Manifolds with non-stable fundamental groups at infinity, III

C R GUILBAULT
F C TINSLEY

We continue our study of ends non-compact manifolds. The over-arching aim is to provide an appropriate generalization of Siebenmann's famous collaring theorem that applies to manifolds having non-stable fundamental group systems at infinity. In this paper a primary goal is finally achieved; namely, a complete characterization of pseudo-collarability for manifolds of dimension at least 6.

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1 Introduction

This is the third in a series of papers aimed at generalizing Siebenmann's famous PhD thesis [13] so that the results apply to manifolds with nonstable fundamental groups at infinity. Siebenmann's work provides necessary and sufficient conditions for an open manifold of dimension ≥ 6 to contain an open collar neighborhood of infinity, ie, a manifold neighborhood of infinity N such that $N \approx \partial N \times [0,1)$. Clearly, a stable fundamental group at infinity is necessary in order for such a neighborhood to exist. Hence, our first task was to identify a useful, but less rigid, 'end structure' to aim for. We define a manifold N^n with compact boundary to be a *homotopy collar* provided $\partial N^n \hookrightarrow N^n$ is a homotopy equivalence. Then define a *pseudo-collar* to be a homotopy collar which contains arbitrarily small homotopy collar neighborhoods of infinity. An open manifold (or more generally, a manifold with compact boundary) is *pseudo-collarable* if it contains a pseudo-collar neighborhood of infinity. Obviously, an open collar is a special case of a pseudo-collar. Guilbault [7] contains a detailed discussion of pseudo-collars, including motivation for the definition and a variety of examples—both pseudo-collarable and non-pseudo-collarable. In addition, a set of three conditions (see below) necessary for pseudo-collarability—each analogous to a condition from Siebenmann's original theorem—was identified there. A primary goal became establishment of the sufficiency of these conditions. At the time [7] was written, we were only partly successful at attaining that goal. We obtained an existence theorem for pseudo-collars, but only by making an additional assumption regarding the second homotopy group at infinity. In this paper we eliminate that hypothesis; thereby obtaining the following complete characterization.

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Theorem 1.1 (Pseudo-collarability Characterization Theorem) *A one ended n–manifold M^n $(n \geq 6)$ with compact boundary is pseudo-collarable iff each of the following conditions holds:*

(1) *M^n is inward tame at infinity,*

(2) *$\pi_1(\varepsilon(M^n))$ is perfectly semistable, and*

(3) *$\sigma_\infty(M^n) = 0 \in \widetilde{K}_0(\pi_1(\varepsilon(M^n)))$.*

Remark 1 While it is convenient to focus on one ended manifolds, the above theorem actually applies to all manifolds with compact boundary. This is true because an inward tame manifold with compact boundary has only finitely many ends. (See Section 3 of Guilbault–Tinsley [8].) Hence, Theorem 1.1 may be applied to each end individually. Manifolds with non-compact boundaries are an entirely different story and will not be discussed here. A detailed discussion of that situation will be provided in Guilbault [5].

Remark 2 A side benefit of our new proof is the inclusion of the $n = 6$ case. In fact, our proof is also valid in dimension five when all of the groups involved are 'good' in the sense of Freedman and Quinn [4]. In that dimension the pseudo-collar structure obtained is purely topological—as opposed to PL or smooth. This parallels the status of Siebenmann's theorem in dimension 5. We discuss dimensions ≤ 4 at the end of this section.

The condition of *inward tameness* means that each neighborhood of infinity can be pulled into a compact subset of itself, or equivalently, that M^n contains arbitrarily small neighborhoods of infinity which are finitely dominated. Next let $\pi_1(\varepsilon(M^n))$ denote the inverse system of fundamental groups of neighborhoods of infinity. Such a system is *semistable* if it is equivalent to a system in which all bonding maps are surjections. If, in addition, it can be arranged that the kernels of these bonding maps are perfect groups, then the system is *perfectly semistable*. The obstruction $\sigma_\infty(M^n) \in \widetilde{K}_0(\pi_1(\varepsilon(M^n)))$ vanishes precisely when each (clean) neighborhood of infinity has finite homotopy type. More detailed formulations of these definitions will be given in Section 2.

Conditions (1)–(3) correspond directly to the three conditions identified by Siebenmann as necessary and sufficient for the existence of an actual open collar neighborhood of infinity for manifolds of dimension ≥ 6. (His original version combined the first two into a single assumption.) Indeed, condition (1) is precisely one of his conditions, condition (2) is a relaxation of his π_1–stability condition, and condition (3) is the natural reformulation of his third condition to the situation where $\pi_1\,(\varepsilon(M^n))$ is not necessarily stable.

In the second paper of this series [8], we focused our attention on the interdependence of conditions (1)–(3). Specifically, it seemed that condition (2) might be implied by condition (1), or by a combination of (1) and (3). This turned out to be partly true. We showed that, for manifolds with compact boundary, inward tameness implies π_1–semistability. However, that paper also presents examples satisfying both (1) and (3) which do not have *perfectly* semistable fundamental group at infinity—and thus are not pseudo-collarable. Those results solidified conditions (1)–(3) as the best hope for a complete characterization of manifolds with pseudo-collarable ends.

The proof of the Pseudo-collar Characterization Theorem is based upon the proof of the 'Main Existence Theorem' of [7], which was based on Siebenmann's original work. The primary task of this paper is to redo the final step of our earlier proof without assuming π_2–semistability. In an interesting twist, our new strategy results in a proof that more closely resembles Siebenmann's original argument than its predecessor. Even so, the reader would be well served to have a copy of [7] available.

Remark 3 When $\pi_1(\varepsilon(M^n))$ is stable, conditions (1)–(3) become identical to Siebenmann's conditions. Thus, an application of [13] tells us that every pseudo-collar with stable fundamental group at infinity contains a genuine collar. This fact can also be obtained by a relatively simple direct argument. Thus, one may view Siebenmann's theorem as a special case of the Pseudo-collarability Characterization Theorem. For completeness, we have included that direct argument as Proposition 2.3 in the following section.

