Geometry & Topology 10 (2006) 1157–1172 **1157**

Siegel–Veech constants in $\mathcal{H}(2)$

SAMUEL LELIÈVRE

Abelian differentials on Riemann surfaces can be seen as translation surfaces, which are flat surfaces with cone-type singularities. Closed geodesics for the associated flat metrics form cylinders whose number under a given maximal length was proved by Eskin and Masur to generically have quadratic asymptotics in this length, with a common coefficient constant for the quadratic asymptotics called a Siegel–Veech constant which is shared by almost all surfaces in each moduli space of translation surfaces.

Square-tiled surfaces are specific translation surfaces which have their own quadratic asymptotics for the number of cylinders of closed geodesics. It is an interesting question whether the Siegel–Veech constant of a given moduli space can be recovered as a limit of individual constants of square-tiled surfaces in this moduli space. We prove that this is the case in the moduli space $\mathcal{H}(2)$ of translation surfaces of genus two with one singularity.

30F30; 53C22

1 Introduction

1.1 Geodesics on the torus

On the standard torus $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the number $N(L)$ of maximal families of parallel simple closed geodesics of length not exceeding L is well-known (and easily seen) to grow quadratically in L, with

$$N(L) \sim \frac{1}{2\,\zeta(2)} \cdot \pi L^2,$$

which is half of the asymptotic for the number of primitive lattice points in a disc of radius L. The factor one-half comes from counting unoriented rather than oriented geodesics.

By convention, the corresponding *Siegel–Veech constant* is $1/(2\,\zeta(2))$. Note that it is the coefficient of πL^2, not of L^2, in the asymptotic.

Published: 12 September 2006 DOI: 10.2140/gt.2006.10.1157

Marking the origin of the torus (artificially considering it as a singularity or saddle), the number of geodesic segments of length at most L joining the saddle to itself coincides with the number of families of simple closed geodesics.

1.2 Geodesics on translation surfaces

It is a standard fact that abelian differentials on Riemann surfaces can be seen as translation surfaces. On translation surfaces of genus at least 2, countings of closed geodesics or saddle connections similar to those just described for the torus can be made.

There, the countings of saddle connections and of cylinders of simple closed geodesics do not coincide, but their growth rates remain quadratic. This is made more precise by several related results.

Masur [10; 11] proved that for every translation surface, there exist positive constants c and C such that the counting functions of saddle connections and of maximal cylinders of closed geodesics satisfy

$$c \cdot \pi L^2 \leqslant N_{\mathrm{cyl}}(L) \leqslant N_{\mathrm{sc}}(L) \leqslant C \cdot \pi L^2$$

for large enough L.

Veech [14] proved that on a square-tiled surface (and on any Veech surface) there are in fact *exact quadratic asymptotics*; Gutkin and Judge [7] gave a different proof.

Another proof for the upper quadratic bounds for $N_{\mathrm{cyl}}(L)$ and $N_{\mathrm{sc}}(L)$ was given by Vorobets [15]. Eskin and Masur [3] gave yet another one and proved that for each connected component of each stratum of each moduli space of normalised (area 1) abelian or quadratic differentials, there are constants c_{sc} and c_{cyl} such that *almost every surface* in the component has $N_{\mathrm{sc}}(L) \sim c_{\mathrm{sc}}\pi L^2$ and $N_{\mathrm{cyl}}(L) \sim c_{\mathrm{cyl}}\pi L^2$.

It is an interesting open problem whether *all* translation surfaces have exact quadratic asymptotics for countings of saddle connections and of cylinders of closed geodesics.

The particular constants for many Veech surfaces have been computed explicitly by Veech [14], Vorobets [15], Gutkin and Judge [7], and Schmoll [13]. Constants for some families of non-Veech surfaces were also given by Eskin, Masur and Schmoll [4] and Eskin, Marklof and Morris [2]. The generic constants for the connected components of all strata of abelian differentials were computed by Eskin, Masur and Zorich [5].

In general, the particular constants for Veech surfaces do not coincide with the generic constants of the strata where they live.

There is another subtle difference between Veech surfaces and generic surfaces. Define cylinders as *regular* if their boundary components both consist of a single saddle connection. In any connected component of stratum in genus at least 2, a generic surface has no irregular cylinders while on Veech surfaces, countings of irregular cylinders have quadratic asymptotics.

However we will prove that on the stratum $\mathcal{H}(2)$ of translation surfaces of genus 2 with one singularity, the individual 'quadratic constants' for *regular* cylinders on square-tiled surfaces retreive the generic Siegel–Veech constant of $\mathcal{H}(2)$ as a limit. See Theorem 1 in Section 1.3 for a precise statement.

1.3 Setting and main result

In this paper, we are concerned with the stratum $\mathcal{H}(2)$ consisting of genus–2 abelian differentials with a double zero, or in other words, translation surfaces of genus 2 with one singularity (of angle 6π).

