Thin buildings

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Let X be a building of uniform thickness $q+1$. L^2–Betti numbers of X are reinterpreted as von-Neumann dimensions of weighted L^2–cohomology of the underlying Coxeter group. The dimension is measured with the help of the Hecke algebra. The weight depends on the thickness q. The weighted cohomology makes sense for all real positive values of q, and is computed for small q. If the Davis complex of the Coxeter group is a manifold, a version of Poincaré duality allows to deduce that the L^2–cohomology of a building with large thickness is concentrated in the top dimension.

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Introduction

Let (G, B, N, S) be a BN–pair, and let X be the associated building (notation as in Brown [2, Chapter 5]). There are many geometric realizations of X. We consider the one introduced by Davis in [4]. Then X is a locally finite simplicial complex, acted upon by G. The action has a fundamental domain with stabiliser B. The standard choice of such a domain is called the Davis chamber. We can and will assume that G is a closed subgroup of the group $\mathrm{Aut}(X)$ of simplicial automorphisms of X (in the compact-open topology). If this is not the case, one can pass to the quotient of G by the kernel of the G–action on X (that quotient is a subgroup of $\mathrm{Aut}(X)$), and then take its closure in $\mathrm{Aut}(X)$.

Let $L^2C^i(X)$ be the space of i–cochains on X which are square-summable with respect to the counting measure on the set $X^{(i)}$ of i–simplices in X. Then the coboundary map $\delta^i\colon L^2C^i(X) \to L^2C^{i+1}(X)$ is a bounded operator. The reduced L^2–cohomology of X is defined to be $L^2H^i(X) = \ker\delta^i/\overline{\mathrm{im}\delta^{i-1}}$. This is a Hilbert space, carrying a unitary G–representation. Using the von Neumann G–dimension one defines $L^2b^i(X) = \dim_G L^2H^i(X)$. We are interested in calculating these Betti numbers. (This problem was considered by Dymara and Januszkiewicz in [8] and by Davis and Okun in [6].)

The first step is to pass from the cochain complex $(L^2C^*(X), \delta)$ to a smaller complex of B–invariants: $(L^2C^*(X)^B, \delta)$. Now $L^2C^i(X)^B$ can be identified with a space

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of cochains on $X/B = \Sigma$—the Davis complex of the Weyl group W of the building. However, a simplex $\sigma \in \Sigma$ has a preimage in X consisting of $q^{d(\sigma)}$ simplices, where $q+1$ is the thickness of the building and $d(\sigma)$ is the distance from σ to the chamber stabilised by B. Therefore a cochain f on Σ represents a square-summable B–invariant cochain if and only if it satisfies $\sum_\sigma |f(\sigma)|^2 q^{d(\sigma)} < \infty$; we denote the space of such cochains $L^2_q C^*(\Sigma)$. The complex $(L^2_q C^*(\Sigma), \delta)$ and its (reduced) cohomology $L^2_q H^*(\Sigma)$ are acted upon by the Hecke algebra $\mathbf{C}[B\backslash G/B]$. A suitable von Neumann completion of the latter can be used to measure the dimension of $L^2_q H^i(\Sigma)$, yielding Betti numbers $L^2_q b^i(\Sigma)$. It turns out that $L^2_q b^i(\Sigma) = L^2 b^i(X)$. In particular, the L^2–Betti numbers of a building depend only on W and on q.

The good news is that the complex $(L^2_q(\Sigma), \delta)$, the Hecke algebra and the Betti numbers $L^2_q b^i(\Sigma)$ can be defined for all real $q > 0$, in a uniform manner which for integer values of q gives exactly the objects discussed above. It turns out that for small q (namely for $q < \rho_W$, where ρ_W is the logarithmic growth rate of W) the Betti numbers $L^2_q b^i(\Sigma)$ are 0 except for $i = 0$. Since $\rho_W \leq 1$, this result says nothing about actual buildings. However, in Section 6 we prove a version of Poincaré duality, saying that if Σ is a manifold of dimension n, then $L^2_q b^i(\Sigma) = L^2_{1/q} b^{n-i}(\Sigma)$. Thus, if the Davis complex of the Weyl group of a building (ie, an apartment in the Davis realization of the building) is an n–manifold, and if $q > \frac{1}{\rho_W}$, then the L^2–Betti numbers of the building vanish except for $L^2 b^n(X)$.

Examples of buildings to which our method applies can be constructed from flag triangulations of spheres. Davis associates a right-angled Coxeter group to any such triangulation; this right-angled Coxeter group is the Weyl group of a family of buildings with manifold apartments, parametrised by thickness. Let us mention that the argument applies also to Euclidean buildings, yielding another calculation of their L^2–Betti numbers.

In a forthcoming paper (Davis–Dymara–Januszkiewicz–Okun [5]) the L^2–Betti numbers of all buildings satisfying $q > \frac{1}{\rho_W}$ are calculated.

The definitions, results and arguments of this paper go through, with appropriate reading, in the multi-parameter case. A detailed account of the multi-parameter setting is given in [5].

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0 Integer thickness

Let (W, S) be a Coxeter system. Let Δ be a simplex with codimension 1 faces labelled by elements of S, and let Δ' be its first barycentric subdivision. Each $T \subseteq S$ generates a subgroup W_T of W called a special subgroup; also, T corresponds to a face Δ_T of Δ (the intersection of codimension 1 faces labelled by elements of T). The Davis chamber D is the subcomplex of Δ' spanned by barycentres of faces Δ_T for which W_T is finite ($\mathcal{F}$ will denote the set of subsets $T \subseteq S$ such that W_T is finite). To every $T \subseteq S$ we assign a face of the Davis chamber: $D_T = D \cap \Delta_T$. The Davis realization Σ of the Coxeter complex is $W \times D/\sim$, where $(w, p) \sim (u, q)$ if and only if for some T we have $p = q \in D_T$ and $w^{-1}u \in W_T$. The action of W on the first factor descends to an action on Σ. We denote the image of σ under the action of w by $w\sigma$, and the W–orbit of σ in Σ by $W\sigma$. The images of $w \times D$ in Σ are called chambers. The action of W on Σ is simply transitive on the set of chambers.

A Tits building X_{Tits} with Weyl group W is a set with a W–valued distance function d, satisfying certain conditions (see Ronan [11]). Its Davis incarnation is $X = X_{Tits} \times D/\sim$, where $(x, p) \sim (y, q)$ if and only if for some T we have $p = q \in D_T$ and $d(x, y) \in W_T$. The images of $x \times D$ in X are called chambers.

We will consider only buildings of uniformly bounded thickness, ie, such that for some constant $N > 0$, any $s \in S$ and any $x \in X_{Tits}$ there are no more than N elements $y \in X_{Tits}$ satisfying $d(x, y) = s$. If this number of s–neighbours of x is equal to q for all pairs (x, s), then we say that the building has uniform thickness $q + 1$. We denote such building $X(q)$ (for a right-angled Coxeter group it is unique).

Uniformly bounded thickness is equivalent to X being uniformly locally finite. Thus we can consider (reduced) L^2–(co)homology of X. This is obtained from the complex of L^2 (co)chains on X with the usual (co)boundary operators ∂, δ. These operators are in fact adjoint to each other, so that the (co)homology can be identified with $L^2\mathcal{H}^*(X)$, the space of harmonic (co)chains ("reduced" means that we divide the kernel by the closure of the image).

Assume now that X_{Tits} comes from a BN–pair in a group G. Then G acts by simplicial automorphisms on X. We can assume that G acts faithfully and is locally compact (possibly taking the closure of its image in $\mathrm{Aut}(X)$ in the compact-open topology). We use G to measure the size of $L^2\mathcal{H}^i(X)$ via the von Neumann dimension. To do this, we first express $L^2C^i(X)$ as $\oplus_{\sigma^i \subset D} L^2(G\sigma^i)$. Then we notice that $L^2(G\sigma^i)$ is naturally isomorphic to $L^2(G)^{G_{\sigma^i}}$ (where G_{σ^i} is the stabiliser of σ^i in G). It is convenient to multiply this isomorphism by a suitable scalar factor in order to make it isometric. Then the space $L^2(G)^{G_{\sigma^i}}$ is embedded into $L^2(G)$, giving us finally an embedding of left G–modules $L^2C^i(X) \hookrightarrow \oplus_{\sigma^i \subset D} L^2(G)$. In particular, $L^2\mathcal{H}^i(X)$ is now embedded as a left G–module in $\oplus_{\sigma^i \subset D} L^2(G)$; we can consider the orthogonal projection onto this subspace, and define $L^2b^i(X)$ to be the von Neumann trace of that projection. Let B be the stabiliser of D in G. For each $\sigma^i \subset D$ we have a vector $\mathbf{1}_\sigma$ in $\oplus_{\sigma^i \subset D} L^2(G)$, having σ th component $\mathbf{1}_B$ and other components 0. The projection onto $L^2\mathcal{H}^i(X)$ is given by a matrix whose σ th row gives the projection of $\mathbf{1}_\sigma$ on $L^2\mathcal{H}^i(X)$, expressed as an element of $\oplus_{\sigma^i \subset D} L^2(G)$ (while applying this matrix we understand multiplication as convolution). Notice that both $\mathbf{1}_\sigma$ and the space $L^2\mathcal{H}^i(X)$ are B–invariant; so therefore will be the projection of $\mathbf{1}_\sigma$ on $L^2\mathcal{H}^i(X)$.

1 Real thickness

For a $w \in W$ we denote by $d(w)$ the length of a shortest word in the generators S representing w. For a chamber $c = w \times D$ of Σ we put $d(c) = d(w)$. For every simplex $\sigma \subset \Sigma$ there is a unique chamber $c \supseteq \sigma$ with smallest $d(c)$; we put $d(\sigma) = d(c)$.

