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Obstructions to special Lagrangian desingularizations and the Lagrangian prescribed boundary problem

MARK HASKINS
TOMMASO PACINI

We exhibit infinitely many, explicit special Lagrangian isolated singularities that admit no asymptotically conical special Lagrangian smoothings. The existence/nonexistence of such smoothings of special Lagrangian cones is an important component of the current efforts to understand which singular special Lagrangians arise as limits of smooth special Lagrangians.

We also use soft methods from symplectic geometry (the relative version of the h–principle for Lagrangian immersions) and tools from algebraic topology to prove (both positive and negative) results about Lagrangian desingularizations of Lagrangian submanifolds with isolated singularities; we view the (Maslov-zero) Lagrangian desingularization problem as the natural soft analogue of the special Lagrangian smoothing problem.

53D12, 53C38; 53C42

1 Introduction

Let M be a Calabi–Yau manifold of complex dimension n with Kähler form ω and nonzero parallel holomorphic n–form Ω. Suitably normalized, $\mathrm{Re}\,\Omega$ is a calibrated form whose calibrated submanifolds are called special Lagrangian (SL) submanifolds; see Harvey and Lawson [22]. SL submanifolds are thus a very natural class of volume-minimizing submanifolds in Calabi–Yau manifolds. SL submanifolds also appear in string theory as "supersymmetric cycles" and play a fundamental role in the Strominger–Yau–Zaslow approach to mirror symmetry [48]. In SYZ-related work and also in other problems in special Lagrangian geometry, singular SL objects play a fundamental role. This has motivated a considerable amount of recent work devoted to singular SL submanifolds [9; 18; 19; 20; 23; 24; 25; 28; 29; 30; 31; 32; 33; 34; 39].

So far, the most fruitful approach to understanding singular SL objects has been to focus on special types of singularities. For example, one might study ruled SL 3–folds, U(1)–invariant SL 3–folds, or SL 3–folds with isolated conical singularities. In particular, a *special Lagrangian n–fold with isolated conical singularities* is a singular

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SL n–fold with an isolated set of singular points each of which is modelled on a SL cone in $\mathbb{C}^n$. We call such a singular SL n–fold a *special Lagrangian conifold*. In a series of five papers, Joyce developed the basic foundations of a theory of compact SL conifolds in (almost) Calabi–Yau manifolds [29].

In particular, Joyce studied the desingularization theory of SL conifolds. The desingularization theory has both local and global aspects; since the global aspects are described in detail in [29], it remains to understand the local smoothing question. More concretely, given a SL cone C in $\mathbb{C}^n$ we would like to understand all asymptotically conical SL (ACSL) n–folds in $\mathbb{C}^n$ which at infinity approach the given cone C. We call this the *ACSL smoothing problem*.

The main aim of this paper is to study the ACSL smoothing problem and various closely related questions. To begin with we prove a number of nonsmoothing results. For example, we exhibit an explicit SL cone C in $\mathbb{C}^6$ whose link is a 5–manifold which is not nullcobordant. Hence C is an isolated SL singularity which for topological reasons cannot admit any smoothing whatsoever.

We then show that there are *infinitely many* explicit SL cones which admit no ACSL smoothings (with rate $\lambda \leq 0$ – see equation (2–5) for the precise definition). On the other hand, infinitely many of these SL cones have links which are nullcobordant. Therefore there is no differential-topological obstruction to smoothing these singularities; in fact, our obstruction to ACSL smoothings arises from analytic constraints implicit in the SL condition.

Given such an SL cone C, it is natural to ask whether C admits Lagrangian rather than special Lagrangian smoothings. In fact, since any SL submanifold has zero Maslov class, the natural weakening of the SL smoothing question to the Lagrangian category is to ask whether C admits Maslov-zero Lagrangian smoothings. Therefore the main focus of the paper is on the Lagrangian-topological aspects of these smoothing questions, viewed as a preliminary step toward a better understanding of the existence/nonexistence of ACSL smoothings and SL desingularizations of SL conifolds.