Theorem 1 *Consider a sequence S_n of area–1 surfaces in $\mathcal{H}(2)$, each tiled by some prime number p_n of squares, with $p_n \to \infty$. Then the constants in the quadratic asymptotics for regular cylinders of closed geodesics on the surfaces S_n tend to $(10/3)(1/(2\zeta(2)))$, the Siegel–Veech constant of $\mathcal{H}(2)$ for cylinders of closed geodesics.*

Remark It is possible to adapt our calculations to show that the constants in the quadratic asymptotics for *irregular* cylinders of closed geodesics on the surfaces S_n in the theorem tend to 0, so that the constants in the quadratic asymptotics for *all* cylinders (both regular and irregular) tend to the generic constant of the stratum $\mathcal{H}(2)$ as well.

Remark We believe that the assumption that the number of squares tiling the surfaces is prime is unnecessary, but we have not yet been able to adapt the calculations to show the convergence of Siegel–Veech constants in the case of nonprime numbers of tiles.

The proof of the theorem relies on fine estimates presented in Section 3.1.

Pierre Arnoux pointed out to us the analogy to a result of C Faivre on Lévy constants of quadratic numbers. See Faivre [6] or Dal'Bo and Peigné [1].

1.4 Acknowledgments

The author wishes to thank Anton Zorich for guiding him to this problem, Pascal Hubert, Joël Rivat and Emmanuel Royer for useful conversations, and Cécile Dartyge and Gérald Tenenbaum who helped him with the estimates in Section 3.1. He also thanks the referee for useful remarks.

This research was carried out in Montpellier for the most part; some intuition was gained from computer calculations (programmed in Caml Light) run using the Medicis server at École polytechnique.

2 Preliminaries

The stratum $\mathcal{H}(2)$ is the simplest stratum of abelian differentials after the (well-understood) stratum of abelian differentials on tori. As every stratum, it admits a natural $\mathrm{SL}(2,\mathbb{R})$ action, and we will recall here some facts concerning the orbits of certain special points of $\mathcal{H}(2)$, the square-tiled surfaces.

A square-tiled surface is a ramified translation cover of the standard torus with only one branch point. The number of square tiles is the number of sheets of the covering or the degree of the corresponding covering map to the standard torus. A square-tiled surface is called *primitive* if this covering map does not factor through a covering of a larger torus with only one branch point.

2.1 Orbits of square-tiled surfaces

By a theorem of McMullen [12], in $\mathcal{H}(2)$, primitive n–square-tiled surfaces for $n > 3$ are in a single $\mathrm{SL}(2,\mathbb{R})$–orbit if n is even, and in exactly two $\mathrm{SL}(2,\mathbb{R})$–orbits if n is odd (see Hubert and Lelièvre [8] for the prime n case). We will denote these orbits by $\mathcal{A}_n$ and $\mathcal{B}_n$ for odd n and by $\mathcal{E}_n$ for even n.

The integer points in these orbits are primitive n–square-tiled surfaces, and they form $\mathrm{SL}(2,\mathbb{Z})$–orbits which we will denote respectively by A_n, B_n and E_n. The number of primitive n–square-tiled surfaces in $\mathcal{H}(2)$ is thus the cardinality of E_n when n is even and the sum of the cardinalities of A_n and B_n when n is odd. This number is given in [4, Lemma 4.11] to be asymptotic to

$$\frac{3}{8}n^3\prod_{p|n}\Big(1-\frac{1}{p^2}\Big).$$

Formulas for the separate countings of A_n and B_n conjectured by Hubert and Lelièvre [8] are established by Lelièvre and Royer [9] to be respectively

$$a_n = \frac{3}{16}(n-1)\,n^2 \prod_{p|n}\Big(1-\frac{1}{p^2}\Big) \quad \text{and} \quad b_n = \frac{3}{16}(n-1)\,n^2 \prod_{p|n}\Big(1-\frac{1}{p^2}\Big).$$

If n tends to infinity within the set of prime numbers, both a_n and b_n are asymptotic to $(3/16)\,n^3$.

The natural definition of primitive n–square-tiled surfaces gives them area n (each square tile has area 1), but it is sometimes useful to consider the corresponding unit area surfaces by applying the natural projection from $\mathcal{H}(2)$ to the unit hyperboloid $\mathcal{H}_1(2)$.

2.2 Cusps

Each square-tiled surface in the stratum $\mathcal{H}(2)$ decomposes into either one or two horizontal cylinders, and can be given as coordinates the heights, widths and twist parameters of these cylinders; see [4] or [8]. Here we are interested in *regular* cylinders of closed geodesics, which exist only in *two-cylinder* decompositions (in one-cylinder decompositions, the unique cylinder has three saddle connections on each boundary component).

The decompositions into cylinders provide a way to parametrise square-tiled surfaces (by the heights, widths and twist parameters of their cylinders). These parameters are very convenient to describe the action of $\mathcal{U} = \{\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$, which only affects the twist parameters.

The following lemma puts together Lemmas 2.4, 2.5 and 3.1 of [8]. The notations $\wedge$ and $\vee$ are used for gcd and lcm, respectively.