Remark 4 For irreducible 3–dimensional manifolds with compact boundary the assumption of inward tameness, by itself, implies the existence of an open collar neighborhood of infinity, Tucker [15]. By contrast, in dimension 4, Kwasik and Schultz [9] have given examples where Siebenmann's Collaring Theorem fails. Since these examples have 'good' fundamental groups at infinity, Proposition 2.3 (or a quick review of [9]) shows that Theorem 1.1 also fails in dimension 4.

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2 Definitions and terminology

In this section we briefly review most of the terminology and notation needed in the remainder of the paper. It is divided into two subsections—the first devoted to inverse sequences of groups and the second to the topology of ends of manifolds.

2.1 Algebra of inverse sequences

Throughout this section all arrows denote homomorphisms, while arrows of the type $\twoheadrightarrow$ or $\twoheadleftarrow$ denote surjections. The symbol $\cong$ denotes isomorphisms.

Let

$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \cdots$$

be an inverse sequence of groups and homomorphisms. A *subsequence* of $\{G_i, \lambda_i\}$ is an inverse sequence of the form

$$G_{i_0} \xleftarrow{\lambda_{i_0+1}\circ\cdots\circ\lambda_{i_1}} G_{i_1} \xleftarrow{\lambda_{i_1+1}\circ\cdots\circ\lambda_{i_2}} G_{i_2} \xleftarrow{\lambda_{i_2+1}\circ\cdots\circ\lambda_{i_3}} \cdots .$$

In the future we will denote a composition $\lambda_i \circ \cdots \circ \lambda_j$ $(i \le j)$ by $\lambda_{i,j}$.

Sequences $\{G_i, \lambda_i\}$ and $\{H_i, \mu_i\}$ are *pro-equivalent* if, after passing to subsequences, there exists a commuting diagram:

$$\begin{tikzcd}
G_{i_0} & & G_{i_1} \arrow[ll, "\lambda_{i_0+1,i_1}"'] \arrow[dl] & & G_{i_2} \arrow[ll, "\lambda_{i_1+1,i_2}"'] \arrow[dl] & & \cdots \arrow[ll, "\lambda_{i_2+1,i_3}"'] \arrow[dl] \\
& H_{j_0} \arrow[ul] & & H_{j_1} \arrow[ll, "\mu_{j_0+1,j_1}"'] \arrow[ul] & & H_{j_2} \arrow[ll, "\mu_{j_1+1,j_2}"'] \arrow[ul] & \cdots
\end{tikzcd}.$$

Clearly an inverse sequence is pro-equivalent to any of its subsequences. To avoid tedious notation, we often do not distinguish $\{G_i, \lambda_i\}$ from its subsequences. Instead we simply assume that $\{G_i, \lambda_i\}$ has the desired properties of a preferred subsequence—often prefaced by the words 'after passing to a subsequence and relabelling'.

The *inverse limit* of a sequence $\{G_i, \lambda_i\}$ is a subgroup of $\prod G_i$ defined by

$$\varprojlim \{G_i, \lambda_i\} = \left\{ (g_0, g_1, g_2, \cdots) \in \prod_{i=0}^{\infty} G_i \,\middle|\, \lambda_i(g_i) = g_{i-1} \right\}.$$

Notice that for each i, there is a *projection homomorphism* $p_i\colon \varprojlim \{G_i, \lambda_i\} \to G_i$. It is a standard fact that pro-equivalent inverse sequences have isomorphic inverse limits.

An inverse sequence $\{G_i, \lambda_i\}$ is *stable* if it is pro-equivalent to an inverse sequence $\{H_i, \mu_i\}$ for which each μ_i is an isomorphism. A more usable formulation is that $\{G_i, \lambda_i\}$ is stable if, after passing to a subsequence and relabelling, there is a commutative diagram of the form

$$(*) \qquad \begin{tikzcd}
G_0 & & G_1 \arrow[ll, "\lambda_1"'] \arrow[dl] & & G_2 \arrow[ll, "\lambda_2"'] \arrow[dl] & & G_3 \arrow[ll, "\lambda_3"'] \arrow[dl] & \cdots \arrow[l, "\lambda_4"'] \\
& im(\lambda_1) \arrow[ul] & & im(\lambda_2) \arrow[ll] \arrow[ul] & & im(\lambda_3) \arrow[ll] \arrow[ul] & \cdots \arrow[l]
\end{tikzcd}$$

where each bonding map in the bottom row (obtained by restricting the corresponding λ_i) is an isomorphism. If $\{H_i, \mu_i\}$ can be chosen so that each μ_i is an epimorphism, we say that our inverse sequence is *semistable* (or *Mittag–Leffler,* or *pro-epimorphic*). In this case, it can be arranged that the restriction maps in the bottom row of $(*)$ are epimorphisms. Similarly, if $\{H_i, \mu_i\}$ can be chosen so that each μ_i is a monomorphism, we say that our inverse sequence is *pro-monomorphic*; it can then be arranged that the restriction maps in the bottom row of $(*)$ are monomorphisms. It is easy to see that an inverse sequence that is semistable and pro-monomorphic is stable.

Recall that a *commutator* element of a group H is an element of the form $x^{-1}y^{-1}xy$ where $x, y \in H$; and the *commutator subgroup* of H, denoted $[H, H]$, is the subgroup generated by all of its commutators. The group H is *perfect* if $[H, H] = H$. An inverse sequence of groups is *perfectly semistable* if it is pro-equivalent to an inverse sequence

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots$$

of finitely presentable groups and surjections where each $\ker(\lambda_i)$ perfect. The following shows that inverse sequences of this type behave well under passage to subsequences.