For a real number $t > 0$ we equip the set $\Sigma^{(i)}$ of i–simplices in Σ with the measure $\mu_t(\sigma) = t^{d(\sigma)}$. We also pick (arbitrarily) orientations of simplices in D, and extend them W–equivariantly to orientations of all simplices in Σ. This allows us to identify chains and cochains with functions. We put

$$L^2_tC^i(\Sigma) = L^2_tC_i(\Sigma) = L^2(\Sigma^{(i)}, \mu_t).$$

We now define $\delta^i\colon L^2_tC^i(\Sigma) \to L^2_tC^{i+1}(\Sigma)$ by

$$\delta^i(f)(\tau^{i+1}) = \sum_{\sigma^i \subset \tau^{i+1}} [\tau : \sigma] f(\sigma)$$

and $\partial^t_i\colon L^2_tC_i(\Sigma) \to L^2_tC_{i-1}(\Sigma)$ by

$$\partial^t_i(f)(\eta^{i-1}) = \sum_{\sigma^i \supset \eta^{i-1}} [\eta : \sigma] t^{d(\sigma)-d(\eta)} f(\sigma)$$

(here $[\alpha : \beta] = \pm 1$ tells us whether orientations of α and β agree or not). We have

$$\begin{aligned}\langle \delta^i(f), g\rangle_t &= \sum_{\tau^{i+1}} \Big(\sum_{\sigma^i \subset \tau^{i+1}} [\tau : \sigma] f(\sigma) \overline{g(\tau)} t^{d(\tau)} \Big) \\ &= \sum_{\sigma^i} f(\sigma) \overline{\Big(\sum_{\tau^{i+1} \supset \sigma^i} [\tau : \sigma] t^{d(\tau)-d(\sigma)} g(\tau) \Big)} t^{d(\sigma)} = \langle f, \partial^t_i(g)\rangle_t.\end{aligned}$$

That is, $\delta^* = \partial^t$ as operators on $L^2_t C^*(\Sigma)$. It follows that $(\partial^t)^2 = 0$ (since $\delta^2 = 0$), and we can consider (reduced) L^2_t–(co)homology:

$$L^2_t H^i(\Sigma) = \ker \delta^i / \overline{\operatorname{im} \delta^{i-1}}, \quad L^2_t H_i(\Sigma) = \ker \partial^t_i / \overline{\operatorname{im} \partial^t_{i+1}}$$

Since $\delta^* = \partial^t$, $(\partial^t)^* = \delta$ we have $L^2_t C^i(\Sigma) = \ker \partial^t_i \oplus \overline{\operatorname{im} \delta^{i-1}} = \ker \delta^i \oplus \overline{\operatorname{im} \partial^t_{i+1}}$ (orthogonal direct sums). It follows that

$$L^2_t H^i(\Sigma) \simeq L^2_t \mathcal{H}^i(\Sigma) \simeq L^2_t H_i(\Sigma),$$

where $L^2_t \mathcal{H}^i(\Sigma)$ is the space $\ker \delta^i \cap \ker \partial^t_i$ of harmonic i–cochains.

Remark Suppose that $X(q)$ is a building associated to a BN–pair, with Weyl group W. Then the B–invariant part of the L^2 cochain complex of $X(q)$ is isomorphic to $L^2_q C^*(\Sigma)$.

2 Hecke algebra

We deform the usual scalar product on $\mathbf{C}[W]$ into $\langle\, , \rangle_t$:

$$\Big\langle \sum_{w \in W} a_w \delta_w, \sum_{w \in W} b_w \delta_w \Big\rangle_t = \sum_{w \in W} a_w \overline{b_w} t^{d(w)}. \tag{2–1}$$

We also correspondingly deform the multiplication into the following Hecke t–multiplication: for $w \in W$, $s \in S$ we put

$$\delta_w \delta_s = \begin{cases} \delta_{ws} & \text{if } d(ws) > d(w); \\ t\delta_{ws} + (t-1)\delta_w & \text{if } d(ws) < d(w). \end{cases} \tag{2–2}$$

This extends to a $\mathbf{C}$–bilinear associative multiplication on $\mathbf{C}[W]$ (see Bourbaki [1]). Using (2–2) and induction on $d(v)$ one easily shows

$$\delta_w \delta_v = \delta_{wv} \qquad \text{if } d(wv) = d(w) + d(v). \tag{2–3}$$

We keep the involution on $\mathbf{C}[W]$ independent of t:

$$\Big(\sum_{w \in W} a_w \delta_w \Big)^* = \sum_{w \in W} \overline{a_{w^{-1}}} \delta_w. \tag{2–4}$$

Proposition 2.1 *The above scalar product, multiplication and involution define a Hilbert algebra structure on $\mathbf{C}[W]$ (in the sense of Dixmier [7, A.54]); we use the notation $\mathbf{C}_t[W]$ to indicate this structure.*

Proof We begin with involutivity: $(xy)^* = y^* x^*$. One checks it using (2–2) and (2–3) for $x = \delta_w$, $y = \delta_s$ considering two cases: $d(ws) < d(w)$, $d(ws) > d(w)$. Then one checks it for $x = \delta_w$, $y = \delta_u$ by induction on $d(u)$. Finally, by $\mathbf{C}$–bilinearity of multiplication, the result extends to general x, y. From involutivity and (2–2) we immediately get

$$\delta_s \delta_w = \begin{cases} \delta_{sw} & \text{if } d(sw) > d(w); \\ t\delta_{sw} + (t-1)\delta_w & \text{if } d(sw) < d(w). \end{cases} \tag{2–5}$$

We now recall and prove the conditions (i)–(iv) of [7] defining a Hilbert algebra.

(i) $\langle x, y\rangle_t = \langle y^*, x^*\rangle_t$.

This is a straightforward calculation (using $d(w) = d(w^{-1})$).

(ii) $\langle xy, z\rangle_t = \langle y, x^* z\rangle_t$.

Due to linearity it is enough to check (ii) in the case $y = \delta_w$, $z = \delta_u$, $x = \delta_v$. First one treats the case $v = s \in S$, directly using (2–5); this requires four sub-cases, depending on comparison of $d(sw)$ with $d(w)$ and $d(su)$ with $d(u)$. Then one performs an easy induction on $d(v)$.

(iii) For every $x \in \mathbf{C}_t[W]$ the map $\mathbf{C}_t[W] \ni y \mapsto xy \in \mathbf{C}_t[W]$ is continuous.

One checks first that $y \mapsto \delta_s y$ is continuous, directly using (2–5). Continuity of $y \mapsto xy$ for arbitrary $x \in \mathbf{C}_t[W]$ follows, because compositions and linear combinations of continuous maps are continuous.

(iv) The set $\{xy \mid x, y \in \mathbf{C}_t[W]\}$ is dense in $\mathbf{C}_t[W]$.

This is immediate, since we have a unit element δ_1 in $\mathbf{C}_t[W]$. □

Corollary 2.2 *The coefficient of δ_1 in ab is equal to $\langle a, b^*\rangle_t$.*

Proof That coefficient is equal to $\langle ab, \delta_1\rangle_t$, which by (ii) and (i) is $\langle b, a^*\rangle_t = \langle a, b^*\rangle_t$. □

As in [7, A.54], we get two von Neumann algebras U_t, V_t: they are weak closures of $\mathbf{C}_t[W]$ acting on its completion L^2_t by left (respectively right) multiplication.

As in [7, A.57], we put $\mathbf{C}_t[W]'$ to be the algebra of all bounded elements of L^2_t; bounded means that left (or, equivalently, right) multiplication by the element is bounded on $\mathbf{C}_t[W]$ (so, extends to a bounded operator on L^2_t and defines an element of U_t or V_t).

As in [7, A.60], we have natural traces tr on U_t, V_t: if $B \in U_t$ (or $B \in V_t$) is self-adjoint and positive, we ask whether $B^{\frac{1}{2}} = a\cdot$ (resp. $B^{\frac{1}{2}} = \cdot a$) for an $a \in \mathbf{C}_t[W]'$. If it is so, we put $\operatorname{tr} B = \|a\|_t^2$; otherwise we put $\operatorname{tr} B = +\infty$. The $a = \sum_{w\in W} a_w \delta_w$ we are asking for is self-adjoint: $a_w = \overline{a_{w^{-1}}}$, so that by Corollary 2.2 $\|a\|_t^2$ is equal to the coefficient of δ_1 in a^2. Thus B is the multiplication by the bounded self-adjoint element $b = a^2$, and $\operatorname{tr} B$ is equal to the coefficient of δ_1 in b.

Suppose now that we are given a closed subspace Z of $\oplus_{i=1}^{l} L_t^2$, such that the orthogonal projection P_Z onto Z is an element of $M_{l\times l} \otimes V_t$. To calculate the trace of this projection we first need to identify P_Z as a matrix. So, we take the standard basis $\{e_i\}$ of $\oplus_{i=1}^{l} L_t^2$ (e_i has δ_1 as the i th coordinate, and other coordinates 0), and apply P_Z to it. We expand the results in the basis $\{e_i\}$: let $a_i^j \in L_t^2$ be the j th coordinate of $P_Z(e_i)$. Then we take the coefficient of δ_1 in a_i^i and sum over i. The number we get is the trace of P_Z.

3 L_t^2–Betti numbers

It will be convenient to identify L_t^2 with $L^2(W,\nu_t)$, where $\nu_t(w) = t^{d(w)}$. For any Coxeter group Γ (we have W as well as its subgroups W_T in mind) the generating function of Γ is defined by $\Gamma(x) = \sum_{\gamma\in\Gamma} x^{d(\gamma)}$. For a finite Γ it is a polynomial, in general it is a rational function. We denote by ρ_Γ the radius of convergence of the series defining $\Gamma(x)$.

As in the case of buildings (Section 0), we have $L_t^2 C^i(\Sigma) = \bigoplus_{\sigma^i\subset D} L^2(W\sigma^i,\mu_t)$. Now $L^2(W\sigma^i,\mu_t)$ can be identified with $L^2(W,\nu_t)^{W_{T(\sigma)}}$ (where $T(\sigma)$ is the largest subset of S such that $\sigma \subseteq D_{T(\sigma)}$) via the map ϕ given by $\phi(f)(w) = \frac{1}{\sqrt{W_{T(\sigma)}(t)}} f(w\sigma)$ (we distorted the natural map by the factor $\frac{1}{\sqrt{W_{T(\sigma)}(t)}}$ in order to make it isometric). Finally, $L^2(W,\nu_t)^{W_{T(\sigma)}}$ is a subspace of $L^2(W,\nu_t) = L_t^2$, so that we get an isometric embedding

$$\Phi\colon L_t^2 C^i(\Sigma) \hookrightarrow \bigoplus_{\sigma^i\subset D} L_t^2 = C^i(D)\otimes L_t^2.$$

Let $\mathcal{L}$ denote the algebra U_t acting diagonally on the left on $\oplus_{\sigma\subset D} L_t^2 = C^*(D)\otimes L_t^2$; let $\mathcal{R}$ be $\operatorname{End}(C^*(D))\otimes V_t$ acting on the same space on the right. The von Neumann algebras $\mathcal{L}$ and $\mathcal{R}$ are commutants of each other.