Lemma 1 *Let S be a primitive n–square-tiled surface, and denote by $\mathcal{D}$, resp. D, its orbit under $\mathrm{SL}(2,\mathbb{R})$, resp. $\mathrm{SL}(2,\mathbb{Z})$. Then D is the set of primitive n–square-tiled surfaces in $\mathcal{D}$ and the cusps of $\mathcal{D}$ are in bijection with the $\mathcal{U}$–orbits in D.*

If S has two cylinders, with h_i, w_i and t_i $(i = 1, 2)$ as height, width and twist parameters, then its cusp width (the cardinality of its $\mathcal{U}$–orbit) is

$$\mathrm{cw}(S) = \frac{w_1}{w_1 \wedge h_1} \vee \frac{w_2}{w_2 \wedge h_2} \quad \Big(= \frac{w_1}{w_1 \wedge h_1} \times \frac{w_2}{w_2 \wedge h_2} \text{ for prime } n\Big).$$

The surface S' with $h_i' = h_i$, $w_i' = w_i$, and $t_i' = t_i \bmod (w_i \wedge h_i)$ is a "canonical" representative of the $\mathcal{U}$–orbit of S. Each cusp thus has a unique representative with $0 \leqslant t_i' < w_i \wedge h_i$.

We also recall that each direction of rational slope on a given square-tiled surface S gives rise to a decomposition of S in cylinders of closed geodesics, and this direction can be associated to one of the cusps of the $\mathrm{SL}(2,\mathbb{R})$–orbit of S.

Note that these cusps can also be understood as cusps of $\Gamma(S)$, the Veech group, or stabiliser under $\mathrm{SL}(2,\mathbb{R})$ of S. Algebraically this means conjugacy classes of maximal parabolic subgroups; geometrically the 'cusps' of the quotient surface $\Gamma(S)\backslash\mathbf{H}$.

2.3 A formula for the constants

Here, we establish a formula for the constants for which we will compute estimates in Section 3.

Lemma 2 *The number $N_{\mathrm{reg}}(L)$ of regular cylinders of closed geodesics of length at most L on a unit area square-tiled surface S has the following asymptotics:*

$$N_{\mathrm{reg}}(L) \sim \frac{n}{\#D} \sum_{\substack{\mathcal{C}_j \text{ two-cyl} \\ \text{cusp of } S}} \frac{\mathrm{cw}(\mathcal{C}_j)}{w_1^2} \frac{1}{2\zeta(2)} \pi L^2.$$

Remark Following tradition we write the asymptotic as a multiple of πL^2 rather than just L^2 and write $(1/(2\zeta(2)))\pi L^2$ instead of $(3/\pi)L^2$ to bring out the analogy with the corresponding formula for the torus.

Proof We deduce this formula from the material reviewed in Section 2.1–Section 2.2, and from Section 3 of [14] to which we refer freely here both for notation and results.

For any finite covolume subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$, Veech introduces a complete set $\{\Lambda_j\}_{1\leqslant j\leqslant r}$ of representatives of the maximal parabolic subgroups of Γ. We will also refer to the cusps $\mathcal{C}_j$. He defines $\Lambda_0 = \{\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right): k \in \mathbb{Z}\}$ which we denoted by $\mathcal{U}$.

Then for each j he selects $\beta_j \in \mathrm{SL}(2,\mathbb{R})$ conjugating Λ_0 to Λ_j, ie, $\beta_j^{-1}\Lambda_j\beta_j = \Lambda_0$. When Γ is the Veech group of a translation surface S, this amounts to representing the cusp j by the surface $\beta_j^{-1}S$. Indeed, $\beta_j S$ has Veech group $\beta_j^{-1}\Gamma\beta_j$.

These 'representatives' $\beta_j^{-1}S$ of the cusps have width 1. For square-tiled surfaces, $\Gamma(S)$ is always a subgroup of $\mathrm{SL}(2,\mathbb{Z})$, so it is also usual to conjugate inside $\mathrm{SL}(2,\mathbb{Z})$ to the group generated by some $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ rather than to Λ_0 itself, thus keeping track of the cusp width (the adequate k).

Let us illustrate the difference on an example.

Consider the surface S pictured on the left of Figure 1 made of seven squares $s_1, \ldots, s_7$ forming a horizontal cylinder where the right edge of each s_i is glued to the left edge of s_{i+1} (indices being understood modulo 7), and where the top edges of squares s_1 to s_7 are respectively glued to the bottom edges of squares s_3 to s_6, s_1 to s_2, and s_7.

Consider the direction of the first diagonal. In this direction the surface S decomposes into cylinders of closed geodesics as illustrated on the right of Figure 1 (parallel sides of same length are identified).