Lemma 2.1 *A composition of surjective group homomorphisms, each having perfect kernels, has perfect kernel. Thus, if an inverse sequence of surjective group homomorphisms has the property that the kernel of each bonding map is perfect, then each of its subsequences also has that property.*

Proof See [7, Lemma 1]. □

2.2 Topology of ends of manifolds

Throughout this paper, $\approx$ will represent homeomorphism, while $\simeq$ will indicate homotopic maps or homotopy equivalent spaces. The word *manifold* means *manifold with (possibly empty) boundary*. A manifold is *open* if it is non-compact and has no boundary. We will restrict our attention to manifolds with compact boundaries. This prevents the 'topology at infinity' of our manifold from getting entangled with the topology at infinity of its boundary. Manifolds with noncompact boundaries will be addressed in [5].

For convenience, all manifolds are assumed to be PL. Analogous results may be obtained for smooth or topological manifolds in the usual ways. Occasionally we will observe that a theorem remains valid in dimension 4 or 5. Results of this sort usually require the purely topological 4–dimensional techniques developed by Freedman [4]; thus, the corresponding conclusions are only topological. The main focus of this paper, however, is on dimensions ≥ 6.

Let M^n be a manifold with compact (possibly empty) boundary. A set $N \subset M^n$ is a *neighborhood of infinity* if $\overline{M^n - N}$ is compact. A neighborhood of infinity N is *clean* if

- N is a closed subset of M^n,
- $N \cap \partial M^n = \varnothing$, and
- N is a codimension 0 submanifold of M^n with bicollared boundary.

It is easy to see that each neighborhood of infinity contains a clean neighborhood of infinity.

We say that M^n *has* k *ends* if it contains a compactum C such that, for every compactum D with $C \subset D$, $M^n - D$ has exactly k unbounded components, ie, k components with noncompact closures. When k exists, it is uniquely determined; if k does not exist, we say M^n has *infinitely many ends*. If M^n is k–ended, then it contains a clean neighborhood of infinity N consisting of k connected components, each of which is a one ended manifold with compact boundary. Thus, when studying manifolds with finitely many ends, it suffices to understand the *one ended* situation. That is the case in this paper, where our standard hypotheses ensure finitely many ends. See [8, Proposition 3.1].

A connected clean neighborhood of infinity with connected boundary is called a *0–neighborhood of infinity*. If N is clean and connected but has more than one boundary component, we may choose a finite collection of disjoint properly embedded arcs in N that connect those components. Deleting from N the interiors of regular neighborhoods of these arcs produces a 0–neighborhood of infinity $N_0 \subset N$.

A nested sequence $N_0 \supset N_1 \supset N_2 \supset \cdots$ of neighborhoods of infinity is *cofinal* if $\bigcap_{i=0}^{\infty} N_i = \varnothing$. For any one ended manifold M^n, one may easily obtain a cofinal sequence of 0–neighborhoods of infinity.

We say that M^n is *inward tame* at infinity if, for arbitrarily small neighborhoods of infinity N, there exist homotopies $H\colon N \times [0,1] \to N$ such that $H_0 = id_N$ and $\overline{H_1(N)}$ is compact. Thus inward tameness means each neighborhood of infinity can be pulled into a compact subset of itself.

Recall that a space X is *finitely dominated* if there exists a finite complex K and maps $u\colon X \to K$ and $d\colon K \to X$ such that $d \circ u \simeq id_X$. The following lemma uses this notion to offer equivalent formulations of 'inward tameness'.

Lemma 2.2 [8, Lemma 2.4] *For a manifold M^n, the following are equivalent.*

(1) *M^n is inward tame at infinity.*

(2) *Each clean neighborhood of infinity in M^n is finitely dominated.*

(3) *For each cofinal sequence $\{N_i\}$ of clean neighborhoods of infinity, the inverse sequence*

$$N_0 \overset{j_1}{\hookleftarrow} N_1 \overset{j_2}{\hookleftarrow} N_2 \overset{j_3}{\hookleftarrow} \cdots$$

is pro-homotopy equivalent to an inverse sequence of finite polyhedra.

Given a nested cofinal sequence $\{N_i\}_{i=0}^{\infty}$ of connected neighborhoods of infinity, base points $p_i \in N_i$, and paths $\alpha_i \subset N_i$ connecting p_i to p_{i+1}, we obtain an inverse sequence:

$$\pi_1(N_0, p_0) \overset{\lambda_1}{\longleftarrow} \pi_1(N_1, p_1) \overset{\lambda_2}{\longleftarrow} \pi_1(N_2, p_2) \overset{\lambda_3}{\longleftarrow} \cdots .$$

Here, each $\lambda_{i+1}\colon \pi_1(N_{i+1}, p_{i+1}) \to \pi_1(N_i, p_i)$ is the homomorphism induced by inclusion followed by the change of base point isomorphism determined by α_i. The obvious singular ray obtained by piecing together the α_i's is often referred to as the *base ray* for the inverse sequence. Provided the sequence is semistable, one can show that its pro-equivalence class does not depend on any of the choices made above. We refer to the pro-equivalence class of this sequence as the *fundamental group system at infinity* for M^n and denote it by $\pi_1(\varepsilon(M^n))$. We denote the inverse limit of this sequence by $\check{\pi}_1(\varepsilon(M^n))$. (In the absence of semistability, the pro-equivalence class of the inverse sequence depends on the choice of base ray, and hence, this choice becomes part of the data.) It is easy to see how the same procedure may also be used to define $\pi_k(\varepsilon(M^n))$ and $\check{\pi}_k(\varepsilon(M^n))$ for $k > 1$.