Lemma 3.1 *The projection of L^2_t onto $L^2(W\sigma,\mu_t)=L^2(W,\nu_t)^{W_{T(\sigma)}}$ is given by the right Hecke t–multiplication by*

$$p_{T(\sigma)}=\frac{1}{W_{T(\sigma)}(t)}\sum_{w\in W_{T(\sigma)}}\delta_w. \tag{3–1}$$

Proof Put $T=T(\sigma)$. The subspace onto which we project consists of those elements of L^2_t which are right W_T–invariant; this is equivalent to being invariant under right Hecke t–multiplication by $\frac{1}{1+t}(\delta_1+\delta_s)$ for all $s\in T$ (to check this one splits W into pairs $\{w,ws\}$, and calculates for each pair separately using (2–2)). As a result, this subspace is $\mathcal{L}$–invariant, so that the projection P_T onto it is an element of $\mathcal{R}$. It follows that P_T is given by right Hecke t–multiplication by $P_T(\delta_1)$. The latter is clearly of the form $C\sum_{w\in W_T}\delta_w$, where C is a constant such that

$$\langle\delta_1-C\sum_{w\in W_T}\delta_w, C\sum_{w\in W_T}\delta_w\rangle_t=0.$$

This gives $C=\|\sum_{w\in W_T}\delta_w\|_t^{-2}=\left(\sum_{w\in W_T}t^{d(w)}\right)^{-1}=\frac{1}{W_T(t)}$. □

Lemma 3.2 *$\mathcal{L}$ preserves the subspace $L^2_tC^i(\Sigma)\subset C^*(D)\otimes L^2_t$ and commutes with δ and ∂^t.*

Proof The first claim follows from Lemma 3.1 (and actually was a step in the proof of that lemma). To prove the second part notice that δ is an element of $\mathcal{R}$: the matrix with V_t–coefficients describing δ has non-zero $\sigma\tau$–entry if and only if σ is a codimension 1 face of τ; the entry is then $\sqrt{\frac{W_{T(\sigma)}(t)}{W_{T(\tau)}(t)}}\delta_1$. It follows that δ commutes with $\mathcal{L}$. So therefore does its adjoint ∂^t. □

Corollary 3.3 *$L^2_tC^i(\Sigma)$, $L^2_t\mathcal{H}^i(\Sigma)$, $\ker\delta^i$, $\ker\partial^t_i$, $\overline{\operatorname{im}\delta^i}$, $\overline{\operatorname{im}\partial^i_t}$ are $\mathcal{L}$–invariant; therefore, orthogonal projections onto these spaces belong to $\mathcal{R}$.*

We use tr to denote the tensor product of the usual matrix trace on $\operatorname{End}(C^*(D))$ and the von Neumann trace on V_t as described in Section 2. We put

$$b^i_t=L^2_tb^i(\Sigma)=\operatorname{tr}\Big(\text{projection onto } L^2_t\mathcal{H}^i(\Sigma)\Big) \tag{3–2}$$

$$c^i_t=L^2_tc^i(\Sigma)=\operatorname{tr}\Big(\text{projection onto } L^2_tC^i(\Sigma)\Big) \tag{3–3}$$

$$\chi_t=\sum_i(-1)^ib^i_t=\sum_i(-1)^ic^i_t. \tag{3–4}$$

The sums in (3–4) give the same value by the standard algebraic topology argument. It follows from Lemma 3.1 that $c_t^i = \sum_{\sigma^i \subset D} \frac{1}{W_{T(\sigma)}(t)}$. Grouping together simplices σ with the same $T(\sigma)$ and using formula (5) from Charney–Davis [3] we obtain the following result (see Serre [12]).

Corollary 3.4

$$\chi_t = \frac{1}{W(t)}$$

Theorem 3.5 *Suppose that $X(q)$ is a building associated to a BN–pair, with Weyl group W. Then $L^2 b^i(X(q)) = b_q^i$.*

Proof For $t = q$, $L_t^2 C^i(\Sigma)$ coincides with the space of B–invariant elements of $L^2 C^i(X(q))$. By the concluding remarks of Section 0, the matrix of the projection onto $L^2 \mathcal{H}^i(X(q))$ has B–invariant entries—so that it coincides with the one we use to define b_t^i. Hence the conclusion. □

Suppose now that the pair $(D, \partial D = D \cap \partial\Delta)$ is a generalised homology n–disc (ie, it is a homology manifold with boundary, with relative homology groups the same as those of an n–disc modulo its boundary). Then each $D_T = D \cap \Delta_T$ is also a homology $(n - |T|)$–disc (for $T \in \mathcal{F}$). We can now use wD_T, $w \in W$, $T \in \mathcal{F}$, as a homology cellular structure on Σ (denoted Σ_{ghd}). The cell D_T has the form of an o_T–centred cone; we put $d(wD_T) = d(wo_T)$, and define μ_t, (co)chain complexes, the embedding Φ, the U_t–module structure and the numbers $b_t^i(\Sigma_{ghd})$ in essentially the same way as for the original triangulation of Σ.

4 Dual cells

So far we used the triangulation of Σ which originated from the barycentric subdivision of a simplex. We will use notation Σ_{st} to remind that we have this standard triangulation in mind. In this section we will describe another cell structure on Σ. It will make our discussion of Poincaré duality in Section 6 look pretty standard.

To each $T \in \mathcal{F}$ we associate a face Δ_T of Δ, whose barycentre o_T is a vertex of the Davis chamber D. We define $\langle T\rangle$ as the union of all simplices $\sigma \subset \Sigma$ such that $\sigma \cap D_T = o_T$ (recall that $D_T = D \cap \Delta_T$). As a simplicial complex, $\langle T\rangle$ is an o_T–centred cone over Σ_T; since T is such that W_T is finite, Σ_T is a sphere and $\langle T\rangle$ is a disc of dimension $|T|$. The boundary of $\langle T\rangle$ is cellulated by $w\langle U\rangle$, for all possible $T \subset U \subseteq S$, $w \in W_T$. The complex Σ cellulated by $w\langle T\rangle$, over all $w \in W$,

$T \in \mathcal{F}$, is a cellular complex that we denote Σ_d. The cells of Σ_d will be called *dual cells*. The name *Coxeter blocks* is also used (Davis [4]).

We now put $d(w\langle T\rangle) = d(wo_T)$, and define the measures μ_t on the set $\Sigma_d^{(i)}$ of i–dimensional cells of Σ_d by $\mu_t(\langle a\rangle) = t^{d(\langle a\rangle)}$. Then

$$L_t^2 C^i(\Sigma_d) = L_t^2 C_i(\Sigma_d) \simeq L^2(\Sigma_d^{(i)}, \mu_t),$$

We now define $\delta^i\colon L_t^2 C^i(\Sigma_d) \to L_t^2 C^{i+1}(\Sigma_d)$ by

$$\delta^i(f)(\langle\tau\rangle^{i+1}) = \sum_{\langle\sigma\rangle^i \subset \langle\tau\rangle^{i+1}} [\langle\tau\rangle : \langle\sigma\rangle] f(\langle\sigma\rangle)$$

and $\partial_i^t\colon L_t^2 C_i(\Sigma_d) \to L_t^2 C_{i-1}(\Sigma_d)$ by

$$\partial_i^t(f)(\langle\eta\rangle^{i-1}) = \sum_{\langle\sigma\rangle^i \supset \langle\eta\rangle^{i-1}} [\langle\eta\rangle : \langle\sigma\rangle] t^{d(\langle\sigma\rangle) - d(\langle\eta\rangle)} f(\langle\sigma\rangle).$$

The discussion from Section 1 can be continued, and supplies us with $L_t^2\mathcal{H}^i(\Sigma_d)$. Now we wish to bring in the Hecke algebra. We pick (arbitrarily) orientations of the cells $\langle T\rangle$ ($T \in \mathcal{F}$), and extend these to orientations of all cells in Σ_d as follows: $w\langle T\rangle$ is the oriented cell which is the image of the oriented cell $\langle T\rangle$ by w, with orientation changed by a factor of $(-1)^{d(w)}$. Using these orientations, we identify $L_t^2 C^*(\Sigma_d)$ with $\oplus_{T\in\mathcal{F}} L^2(W\langle T\rangle, \mu_t)$. For every $T \in \mathcal{F}$ we define a map $\psi_T\colon L^2(W\langle T\rangle, \mu_t) \to L_t^2$ by the formula

$$\psi_T(f) = \sum_{w\in W^T} f(w\langle T\rangle)(-1)^{d(w)} \sqrt{W_T(t^{-1})}\, \delta_w h_T, \tag{4–1}$$

where $W^T = \{w \in W \mid \forall u \in W_T, d(wu) \geq d(w)\}$ (the set of T–reduced elements), and

$$h_T = \frac{1}{W_T(t^{-1})} \sum_{u\in W_T} (-t)^{-d(u)} \delta_u. \tag{4–2}$$

Putting together these maps we get a map $\Psi\colon L_t^2 C^*(\Sigma_d) \to \oplus_{T\in\mathcal{F}} L_t^2$.

Lemma 4.1 (1) *For all $s \in T$ we have $\delta_s h_T = -h_T$.*

(2) *For all $u \in W_T$ we have $\delta_u h_T = (-1)^{d(u)} h_T$.*

(3) *For all $U \subseteq T$ we have $h_U h_T = h_T$.*

Proof (1) Let $w \in W$ be such that $d(sw) > d(w)$. Then $\delta_s\delta_w = \delta_{sw}$ (by (2–3)). We then have

$$\delta_s(\delta_w - \frac{1}{t}\delta_{sw}) = \delta_{sw} - \frac{1}{t}(\delta_s\delta_s)\delta_w = \delta_{sw} - \frac{1}{t}(t\delta_1 + (t-1)\delta_s)\delta_w$$
$$= (1 - \frac{t-1}{t})\delta_{sw} - \delta_w = -(\delta_w - \frac{1}{t}\delta_{sw})$$

Since h_T is a linear combination of expressions of the form $\delta_w - \frac{1}{t}\delta_{sw}$, (1) follows.

(2) Follows from (1) by induction on $d(u)$.

(3)
$$h_U h_T = \frac{1}{W_U(t^{-1})}\sum_{u\in W_U}(-t)^{-d(u)}\delta_u h_T$$
$$= \frac{1}{W_U(t^{-1})}\sum_{u\in W_U}(-t)^{-d(u)}(-1)^{d(u)} h_T$$
$$= \frac{1}{W_U(t^{-1})}\left(\sum_{u\in W_U} t^{-d(u)}\right) h_T = h_T$$
□

Lemma 4.2 (1) *For every $T \in \mathcal{F}$ the map ψ_T is an isometric embedding.*

(2) *The orthogonal projection of L^2_t onto the image of ψ_T is given by right Hecke t–multiplication by h_T.*

Proof (1) The squared norm of a summand from the right hand side of (4–1) is

$$\| f(w\langle T\rangle)(-1)^{d(w)}\sqrt{W_T(t^{-1})}\,\delta_w h_T\|_t^2 = |f(w\langle T\rangle)|^2 W_T(t^{-1})\|\delta_w h_T\|_t^2.$$

Since w is T–reduced, we have $\delta_w\delta_u = \delta_{wu}$ for all $u \in W_T$. Therefore

$$\|\delta_w h_T\|_t^2 = \left\|\frac{1}{W_T(t^{-1})}\sum_{u\in W_T}(-t)^{-d(u)}\delta_{wu}\right\|_t^2 = \left|\frac{1}{W_T(t^{-1})}\right|^2\sum_{u\in W_T}|-t|^{-2d(u)}t^{d(wu)}$$
$$= t^{d(w)}\frac{1}{W_T(t^{-1})^2}\sum_{u\in W_T}t^{-d(u)} = t^{d(w)}\frac{1}{W_T(t^{-1})}.$$

(2) Due to $h_T h_T = h_T$ and $h_T^* = h_T$, right Hecke t–multiplication by h_T is an orthogonal projection. Let $w \in W$; write $w = vu$ where $u \in W_T$ and v is T–reduced. Then $\delta_w h_T = \delta_v\delta_u h_T = (-1)^{d(u)}\delta_v h_T$. This shows that image of the space of finitely supported functions (on $W\langle T\rangle$) under ψ_T is equal to the image of the space of finitely supported functions (on W) under right Hecke t–multiplication by h_T. Since ψ_T is isometric, the L^2_t–completions of these images also coincide. □

Denote by $\mathcal{L}$ the algebra U_t acting diagonally on the left on $\oplus_{T\in\mathcal{F}} L^2_t$, and by $\mathcal{R}$ its commutant $M_{|\mathcal{F}|}(\mathbf{C}) \otimes V_t$ (acting on the right). It follows from Lemma 4.2 that the image of Ψ is $\mathcal{L}$–invariant. In other words, we have a U_t–module structure on $L^2_t C^*(\Sigma_d)$, defined by the condition that the isometric embedding $\Psi\colon L^2_t C^*(\Sigma_d) \to \oplus L^2_t$ is a morphism of U_t–modules. Thus, we think of $L^2_t C^*(\Sigma_d)$ as of a submodule of $\oplus_{T\in\mathcal{F}} L^2_t$.

