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On the chain-level intersection pairing for PL manifolds

J E MCCLURE

Let M be a compact oriented PL manifold and let $C_* M$ be its PL chain complex. The domain of the chain-level intersection pairing is a subcomplex of $C_* M \otimes C_* M$. We prove that the inclusion map from this subcomplex to $C_* M \otimes C_* M$ is a quasi-isomorphism. An analogous result is true for the domain of the iterated intersection pairing. Using this, we show that the intersection pairing gives $C_* M$ a structure of partially defined commutative DGA, which in particular implies that $C_* M$ is canonically quasi-isomorphic to an E_∞ chain algebra.

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1 Introduction

Let M be a compact oriented PL manifold. The chain-level intersection pairing was introduced by Lefschetz [8] as a tool for constructing the intersection pairing on the homology of M. A version of the chain-level intersection pairing is a basic ingredient in Chas and Sullivan's construction [2] of a Batalin–Vilkovisky structure on the homology of the free loop space of M.

For a complete understanding of the chain-level intersection pairing, it seems helpful to have the following theorem. Let $C_* M$ be the PL chain complex of M (see Section 3 for the definition). Let us say that a subcomplex of a chain complex is *full* if the inclusion map is a quasi-isomorphism.

Theorem 1.1 *The domain of the chain-level intersection pairing is a full subcomplex of $C_* M \otimes C_* M$.*

It might seem at first that something like Theorem 1.1 would have been needed already by Lefschetz to define the intersection pairing on homology, but for that purpose two weaker facts would suffice:

(i) For any cycles C and D in $C_* M$, the chain $C \otimes D$ is homologous to an element in the domain of the intersection pairing.

(ii) If C', D' are two other cycles with $C' \otimes D'$ homologous to $C \otimes D$, then the difference $C' \otimes D' - C \otimes D$ is the boundary of an element in the domain of the

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intersection pairing. (This is needed to show that the intersection pairing on homology is well-defined.)

Theorem 1.1 is harder to prove than (i) and (ii) because (among other reasons) a cycle in $C_*M \otimes C_*M$ cannot in general be written in the form $\sum C_i \otimes D_i$ with C_i and D_i cycles.

One goal of this paper is to prove Theorem 1.1 and, more generally, the analogous statement for the k–fold iterate of the intersection pairing; see Proposition 12.3 and Remark 12.4.

It seems useful to go farther and to show that the intersection pairing gives C_*M a structure of "partially defined commutative DGA;" this is the second (and main) goal of this paper (see Theorem 12.1). Combining this with Remark 12.2 and [17, Theorem 1] shows in particular that C_*M is canonically quasi-isomorphic to an E_∞ chain algebra.

The third goal of this paper is to give a new treatment of the chain-level intersection pairing, based on the account of Goresky and MacPherson [5] but with some improvements.

The results of this paper will be applied by the author [13] to prove two theorems about the Chas–Sullivan operations. Let LM be the free loop space of M, let S_* denote the singular chain functor and let $\mathcal{F}$ be the framed little 2-disks operad as in [4].

Theorem A *The Batalin–Vilkovisky structure on the homology of LM is induced by a natural action of an operad quasi-isomorphic to $S_*\mathcal{F}$ on a chain complex quasi-isomorphic to $S_*(LM)$.*

(Theorem A is the analog for $H_*(LM)$ of Deligne's Hochschild cohomology conjecture; see Markl, Shnider and Stasheff [11, Section I.1.19].)

Theorem B *The Eilenberg–Moore spectral sequence converging to the homology of LM is a spectral sequence of Batalin–Vilkovisky algebras.*

The paper is organized as follows.

Section 2 gives a brief discussion of the definitions of the chain-level intersection pairing given in [8], [9] and [5] and explains why these versions of the definition are not convenient as a starting point for proving Theorem 1.1.

Section 3 recalls (from [5]) the definition of the PL chain complex of a PL space. Section 4 recalls (also from [5]) a method for making chain-level constructions by using relative homology. Section 5 constructs the umkehr (that is, "reverse") map in relative homology induced by a PL map between compact oriented PL manifolds. In Section 6 a chain-level umkehr map is deduced from this using the method of Section 4. Section 7 recalls the definition of exterior product for PL chains.

In Section 8, the chain-level intersection pairing is defined as the composite of the exterior product and the chain-level umkehr map induced by the diagonal map; the motivation for this definition is that the intersection of two subsets of a set S can be identified with the intersection of their Cartesian product with the diagonal in $S \times S$.

Section 9 gives the formal definition of "partially defined commutative algebra." I use Leinster's concept of homotopy algebra [10] for this purpose rather than the Kriz–May definition of partial algebra [7] (but I will use the term "Leinster partial algebra" instead of "homotopy algebra," since the latter term seems excessively generic). The reason for using Leinster's definition is that it is simpler and more intuitive. It will be shown in [13] that the Kriz–May definition is a special case of the Leinster definition (see Remark 9.5(b) below).

Section 10–Section 13 give the proof that the intersection pairing and its iterates determine a Leinster partial commutative DGA structure on $C_* M$. The proof uses a "moving lemma" (Lemma 13.5) which is proved in Section 14 by means of a general-position result (Proposition 14.6) that may be of independent interest. Proposition 14.6 is proved in Section 15 and Section 16.

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2 The Lefschetz and Goresky–MacPherson definitions of the chain-level intersection pairing

This section is not needed logically for the rest of the paper; it is offered as motivation for Section 3–Section 8. The reader may also find it helpful to consult Steenrod's account of Lefschetz's work on the intersection pairing [16, pages 28–30].

This section uses some technical terms which will be defined in Section 3–Section 7.

Lefschetz's first account of the chain-level intersection pairing $C \cdot D$ was in [8]. In this paper he uses the obvious definition: if $C = \sum m_i \sigma_i$ and $D = \sum n_i \tau_i$ then

$$C \cdot D = \sum \pm m_i n_j \, \sigma_i \cap \tau_j, \tag{1}$$

where the signs are determined by the orientations of σ_i, τ_j and M. This formula does not in fact give a chain unless all of the intersections $\sigma_i \cap \tau_j$ have the same dimension, so some restriction on the pair (C, D) is necessary. Generically, the intersection of a p–dimensional PL subspace and a q–dimensional PL subspace has dimension $\le p + q - \dim M$; pairs of PL subspaces with this property are said to be in *general position*. Lefschetz restricts the domain of the intersection pairing to pairs (C, D) for which all of the pairs (σ_i, τ_j) are in general position, and he interprets terms $\sigma_i \cap \tau_j$ which are in dimension less than $\dim C + \dim D - \dim M$ as 0.

In order to prove the crucial formula

$$\partial(C \cdot D) = (\partial C) \cdot D \,\pm\, C \cdot \partial D, \tag{2}$$

Lefschetz imposes a further restriction on the domain of the intersection pairing: he requires that all of the pairs $(\partial\sigma_i, \tau_j)$ and $(\sigma_i, \partial\tau_j)$ should also be in general position.[1] This assumption allows him to prove equation (2) by working with one pair of simplices at a time and extending additively.

This definition has the disadvantage that the domain of the intersection pairing is not invariant under subdivision. For example, if σ and τ are 1–simplices in a 2–manifold intersecting at a point in the interior of both, then the pair (σ, τ) is in the domain, but if we subdivide σ and τ at the intersection point we obtain a pair of chains $(\sigma' + \sigma'', \tau' + \tau'')$ which is not in the domain (because for example the pair $(\partial\sigma', \tau')$ is not in general position).[2] This phenomenon is general: if (C, D) is in the domain of this version of the intersection pairing with $C \cdot D \neq 0$ then there will always be a subdivision in which the pair of chains determined by C and D is not in the domain.

Lefschetz returned to the chain-level intersection pairing in [9, Section IV.6]. He gave a formula more general than (1) (equation (46) on page 212) in which the coefficients are "looping coefficients" [9, Section IV.5]. This allowed him to enlarge the domain of the intersection pairing as follows: if we write $\mathrm{supp}(C)$ for the union of the simplices that occur in C, then $C \cdot D$ is defined when the three pairs $(\mathrm{supp}(C), \mathrm{supp}(D))$, $(\mathrm{supp}(\partial C), \mathrm{supp}(D))$, $(\mathrm{supp}(C), \mathrm{supp}(\partial D))$ are in general position; note that this condition is invariant under subdivision.

[1] If C and D are chains on the *same triangulation* this condition forces $C \cdot D$ to be 0, because σ_i and τ_j will intersect along a common face and therefore $\sigma_i \cap \tau_j$ will be contained in $\partial\sigma_i \cap \tau_j$.

[2] Note also that, if the intersection point is P, then formula (1) gives $\sigma \cdot \tau = \pm P$ but $(\sigma' + \sigma'') \cdot (\tau' + \tau'') = \pm 4P$.

The "looping coefficients" used in Lefschetz's second definition are tricky to define explicitly [9, Subsection 58 on page 216]. The theory has been worked out carefully in Keller [6] (which I have not had an opportunity to consult) and seems to be rather complicated (see the Math Review).

The chain-level intersection pairing became temporarily obsolete when the cup product was discovered and it was noticed that the intersection pairing in homology could be defined using only Poincaré duality and the cup product, without any recourse to the chain level.

Goresky and MacPherson returned to the chain-level intersection pairing as a tool for constructing an intersection pairing in intersection homology [5, Section 2]. They introduced the PL chain complex C_*M (as the direct limit of simplicial chains under subdivision; see Section 3) and defined the intersection pairing (which they denoted by $\cap$) on a certain subset of $C_*M \times C_*M$ by means of an elegant construction in which the procedure of the previous paragraph is reversed: the chain-level intersection pairing is derived from the relative versions of Poincaré duality and the cup product. Their version of the chain-level intersection pairing is probably equivalent to Lefschetz's second definition.

