological index is defined, as in the untwisted case, by push-forward from a twisted K class of compact support on the cotangent bundle. The proof we give of the equality of the analytic and the topological index is a generalization of the axiomatic proof of the families index theorem of [2]. This is done by proving that all normalized, functorial index maps, in twisted K–theory, satisfying excision and multiplicativity coincide and then showing that the analytic index and the topological index satisfy these conditions.

Twisted K–theory arises naturally when one considers the Thom isomorphism for a real Riemannian vector bundle, E, in the even-rank case and when X is compact. The compactly supported K–theory of E, $K_c(E)$, is then isomorphic to the twisted K–theory of X, $K(C(X,\mathrm{Cl}(E)))$, where $\mathrm{Cl}(E)$ denotes the Clifford algebra bundle of E, and $C(X,\mathrm{Cl}(E))$ denotes the algebra of continuous sections. There is a similar statement with a shift in degree for odd-dimensional vector bundles. The twisting of families of Dirac operators by projective vector

[1] Generally these are called *gauge bundles* in the physics literature

bundles provides many examples of elliptic families and, as in the untwisted case, the Chern character has a more explicit formula.

Recently, there has been considerable interest in twisted K–theory by physicists, with elements of twisted K–theory interpreted as charges of D–branes in the presence of a background field; cf [1], [21], [11].

There is an alternative approach to the twisted index theorem, not carried out in detail here. This is to realize the projective family as an ordinary equivariant family of elliptic pseudodifferential operators on an associated principal $\mathrm{PU}(n)$–bundle. Then the analytic index and the topological index are elements in the $\mathrm{U}(n)$–equivariant K–theory of this bundle. Their equality follows from the equivariant index theorem for families of elliptic operators as in [20]. The proof can then be completed by showing that the various definitions of the analytic index and of the topological index agree for projective families of elliptic operators.

A subsequent paper will deal with the general case of the twisted index theorem when the twisting 3–cocycle is not necessarily torsion. Then there is no known finite-dimensional description of twisted K–theory, and even to formulate the index theorem requires a somewhat different approach.

The paper is organized as follows. A review of twisted K–theory, with an emphasis on the interpretation of elements of twisted K–theory as differences of projective vector bundles is given in Section 1. The definition of general projective families of pseudodifferential operators is explained in Section 2, leading to the definition of the analytic index, in the elliptic case, as an element in twisted K–theory. The definition of the topological index is given in Section 3 and Section 4 contains the proof of the equality of these two indices. In Section 5 the Chern character of the analytic index is computed and in Section 6 the determinant bundle is discussed in this context. Finally, Section 7 contains a brief description of Dirac operators.

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1 Review of twisted K–theory

General references for most of the material summarized here are [17, 18].

1.1 Brauer groups and the Dixmier–Douady invariant

We begin by reviewing some results due to Dixmier and Douady, [6]. Let X be a smooth manifold, let $\mathcal{H}$ denote an infinite-dimensional, separable, Hilbert space and let $\mathcal{K}$ be the C^*–algebra of compact operators on $\mathcal{H}$. Let $\mathrm{U}(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$ endowed with the strong operator topology and let $\mathrm{PU}(\mathcal{H}) = \mathrm{U}(\mathcal{H})/\mathrm{U}(1)$ be the projective unitary group with the quotient space topology, where $\mathrm{U}(1)$ consists of scalar multiples of the identity operator on $\mathcal{H}$ of norm equal to 1. Recall that, if G is a topological group, principal G bundles over X are classified up to isomorphism by the first cohomology of X with coefficients in the sheaf of germs of continuous functions from X to G, $H^1(X, \underline{G})$, where the transition maps of any trivialization of a principal G bundle over X define a cocycle in $Z^1(X, \underline{G})$ with fixed cohomology class. The exact sequence of sheaves of groups,

$$1 \longrightarrow \underline{\mathrm{U}(1)} \longrightarrow \underline{\mathrm{U}(\mathcal{H})} \longrightarrow \underline{\mathrm{PU}(\mathcal{H})} \longrightarrow 1$$

gives rise to the long exact sequence of cohomology groups,

$$\ldots \longrightarrow H^1(X, \underline{\mathrm{U}(\mathcal{H})}) \longrightarrow H^1(X, \underline{\mathrm{PU}(\mathcal{H})}) \xrightarrow{\delta_1} H^2(X, \underline{\mathrm{U}(1)}) \longrightarrow 1. \qquad (2)$$

Since $\mathrm{U}(\mathcal{H})$ is contractible in the strong operator topology, the sheaf $\underline{\mathrm{U}(\mathcal{H})}$ is soft and the sheaf cohomology vanishes, $H^1(X, \underline{\mathrm{U}(\mathcal{H})}) = \{0\}$. Equivalently, every Hilbert bundle over X is trivializable in the strong operator topology. In fact, Kuiper [10] proves the stronger result that $\mathrm{U}(\mathcal{H})$ is contractible in the operator norm topology, so the same conclusion holds in this sense too. It follows from (2) that δ_1 is an isomorphism. That is, principal $\mathrm{PU}(\mathcal{H})$ bundles over X are classified up to isomorphism by $H^2(X, \underline{\mathrm{U}(1)})$. From the exact sequence of groups,

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathrm{U}(1) \longrightarrow 1$$

we obtain the long exact sequence of cohomology groups,

$$\cdots \longrightarrow H^2(X, \underline{\mathbb{R}}) \longrightarrow H^2(X, \underline{\mathrm{U}(1)}) \xrightarrow{\delta_2} H^3(X, \mathbb{Z}) \longrightarrow H^3(X, \underline{\mathbb{R}}) \longrightarrow \cdots .$$

Now $H^j(X, \underline{\mathbb{R}}) = \{0\}$ for $j > 0$ since $\underline{\mathbb{R}}$ is a fine sheaf, therefore δ is also an isomorphism. That is, principal $\mathrm{PU}(\mathcal{H})$ bundles over X are also classified up to isomorphism by $H^3(X, \mathbb{Z})$. The class $\delta_2\delta_1([P]) \in H^3(X, \mathbb{Z})$ is called the *Dixmier–Douady class* of the principal $\mathrm{PU}(\mathcal{H})$ bundle P over X, where $[P] \in H^1(X, \underline{\mathrm{PU}(\mathcal{H})})$.

For $g \in \mathrm{U}(\mathcal{H})$, let $Ad(g)$ denote the automorphism $T \longrightarrow gTg^{-1}$ of $\mathcal{K}$. As is well-known, Ad is a continuous homomorphism of $\mathrm{U}(\mathcal{H})$, given the strong operator topology, onto $\mathrm{Aut}(\mathcal{K})$ with kernel the circle of scalar multiples of the identity where $\mathrm{Aut}(\mathcal{K})$ is given the point-norm topology, that is the topology of pointwise convergence of functions on $\mathcal{K}$, cf [15], Chapter 1. Under this homomorphism we may identify $\mathrm{PU}(\mathcal{H})$ with $\mathrm{Aut}(\mathcal{K})$. Thus

Proposition 1 (Dixmier–Douady [6]) *The isomorphism classes of locally trivial bundles over X with fibre $\mathcal{K}$ and structure group* $\mathrm{Aut}(\mathcal{K})$ *are parametrized by $H^3(X,\mathbb{Z})$.*

Since $\mathcal{K}\otimes\mathcal{K}\cong\mathcal{K}$, the isomorphism classes of locally trivial bundles over X with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$ form a group under the tensor product, where the inverse of such a bundle is the conjugate bundle. This group is known as the *infinite Brauer group* and is denoted by $\mathrm{Br}^\infty(X)$ (cf [14]). So, essentially as a restatement of Proposition 1

$$\mathrm{Br}^\infty(X)\cong H^3(X,\mathbb{Z}) \tag{3}$$

where the cohomology class in $H^3(X,\mathbb{Z})$ associated to a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$ is again called the Dixmier–Douady invariant of $\mathcal{E}$ and is denoted by $\delta(\mathcal{E})$.

In this paper, we will be concerned mainly with torsion classes in $H^3(X,\mathbb{Z})$. Let $\mathrm{tor}(H^3(X,\mathbb{Z}))$ denote the subgroup of torsion elements in $H^3(X,\mathbb{Z})$. Suppose now that X is compact. Then there is a well-known description of $\mathrm{tor}(H^3(X,\mathbb{Z}))$ in terms of locally trivial bundles of finite-dimensional Azumaya algebras over X, [7]. Recall that an Azumaya algebra of rank m is an algebra which is isomorphic to the algebra of $m\times m$ matrices, $M_m(\mathbb{C})$.

Definition An *Azumaya bundle* over a manifold X is a vector bundle with fibres which are Azumaya algebras and which has local trivialization reducing these algebras to $M_m(\mathbb{C})$.

Remark The terminology 'Azumaya bundle' is not the only one in the literature for this notion. The oldest is that of "n–homogeneous C^*–algebra" due to Kaplansky, cf [9], page 236. These are by definition C^*–algebras all of whose irreducible representations have dimension equal to n. A highly nontrivial theorem of Takesaki–Tomiyama, [19] shows that these algebras are Azumaya algebras over a commutative algebra and hence they identify with the space of continuous sections of a bundle of finite dimensional simple algebras. (A general reference is Diximier's book on C^*–algebras, [5]. For the reader's benefit, we recall that an Azumaya algebra $\mathcal{A}$ over a ring R is an algebra that is a finitely generated, projective algebra over R such that $\mathcal{A}\otimes\mathcal{A}^{op}\simeq\mathrm{End}_R(\mathcal{A})$.)

An example of an Azumaya bundle over X is the algebra, $\mathrm{End}(E)$, of all endomorphisms of a vector bundle E over X. Two Azumaya bundles $\mathcal{E}$ and $\mathcal{F}$ over X are said to be equivalent if there are vector bundles E and F over X such that $\mathcal{E}\otimes\mathrm{End}(E)$ is isomorphic to $\mathcal{F}\otimes\mathrm{End}(F)$. In particular, an Azumaya bundle of the form $\mathrm{End}(E)$ is equivalent to $C(X)$ for any vector bundle E over X. The group of all equivalence classes of Azumaya bundles over X is called the Brauer group of X and is denoted by $\mathrm{Br}(X)$. We will denote by $\delta'(\mathcal{E})$ the class in $\mathrm{tor}(H^3(X,\mathbb{Z}))$ corresponding to the Azumaya bundle $\mathcal{E}$ over X. It is constructed using the same local description as above but with $\mathcal{H}$ now finite dimensional. Serre's theorem, [8] asserts that

$$\mathrm{Br}(X)\cong\mathrm{tor}(H^3(X,\mathbb{Z})). \tag{4}$$

This is also the case if we consider smooth Azumaya bundles.

Another important class of Azumaya algebras arises from the bundles of Clifford algebras of vector bundles. For a real vector bundle, with fibre metric, the associated bundle of complexified Clifford algebras is an Azumaya bundle for even rank and for odd rank is a direct sum of two Azumaya bundles. In the even-rank case, the Dixmier–Douady invariant of this Azumaya bundle is the third integral Stiefel–Whitney class, the vanishing of which is equivalent to the existence of a $\mathrm{spin}^{\mathbb{C}}$ structure on the bundle.