Lemma 4.3 *The map $\delta\colon L^2_t C^*(\Sigma_d) \to L^2_t C^*(\Sigma_d)$ is (a restriction of) an element of $\mathcal{R}$. For $U \subset T \in \mathcal{F}$ satisfying $|T| = |U| + 1$, the UT–entry of this element is*

$$[\langle T\rangle : \langle U\rangle]\sqrt{\frac{W_T(t^{-1})}{W_U(t^{-1})}}\, h_T$$

Proof Consider a pair of cells $w\langle U\rangle$, $w\langle T\rangle$. We have $[w\langle T\rangle : w\langle U\rangle] = [\langle T\rangle : \langle U\rangle]$. We can assume that w is U–reduced, and write it as vu, where v is T–reduced and $u \in W_T$. Let $f \in L^2_t C^{\dim\langle U\rangle}(\Sigma_d)$. The summand in $\psi_U(f)$ corresponding to the cell $w\langle U\rangle$ is

$$f(w\langle U\rangle)(-1)^{d(w)}\sqrt{W_U(t^{-1})}\,\delta_w h_U.$$

The summand in $\psi_T(\delta f)$ corresponding to the contribution of $f(w\langle U\rangle)$ to $(\delta f)(w\langle T\rangle)$ is

$$[\langle T\rangle : \langle U\rangle] f(w\langle U\rangle)(-1)^{d(v)}\sqrt{W_T(t^{-1})}\,\delta_v h_T.$$

Now $\delta_w h_U h_T = \delta_w h_T = \delta_v \delta_u h_T = (-1)^{d(u)}\delta_v h_T$, and the lemma follows. □

Corollary 4.4 *The subspaces $L^2_t C^i(\Sigma_d)$, $L^2_t \mathcal{H}^i(\Sigma_d)$, $\ker \delta^i$, $\ker \partial^t_i$, $\overline{\operatorname{im}\delta^i}$ and $\overline{\operatorname{im}\partial^i_t}$ of $\oplus_{T\in\mathcal{F}} L^2_t$ are $\mathcal{L}$–invariant; therefore, orthogonal projections onto these spaces are elements of $\mathcal{R}$.*

5 Invariance

In this section we prove that $L^2_t H^*(\Sigma_d) \simeq L^2_t H^*(\Sigma_{st})$ ($\simeq L^2_t H^*(\Sigma_{ghd})$, if the latter exists) as U_t–modules. It will be convenient for us to work with homology rather than cohomology; since both are isomorphic to the U_t–module of harmonic cochains, it makes no difference.

We start by fixing orientation conventions. Let us pick arbitrary orientations of the dual cells $\langle T\rangle$ for all $T \in \mathcal{F}$. We extend these orientations to all dual cells as in Section 4 ($w\langle T\rangle$ is oriented by $(-1)^{d(w)}$ times the orientation of $\langle T\rangle$ pushed forward by w). For $T \in \mathcal{F}$ of cardinality k, let $\langle T\rangle \cap D^{(k)}$ be the set of all k–simplices of Σ_{st} contained in $\langle T\rangle \cap D$. We orient every element of $\langle T\rangle \cap D^{(k)}$ by the restriction of the chosen orientation of $\langle T\rangle$. We then extend these orientations W–equivariantly (to a part of Σ_{st}), and put arbitrary equivariant orientations on the rest of Σ_{st}. Notice that if a k–simplex σ is contained in $w\langle T\rangle$ (where T has cardinality k), then the orientation of σ agrees with $(-1)^{d(\sigma)}$ times that of $w\langle T\rangle$. Orientations being chosen, we treat (co)chains as functions on the set of cells/simplices.

We define a topological embedding of Hilbert spaces $\theta\colon L^2_tC^*(\Sigma_d) \to L^2_tC^*(\Sigma_{st})$.

Definition Let $f \in L^2_tC^k(\Sigma_d)$, $\sigma \in \Sigma^{(k)}_{st}$.

(1) If there exists $\langle\alpha\rangle \in \Sigma^{(k)}_d$ such that $\sigma \subseteq \langle\alpha\rangle$ (there is at most one such $\langle\alpha\rangle$), then

$$\theta f(\sigma) = (-1)^{d(\sigma)} t^{d(\langle\alpha\rangle)-d(\sigma)} f(\langle\alpha\rangle).$$

(2) If there is no $\langle\alpha\rangle$ as in (1), we put $\theta f(\sigma) = 0$.

Lemma 5.1

$$\partial^t\theta = \theta\partial^t.$$

Proof We will show that for all $f \in L^2_tC^k(\Sigma_d)$, $\sigma \in \Sigma^{(k)}_{st}$ we have $\partial^t\theta f(\sigma) = \theta\partial^t f(\sigma)$. There are two cases to consider.

(1) Suppose that there exists $\langle\alpha\rangle \in \Sigma^{(k)}_d$ such that $\sigma \subseteq \langle\alpha\rangle$. Then

$$\begin{aligned}
\theta\partial^t f(\sigma) &= (-1)^{d(\sigma)} t^{d(\langle\alpha\rangle)-d(\sigma)} \partial^t f(\langle\alpha\rangle)\\
&= (-1)^{d(\sigma)} t^{d(\langle\alpha\rangle)-d(\sigma)} \sum_{\langle\beta\rangle^{k+1}\supset\langle\alpha\rangle} [\langle\beta\rangle : \langle\alpha\rangle] t^{d(\langle\beta\rangle)-d(\langle\alpha\rangle)} f(\langle\beta\rangle)\\
&= (-1)^{d(\sigma)} \sum_{\langle\beta\rangle^{k+1}\supset\langle\alpha\rangle} [\langle\beta\rangle : \langle\alpha\rangle] t^{d(\langle\beta\rangle)-d(\sigma)} f(\langle\beta\rangle). \qquad (5–1)
\end{aligned}$$

On the other hand,

$$\partial^t\theta f(\sigma) = \sum_{\tau^{k+1}\supset\sigma} [\tau : \sigma] t^{d(\tau)-d(\sigma)} \theta f(\tau). \qquad (5–2)$$

Notice that if $\theta f(\tau) \neq 0$ then there exists a dual cell $\langle\beta\rangle^{k+1} \supset \tau$. Such $\langle\beta\rangle$ is unique and $\langle\tau\rangle$ is the only $(k+1)$–simplex in $\langle\beta\rangle$ with face $\langle\sigma\rangle$. Therefore (5–2) equals

$$\sum_{\langle\beta\rangle^{k+1}\supset\langle\alpha\rangle} [\tau : \sigma] t^{d(\tau)-d(\sigma)} (-1)^{d(\tau)} t^{d(\langle\beta\rangle)-d(\tau)} f(\langle\beta\rangle)$$

$$= \sum_{\langle\beta\rangle^{k+1}\supset\langle\alpha\rangle} [\tau:\sigma](-1)^{d(\tau)} t^{d(\langle\beta\rangle)-d(\sigma)} f(\langle\beta\rangle). \tag{5–3}$$

Now (5–3) and (5–1) are equal because $[\tau:\sigma] = (-1)^{d(\tau)}(-1)^{d(\sigma)}[\langle\beta\rangle : \langle\alpha\rangle]$.

(2) The smallest dual cell $\langle\alpha\rangle$ containing σ is of dimension $m > k$. Then $\theta\partial^t f(\sigma) = 0$. On the other hand,

$$\partial^t\theta f(\sigma) = \sum_{\tau^{k+1}\supset\sigma} [\tau:\sigma] t^{d(\tau)-d(\sigma)} \theta f(\tau).$$

Let $\tau^{k+1}\supset\sigma$, and let $\langle\beta\rangle\supset\langle\alpha\rangle$ be the smallest dual cell containing τ. If $\theta f(\tau)\neq 0$, then $\dim\langle\beta\rangle = k+1$, which forces $\langle\beta\rangle = \langle\alpha\rangle$ and $m = \dim\langle\alpha\rangle = k+1$. Thus, we are reduced to the case $m = k+1$. In this case, there are exactly two simplices $\sigma_\pm \in \Sigma_{st}^{(k+1)}$, $\sigma_\pm \subset \langle\alpha\rangle$, $\sigma_\pm \supset \sigma$. Since $\sigma_\pm$ is oriented by $(-1)^{d(\sigma_\pm)}$ times the orientation of $\langle\alpha\rangle$, we have

$$(-1)^{d(\sigma_+)}[\sigma_+ : \sigma] = -(-1)^{d(\sigma_-)}[\sigma_- : \sigma]. \tag{5–4}$$

Therefore

$$\begin{aligned}
\partial^t\theta f(\sigma) &= [\sigma_+ : \sigma] t^{d(\sigma_+)-d(\sigma)} \theta f(\sigma_+) + [\sigma_- : \sigma] t^{d(\sigma_-)-d(\sigma)} \theta f(\sigma_-) \\
&= [\sigma_+ : \sigma] t^{d(\sigma_+)-d(\sigma)} (-1)^{d(\sigma_+)} t^{d(\langle\alpha\rangle)-d(\sigma_+)} f(\langle\alpha\rangle) \\
&\quad + [\sigma_- : \sigma] t^{d(\sigma_-)-d(\sigma)} (-1)^{d(\sigma_-)} t^{d(\langle\alpha\rangle)-d(\sigma_-)} f(\langle\alpha\rangle) \\
&= ((-1)^{d(\sigma_+)}[\sigma_+ : \sigma] + (-1)^{d(\sigma_-)}[\sigma_- : \sigma]) t^{d(\langle\alpha\rangle)} f(\langle\alpha\rangle) \\
&= 0. \qquad (5\text{–}5)
\end{aligned}$$