More specifically, in this paper we address questions such as the following:

(1) Let C be a SL cone in $\mathbb{C}^n$. Does there exist a smooth complete oriented immersed Lagrangian submanifold Y which coincides with C outside the sphere $\mathbb{S}^{2n-1}$? What can we say about the topology of Y? Can we find such a Y with zero Maslov class?

(2) Let X be a SL conifold in an (almost) Calabi–Yau manifold. Does there exist a Maslov-zero Lagrangian submanifold Y which coincides with X away from a neighbourhood of its singularities?

We think of these two questions as soft analogues of the SL desingularization problem. Notice, in particular, that in questions (1) and (2) the smoothings coincide with the original singular object outside a compact set. Real analyticity would prevent this from happening in the SL case: for example, a complete ACSL smoothing of a SL n–cone C will never coincide with C in an open set (or even intersect it along a $(n-1)$–dimensional submanifold). We can take this behaviour into account by prescribing other types of asymptotics on the "ends" of Y. Hence one could also ask the following as a soft analogue of the existence question for ACSL smoothings.

(3) Let C be a SL cone in $\mathbb{C}^n$ with link Σ. Suppose we are given a Maslov-zero Lagrangian conical end, ie a Maslov-zero Lagrangian immersion f of $\Sigma\times[1,\infty)$ which is asymptotic to C at infinity. What Maslov-zero Lagrangian "fillings" of the conical end f exist?

All these Lagrangian smoothing problems can be viewed as particular cases of a certain boundary value problem for Lagrangian submanifolds which we call the *Prescribed Boundary Problem* and study in Section 4. The Prescribed Boundary Problem is related to but different from the notion of Lagrangian cobordism groups introduced by Arnold [1] and studied in greater depth by Audin [3; 4].

At least in the category of *immersed* Lagrangian submanifolds it turns out that the Prescribed Boundary Problem is a soft problem, in the sense that it obeys an h–principle; using the relative version of the Gromov–Lees h–principle for Lagrangian immersions we reduce the solvability of the Prescribed Boundary Problem to the problem of extending a certain map from the boundary of a manifold with boundary to its interior. This converts the solvability question into a problem in homotopy theory, which we use obstruction theory to study.

We show that in general there are obstructions to solving the Prescribed Boundary Problem and therefore obstructions to smoothing our Lagrangian singularities. To illustrate this we give explicit examples of initial data for which the Prescribed Boundary Problem has no solutions. Moreover, in low dimensions we can identify the obstructions very concretely and in particular identify situations in which all the obstructions vanish. An easy corollary of this is that any SL conifold in dimensions 2 or 3 admits infinitely many topologically distinct oriented Maslov-zero Lagrangian desingularizations (Theorem 6.7 and Corollary 6.11). In particular, any SL cone in $\mathbb{C}^2$ and $\mathbb{C}^3$ admits infinitely many topologically distinct oriented Maslov-zero smoothings (Corollary 6.9).

The rest of this paper is organized as follows:

In Section 2 we prove various nonsmoothability results for special Lagrangian singularities. Section 2.1 serves as an introduction to smoothing problems for isolated conical singularities of minimal varieties, recalling known smoothing results for volume-minimizing hypersurfaces. Section 2.2, Section 2.3 and Section 2.4 recall basic definitions and facts about Lagrangian/special Lagrangian geometry, special Lagrangian cones and asymptotically conical special Lagrangian n–folds respectively. In Section 2.5 we prove a first nonsmoothability result, Theorem 2.2. After recalling some basic results about G–invariant SL n–folds in Section 2.6, in Section 2.7 we prove a further nonsmoothability result, Theorem 2.11.

In Section 3.1–Section 3.3 we recall the Gromov–Lees h–principle for Lagrangian immersions of closed manifolds [17; 38], focusing on how it reduces an *a priori* geometric question to a topological one. Section 3.4 discusses the Maslov class of a Lagrangian immersion in $\mathbb{C}^n$ and uses the Lagrangian h–principle to prove a Maslov-class realizability result, Proposition 3.19. Section 3.5 gives examples to demonstrate how the h–principle provides an effective tool to prove both existence and nonexistence results for Lagrangian immersions. The material of this section is well-known but is included both as an introduction to the h–principle for h–principle novices and to establish notation, terminology and some results that are needed in Section 4. Those already comfortable with the Lagrangian h–principle are advised to skim this section and proceed to Section 4.