In order to prove (or even state) Theorem 1.1 it is necessary to extend the domain of the intersection pairing from a subset of $C_*M \times C_*M$ to a subset of $C_*M \otimes C_*M$. The obvious way to do this would be to consider elements

$$\sum C_i \otimes D_i$$

in which every pair (C_i, D_i) is in the domain of the Goresky–MacPherson intersection pairing $\cap$ and to define the intersection pairing on such an element to be

$$\sum C_i \cap D_i.$$

But it is not at all clear that this is well defined, and it also is not clear how to determine when an element of $C_*M \otimes C_*M$ has the required form (which would make it difficult to show that the domain of the operation is a full subcomplex).

The definition to be given in Section 8 resolves both of these issues by defining the intersection pairing (up to a dimension shift) as the composite of the exterior product

$$\varepsilon\colon C_*M \otimes C_*M \to C_*(M \times M)$$

(see Section 7) and the chain-level umkehr map

$$\Delta_!\colon C_*^{\Delta}(M \times M) \to C_*M$$

induced by the diagonal (see Section 6); here $C_*^\Delta(M\times M)$ denotes the set of chains E in $C_*(M\times M)$ for which both E and ∂E are in general position with respect to the diagonal. With this definition, the domain of the intersection pairing (up to a dimension shift) is

$$\varepsilon^{-1}(C_*^\Delta(M\times M)).$$

The analog of equation (2) is immediate from the fact that ε and $\Delta_!$ are chain maps.

3 PL chains

We begin by reviewing some basic definitions.

A *simplicial complex* K is a set of simplices in $\mathbb{R}^n$ (for some n) with two properties: every face of a simplex in K is in K and the intersection of two simplices in K is a common face. (A face of a simplex σ is the simplex spanned by some subset of the vertices of σ.)

The *simplicial chain complex of* K, denoted c_*K, is defined by letting c_pK be generated by pairs (σ,o), where σ is a p–simplex of K and o is an orientation of σ, subject to the relation $(\sigma,o)=-(\sigma,-o)$ where $-o$ denotes the opposite orientation. We leave it as an exercise to formulate the definition of the boundary map ∂ (or see Spanier [15, page 159]). If we choose orientations for the simplices of K (with no requirement of consistency among the orientations) then every nonzero element c of c_*K can be written uniquely in the form $\sum n_i\sigma_i$ with all $n_i\neq 0$.

The *realization* of K, denoted $|K|$, is the union of the simplices of K.

A *subdivision* of K is a simplicial complex L with two properties: $|L|=|K|$ and every simplex of L is contained in a simplex of K.

The *subdivision category* of K has an object for each subdivision L of K and a morphism $L\to L'$ whenever L' is a subdivision of L.

If L' is a subdivision of L there is an induced monomorphism $c_*L\to c_*L'$ which takes (σ,o) to $\sum(\tau,o_\tau)$, where the sum runs over all $\tau\in L'$ which are contained in σ and have the same dimension as σ, and o_τ is the orientation induced by o. This makes c_* a covariant functor on the subdivision category of K.

A subspace X of $\mathbb{R}^n$ will be called a *PL space* if there is a simplicial complex K with $X=|K|$. K will be called a *triangulation* of X; note that X determines K up to subdivision by Bryant [1, page 222].

The *PL chain complex* of a PL space $|K|$, denoted $C_*|K|$, is the direct limit

$$\operatorname*{colim}_L c_*L$$

taken over the subdivision category of K.

Remark 3.1 This definition is taken from Goresky and MacPherson [5, Subsection 1.2], which seems to be the first place where the PL chain complex was defined.

Note that the direct system defining $C_*|K|$ is a rather simple one: the subdivision category is a directed set (because any two subdivisions have a common refinement [1, page 222]), and all of the maps $c_*L \to c_*L'$ are monomorphisms. It follows that each of the maps $c_*L \to C_*|K|$ is a monomorphism.

Remark 3.2 The homology of c_*L is canonically isomorphic to the singular homology of $|K|$ by [15, Theorems 4.3.8 and 4.4.2]; since homology commutes with colimits over directed sets, the homology of $C_*|K|$ is also canonically isomorphic to the singular homology of $|K|$.

Now let C be a nonzero element of $C_*|K|$. There is a subdivision L of K with C in c_*L, so (after choosing orientations for the simplices in L) we can write $C = \sum n_i\sigma_i$ where the σ_i are simplices in L and the n_i are nonzero. We define the *support of* C, denoted $\mathrm{supp}(C)$, to be $\bigcup \sigma_i$; this is independent of the choice of L. The support of 0 is defined to be the empty set.

4 A useful lemma

Let K be a simplicial complex. A *subcomplex* of K is a subset K' of K with the property that every face of every simplex in K' is also in K'.

A *PL subspace* of $|K|$ is a space of the form $|L|$ where L is a subcomplex of a subdivision of K.

The next lemma is taken from Section 1.2 of [5]; it gives a way of using relative homology to make chain-level constructions.

Lemma 4.1 *Let K be a simplicial complex and let A and B be PL subspaces of $|K|$ such that $B \subset A$ and $\dim B \le \dim A - 1$. Let $p = \dim A$.*

(a) *There is a natural isomorphism $\alpha_{A,B}$ from $H_p(A,B)$ to the abelian group*

$$\{\, C \in C_p(K) : \mathrm{supp}(C) \subset A \text{ and } \mathrm{supp}(\partial C) \subset B \,\}.$$

(b) *The following diagram commutes:*

$$\begin{tikzcd}
H_p(A,B) \arrow[r, "\alpha_{A,B}"] \arrow[d, "\partial"'] & \{\, C \in C_p(K) \,:\, \mathrm{supp}(C) \subset A \text{ and } \mathrm{supp}(\partial C) \subset B \,\} \arrow[d, "\partial"] \\
H_{p-1}(B,\varnothing) \arrow[r, "\alpha_{B,\varnothing}"] & \{\, D \in C_{p-1}(K) \,:\, \mathrm{supp}(D) \subset B \text{ and } \partial D = 0 \,\}
\end{tikzcd}$$

Proof For part (a), note that $H_p(A,B)$ is isomorphic to the p–th homology of the complex C_*A/C_*B, and this in turn is isomorphic to the quotient of the relative cycles by the relative boundaries. The set specified in the lemma is the set of relative cycles, while the set of relative boundaries is $\partial(C_{p+1}A) + C_pB$, which is zero because of the hypotheses. Part (b) is immediate from the definitions. $\square$

5 An umkehr map in relative homology

A *PL map* from $|K|$ to $|K'|$ is a continuous function f with the property that, for some subdivision L of K, the restriction of f to each simplex of L is an affine map with image in a simplex of K'.

A *PL homeomorphism* is a PL map which is a homeomorphism.

An *m–dimensional PL manifold* is a PL space M with the property that each point of M is contained in the interior of a PL subspace which is PL homeomorphic to the m simplex.

Let M be a compact oriented m–dimensional PL manifold and let A and B be PL subspaces of M with $B \subset A$. Let N be a compact oriented PL manifold of dimension n and let $f\colon N \to M$ be a PL map. Let $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$.

We want to construct a map

$$f_!\colon H_*(A,B) \to H_{*+n-m}(A',B') \tag{3}$$

(one should think of this as taking a homology class to its inverse image with respect to f).

Let (U',V') be an open pair in N with $A' \subset U'$ and $B' \subset V'$. Choose an open pair (U,V) in M with $A \subset U$, $B \subset V$, $f^{-1}(U) \subset U'$ and $f^{-1}(V) \subset V'$ (for example, we can let $U = M - f(N-U')$ and $V = M - f(N-V')$). Consider the composite

$$H_*(A,B) \to H_*(U,V) \cong \check{H}^{m-*}(M-V, M-U) \xrightarrow{f^*} \check{H}^{m-*}(N-V', N-U') \cong H_{*+n-m}(U',V'),$$

where the second and fourth maps are Poincaré–Lefschetz duality isomorphisms; see Dold [3, Proposition VIII.7.2]. By the naturality of the cap product [3, VIII.7.6] this composite is independent of the choice of (U, V) and is natural with respect to (U', V'). We therefore get a map

$$H_*(A, B) \to \lim H_{*+n-m}(U', V')$$

where the inverse limit is taken over all open pairs $(U', V') \supset (A', B')$. This inverse limit is isomorphic to $H_{*+n-m}(A', B')$ by [3, Exercise 4 at the end of Section VIII.13]; here we use the fact that the realization of a simplicial complex is an ENR (see for example [3, Proposition IV.8.12]). This completes the construction of the map (3).

For use in the next section, we need:

Lemma 5.1 *The following diagram commutes:*

$$\begin{tikzcd} H_*(A, B) \arrow[r, "f_!"] \arrow[d, "\partial"'] & H_{*+n-m}(A', B') \arrow[d, "\partial"] \\ H_{*-1}(B) \arrow[r, "f_!"] & H_{*+n-m-1}(B') \end{tikzcd}$$

Proof This follows easily from [3, VII.12.22]. □

6 An umkehr map at the chain level

Let M, N and $f\colon N \to M$ be as in the previous section.

We say that a PL subspace A of M is in *general position* with respect to f if

$$\dim(f^{-1}(A)) \le \dim A + n - m.$$

(The dimension of the empty set is defined to be $-\infty$, so if $f^{-1}(A)$ is empty then A is in general position.)

Remark 6.1 For later use we make two observations.

(a) Suppose that f is a composite gh, that A is in general position with respect to g, and that $g^{-1}(A)$ is in general position with respect to h. Then A is in general position with respect to f.

(b) Suppose that N is a Cartesian product $M \times M_1$ and $f\colon N \to M$ is the projection. Then every A is in general position with respect to f.

A p–chain C in C_*M is said to be in general position with respect to f if

$$\dim(f^{-1}(\mathrm{supp}(C))) \le p+n-m.$$

Let $C_*^f M$ be the set of all chains $C \in C_*M$ for which both C and ∂C are in general position with respect to f. Note that $C_*^f M$ is a subcomplex of C_*M.