Thus we see that there are two descriptions of $\mathrm{tor}(H^3(X,\mathbb{Z}))$, one in terms of Azumaya bundles over X, and the other as a special case of locally trivial bundles over X with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$. These two descriptions are related as follows. Given an Azumaya bundle $\mathcal{E}$ over X, the tensor product $\mathcal{E}\otimes\mathcal{K}$ is a locally trivial bundle over X with fibre $M_m(\mathbb{C})\otimes\mathcal{K}\cong\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$, such that $\delta'(\mathcal{E})=\delta(\mathcal{E}\otimes\mathcal{K})$. Notice that the algebras $C(X,\mathcal{E})$ and $C(X,\mathcal{E}\otimes\mathcal{K})=C(X,\mathcal{E})\otimes\mathcal{K}$ are Morita equivalent. Moreover if $\mathcal{E}$ and $\mathcal{F}$ are equivalent as Azumaya bundles over X then $\mathcal{E}\otimes\mathcal{K}$ and $\mathcal{F}\otimes\mathcal{K}$ are isomorphic over X, as locally trivial bundles with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$. To see this, recall that by the assumed equivalence there are vector bundles E and F over X such that $\mathcal{E}\otimes\mathrm{End}(E)$ is isomorphic to $\mathcal{F}\otimes\mathrm{End}(F)$. Tensoring both bundles with $\mathcal{K}(\mathcal{H})$, we see that $\mathcal{E}\otimes\mathcal{K}(E\otimes\mathcal{H})$ is isomorphic to $\mathcal{F}\otimes\mathcal{K}(F\otimes\mathcal{H})$, where $\mathcal{K}(E\otimes\mathcal{H})$ and $\mathcal{K}(F\otimes\mathcal{H})$ are the bundles of compact operators on the infinite dimensional Hilbert bundles $E\otimes\mathcal{H}$ and $F\otimes\mathcal{H}$, respectively. By the contractibility of the unitary group of an infinite dimensional Hilbert space in the strong operator topology, the infinite dimensional Hilbert bundles $E\otimes\mathcal{H}$ and $F\otimes\mathcal{H}$ are trivial, and therefore both $\mathcal{K}(E\otimes\mathcal{H})$ and $\mathcal{K}(F\otimes\mathcal{H})$ are isomorphic to the trivial bundle $X\times\mathcal{K}$. It follows that $\mathcal{E}$ and $\mathcal{F}$ are equivalent Azumaya bundles over X, if and only if $\mathcal{E}\otimes\mathcal{K}$ and $\mathcal{F}\otimes\mathcal{K}$ are isomorphic, as asserted.

Recall that a C^*–algebra A is said to be *stably unital* if there is a sequence of projections $p_n \in A \otimes \mathcal{K}$ such that in the strong topology $Tp_n \longrightarrow T$ for each $T \in A \otimes \mathcal{K}$. In particular $\mathcal{K}$ itself is stably unital since each compact operator can be approximated by finite rank operators. It follows that any unital algebra is stably unital.

Lemma 1 *If X is a compact manifold and $\mathcal{E}$ is a locally trivial bundle over X with fibre $\mathcal{K}$ and structure group* $\mathrm{Aut}(\mathcal{K})$ *then $C(X, \mathcal{E})$, the C^*–algebra of continuous sections of $\mathcal{E}$, is stably unital if and only if its Dixmier–Douady invariant is a torsion element in $H^3(X, \mathbb{Z})$.*

Proof The assumption that $C(X, \mathcal{E})$ is stably unital where $\mathcal{E}$ has fibres isomorphic to $\mathcal{K}$, implies in particular that there is a non-trivial projection $p \in C(X, \mathcal{E})$. This must be of finite rank in each fibre. The C^*–algebra $pC(X, \mathcal{E})p$ is a corner in $C(X, \mathcal{E})$ in the sense of Rieffel, [16], so these algebras are Morita equivalent. They are both continuous trace C^*–algebras with the same spectrum, which is equal to X, and therefore the same Dixmier–Douady invariant by the classification of continuous trace C^*–algebras [15]. Since $pC(X, \mathcal{E})p$ is the C^*–algebra of sections of an Azumaya bundle over X, the Dixmier–Douady invariant is a torsion element in $H^3(X, \mathbb{Z})$.

The converse is really just Serre's theorem; given $\Theta \in \mathrm{tor}(H^3(X, \mathbb{Z}))$ there is a principal $\mathrm{PU}(m)$ bundle over X whose Dixmier–Douady invariant is Θ where the order of Θ necessarily divides m. The associated Azumaya bundle $\mathcal{A}$ also has the same Dixmier–Douady invariant. Then $\mathcal{E} = \mathcal{A} \otimes \mathcal{K}$ is a locally trivial bundle over X with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$ and with Dixmier–Douady invariant $\delta(\mathcal{E}) = \Theta$. So there is a non-trivial projection $p_1 \in C(X, \mathcal{E})$ such that $p_1 C(X, \mathcal{E}) p_1 = C(X, \mathcal{A})$. In fact, one can define a nested sequence of Azumaya bundles, $\mathcal{A}_j$, over X defined by $\mathcal{A}_j = \mathcal{A} \otimes M_j(\mathbb{C})$ together with the corresponding projections $p_j \in C(X, \mathcal{E})$ such that $p_j C(X, \mathcal{E}) p_j = C(X, \mathcal{A}_j)$ for all $j \in \mathbb{N}$. Then $\{p_j\}_{j \in \mathbb{N}}$ is an approximate identity of projections in $C(X, \mathcal{E})$, that is, $C(X, \mathcal{E})$ is stably unital. □

Remark This argument shows that for any torsion class $\Theta \in \mathrm{tor}(H^3(X, \mathbb{Z}))$ there is a *smooth* Azumaya bundle with Dixmier–Doaudy invariant Θ.

1.2 Twisted K–theory

Let X be a manifold and let $\mathcal{J}$ be a locally trivial bundle of algebras over X with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$. Two such bundles are isomorphic if and only if they have the same Dixmier–Douady invariant $\delta(\mathcal{J}) \in H^3(X,\mathbb{Z})$. The *twisted K–theory* of X (with compact supports) has been defined by Rosenberg [18] as

$$K_c^j(X,\mathcal{J}) = K_j(C_0(X,\mathcal{J})) \quad j=0,1, \tag{5}$$

where and $K_\bullet(C_0(X,\mathcal{J}))$ denotes the topological K–theory of the C^*–algebra of continuous sections of $\mathcal{J}$ that vanish outside a compact subset of X. In case X is compact we use the notation $K^j(X,\mathcal{J})$. The space $K^j(X,\mathcal{J})$ or $K_c^j(X,\mathcal{J})$ is an abelian group. It is tempting to think of the twisted K–theory of X as determined by the class $\Theta = \delta(\mathcal{J})$. However, this is not strictly speaking correct; whilst it is the case that any other choice $\mathcal{J}'$ such that $\delta(\mathcal{J}') = \Theta$ is isomorphic to $\mathcal{J}$ and therefore there is an isomorphism $K^j(X,\mathcal{J}) \cong K^j(X;\mathcal{J}')$ this isomorphism itself is not unique, nor is its homotopy class. However both the abelian group structure and the module structure on $K^0(X;\mathcal{J})$ over $K^0(X)$ arising from tensor product are natural, ie, are preserved by such isomorphisms. With this caveat one can use the notation $K^j(X,\Theta)$ to denote the twisted K–theory with Dixmier–Douady invariant $\Theta \in H^3(X,\mathbb{Z})$.

In the case of principal interest here, when $\Theta \in \operatorname{tor}(H^3(X,\mathbb{Z}))$ we will take $\mathcal{J} = \mathcal{K}_{\mathcal{A}} = \mathcal{A} \otimes \mathcal{K}$ where $\mathcal{A}$ is an Azumaya bundle and use the notation

$$K_c^j(X,\mathcal{A}) = K_j(C_0(X,\mathcal{A}\otimes\mathcal{K})) = K_j(C_0(X,\mathcal{J})) \quad j=0,1 \tag{6}$$

and $K^j(X,\mathcal{A})$ in the compact case. As noted above, if $\mathcal{A}$ and $\mathcal{A}'$ are two Azumaya bundles with the same Dixmier–Douady invariant, then as bundles of algebras, $\mathcal{A}\otimes\operatorname{End}(V) \equiv \mathcal{A}'\otimes\operatorname{End}(W)$ for some vector bundles V and W. Whilst there is no natural isomorphism of $\operatorname{End}(V)\otimes\mathcal{J}$ and $\mathcal{J}$, these bundles of algebras are isomorphic. Moreover Π_0 of the group induced by diffeomorphisms on $\mathcal{J}$ is naturally isomorphic to $H^2(X,\mathbb{Z})$. It follows that the isomorphism between $K^0(X,\mathcal{A})$ and $K^0(X,\mathcal{A}')$ is determined up to the action of the image of $H^2(X,\mathbb{Z})$, as isomorphism classes, in $K^0(X)$ acting on $K^0(X,\mathcal{A})$ through the module structure.

There are alternate descriptions of $K^0(X,\mathcal{A})$. A description in terms of the twisted index map is mentioned in [18] and we give a complete proof here. Let $Y_{\mathcal{A}}$ be the principal $\operatorname{PU}(\mathcal{H}) = \operatorname{Aut}(\mathcal{K})$ bundle over X associated to $\mathcal{K}_{\mathcal{A}} = \mathcal{A}\otimes\mathcal{K}$ and let $\operatorname{Fred}_{\mathcal{A}} = (Y_{\mathcal{A}} \times \operatorname{Fred}(\mathcal{H}))\,/\,\operatorname{PU}(\mathcal{H})$ be the bundle of twisted Fredholm operators where $\operatorname{Fred}(\mathcal{H})$ denotes the space of Fredholm operators on $\mathcal{H}$.

We shall use the short exact sequence of C^*–algebras,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{K} \longrightarrow 0$$

where $\mathcal{B}/\mathcal{K}$ is the Calkin algebra. It gives rise to the short exact sequence of C^*–algebras of sections,

$$0 \longrightarrow C(X,\mathcal{K}_{\mathcal{A}}) \longrightarrow C(X,\mathcal{B}_{\mathcal{A}}) \longrightarrow C(X,(\mathcal{B}/\mathcal{K})_{\mathcal{A}}) \longrightarrow 0 \tag{7}$$

where the bundles $\mathcal{B}_{\mathcal{A}}$ and $(\mathcal{B}/\mathcal{K})_{\mathcal{A}}$ are also associated to $Y_{\mathcal{A}}$.

Lemma 2 *Let X be a compact manifold and $\mathcal{A}$ an Azumaya bundle over X. Then the twisted K–theory $K^0(X,\mathcal{A})$ is isomorphic to the Grothendieck group of Murray–von Neumann equivalence classes of projections in $C(X,\mathcal{K}_\mathcal{A})$. Similarly, $K^0(X,\mathcal{B}_\mathcal{A})$ is isomorphic to the Grothendieck group of Murray–von Neumann equivalence classes of finite rank projections in $C(X,\mathcal{B}_\mathcal{A})$.*

Proof This is a consequence of Lemma 1 above, which asserts that $C(X,K_\mathcal{A})$ is stably unital, and of Proposition 5.5.5. in [6]. □

Proposition 2 *For any compact maniold X, there is a twisted index map, defined explicitly in* (10) *below, giving an isomorphism.*

$$\text{index}\colon\ \pi_0(C(X,\mathrm{Fred}_\mathcal{A})) \xrightarrow{\ \sim\ } K^0(X,\mathcal{A}). \tag{8}$$

Proof First we show that $K_0(C(X,\mathcal{B}_\mathcal{A}))=\{0\}$. We remind the reader that

$$\mathrm{GL}(n,\mathcal{B})\simeq\mathrm{GL}(\mathcal{B})$$

is connected, so $K_1(\mathcal{B})=0$. Also $K_0(\mathcal{B})=0$ because all infinite projections in $\mathcal{B}$ are equivalent to I, so all finite projections are equivalent to 0.

A finite projection in $C(X,\mathcal{B}_\mathcal{A})$ is a cross section of finite projections, $x\longmapsto P_x, x\in X$; see Lemma 2. The set of partial isometries at x connecting I to $I-P_x$ is a principal homogeneous space for the full unitary group and hence contractible. A cross section of partial isometries makes I equivalent to $I-P$ and P equivalent to 0. Hence $K_0(C(X,\mathcal{B}_\mathcal{A}))=0$.

We leave to the reader the proof that $K_1(C(X,\mathcal{B}_\mathcal{A}))=\{0\}$, using the fact that $GL(\mathcal{B})$ is contractible.