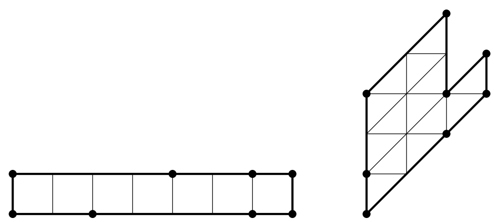

Figure 1

We get to the standard square-tiled representative of the cusp corresponding to that direction by applying the matrix $M = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ (the matrix in $\mathrm{SL}(2,\mathbb{Z})$ which sends $(1,1)$ to $(1,0)$ and $(0,1)$ to itself). A choice of β_j^{-1} is $\left(\begin{smallmatrix} 1/\sqrt{3} & 0 \\ 0 & \sqrt{3} \end{smallmatrix}\right) \cdot M$. This sends $3\cdot(1,1)$ to $(1,0)$.

Figure 2 represents $\beta_j^{-1}\, S$ on the left and $M\, S$ on the right.

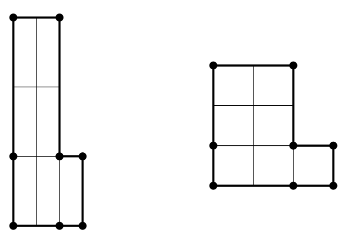

Figure 2

Veech defines ξ_j to be the vector $\beta_j \cdot (1,0)$. And for each cylinder of closed geodesics in the direction of Λ_j, calling v the holonomy vector for this cylinder, he associates to the cylinder the constant $c_j(v) = \|\xi_j\| / \|v\|$.

In our notation, for a surface tiled by unit squares, if v is the holonomy vector of cylinder i of cusp $\mathcal{C}_j$, we have $c_j(v) = \sqrt{\mathrm{cw}(\mathcal{C}_j)}/w_i$, where w_i is the width of this cylinder and $\mathrm{cw}(\mathcal{C}_j)$ the width of this cusp.

Veech's formula for the asymptotics [14, formula (3.11)] is:

$$N(L) \sim \mathrm{vol}(\,\Gamma(S)\backslash\mathbf{H}\,)^{-1}\left(\sum_{j=1}^{r}\left(\sum_{v} c_j(v)^2\right)\right)L^2.$$

So the contribution of a given cylinder of a cusp $\mathcal{C}_j$ to the coefficient of the quadratic asymptotics is $\mathrm{vol}(\,\Gamma(S)\backslash\mathbf{H}\,)^{-1}c_j(v)^2$, where v is the holonomy vector of this cylinder.

If we are concerned with regular cylinders of closed geodesics for square-tiled surfaces in $\mathcal{H}(2)$, we need only consider cylinder 1 of two-cylinder cusps.

The volume of the quotient $\Gamma(S)\backslash\mathbf{H}$ equals the index of $\Gamma(S)$ in $\mathrm{SL}(2,\mathbb{Z})$ times the volume of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbf{H}$; and the index equals the cardinality of the $\mathrm{SL}(2,\mathbb{Z})$–orbit D of S, while the volume of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbf{H}$ is $\pi/3$.

The last thing to observe is the effect of scaling a surface. Consider a surface S where quadratic asymptotics $N(L) = c\cdot\pi L^2$ hold, and scale S by a scale factor r. On rS, the asymptotics become $N(L) = c\cdot\pi(L/r)^2$.

A square-tiled surface S of area 1 is a scaled-down version by $1/\sqrt{n}$ of a surface tiled by n unit squares; this scaling changes the asymptotic by a factor n.

This completes the proof of the formula in Lemma 2. □

Let us denote by $\tilde{c}(S)$ the quantity

$$\frac{n}{\#D}\sum_{\substack{\mathcal{C}_j \text{ two-cyl}\\ \text{cusp of } S}}\frac{\mathrm{cw}(\mathcal{C}_j)}{w_1^2}.$$

Our aim is now to prove that $\tilde{c}(S)$ tends to $10/3$ as the number of square tiles of S tends to infinity staying prime. This will establish Theorem 1.

As a first step for this, using the description of the two-cylinders cusps in the orbits of square-tiled surfaces (see [8]), and renaming w_1, w_2, h_1, h_2 as a, b, h, y respectively, we get:

- for S in orbit A_n,

$$\tilde{c}(S) = \frac{n}{a_n}\Bigg(\sum_{\substack{a,b,h,y\geqslant 1\\ ah+by=n\\ h\wedge y=1\\ h,\,y \text{ odd}\\ a<b}}\frac{ab}{a^2} + \frac{1}{2}\sum_{\substack{a,b,h,y\geqslant 1\\ ah+by=n\\ h\wedge y=1\\ a\not\equiv b \mod 2\\ h\not\equiv y \mod 2\\ a<b}}\frac{ab}{a^2}\Bigg)$$

- for S in orbit B_n,

$$\widetilde{c}(S) = \frac{n}{b_n}\Big(\sum_{\substack{a,b,h,y\geqslant 1\\ ah+by=n\\ h\wedge y=1\\ a,\,b \text{ odd}\\ a<b}} \frac{ab}{a^2} + \frac{1}{2} \sum_{\substack{a,b,h,y\geqslant 1\\ ah+by=n\\ h\wedge y=1\\ a\not\equiv b \mod 2\\ h\not\equiv y \mod 2\\ k<\ell}} \frac{ab}{a^2}\Big)$$