□

Lemma 5.2 *θ is a morphism of U_t–modules.*

Proof The U_t–module structures on $L^2_t C^k(\Sigma_d)$ and on $L^2_t C^k(\Sigma_{st})$ are defined via embeddings Ψ and Φ. We will compare Ψ and $\Phi\circ\theta$. Let $f \in L^2_t C^k(\Sigma_d)$; $\Psi(f)$ is a collection of $\psi_T(f)$, where

$$\psi_T(f) = \sum_{w\in W^T} f(w\langle T\rangle)(-1)^{d(w)} \sqrt{W_T(t^{-1})}\,\delta_w h_T. \tag{5–6}$$

The part of θf corresponding to $\psi_T(f)$ is supported by the set of W–translates of simplices $\sigma \in \langle T\rangle \cap D^{(k)}$, and is mapped by Φ into $\oplus_{\sigma\in\langle T\rangle\cap D^{(k)}} L^2_t$. The component indexed by σ is $\sum_{w\in W} \theta f(w\sigma)\delta_w$ (notice that the stabiliser of σ is trivial), ie,

$$\sum_{w\in W} (-1)^{d(w\langle T\rangle)} t^{d(w\langle T\rangle)-d(w\sigma)} f(w\langle T\rangle)\delta_w. \tag{5–7}$$

Comparing (5–6) and (5–7) with the help of (4–2), we get that $\psi_T(f)$ agrees with (every component of) the corresponding part of $\Phi(\theta f)$, up to a multiplicative factor of $\sqrt{W_T(t^{-1})}$. This implies the lemma. □

Theorem 5.3 *The map θ induces an isomorphism of U_t–modules $L^2_t H_*(\Sigma_d) \simeq L^2_t H_*(\Sigma_{st})$.*

Proof Lemmas 5.1 and 5.2 imply that θ induces a morphism of U_t–modules on homology. We have to check that it is an isomorphism of vector spaces.

Let K_* be the image of θ. It is a subcomplex of $(L^2_t C_*(\Sigma_{st}), \partial^t)$. A k–chain $c \in L^2_t C_*(\Sigma_{st})$ is in K_* if and only if the following two conditions hold:

(1) c is supported by the union of k–dimensional dual cells: $\bigcup \Sigma_d^{(k)}$;

(2) if $\sigma^k, \tau^k \subseteq \langle \alpha \rangle^k$, then $c(\sigma) = (-t)^{d(\tau)-d(\sigma)} c(\tau)$.

We need to show that the inclusion $K_* \hookrightarrow L^2_t C_*(\Sigma_{st})$ induces an isomorphism on (reduced) homology.

Let $m_t \colon L^2_t C_*(\Sigma_{st}) \to L^2_{t^{-1}} C_*(\Sigma_{st})$ be the isomorphism (of Hilbert spaces) $m_t f(\sigma) = t^{d(\sigma)} f(\sigma)$. Instead of working directly with K_*, $L^2_t C_*(\Sigma_{st})$ and ∂^t, we will work with $L_* = m_t(K_*)$, $E_* = L^2_{t^{-1}} C_*(\Sigma_{st}) = m_t(L^2_t C_*(\Sigma_{st}))$ and $\partial = m_t \partial^t m_t^{-1}$. The advantage is that

$$\begin{aligned}
\partial g(\sigma) &= m_t \partial^t m_t^{-1} g(\sigma) = t^{d(\sigma)} \partial^t m_t^{-1} g(\sigma) \\
&= t^{d(\sigma)} \sum_{\tau^{k+1} \supset \sigma} [\tau : \sigma] t^{d(\tau)-d(\sigma)} m_t^{-1} g(\tau) \qquad (5\text{–}8) \\
&= \sum_{\tau^{k+1} \supset \sigma} [\tau : \sigma] t^{d(\tau)} t^{-d(\tau)} g(\tau) = \sum_{\tau^{k+1} \supset \sigma} [\tau : \sigma] g(\tau).
\end{aligned}$$

To check whether $c \in E_*$ is in L_* we use (1) and the following version of (2):

(2′) if $\sigma^k, \tau^k \subseteq \langle \alpha \rangle^k$, then $c(\sigma) = (-1)^{d(\tau)-d(\sigma)} c(\tau)$.

Lemma 5.4 *Let $c \in E_k$. If $\partial c \in L_*$, then there exists a $d \in E_{k+1}$ such that $c - \partial d \in L_*$. Moreover, there is a constant C depending only on W and t such that d can be chosen so that $\|d\| \le C \|c\|$.*

Proof Each dual cell $\langle \alpha \rangle$ is a disc; we denote by $\operatorname{int}\langle \alpha \rangle$ its interior, and by $\operatorname{bd}\langle \alpha \rangle$ its boundary. We construct, by descending induction on m ($m \ge k$), cochains $d_m \in E_{k+1}$ such that $c - \partial d_m$ is supported by the union of dual cells of dimensions at most m.

For $m \geq \dim \Sigma$ we put $d_m = 0$. Suppose that d_m is already constructed, where $m > k$. For every dual m–cell $\langle\alpha\rangle$, let c_α be the restriction of $c - \partial d_m$ to $\langle\alpha\rangle$ (ie, if $c - \partial d_m = \sum a_\sigma \sigma$, then $c_\alpha = \sum_{\sigma \subseteq \langle\alpha\rangle} a_\sigma \sigma$). Let $\sigma^k \cap \operatorname{int}\langle\alpha\rangle \neq \varnothing$. Then σ appears in ∂c_α and in $\partial c = \partial(c - \partial d)$ with the same coefficient, due to the inductive assumption. But, since $\partial c \in L_*$, this coefficient is 0. As a result, $c_\alpha \in Z_k(\langle\alpha\rangle, \operatorname{bd}\langle\alpha\rangle)$. Since $H_k(\langle\alpha\rangle, \operatorname{bd}\langle\alpha\rangle) = 0$ (recall that $m = \dim \langle\alpha\rangle > k$), we can find $d_\alpha \in C_{k+1}(\langle\alpha\rangle)$ such that $c_\alpha - \partial d_\alpha \in C_k(\operatorname{bd}\langle\alpha\rangle)$. Moreover, we can choose d_α so that $\|d_\alpha\| \leq C_1 \|c_\alpha\|$, for some constant C_1 depending only on W and t. Due to uniform local finiteness of Σ, we deduce $\|\sum_{\langle\alpha\rangle} d_\alpha\| \leq C_2 \|c\|$ for some constant C_2. We put $d_{m-1} = d_m + \sum_{\langle\alpha\rangle \in \Sigma_d^{(m)}} d_\alpha$, and $d = d_k$.

The estimate $\|d\| \leq C\|c\|$ clearly follows from the construction. The chain $c - \partial d = \sum b_\sigma \sigma$ is supported by the union of dual cells of dimensions at most k. Let us check that it satisfies the condition $(2')$. Suppose that $\sigma^{k-1} \cap \operatorname{int}\langle\alpha\rangle^k \neq \varnothing$. There are exactly two k–simplices $\sigma_\pm \subset \langle\alpha\rangle$ such that $\sigma \subset \sigma_\pm$. The coefficient of σ in $\partial(c - \partial d) = \partial c$ is 0 (because $\partial c \in L_*$), and, on the other hand, is equal to $[\sigma_+ : \sigma] b_{\sigma_+} + [\sigma_- : \sigma] b_{\sigma_-}$. Using (5–4) we get $b_{\sigma_+} = (-1)^{d(\sigma_+) - d(\sigma_-)} b_{\sigma_-}$. This holds for all σ^{k-1} satisfying $\sigma^{k-1} \cap \operatorname{int}\langle\alpha\rangle^k \neq \varnothing$, which implies that $c - \partial d$ satisfies $(2')$. Hence $c - \partial d \in L_*$. The lemma is proved. □

We are ready to check that the inclusion $\iota\colon L_* \hookrightarrow E_*$ induces an isomorphism ι_* on (reduced) homology. To show that ι_* is surjective, suppose that $c \in E_*$ is closed: $\partial c = 0$. Then $\partial c \in L_*$, and, by Lemma 5.4, there exists $d \in E_*$ such that $c - \partial d \in L_*$. We get $[c] = \iota_*[c - \partial d]$.

To show that ι_* is 1–1, suppose that $l \in L_*$, $\partial l = 0$ and $\iota_*[l] = 0$, ie, $l = \lim \partial e_n$ for some sequence of $e_n \in E_*$. Applying Lemma 5.4 to $c = l - \partial e_n$, we get that there exist $f_n \in E_*$, $f_n \to 0$ such that $l - \partial e_n - \partial f_n \in L_*$. But, since $l \in L_*$, we deduce that $\partial(e_n + f_n) \in L_*$. Now we apply Lemma 5.4 to $c = e_n + f_n$ to get $g_n \in E_*$ such that $h_n = e_n + f_n - \partial g_n \in L_*$. We have

$$\partial h_n = \partial e_n + \partial f_n - \partial\partial g_n.$$

The last term is 0, the middle term converges to 0 since ∂ is bounded and $f_n \to 0$, so that, finally,

$$\lim \partial h_n = \lim \partial e_n = l.$$

This means that $[l] = 0$ in $H_*(L_*)$.

We have shown that $(L_*, \partial) \hookrightarrow (E_*, \partial)$ induces an isomorphism on homology. Therefore so does the inclusion $(K_*, \partial^t) \hookrightarrow (L^2_t C_*(\Sigma_{st}), \partial^t)$. The theorem follows. □

Let us now assume that D is a generalised homology disc. Then, along the same lines as above, one shows $L^2_t H^*(\Sigma_{st}) \simeq L^2_t H^*(\Sigma_{ghd})$ (as U_t–modules). More precisely, one defines $\theta\colon L^2_t H^*(\Sigma_{ghd}) \to L^2_t H^*(\Sigma_{st})$ by $\theta f(\sigma) = f(\alpha)$ if $\sigma^k \subseteq \alpha^k \in \Sigma^{(k)}_{ghd}$, and $\theta f(\sigma) = 0$ if no such α^k exists. The proof of $\partial^t \theta = \theta \partial^t$ is similar to that of Lemma 5.1, and it is clear that θ is a U_t–morphism. A chain $c \in L^2_t C_k(\Sigma_{st})$ is in the image K_* of θ if and only if

(1) c is supported by $\bigcup \Sigma^{(k)}_{ghd}$;

(2) if $\sigma^k, \tau^k \subseteq \alpha^k \in \Sigma^{(k)}_{ghd}$, then $c(\sigma) = c(\tau)$.