We want to construct a chain map

$$f_!\colon C_*^f M \to C_{*+n-m}N.$$

So let $C \in C_q^f M$. Let $[C]$ be the homology class of C in $H_q(\mathrm{supp}(C), \mathrm{supp}(\partial C))$. Let T be the abelian group

$$\{\, D \in C_{q+n-m}N \mid \mathrm{supp}(D) \subset f^{-1}(\mathrm{supp}(C)) \text{ and } \mathrm{supp}(\partial D) \subset f^{-1}(\mathrm{supp}(\partial C)) \,\}.$$

We define $f_!(C)$ to be the image of $[C]$ under the following composite:

$$H_q(\mathrm{supp}(C), \mathrm{supp}(\partial C)) \xrightarrow{f_!} H_{q+n-m}(f^{-1}(\mathrm{supp}(C)), f^{-1}(\mathrm{supp}(\partial C)))$$
$$\cong T \hookrightarrow C_{q+n-m}N$$

Here the first map was constructed in Section 5 and the isomorphism is from Lemma 4.1 (which applies because of the hypothesis that both C and ∂C are in general position with respect to f). $f_!$ is a chain map by Lemma 4.1(b) and Lemma 5.1.

Remark 6.2 Note that, by definition of T, we have $\mathrm{supp}(f_!(C)) \subset f^{-1}(\mathrm{supp}(C))$.

7 The exterior product for PL chains

Let σ_1 and σ_2 be simplices. It is easy to see that $\sigma_1 \times \sigma_2$ is a PL space; that is, there is a simplicial complex J with $|J| = \sigma_1 \times \sigma_2$. Note that there is no canonical way to choose J, but that any two choices of J have a common subdivision.

It follows that the product of any two PL spaces is a PL space.

Let $|K_1|$ and $|K_2|$ be PL spaces. We want to construct a map

(4) $$\varepsilon\colon C_*|K_1| \otimes C_*|K_2| \to C_*(|K_1| \times |K_2|),$$

called the *exterior product*.

As a first step, let L_1 and L_2 be subdivisions of K_1 and K_2 respectively. We will define a map

(5) $$\varepsilon'\colon c_*L_1 \otimes c_*L_2 \to C_*(|K_1| \times |K_2|)$$

(see Section 3 for the definition of c_*).

It suffices to define ε' on generators, so for $i = 1, 2$ let σ_i be a simplex of L_i with orientation o_i. Let J be a simplicial complex with

$$|J| = \sigma_1 \times \sigma_2.$$

Then $\varepsilon'((\sigma_1, o_1) \otimes (\sigma_2, o_2))$ is defined to be

$$\sum (\tau, o_\tau)$$

where τ runs through the simplices of J with dimension $\dim \sigma_1 + \dim \sigma_2$, and o_τ is the orientation of τ induced by $o_1 \times o_2$.

The maps ε' are consistent as L_1 and L_2 vary; passage to colimits gives the map ε.

Remark 7.1 (a) It is easy to check that ε is a monomorphism.

(b) The quasi-isomorphism relating c_* to singular chains [15, Theorems 4.3.8 and 4.4.2] takes ε to the Eilenberg–MacLane shuffle product [3, VI.12.26.2]. Since the latter is a quasi-isomorphism, so is ε.

(c) For singular chains, the shuffle product has an explicit natural homotopy inverse, namely the Alexander–Whitney map [3, VI.12.26.2]. Unfortunately the Alexander–Whitney map requires an ordering of the vertices of each simplex, so it seems to have no analog for PL chains.

8 The chain-level intersection pairing

We now have the ingredients needed to define the chain-level intersection pairing.

Let M be a compact oriented PL manifold of dimension m and let Δ be the diagonal map from M to $M \times M$. As in Section 6, let $C_*^\Delta(M \times M)$ be the subcomplex of $C_*(M \times M)$ consisting of chains C for which both C and ∂C are in general position with respect to Δ.

It is convenient to shift degrees so that the intersection pairing preserves degree. For a chain complex C_* and an integer n, we will write $\Sigma^n C_*$ for the n–fold suspension of C_*, that is, the chain complex with C_i in degree $i + n$.

Define $$G_2 \subset \Sigma^{-2m}(C_* M \otimes C_* M)$$

to be $\Sigma^{-2m}(\varepsilon^{-1}(C_*^\Delta(M \times M)))$, where ε is the exterior product (the G stands for "general position" and the subscript 2 will be explained in Section 10).

The chain-level intersection pairing μ is the composite

$$G_2 \xrightarrow{\varepsilon} \Sigma^{-2m} C_*^{\Delta}(M \times M) \xrightarrow{\Delta_!} \Sigma^{-m} C_* M.$$

Remark 8.1 It is not difficult to check that, if C and D are chains for which the Goresky–MacPherson intersection pairing $C \cap D$ is defined [5, pages 141–142], then (up to the dimension shifts in the definitions of G_2 and μ) $C \otimes D$ is in G_2 and $\mu(C \otimes D) = C \cap D$.

9 Leinster partial commutative DGAs

Our main goal in the rest of the paper is to show that the chain-level intersection pairing and its iterates determine a partially defined commutative DGA structure on $\Sigma^{-m} C_* M$.

First we need a precise definition of "partially defined commutative DGA." We will use the definition given by Leinster in [10, Section 2.2] (but note that Leinster uses the term "homotopy algebra" instead of "partial algebra").

Notation 9.1 (i) For $k \geq 1$ let $\bar{k}$ denote the set $\{1, \ldots, k\}$. Let $\bar{0}$ be the empty set.

(ii) Let Φ be the full subcategory of Set with objects $\bar{k}$ for $k \geq 0$.

(iii) Given a functor A defined on Φ, write A_k (instead of $A(\bar{k})$) for the value of A at $\bar{k}$.

Disjoint union gives a functor $\coprod\colon \Phi \times \Phi \to \Phi$. In particular, if A is a functor defined on Φ then the functor $A \circ \coprod$ on $\Phi \times \Phi$ takes $(\bar{k}, \bar{l})$ to A_{k+l}.

Notation 9.2 (i) Let Ch denote the category of chain complexes.

(ii) Let $(\mathbb{Z}, 0)$ denote the chain complex which has $\mathbb{Z}$ in dimension 0 and 0 in all other dimensions.

Definition 9.3 A Leinster partial commutative DGA is a functor A from Φ to Ch together with chain maps

$$\xi_{k,l}\colon A_{k+l} \to A_k \otimes A_l \quad \text{for each } k, l \qquad \text{and} \qquad \xi_0\colon A_0 \to (\mathbb{Z}, 0)$$

such that the following conditions hold.

(i) The collection $\xi_{k,l}$ is a natural transformation from $A \circ \coprod$ to $A \otimes A$ (considered as functors from $\Phi \times \Phi$ to Ch).

(ii) The following diagram commutes for all k, l, n:

$$\begin{tikzcd} A_{k+l+n} \arrow[r, "\xi_{k+l,n}"] \arrow[d, "\xi_{k,l+n}"'] & A_{k+l} \otimes A_n \arrow[d, "\xi_{k,l}\otimes 1"] \\ A_k \otimes A_{l+n} \arrow[r, "1\otimes\xi_{l,n}"] & A_k \otimes A_l \otimes A_n \end{tikzcd}$$

(iii) Let $\tau\colon \overline{k+l} \to \overline{k+l}$ be the block permutation that transposes $\{1,\dots,k\}$ and $\{k+1,\dots,k+l\}$. The following diagram commutes for all k, l:

$$\begin{tikzcd} A_{k+l} \arrow[r, "\xi_{k,l}"] \arrow[d, "\tau_*"'] & A_k \otimes A_l \arrow[d, "\cong"] \\ A_{k+l} \arrow[r, "\xi_{l,k}"] & A_l \otimes A_k \end{tikzcd}$$

(iv) The following diagram commutes for all k:

$$\begin{tikzcd} A_k \arrow[r, "\xi_{0,k}"] \arrow[dr, "\cong"'] & A_0 \otimes A_k \arrow[d, "\xi_0\otimes 1"] \\ & \mathbb{Z} \otimes A_k \end{tikzcd}$$

(v) ξ_0 and each $\xi_{k,l}$ are quasi-isomorphisms.

Remark 9.4 (a) An ordinary commutative DGA B determines a Leinster partial commutative DGA with $A_k = B^{\otimes k}$.

(b) Conversely, the proof of [17, Theorem 1] can be modified to show that Leinster partial commutative DGAs can be functorially replaced by quasi-isomorphic E_∞ DGAs.

Remark 9.5 (a) Definition 9.3 is the precise analog, for the category Ch, of Segal's Γ–spaces [14]. This is not immediately obvious, since a Γ–space is a functor on the category $\mathcal{F}$ of *based* finite sets [12, Remark 1.4]; the point is that the maps $\xi_{k,l}$ in Definition 9.3 encode the same information as the projection maps in Segal's definition.

(b) It will be shown in [13] that partial commutative DGAs in the sense of Kriz and May [7, Section II.2] are the same thing as Leinster partial commutative DGAs in which all of the maps $\xi_{k,l}$ are monomorphisms.

10 The functor G

As a first step in showing that the intersection pairing on $\Sigma^{-m}C_*M$ extends to a Leinster partial commutative DGA structure, we define a functor G from Φ to Ch with $G_1 = \Sigma^{-m}C_*M$. The G stands for "general position."

G_2 has already been defined in Section 8. To define G_k for $k \geq 3$ we need a definition.

Let $R\colon \bar{k} \to \bar{k}'$ be any map. Define

$$R^*\colon M^{k'} \to M^k$$

to be the composite

$$M^{k'} = \mathrm{Map}(\bar{k}', M) \to \mathrm{Map}(\bar{k}, M) = M^k$$

where the second arrow is induced by R. Thus the projection of $R^*(x_1, \dots, x_{k'})$ on the i–th factor is $x_{R(i)}$.

If $R\colon \bar{k} \twoheadrightarrow \bar{k}'$ is a surjection then we think of R^* as a generalized diagonal map. For example, if k' is 1 and R is the constant map then $R^*\colon M \to M^k$ is the usual diagonal map.