Now consider the six term exact sequence in K–theory

$$\begin{tikzcd}
K^1(X,\mathcal{A}) \arrow[r] & K_1(C(X,\mathcal{B}_\mathcal{A})) \arrow[r] & K_1(C(X,(\mathcal{B}/\mathcal{K})_\mathcal{A})) \arrow[d, "\mathrm{index}'"] \\
K_0(C(X,(\mathcal{B}/\mathcal{K})_\mathcal{A})) \arrow[u] & K_0(C(X,\mathcal{B}_\mathcal{A})) \arrow[l] & K^0(X,\mathcal{A}) \arrow[l]
\end{tikzcd} \tag{9}$$

arising from (8).

From this, using (9), we get $K_1(C(X,\mathcal{B}_X)\mathcal{A})\simeq K^0\,(X,\mathcal{A})$. However, note that by definition

$$\begin{aligned}
K_1(C(X,(\mathcal{B}/\mathcal{K})_\mathcal{A})) &= \varinjlim\ \pi_0\,(\mathrm{GL}(n,C(X,(\mathcal{B}/\mathcal{K})_\mathcal{A})))\\
&= \varinjlim\ \pi_0(C(X,\mathrm{GL}(n,\mathcal{B}/\mathcal{K})_\mathcal{A}))
\end{aligned}$$

where $\mathrm{GL}(n, A)$ denotes the group of invertible $n \times n$ matrices with entries in the C^*–algebra A. In the case of the Calkin algebra, $\mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \cong \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ and $\mathcal{K}(\mathcal{H}) \otimes M_n(\mathbb{C}) \cong \mathcal{K}(\mathcal{H} \otimes \mathbb{C}^n)$ from which it follows that

$$\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \otimes M_n(\mathbb{C}) \cong \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)/\mathcal{K}(\mathcal{H} \otimes \mathbb{C}^n).$$

Therefore $\mathrm{GL}(n, \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})) \cong \mathrm{GL}(1, \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)/\mathcal{K}(\mathcal{H} \otimes \mathbb{C}^n))$, so

$$K_1(C(X, (\mathcal{B}/\mathcal{K})_{\mathcal{A}})) \cong \pi_0(C(X, \mathrm{GL}(1, \mathcal{B}/\mathcal{K})_{\mathcal{A}})),$$

and we obtain the isomorphism

$$\mathrm{index}' \colon\ \pi_0(C(X, \mathrm{GL}(1, \mathcal{B}/\mathcal{K})_{\mathcal{A}})) \longrightarrow K^0(X, \mathcal{A}). \tag{10}$$

□

Thus when Θ is a torsion class, the corresponding twisted K–theory can be described just as if the C^*–algebra $C(X, \mathcal{K}_{\mathcal{A}})$ had a unit. In this case the index isomorphism can also be written in familiar form.

Proposition 3 *Suppose X is a compact manifold and $\mathcal{A}$ is an Azumaya bundle over X. Then given a section $s \in C(X, \mathrm{Fred}_{\mathcal{A}})$ there is a section $t \in C(X, \mathcal{K}_{\mathcal{A}})$ such that $\mathrm{index}(s + t) = p_1 - p_2$, where p_1, p_2 are projections in $C(X, \mathcal{K}_{\mathcal{A}})$ representing the projection onto the kernel of $(s + t)$ and the projection onto the kernel of the adjoint $(s + t)^*$ respectively. Moreover the map*

$$\begin{aligned} \pi_0(C(X, \mathrm{Fred}_{\mathcal{A}})) \quad &\longrightarrow \quad K^0(X, \mathcal{A}) \\ [[s]] \quad &\longrightarrow \quad [\mathrm{index}(s + t)] \end{aligned}$$

is well defined and is independent of the choice of $t \in C(X, \mathcal{K}_{\mathcal{A}})$.

1.3 Projective vector bundles

Consider the short exact sequence of groups

$$\mathbb{Z}_n \longrightarrow \mathrm{SU}(n) \xrightarrow{\pi} \mathrm{PU}(n),\ n \in \mathbb{N} \tag{11}$$

and the associated (determinant) line bundle L over $\mathrm{PU}(n)$. The fiber at $p \in \mathrm{PU}(n)$ is

$$\begin{aligned} L_p = \{(a, z) \in \mathrm{SU}(n) \times \mathbb{C}; \pi(a) = p\}/\sim, \\ (a, z) \sim (a', z') \text{ if } a' = ta,\ z' = tz,\ t \in \mathbb{Z}_n. \end{aligned} \tag{12}$$

This is a primitive line bundle over $\mathrm{PU}(n)$ in the sense that there is a natural $\mathrm{SU}(n)$ action on the total space of L,

$$l_a \colon\ (g, z) \longmapsto (ga, z) \tag{13}$$

which induces a natural isomorphism

$$L_{pq} \equiv L_p \otimes L_q \; \forall \; p, q \in \mathrm{PU}(n). \tag{14}$$

Indeed, if $a \in \mathrm{SU}(n)$ and $\pi(a) = p$ then $l_q \colon L_b \longrightarrow L_{ab}$ and $l_a \colon L_{\mathrm{Id}} \equiv \mathbb{C} \longrightarrow L_q$ combine to give an isomorphism (14) which is independent of choices. From the definition there is an injection

$$i_q \colon \pi^{-1}(q) \hookrightarrow L_q, \; q \in \mathrm{PU}(n) \tag{15}$$

mapping a to the equivalence class of $(a, 1)$. Thus $a \in \mathrm{SU}(n)$ fixes a trivialization $e_a \colon L_{\pi(a)} \longrightarrow \mathbb{C}$, determined by $(a, 1) \longmapsto 1$.

Let $P = P(\mathcal{A})$ be the $\mathrm{PU}(n)$ bundle associated to an Azumaya bundle. Thus the fiber P_x at $x \in X$ consists of the algebra isomomorphisms of $\mathcal{A}_x$ to $M(n, \mathbb{C})$. By a *projective* vector bundle over X, associated to $\mathcal{A}$, we shall mean a complex vector bundle E over $P(\mathcal{A})$ with a smooth family of linear isomorphisms

$$\gamma_p \colon p^*E \longrightarrow E \otimes L_p^{-1}, \; p \in \mathrm{PU}(n) \tag{16}$$

satisfying the compatibility condition

$$\gamma_{pp'} = \gamma_p \circ \gamma_{p'} \tag{17}$$

in the sense that on the right $\gamma_p \colon p^*E \otimes L_{p'}^{-1} \longrightarrow E \otimes L_p^{-1} \otimes L_{p'}^{-1} \longrightarrow E \otimes L_{pp'}^{-1}$ using (14). In fact γ lifts to an action of $\mathrm{SU}(n)$:

$$\tilde{\gamma}_a \colon p^*E \longrightarrow E, \; \tilde{\gamma}_a = (\mathrm{Id} \otimes e_a)\gamma_{\pi(a)}. \tag{18}$$

Thus projective vector bundles are just a special case of $\mathrm{SU}(n)$–equivariant vector bundles over P. This has also been studied in the case when P is a bundle gerbe [12] in which case E is known as a bundle gerbe module [4].

A bundle homomorphism between two projective bundles E and F is itself projective if it intertwines the corresponding isomorphisms (16). Since there is a natural isomorphism $\hom(E_m, F_m) \equiv \hom(E_m \otimes L, F_m \otimes L)$ for any complex line L, the identifications γ_p act by conjugation on $\hom(E, F)$ and give it the structure of a $\mathrm{PU}(n)$–invariant bundle. Thus the invariant sections, the projective homomorphisms, are the sections of a bundle $\hom_{\mathrm{proj}}(E, F)$ over X.

Just as a family of finite rank projections, forming a section of $\mathcal{C}(X, \mathcal{K})$, fixes a vector bundle over X, so a section of a model twisted bundle $\mathcal{A} \otimes \mathcal{K}$, with values in the projections, fixes a projective vector bundle over X. Thus, if $m \in P_x$ then by definition $m \colon \mathcal{A}_x \longrightarrow M(n, \mathbb{C})$ is an algebra isomorphism. The projection $\mu_x \in \mathcal{A}_x \otimes \mathcal{K}$ thus becomes a finite rank projection in $M(n, \mathbb{C}) \otimes \mathcal{K}$. Using a fixed identification

$$R \colon \mathbb{C}^n \otimes \mathcal{H} \longrightarrow \mathcal{H} \tag{19}$$

and the induced identification $\mathrm{Ad}(R)$ of $M_n(\mathbb{C})\otimes\mathcal{K}$ and $\mathcal{K}(\mathcal{H})$, this projection may be identified with its range $E_m \subset \mathcal{H}$. The continuity of this operation shows that the E_m form a vector bundle E over P. To see that E is a projective vector bundle observe that under the action of $p \in \mathrm{PU}(n)$ on P_x, replacing m by mp, E_m is transformed to $E_{m'} = R(a\otimes\mathrm{Id})E_m$ where $a \in \mathrm{SU}(n)$ is a lift of p, $\pi(a) = p$. Using the choice of a to trivialize L_p, the resulting linear map

$$\gamma_p\colon\ E_m \longrightarrow E_{m'} \otimes L_p^{-1} \tag{20}$$

is independent of choices and satisfies (17).

The direct sum of two projective vector bundles over P is again a projective bundle so there is an associated Grothendieck group of projective K–theory over P, $K^0_{\mathrm{proj}}(P)$. The discussion above of the equivalence of projections in $C(X;\mathcal{A}\otimes\mathcal{K})$ and projective vector bundles then shows the natural equality of the corresponding Grothendieck groups, just as in the untwisted case.

Lemma 3 *If P is the principal $\mathrm{PU}(n)$ bundle over X associated to an Azumaya bundle, then $K^0_{\mathrm{proj}}(P)$ is canonically isomorphic to $K^0(X;\mathcal{A})$.*

Similar conclusions hold for K–theory with compact supports if the base is not compact; we denote the twisted K–groups with compact support $K_c(X,\mathcal{A})$.

A projective vector bundle may be specified by local trivializations relative to a trivialization of $\mathcal{A}$ and we proceed to discuss the smoothness and equivalence of such trivializations.

Consider a 'full' local trivialization of the Azumaya bundle $\mathcal{A}$ over a good open cover $\{U_a\}_{a\in A}$ of the base X. Thus, there are algebra isomorphisms

$$F_a\colon\ \mathcal{A}\big|_{U_a} \longrightarrow U_a \times M(n,\mathbb{C})$$

with lifted transition maps, chosen to be to continuous (or smooth if the G_{ab} are smooth)

$$G_{ab}\colon\ U_{ab} = U_a \cap U_b \longrightarrow \mathrm{SU}(n) \text{ such that } G_{ab} \equiv F_a \circ F_b^{-1} \text{ in } \mathrm{PU}(n). \tag{21}$$

That is, the transition maps for $\mathcal{A}$ over U_{ab} are given by the adjoint action of the G_{ab}. Thus the Dixmier–Douady cocycle associated to the trivialization is

$$\theta_{abc} = G_{ab}G_{bc}G_{ca}\colon\ U_{abc} \longrightarrow \mathbb{Z}_n \subset \mathrm{U}(1) \subset \mathbb{C}^*. \tag{22}$$

Such a choice of full local trivialization necessarily gives a local trivialization of the associated $\mathrm{PU}(n)$ bundle, P, and also gives a local trivialization of the determinant bundle L over P. The Dixmier–Douady class of $\mathcal{A}$ is defined as the cohomology class $[\theta] \in H^2(X,\underline{\mathrm{U}(1)}) \cong H^3(X,\mathbb{Z})$. Whereas it is true that one can pick a "constant" $\mathbb{Z}_n$–valued, 2–cocycle that represents the Dixmier–Douady class as shown above, the equivalence relation (ie cohomology class) is not in $H^2(X,\mathbb{Z}_n)$, but rather in the sheaf cohomology $H^2(X,\underline{\mathrm{U}(1)})$ (which is isomorphic to $H^3(X,\mathbb{Z})$).