These conditions do not change under m_t, and the rest of the proof of Theorem 5.3 can be repeated with dual cells replaced by cells of Σ_{ghd} (the only other change will be $[\sigma_+ : \sigma] = -[\sigma_- : \sigma]$ instead of the more complicated (5–4)). We get

Theorem 5.5 *Let $(D, \partial D)$ be a generalised homology disc. Then we have the following isomorphisms of (graded) U_t–modules: $L^2_t H^*(\Sigma_{ghd}) \simeq L^2_t H^*(\Sigma_{st}) \simeq L^2_t H^*(\Sigma_d)$.*

6 Poincaré Duality

Let us define a map $D\colon L^2_t \to L^2_{t^{-1}}$ by

$$D\left(\sum a_w \delta_w\right) = \sum (-t)^{d(w)} a_w \delta_w. \tag{6–1}$$

Direct calculation shows that D is an isometric isomorphism of Hilbert spaces. Notice that D maps $\mathbf{C}_t[W]$ onto $\mathbf{C}_{t^{-1}}[W]$. It is easy to check that D preserves the relations defining Hecke multiplication: if $d(ws) > d(w)$, then

$$D(\delta_w \delta_s) = D(\delta_{ws}) = (-t)^{d(ws)} \delta_{ws} = (-t)^{d(w)} \delta_w (-t\delta_s) = D(\delta_w) D(\delta_s);$$

if $d(ws) < d(w)$, then

$$\begin{aligned} D(\delta_w \delta_s) &= D(t\delta_{ws} + (t-1)\delta_w) = t(-t)^{d(ws)} \delta_{ws} + (t-1)(-t)^{d(w)} \delta_w \\ &= (-t)^{d(w)+1} t^{-1} \delta_{ws} + (-t)^{d(w)+1} (t^{-1} - 1)\delta_w = (-t)^{d(w)} \delta_w (-t)\delta_s \\ &= D(\delta_w) D(\delta_s). \end{aligned}$$

Hence, D restricts to an isometric isomorphism of Hilbert algebras $\mathbf{C}_t[W]$ and $\mathbf{C}_{t^{-1}}[W]$. In particular, D preserves products: for all $x, y \in \mathbf{C}_t[W]$, we have $D(xy) = D(x)D(y)$. Passing to limits with y in the norm $\|\cdot\|_t$, we deduce that the map $D\colon L^2_t \to L^2_{t^{-1}}$ is a morphism of left modules over the algebra morphism

$D: \mathbf{C}_t[W] \to \mathbf{C}_{t^{-1}}[W]$. Then passing to limits with x in the weak operator topology, we deduce that $D: L^2_t \to L^2_{t^{-1}}$ is a morphism of left modules over the von Neumann algebra isomorphism $D: U_t \to U_{t^{-1}}$. Analogous statements hold for the right module structures. Finally, since D preserves the coefficient of δ_1, it preserves dimensions of (left) submodules of L^2_t.

Theorem 6.1 *Suppose that the pair $(D, \partial D)$ is a generalised homology n–disc. Then $b^i_t = b^{n-i}_{t^{-1}}$.*

Proof There is a bijection $D_T \leftrightarrow \langle T\rangle$, where $T \in \mathcal{F}$; it can be unambiguously extended to $wD_T \leftrightarrow w\langle T\rangle$, a natural bijection between i–cells of Σ_{ghd} and $(n-i)$–cells of Σ_d. When w and T are not specified we write simply $\sigma \leftrightarrow \langle\sigma\rangle$. A property of this bijection which is crucial for us is: the codimension 1 faces of $\langle\tau^{i-1}\rangle$ are $\langle\sigma^i\rangle$, for $\sigma \supseteq \tau$. Let us pick orientations of all faces D_T of D, and extend them equivariantly to orientations of all cells η in Σ_{ghd}. Then we orient each dual cell $\langle\eta\rangle$ dually to the chosen orientation of η (dually with respect to a chosen orientation of Σ). These orientations are of the kind considered in Section 4. With these choices we have $[\langle\sigma\rangle : \langle\tau\rangle] = \pm[\sigma : \tau]$, with the sign depending only on the dimensions of σ, τ (and on n, which is fixed in our discussion).

We define the duality map $\mathcal{D}: L^2_t C^*(\Sigma_{ghd}) \to L^2_{t^{-1}} C^{n-*}(\Sigma_d)$ by

$$\mathcal{D}f(\langle\sigma\rangle) = t^{d(\sigma)} f(\sigma). \tag{6–2}$$

The map $\mathcal{D}$ is an isometry of Hilbert spaces. We will now check that $\delta^{n-i}\mathcal{D} = \pm\mathcal{D}\partial^t_i$ (the sign depending only on i, n):

$$\delta(\mathcal{D}f)(\langle\tau^{i-1}\rangle) = \sum_{\sigma^i \supset \tau^{i-1}} [\langle\sigma\rangle : \langle\tau\rangle](\mathcal{D}f)(\langle\sigma\rangle) = \pm \sum_{\sigma^i \supset \tau^{i-1}} [\sigma : \tau] t^{d(\sigma)} f(\sigma)$$

while

$$\mathcal{D}(\partial^t f)(\langle\tau^{i-1}\rangle) = t^{d(\tau)}(\partial^t f)(\tau^{i-1}) = t^{d(\tau)} \sum_{\sigma^i \supset \tau^{i-1}} [\sigma : \tau] t^{d(\sigma)-d(\tau)} f(\sigma)$$

which proves what we wanted. It follows that $\mathcal{D}$ intertwines also the adjoint operators; consequently, it restricts to an isomorphism $\mathcal{D}: L^2_t \mathcal{H}^*(\Sigma_{ghd}) \to L^2_{t^{-1}} \mathcal{H}^{n-*}(\Sigma_d)$.

We still have to check that the Hecke dimensions of these spaces are the same.

To this end, let us now consider $L^2_t C^*(\Sigma_{ghd})$ as a subspace of $\oplus_{T\in\mathcal{F}} L^2_t$ via the embedding Φ_t (see Section 3), and $L^2_{t^{-1}} C^{n-*}(\Sigma_d)$ as a subspace of $\oplus_{T\in\mathcal{F}} L^2_{t^{-1}}$ via the embedding $\Psi_{t^{-1}}$ (see Section 4). We will check that $\mathcal{D}$ can be regarded as the restriction of the map D (applied componentwise in $\oplus_{T\in\mathcal{F}} L^2_t$); it will follow that $\mathcal{D}$ preserves dimensions. Let $f \in L^2(WD_T, \mu_t)$ be a part of a cochain on Σ_{ghd}. Then

$$\phi_T(f) = \sqrt{W_T(t)} \sum_{w\in W^T} f(wD_T)\delta_w p_T(t),$$

where $p_T(t) = \frac{1}{W_T(t)} \sum_{u\in W_T} \delta_u$. Since

$$\begin{aligned} D(p_T(t)) &= \frac{1}{W_T(t)} \sum_{u\in W_T} (-t)^{d(u)}\delta_u \\ &= \frac{1}{W_T((t^{-1})^{-1})} \sum_{u\in W_T} (-t^{-1})^{-d(u)}\delta_u = h_T(t^{-1}), \end{aligned}$$

we have

$$D(\phi_T(f)) = \sum_{w\in W^T} f(wD_T)\sqrt{W_T(t)}(-t)^{d(w)}\delta_w h_T(t^{-1}). \tag{6–3}$$

On the other hand, $(\mathcal{D}f)(w\langle T\rangle) = t^{d(w\langle T\rangle)} f(wD_T)$, and

$$\psi_T(\mathcal{D}f) = \sum_{w\in W^T} t^{d(w\langle T\rangle)} f(wD_T)(-1)^{d(w)}\sqrt{W_T(t)}\,\delta_w h_T(t^{-1}). \tag{6–4}$$

Since for $w\in W^T$ we have $d(w\langle T\rangle) = d(w)$, (6–3) and (6–4) are equal. □

Remark The above proof shows that $\mathcal{D}$ is an isomorphism of the U_t–module $L^2_t\mathcal{H}^*(\Sigma_{ghd})$ and the $U_{t^{-1}}$–module $L^2_{t^{-1}}\mathcal{H}^{n-*}(\Sigma_d)$, over the algebra isomorphism $D\colon U_t \to U_{t^{-1}}$.

7 Calculation of b^0_t

Theorem 7.1 *For $t < \rho_W$ we have $b^0_t = \frac{1}{W(t)}$; for $t \ge \rho_W$ we have $b^0_t = 0$.*

Proof We will use the cell structure Σ_d. Vertices of Σ_d are located at the centres of chambers wD, thus they are in bijection with W. We embed $L^2_tC^0(\Sigma_d)$ into L^2_t by $(\Psi c)(w) = (-1)^{d(w)}c(w\langle\varnothing\rangle)$. This embedding maps all harmonic 0–cochains to constant functions, multiples of $\mathbf{1}(w) = 1$. The square of the norm of $\mathbf{1}$ is $\sum_{w\in W} t^{d(w)}$. It is finite and equal to $W(t)$ for $t < \rho_W$, and infinite if $t \ge \rho_W$. The latter means that for $t \ge \rho_W$ we have $L^2_t\mathcal{H}^0(\Sigma_d) = 0$.

To find b^0_t for $t < \rho_W$ we need to identify the projection of δ_1 on $L^2_t\mathcal{H}^0(\Sigma_d)$; it is $C\mathbf{1}$, where

$$\langle \delta_1 - C\mathbf{1}, \mathbf{1}\rangle_t = 0.$$

This gives $C = \|\mathbf{1}\|_t^{-2} = \frac{1}{W(t)}$. In accordance with the procedure described at the end of Section 2, we find $b_t^0 = C = \frac{1}{W(t)}$. □

In view of Corollary 3.4, the above result makes it plausible to suspect that for $t < \rho_W$ we have $b_t^{>0} = 0$. In the next section we prove that this is true for right angled Coxeter groups.

8 Mayer–Vietoris sequence

In this section we limit our attention to right angled Coxeter groups. "Right angled" means that whenever two generators $s, s' \in S$ are related in the standard presentation, they in fact commute. If we join each pair of commuting generators by an edge, we get a graph with the set of vertices S. It is convenient to fill it, gluing in a simplex whenever we can see its 1–skeleton in the graph. The resulting simplicial complex is denoted L, and the Coxeter group W_L. The Davis chamber D can be identified with the cone CL' over the first barycentric subdivision of L. We say that a subcomplex $K \subseteq L$ is full, if whenever it contains all vertices of a simplex of L, it contains the simplex as well. Full subcomplexes K correspond to subsets of S and thus to special subgroups W_K of W_L. The Davis complex of W_K is naturally embedded in Σ_{W_L}: we first embed $D_K = CK'$ in $D_L = CL'$, and then extend W_K–equivariantly. We abbreviate Σ_{W_L} to Σ_L.