Let ε_k denote the k–fold exterior product

$$(C_*M)^{\otimes k} \hookrightarrow C_*(M^k).$$

Definition 10.1 Define G_0 to be $\mathbb{Z}$ and G_1 to be $\Sigma^{-m}C_*M$. For $k \geq 2$ define G_k to be the subcomplex of $\Sigma^{-mk}((C_*M)^{\otimes k})$ consisting of the elements $\Sigma^{-mk}C$ for which both $\varepsilon_k(C)$ and $\varepsilon_k(\partial C)$ are in general position with respect to all generalized diagonal maps, that is,

$$G_k = \bigcap_{k'<k} \ \bigcap_{R\colon \bar{k} \twoheadrightarrow \bar{k}'} \Sigma^{-mk}(\varepsilon_k^{-1}(C_*^{R^*} M^k)).$$

Remark 10.2 One might at first expect a simpler definition of G_k, in which general position is required only with respect to the ordinary diagonal $M \to M^k$. The more complicated definition given here is needed for Lemma 11.1.

Lemma 10.3 *If $\Sigma^{-mk}C$ is in G_k then both $\varepsilon_k(C)$ and $\varepsilon_k(\partial C)$ are in general position with respect to all maps R^*.*

Proof Any R factors as R_1R_2, where R_1 is an inclusion and R_2 is a surjection. Then $R^* = R_2^*R_1^*$, and R_1^* is a composite of projection maps. The lemma now follows from Remark 6.1(a) and Remark 6.1(b). □

It remains to define the effect of G on morphisms in Φ. First we need three lemmas.

Lemma 10.4 *Let $N_1 \xrightarrow{g} N_2 \xrightarrow{f} N_3$ be a diagram of compact oriented PL manifolds and PL maps. Let $C \in C_*^f N_3 \cap C_*^{fg} N_3$. Then*

$$\text{(a)}\quad f_! C \in C_*^g N_2 \qquad \text{and} \qquad \text{(b)}\quad g_! f_! C = (fg)_! C.$$

Proof Part (a) is immediate from Remark 6.2 and (b) follows from the definitions. □

Lemma 10.5 *Let $f\colon N_1 \to M_1$ and $g\colon N_2 \to M_2$ be PL maps between compact oriented PL manifolds. Then*

(a) *the exterior product*

$$\varepsilon\colon C_* M_1 \otimes C_* M_2 \to C_*(M_1 \times M_2)$$

takes $C_*^f M_1 \otimes C_*^g M_2$ to $C_*^{f\times g}(M_1 \times M_2)$, and

(b) *the diagram*

$$\begin{array}{ccc}
C_p^f M_1 \otimes C_q^g M_2 & \xrightarrow{\quad\varepsilon\quad} & C_{p+q}^{f\times g}(M_1 \times M_2) \\
\Big\downarrow{\scriptstyle f_! \otimes g_!} & & \Big\downarrow{\scriptstyle (f\times g)_!} \\
C_{p+n_1-m_1} N_1 \otimes C_{q+n_2-m_2} N_2 & \xrightarrow{\varepsilon} & C_{p+q+n_1+n_2-m_1-m_2}(N_1 \times N_2)
\end{array}$$

commutes for all p and q, where m_i (resp. n_i) is the dimension of M_i (resp. N_i).

Proof Part (a) is obvious from the definitions. Part (b) follows from [3, VII.12.17]. □

Lemma 10.6 *Let $R\colon \bar{k} \to \bar{k'}$ be any map. Then $(R^*)_! \circ \varepsilon_k$ takes G_k to $\varepsilon_{k'}(G_{k'})$.*

Proof We need to prove two things: that

$$(R^*)_!(\varepsilon_k(G_k)) \subset \Sigma^{-mk'} \varepsilon_{k'}(C_*(M)^{\otimes k'})$$

and that

$$(R^*)_!(\varepsilon_k(G_k)) \subset \Sigma^{-mk'} C_*^{S^*}(M^{k'})$$

for all surjections $S\colon \bar{k'} \to \bar{k''}$. The first follows from Lemma 10.5 and the second from Lemma 10.3 and Lemma 10.4(a). □

We can now define the effect of G on morphisms by letting

$$G_R\colon G_k \to G_{k'}$$

be $\varepsilon_{k'}^{-1} \circ (R^*)_! \circ \varepsilon_k$ (recall that $\varepsilon_{k'}$ is a monomorphism). Lemma 10.4(b) implies that $G_{R\circ S} = G_R \circ G_S$ for all R and S.

11 The maps $\xi_{k,l}$

In order to construct the maps

$$\xi_{k,l}\colon G_{k+l} \to G_k \otimes G_l$$

it suffices to prove the following lemma.

Lemma 11.1 *The inclusion*

$$G_{k+l} \hookrightarrow \Sigma^{-m(k+l)}(C_*M)^{\otimes(k+l)} \cong \Sigma^{-mk}(C_*M)^{\otimes k} \otimes \Sigma^{-ml}(C_*M)^{\otimes l}$$

has its image in $G_k \otimes G_l$*.*

We can then define $\xi_{k,l}$ to be the inclusion $G_{k+l} \hookrightarrow G_k \otimes G_l$.

In order to prove Lemma 11.1 we need a criterion for deciding when an element of $\Sigma^{-mk}(C_*M)^{\otimes k} \otimes \Sigma^{-ml}(C_*M)^{\otimes l}$ is in $G_k \otimes G_l$; we will build up to this in stages, culminating in Corollary 11.4.

Let K be a triangulation of M^k. Choose orientations for the simplices of K (with no consistency required among the choices). Recall that (since orientations have been chosen) c_pK is the free abelian group generated by the p–simplices of K. Let $R\colon \bar{k} \to \bar{k}'$ be any map and define $c_p(K,R)$ to be the free abelian group generated by the p–simplices of K that are *not* in general position with respect to R^*. Let

$$\Upsilon_p^{K,R}\colon c_pK \to c_p(K,R)$$

be the homomorphism which is the identity on $c_p(K,R)$ and which takes the p–simplices that are in general position with respect to R^* to 0. Let

$$\Psi_p^{K,R}\colon c_pK \to c_p(K,R) \oplus c_{p-1}(K,R)$$

be $\Upsilon_p^{K,R} + \Upsilon_{p-1}^{K,R} \circ \partial$.

As an immediate consequence of the definitions, we have:

Lemma 11.2 (a) *An element of c_pK is in general position with respect to R^* if and only if it is in the kernel of $\Upsilon_p^{K,R}$.*

(b) *An element of c_pK is in $C_*^{R^*}(M^k)$ (that is, the element and its boundary are both in general position with respect to R^*) if and only if it is in the kernel of $\Psi_p^{K,R}$.*

(c) *An element of c_pK is in*

$$\bigcap_{k'<k} \bigcap_{R\colon \bar{k} \twoheadrightarrow \bar{k}'} C_*^{R^*}(M^k)$$

if and only if it is in the kernel of

$$\sum_R \Psi_p^{K,R}\colon c_p K \to \bigoplus_R (c_p(K,R) \oplus c_{p-1}(K,R)).$$

Next let L be a triangulation of M^l. We would like to characterize the elements of $\Sigma^{-mk} c_p K \otimes \Sigma^{-ml} c_q L$ that are in $G_k \otimes G_l$. First we need some algebra.

Lemma 11.3 *For exact sequences of abelian groups*

$$0 \to A \to B \xrightarrow{g} C \qquad \text{and} \qquad 0 \to D \to E \xrightarrow{h} F$$

with C and F torsion free, $A \otimes D$ is the kernel of

$$g \otimes 1 + 1 \otimes h\colon B \otimes E \to (C \otimes E) \oplus (B \otimes F).$$

Proof We may assume without loss of generality that g and h are surjective. Then the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \otimes D \arrow[r] \arrow[d] & B \otimes D \arrow[r] \arrow[d] & C \otimes D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes E \arrow[r] \arrow[d] & B \otimes E \arrow[r] \arrow[d] & C \otimes E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes F \arrow[r] \arrow[d] & B \otimes F \arrow[r] \arrow[d] & C \otimes F \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

has exact rows and columns. The lemma follows by an easy diagram chase. □

Corollary 11.4 *Let $C \in (C_* M)^{\otimes k} \otimes (C_* M)^{\otimes l}$ be such that $(\varepsilon_k \otimes \varepsilon_l)(C)$ is in $c_* K \otimes c_* L$. Then $\Sigma^{-m(k+l)} C$ is in $G_k \otimes G_l$ if and only if $(\varepsilon_k \otimes \varepsilon_l)(C)$ is in the kernel of*

$$\sum_{R\colon \bar{k} \to \bar{k}'} \Psi_p^{K,R} \otimes 1 + \sum_{S\colon \bar{l} \to \bar{l}'} 1 \otimes \Psi_q^{L,S}$$

Proof of Lemma 11.1 Let $\Sigma^{-m(k+l)} C \in G_{k+l}$. Then there are triangulations K of M^k and L of M^l such that $(\varepsilon_k \otimes \varepsilon_l)(C) \in c_* K \otimes c_* L$. Let $R\colon \bar{k} \to \bar{k}'$. Then both

$\varepsilon_{k+l}(C)$ and $\varepsilon_{k+l}(\partial C)$ are in general position with respect to $(R\times 1)^*$ (by definition of G_{k+l}), and it is easy to see that this implies

$$(\Psi_p^{K,R}\otimes 1)(\varepsilon_k\otimes\varepsilon_l)(C)=0.$$

Similarly
$$(1\otimes\Psi_q^{L,S})(\varepsilon_k\otimes\varepsilon_l)(C)=0$$

for all S. Thus $\Sigma^{-m(k+l)}C$ is in $G_k\otimes G_l$ by Corollary 11.4. □

12 The main theorem

Theorem 12.1 *The functor G defined in Section 10 with the maps $\xi_{k,l}$ defined in Section 11 is a Leinster partial commutative DGA.*

Remark 12.2 Since the maps $\xi_{k,l}$ are monomorphisms, G is also a Kriz–May partial commutative DGA (see Remark 9.5(b)).

To prove Theorem 12.1, we need to verify the five parts of Definition 9.3. Part (i) follows easily from the definitions and Lemma 10.5. Parts (ii)–(iv) are immediate from the definition of $\xi_{k,l}$. Part (v) is an easy consequence of the following result, which will be proved in the next section.