Projective vector bundle data, associated to such a full trivialization of $\mathcal{A}$, consists of complex vector bundles E_a, of some fixed rank k, and transition maps $Q_{ab}\colon E_b \longrightarrow E_a$ over each U_{ab} satisfying the weak cocycle condition

$$Q_{ab}Q_{bc} = \theta_{abc}Q_{ac} \tag{23}$$

where θ is given by (22). Two sets of such data E_a, Q_{ab} and E'_a, Q'_{ab} over the same cover are *equivalent* if there are bundle isomorphisms $T_a\colon E_a \longrightarrow E'_a$ such that $Q'_{ab} = T_aQ_{ab}T_b^{-1}$ over each U_{ab}.

Associated with the trivialization of $\mathcal{A}$ there is a particular set of projective vector bundle data given by the trivial bundles $\mathbb{C}^n$ over the U_a and the transition maps G_{ab}. We will denote this data as E_τ where τ denotes the trivialization of $\mathcal{A}$. The Azumaya algebra $\mathcal{A}$ may then be identified with $\hom_{\mathrm{proj}}(E_\tau, E_\tau)$ for any of the projective vector bundles E_τ.

Lemma 4 *Projective vector bundle data with respect to a full trivialization of an Azumaya bundle lifts to define a projective vector bundle over the associated principal* $\mathrm{PU}(n)$ *bundle; all projective bundles arise this way and projective isomorphisms of projective bundles corresponds to equivalence of the projective vector bundle data.*

Proof The given trivialization of $\mathcal{A}$, over each U_a defines a section of $s_a\colon U_a \longrightarrow P = P(\mathcal{A})$ over U_a. Using this section we may lift the bundle E_a to the image of the section and then extend it to a bundle $E^{(a)}$ on the whole of $P\big|_{U_a}$ which is projective, namely by setting

$$E(a)_{s_a(x)p} = E_a(x)\otimes L_p \;\forall\; p\in \mathrm{PU}(n),\; x\in U_a \tag{24}$$

and taking the ismorphism γ_p over U_a to be given by the identity on $E_a(x)$. Over each intersection we then have an isomorphism

$$Q_{ab}(x)\colon\; E_a(x) = E^{(a)}(s_a(x)) \longrightarrow E^{(b)}(s_b(x)) = E_b(x).$$

Now, from the trivialization of P we have $s_b(x) = s_a(x)g_{ab}(x)$ where $G_{ab}\colon U_{ab} \longrightarrow \mathrm{SU}(n)$ and g_{ab} is the projection of G_{ab} into $\mathrm{PU}(n)$. The choice of G_{ab} therefore also fixes a trivialization of the determinant bundle $L_{g_{ab}(x)} \longrightarrow \mathbb{C}$. Since $E^{(b)}(s_b(x))$ is identified with $E^{(b)}(s_a(x))\otimes L^{-1}_{g_{ab}(x)}$ by the primitivity, this allows Q_{ab} to be interpreted as the transition map from $E^{(a)}$ to $E^{(b)}$ over the preimage of U_{ab}. Furthermore the weak cocycle condition (23) now becomes the cocycle condition guaranteeing that the $E^{(a)}$ combine to a globally defined, projective, bundle over P.

This argument can be reversed to construct projective vector bundle data from a projective vector bundle over P and a similar argument shows that projective bundle isomorphisms correspond to isomorphisms of the projective vector bundle data. □

Thus we may simply describe 'projective vector bundle data' as a local trivialization of the corresponding projective bundle, where this also involves the choice of a full local trivialization of $\mathcal{A}$. The projective vector bundle data E_τ associated to a full trivialization of $\mathcal{A}$ thus determines a projective vector bundle, which we may also denote by E_τ, over $P(\mathcal{A})$. In particular there are projective vector bundles of arbitrarily large rank over P.

Many of the standard results relating K–theory to vector bundles carry over to the twisted case. We recall two of these which are important for the proof of the index theorem. Lemma 5 below essentially follows from definitions, the morphism $C_0(U,\mathcal{A}) \to C(M,\mathcal{A})$, and the functoriality of K–theory.

Lemma 5 *If $U \subset X$ is an open set of a compact manifold then for any Azumaya bundle $\mathcal{A}$, over X, there is an extension map*

$$K^0_c(U,\mathcal{A}_U) \longrightarrow K^0(X,\mathcal{A}). \tag{25}$$

Proof This proceeds in essentially the usual way. An element of $K^0_c(U,\mathcal{A}_U)$ is represented in terms of a full local trivialization of $\mathcal{A}$ by a pair of sets of projective vector data E_1, E_2 over U and a given bundle isomorphism between them outside a compact set, $c\colon E_1 \longrightarrow E_2$ over $U\setminus K$. If E_τ is the projective vector bundle data associated to the trivialization of $\mathcal{A}$ then we may embed E_2 as a projective subbundle of E^q_τ for some integer q. To see this, first choose an embedding e of E_2 as a subbundle of E^l_τ over P for some large l. Then choose a full trivialization of $\mathcal{A}$ and a partition of unity ψ_a subordinate on X to the open U_a. Over the premimage of each U_a, e_a can be extended uniquely to a projective embedding e_a of E in E^l_τ. The global map formed by the sum over a of the $\psi_a e_a$ gives a projective embedding into E^q_τ where $q = lN$ and N is the number of sets in the open cover. Then $E^q_\tau = E_2 \oplus F$ for some complementary projective bundle F over U. The data $E_1 \oplus F$ over U and E^q_τ over $Z\setminus K$ with the isomorphism $c \oplus \mathrm{Id}_F$ over $U\setminus K$ determine a projective vector bundle over Z. The element of $K^0(X;\mathcal{A})$ represented by the pair consisting of this bundle and E^q_τ is independent of choices, so defines the extension map. □

Proposition 4 *For any real vector bundle $\pi\colon\ V \longrightarrow X$ and Azumaya bundle $\mathcal{A}$ over X a section over the sphere bundle, SV, of V of the isomorphism bundle of the lifts of two projective vector bundles over X, determines an element of $K^0_c(V, \pi^*\mathcal{A})$; all elements arise this way and two isomorphisms give the same element if they are homotopic after stabilization with the identity isomorphisms of projective bundles.*

Proof The proof is the same as in the untwisted case. □

If W is a vector bundle over X and E is a projective bundle over a $\mathrm{PU}(n)$ bundle over X then $E \otimes \pi^*W$ is naturally a projective bundle. This operation extends to make $K^0_{\mathrm{proj}}(P)$ a module over (untwisted) $K^0(X)$ and hence, in view of Lemma 3 gives a module structure

$$K^0(X) \times K^0(X; \mathcal{A}) \longrightarrow K^0(X; \mathcal{A}). \tag{26}$$

1.4 The Chern character of projective vector bundles

The Chern character

$$\mathrm{Ch}_{\mathcal{A}}\colon\ K^0(X; \mathcal{A}) \longrightarrow H^{\mathrm{ev}}(X; \mathbb{Q}) \tag{27}$$

may be defined in one of several equivalent ways. It is known that $K^0(X)$ and $K^0(X, \mathcal{A})$ are isomorphic after tensoring with $\mathbb{Q}$, [8] (see also [21], [13]). Since the Chern Character on $K^0(X)$ factors through $K^0(X) \otimes \mathbb{C}$ it is also defined in the twisted case.

To define (27) directly using the Chern–Weil approach we note that the local constancy of the Cěch 2–cocycle θ in (23) allows a connection to be defined directly on such projective vector bundle data. That is, despite the failure of the usual cocycle condition, there exist connections ∇^a on each of the bundles E_a which are identified by the Q_{ab}. To see this, simply take arbitary connections $\tilde{\nabla}^a$ on each of the E_a and a partition of unity ϕ_a subordinate to the cover. Now define a new connection on E_a by

$$\nabla^a = \phi_a \nabla^a + \sum_{b \neq a} \phi_b Q^*_{ab} \nabla^b.$$

These are consistent under the transition maps. Thus the curvature of this collective connection is a well-defined section of the endomorphism bundle of the given projective vector bundle data. As such the usual Chern–Weil arguments apply and give the Chern character (27). Lifted to P this connection gives a projective connection on the lift of E to a projective bundle; this also allows the Chern character to be defined directly on $K^0_{\mathrm{proj}}(P)$.

Either of these approaches to the Chern character show that it distributes over the usual Chern character on $K^0(X)$ under the action (26). In particular it follows that (27) is an isomorphism over $\mathbb{Q}$.

We may also define the Chern character by reducing to the standard case by taking tensor powers. Thus if E is a projective vector bundle associated to the Azumaya bundle $\mathcal{A}$ and n is the order of the associated Dixmier–Douady invariant then

$$\phi^* E^{\otimes n} \cong \pi_P^* E^{\otimes n}$$

and therefore $E^{\otimes n} = \pi^*(F)$ for some vector bundle $F \to X$.

Observe that the Chern character of E satisfies,

$$\mathrm{Ch}(E^{\otimes n}) = \mathrm{Ch}(E)^n = \pi^*(\mathrm{Ch}(F)). \tag{28}$$

We claim that $\mathrm{Ch}(E) = \pi^*(\Lambda)$ for some cohomology class $\Lambda \in H^{even}(X, \mathbb{Q})$. First observe that (28) implies that the degree zero term, which is a constant term, is of the desired form. Next assume that the degree $2k$ component of the Chern character satisfies $\mathrm{Ch}_k(E) = \pi^*(\Lambda_k)$ for some cohomology class $\Lambda_k \in H^{2k}(X, \mathbb{Q})$. Then (28) implies that the degree (2k+2) component $\mathrm{Ch}_{k+1}(E^{\otimes n}) = \pi^*(\mathrm{Ch}_{k+1}(F))$. But the left hand side is of the form of the Chern character,

$$\mathrm{Ch}_{k+1}(E^{\otimes n}) = a_0 \,\mathrm{Ch}_{k+1}(E) + \sum_{|I|=k+1, r>1} a_I \,\mathrm{Ch}_I(E) \tag{29}$$

where $I = (i_1, \ldots, i_r)$, $|I| = i_1 + \cdots + i_r$, $\mathrm{Ch}_I(E) = \mathrm{Ch}_{i_1}(E) \cup \cdots \cup \mathrm{Ch}_{i_r}(E)$ and $a_0, a_I \in \mathbb{Q}$ are such that $a_0 \neq 0$. By the induction hypothesis, we deduce that $\mathrm{Ch}_{k+1}(E)$ is of the form $\pi^*(\Lambda_{k+1})$ for some cohomology class $\Lambda_{k+1} \in H^{2k+2}(X, \mathbb{Q})$. This proves the claim.

Then the Chern character of the projective vector bundle E above is given by

$$\mathrm{Ch}_{\mathcal{A}}(E) = \Lambda \in H^{even}(X, \mathbb{Q}). \tag{30}$$

That is, the lift of the Chern character of the projective vector bundle E to P coincides with the ordinary Chern character of E. The following properties of the Chern character of projective vector bundles follow from the corresponding properties of the Chern character of vector bundles.

Lemma 6 *Let $\mathcal{A}$ be an Azumaya bundle over X and P be the principal $\mathrm{PU}(n)$ bundle associated to $\mathcal{A}$. Let $E \to P$ be a projective vector bundle over X, associated to $\mathcal{A}$, then the Chern character defined as in* (30) *above has the following properties.*

(1) *If $E' \to X$ is another projective vector bundle, then*

$$\mathrm{Ch}_{\mathcal{A}}(E \oplus E') = \mathrm{Ch}_{\mathcal{A}}(E) + \mathrm{Ch}_{\mathcal{A}}(E'),$$

so the Chern character is a homomorphism

$$\mathrm{Ch}_{\mathcal{A}}\colon\ K^0(X, \mathcal{A}) \to H^{even}(X, \mathbb{Q}).$$

(2) *The degree 2 component of the Chern character* $\mathrm{Ch}_{\mathcal{A}}(E)$ *coincides with the first Chern class of the determinant line bundle* $\det(E) \to X$.