Let $L = L_1 \cup L_2$, where L_1, L_2 and (consequently) $L_0 = L_1 \cap L_2$ are full subcomplexes of L. We embed W_{L_i} into W_L, and Σ_{L_i} into Σ_L; then $\Sigma_L = W_L\Sigma_{L_1} \cup W_L\Sigma_{L_2}$, $W_L\Sigma_{L_1} \cap W_L\Sigma_{L_2} = W_L\Sigma_{L_0}$. We have a short exact sequence of cochain complexes

$$0 \to L^2_tC^*(\Sigma_L) \to L^2_tC^*(W_L\Sigma_{L_1}) \oplus L^2_tC^*(W_L\Sigma_{L_2}) \to L^2_tC^*(W_L\Sigma_{L_0}) \to 0,$$

from which we get the long Mayer–Vietoris sequence:

$$\begin{aligned}(8\text{–}1)\qquad \ldots \to L^2_tH^{i-1}(W_L\Sigma_{L_0}) \to & L^2_tH^i(\Sigma_L) \to \\ \to L^2_tH^i(W_L\Sigma_{L_1}) \oplus & L^2_tH^i(W_L\Sigma_{L_2}) \to L^2_tH^i(W_L\Sigma_{L_0}) \to \ldots\end{aligned}$$

Since we work with reduced cohomology, this sequence is only weakly exact (the kernels are closures of the images), see Lück [9, 1.22]. Still, if a term is preceded and followed by zero terms it has to be zero. Notice that $W_L\Sigma_{L_i}$ is the disjoint union of $w\Sigma_{L_i}$, where w runs through a set of representatives of W_{L_i}–cosets in W_L. The L^2_t norm on $w\Sigma_{L_i}$ is $t^{d/2}$ times the L^2_t norm on Σ_{L_i}, where d is the length of the shortest element of wW_{L_i}. In particular, if $L^2_tH^p(\Sigma_{L_i}) = 0$, then $L^2_tH^p(W_L\Sigma_{L_i}) = 0$.

Corollary 8.1 *Suppose that $b_t^{>0}(\Sigma_{L_i}) = 0$ for $i = 0, 1, 2$. Then $b_t^{>1}(\Sigma_L) = 0$.*

Theorem 8.2 *Let W be a right angled Coxeter group. For $t < \rho_W$ we have $b_t^0 = \chi_t = \frac{1}{W(t)}$ and $b_t^{>0} = 0$.*

Proof Let $W = W_L$. We argue by induction on the number of vertices of L.

(1) If L is a simplex, then $\Sigma_{L,d}$ is a cube; its L_t^2 cohomology coincides with the usual cohomology and is concentrated in dimension 0.

(2) If L is not a simplex, we can find two vertices $a, b \in L$ not connected by an edge; we put $L_1 = \bigcup\{\sigma \mid a \notin \sigma\}$, $L_2 = \bigcup\{\sigma \mid b \notin \sigma\}$ and $L_0 = L_1 \cap L_2$. These have fewer vertices than L, and so $L_t^2 H^{>0}(\Sigma_{L_i}) = 0$ for $t < \rho(W_{L_i})$ $(i = 0, 1, 2)$. Since $L_i \subset L$, we have $\rho(W_{L_i}) \geq \rho(W_L)$. Therefore we have $L_t^2 H^{>0}(\Sigma_{L_i}) = 0$ for $t < \rho(W_L)$. It follows from Corollary 8.1 that $L_t^2 H^{>1}(\Sigma_L) = 0$ (still for $t < \rho(W_L)$), while from Corollary 3.4 and Theorem 7.1 we conclude that

$$b_t^0(\Sigma_L) = \chi_t(\Sigma_L) = b_t^0(\Sigma_L) - b_t^1(\Sigma_L).$$

Thus $b_t^1(\Sigma_L) = 0$. □

Corollary 8.3 *Assume that L is a generalised homology $(n-1)$–sphere (ie, $(D, \partial D)$ is a generalised homology n–disc); then for $t < \frac{1}{\rho(W_L)}$ we have $b_t^n = 0$, while for $t > \frac{1}{\rho(W_L)}$ the L_t^2–cohomology is concentrated in dimension n and $b_t^n = (-1)^n \chi_t = \frac{(-1)^n}{W_L(t)}$.*

Proof This follows from Theorems 8.2 and 7.1 via Poincaré duality (Theorem 6.1). □

Proposition 8.4 *Let $K \subset L$ be a full subcomplex. The dimension of the $U_t(W_L)$–module $L_t^2 H^q(W_L \Sigma_K)$ is the same as the dimension of the $U_t(W_K)$–module $L_t^2 H^q(\Sigma_K)$ (ie, it is equal to $b_t^q(\Sigma_K)$).*

Proof A harmonic q–cochain on $W_L \Sigma_K = \bigcup\{w\Sigma_K \mid w \in W_L\}$ is the same thing as a collection of harmonic q–cochains on $w\Sigma_K$. In order to calculate dimensions, we embed everything in $V = \oplus_{\sigma \subset D_L} L_t^2(W_L)$. Let $\mathbf{1}_\sigma \in V$ have δ_1 as its coordinate with index σ, and 0 on all other coordinates. As we project $\mathbf{1}_{\sigma^q}$ on $L_t^2 \mathcal{H}^q(W_L \Sigma_K)$, we get in fact a harmonic cochain supported on Σ_K—harmonic cochains supported on other components of $W_L \Sigma_K$ are orthogonal to $\mathbf{1}_{\sigma^q}$, so also to its projection. We can as well project $\mathbf{1}_{\sigma^q}$ on $L_t^2 \mathcal{H}^q(\Sigma_K)$ inside $\oplus L_t^2(W_K)$, so that the projection matrices are the same (apart for the case $\sigma \not\subset K$, which gives 0 in the first case and does not appear in the second), and traces coincide. □

9 Chain homotopy contraction

In this section we will describe a simplicial version of the geodesic contraction of Σ with respect to the Moussong metric. We will consider the chain complex $C_*(\Sigma_{st})$ equipped with the boundary operator ∂ given by (5–8). Henceforth we write Σ for Σ_{st}, and we denote by b the barycentre of the basic chamber D. Recall that Σ can be equipped with a W–invariant, $CAT(0)$ metric d_M, the Moussong metric (Moussong [10]). From now on, all balls, geodesics etc. will be considered with respect to d_M (unless explicitly stated otherwise). Besides $CAT(0)$, the following property of the Moussong metric will be useful for us: for every $R > 0$ there exists a constant $N(R)$ such that any ball of radius R in Σ intersects at most $N(R)$ chambers.

Theorem 9.1 *There exists a linear map $H\colon C_*(\Sigma) \to C_{*+1}(\Sigma)$, and constants C, R, with the following properties:*

(a) *if $v \in \Sigma^{(0)}$, then $\partial H(v) = v - b$;*

(b) *if σ is a simplex of positive dimension, then $\partial H(\sigma) = \sigma - H(\partial\sigma)$;*

(c) *for every simplex σ, $\|H(\sigma)\|_{L^\infty} < C$;*

(d) *if γ is a geodesic from a vertex of a simplex σ to b, then $\operatorname{supp}(H(\sigma)) \subseteq B_R(\operatorname{image}(\gamma))$.*

Proof We will construct, for all integers $i \geq 0$, linear maps $h_i\colon C_*(\Sigma) \to C_*(\Sigma)$, $H_i\colon C_*(\Sigma) \to C_{*+1}(\Sigma)$ such that:

(1) $h_0 = \mathrm{id}$;

(2) $\partial h_i = h_i \partial$;

(3) $\partial H_i = h_i - H_i\partial - h_{i+1}$;

(4) $\exists C_k, \forall \sigma \in \Sigma^{(k)}, \forall i \geq 0, \|H_i(\sigma)\|_{L^\infty} < C_k$ and $\|h_i(\sigma)\|_{L^\infty} < C_k$;

(5) $\exists R_k, \forall \sigma \in \Sigma^{(k)}, \forall i \geq 0$, if γ is a geodesic from a vertex of σ to b, then $\operatorname{supp}(h_i(\sigma))$, $\operatorname{supp}(H_{i-1}(\sigma))$ (if $i > 0$) and $\operatorname{supp}(H_i(\sigma))$ are contained in the ball $B_{R_k}(\gamma(i))$ (or in $B_{R_k}(b)$, if $i > \operatorname{length}(\gamma)$);

(6) if $i \geq \operatorname{diam}(\sigma \cup \{b\})$, then $h_i(\sigma) = 0$ (unless $\dim \sigma = 0$, in which case $h_i(\sigma) = b$) and $H_i(\sigma) = 0$.

The construction will be by induction on the chain degree k. Throughout this proof, we will say that a family of chains is uniformly bounded if they have uniformly bounded support diameters and L^∞ norms. Let A be the length of the longest edge in Σ.

(1) $k = 0$

Let $v \in \Sigma^{(0)}$, let $\gamma_v\colon [0,l] \to \Sigma$ be a geodesic such that $\gamma_v(0) = v$, $\gamma_v(l) = b$. We put $h_0(v) = v$, $h_i(v) = b$ if $i \geq l$, and we choose a vertex within distance A from $\gamma_v(i)$ and declare it to be $h_i(v)$ in the remaining cases. We have $d(h_i(v), h_{i+1}(v)) \leq 1 + 2A$. Now, up to the action of W, there are only finitely many pairs of vertices (y,z) satisfying $d(y,z) < 1 + 2A$. In every W–orbit of such pairs we choose a pair (y,z) and we fix a 1–chain $H(y,z)$, $\partial H(y,z) = y - z$; we then extend H to the W–orbit of (y,z) using the W–action (making choices if stabilisers are non-trivial). In the case $y = z$ we choose $H(y,y) = 0$. Notice that the chosen 1–chains H are uniformly bounded. Finally, we put $H_i(v) = H(h_i(v), h_{i+1}(v))$.

(2) $k \to (k+1)$

Let $\sigma \in \Sigma^{(k+1)}$. Then, due to (2), $\partial h_i(\partial\sigma) = h_i(\partial\partial\sigma) = 0$. Thus, $h_i(\partial\sigma)$ is a cycle. Moreover, we claim that as we vary σ, the cycles $h_i(\partial\sigma)$ are uniformly bounded. In fact, as a consequence of (5), every simplex in the support of $h_i(\partial\sigma)$ is within R_k of one of the points $\gamma_v(i)$, where v runs through the vertices of σ, and, by $CAT(0)$ comparison, the $k+2$ points $\gamma_v(i)$ are within $2A$ of each other. Whence uniform boundedness of supports of $h_i(\partial\sigma)$. Uniform boundedness of L^∞ norms follows from (4). Up to the W–action on $C_k(\Sigma)$, there are only finitely many possible values of $h_i(\partial\sigma)$. As in step 1, we fix $(k+1)$–chains $h_i(\sigma)$, $\partial h_i(\sigma) = h_i(\partial\sigma)$, so that they are uniformly bounded (and are 0 whenever $h_i(\partial\sigma) = 0$).

To define $H_i(\sigma)$, we consider the chain $h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)$. It is a cycle:

$$\partial(h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)) = \partial h_i(\sigma) - \partial H_i(\partial\sigma) - \partial h_{i+1}(\sigma)$$

$$= h_i(\partial\sigma) - \big(h_i(\partial\sigma) - H_i(\partial\partial\sigma) - h_{i+1}(\partial\sigma)\big) - h_{i+1}(\partial\sigma) = 0.$$

Again, all such chains (as we vary σ) are uniformly bounded, and we can choose $H_i(\sigma)$, satisfying $\partial H_i(\sigma) = h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)$, in a uniformly bounded way. As before, we put $H_i(\sigma) = 0$ whenever we have to chose it so that it has boundary 0 (so as to satisfy (6)).