Proposition 12.3 *The inclusion*

$$G_k\hookrightarrow\Sigma^{-mk}(C_*M)^{\otimes k}$$

is a quasi-isomorphism for all k.

Remark 12.4 When $k=2$ this is Theorem 1.1 of the introduction, up to the dimension shift introduced in Section 8.

13 Proof of Proposition 12.3

Throughout this section and the next we fix an integer $k\geq 2$.

A *PL homotopy* is a PL map $h\colon X\times I\to Y$, where X and Y are PL spaces and I is the interval $[0,1]$ with its usual PL structure.

It will be convenient to have notation for the standard inclusion maps $X\to X\times I$. We write i_0 (resp. i_1) for the map which takes x to $(x,0)$ (resp. $(x,1)$).

We need a supply of PL homotopies that preserve the image of

$$\varepsilon_k\colon (C_*M)^{\otimes k}\hookrightarrow C_*(M^k).$$

Definition 13.1 Suppose that we are given a number l with $1 \le l \le k$ and a PL homotopy
$$\phi\colon M \times I \to M.$$
The *l–th factor PL homotopy* determined by this data is the composite
$$M^k \times I \cong M^{l-1} \times (M \times I) \times M^{k-l} \xrightarrow{1\times\phi\times 1} M^k.$$

Let ι be the canonical element of $C_1(I)$.

Lemma 13.2 *Let $h\colon M^k \times I \to M^k$ be an l–th factor PL homotopy for some l and let C be in the image of $\varepsilon_k\colon (C_*M)^{\otimes k} \hookrightarrow C_*(M^k)$. Then*

(a) *$(h \circ i_1)_*(C)$ is in the image of ε_k, and*

(b) *$h_*(\varepsilon(C \otimes \iota))$ is in the image of ε_k.*

Proof This is an easy consequence of the definitions. □

For the proof of Proposition 12.3 we will use a filtration of $\Sigma^{-mk}(C_*M)^{\otimes k}$.

Definition 13.3 (i) For $0 \le j \le k$ define Λ_j to be the set of all surjections $R\colon \bar{k} \twoheadrightarrow \bar{k}'$ such that for each $i > j$ the set $R^{-1}(R(i))$ has only one element.

(ii) For $0 \le j \le k$ define G_k^j to be the subcomplex of $\Sigma^{-mk}(C_*M)^{\otimes k}$ consisting of the chains C for which both $\varepsilon_k(C)$ and $\varepsilon_k(\partial C)$ are in general position with respect to R^* for all $R \in \Lambda_j$.

Thus we have a filtration
$$G_k = G_k^k \subset G_k^{k-1} \subset \cdots \subset G_k^0 = \Sigma^{-mk}(C_*M)^{\otimes k}.$$
Proposition 12.3 follows immediately from:

Proposition 13.4 *For each $1 \le j \le k$ the inclusion $G_k^j \subset G_k^{j-1}$ is a quasi-isomorphism.*

For this we need a lemma which will be proved in Section 14.

Lemma 13.5 *Suppose that $D \in \Sigma^{mk} G_k^{j-1}$ and $\partial D \in \Sigma^{mk} G_k^j$. Then there is a j–th factor homotopy*
$$h\colon M^k \times I \to M^k$$
such that

(a) *$h \circ i_0$ is the identity,*

(b) *the chains $(h \circ i_1)_*(\varepsilon_k D)$, $(h \circ i_1)_*(\varepsilon_k(\partial D))$ and $h_*(\varepsilon(\partial D \otimes \iota))$ are in general position with respect to R^* for all $R \in \Lambda_j$, and*

(c) *the chain $h_*(\varepsilon(D \otimes \iota))$ is in general position with respect to R^* for all $R \in \Lambda_{j-1}$.*

Proof of Proposition 13.4 We have to show two things:

(i) For D a cycle in $\Sigma^{mk} G_k^{j-1}$, there is a cycle C in $\Sigma^{mk} G_k^j$ homologous to D.

(ii) If C is a cycle in $\Sigma^{mk} G_k^j$ which is the boundary of an element of $\Sigma^{mk} G_k^{j-1}$ then C is the boundary of an element of $\Sigma^{mk} G_k^j$.

To show (i), choose a homotopy h as in Lemma 13.5.

Lemma 13.2 implies that $(h \circ i_1)_*(\varepsilon_k D)$ is in the image of ε_k, so we may define

$$C = \varepsilon_k^{-1}((h \circ i_1)_*(\varepsilon_k D)).$$

C is a cycle, and Lemma 13.5(b) implies that C is in $\Sigma^{mk} G_k^j$.

Lemma 13.2 also implies that $h_*(\varepsilon(D \otimes \iota))$ is in the image of ε_k, so we may define

$$E = \varepsilon_k^{-1}(h_*(\varepsilon(D \otimes \iota))).$$

Lemma 13.5(c) implies that E is in $\Sigma^{mk} G_k^{j-1}$.

Let $\kappa, \lambda \in C_0 I$ be the 0–chains associated to $0, 1 \in I$; then $\partial\iota = \lambda - \kappa$. Now

$$\begin{aligned}
\varepsilon_k(\partial E) &= \partial(h_*(\varepsilon(D \otimes \iota))) \\
&= h_*(\varepsilon(\partial D \otimes \iota)) + (-1)^{|D|} h_*(\varepsilon(D \otimes \lambda)) - (-1)^{|D|} h_*(\varepsilon(D \otimes \kappa)) \\
&= 0 + (-1)^{|D|}(h \circ i_1)_*(\varepsilon_k D) - (-1)^{|D|}(h \circ i_0)_*(\varepsilon_k D) \\
&= (-1)^{|D|} \varepsilon_k(C - D).
\end{aligned}$$

Since ε_k is a monomorphism, this implies that C is homologous to D.

To show (ii), let $D \in \Sigma^{mk} G_k^{j-1}$ with $\partial D = C$. Choose a homotopy h as in Lemma 13.5. Then $(h \circ i_1)_*(\varepsilon_k D)$ and $h_*(\varepsilon(\partial D \otimes \iota))$ are in the image of ε_k by Lemma 13.2, so we may define

$$E_1 = \varepsilon_k^{-1}((h \circ i_1)_*(\varepsilon_k D)) \qquad \text{and} \qquad E_2 = \varepsilon_k^{-1}(h_*(\varepsilon(\partial D \otimes \iota))).$$

Lemma 13.5(b) implies that E_1 and E_2 are in $\Sigma^{mk} G_k^j$. Now

$$\begin{aligned}\varepsilon_k(\partial E_2) &= (-1)^{|D|+1}(h_*(\varepsilon(\partial D \otimes \lambda)) - h_*(\varepsilon(\partial D \otimes \kappa))) \\ &= (-1)^{|D|+1}((h \circ i_1)_*(\varepsilon_k \partial D) - (h \circ i_0)_*(\varepsilon_k \partial D)) \\ &= (-1)^{|D|+1}\varepsilon_k(\partial E_1 - C).\end{aligned}$$

Since ε_k is a monomorphism, this implies $\partial((-1)^{|D|} E_2 + E_1) = C$. □

14 Proof of Lemma 13.5

We will assume that $j = k$, since the other cases are essentially the same and the notation is simpler in this case. So suppose we are given a D satisfying:

Assumption 14.1 (i) D is in $\Sigma^{mk} G_k^{k-1}$.

(ii) ∂D is in $\Sigma^{mk} G_k^k$.

With the assumption that $j = k$, Lemma 13.5 specializes to:

Lemma 14.2 *There is a k–th factor homotopy $h\colon M^k \times I \to M^k$ such that*

(a) *$h \circ i_0$ is the identity,*

(b) *the chains $(h \circ i_1)_*(\varepsilon_k D)$, $(h \circ i_1)_*(\varepsilon_k(\partial D))$ and $h_*(\varepsilon(\partial D \otimes \iota))$ are in general position with respect to R^* for all $R\colon \overline{k} \twoheadrightarrow \overline{k}'$, and*

(c) *the chain $h_*(\varepsilon(D \otimes \iota))$ is in general position with respect to R^* for all $R \in \Lambda_{k-1}$.*

Remark 14.3 Since R^* is 1-1, the definition of general position simplifies somewhat: a chain C is in general position with respect to R^* if and only if

$$\dim(\operatorname{supp}(C) \cap \operatorname{im}(R^*)) \leq \dim C + (k' - k)m.$$

Choose a triangulation K of M such that $D \in (c_* K)^{\otimes k}$.

Notation 14.4 Let $\tau_1, \ldots, \tau_r$ be the simplices of K.

We fix orientations for $\tau_1, \ldots, \tau_r$ (with no requirement of consistency among the choices); this allows us to think of the τ_j as generators of $c_* K$.

Since D is in $(c_* K)^{\otimes k}$ it can be written as a sum

$$D = \sum_{\mathbf{a}} n_{\mathbf{a}}\, \tau_{a_1} \otimes \cdots \otimes \tau_{a_k}, \tag{6}$$

where $\mathbf{a}$ runs through multi-indices $(a_1,\ldots,a_k)\in\{1,\ldots,r\}^k$ and $n_{\mathbf{a}}\in\mathbb{Z}$. Then

$$\operatorname{supp}(\varepsilon_k D)=\bigcup_{n_{\mathbf{a}}\neq 0}\tau_{a_1}\times\cdots\times\tau_{a_k}.$$

Similarly, ∂D can be written as

$$\partial D=\sum_{\mathbf{a}} n'_{\mathbf{a}}\,\tau_{a_1}\otimes\cdots\otimes\tau_{a_k}, \tag{7}$$

and we have

$$\operatorname{supp}(\varepsilon_k(\partial D))=\bigcup_{n'_{\mathbf{a}}\neq 0}\tau_{a_1}\times\cdots\times\tau_{a_k}. \tag{8}$$

With this notation, we can spell out the meaning of Assumption 14.1:

Lemma 14.5 (a) *If S is a subset of $\overline{k-1}$ and $\mathbf{a}$ is a multi-index with $n_{\mathbf{a}}\neq 0$ then*

$$\dim\Big(\bigcap_{i\in S}\tau_{a_i}\Big)\le\sum_{i\in S}\dim\tau_{a_i}+(1-|S|)m,$$

where $|S|$ is the cardinality of S.