The idea is to group two-cylinder cusps sharing the same parameters w_1, w_2, h_1, h_2. Then the sum of the cusp widths adds up to $w_1 w_2$ (for nonprime n some values of the twist parameters could correspond to nonprimitive surfaces, but for prime n all surfaces with n tiles are primitive). All surfaces with h_1 and h_2 odd are in orbit A, all surfaces with w_1 and w_2 odd are in orbit B, and those with mixed parities for w_i and h_i are half in orbit A half in orbit B.

3 Asymptotics for a large prime number of squares

We need to estimate quantities of the type

$$\widetilde{c}(D_n) = \frac{n}{\#D_n} \sum \frac{ab}{a^2}$$

where the sum is over positive integers a, b, h, y satisfying conditions as above.

3.1 A simpler sum

Since $\#D$, for prime n, is asymptotically $(3/16)n^3$, we first replace $n/\#D_n$ by $1/n^2$. Second, we momentarily drop the parity conditions; we will reintroduce them in the following subsections. Last, we drop the condition $a<b$; we will explain later why this does not change the asymptotic.

So we first consider the following simplified sum:

$$S(n) = \sum_{a\geqslant 1} \frac{1}{a^2} \sum_{b\geqslant 1} \sum_{\substack{h\geqslant 1,\, y\geqslant 1\\ ah+by=n}} \frac{ab}{n^2}.$$

Denote the sum over b by $S(n,a)$. Introducing the variable $m=by$,

$$S(n,a) = \sum_{\substack{1\leqslant m\leqslant n-a\\ m\equiv n \pmod a}} \sum_{b|m} \frac{ab}{n^2} = \frac{a}{n^2}\cdot F(n-a,n,a)$$

where
$$F(x,k,q) = \sum_{\substack{1 \leqslant m \leqslant x \\ m \equiv k \pmod q}} \sum_{b|m} b.$$

The following asymptotics hold for $F(x,k,q)$, $S(n,a)$ and $S(n)$.

Lemma 3 *For $k \wedge q = 1$ and $x \to \infty$,*
$$F(x,k,q) = \frac{x^2}{q} \cdot \frac{\pi^2}{12} \prod_{p|q} \Bigl(1 - \frac{1}{p^2}\Bigr) + O_q(x \log x).$$

Lemma 4 *For n prime, $S(n,a) \to \dfrac{\pi^2}{12} \prod_{p|a} \Bigl(1 - \dfrac{1}{p^2}\Bigr)$ as $n \to \infty$.*

Lemma 5 *For n prime, $S(n) \to \dfrac{5}{4}$ as $n \to \infty$.*

Proof of Lemma 3 If m is prime to q, denote by $\bar{m}$ the integer in $\{0, \dots, q-1\}$ such that $\bar{m}m \equiv 1 \pmod q$, and by $u = u(m,k,q)$ the integer in $\{0, \dots, q-1\}$ such that $u \equiv \bar{m}k \pmod q$; error terms depend on q.

$$\begin{aligned}
F(x,k,q) &= \sum_{\substack{1 \leqslant md \leqslant x \\ md \equiv k \pmod q}} d \\
&= \sum_{\substack{1 \leqslant m \leqslant x \\ m \wedge q = 1}} \sum_{\substack{1 \leqslant d \leqslant x/m \\ d \equiv \bar{m}k \pmod q}} d \\
&= \sum_{\substack{1 \leqslant m \leqslant x \\ m \wedge q = 1}} \sum_{\substack{1 \leqslant d \leqslant x/m \\ d \equiv u \pmod q}} d \\
&= \sum_{\substack{1 \leqslant m \leqslant x \\ m \wedge q = 1}} \sum_{1 \leqslant u + \lambda q \leqslant x/m} (u + \lambda q) \\
&= \sum_{\substack{1 \leqslant m \leqslant x \\ m \wedge q = 1}} \Biggl(\Biggl(\sum_{1 \leqslant \lambda \leqslant \frac{1}{q}(\frac{x}{m} - u)} \lambda q \Biggr) + O\Bigl(\frac{x}{m}\Bigr) \Biggr) \\
&= \sum_{\substack{1 \leqslant m \leqslant x \\ m \wedge q = 1}} \Bigl(\frac{1}{2} q \Bigl(\frac{x}{qm}\Bigr)^2 + O\Bigl(\frac{x}{m}\Bigr) + O(1) \Bigr) \\
&= \frac{x^2}{2q} \sum_{\substack{1 \leqslant m \leqslant x \\ m \wedge q = 1}} \frac{1}{m^2} + O(x \log x)
\end{aligned}$$

To sum only over the integers m with $m \wedge q = 1$, we can sum over all m with a factor $\mu(m \wedge q)$, so that all terms cancel out except the ones we want.