In Section 4.1–Section 4.2 we describe the *Lagrangian Cobordism Problem*, some obstructions to solving it and a refinement of it which we call the *Prescribed Boundary Problem*. Both problems are boundary value problems for Lagrangian immersions in $\mathbb{C}^n$ of compact manifolds with boundary. In Section 4.3 we show how to use a relative version of the h–principle for Lagrangian immersions (Theorem 4.6) to analyze the solvability of the Prescribed Boundary Problem. As a result we reduce the solvability of the Prescribed Boundary Problem to a problem in algebraic topology (see Lemma 4.11); namely, does a certain map with values in GL(n, $\mathbb{C}$) extend continuously from the boundary to the interior? In Section 4.4 we recall a standard obstruction theory framework for analyzing extension problems and apply it to the case-at-hand. We pay particular attention to cases in which either all obstructions vanish or where the obstructions can be described particularly concretely. For example, in less than 6 complex dimensions there are exactly two obstructions to extending the map in question; in Theorem 4.16 we write down these obstructions explicitly. One obstruction is closely related to the Maslov class of the initial data; moreover in dimensions 2 and 3 the other obstruction vanishes. This allows us to prove particularly nice results for Maslov-zero initial data in these dimensions.

In Section 5.1–Section 5.3 we apply the results of Section 4 to study the solvability of the Prescribed Boundary Problem in complex dimensions 5 and less, taking care to show how our Assumptions A–C can be satisfied. The main results are Theorem 5.4, Theorem 5.8 and Theorem 5.12.

In Section 6 we return to the soft Lagrangian desingularization problems and address Questions (1)–(3) of Section 1 using the results of Section 5. We introduce a class of singular oriented Lagrangian submanifolds called *oriented Lagrangian submanifolds with exact isolated singularities*. This class includes the SL conifolds studied by Joyce in the series of papers summarized in [29]. In Theorem 6.7 we add the Maslov-zero assumption and apply our results from Section 5 to prove desingularization and smoothing results for Lagrangian submanifolds of this type in $\mathbb{C}^2$ and $\mathbb{C}^3$. Section 6.2 extends these results to almost Calabi–Yau manifolds, while Section 6.3 addresses Question (3) above.

In Appendix A we compare the work of this paper to existing work in the Lagrangian and special Lagrangian literature.

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2 Nonsmoothable special Lagrangian singularities

The singularities of area-minimizing surfaces and codimension 1 volume-minimizing objects are by now fairly well-understood. In higher dimension/codimension, examples show that a far wider range of singular behaviour is possible and as a consequence many codimension one results fail in higher codimension. However, for special types of volume-minimizing objects, particularly *calibrated submanifolds* (or currents), one might still hope that some codimension one features survive. In the case of smoothability of special Lagrangian singularities, however, we will see shortly that such optimism would be misplaced.

2.1 Smoothability of isolated conical singularities of minimal varieties

Let C be a minimal hypercone in $\mathbb{R}^n$ with $\mathrm{Sing}(C) = (0)$. Then the link of the cone, $\Sigma = C \cap \mathbb{S}^{n-1}$, is an embedded smooth compact $(n-2)$–dimensional minimal manifold of $\mathbb{S}^{n-1}$. It follows from the Maximum Principle that Σ must be connected. Since Σ is an embedded hypersurface of $\mathbb{S}^{n-1}$ and the tangent bundle of $\mathbb{S}^{n-1}$ is stably trivial, it follows that the tangent bundle of Σ is also stably trivial. This implies that the link of any minimal hypercone is nullcobordant (see Section 2.5) and therefore admits some (topological) smoothing.