(b) *The same inequality holds if S is a subset of $\bar{k}$ and $\mathbf{a}$ is a multi-index with $n'_{\mathbf{a}}\neq 0$.*

Proof For part (a), let $k'=k-|S|+1$ and let $R\colon \bar{k}\twoheadrightarrow\bar{k}'$ be any surjection which takes S to 1 and is 1-1 on the rest of $\bar{k}$. Note that $\operatorname{supp}(\varepsilon_k D)\cap\operatorname{im}(R^*)$ is homeomorphic to the subspace

$$\bigcup_{n_{\mathbf{a}}\neq 0}\Big(\bigcap_{i\in S}\tau_{a_i}\times\prod_{i\notin S}\tau_{a_i}\Big)$$

of $M^{k'}$. By Assumption 14.1(i) and Remark 14.3 we have

$$\dim\Big(\bigcap_{i\in S}\tau_{a_i}\times\prod_{i\notin S}\tau_{a_i}\Big)\le\dim D+(1-|S|)m \tag{9}$$

for all $\mathbf{a}$ with $n_{\mathbf{a}}\neq 0$. But

$$\dim\Big(\bigcap_{i\in S}\tau_{a_i}\times\prod_{i\notin S}\tau_{a_i}\Big)=\dim\Big(\bigcap_{i\in S}\tau_{a_i}\Big)+\sum_{i\notin S}\dim\tau_{a_i},$$

and

$$\dim D=\sum_{i=1}^{k}\dim\tau_{a_i}.$$

Combining these equations with inequality (9) completes the proof of part (a). The proof of part (b) is similar. □

Next we need a general-position result that will be proved in Section 15–Section 16.

Proposition 14.6 *Let M be a compact PL manifold of dimension m and let K be a triangulation of M. Then there is a PL homotopy*

$$\phi\colon M \times I \to M$$

with the following properties.

(a) $\phi \circ i_0$ *is the identity.*

(b) *If σ and τ are simplices of K then $(\phi \circ i_1)(\sigma)$ and τ are in general position, ie*

$$\dim((\phi \circ i_1)(\sigma) \cap \tau) \leq \dim((\phi \circ i_1)(\sigma)) + \dim \tau - m.$$

(c) *If σ and τ are any simplices of K then*

$$\dim(\phi(\sigma \times I) \cap \tau) \leq \max(\dim \sigma + 1 + \dim \tau - m, \dim(\sigma \cap \tau)).$$

We choose
$$h\colon M^k \times I \to M^k$$
to be the k–th factor homotopy determined by the PL homotopy ϕ supplied by the Proposition.

Now let $R\colon \bar{k} \twoheadrightarrow \bar{k}'$ be a surjection. To prove Lemma 14.2 we need to show that the chains $(h \circ i_1)_*(\varepsilon_k D)$, $(h \circ i_1)_*(\varepsilon_k(\partial D))$, $h_*(\varepsilon(\partial D \otimes \iota))$, and (if $R \in \Lambda_{k-1}$) $h_*(\varepsilon(D \otimes \iota))$ are in general position with respect to R^*. We will give the proof for $h_*(\varepsilon(\partial D \otimes \iota))$; the other cases are similar and easier.

Denote the set $R^{-1}(R(k))$ by Q.

First observe that $\mathrm{supp}(h_*(\varepsilon(\partial D \otimes \iota)))$ is the union of the $\dim D$ dimensional simplices of $h(\mathrm{supp}(\varepsilon_k D) \times I)$. In particular, it is contained in the union over all $\mathbf{a}$ such that $n'_{\mathbf{a}} \neq 0$ of

$$\tau_{a_1} \times \cdots \times \tau_{a_{k-1}} \times \phi(\tau_{a_k} \times I).$$

It follows that $\mathrm{supp}(h_*(\varepsilon(\partial D \otimes \iota))) \cap \mathrm{im}(R^*)$ is homeomorphic to a PL subspace of the union over all $\mathbf{a}$ such that $n'_{\mathbf{a}} \neq 0$ of

$$\Big(\prod_{j \neq R(k)} \bigcap_{i \in R^{-1}(j)} \tau_{a_i}\Big) \times \Big(\phi(\tau_{a_k} \times I) \cap \bigcap_{i \in Q-\{k\}} \tau_{a_i}\Big),$$

and thus we have

$$\dim(\operatorname{supp}(h_*(\varepsilon(\partial D\otimes\iota)))\cap\operatorname{im}(R^*))\\ \le \max_{n'_{\mathbf{a}}\neq 0}\Big(\sum_{j\neq R(k)}\dim\Big(\bigcap_{i\in R^{-1}(j)}\tau_{a_i}\Big)+\dim\Big(\phi(\tau_{a_k}\times I)\cap\bigcap_{i\in Q-\{k\}}\tau_{a_i}\Big)\Big).$$

It therefore suffices by Remark 14.3 to show that for all $\mathbf{a}$ with $n'_{\mathbf{a}}\neq 0$,

$$\sum_{j\neq R(k)}\dim\Big(\bigcap_{i\in R^{-1}(j)}\tau_{a_i}\Big)+\dim\Big(\phi(\tau_{a_k}\times I)\cap\bigcap_{i\in Q-\{k\}}\tau_{a_i}\Big)\\ \le \dim(h_*(\varepsilon(\partial D\otimes\iota)))+(k'-k)m. \tag{10}$$

Fix a multi-index $\mathbf{a}$ with $n'_{\mathbf{a}}\neq 0$. By Proposition 14.6(c) we know that one of the two following inequalities holds:

$$\dim\Big(\phi(\tau_{a_k}\times I)\cap\bigcap_{i\in Q-\{k\}}\tau_{a_i}\Big)\le \dim\tau_{a_k}+1+\dim\Big(\bigcap_{i\in Q-\{k\}}\tau_{a_i}\Big)-m \tag{11}$$

$$\dim\Big(\phi(\tau_{a_k}\times I)\cap\bigcap_{i\in Q-\{k\}}\tau_{a_i}\Big)\le \dim\Big(\tau_{a_k}\cap\bigcap_{i\in Q-\{k\}}\tau_{a_i}\Big) \tag{12}$$

If (12) holds then

$$\begin{aligned}\sum_{j\neq R(k)}\dim\Big(\bigcap_{i\in R^{-1}(j)}\tau_{a_i}\Big)&+\dim\Big(\phi(\tau_{a_k}\times I)\cap\bigcap_{i\in Q-\{k\}}\tau_{a_i}\Big)\\ &\le \sum_{j\neq R(k)}\dim\Big(\bigcap_{i\in R^{-1}(j)}\tau_{a_i}\Big)+\dim\Big(\tau_{a_k}\cap\bigcap_{i\in Q-\{k\}}\tau_{a_i}\Big)\\ &=\sum_{j=1}^{k'}\dim\Big(\bigcap_{i\in R^{-1}(j)}\tau_{a_i}\Big)\\ &\le \dim(\operatorname{supp}(\varepsilon_k(\partial D))\cap\operatorname{im}(R^*))\quad\text{by equation (8)}\\ &\le \dim(\varepsilon_k(\partial D))+(k'-k)m\quad\text{by Assumption 14.1(ii)}\\ &< \dim(h_*(\varepsilon(\partial D\otimes\iota)))+(k'-k)m\end{aligned}$$

so inequality (10) holds in this case.

If (11) holds then we have

$$
\begin{aligned}
&\sum_{j\neq R(k)} \dim\Big(\bigcap_{i\in R^{-1}(j)} \tau_{a_i}\Big) + \dim\Big(\phi(\tau_{a_k}\times I)\cap \bigcap_{i\in Q-\{k\}} \tau_{a_i}\Big) \\
&\qquad \le \sum_{j\neq R(k)} \Big(\sum_{i\in R^{-1}(j)} \dim\tau_{a_i} + (1-|R^{-1}(j)|)m\Big) + \dim(\phi(\tau_{a_k}\times I)) \\
&\qquad\qquad + \dim\Big(\bigcap_{i\in Q-\{k\}} \tau_{a_i}\Big) - m \quad \text{by Lemma 14.5(b) and inequality (11)} \\
&\qquad \le \Big(\sum_{i\notin R^{-1}(k)} \dim\tau_{a_i}\Big) + (k'-1-k+|Q|)m + \dim\tau_{a_k} + 1 \\
&\qquad\qquad + \dim\Big(\bigcap_{i\in Q-\{k\}} \tau_{a_i}\Big) - m \\
&\qquad \le \Big(\sum_{i\notin R^{-1}(k)} \dim\tau_{a_i}\Big) + (k'-1-k+|Q|)m + \dim\tau_{a_k} + 1 \\
&\qquad\qquad + \Big(\sum_{i\in Q-\{k\}} \tau_{a_i}\Big) + (1-|Q|+1)m - m \quad \text{by Lemma 14.5(b)} \\
&\qquad = \Big(\sum_{i=1}^{k} \dim\tau_{a_i}\Big) + 1 + (k'-k)m \\
&\qquad = \dim(\partial D) + 1 + (k'-k)m \quad \text{by equation (7)} \\
&\qquad = \dim(h_*(\varepsilon(\partial D\otimes\iota))) + (k'-k)m
\end{aligned}
$$

which proves inequality (10) in this case.

Thus we have shown that $h_*(\varepsilon(\partial D\otimes\iota))$ is in general position with respect to R^*.

15 Background for the proof of Proposition 14.6

First we have two simple facts about affine geometry which are the heart of the proof. Recall that the *affine span* of a subset of $\mathbb{R}^n$ is the smallest affine subspace containing it.

Lemma 15.1 *Let σ and τ be simplices in $\mathbb{R}^n$ such that the affine span of $\sigma\cup\tau$ is all of $\mathbb{R}^n$. Then σ and τ are in general position.*

Proof Let U (resp. V) be the affine span of σ (resp. τ). If $U\cap V$ is empty the statement is obvious. Otherwise we can choose a point in $U\cap V$ and move it to the origin by a translation; then U and V become ordinary subspaces which span $\mathbb{R}^n$ and we have

$$\dim(U \cap V) = \dim U + \dim V - n,$$

which proves the lemma. □

Notation 15.2 If σ is a simplex in $\mathbb{R}^n$ and u is an element of $\mathbb{R}^n$ which is not in σ, the convex hull of σ and u will be denoted by $\langle \sigma, u \rangle$.