2 The analytic index

For two ordinary vector bundles, $E^{\pm}$, over a compact manifold Z, the space $\Psi^m(Z; E^+, E^-)$ of pseudodifferential operators of order m mapping $\mathcal{C}^\infty(Z; E^+)$ to $\mathcal{C}^\infty(Z; E^-)$ may be identified naturally with the tensor product

$$\Psi^m(Z; E^+, E^-) = \Psi^m(Z) \otimes_{\mathcal{C}^\infty(Z^2)} \mathcal{C}^\infty(Z^2; \mathrm{Hom}(E^+, E^-)) \tag{31}$$

where $\mathrm{Hom}(E^+, E^-)$ is the 'big' homomorphism bundle over Z^2 which has fiber $\hom(E^+_z, E^-_{z'})$ at (z', z) and $\Psi^m(Z)$ is the space of pseudodifferential operators acting on functions. The latter is a module over $\mathcal{C}^\infty(Z^2)$ through its realization as a space of Schwartz kernels. In particular

$$\Psi^m(Z; E, E) = \Psi^m(Z; E) = \Psi^m(Z) \otimes_{\mathcal{C}^\infty(Z^2)} \mathcal{C}^\infty(Z^2; \mathrm{Hom}(E))$$

when the two bundles coincide.

For a fibration $\phi\colon\ M \longrightarrow X$, with compact boundaryless fibres, the bundle of pseudodifferential operators acting on sections of vector bundles over the total space may be similarly defined. Note that the operators act fibre-wise and so commute with multiplication by functions on the base. If $\mathrm{Hom}(M^2_\phi, E^+, E^-)$ denotes the bundle over the fibre product, which is the 'big' homomorphism bundle on each fibre, then again

$$\Psi^m(M/X; E^+, E^-) = \Psi^m(M/X) \otimes_{\mathcal{C}^\infty(M^2_\phi)} \mathcal{C}^\infty(M^2_\phi; \mathrm{Hom}(E^+, E^-))$$

is the bundle of operators to which the usual families index theorem applies. Here $\Psi^m(M/X)$ is the bundle of pseudodifferential operators acting on functions on the fibres.

Now, let $\mathcal{A}$ be an Azumaya bundle over the base of the fibration ϕ. Consider a projective vector bundle, E, over the lift to M of the principal $\mathrm{PU}(N)$ bundle associated to $\mathcal{A}$. Given a local trivialization of $\mathcal{A}$ there is a bundle trivialization of E with respect to the lift of the trivialization to M. We shall call this a *basic bundle trivialization.*

Lemma 7 *If E^+ and E^- are two projective vector bundles over the lift to M of the* $\mathrm{PU}(N)$ *bundle associated to an Azumaya bundle on X then the big homomorphism bundles* $\mathrm{Hom}(Q_a^+, Q_a^-)$, *arising from basic bundle trivializations of the $E^\pm$ define a vector bundle* $\mathrm{Hom}(E^+, E^-)$ *over* M_ϕ^2.

Proof As already noted, a local trivialization of $\mathcal{A}$ over the base gives a trivialization of the associated $\mathrm{PU}(N)$ bundle, and hence of its lift to M. This leads to bundle trivializations $Q_a^\pm$, over the elements $\phi^{-1}(U_a)$ of this open cover, of the projective bundles $E^\pm$. Since the transition maps act by the adjoint action, the scalar factors cancel and the 'big' homomorphism bundles between the $Q_a^\pm$ now patch to give a global bundle $\mathrm{Hom}(E^+, E^-)$ over M_ϕ^2. □

If $E_i,\ i = 1, 2, 3,$ are three such projective bundles then, just as for the usual homomorphism bundles, there is a bilinear product map

$$\mathrm{Hom}(E_1, E_2) \otimes \mathrm{Hom}(E_2, E_3)\big|_C \longrightarrow \psi^* \mathrm{Hom}(E_1, E_2) \tag{32}$$

where C is the central fiber diagonal in $M_\phi^2 \times M_\phi^2$ and $\psi\colon C \longrightarrow M_\phi^2$ is projection off the middle factor. This reduces to the composition law for $\mathrm{hom}(E_i, E_j) = \mathrm{Hom}(E_i, E_j)\big|_{\mathrm{Diag}}$ on the diagonal.

We may now simply define the algebra of twisted (fiber-wise) pseudodifferential operators as

$$\Psi^m(M/X; E^+, E^-) = \Psi^m(M/X) \otimes_{\mathcal{C}^\infty(M_\phi^2)} \mathcal{C}^\infty(M_\phi^2; \mathrm{Hom}(E^+, E^-)). \tag{33}$$

Restricted to open sets in the base over which $\mathcal{A}$ is trivialized, this reduces to the standard definition. Thus, the symbol sequence remains exact

$$\begin{aligned} 0 \longrightarrow \Psi^{m-1}(M/X; E^+, E^-) \hookrightarrow \\ \Psi^m(M/X; E^+, E^-) \xrightarrow{\sigma_m} S^{[m]}(S^*(M/X); \phi^* \mathrm{hom}(E^+, E^-)) \longrightarrow 0 \end{aligned} \tag{34}$$

with the proof essentially unchanged. Here $S^{[m]}(S^*(M/X); \rho^* \mathrm{hom}(E^+, E^-))$ is the quotient space of symbolic sections of order m, by symbolic sections of order $m-1$, of $\rho^* \mathrm{hom}(E^+, E^-)$ as a bundle over $S^*(M/X)$, the fibre cosphere bundle, $\rho\colon S(M/X) \longrightarrow M$ being the projection. Similarly the usual composition properties carry over to this twisted case, since they apply to the local families. For any three projective vector bundles $E_i,\ i = 1, 2, 3,$ over the lift of the same $\mathrm{PU}(N)$ bundle from the base

$$\begin{aligned} \Psi^m(M/X; E_2, E_3) \circ \Psi^{m'}(M/X; E_1, E_2) \subset \Psi^{m+m'}(M/X; E_1, E_3), \\ \sigma_{m+m'}(A \circ B) = \sigma_m(A) \circ \sigma_{m'}(B). \end{aligned} \tag{35}$$

.

For any basic bundle trivialization of a projective vector bundle with respect to a local trivialization of $\mathcal{A}$ the spaces of sections of the local bundles form infinite-dimensional projective bundle data over the base, associated to the same trivialization of $\mathcal{A}$. More generally, for any fixed real number, m, the spaces of Sobolev sections of order m over the fibres form projective Hilbert bundle data over the base; we will denote the corresponding projective bundle $H^m(M/X;E)$. The boundedness of pseudodifferential operators on Sobolev spaces then shows that any $A \in \Psi^m(M/X;E^+,E^-)$ defines a bounded operator

$$A\colon\ H^{m_1}(M/X,E^+) \longrightarrow H^{m_2}(M/X;E^-) \text{ provided } m_1 \geq m_2+m. \tag{36}$$

If $m_1 > m_2 + m$ this operator is compact.

It is possible to choose quantization maps as in the untwisted case. To do so, choose basic bundle trivializations and quantization maps, that is right inverses for the symbol map, for the local bundles $Q^{\pm}_a$. Using a partition of unity on the base this gives a global quantization map:

$$\begin{gathered} q_m\colon\ S^{[m]}(S^*(M/X);\rho^*\hom(E^+,E^-)) \longrightarrow \Psi^m(M/X;E^+,E^-), \\ \sigma_m\circ q_m = \mathrm{Id},\ q_m\circ\sigma_m - \mathrm{Id}\colon\ \Psi^m(M/X;E^+,E^-) \longrightarrow \Psi^{m-1}(M/X;E^+,E^-). \end{gathered} \tag{37}$$

By definition a projective family in $\Psi^m(M/X;E^+,E^-)$ is elliptic if σ_m is invertible, with inverse in $S^{[m]}(S^*(M/X);\rho^*\hom(E^-,E^+))$. Directly from the symbolic properties of the algebra, this is equivalent to there being a parameterix $B \in \Psi^{-m}(M/X;E^-,E^+)$ such that $A\circ B - \mathrm{Id} \in \Psi^{-1}(M/X;E^-)$ and $B\circ A - \mathrm{Id} \in \Psi^{-1}(M/X;E^+)$. These 'error terms' give compact maps, for $m_1 = m_2 + m$, in (36). Thus the elliptic family consists of Fredholm operators. It follows from the discussion in Section 1 that the family defines a twisted $K-$ class using (8). To see this class more concretely, as in the untwisted case, we may perturb the family so that the index bundle gives projective vector bundle data with respect to the given trivialization of $\mathcal{A}$. Locally in the base a bundle map from an auxilliary vector bundle, over the base, may be added to make the family surjective. Choosing this bundle to be part of (some large power) of projective vector bundle data these local maps may be made into global smooth homomorphism into the image bundle

$$f\colon\ E^N_\tau \longrightarrow \mathcal{C}^\infty(M/X;E^-) \text{ such that } P+f \text{ is surjective.} \tag{38}$$

This necessarily stabilizes the null bundle to projective vector bundle data with respect to the trivialization and we set

$$\mathrm{index}_a(P) = \left[\ker(P+f) - E^N_\tau\right] \in K^0(X,\mathcal{A}). \tag{39}$$

As in the untwisted case this class can be seen to be independent of the precise stabilization used and to be homotopy invariant. In fact adding a further stabilizing bundle is easily seen to leave the index unchanged and stabilizing the family with the additional parameter of a homotopy shows the homotopy invariance.

Proposition 5 *For a fibration* (1), *Azumaya bundle* $\mathcal{A}$ *over* X *and projective bundles* $E^{\pm}$, *there is a quantization of a given ismorphism* b *of the lifts of these bundles to* $S^*(X/M)$ *for which the null spaces, and hence also the null spaces of the adjoint family, define a projective bundle over the base so that the difference class* $\mathrm{index}_a(b) \in K^0(X, \mathcal{A})$ *depends only on the class of* b *in* $K^0(T(X/M), \rho^*\phi^*\mathcal{A})$ *and so defines the analytic index homomorphism*

$$\mathrm{index}_a\colon\ K(T(M/X), \rho^*\phi^*\mathcal{A}) \longrightarrow K(X; \mathcal{A}). \tag{40}$$

Proof The stabilization discussed above associates to an elliptic family with principal symbol b an element of $K(X, \mathcal{A})$. This class is independent of the stabilization used to define it and is similarly independent of the quantization chosen, since two such families differ by a family of compact operators. Clearly the element is unchanged if the symbol, or operator, is stabilized by the identity on some other primitve vector bundle defined over the lift of the same $\mathrm{PU}(N)$ bundle. Furthermore the homotopy invariance of the index and the existence of a quantization map show that the element depends only on the homotopy class of the symbol. The additivity of the index under composition and the multiplicativity of the symbol map then shows that the resulting map (40) is a homomorphism. □

3 The topological index

In this section we define the topological index map for a fibration of compact manifolds (1)

$$\mathrm{index}_t\colon\ K_c(T(M/X), \rho^*\phi^*\mathcal{A}) \longrightarrow K^0(X, \mathcal{A}) \tag{41}$$

where $\rho\colon\ T(X/M) \longrightarrow M$ is the projection and $\mathcal{A}$ is an Azumaya bundle over X.

We first recall some functorial properties of twisted K–theory. Let $f\colon\ Y \longrightarrow Z$ be a smooth map between compact manifolds. Then the pullback map,

$$f^!\colon\ K(Z, \mathcal{A}) \longrightarrow K(Y, f^*\mathcal{A}),$$

for any Azumaya bundle $\mathcal{A}$, is defined as follows. Let V be finite dimensional projective vector bundle data over Z, associated with a trivialization of $\mathcal{A}$.

Then f^*V is projective vector bundle data over Y associated to the lifted trivialization and the resulting class in K–theory is independent of choices, so defines $f^!$. Alternatively, if s is a section of the twisted Fredholm bundle of $\mathcal{A}\otimes\mathcal{K}$ over Z, then the pullback f^*s is a section of the corresponding twisted Fredholm bundle over Y. The pull-back map may also be defined directly in terms of the pull-back of projective bundles from the $\mathrm{PU}(N)$ bundle associated to $\mathcal{A}$ over Z to its pull-back over Y.