$$\begin{aligned} F(x,k,q) &= \frac{x^2}{2q} \sum_{d|q} \Big(\frac{\mu(d)}{d^2} \sum_{m \leqslant x/d} \frac{1}{m^2} \Big) + O(x \log x) \\ &= \frac{x^2}{2q} \sum_{d|q} \frac{\mu(d)}{d^2} \Big(\frac{\pi^2}{6} + O(1/x) \Big) + O(x \log x) \\ &= \frac{x^2}{q} \cdot \frac{\pi^2}{12} \prod_{p|q} \Big(1 - \frac{1}{p^2} \Big) + O(x \log x) \qquad \square \end{aligned}$$

Proof of Lemma 4 The limit follows immediately from Lemma 3 by a dominated convergence argument (similar arguments were used in [8, Section 7]). $\square$

Proof of Lemma 5 This is a consequence of Lemma 4 by the following observation.

$$\begin{aligned} \sum_{a \geqslant 1} \frac{1}{a^2} \prod_{p|a} \Big(1 - \frac{1}{p^2} \Big) &= \prod_{p} (1 + \sum_{\nu \geqslant 1} p^{-2\nu} (1 - p^{-2\nu})) = \prod_{p} (1 + p^{-2}) \\ &= \prod_{p} \frac{1 - p^{-4}}{1 - p^{-2}} = \frac{\zeta(2)}{\zeta(4)} = \frac{\pi^2/6}{\pi^4/90} = \frac{15}{\pi^2} \qquad \square \end{aligned}$$

3.2 Sums with specified parities

We introduce subsums of $S(n)$ for specified parities of the parameters.

The observation we just made will need to be completed by the following one.

$$\sum_{\substack{a \geqslant 1 \\ a \text{ even}}} \frac{1}{a^2} \prod_{p|a} \Big(1 - \frac{1}{p^2} \Big) = \sum_{\substack{a \geqslant 1 \\ a \text{ odd}}} \frac{1}{4a^2} \frac{3}{4} \prod_{p|a} \Big(1 - \frac{1}{p^2} \Big) + \sum_{\substack{a \geqslant 1 \\ a \text{ even}}} \frac{1}{4a^2} \prod_{p|a} \Big(1 - \frac{1}{p^2} \Big)$$

so that $$\sum_{\substack{a \geqslant 1 \\ a \text{ odd}}} \frac{1}{a^2} \prod_{p|a} \Big(1 - \frac{1}{p^2} \Big) = \frac{12}{\pi^2} \quad \text{and} \quad \sum_{\substack{a \geqslant 1 \\ a \text{ even}}} \frac{1}{a^2} \prod_{p|a} \Big(1 - \frac{1}{p^2} \Big) = \frac{3}{\pi^2}$$

3.2.1 Odd widths We now consider the sum over odd a and b:

$$S^{\text{ow}}(n) = \sum_{\substack{a \geqslant 1 \\ a \text{ odd}}} \frac{1}{a^2} \sum_{\substack{b \geqslant 1 \\ b \text{ odd}}} \sum_{\substack{h \geqslant 1, y \geqslant 1 \\ ah + by = n}} \frac{ab}{n^2}$$

We proceed as for the sum $S(n)$, putting

$$F^{\mathrm{ow}}(x,k,q) = \sum_{\substack{1 \leqslant m \leqslant x \\ m \equiv k \pmod q}} \sum_{\substack{b \mid m \\ b \text{ odd}}} b,$$

$$S^{\mathrm{ow}}(n,a) = \frac{a}{n^2} \cdot F^{\mathrm{ow}}(n-a,n,a) \quad \text{and} \quad S^{\mathrm{ow}}(n) = \sum_{\substack{a \geqslant 1 \\ a \text{ odd}}} \frac{1}{a^2} S^{\mathrm{ow}}(n,a).$$

Lemma 6 *The following asymptotics hold for $F^{\mathrm{ow}}(x,k,q)$, $S^{\mathrm{ow}}(n,a)$ and $S^{\mathrm{ow}}(n)$.*

For odd q, odd k, and $x \to \infty$,

$$F^{\mathrm{ow}}(x,k,q) = \frac{x^2}{q} \frac{\pi^2}{24} \prod_{p \mid q} \Big(1 - \frac{1}{p^2}\Big) + O(x \log x).$$

For odd a,

$$S^{\mathrm{ow}}(n,a) \xrightarrow[n \text{ prime}]{n \to \infty} \frac{\pi^2}{24} \prod_{p \mid a} \Big(1 - \frac{1}{p^2}\Big).$$