Lemma 15.3 *Let σ and τ be simplices in $\mathbb{R}^n$. Let u be a point which is not in the affine span of $\sigma \cup \tau$. Then*

$$\langle \sigma, u \rangle \cap \tau = \sigma \cap \tau.$$

Proof Let $v \in \langle \sigma, u \rangle \cap \tau$. Since $v \in \langle \sigma, u \rangle$, we can write v in the form $\alpha u + (1-\alpha)s$, with $s \in \sigma$. If α were nonzero we would have

$$u = \frac{1}{\alpha} v - \frac{1-\alpha}{\alpha} s.$$

Since $v \in \tau$, this would imply that u is in the affine span of $\sigma \cup \tau$. Therefore α must be 0, so v is in σ, and hence in $\sigma \cap \tau$, which proves the lemma. □

Next we recall a well-known way of triangulating $\sigma \times I$. By an *ordered* simplex we will mean a simplex with a total ordering of its vertices.

Lemma 15.4 *Let σ be an ordered simplex and let $v_0 < \cdots < v_l$ be the ordering of its vertices. For $0 \le i \le l$ let $\sigma[i] \subset \sigma \times I$ be the convex hull of*

$$\{(v_j, 0) \mid j \le i\} \cup \{(v_j, 1) \mid j \ge i\}.$$

Then each $\sigma[i]$ is an $(l+1)$–simplex.

Also, the set L whose elements are the $\sigma[i]$ and their faces is a triangulation of $\sigma \times I$.

Remark 15.5 With the notation of Lemma 15.4, let τ be the simplex spanned by $v_0, \ldots, v_{l-1}$. Then

$$\sigma[i] = \langle \tau[i], (v_l, 1) \rangle \quad \text{for each } i < l, \qquad \text{and} \qquad \sigma[l] = \langle \sigma \times \{0\}, (v_l, 1) \rangle.$$

Finally, we need a tool for extending PL maps and homotopies.

Construction 15.6 Let ρ be a simplex in $\mathbb{R}^n$ and let u be an element of $\mathbb{R}^n$ which is not in ρ.

Let Ω be a PL space with a PL homeomorphism $\omega\colon \Omega \to \Delta^m$.

(i) Let $f\colon \rho \to \Omega$ be a PL map and w an element of Ω. We can extend f to a PL map

$$\bar{f}\colon \langle \rho, u\rangle \to \Omega$$

by the formula

$$\bar{f}(\alpha x + (1-\alpha)u) = \omega^{-1}(\alpha\omega(f(x)) + (1-\alpha)\omega(w)).$$

(ii) Next suppose we are given an ordering of the vertices of ρ; extend this to $\langle \rho, u\rangle$ by letting u be the maximal element. Let $\phi\colon \rho \times I \to \Omega$ be a PL homotopy and let z and z' be elements of Ω. We can extend ϕ to a PL homotopy

$$\bar{\phi}\colon \langle \rho, u\rangle \times I \to \Omega$$

as follows. Let $l-1$ be the dimension of ρ. For $i < l$ we have

$$\langle \rho, u\rangle[i] = \langle \rho[i], (u,1)\rangle$$

by Remark 15.5. ϕ is already defined on $\rho[i]$, and we can extend it to $\langle \rho[i], (u,1)\rangle$ by using the construction in part (i). For $i = l$, Remark 15.5 gives

$$\langle \rho, u\rangle[l] = \langle \langle \rho, u\rangle \times \{0\}, (u,1)\rangle = \langle \langle \rho \times \{0\}, (u,0)\rangle, (u,1)\rangle.$$

ϕ is already defined on $\rho \times \{0\}$, and we can extend it to $\langle \langle \rho \times \{0\}, (u,0)\rangle, (u,1)\rangle$ by applying the construction in part (i) twice, taking $(u,0)$ to z and $(u,1)$ to z'.

16 Proof of Proposition 14.6

By the definition of PL manifold, there is a collection of PL subspaces $\Omega_i \subset M$ such that each Ω_i is PL homeomorphic to Δ^m and the interiors $\mathrm{int}(\Omega_i)$ cover M. Choose PL homeomorphisms

$$\omega_i\colon \Omega_i \to \Delta^m.$$

Recall that by the definition in Section 3, a PL space is given as a subspace of some $\mathbb{R}^n$, and therefore inherits a metric. In particular, this is true for the PL manifold M. Let us denote the metric on M by d and the standard norm on $\mathbb{R}^m$ by $||\ ||$.

Definition 16.1 (i) For each Ω_i, choose numbers γ_i and δ_i with

$$||\omega_i(x) - \omega_i(y)|| \le \gamma_i\, d(x,y) \qquad \text{and} \qquad d(x,y) \le \delta_i\, ||\omega_i(x) - \omega_i(y)||$$

for all $x, y \in \Omega_i$ (such numbers exist because ω_i and its inverse are PL maps).

(ii) Let λ be the greater of $\max_i \gamma_i \delta_i$ and 1.

Definition 16.2 Let η be the Lebesgue number of the covering $\{\Omega_i\}$ (with respect to the metric d).

Next observe that if Proposition 14.6 holds for some subdivision of K then it holds for K. Choose a subdivision L of K such that

(i) each Ω_i is a union of simplices of L,

(ii) the restriction of each ω_i to each simplex of L in Ω_i is affine,

(iii) the diameter of each simplex of L is less than $\frac{\eta}{2}$.

It suffices to prove Proposition 14.6 for the triangulation L.

Choose an ordering $v_1, \dots, v_s$ for the vertices of L.

Definition 16.3 For $1 \le p \le s$ let A_p be the union of the simplices of L whose vertices are in the set $\{v_1, \dots, v_p\}$. Let A_0 be the empty set.

Note that A_s is M.

We will construct, by induction over p with $0 \le p \le s$, a PL homotopy

$$\phi_p \colon A_p \times I \to M$$

with the following properties:

(1) The restriction of ϕ_p to $A_{p-1} \times I$ is ϕ_{p-1}.

(2) $\phi_p \circ i_0$ is the inclusion map of A_p into M.

(3) For each $x \in A_p, t \in I$ we have

$$d(\phi_p(x,t), x) \le \frac{\eta}{2\lambda^{s-p}}.$$

(4) If σ is a simplex of L in A_p and τ is any simplex of L then $\phi(\sigma \times \{1\})$ and τ are in general position, and

$$\dim(\phi_p(\sigma \times I) \cap \tau) \le \max(\dim \sigma + 1 + \dim \tau - m, \dim(\sigma \cap \tau)).$$

This will complete the proof of Proposition 14.6, because the homotopy ϕ_s will have the required properties.

The first step of the induction (the case $p = 0$) is trivial. Suppose that ϕ_{p-1} has been constructed.

Notation 16.4 Denote the simplices of L which are in A_p but not A_{p-1} by

$$\pi_1, \dots, \pi_t.$$

For each π_j, let ρ_j be the face opposite v_p; thus

$$\pi_j = \langle \rho_j, v_p \rangle.$$

Combining property (iii) of the triangulation L with property (3) of ϕ_{p-1} and the fact that $\lambda \geq 1$, we see that for each j the diameter of the set

$$\pi_j \cup \phi_{p-1}(\rho_j \times I)$$

is less than η. It follows that for each j we can choose a number $i(j)$ with

$$\pi_j \cup \phi_{p-1}(\rho_j \times I) \subset \mathrm{int}(\Omega_{i(j)}). \tag{13}$$

Notation 16.5 Let Ξ denote the intersection of the sets $\mathrm{int}(\Omega_{i(j)})$.

Note that Ξ is nonempty (for example, it contains v_p).

If z' is any point in Ξ we can apply Construction 15.6(ii) (with $\Omega = \Omega_{i(j)}$ and $z = v_p$) to extend ϕ_{p-1} over all $\pi_j \times I$ simultaneously. The resulting homotopy ϕ_p will automatically satisfy properties (1) and (2) above. We next state two lemmas which will show that there is a z' for which properties (3) and (4) hold.

Lemma 16.6 *ϕ_p satisfies property (3) if z' is in the open ball B of radius $\frac{\eta}{2\lambda^{s-p+1}}$ around v_p.*

In order to verify property (4) for all simplices σ of L which are in A_p, it suffices to consider the simplices which are in A_p but not in A_{p-1} (that is, the simplices $\pi_1, \dots, \pi_t$) since the inductive hypothesis ensures that property (4) holds for all simplices of A_{p-1}.

Lemma 16.7 *For each π_j and for each simplex τ of L, there is an open set $U_{j,\tau}$ which is dense in Ξ such that if z' is in $U_{j,\tau}$ then*

(a) *$\phi_p(\pi_j \times \{1\})$ and τ are in general position, and*

(b) $\dim(\phi_p(\pi_j \times I) \cap \tau) \leq \max(\dim \pi_j + 1 + \dim \tau - m, \dim(\pi_j \cap \tau))$.

Before proving Lemma 16.6 and Lemma 16.7 we observe that the set

$$U = B \cap \bigcap_{j,\tau} U_{j,\tau}$$

will be dense in $B \cap \Xi$ (and in particular nonempty), and if z' is in U then ϕ_p will satisfy properties (1)–(4), which completes the inductive step and thereby the proof of Proposition 14.6.

Proof of Lemma 16.6 Let $x \in \pi_j$ and $t \in I$. Let l be the dimension of π_j. With the notation of Lemma 15.4, we have $(x,t) \in \pi_j[e]$ for some e with $0 \le e \le l$. There are two cases to consider: $e < l$ and $e = l$.