Lemma 8 *For any Azumaya bundle there is a canonical isomorphism*

$$j_!\colon\ K(X,\mathcal{A}) \cong K_c(X\times\mathbb{R}^{2N}, p_1^*\mathcal{A})$$

determined by Bott periodicity.

Proof Recall that $K_c(X\times\mathbb{R}^{2N}, p_1^*\mathcal{A}) \cong K(C_0(X\times\mathbb{R}^{2N}, \mathcal{E}_{p_1^*\mathcal{A}}))$. Now $\mathcal{E}_{p_1^*\mathcal{A}} \cong p_1^*\mathcal{E}_{\mathcal{A}}$, so that $C_0(X\times\mathbb{R}^{2N}, \mathcal{E}_{p_1^*\mathcal{A}}) \cong C(X,\mathcal{E}_{\mathcal{A}})\widehat{\otimes} C_0(\mathbb{R}^{2N})$. Thus,

$$K_c(X\times\mathbb{R}^{2N}, p_1^*\mathcal{A}) \cong K(C(X,\mathcal{E}_{\mathcal{A}})\otimes C_0(\mathbb{R}^{2N})).$$

Together with Bott periodicity, $K(C(X,\mathcal{E}_{\mathcal{A}})\otimes C_0(\mathbb{R}^{2N})) \cong K(X,\mathcal{A})$, this proves the lemma. □

If $\phi\colon\ M \longrightarrow X$ is our basic fibre bundle of compact manifolds we know that there is an embedding $f\colon\ M \longrightarrow X\times\mathbb{R}^N$, cf [2] Section 3. Then the *fibre-wise* differential is an embedding $Df\colon\ T(M/X) \longrightarrow X\times\mathbb{R}^{2N}$ with complex normal bundle. In the untwisted case we have, via the Thom isomorphism, $Df_!\colon\ K_c(T(M/X)) \longrightarrow K_c(X\times\mathbb{R}^{2N})$. By anology with the case of compact $\mathrm{spin}^{\mathbb{C}}$ manifolds, we call this the Gysin map.

We explain the extension to the twisted case. So again let $\mathcal{A}$ be an Azumaya algebra over X with $Y = T(M/X)$. Let E be the (complex) normal bundle to the imbedding $i\colon\ M \longrightarrow X\times\mathbb{R}^N$, and let $\mathcal{A}_E$ be the lift of $\mathcal{A}$ to E. Then

$$\begin{aligned} i_!\colon\ K_c(Y,\mathcal{A}) &\longrightarrow K_c(E,\mathcal{A}_E),\\ \xi &\longmapsto (\pi^*\xi, \pi^*G)\otimes(\pi^*S^+, \pi^*S^-, c(v)) \end{aligned} \tag{42}$$

where $\xi = (\xi^+, \xi^-)$ is pair of projective vector bundle data over X, associated to a local trivialization of $\mathcal{A}$, with $G\colon\ \xi^+ \longrightarrow \xi^-$ a projective bundle map between them which is an isomorphism outside a compact set and $(\pi^*S^+, \pi^*S^-, c(v))$ is the usual Thom class of the complex vector bundle E. On the the right hand side the the graded pair of projective vector bundle data is

$$(\pi^*\xi^+\otimes\pi^*S^+ \oplus \pi^*\xi^-\otimes\pi^*S^-, \pi^*\xi^+\otimes\pi^*S^- \oplus \pi^*\xi^-\otimes\pi^*S^+)$$

with map between them being

$$\begin{bmatrix} G & c(v) \\ c(v) & G^{-1} \end{bmatrix}, \ v \in E.$$

This is an isomorphism outside a compact subset of E and defines a class in $K_c(E, \mathcal{A}_E)$ which is independent of choices. The Thom isomorphism in this context, cf [7], asserts that $i_!$ is an isomorphism.

Now, E is diffeomorphic to a tubular neighborhood $\mathcal{U}$ of the image of Y; let $\Phi\colon \mathcal{U} \longrightarrow E$ denote this diffeomorphism. By the Thom isomorphism above

$$i_!\colon K_c(Y, i^*\mathcal{A}) \longrightarrow \mathcal{K}_c(E, \mathcal{A}_E) \cong \mathcal{K}_c(\mathcal{U}, \mathcal{A}'),$$

where $\mathcal{A}' = \Phi^*(\mathcal{A}_E)$. Using Lemma 5, the inclusion of the open set U in $X \times R^{2N}$ induces a map $K_c(\mathcal{U}, \mathcal{A}') \longrightarrow K_c(X \times R^{2N}, \pi_1^*\mathcal{A})$ where $\pi_1\colon X \times \mathbb{R}^{2N} \longrightarrow X$ is the projection. The composition of these maps defines the Gysin map. In particular we get the Gysin map in twisted K–theory,

$$Df_!\colon K_c(T(M/X), \rho^*\pi^*\mathcal{A}) \longrightarrow K_c(X \times \mathbb{R}^{2N}, \pi_1^*\mathcal{A})$$

where $\rho\colon T(M/X) \longrightarrow X$ is the projection map. Since $\pi = \pi_1 \circ f$ it follows that $Df^*\pi_1^*\mathcal{A} = \rho^*\pi^*\mathcal{A}$. Now define the *topological index*, (41) as the map

$$\mathrm{index}_t = j_!^{-1} \circ Df_!\colon K_c(T(M/X), \rho^*\phi^*\mathcal{A}) \longrightarrow K(X, \mathcal{A}),$$

where we apply Lemma 8 to see that the inverse $j_!^{-1}$ exists.

4 Proof of the index theorem in twisted K–theory

We follow the axiomatic approach of Atiyah–Singer to prove that the analytic index and the topological index coincide.

Definition An *index map* is a homomorphism

$$\mathrm{index}\colon K_c(T(M/X), \rho^*\pi^*\mathcal{A}) \longrightarrow K(X, \mathcal{A}), \tag{43}$$

satisfying the following:

(1) (Functorial axiom) If M and M' are two fibre bundles with compact fibres over X and $f\colon M \longrightarrow M'$ is a diffeomorphism which commutes with the projection maps $\phi\colon M \longrightarrow X$ and $\phi'\colon M' \longrightarrow X$ then the diagram

$$\begin{tikzcd} K_c(T(M/X), \rho^*\phi^*\mathcal{A}) \arrow[dr, "\mathrm{index}"'] & & K_c(T(M'/X), \rho^*(\phi')^*\mathcal{A}) \arrow[ll, "f^!"'] \arrow[dl, "\mathrm{index}"] \\ & K(X, \mathcal{A}) & \end{tikzcd} \tag{44}$$

is commutative.

(2) (Excision axiom) Let $\phi\colon M \longrightarrow X$ and $\phi'\colon M' \longrightarrow X$ be two fibre bundles of compact manifolds, and let $\alpha : \mathcal{U} \subset M$ and $\alpha' : \mathcal{U}' \subset M'$ be two open sets with a diffeomorphism $g\colon \mathcal{U} \cong \mathcal{U}'$ satisfying $\phi' \circ g = \phi$ used to identify them, then the diagram

$$\begin{tikzcd} & K_c(T(M/X), \rho^*\phi^*\mathcal{A}) \arrow[dr, "\mathrm{index}"] & \\ K_c(T(\mathcal{U}/\phi(\mathcal{U})), \rho^*\alpha^*\phi^*\mathcal{A}) \arrow[ur, "\alpha_!"] \arrow[dr, "(\alpha')_!"'] & & K(X, \mathcal{A}) \\ & K_c(T(M'/X), \rho^*(\phi')^*\mathcal{A}) \arrow[ur, "\mathrm{index}'"'] & \end{tikzcd} \tag{45}$$

is commutative.

(3) (Multiplicativity axiom) Let V be a real vector space and suppose that $i\colon M \longrightarrow X \times V$ is an embedding which intertwines the projection maps $\phi\colon M \longrightarrow X$ and $\pi_1\colon X \times V \longrightarrow X$, ie $\pi_1 \circ i = \phi$; the fibrewise differential $i_*\colon T(M/X) \longrightarrow X \times TV$ also intertwines the projections. The one-point compactification $S(V \oplus \mathbb{R})$ of V is a sphere with the canonical inclusion $e\colon TV \longrightarrow TS(V \oplus \mathbb{R})$ inducing the inclusion $e' = \mathrm{Id} \times e\colon X \times TV \longrightarrow X \times TS(V \oplus \mathbb{R})$. Then the diagram

$$\begin{tikzcd} & K_c(X \times TV, \pi_1^*\mathcal{A}) \arrow[d, "(e')_!"] \\ K_c(T(M/X), \rho^*\phi^*\mathcal{A}) \arrow[ur, "(i_*)_!"] \arrow[dr, "\mathrm{index}"'] & K_c(X \times TS(V \oplus \mathbb{R}), \pi_1^*\mathcal{A}) \arrow[d, "\mathrm{index}"] \\ & K(X, \mathcal{A}) \end{tikzcd} \tag{46}$$

commutes.

(4) (Normalization axiom) If the fibre bundle of compact manifolds $\phi\colon M \longrightarrow X$ has single point fibres, then the index map

$$\mathrm{index}\colon K_c(T(M/X), \rho^*\pi^*\mathcal{A}) = K(X, \mathcal{A}) \longrightarrow K(X, \mathcal{A}) \tag{47}$$

is the identity map.

The next theorem asserts in particular that such an index map does exist.

Theorem 4.1 *The topological index,* index_t*, is an index map.*

Proof We proceed to check the axioms above in turn.

If $f\colon M \longrightarrow M'$ is a diffeomorphism as in the statement of the functoriality axiom, let $i\colon M' \longrightarrow X \times V$ be an embedding commuting with the projections, where V is a finite dimensional vector space. Then $i \circ f\colon M \longrightarrow X \times V$ is also such an embedding. Using these maps, we may identify the topological index maps as $\mathrm{index}_t' = j_!^{-1} \circ (i_*)_!$ and $\mathrm{index}_t = j_!^{-1} \circ ((i \circ f)_*)_! = j_!^{-1} \circ (i_*)_! \circ (f_*)_! = \mathrm{index}_t' \circ (f_*)_!$, where $j\colon X \hookrightarrow X \times V$ is the zero section embedding. Then the diagram

$$\begin{tikzcd}[column sep=small]
K_c(T(M/X), \rho^*\phi^*\mathcal{A}) \arrow[rr, "((f_*)_!)^{-1}"] \arrow[dr, "\mathrm{index}_t"'] & & K_c(T(M'/X), \rho^*\phi'^*\mathcal{A}) \\
& K(X, \mathcal{A}) \arrow[ur, "\mathrm{index}_t"'] &
\end{tikzcd}$$

commutes. Since f is a diffeomorphism, $(f_*)_!)^{-1} = f^!$, which establishes that index_t is functorial.

Next consider the excision axiom and let $i'\colon M' \longrightarrow X \times V$ be an embedding, and $j\colon X \hookrightarrow X \times V$ be the zero section embedding. Then $\mathrm{index}_t' = j_!^{-1} \circ (i_*)_!$, so that the relevant map in the lower part of the diagram (45) is $j_!^{-1} \circ (i_*)_! \circ (\alpha_*')_! = j_!^{-1} \circ ((i \circ \alpha')_*)_!$. But this is merely the topological index $K_c(T(\mathcal{U}/\pi(\mathcal{U})), \rho^*\alpha^*\pi^*\mathcal{A}) \longrightarrow K(X, \mathcal{A})$. That it agrees with the relevant map in the upper part of the diagram, follows from the fact that the topological index is well defined and independent of the choice of embedding. Thus the topological index satisfies the excision property.