In [16], Wall shows that each finitely dominated connected space X determines a well-defined element $\sigma(X)$ lying in $\widetilde{K}_0(\mathbb{Z}[\pi_1 X])$ (the group of stable equivalence classes of finitely generated projective $\mathbb{Z}[\pi_1 X]$–modules under the operation induced by direct sum) that vanishes if and only if X has the homotopy type of a finite complex. Given a nested cofinal sequence $\{N_i\}_{i=0}^{\infty}$ of connected clean neighborhoods of infinity in an inward tame manifold M^n, we have a Wall obstruction $\sigma(N_i)$ for each i. These may be combined into a single obstruction

$$\begin{aligned}\sigma_\infty(M^n) &= (-1)^n\,(\sigma(N_0), \sigma(N_1), \sigma(N_2), \cdots)\\ &\in \widetilde{K}_0\left(\pi_1\left(\varepsilon\left(M^n\right)\right)\right) \equiv \varprojlim \widetilde{K}_0(\mathbb{Z}[\pi_1 N_i])\end{aligned}$$

that is well-defined and which vanishes if and only if each clean neighborhood of infinity in M^n has finite homotopy type. See Chapman and Siebenmann [1] for details.

We conclude this section by providing a direct proof of the following:

Proposition 2.3 *Every pseudo-collar of dimension ≥ 5 with stable fundamental group contains an open collar neighborhood of infinity. In dimension 4 this remains true in the topological category provided the fundamental group at infinity is good.*

Proof If N^n is a connected pseudo-collar, then by definition there exists a cofinal sequence $N^n = N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ of homotopy collar neighborhoods of infinity. Letting $W_i = N_i - \overset{\circ}{N}_{i+1}$ for each i, divides N^n into a countable sequence $\{(W_i, \partial N_i, \partial N_{i+1})\}$ of cobordisms with the property that each $\partial N_i \hookrightarrow W_i$ is a homotopy equivalence. (We call these *one-sided* h–cobordisms.) Although $\partial N_{i+1} \hookrightarrow W_i$ needn't be a homotopy equivalence, an argument involving duality in the universal cover of W_i (see [8, Theorem 2.5]), implies that $\pi_1(\partial N_{i+1}) \to \pi_1(W_i)$ is surjective for each i. By commutativity of

$$\begin{array}{ccc} \pi_1(\partial N_{i+1}) & \twoheadrightarrow & \pi_1(W_i) \\ \cong\downarrow & & \cong\downarrow \\ \pi_1(N_{i+1}) & \longrightarrow & \pi_1(N_i) \end{array}$$

each bonding map in the following sequence is surjective

$$\pi_1(N_1) \twoheadleftarrow \pi_1(N_2) \twoheadleftarrow \pi_1(N_3) \twoheadleftarrow \cdots.$$

The only way that an inverse sequence of surjections can be stable is that eventually all bonding homomorphisms are isomorphisms. Choose i_0 sufficiently large that $\pi_1(N_{i+1}) \to \pi_1(N_i)$ is an isomorphism for all $i \geq i_0$. Then for each $i \geq i_0$, $(W_i, \partial N_i, \partial N_{i+1})$ is a genuine h–cobordism. If each is a product, we may piece the product structures together to obtain an open collar structure on N_{i_0}. Otherwise we will apply the 'weak h–cobordism theorem' (Stallings [14] or Connell [3]) to rechoose the cobordisms so that each is a product. To accomplish this, assume that $n \geq 5$. Then, by the weak h–cobordism theorem, there is a homeomorphism $h\colon \partial N_{i_0} \times [0,1) \to W_{i_0} - \partial N_{i_0+1}$. Choose t close to 1; then replace N_{i_0+1} with $N'_{i_0+1} = N_{i_0} - h\left(\partial N_{i_0} \times [0,t)\right)$ and W_{i_0} with $W'_{i_0} = N_{i_0} - \overset{\circ}{N}{}'_{i_0+1} = h(\partial N_i \times [0,t])$. Next apply the same procedure to N'_{i_0+1}, N_{i_0+2} and the h–cobordism between $\partial N'_{i_0+1}$ and ∂N_{i_0+2} to rechoose that cobordism so that it is a product. Continue this procedure inductively to arrange that all of the h–cobordisms are products. If $n = 4$, the weak h–cobordism theorem (and hence, the above proof) is still valid provided the common fundamental group is good and the conclusion we seek is only topological [6, Corollary 3.5]. $\square$

3 One sided h–cobordisms and the Plus Construction

As noted above, a pseudo-collar structure on the end of a manifold allows one to express that end as a countable union of compact 'one-sided h–cobordisms' $\{(W_i, A_i, B_i)\}_{i=1}^{\infty}$ in the sense that each inclusion $A_i \hookrightarrow W_i$ is a homotopy equivalence. For any such cobordism, $\pi_1(B_i) \to \pi_1(W_i)$ is surjective and has perfect kernel (again see [8, Theorem 2.5]). Quillen's famous 'plus construction' ([11] or [4, Section 11.1]) provides a partial converse to that observation.

Theorem 3.1 (The Plus Construction) *Let B be a closed $(n-1)$–manifold $(n \geq 6)$ and $h\colon \pi_1(B) \to H$ a surjective homomorphism onto a finitely presented group such that $\ker(h)$ is perfect. Then there exists a compact n–dimensional cobordism (W, A, B) such that $\ker(\pi_1(B) \to \pi_1(W)) = \ker h$, and $A \hookrightarrow W$ is a simple homotopy equivalence. These properties determine W uniquely up to homeomorphism rel B. If $n = 5$, this result is still valid (in the topological category) provided the group H is good.*

When a one-sided h–cobordism has trivial Whitehead torsion, ie, when the corresponding homotopy equivalence is simple, we refer to it as a *plus cobordism*.

In [7] techniques borrowed from the proof of Theorem 3.1 were used in obtaining pseudo-collar structures. In the current paper we apply the plus construction itself to isolate the difficulties caused by non-stability of the fundamental group. The key technical tool is the following:

Theorem 3.2 (The Embedded Plus Construction) *Let R be a connected manifold of dimension ≥ 6; B be a closed component of ∂R; and*

$$G \subseteq \ker(\pi_1(B) \to \pi_1(R))$$

a perfect group which is the normal closure in $\pi_1(B)$ of a finite set of elements. Then there exists a plus cobordism (W, A, B) embedded in R which is the identity on B for which $\ker(\pi_1(B) \to \pi_1(W)) = G$. If $n = 5$ and $\pi_1(B)/G$ is good, the conclusion is still valid provided we work in the topological category.