In the first case, Remark 15.5 allows us to write (x,t) as

$$\alpha\,(y,t') + (1-\alpha)\,(v_p,1)$$

with $0 \le \alpha \le 1$, $y \in \rho_j$ and $t' \in I$. By Construction 15.6(ii) we have

$$\omega_{i(j)}(\phi_p(x,t)) = \alpha\,\omega_{i(j)}(\phi_{p-1}(y,t')) + (1-\alpha)\,\omega_{i(j)}(z'), \tag{14}$$

and by property (ii) of the triangulation L we have

$$\omega_{i(j)}(x) = \alpha\,\omega_{i(j)}(y) + (1-\alpha)\,\omega_{i(j)}(v_p). \tag{15}$$

Now we have

$$\begin{aligned} d(\phi_p(x,t),x) &\le \delta_{i(j)}\,||\omega_{i(j)}(\phi_p(x,t)) - \omega_{i(j)}(x)|| \quad \text{by Definition 16.1(i)} \\ &\le \delta_{i(j)}(\alpha\,||\omega_{i(j)}(\phi_{p-1}(y,t')) - \omega_{i(j)}(y)|| \\ &\qquad + (1-\alpha)\,||\omega_{i(j)}(z') - \omega_{i(j)}(v_p)||) \\ &\qquad \text{by equations (14) and (15)} \\ &\le \delta_{i(j)}\gamma_{i(j)}(\alpha\,d(\phi_{p-1}(y,t'),y) + (1-\alpha)\,d(z',v_p)) \\ &\qquad \text{by Definition 16.1(i)} \\ &\le \lambda\left(\alpha\frac{\eta}{2\lambda^{s-p+1}} + (1-\alpha)\,d(z',v_p)\right) \\ &\qquad \text{by Definition 16.1(ii) and property (3) of } \phi_{p-1} \end{aligned}$$

and this will be $\le \frac{\eta}{2\lambda^{s-p}}$ if $d(z',v_p) \le \frac{\eta}{2\lambda^{s-p+1}}$.

For the second case of property (3) we have $e = l$, and Remark 15.5 gives

$$(x,t) = \alpha\,(y,0) + (1-\alpha)\,(v_p,1)$$

for some $y \in \pi_j$. Equation (15) is still valid, and equation (14) is replaced by

$$\omega_{i(j)}(\phi_p(x,t)) = \alpha\, \omega_{i(j)}(y) + (1-\alpha)\, \omega_{i(j)}(z'). \tag{16}$$

Now we have

$$\begin{aligned} d(\phi_p(x,t),x) &\le \delta_{i(j)}\, ||\omega_{i(j)}(\phi_p(x,t)) - \omega_{i(j)}(x)|| \quad \text{by Definition 16.1(i)} \\ &\le \delta_{i(j)}(1-\alpha)\, ||\omega_{i(j)}(z') - \omega_{i(j)}(v_p)|| \quad \text{by equations (15) and (16)} \\ &\le \delta_{i(j)}\gamma_{i(j)} d(z',v_p) \quad \text{by Definition 16.1(i)} \\ &\le \lambda d(z',v_p) \quad \text{by Definition 16.1(ii)} \end{aligned}$$

For this to be $\le \frac{\eta}{2\lambda^{s-p}}$ it again suffices to have $d(z',v_p) \le \frac{\eta}{2\lambda^{s-p+1}}$. □

Proof of Lemma 16.7 We begin by considering the condition in part (a). For this we need a precise description of the subspace $\phi_p(\pi_j \times \{1\})$. Recall that ρ_j denotes the face of π_j opposite to the vertex v_p. By statement (13), the set $\phi_{p-1}(\rho_j \times \{1\})$ is contained in $\Omega_{i(j)}$. The image of $\phi_{p-1}(\rho_j \times \{1\})$ under the PL homeomorphism

$$\omega_{i(j)}\colon \Omega_{i(j)} \to \Delta^m$$

is a union of simplices which we will denote by $\chi_1, \dots, \chi_u$. By Construction 15.6(ii), the subspace $\phi_p(\pi_j \times \{1\})$ is $\omega_{i(j)}^{-1}$ of the union of the simplices

$$\langle \chi_1, \omega_{i(j)}(z')\rangle, \dots, \langle \chi_u, \omega_{i(j)}(z')\rangle.$$

In order for z' to satisfy the condition in Lemma 16.7(a), each of the pairs

$$(\langle \chi_q, \omega_{i(j)}(z')\rangle, \omega_{i(j)}(\tau))$$

must be in general position (note that $\omega_{i(j)}(\tau)$ is a simplex by property (ii) of the triangulation L). By Lemma 15.1, this condition is automatically satisfied (with no restriction on z') by those pairs for which the affine span of $\chi_q \cup \omega_{i(j)}(\tau)$ is all of $\mathbb{R}^m$. For the remaining pairs, it suffices by Lemma 15.3 that $\omega_{i(j)}(z')$ should not be in the affine span of $\chi_q \cup \omega_{i(j)}(\tau)$ (note that χ_q and $\omega_{i(j)}(\tau)$ are in general position because we have assumed that ϕ_{p-1} satisfies property (4)). Since this affine span is nowhere dense, the set of allowable z' for each such pair is an open set V_q which is dense in Ξ. The intersection of the V_q is an open set V which is dense in Ξ and if $z' \in V$ then the condition in Lemma 16.7(a) is satisfied.

For part (b), let l denote the dimension of π_j. With the notation of Lemma 15.4 and Remark 15.5, we have

$$\pi_j \times I = \Big(\bigcup_{0\le e<l} \langle \rho_j[e], (v_p,1)\rangle\Big) \cup \langle \pi_j \times \{0\}, (v_p,1)\rangle.$$

For each $e < l$ the image of $\phi_{p-1}(\rho_j[e])$ under $\omega_{i(j)}$ is a union of simplices which we will denote by $\psi_1^e, \ldots$. By Construction 15.6(ii), $\phi_p(\langle \rho_j[e], (v_p, 1)\rangle)$ is $\omega_{i(j)}^{-1}$ of the union of the simplices

$$\langle \psi_1^e, \omega_{i(j)}(z')\rangle, \ldots$$

and $\phi_p(\langle \pi_j \times \{0\}, (v_p, 1)\rangle)$ is $\omega_{i(j)}^{-1}$ of the simplex

$$\langle \omega_{i(j)}(\pi_j), \omega_{i(j)}(z')\rangle.$$

In order for z' to satisfy the condition in Lemma 16.7(b), we must have

$$\text{(17)}\quad \dim(\langle \psi_q^e, \omega_{i(j)}(z')\rangle \cap \omega_{i(j)}(\tau)) \le \max(\dim \pi_j + 1 + \dim \tau - m, \dim(\pi_j \cap \tau))$$

for all q, and

$$\text{(18)}\quad \dim(\langle \omega_{i(j)}(\pi_j), \omega_{i(j)}(z')\rangle \cap \omega_{i(j)}(\tau)) \le \max(\dim \pi_j + 1 + \dim \tau - m, \dim(\pi_j \cap \tau)).$$

Since we have assumed that ϕ_{p-1} satisfies property (4), we have that for all q,

$$\text{(19)}\qquad \dim(\psi_q^e \cap \omega_{i(j)}(\tau)) \le \max(\dim \rho_j + 1 + \dim \tau - m, \dim(\rho_j \cap \tau)).$$

To prove inequality (17) we must consider two cases: either the affine span of the union $\psi_q^e \cup \omega_{i(j)}(\tau)$ is all of $\mathbb{R}^m$ or it is not. In the first case we have (with no restriction on z')

$$\begin{aligned}
\dim(\langle \psi_q^e, \omega_{i(j)}(z')\rangle \cap \omega_{i(j)}(\tau)) &\le \dim(\langle \psi_q^e, \omega_{i(j)}(z')\rangle) + \dim(\omega_{i(j)}(\tau)) - m \\
&\qquad \text{by Lemma 15.1} \\
&\le \dim \psi_q^e + 1 + \dim \tau - m \\
&\le \dim \rho_j + 2 + \dim \tau - m \\
&= \dim \pi_j + 1 + \dim \tau - m.
\end{aligned}$$

In the second case we assume that $\omega_{i(j)}(z')$ is not in the affine span of $\psi_q^e \cup \omega_{i(j)}(\tau)$. Then we have

$$\begin{aligned}
\dim(\langle \psi_q^e, \omega_{i(j)}(z')\rangle \cap \omega_{i(j)}(\tau)) &= \dim(\psi_q^e \cap \omega_{i(j)}(\tau)) \quad \text{by Lemma 15.3} \\
&\le \max(\dim \rho_j + 1 + \dim \tau - m, \dim(\rho_j \cap \tau)) \\
&\qquad \text{by inequality (19)} \\
&\le \max(\dim \pi_j + 1 + \dim \tau - m, \dim(\pi_j \cap \tau)).
\end{aligned}$$

To prove inequality (18) we again consider two cases: either the affine span of the union $\omega_{i(j)}(\pi_j) \cup \omega_{i(j)}(\tau)$ is all of $\mathbb{R}^m$ or it is not. In the first case we have (with no restriction on z')

$$\begin{aligned}\dim(\langle\omega_{i(j)}(\pi_j),\omega_{i(j)}(z')\rangle \cap \omega_{i(j)}(\tau)) &\le \dim(\langle\omega_{i(j)}(\pi_j),\omega_{i(j)}(z')\rangle) + \dim(\omega_{i(j)}(\tau)) - m \\ &\qquad \text{by Lemma 15.1} \\ &\le \dim \pi_j + 1 + \dim \tau - m.\end{aligned}$$

In the second case we assume that $\omega_{i(j)}(z')$ is not in the affine span of the union $\omega_{i(j)}(\pi_j) \cup \omega_{i(j)}(\tau)$. Then we have

$$\begin{aligned}\dim(\langle\omega_{i(j)}(\pi_j),\omega_{i(j)}(z')\rangle \cap \omega_{i(j)}(\tau)) &= \dim(\omega_{i(j)}(\pi_j) \cap \omega_{i(j)}(\tau)) \\ &\qquad \text{by Lemma 15.3} \\ &= \dim(\pi_j \cap \tau).\end{aligned}$$

To sum up, there is a dense open subset W of Ξ such that if $z' \in W$ then the condition of Lemma 16.7(b) is satisfied.

Finally, let $U_{j,\tau}$ be $V \cap W$. □

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