Finally,

$$S^{\mathrm{ow}}(n) \xrightarrow[n \text{ prime}]{n \to \infty} \frac{1}{2}.$$

Proof

$$\begin{aligned}
F^{\mathrm{ow}}(x,k,q) &= \sum_{t \geqslant 0} \sum_{\substack{1 \leqslant m \leqslant x/2^t \\ 2^t m \equiv k \pmod q \\ m \equiv 1 \pmod 2}} \sum_{b \mid m} b \\
&= \sum_{t \geqslant 0} \Bigg(\frac{(x/2^t)^2}{2q} \frac{\pi^2}{12} \prod_{p \mid 2q} \Big(1 - \frac{1}{p^2}\Big) + O((x/2^t) \log(x/2^t)) \Bigg) \\
&= \frac{x^2}{q} \frac{1}{1-1/4} \frac{\pi^2}{24} \Big(1 - \frac{1}{2^2}\Big) \prod_{p \mid q} \Big(1 - \frac{1}{p^2}\Big) + O(x \log x) \\
&= \frac{x^2}{q} \frac{\pi^2}{24} \prod_{p \mid q} \Big(1 - \frac{1}{p^2}\Big) + O(x \log x)
\end{aligned}$$

□

3.2.2 Odd heights We now consider the sum over odd h and y:

$$S^{\mathrm{oh}}(n) = \sum_{a \geqslant 1} \frac{1}{a^2} \sum_{b \geqslant 1} \sum_{\substack{h \geqslant 1, y \geqslant 1 \\ h,\, y \text{ odd} \\ ah + by = n}} \frac{ab}{n^2}.$$

Proceeding as previously, we are led to introduce

$$F^{\mathrm{oh}}(x,k,q) = \sum_{\substack{1 \leqslant m \leqslant x \\ m \equiv k+q \pmod{2q}}} \sum_{\substack{b|m \\ m/b \text{ odd}}} b \quad \text{and} \quad S^{\mathrm{oh}}(n,a) = \frac{a}{n^2} \cdot F^{\mathrm{oh}}(n-a,n,a),$$

and to write
$$S^{\mathrm{oh}}(n) = \sum_{a \geqslant 1} \frac{1}{a^2} S^{\mathrm{oh}}(n,a).$$

Lemma 7 *The following asymptotics hold for $F^{\mathrm{oh}}(x,k,q)$, $S^{\mathrm{oh}}(n,a)$ and $S^{\mathrm{oh}}(n)$.*

For even q, odd k, and $x \to \infty$,
$$F^{\mathrm{oh}}(x,k,q) = \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}\Big(1-\frac{1}{p^2}\Big) + O(x\log x).$$
For odd q, odd k, and $x \to \infty$,
$$F^{\mathrm{oh}}(x,k,q) = \frac{x^2}{q}\frac{\pi^2}{32}\prod_{p|q}\Big(1-\frac{1}{p^2}\Big) + O(x\log x).$$
For even a,
$$S^{\mathrm{oh}}(n,a) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{\pi^2}{24}\prod_{p|a}\Big(1-\frac{1}{p^2}\Big).$$
For odd a,
$$S^{\mathrm{oh}}(n,a) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{\pi^2}{32}\prod_{p|a}\Big(1-\frac{1}{p^2}\Big).$$
Finally,
$$S^{\mathrm{oh}}(n) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{1}{2}.$$

Proof For even q and odd k:

$$\begin{aligned} F^{\mathrm{oh}}(x,k,q) = \sum_{\substack{1 \leqslant m \leqslant x \\ m \equiv k+q \pmod{2q}}} \sum_{b|m} b &= \frac{x^2}{2q}\frac{\pi^2}{12}\prod_{p|2q}\Big(1-\frac{1}{p^2}\Big) + O(x\log x) \\ &= \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}\Big(1-\frac{1}{p^2}\Big) + O(x\log x) \end{aligned}$$

For odd q and odd k:

$$\begin{aligned}
F^{\mathrm{oh}}(x,k,q) &= \sum_{t\geqslant 1} \sum_{\substack{1\leqslant m\leqslant x/2^t \\ 2^t m\equiv k+q \pmod{2q} \\ m \text{ odd}}} \sum_{b|m} 2^t b \\
&= \sum_{t\geqslant 1} 2^t \sum_{\substack{1\leqslant m\leqslant x/2^t \\ 2^{t-1} m\equiv (k+q)/2 \pmod{q} \\ m \text{ odd}}} \sum_{b|m} b \\
&= \sum_{t\geqslant 1} 2^t \frac{(x/2^t)^2}{2q} \frac{\pi^2}{12} \prod_{p|2q} \Big(1-\frac{1}{p^2}\Big) + O(x\log x) \\
&= \sum_{t\geqslant 1} \frac{1}{2^t} \frac{x^2}{q} \frac{\pi^2}{24} \Big(1-\frac{1}{2^2}\Big) \prod_{p|q} \Big(1-\frac{1}{p^2}\Big) + O(x\log x) \\
&= \frac{x^2}{q} \frac{\pi^2}{32} \prod_{p|q} \Big(1-\frac{1}{p^2}\Big) + O(x\log x) \qquad \square
\end{aligned}$$

3.2.3 Mixed parities Dealing with the even–odd sums as above would be most cumbersome; this is fortunately not necessary. Indeed, $S(n) = S^{\mathrm{ow}}(n) + S^{\mathrm{oh}}(n) + S^{\mathrm{eo}}(n)$, and we know the limits of $S(n)$, $S^{\mathrm{ow}}(n)$ and $S^{\mathrm{oh}}(n)$ when n tends to infinity staying prime, so we have

$$S^{\mathrm{eo}}(n) \xrightarrow[\substack{n\to\infty \\ n \text{ prime}}]{} \frac{1}{4}.$$

3.3 Asymptotics for orbits A and B

We end by showing that the obtained limit is unchanged by adding the condition $a < b$.