Proof Let (W, A, B) be the plus cobordism promised by Theorem 3.1 for the homomorphism $\pi_1(B) \to \pi_1(W)$. We will show (indirectly) how to embed this cobordism into R in the appropriate manner. Let $R_1 = R \cup_B W$, the manifold obtained by gluing a copy of W to R along a common copy of B.

Claim 1 $R \hookrightarrow R_1$ is a simple homotopy equivalence.

Since $A \hookrightarrow W$ is a simple homotopy equivalence, duality implies that $B \hookrightarrow W$ is a $\mathbb{Z}[\pi_1 W]$–homology equivalence; in other words, $H_*\left(\widetilde{W}, \widehat{B}\right) = 0$, where $\widehat{B}$ is the preimage of B under the universal covering projection $\widetilde{W} \to W$. By Van Kampen's theorem, $R \hookrightarrow R_1$ induces a π_1–isomorphism, so $\widetilde{R}$ may be viewed as a subset of $\widetilde{R}_1$. So excision implies that $H_*\left(\widetilde{R}_1, \widetilde{R}\right) \cong H_*\left(\widetilde{W}, \widehat{B}\right) = 0$. From there we see that $\pi_j\left(R_1, R\right)$ is trivial for all j, so $R \hookrightarrow R_1$ is a homotopy equivalence.

The Duality Theorem for Whitehead torsion (Milnor [10, page 394]) may be used to check that this homotopy equivalence is simple. In particular, even though $B \hookrightarrow W$ is not a homotopy equivalence, it is a $\mathbb{Z}[\pi_1 W]$–homology equivalence, and thus determines an element of $Wh(\pi_1 W)$. But $A \hookrightarrow W$ is a simple homotopy equivalence, so by duality, both of these inclusions have trivial torsion. By an application of the Sum Theorem for Whitehead torsion (Cohen [2, Section 23]), $R \hookrightarrow R_1$ also determines the trivial element of $Wh(\pi_1 R)$, so we have the desired simple homotopy equivalence. Let

$$\begin{aligned} X^{n+1} &= \left(R \times \left[0, \frac{1}{2}\right]\right) \cup_{R \times \{\frac{1}{2}\}} \left(R_0 \times \left[\frac{1}{2}, 1\right]\right), \\ R_0 &= R \times \{0\}, \text{ and} \\ Q &= (R_0 \times \{1\}) \cup \left(A \times \left[\frac{1}{2}, 1\right]\right) \cup \left(W \times \left\{\frac{1}{2}\right\}\right) \end{aligned}$$

See Figure 1.

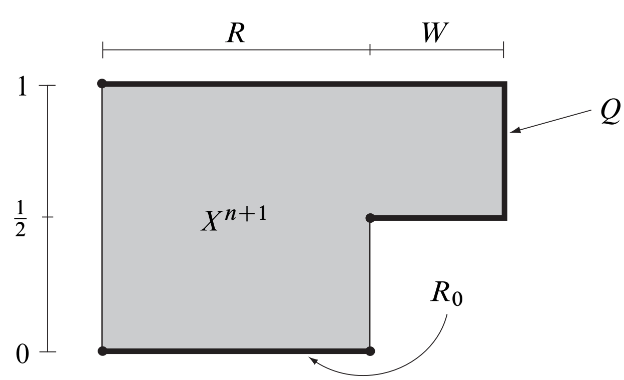

Figure 1: The cobordism $\left(X^{n+1}, R_0, Q\right)$

Claim 2 $\left(X^{n+1}, R_0, Q\right)$ is an s–cobordism.

First note that $\left(X^{n+1}, R_0\right) \simeq (R_1, R)$, so by Claim 1, $R_0 \hookrightarrow X^{n+1}$ is a homotopy equivalence. Next observe that $R_0 \times \{1\} \hookrightarrow X^{n+1}$ and $R_0 \times \{1\} \hookrightarrow Q$ are both homotopy equivalences. The first of these is obvious, while the second follows from the fact that $A \times \left\{\frac{1}{2}\right\} \hookrightarrow W \times \left\{\frac{1}{2}\right\}$ is a homotopy equivalence. It follows that $Q \hookrightarrow X^{n+1}$ is a homotopy equivalence.

To show that $R_0 \hookrightarrow X^{n+1}$ has trivial torsion, factor the inclusion map as $R_0 \hookrightarrow R \times [0,1] \hookrightarrow X^{n+1}$. The first of these inclusions is obviously a simple homotopy equivalence, and the second is a simple homotopy equivalence by an easy application of Claim 1; thus Claim 2 follows.

By the relative s–cobordism theorem (Rourke and Sanderson [12, Chapter 6]), X^{n+1} is a product, so there is a homeomorphism (rel boundary) from Q onto R_0. The image of $W \times \left\{\frac{1}{2}\right\}$ under this homeomorphism provides the desired plus cobordism (W, A, B) embedded in R_0. □

4 Proof of Theorem 1.1

We now move to the proof of our main theorem. For a full understanding, the reader should be familiar with the proof of the Main Existence Theorem of [7] up to the last few pages—which our current argument will replace. Those familiar with [13] will understand the key points. We begin with a brief review.

Start by assuming only that M^n is a one ended manifold with compact boundary, and that n is at least 5. (In [7] we took the traditional route and assumed M^n was an open manifold; but this is unnecessary as long as ∂M^n is compact.) Recall that a 0–neighborhood of infinity is a *generalized* 1*–neighborhood of infinity* provided $\pi_1(\partial U) \to \pi_1(U)$ is an isomorphism. If, in addition, $\pi_i(U, \partial U) = 0$ for all $i \leq k$, then U is a *generalized* k*–neighborhood of infinity*.