Now that we have a family of maps satisfying (1)–(6), we put $H(\sigma) = \sum_{i\geq 0} H_i(\sigma)$. The sum is always finite because of (6). The conditions (a)–(d) are easy to check: (a) and (b) follow from (1), (3) and (6); (c) follows from (4) and (5): since the supports of $H_i(\sigma)$ are uniformly bounded and "move along" a geodesic γ with constant speed as i grows, only a uniformly finite number of $H_i(\sigma)$ contribute to a coefficient of a fixed simplex τ in the chain $H(\sigma)$; moreover, because of (4), each contribution is smaller than $C_{\dim\sigma}$; (d) is a consequence of (5). □

10 Vanishing below ρ

Let H be a map as in Theorem 9.1.

Theorem 10.1 *Suppose that $t > \frac{1}{\rho_W}$. Then the map H extends to a bounded operator $H\colon L^2_t C_*(\Sigma) \to L^2_t C_{*+1}(\Sigma)$.*

Proof Unspecified summations will be over $\Sigma^{(k)}$. N_k will denote the number of k–simplices in a chamber.

Let $a = \sum a_\sigma \sigma \in L^2_t C_k(\Sigma)$. We know that for every simplex σ, $\|H(\sigma)\|_{L^\infty} < C$. Also

$$\sum |a_\sigma| = \sum |a_\sigma| t^{d(\sigma)/2} t^{-d(\sigma)/2} \le \left(\sum |a_\sigma|^2 t^{d(\sigma)}\right)^{1/2} \left(\sum t^{-d(\sigma)}\right)^{1/2}$$
$$\le \|a\|_t \left(N_k W(t^{-1})\right)^{1/2} < +\infty,$$

so that $\sum a_\sigma H(\sigma)$ is pointwise convergent to a chain $H(a) \in L^\infty C_{k+1}(\Sigma)$. We want to estimate $\|H(a)\|_t$. Let us write $\tau \prec \sigma$ if τ appears with non-zero coefficient in $H(\sigma)$. We have $|H(a)_\tau| \le \sum_{\sigma|\tau\prec\sigma} C|a_\sigma|$, so that

$$\sum |H(a)_\tau|^2 t^{d(\tau)} \le C^2 \sum_\tau \Bigg(\sum_{\sigma|\tau\prec\sigma} |a_\sigma|\Bigg)^2 t^{d(\tau)}$$

$$\le C^2 \sum_\tau \Bigg(\sum_{\sigma|\tau\prec\sigma} |a_\sigma| t^{d(\sigma)/2} t^{-\alpha\left(\frac{d(\sigma)-d(\tau)}{2}\right)} t^{-\beta\left(\frac{d(\sigma)-d(\tau)}{2}\right)}\Bigg)^2 \tag{10–1}$$

$$\le C^2 \sum_\tau \Bigg(\sum_{\sigma|\tau\prec\sigma} |a_\sigma|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)}\Bigg)\Bigg(\sum_{\sigma|\tau\prec\sigma} (t^{-\beta})^{d(\sigma)-d(\tau)}\Bigg).$$

Here α, β are positive numbers chosen so that $\alpha+\beta = 1$, $t^{-\beta} < \rho_W$.

Claim *There exists a constant C', independent of τ, such that*

$$\sum_{\sigma|\tau\prec\sigma} (t^{-\beta})^{d(\sigma)-d(\tau)} \le C' W(t^{-\beta}).$$

Proof Recall that A is the length of the longest edge in Σ, and $N(r)$ is the maximal number of chambers intersecting a ball of radius r. The claim follows from two observations.

(1) For $w_0 \in W$ let $E(w_0) = \{w \in W \mid d(w) = d(w_0) + d(w_0^{-1}w)\}$. In more geometric terms, $E(w_0)$ is the set of all w such that some gallery connecting D and wD passes through $w_0 D$. We have

$$\sum_{w \in E(w_0)} (t^{-\beta})^{d(w)-d(w_0)} = \sum_{w \in E(w_0)} (t^{-\beta})^{d(w_0^{-1}w)} \leq \sum_{w \in W} (t^{-\beta})^{d(w)} = W(t^{-\beta}).$$

(2) If $\tau \prec \sigma$, then τ is at distance at most R from a geodesic γ joining (a vertex of) σ and b. Let us consider the union U of all galleries joining D and a fixed chamber D' containing σ. Then U is the intersection of all half-spaces containing D and D' (see Ronan [11]). Since half-spaces are geodesically convex in d_M, we have $\gamma \subseteq U$. Consequently, every point of γ lies in a gallery joining D' and D. Therefore, if we put $B(\tau) = \{w_0 \mid w_0 D \cap B_R(\tau) \neq \varnothing\}$, then we have $\{\sigma \mid \tau \prec \sigma\} \subseteq \bigcup_{w_0 \in B(\tau)} E(w_0)D$.

Putting these together,

$$\begin{aligned}\sum_{\sigma \mid \tau \prec \sigma} (t^{-\beta})^{d(\sigma)-d(\tau)} &\leq \sum_{w_0 \in B(\tau)} t^{-\beta(d(w_0)-d(\tau))} \sum_{w \in E(w_0)} N_k (t^{-\beta})^{d(w)-d(w_0)} \\ &\leq \sum_{w_0 \in B(\tau)} t^{-\beta(d(w_0)-d(\tau))} N_k W(t^{-\beta}).\end{aligned}$$

Notice that $|d(w_0) - d(\tau)|$ does not exceed the gallery distance from $w_0 D$ to some chamber containing τ, and is therefore uniformly bounded. Also, the cardinality of $B(\tau)$ is bounded by $N(R + A)$. The claim is proved. □

Using the claim, we can continue the estimate (10–1):

$$\|H(a)\|_t^2 \leq C^2 C' W(t^{-\beta}) \sum_{\tau} \Bigg(\sum_{\sigma \mid \tau \prec \sigma} |a_\sigma|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)} \Bigg).$$

Now

$$\sum_{\tau} \Bigg(\sum_{\sigma \mid \tau \prec \sigma} |a_\sigma|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)} \Bigg) = \sum_{\sigma} \Bigg(|a_\sigma|^2 t^{d(\sigma)} \sum_{\tau \mid \tau \prec \sigma} (t^{-\alpha})^{d(\sigma)-d(\tau)} \Bigg),$$

so that the following lemma is all we need:

Lemma 10.2 *There exists a constant K independent of σ such that*

$$\sum_{\tau \mid \tau \prec \sigma} (t^{-\alpha})^{d(\sigma)-d(\tau)} < K.$$

Proof Since W acts on (Σ, d_M) isometrically, cocompactly and properly discontinuously, the word metric d on W is quasi-isometric to the metric d_M restricted to $W \simeq Wb \hookrightarrow \Sigma$. This implies that there exist constants M, m, L such that for any two points $y, z \in \Sigma$ and any chambers $D_y \ni y$, $D_z \ni z$, we have

$$M d_M(y,z) + L \geq d(D_y, D_z) \geq m d_M(y,z) - L, \tag{10–2}$$

where we put $d(wD, uD) = d(w,u) = d(w^{-1}u)$.

Let v be a vertex of σ, and let $\gamma\colon [0,l] \to \Sigma$ be a geodesic, $\gamma(0) = v$, $\gamma(l) = b$. To each $\tau \prec \sigma$ we can assign one of the points $\gamma(i)$ $(0 \leq i \leq \lfloor l \rfloor)$ in such a way that $d_M(\tau, \gamma(i)) < R+1$. The number of simplices to which we assign a given $\gamma(i)$ does not exceed $N(R+1)N_k$. Suppose that $\gamma(i)$ is assigned to τ. Let D_τ (resp. D_σ) be the chamber containing τ (resp. σ) such that $d(\tau) = d(D, D_\tau)$ (resp. $d(\sigma) = d(D, D_\sigma)$). Let D_i be a chamber containing $\gamma(i)$. We choose D_i so that some gallery from D to D_σ passes through D_i (see part 2 of the proof of the claim above). Using (10–2), we get

$$\begin{aligned}
d(\sigma) - d(\tau) &= d(D, D_\sigma) - d(D, D_\tau) \\
&\geq d(D, D_i) + d(D_i, D_\sigma) - (d(D, D_i) + d(D_i, D_\tau)) \\
&\geq m d_M(\gamma(i), v) - L - (M d_M(\tau, \gamma(i)) + L) \\
&\geq mi - (M(R+1) + 2L) = mi - P,
\end{aligned}$$

where $P = M(R+1) + 2L$. Remember that t^{-1} and, whence, $t^{-\alpha}$ are less than 1. Therefore

$$\begin{aligned}
\sum_{\tau | \tau \prec \sigma} (t^{-\alpha})^{d(\sigma) - d(\tau)} &\leq \sum_{i=0}^{\lfloor l \rfloor} N(R+1)N_k (t^{-\alpha})^{mi-P} \\
&= N(R+1)N_k t^{\alpha P} \sum_{i=0}^{\lfloor l \rfloor} (t^{-\alpha m})^i \\
&\leq N(R+1)N_k t^{\alpha P} \frac{1}{1 - t^{-\alpha m}}.
\end{aligned}$$

This completes the proof of Lemma 10.2 and of Theorem 10.1. □

Theorem 10.3 *Let W be a Coxeter group. For $t < \rho_W$ we have $b_t^0(W) = \chi_t(W) = \frac{1}{W(t)}$ and $b_t^{>0}(W) = 0$.*

Proof Theorems 9.1 and 10.1 imply that in the range $t > \frac{1}{\rho_W}$ we have

$$H_{>0}(L_t^2 C_*(\Sigma), \partial) = 0.$$

Indeed, if $c \in L_t^2 C_k(\Sigma)$, $\partial c = 0$, then $c = \partial H(c) + H(\partial c) = \partial H(c)$, so that $[c] = 0$. It follows that the isomorphic complex $(L_{t^{-1}}^2 C_*(\Sigma), \partial^{t^{-1}})$ also has vanishing homology in degrees > 0 (if $t^{-1} < \rho_W$). Thus, its homology is concentrated in dimension 0, and the zeroth Betti number is equal to the Euler characteristic. □

Corollary 10.4 *Assume that $(D, \partial D)$ is a generalised homology n–disc; then for $t < \frac{1}{\rho_W}$ we have $b_t^n = 0$, while for $t > \frac{1}{\rho_W}$ the L_t^2 cohomology is concentrated in dimension n and $b_t^n = (-1)^n \chi_t = \frac{(-1)^n}{W(t)}$.*

Proof This follows from Theorems 10.3 and 7.1 using Poincaré duality (Theorem 6.1). □

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