The multiplicativity property for the topological index follows from its independence of the choice of embedding, since (46) amounts to the definition of the topological index for the given embedding. The independence of the choice of embedding is established briefly as follows. Let $i_k\colon M \longrightarrow X \times V_k$, $k = 0, 1$ be two embeddings, and $j_k\colon X \hookrightarrow X \times V_k$ be the zero section embedding, $k = 0, 1$. Consider a linear homotopy $I_t\colon M \longrightarrow X \times V_0 \oplus V_1$, $t \in [0,1]$ defined as $I_t(m) = (i_0(m), ti_1(m))$, and the zero section embedding $J\colon M \longrightarrow X \times V_0 \oplus V_1$ defined as $J(m) = (j_0(m), j_1(m))$. By the homotopy invariance of the induced map in K–theory, it follows that $J_!^{-1} \circ (I_{t*})_!$ is independent of t. Using the functorial property of the topological index, one deduces that ${j_0}_!^{-1} \circ (i_{0*})_!$ agrees with $J_!^{-1} \circ (I_{1*})_!$. Now let $\tilde{I}_t\colon M \longrightarrow X \times V_0 \oplus V_1$, $t \in [0,1]$ be defined as $\tilde{I}_t(m) = (ti_0(m), i_1(m))$. The argument above establishes that ${j_1}_!^{-1} \circ (i_{1*})_!$ also agrees with $J_!^{-1} \circ (\tilde{I}_{1*})_! = J_!^{-1} \circ (I_{1*})_!$. Therefore ${j_0}_!^{-1} \circ (i_{0*})_! = {j_1}_!^{-1} \circ (i_{1*})_!$ as claimed.

For the normalization axiom, note that in case $M = X$, if $i\colon X \longrightarrow X \times V$ is an embedding which commutes with the projection maps $\phi\colon X \longrightarrow X$ and $\pi_1\colon X \times V \longrightarrow X$ then i is necessarily the trivial embedding $\mathrm{Id} \times g$ with $g\colon X \longrightarrow V$ constant. Then $\mathrm{index}_t = j_!^{-1} \circ (i_*)_! = \mathrm{Id}$ since $i_* = \mathrm{Id} \times 0$, which shows that the topological index is normalized. □

Theorem 4.2 *There is a unique index map.*

Proof We have already shown that index_t is an index map. Thus it suffices to consider a general index map as in (43) and to show that $\mathrm{index} = \mathrm{index}_t$.

Suppose that $i\colon M \longrightarrow X \times V$ is an embedding which intertwines the projection maps $\phi\colon M \longrightarrow X$ and $\pi_1\colon X \times V \longrightarrow X$. Then, together with the notation of the Multiplicativity Axiom, set $i^+ = e' \circ i_*\colon T(M/X) \longrightarrow X \times TS(V \oplus \mathbb{R})$. Let $0 \in TV$ be the origin and $j\colon \{0\} \longrightarrow TV$ be the inclusion, inducing the inclusion $j' = \mathrm{Id} \times j\colon X \times \{0\} \longrightarrow X \times TV$ and denote the composite inclusion $j^+ = e' \circ j'\colon X \times \{0\} \longrightarrow X \times TS(V \oplus \mathbb{R})$. Then consider the diagram:

$$\begin{tikzcd}
& K_c(T(M/X), \rho^*\phi^*\mathcal{A}) \arrow[dl, "(i_*)_!"'] \arrow[d, "i^+"] \arrow[dr, "\mathrm{index}"] & \\
K_c(X \times TV, \pi_1^*\mathcal{A}) \arrow[r, "e_!"] & K_c(X \times TS(V \oplus \mathbb{R}), \pi_1^*\mathcal{A}) \arrow[r, "\mathrm{index}^S"] & K(X, \mathcal{A}) \\
& K(X, \mathcal{A}) \arrow[ul, "j_!"] \arrow[u, "j^+"'] \arrow[ur, "\mathrm{index}"'] &
\end{tikzcd}$$

The left side of this diagram commutes by the excision property and the right side by the multiplicative property. By the normalization property, the composite map $\mathrm{index} \circ j^+$ is the identity mapping, so

$$\begin{aligned}
\mathrm{index} &= \mathrm{index}^S \circ i_!^+ = \mathrm{index}^S \circ e_! \circ i_! = \mathrm{index}^S \circ j_!^+ \circ j_!^{-1} \circ i_! \\
&= \mathrm{index}' \circ j_!^{-1} \circ i_! = j_!^{-1} \circ i_! = \mathrm{index}_t .
\end{aligned}$$
□

The following theorem completes the proof of the index theorem in twisted K–theory.

Theorem 4.3 *The analytic index* index_a *is an index map.*

Proof Again we consider the axioms for an index map in order.

The invariance of the algebra of pseudodifferential operators under diffeomorphism, and the naturality in this sense of the symbol map, show that under the hypotheses of the functoriality axiom, there is an isomorphism of short exact sequences (34):

$$\begin{array}{ccccc} \Psi^{-1}(M'/X,\mathcal{E}) & \longrightarrow & \Psi^{0}(M'/X,\mathcal{E}) & \longrightarrow & S^{[0]}(T(M'/X),\rho^*\operatorname{End}(\mathcal{E})) \\ \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle f^*} \\ \Psi^{-1}(M/X,\mathcal{E}) & \longrightarrow & \Psi^{0}(M/X,\mathcal{E}) & \longrightarrow & S^{[0]}(T(M/X),\rho^*\operatorname{End}(\mathcal{E})). \end{array} \tag{48}$$

Since the analytic index is by definition the boundary map in the associated 6–term exact sequence in K–theory, we see that $\mathrm{index}_a(f^*[p]) = f^! \,\mathrm{index}_a([p])$, for all $[p] \in K_c(T(M/X),\rho^*\pi^*\mathcal{A})$. This establishes the functoriality of index_a .

For the excision axiom, observe that any element in $K_c(T(\mathcal{U}/\pi(\mathcal{U})),\rho^*i^*\pi^*\mathcal{A})$ may be represented by a pair of projective vector bundle data over $\mathcal{U}$ and a symbol $q \in S^{[0]}(T(\mathcal{U}/\pi(\mathcal{U})$ with the property that q is equal to the identity homomorphism outside a compact set in $\mathcal{U}$. Complementing the second bundle with respect to vector bundle over M, using the discussion following Lemma 3, we may extend both sets of projective vector data to the whole of M, to be equal outside $\mathcal{U}$. This also extends q to an element $p \in S^{[0]}(T(M/X)$ by trivial extension. The exactness in (34) shows that there is a projective family of elliptic pseudodifferential operators P of order zero with symbol equal to p, by use of a partition of unity we may take it to be equal to the identity outside $\mathcal{U}$ in M. Similarly, q also defines an element $p' \in S^{[0]}(T(M'/X)$ and, from the corresponding exact sequence, there is a projective family of elliptic pseudodifferential operators P' equal to the identity outside $\mathcal{U}$ in M'; we may further arrange that $P = P'$ in $\mathcal{U}$. We can construct parametrices Q of P and Q' of P' such that Q is equal to the identity outside $\mathcal{U}$ in M and Q' is equal to the identity outside $\mathcal{U}$ in M' and $Q = Q'$ in $\mathcal{U}$. By the explicit formula for the analytic index in terms of the projective family of elliptic pseudodifferential operators and its parametrix, see Section 3, it follows that the diagram (45) commutes, that is, the analytic index satisfies the excision property.

Under the hypotheses of the multiplicative axiom we need to show for a class $[p] \in K_c(T(M/X),\rho^*\phi^*\mathcal{A})$, represented by a symbol $p \in S^0(T(M/X),\mathcal{E})$, that $\mathrm{index}_a([p]) = \mathrm{index}_a(h_![p])$, where $h\colon T(M/X) \longrightarrow X \times TS(V \oplus \mathbb{R})$ is the embedding that is obtained as the composition $h = e \circ Di$, and $h_!$ is the Gysin map. This is done by first embedding M as the zero section of the compactification of its normal bundle to a sphere bundle. In this case one may argue as in [2], where a family of operators B is constructed on the sphere $S^n = S(\mathbb{R}^n \times \mathbb{R})$ to be $O(n)$ invariant, surjective and have symbol equal to the Thom class. Then B can be extended naturally to act on the fibres of the sphere bundle. Having stabilized P, to a projective family with the given symbol p, (and a finite rank term f) we may lift it, as described in [2], to be an operator acting on the lift of $E^{\pm}$ to the sphere bundle and reducing to P on fibre-constant sections. As in [2] the tensor product of the lifted operator P and B then acts as a Fredholm family

$$\begin{pmatrix} P & B \\ B^* & P^* \end{pmatrix}. \tag{49}$$

between the bundles $E^+ \otimes G^+ \oplus E^- \otimes G^-$ and $E^- \otimes G^+ \oplus E^+ \otimes G^-$. Since P and B commute it follows as in the untwisted case that the null space of this surjective operator is isomorphic to the null space of P. Thus has represents the same index class which proves the desired multiplicativity in this case.

The general case now follows by using the excision property, so the analytic index satisfies the multiplicative property.

The normalization axiom holds by definition; it is important that this is consistent with the proof of the axioms above. □

The equality of the topological and analytic indexes is now an immediate consequence of Theorems 4.1, 4.2 and 4.3:

Theorem 4.4 (The index theorem in K–theory) *Let $\phi\colon M \longrightarrow X$ be a fibre bundle of compact manifolds, let $\mathcal{A}$ be an Azumaya bundle over X and P be a projective family of elliptic pseudodifferential operators acting between two sets of projective vector bundle data associated to a local trivialization of $\mathcal{A}$ and with symbol having class $p \in K_c(T(M/X); \rho^*\phi^*\mathcal{A})$, where $\rho\colon T(M/X) \longrightarrow M$ is the projection then*

$$\operatorname{index}_a(P) = \operatorname{index}_t(p) \in K(X, \mathcal{A}). \tag{50}$$

5 The Chern character of the index bundle

In this section, we compute the Chern character of the index bundle and obtain the cohomological form of the index theorem for projective families of elliptic pseudodifferential operators. In the process, not surprisingly, the torsion information from the Azumaya bundle is lost. We begin with the the basic properties of the Chern character.

The Chern character of projective vector bundles, defined in Section 1.4, gives a homomorphism

$$\mathrm{Ch}_{\mathcal{A}}\colon\ K^0(X,\mathcal{A}) \longrightarrow H^{even}(X,\mathbb{Q}). \tag{51}$$

It satisfies the following properties.

(1) The Chern character is *functorial* under smooth maps in the sense that if $f\colon\ Y \longrightarrow X$ is a smooth map between compact manifolds, then the following diagram commutes:

$$\begin{array}{ccc} K^0(X,\mathcal{A}) & \xrightarrow{f^!} & K^0(Y,f^*\mathcal{A}) \\ \big\downarrow \mathrm{Ch}_{\mathcal{A}} & & \big\downarrow \mathrm{Ch}_{f^*\mathcal{A}} \\ H^{even}(X,\mathbb{Q}) & \xrightarrow{f^*} & H^{even}(Y,\mathbb{Q}). \end{array} \tag{52}$$

(2) The Chern character respects the module structure, of $K^0(X,\mathcal{A})$ over $K^0(X)$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} K^0(X)\times K^0(X,\mathcal{A}) & \longrightarrow & K^0(X,\mathcal{A}) \\ \big\downarrow \mathrm{Ch}\times\mathrm{Ch}_{\mathcal{A}} & & \big\downarrow \mathrm{Ch}_{\mathcal{A}} \\ H^{even}(X,\mathbb{Q})\times H^{even}(X,\mathbb{Q}) & \longrightarrow & H^{even}(X,\mathbb{Q}) \end{array} \tag{53}$$

where the top horizontal arrow is the action of $K^0(X)$ on $K(X,\mathcal{A})$ given by tensor product and the bottom horizontal arrow is given by the cup product.