By the Generalized $(n-3)$–neighborhoods Theorem ([7, Theorem 5]), inward tameness alone allows us to obtain a cofinal sequence $\{U_i\}$ of generalized $(n-3)$–neighborhoods of infinity in M^n. Since [8, Theorem 1.2] assures that $\pi_1(\varepsilon(M^n))$ is semistable, we may also arrange that $\pi_1(U_i) \leftarrow \pi_1(U_{i+1})$ is surjective for all $i \geq 1$. For each i let $R_i = U_i - \mathring{U}_{i+1}$ and consider the collection of cobordisms $\{(R_i, \partial U_i, \partial U_{i+1})\}$. Then

(i) Each inclusion $\partial U_i \hookrightarrow R_i \hookrightarrow U_i$ induces a π_1–isomorphism,

(ii) $\partial U_{i+1} \hookrightarrow R_i$ induces a π_1–epimorphism for each i,

(iii) $\pi_k(R_i, \partial U_i) = 0$ for all $k < n-3$ and all i, and

(iv) Each $(R_i, \partial U_i, \partial U_{i+1})$ admits a handle decomposition based on ∂U_i containing handles only of index $(n-3)$ and $(n-2)$.

The first two observations follow easily from Van Kampen's Theorem. The third is obtained inductively. First note that by the Hurewicz Theorem $\pi_k(R_i, \partial U_i) \cong \pi_k(\widetilde{R}_i, \partial \widetilde{U}_i) \cong H_k(\widetilde{R}_i, \partial \widetilde{U}_i)$, provided that $\pi_j(R_i, \partial U_i)$ is trivial for $j < k$. Then examine the homology long exact sequence for the triple $\left(\widetilde{U}_i, \widetilde{R}_i, \partial \widetilde{U}_i\right)$ to obtain the desired result. See [7, page 561] for details. The fourth observation is obtained from standard handle theoretic techniques (see [12]). In particular, (iii) allows us to eliminate all handles of index $\leq n-4$; then observation (ii) allows us to eliminate 0– and 1–handles from the corresponding dual handle decomposition of R_i based on ∂U_{i+1}.

By observation (iv), each U_i admits an infinite handle decomposition having handles only of index $(n-3)$ and $(n-2)$. Thus, $(U_i, \partial U_i)$ has the homotopy type of a relative CW pair $(K_i, \partial U_i)$ with $\dim (K_i - \partial U_i) \leq n-2$. Therefore, if one of the U_i is a generalized $(n-2)$–neighborhood of infinity, then it is a homotopy collar. Thus, our goal is to improve arbitrarily small U_i to generalized $(n-2)$–neighborhoods of infinity. This must be done in the above context—in particular, condition (ii) must be preserved. We accomplish this by altering the U_i without changing their fundamental groups.

The next key observation is that, for each i, $\pi_{n-2}(U_i, \partial U_i) \cong H_{n-2}(\widetilde{U}_i, \partial \widetilde{U}_i)$ is a finitely generated projective $\mathbb{Z}[\pi_1 U_i]$–module. Moreover, as an element of $\widetilde{K}_0(\mathbb{Z}[\pi_1 U_i])$, $\left[H_{n-2}(\widetilde{U}_i, \partial \widetilde{U}_i)\right] = (-1)^n \sigma(U_i)$, where $\sigma(U_i)$ is the Wall finiteness obstruction for U_i. This is the content of [7, Lemma 13]. As discussed in Section 2, these elements of $\widetilde{K}_0(\mathbb{Z}[\pi_1 U_i])$ determine the obstruction $\sigma_\infty(M^n)$ found in Theorem 1.1. By assuming that $\sigma_\infty(M^n)$ vanishes, we are given that each $H_{n-2}(\widetilde{U}_i, \partial \widetilde{U}_i)$ is a stably free $\mathbb{Z}[\pi_1 U_i]$–module. By carving out finitely many trivial $(n-3)$–handles from each U_i we can arrange that these homology groups are finitely generated free $\mathbb{Z}[\pi_1 U_i]$–modules. This can be done so that each remains a generalized $(n-3)$–neighborhood of infinity, and so that none of the fundamental groups of the neighborhoods of infinity or their boundaries are changed. To save on notation, we continue to denote this improved collection by $\{U_i\}$. See [7, Lemma 14] for details.

By the finite generation of $H_{n-2}(\widetilde{U}_i, \partial \widetilde{U}_i)$, we may assume (after passing to a subsequence of $\{U_i\}$ and relabeling) that $H_{n-2}(\widetilde{R}_i, \partial \widetilde{U}_i) \to H_{n-2}(\widetilde{U}_i, \partial \widetilde{U}_i)$ is surjective for each i. From there the long exact sequence for the triple $\left(\widetilde{U}_i, \widetilde{R}_i, \partial \widetilde{U}_i\right)$ shows that these surjections are, in fact, isomorphisms. As above, we may choose a handle decompositions for the R_i based on ∂U_i having handles only of index $n-3$ and $n-2$.