In fact, Hardt and Simon proved the following smoothing result for volume-minimizing hypercones:

Theorem 2.1 [21, Theorem 2.1] *Let C be a minimizing hypercone C in $\mathbb{R}^n$ with Sing$(C) = (0)$ and as above let Σ denote the link of the cone. Let E denote one of the two components of $\mathbb{R}^n \setminus \overline{C}$. Then E contains a unique oriented embedded real analytic nonsingular volume-minimizing hypersurface S with empty boundary such that $\operatorname{dist}(S, 0) = 1$. The hypersurface S has the following additional properties:*

(i) *For any $v \in E$, the ray $\{\lambda v : \lambda > 0\}$ intersects S in a unique point, and the intersection is transverse.*

(ii) *(Up to orientation) any oriented minimizing hypersurface contained in E coincides with some dilation of S.*

(iii) *S is asymptotically conical in the sense that there exists some $T > 0$ and some compact subset $K \subset S$ so that $S \setminus K$ is diffeomorphic to $(T, \infty) \times \Sigma$ and so that outside K, S is graphical over the cone C for some C^2 graphing function v (which satisfies some decay conditions at infinity – see Hardt and Simon [21, page 106 equation 1.9] for details).*

Moving to higher codimension, in the spirit of the previous paragraphs it is natural to ask whether calibrated cones have canonical desingularizations analogous to those of minimizing hypercones. In particular, for special Lagrangian cones Rick Schoen originally posed this question to the authors at the IPAM Workshop on Lagrangian submanifolds; the authors would therefore like to thank him for suggesting this problem. Unfortunately, it is not true that SL cones admit canonical SL smoothings; we give examples below to show that there are SL cones which admit no SL smoothings whatsoever.

To proceed further first we recall some basic definitions from Lagrangian and special Lagrangian geometry.

2.2 Calibrations and special Lagrangian geometry in $\mathbb{C}^n$

Let (M, g) be a Riemannian manifold. Let V be an oriented tangent p–plane on M, ie a p–dimensional oriented vector subspace of some tangent plane $T_x M$ to M. The restriction of the Riemannian metric to V, $g|_V$, is a Euclidean metric on V which together with the orientation on V determines a natural p–form on V, the volume form vol_V. A closed p–form ϕ on M is a *calibration* [22] on M if for every oriented tangent p–plane V on M we have $\phi|_V \le \mathrm{vol}_V$. Let L be an oriented submanifold of M with dimension p. L is a *ϕ–calibrated submanifold* if $\phi|_{T_xL} = \mathrm{vol}_{T_xL}$ for all $x \in L$.

There is a natural extension of this definition to singular calibrated submanifolds using the language of Geometric Measure Theory and rectifiable currents [22, Section II.1]. The key property of calibrated submanifolds (even singular ones) is that they are *homologically volume minimizing* [22, Theorem II.4.2]. In particular, any calibrated submanifold is automatically *minimal*, ie has vanishing mean curvature.

Let $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$ be standard complex coordinates on $\mathbb{C}^n$ equipped with the Euclidean metric. Let

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j = d\lambda, \text{ where } \lambda = \frac{1}{2}\Big(\sum_{i=1}^{n} x_i dy_i - y_i dx_i\Big), \tag{2–1}$$

be the standard symplectic 2–form on $\mathbb{C}^n$. Recall that an immersion $i\colon L^n \to \mathbb{R}^{2n}$ of an n–manifold L is *Lagrangian* if $i^*\omega = 0$ or, equivalently, if the natural complex structure J on $\mathbb{R}^{2n} = \mathbb{C}^n$ induces an isomorphism between TL and the normal bundle NL.

Define a complex n–form Ω on $\mathbb{C}^n$ by

$$\Omega = dz_1 \wedge \ldots \wedge dz_n. \tag{2–2}$$

Then $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are real n–forms on $\mathbb{C}^n$. $\mathrm{Re}\,\Omega$ is a calibration on $\mathbb{C}^n$ whose calibrated submanifolds we call *special Lagrangian submanifolds* of $\mathbb{C}^n$, or SL n–folds for short. There is a natural extension of special Lagrangian geometry to any Calabi–Yau manifold M by replacing Ω with the natural parallel holomorphic $(n,0)$–form on M.