Theorem 5.1 (The cohomological formula of the index theorem) *For a fibration (1) of compact manifolds and a projective family of elliptic pseudodifferential operators P with symbol class $p\in K_c(T(M/X),\rho^*\phi^*\mathcal{A})$, where $\rho\colon\ T(M/X)\longrightarrow M$ is the projection, then*

$$\mathrm{Ch}_{\mathcal{A}}(\mathrm{index}_a P) = (-1)^n\phi_*\rho_*\left\{\rho^*\mathrm{Td}(T(M/X)\otimes\mathbb{C})\cup\mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\} \tag{54}$$

where the Chern character is denoted $\mathrm{Ch}_{\mathcal{A}}\colon\ K(X,\mathcal{A})\to H^\bullet(M)$ and $\mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}\colon$ $K_c(T(M/X),\rho^*\phi^*\mathcal{A})\to H_c^\bullet(T(M/X))$, n is the dimension of the fibres of $\phi\circ\rho$, $\phi_*\rho_*\colon\ H_c^\bullet(T(M/X))\longrightarrow H^{\bullet-n}(X)$ is integration along the fibre.

This theorem follows rather routinely from the index theorem in K–theory, Theorem 4.4. The key step to getting the formula (54) is the analog of the Riemann–Roch formula in the context of twisted K–theory, which we now discuss.

Let $\pi\colon E \longrightarrow X$ be a spin$^{\mathbb{C}}$ vector bundle over X, $i\colon X \longrightarrow E$ the zero section embedding, F be a complex projective vector bundle over X that is associated to the Azumaya bundle $\mathcal{A}$ on X. Then we compute,

$$\begin{aligned} \mathrm{Ch}_{\pi^*\mathcal{A}}(i_!F) &= \mathrm{Ch}_{\pi^*\mathcal{A}}(i_!1 \otimes \pi^*F) \\ &= \mathrm{Ch}(i_!1) \cup \mathrm{Ch}_{\pi^*\mathcal{A}}(\pi^*F), \end{aligned}$$

where we have used the fact that the Chern character respects the $K^0(X)$–module structure. The standard Riemann–Roch formula asserts that

$$\mathrm{Ch}(i_!1) = i_*\mathrm{Td}(E)^{-1}.$$

Therefore we obtain the following Riemann–Roch formula for twisted K–theory,

$$\mathrm{Ch}_{\pi^*\mathcal{A}}(i_!F) = i_*\left\{\mathrm{Td}(E)^{-1} \cup \mathrm{Ch}_{\mathcal{A}}(F)\right\}. \tag{55}$$

The index theorem in K–theory in Section 5 shows in particular that

$$\mathrm{Ch}_{\mathcal{A}}(\mathrm{index}_a\, P) = \mathrm{Ch}_{\mathcal{A}}(\mathrm{index}_t\, p).$$

Now $\mathrm{index}_t\, p = j_!^{-1} \circ (Di)_!$ where $i\colon M \hookrightarrow X \times V$ is an embedding that commutes with the projections $\phi\colon M \longrightarrow X$ and $\pi_1\colon X \times V \longrightarrow X$, and $j\colon X \hookrightarrow X \times V$ is the zero section embedding. Therefore

$$\mathrm{Ch}_{\mathcal{A}}(\mathrm{index}_t\, p) = \mathrm{Ch}_{\mathcal{A}}(j_!^{-1} \circ (Di)_!p)$$

By the Riemann–Roch formula for twisted K–theory (55),

$$\mathrm{Ch}_{\pi_1^*\mathcal{A}}(j_!F) = j_*\,\mathrm{Ch}_{\mathcal{A}}(F)$$

since $\pi_1\colon X \times V \longrightarrow X$ is a trivial bundle. Since $\pi_{1*}j_*1 = (-1)^n$, it follows that for $\xi \in K_c(X \times V, \pi_1^*\mathcal{A})$, one has

$$\mathrm{Ch}_{\mathcal{A}}(j_!^{-1}\xi) = (-1)^n \pi_{1*}\,\mathrm{Ch}_{\pi_1^*\mathcal{A}}(\xi)$$

Therefore

$$\mathrm{Ch}_{\mathcal{A}}(j_!^{-1} \circ (Di)_!p) = (-1)^n \pi_{1*}\,\mathrm{Ch}_{\pi_1^*\mathcal{A}}((Di)_!p) \tag{56}$$

By the Riemann–Roch formula for twisted K–theory (55),

$$\mathrm{Ch}_{\pi_1^*\mathcal{A}}((Di)_!p) = (Di)_*\left\{\rho^*\mathrm{Td}(N)^{-1} \cup \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\} \tag{57}$$

where N is the complexified normal bundle to the embedding $Di\colon T(M/X) \longrightarrow X \times TV$, that is $N = X \times TV/Di(T(M/X)) \otimes \mathbb{C}$. Therefore $\mathrm{Td}(N)^{-1} = \mathrm{Td}(T(M/X) \otimes \mathbb{C})$ and (57) becomes

$$\mathrm{Ch}_{\pi_1^*\mathcal{A}}((Di)_!p) = (Di)_*\left\{\rho^*\mathrm{Td}(T(M/X) \otimes \mathbb{C}) \cup \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\}.$$

Therefore (56) becomes

$$\begin{aligned} \mathrm{Ch}_{\mathcal{A}}(j_!^{-1} \circ (Di)_!p) &= (-1)^n \pi_{1*}(Di)_*\left\{\rho^*\mathrm{Td}(T(M/X) \otimes \mathbb{C}) \cup \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\} \\ &= (-1)^n \phi_*\rho_*\left\{\rho^*\mathrm{Td}(T(M/X) \otimes \mathbb{C}) \cup \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\} \end{aligned} \tag{58}$$

since $\phi_*\rho_* = \pi_{1*}(Di)_*$. Therefore

$$\mathrm{Ch}_{\mathcal{A}}(\mathrm{index}_t\, p) = (-1)^n \phi_*\rho_*\left\{\rho^*\mathrm{Td}(T(M/X)) \cup \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\}, \tag{59}$$

proving Theorem 5.1.

6 Determinant line bundle of the index bundle

In this section, we define the determinant line bundle of the index bundle of a projective family of elliptic pseudodifferential operators, and compute its Chern class.

We begin with the general construction of the determinant line bundle of a projective vector bundle over X. Let $\mathcal{A}$ be an Azumaya bundle over X and $P \xrightarrow{\pi} X$ be the principal $\mathrm{PU}(n)$ bundle associated to $\mathcal{A}$. Let $E \to P$ be a projective vector bundle over X, associated to $\mathcal{A}$, cf section 1.3. Recall that E satisfies in addition the condition

$$\phi^* E \cong \pi_P^* E \otimes \pi_{\mathrm{PU}(n)}^* L \tag{60}$$

where $\phi\colon \mathrm{PU}(n) \times P \to P$ is the action, $\pi_P\colon \mathrm{PU}(n) \times P \to P$ is the projection onto the second factor, $\pi_{\mathrm{PU}(n)}\colon \mathrm{PU}(n) \times P \to \mathrm{PU}(n)$ is the projection onto the first factor, $L \to \mathrm{PU}(n)$ is the (determinant) line bundle associated to the principal $\mathbb{Z}_n$ bundle $\mathbb{Z}_n \to \mathrm{SU}(n) \to \mathrm{PU}(n)$ as in section 1.3.

Then we observe that

$$\phi^* \Lambda^n(E) \cong \pi_P^* \Lambda^n(E)$$

and therefore $\Lambda^n(E) = \pi^*(F)$ for some line bundle $F \to X$. Define $\det(E) = F$ to be the determinant line bundle of the projective vector bundle E.

In particular, this gives a homomorphism

$$\det\colon K^0(X, \mathcal{A}) \longrightarrow \pi_0(\mathrm{Pic}(X))$$

where $\mathrm{Pic}(X)$ denotes the Picard variety of X, and the components of the Picard variety $\pi_0(\mathrm{Pic}(X))$ consist of the isomorphism classes of complex line bundles over X.

Theorem 6.1 (Chern class of the determinant line bundle of the index bundle) *For a fibration* (1) *of compact manifolds and a projective family of elliptic pseudodifferential operators P with symbol class $p \in K_c(T(M/X), \rho^*\phi^*\mathcal{A})$, where $\rho\colon T(M/X) \longrightarrow M$ is the projection,*

$$\mathrm{c}_1(\det(\mathrm{index}_a\, P)) = \{(-1)^n \phi_* \rho_* \{\rho^* \mathrm{Td}(T(M/X) \otimes \mathbb{C}) \cup \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\}\}^{[2]} \tag{61}$$

where $\mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}\colon K_c(T(M/X), \rho^*\phi^*\mathcal{A}) \longrightarrow H_c^\bullet(T(M/X))$ is the Chern character, c_1 is the first Chern class, N is the dimension of the fibres of $\phi \circ \rho$, $\phi_*\rho_*$ is integration along the fibres mapping $H_c^\bullet(T(M/X))$ to $H^{\bullet - n}(X)$ is and $\{\cdot\}^{[2]}$ denotes the degree 2 component.

Proof The proof of the theorem follows from Theorem 5.1 and the second part of Lemma 6. □

7 Projective families of Dirac operators

Let $\mathrm{Cl}(M/X)$ denote the bundle of Clifford algebras on the fibres of ϕ for some family of fibre metrics. A fiberwise Clifford module on a bundle E over the total space of a fibration is a smooth action of $\mathrm{Cl}(M/X)$ on the bundle. That is to say it is an algebra homomorphism

$$\mathrm{Cl}(M/X) \longrightarrow \mathrm{End}(E). \tag{62}$$

Since the endomorphism bundle of a projective bundle over E, with respect to an Azumaya bundle $\mathcal{A}$, is a vector bundle, this definition can be taken directly over to the projective case. Similarly, the condition that the Clifford module structure be hermitian can be taken over as the condition that (62) be $*$–preserving. The condition that a unitary connection on E be a Clifford connection is then the usual distribution condition, for vertical vector fields,

$$\nabla_X(\mathrm{cl}(\alpha)e_a) = \mathrm{cl}(\nabla_X\alpha)e_a + \mathrm{cl}(\alpha)(\nabla_X e_a) \tag{63}$$

in terms of the Levi–Civita connection on the fibre Clifford bundle and for any sections e_a of the bundle trivialization of E with respect to a full trivialization of $\mathcal{A}$.

The Dirac operator associated to such a unitary Clifford connection on a hermitian projective Clifford module is then given by the usual formula over the open sets U_a of a given full trivialization of $\mathcal{A}$ over the base:

$$\eth_a e_a = \widetilde{\mathrm{cl}}(\nabla e_a) \tag{64}$$

where $\widetilde{\mathrm{cl}}$ is the contraction given by the Clifford action from $T^*(M/X) \otimes E_a$ to E_a. The invariance properties of the connection and Clifford action show that the $\eth_a$ form a projective family of differential operators on the projective bundle E.

As in the untwisted case, if $\eth$ is a Dirac operator in this sense, acting on a vector bundle F over M and E is a projective vector bundle over M, relative to some Azumaya algebra $\mathcal{A}$, then we may twist $\eth$ by choosing a unitary connection ∇^E on E and extending the Clifford module trivially from F to $F \otimes E$ (to act as multiples of the identity on E) and taking the tensor product connection on $F \otimes E$. The resulting Dirac operator is then a projective family as described above.

In particular, if the bundle $T(M/X)$ is spin and we consider the family of Dirac operators along the fibres of ϕ twisted by the projective vector bundle E over M, we deduce from Theorem 4.4 that

$$\mathrm{index}_a(\eth) = \phi_!(E) \in K(X, \mathcal{A})$$

where $\phi_!\colon K(M,\phi^*\mathcal{A}) \to K(X,\mathcal{A})$ is defined as $\phi_! = j_!^{-1} \circ f_!$ in the notation of Section 3. By Theorem 5.1 and standard manipulations of characteristic classes one has

$$\mathrm{Ch}_{\mathcal{A}}(\mathrm{index}_a(\eth)) = \phi_*(\widehat{A}(M/X)\,\mathrm{Ch}_{\phi^*\mathcal{A}}(E)) \in H^\bullet(X).$$

where $\phi_*\colon H^\bullet(M) \to H^{\bullet-\ell}(X)$ denotes integration along the ℓ–dimensional fibres. A similar formula holds more generally, in case $T(M/X)$ is a spin$^{\mathbb{C}}$ bundle, with an extra factor of $\exp(c_1)$ arising from the twisting curvature, see [3].

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