Indeed, since $\#\{(h,y)\colon h\geqslant 1,\ y\geqslant 1,\ ah+by=n\} \leqslant n$, the sum

$$\sum_{b=1}^{a} \sum_{\substack{h\geqslant 1,\, y\geqslant 1 \\ ah+by=n}} \frac{ab}{n^2}$$

is $O(1/n)$, where the constant of the O depends on a.

Putting the previous sections together, $\widetilde{c}(A_n)$ and $\widetilde{c}(B_n)$ have the same asymptotics: $S^{\mathrm{A}}(n) = (16/3)(S^{\mathrm{oh}}(n) + \frac{1}{2} S^{\mathrm{eo}}(n))$ and $S^{\mathrm{B}}(n) = (16/3)(S^{\mathrm{ow}}(n) + \frac{1}{2} S^{\mathrm{eo}}(n))$, so they both tend to $10/3$ as n tends to infinity, n prime.

4 Concluding remarks

Numerical evidence suggests that the convergence to the generic constant of the stratum occurs not only for prime n but for general n; however a proof would involve some complications in the calculations which would make the exposition tedious.

A similar study for the constants that appear in the quadratic asymptotics for the countings of saddle connections could also be made. There, one has to take into consideration both one-cylinder and two-cylinder cusps, and some interesting phenomena can be observed. Numerical calculations suggest that the sum of the contributions of one-cylinder and two-cylinder cusps has a limit, but separate countings for one-cylinder cusps do not have a limit for general n; their asymptotics have fluctuations involving the prime factors of n.

References

[1] **F Dal'bo**, **M Peigné**, *Comportement asymptotique du nombre de géodésiques fermées sur la surface modulaire en courbure non constante*, Astérisque (1996) 111–177 MR1634272

[2] **A Eskin**, **J Marklof**, **D Witte Morris**, *Unipotent flows on the space of branched covers of Veech surfaces*, Ergodic Theory Dynam. Systems 26 (2006) 129–162 MR2201941

[3] **A Eskin**, **H Masur**, *Asymptotic formulas on flat surfaces*, Ergodic Theory Dynam. Systems 21 (2001) 443–478 MR1827113

[4] **A Eskin**, **H Masur**, **M Schmoll**, *Billiards in rectangles with barriers*, Duke Math. J. 118 (2003) 427–463 MR1983037

[5] **A Eskin**, **H Masur**, **A Zorich**, *Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants*, Publ. Math. Inst. Hautes Études Sci. (2003) 61–179 MR2010740

[6] **C Faivre**, *Distribution of Lévy constants for quadratic numbers*, Acta Arith. 61 (1992) 13–34 MR1153919

[7] **E Gutkin**, **C Judge**, *Affine mappings of translation surfaces: geometry and arithmetic*, Duke Math. J. 103 (2000) 191–213 MR1760625

[8] **P Hubert**, **S Lelièvre**, *Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$*, Israel J. Math. 151 (2006) 281–321 MR2214127

[9] **S Lelièvre**, **E Royer**, *Orbitwise countings in $\mathcal{H}(2)$ and quasimodular forms*, to appear in Internat. Math. Res. Notices (2006)

[10] **H Masur**, *Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential*, from: "Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986)", (D Drasin, editor), Math. Sci. Res. Inst. Publ. 10, Springer, New York (1988) 215–228 MR955824

[11] **H Masur**, *The growth rate of trajectories of a quadratic differential*, Ergodic Theory Dynam. Systems 10 (1990) 151–176 MR1053805

[12] **C T McMullen**, *Teichmüller curves in genus two: discriminant and spin*, Math. Ann. 333 (2005) 87–130 MR2169830

[13] **M Schmoll**, *On the asymptotic quadratic growth rate of saddle connections and periodic orbits on marked flat tori*, Geom. Funct. Anal. 12 (2002) 622–649 MR1924375

[14] **W A Veech**, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. 97 (1989) 553–583 MR1005006

[15] **Y B Vorobets**, *Ergodicity of billiards in polygons*, Mat. Sb. 188 (1997) 65–112 MR1462024

Mathematics Institute, University of Warwick
Coventry CV4 7AL, UK

lelievre@maths.warwick.ac.uk

http://carva.org/samuel.lelievre/

Proposed: Walter Neumann — Received: 30 March 2005
Seconded: Benson Farb, Tobias Colding — Revised: 23 February 2006