From now on, let i be fixed. After introducing some trivial $(n-3, n-2)$–handle pairs, an algebraic lemma and some handle slides allows us to obtain a handle decomposition of R_i based on ∂U_i with $(n-2)$–handles $h_1^{n-2}, h_2^{n-2}, \cdots, h_r^{n-2}$ and an integer $s \le r$, such that the subcollection $\{h_1^{n-2}, h_2^{n-2}, \cdots, h_s^{n-2}\}$ is a free $\mathbb{Z}[\pi_1 R_i]$–basis for $H_{n-2}\left(\widetilde{R}_i, \partial \widetilde{U}_i\right)$. Then the corresponding $\mathbb{Z}[\pi_1 R_i]$–cellular chain complex for $(R_i, \partial U_i)$ may be expressed as

$$(\dagger) \qquad 0 \to \left\langle h_1^{n-2}, \cdots, h_s^{n-2} \right\rangle \oplus \left\langle h_{s+1}^{n-2}, \cdots, h_r^{n-2} \right\rangle \xrightarrow{\partial} \left\langle h_1^{n-3}, \cdots, h_t^{n-3} \right\rangle \to 0$$

where $\left\langle h_1^{n-2}, \cdots, h_s^{n-2} \right\rangle$ represents the free $\mathbb{Z}[\pi_1 R_i]$–submodule of $\widetilde{C}_{n-2}$ generated by the corresponding handles; $\left\langle h_{s+1}^{n-2}, \cdots, h_r^{n-2} \right\rangle$ represents the free submodule of $\widetilde{C}_{n-2}$ generated by the remaining $(n-2)$–handles in R_i; and

$$\left\langle h_1^{n-3}, \cdots, h_t^{n-3} \right\rangle = \widetilde{C}_{n-3}$$

is the free module generated by the $(n-3)$–handles in R_i. Moreover,

$$H_{n-2}(\widetilde{R}_i, \partial \widetilde{U}_i) = \ker(\partial) = \left\langle h_1^{n-2}, \cdots, h_s^{n-2} \right\rangle \oplus \{0\},$$

and ∂ takes $\{0\} \oplus \left\langle h_{s+1}^{n-2}, \cdots, h_r^{n-2} \right\rangle$ injectively into $\left\langle h_1^{n-3}, \cdots, h_t^{n-3} \right\rangle$. This is the content of Lemma 15 and the following paragraph in [7].

At this point, we would like to use the fact that $\partial h_j^{n-2} = 0$ for each $j = 1, \cdots, s$ to slide these handles off all of the $(n-3)$–handles. This would be done by repeated use of the Whitney Lemma in $\partial_+(S \cup h_1^{n-3} \cup \cdots \cup h_t^{n-3})$ to remove the collection of attaching spheres $\{\alpha_j^{n-3}\}_{j=1}^s$ from the belt spheres $\left\{\beta_j^2\right\}_{j=1}^t$ of the $(n-3)$–handles. (Here, S is a closed collar neighborhood of ∂U_i and ∂_+ indicates the right-hand boundary.) After that, we would 'carve out' these $(n-2)$–handles—those generating the unwanted $(n-2)$–dimensional homology—in an attempt to obtain a generalized $(n-2)$–neighborhood of infinity. (This process will be discussed in detail later.) Unfortunately, the desired application of the Whitney Lemma is only assured if the collection $\{\alpha_j^{n-3}\}_{j=1}^s$ is π_1–negligible in $\partial_+(S \cup h_1^{n-3} \cup \cdots \cup h_t^{n-3})$, ie, the inclusion

$$\partial_+(S \cup h_1^{n-3} \cup \cdots \cup h_t^{n-3}) - \cup_{j=1}^s \alpha_j^{n-3} \hookrightarrow \partial_+(S \cup h_1^{n-3} \cup \cdots \cup h_t^{n-3})$$

induces a π_1–isomorphism (see [12, page 72]). Moreover, even if the handles that are generating the unwanted homology can be made to miss the $(n-3)$–handles, we still must be sure that carving out these $(n-2)$–handles does not change the fundamental group of our neighborhood of infinity. Otherwise we will have arranged that the relative $\mathbb{Z}[\pi_1 U_i]$–homology of our new neighborhood of infinity is trivial, but $\pi_1(U_i)$ will be the wrong group.

The above two difficulties are related. To avoid them entirely, we would need to know that, in the corresponding dual handle decomposition, the 2–handles dual to $h_1^{n-2}, \cdots, h_s^{n-2}$ do not kill any non-trivial loops when they are attached to ∂U_{i+1}. Since $\pi_1(\partial U_{i+1}) \to \pi_1(R_i)$ is not injective, that scenario seems highly unlikely.

Remark 5 Examples constructed in [8] show that the above problems can indeed occur.

At this point we begin utilizing Condition (2) of Theorem 1.1. According to [7, Theorem 5], the collection $\{U_i\}$ may then be chosen so that each homomorphism $\pi_1(U_{i+1}) \xrightarrow{\lambda_{i+1}} \pi_1(U_i)$ is surjective with perfect kernel. As noted earlier, the inclusions $\partial U_i \hookrightarrow R_i \hookrightarrow U_i$ each induce π_1–isomorphisms. By similar reasoning $\pi_1(\partial U_{i+1}) \to \pi_1(R_i)$ is surjective with the same kernel as λ_{i+1}. Call this kernel K_{i+1}. By a basic theorem from combinatorial group theory (see [13] or [7, Lemma 3]) K_{i+1} is the normal closure of a finite collection of elements of $\pi_1(\partial U_{i+1})$. Thus we may apply Theorem 3.2 to $(R_i, \partial U_i, \partial U_{i+1})$ to obtain a plus cobordism $(W_i, A_i, \partial U_{i+1})$ embedded in R_i which is the identity on ∂U_{i+1} and for which $\ker(\pi_1(\partial U_{i+1}) \to \pi_1(W_i)) = K_{i+1}$. It follows that $\pi_1(W_i) \xrightarrow{\cong} \pi_1(R_i)$

Let $R_i' = \overline{R_i - W_i}$. Since W_i strong deformation retracts onto A_i we have

- $(R_i', \partial U_i) \hookrightarrow (R_i, \partial U_i)$ is a homotopy equivalence of pairs, and
- $\pi_1(A_i) \xrightarrow{\cong} \pi_1(R_i')$.