More generally, let $f\colon L \to \mathbb{C}^n$ be a Lagrangian immersion of the oriented n–manifold L, and Ω be the holomorphic $(n,0)$–form defined in (2–2). Then $f^*\Omega$ is a complex n–form on L satisfying $|f^*\Omega| = 1$ [22, page 89]. Hence we may write

$$f^*\Omega = e^{i\theta}\,\mathrm{vol}_L \quad \text{on } L, \tag{2–3}$$

for some *phase function* $e^{i\theta}\colon L \to \mathbb{S}^1$. We call $e^{i\theta}$ the *phase of the oriented Lagrangian immersion* f. L is a SL n–fold in $\mathbb{C}^n$ if and only if the phase function $e^{i\theta} \equiv 1$. Reversing the orientation of L changes the sign of the phase function $e^{i\theta}$. The differential $d\theta$ is a well-defined closed 1–form on L satisfying [22, page 96]

$$d\theta = \iota_H \omega, \tag{2–4}$$

where H is the mean curvature vector of L. The cohomology class $[d\theta] \in H^1(L,\mathbb{R})$ is closely related to the *Maslov class* of the Lagrangian immersion defined in Section 3.4.

In particular, (2–4) implies that a connected component of L is minimal if and only if the phase function $e^{i\theta}$ is constant. $e^{i\theta} \equiv 1$ is equivalent to L being special Lagrangian; if $e^{i\theta} \equiv e^{i\theta_0}$ for some constant $\theta_0 \in [0,2\pi)$ we will call L, θ_0*–special Lagrangian* or θ_0-SL for short. θ_0-SL n–folds are calibrated submanifolds with respect to the real part of the rotated holomorphic $(n,0)$–form, $\Omega_{\theta_0} = e^{-i\theta_0}\Omega$.

2.3 Special Legendrian submanifolds and special Lagrangian cones

For any compact oriented embedded (but not necessarily connected) submanifold $\Sigma \subset \mathbb{S}^{2n-1}(1) \subset \mathbb{C}^n$ define the *cone on* Σ,

$$C(\Sigma) = \{tx : t \in \mathbb{R}^{\geq 0}, x \in \Sigma\}.$$

A cone C in $\mathbb{C}^n$ (that is a subset invariant under dilations) is *regular* if there exists Σ as above so that $C = C(\Sigma)$, in which case we call Σ the *link* of the cone C. $C'(\Sigma) := C(\Sigma) - \{0\}$ is an embedded smooth submanifold, but $C(\Sigma)$ has an isolated singularity at 0 unless Σ is a totally geodesic sphere.

Let r denote the radial coordinate on $\mathbb{C}^n$ and let X be the Liouville vector field

$$X = \frac{1}{2} r \frac{\partial}{\partial r} = \frac{1}{2} \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}.$$

The unit sphere $\mathbb{S}^{2n-1}$ inherits a natural contact form

$$\gamma = \iota_X \omega|_{\mathbb{S}^{2n-1}} = \sum_{j=1}^{n} x_j\, dy_j - y_j\, dx_j \Big|_{\mathbb{S}^{2n-1}}$$

from its embedding in $\mathbb{C}^n$. A regular cone C in $\mathbb{C}^n$ is Lagrangian if and only if its link Σ is Legendrian. We call a submanifold Σ of $\mathbb{S}^{2n-1}$ *special Legendrian* if the cone over Σ, $C'(\Sigma)$ is special Lagrangian in $\mathbb{C}^n$.

2.4 Asymptotically conical SL n–folds and ACSL smoothings

Roughly speaking, an *asymptotically conical special Lagrangian n–fold L* (ACSL) in $\mathbb{C}^n$ is a nonsingular SL n–fold which tends to a regular SL cone C at infinity. In particular, if L is an ACSL submanifold then $\lim_{t\to 0_+} tL = C$ for some SL cone C. Hence ACSL submanifolds give rise to local models for how nonsingular SL submanifolds can develop isolated singularities modelled on the cone C. Conversely, as in [29], one can use ACSL submanifolds to desingularize SL conifolds.

More precisely, we say that a closed, nonsingular SL submanifold L of $\mathbb{C}^n$ is *asymptotically conical (AC) with decay rate* $\lambda < 2$ *and cone* C if for some compact subset $K \subset L$ and some $T > 0$, there is a diffeomorphism $\phi\colon \Sigma \times (T, \infty) \to L \setminus K$ such that

$$|\nabla^k(\phi - i)| = O(r^{\lambda-1-k}) \quad \text{as } r \to \infty \text{ for } k = 0, 1, \tag{2–5}$$

where $i(r, \sigma) = r\sigma$ and ∇ and $|\cdot|$ are defined using the cone metric g' on C.

Given a regular SL cone C in $\mathbb{C}^n$ an important question is: does C admit an ACSL submanifold L asymptotic to the given SL cone C? We call this the *asymptotically conical special Lagrangian smoothing problem for the SL cone* C or for short the *ACSL smoothing problem*.

From the point of view of the global smoothing theory developed by Joyce in [29] it is natural to ask whether C admits an ACSL smoothing with decay rate $\lambda \leq 0$. We call this the *ACSL smoothing problem with decay*.

We will see shortly that neither version of the ACSL smoothing problem is always solvable.

2.5 Nonsmoothable SL singularities I

Suppose that C is a SL cone with link Σ for which the ACSL smoothing problem is solvable. Then clearly the smooth oriented manifold Σ bounds a compact smooth oriented manifold M. In other words, Σ is *oriented cobordant to the empty set*. This is a special case of the oriented version of the Cobordism Problem solved by Wall [50], following the pioneering work of Thom in the unoriented case [49].

Wall determined the structure of the oriented cobordism ring $\Omega_* = (\Omega_0, \Omega_1, \Omega_2, \dots)$ [50]. He showed that the oriented cobordism groups Ω_n are trivial for $n = 1, 2, 3, 6, 7$ and are nonzero in all other dimensions (see [41, page 203] for a table of Ω_n up to $n = 11$). Hence for $n \in \{1, 2, 3, 6, 7\}$ any oriented n–manifold Σ bounds an oriented $(n+1)$–manifold L.

More generally, Wall [50, page 306] proved that $[\Sigma] = 0$ in the oriented cobordism ring if and only if all the Pontrjagin and Stiefel–Whitney numbers of Σ vanish (see Milnor and Stasheff [41, Chapters 4 and 16] for a definition of these characteristic numbers). For instance it follows from (3–3) that $\mathbb{CP}^2$ does not bound any oriented 5–manifold. On the other hand, if all the Pontrjagin and Stiefel–Whitney classes of $T\Sigma$ are zero then of course so are the Pontrjagin and Stiefel–Whitney numbers. Hence every stably parallelizable manifold bounds.

Thus if we can find a SL cone C whose link Σ is not nullcobordant then for purely topological reasons neither version of the ACSL smoothing problem for C is solvable. We will give such an example below and hence see that there is no obvious SL analogue of the Hardt–Simon desingularization theorem for minimizing hypercones.

Theorem 2.2 *Let A denote an $n \times n$ complex matrix. Define a map ϕ from $SU(n)$ to $Sym(n, \mathbb{C})$, the symmetric $n \times n$ complex matrices, by*

$$A \mapsto \frac{1}{\sqrt{n}} A A^T.$$

Then ϕ passes to a well-defined map $\widehat{\phi}$ on the quotient $SU(n)/SO(n)$. The induced map $\widehat{\phi}\colon SU(n)/SO(n) \to Sym(n, \mathbb{C})$ gives a special Legendrian embedding of the quotient $\Sigma = SU(n)/SO(n)$ into the unit sphere of $Sym(n, \mathbb{C})$ (and hence the cone over Σ is special Lagrangian).

For $n = 3$, $\Sigma = SU(3)/SO(3)$ does not bound any compact 6–manifold and in particular the ACSL smoothing problem with cone $C = C(\Sigma)$ has no solutions.