The first property ensures that the inclusion induced maps

$$H_{n-2}(\widetilde{R}_i', \partial \widetilde{U}_i) \to H_{n-2}(\widetilde{R}_i, \partial \widetilde{U}_i) \to H_{n-2}(\widetilde{U}_i, \partial \widetilde{U}_i)$$

are all isomorphisms; thus, $H_{n-2}(\widetilde{R}_i', \partial \widetilde{U}_i)$ is a free $\mathbb{Z}[\pi_1 U_i]$–module which carries the $\mathbb{Z}[\pi_1 U_i]$–homology of $(U_i, \partial U_i)$. By performing the same procedures on the cobordism $(R_i', \partial U_i, A)$ as we did earlier on $(R_i, \partial U_i, \partial U_{i+1})$ we may obtain a handle decomposition of R_i' based on ∂U_i which has handles only of index $n-3$ and $n-2$. Moreover, we may arrange that the corresponding cellular chain complex is of the form (†) (although the precise numbers of handles may have changed). We adopt that notation without changing the names of the handles.

The second property ensures that, under the dual handle decomposition of R_i', the 2–handles dual to $h_1^{n-2}, \cdots, h_s^{n-2}$ do not kill any non-trivial loops when they are attached to A. This means that the attaching $(n-3)$–spheres of $h_1^{n-2}, \cdots, h_s^{n-2}$ are all π_1–negligible in $\partial_+ \left(S \cup h_1^{n-3} \cup \cdots \cup h_t^{n-3}\right)$. The non-simply connected Whitney Lemma [12, page 72] may now be applied to isotope the attaching spheres of

$h_1^{n-2}, \cdots, h_s^{n-2}$ off all of the belt spheres of the $(n-3)$–handles. Thus, we may assume that $h_1^{n-2}, \cdots, h_s^{n-2}$ are attached directly to S. Let $Q = S \cup \left(h_1^{n-2} \cup \cdots \cup h_s^{n-2}\right)$ and let $V_i = \overline{U_i - Q}$. We will show that V_i is the desired generalized $(n-2)$–neighborhood of infinity. The first issue involves the fundamental group. We wish to observe that the fundamental group has not changed, ie, that $V_i \hookrightarrow U_i$ induces a π_1–isomorphism and that V_i is a generalized 1–neighborhood of infinity.

First, note that $\overline{R_i' - Q}$ may be obtained from ∂V_i by attaching $(n-3)$– and $(n-2)$–handles; in particular $\{h_1^{n-3}, \cdots, h_t^{n-3}\}$ and $\{h_{s+1}^{n-2}, \cdots, h_r^{n-2}\}$. Since $n \geq 6$, $\partial V_i \hookrightarrow \overline{R_i' - Q}$ induces a π_1–isomorphism. By inverting this handle decomposition, it is clear that $A \hookrightarrow \overline{R_i' - Q}$ induces a π_1–surjection. Moreover, since the composition $A \hookrightarrow \overline{R_i' - Q} \hookrightarrow R_i'$ induces a π_1–isomorphism, we also have injectivity; so $A \hookrightarrow \overline{R_i' - Q}$ induces a π_1–isomorphism. This also implies that $\overline{R_i' - Q} \hookrightarrow R_i'$ induces a π_1–isomorphism.

Since U_{i+1} is a generalized 1–neighborhood of infinity, the Van Kampen theorem assures us that $A \hookrightarrow W \cup U_{i+1}$ induces a π_1–isomorphism. Similar arguments then provide the necessary isomorphisms $\pi_1(\partial V_i) \xrightarrow{\cong} \pi_1(V_i)$ and $\pi_1(\partial V_i) \xrightarrow{\cong} \pi_1(U_i)$.

Lastly, we verify that V_i is a generalized $(n-2)$–neighborhood of infinity. Begin with the long exact sequence for the triple $\left(\widetilde{U}_i, \widetilde{Q}, \partial \widetilde{U}_i\right)$.

$$\cdots \to H_k\left(\widetilde{Q}, \partial \widetilde{U}_i\right) \to H_k\left(\widetilde{U}_i, \partial \widetilde{U}_i\right) \to H_k\left(\widetilde{U}_i, \widetilde{Q}\right) \to H_{k-1}\left(\widetilde{Q}, \partial \widetilde{U}_i\right) \to \cdots$$

Since $H_k\left(\widetilde{U}_i, \partial \widetilde{U}_i\right)$ and $H_{k-1}\left(\widetilde{Q}, \partial \widetilde{U}_i\right)$ are trivial for all $k \leq n-3$, $H_k\left(\widetilde{U}_i, \widetilde{Q}\right)$ also vanishes for $k \leq n-3$. If $k = n-2$, the surjectivity of $H_{n-2}\left(\widetilde{Q}, \partial \widetilde{U}_i\right) \to H_{n-2}\left(\widetilde{U}_i, \partial \widetilde{U}_i\right)$ together with the triviality of $H_{n-3}\left(\widetilde{Q}, \partial \widetilde{U}_i\right)$ implies the triviality of $H_{n-2}\left(\widetilde{U}_i, \widetilde{Q}\right)$. But the above π_1–isomorphisms imply that $\left(\widetilde{V}_i, \partial \widetilde{V}_i\right)$ is the preimage of $(V_i, \partial V_i)$ under the covering projection $p\colon \left(\widetilde{U}_i, \partial \widetilde{U}_i\right) \to (U_i, \partial U_i)$. Thus we may excise the interior of $\widetilde{Q}$ from $\widetilde{U}_i$ to show that $H_k\left(\widetilde{V}_i, \partial \widetilde{V}_i\right)$ vanishes for all $k \leq n-2$.

Remark 6 With a few minor refinements, the above argument can be carried out when $n = 5$ provided the Whitney Lemma is valid in the 4–manifold $\partial_+\left(S \cup h_1^{n-3} \cup \cdots \cup h_t^{n-3}\right)$. This explains Remark 2.

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Department of Mathematical Sciences, University of Wisconsin-Milwaukee
Milwaukee, WI 53201, USA
Department of Mathematics, The Colorado College
Colorado Springs, Colorado 80903, USA

craigg@uwm.edu, ftinsley@coloradocollege.edu

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