Proof The fact that the above embedding is special Lagrangian was proven by Cheng [11, Theorem 1.2]. Audin showed that the Stiefel–Whitney number $w_2 w_3(T\Sigma) = 1 \neq 0$ for $n = 3$ [5, page 191]. Since Σ has a nonzero Stiefel–Whitney number it does not bound any smooth 6–manifold. □

Remark 2.3 In fact, Ohnita [43, Theorem 2.2] has recently studied the *stability index* of these (very symmetric) SL cones. He showed that the SL cone over $\mathrm{SU}(3)/\mathrm{SO}(3)$ is *strictly stable* in the sense of [23; 29]. Hence this SL cone would be a good candidate for a "common singularity type". On the other hand what we have just said shows that there can be no SL smoothings of X.

Next we exhibit infinitely many SL cones for which there is no solution to the ACSL smoothing problem with decay. These nonsmoothing results will also show that there are obstructions to the solvability of the ACSL smoothing problem beyond the cobordism class of the link of the cone.

First we need some auxiliary results on SL n–folds with symmetry and moment maps.

2.6 G–invariant SL submanifolds

In symplectic geometry continuous group actions which preserve ω are often induced by a collection of functions, the *moment map* of the action. We describe the situation in $\mathbb{C}^n$, where the situation is particularly simple.

Let (M, J, g, ω) be a Kähler manifold, and let G be a Lie group acting smoothly on M preserving both J and g (and hence ω). This induces a linear map ϕ from the Lie algebra $\mathfrak{g}$ of G to the vector fields on M. Given $x \in \mathfrak{g}$, let $v = \phi(x)$ denote the corresponding vector field on M. Since $\mathcal{L}_v\omega = 0$, it follows from Cartan's formula that $\iota_v\omega$ is a closed 1–form on M. If $H^1(M, \mathbb{R}) = 0$ then there exists a smooth function μ^x on M, unique up to a constant, such that $d\mu^x = \iota_v\omega$. μ^x is called a moment map for x, or a Hamiltonian function for the Hamiltonian vector field v. We can attempt to collect all these functions μ^x, $x \in \mathfrak{g}$, together to make a moment map for the whole action. A smooth map $\mu\colon M \to \mathfrak{g}^*$ is called a *moment map* for the action of G on M if

(i) $\iota_{\phi(x)}\omega = \langle x, d\mu\rangle$ for all $x \in \mathfrak{g}^*$ where $\langle , \rangle$ is the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$;

(ii) $\mu\colon M \to \mathfrak{g}^*$ is equivariant with respect to the G–action on M and the coadjoint G–action on $\mathfrak{g}^*$.

In general there are obstructions to a symplectic G–action admitting a moment map. The subsets $\mu^{-1}(c)$ are the *level sets* of the moment map. The *centre* $Z(\mathfrak{g}^*)$ is the subspace of $\mathfrak{g}^*$ fixed by the coadjoint action of G. Property (ii) of μ implies that a level set $\mu^{-1}(c)$ is G–invariant if and only if $c \in Z(\mathfrak{g}^*)$.

When a moment map for a symplectic group action does exist, it is a very useful tool for studying G–invariant isotropic submanifolds of (M, ω). Using property (i) above it is easy to see that any G–invariant isotropic submanifold of (M, ω) must be contained in some level set $\mu^{-1}(c)$ of the moment map. Using property (ii) one can check that $c \in Z(\mathfrak{g}^*)$. Hence we have the useful fact that: any G–invariant isotropic submanifold of (M, ω) is contained in the level set $\mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.

The group of automorphisms of $\mathbb{C}^n$ preserving g, ω and Ω is $\mathrm{SU(n)} \ltimes \mathbb{C}^n$, where $\mathbb{C}^n$ acts by translations. Since $\mathbb{C}^n$ is simply connected, each element x in the Lie algebra $\mathfrak{su}(n) \ltimes \mathbb{C}^n$ has a moment map $\mu^x\colon \mathbb{C}^n \to \mathbb{R}$. More concretely, we can describe these functions as follows: a function is said to be a *harmonic Hermitian quadratic* if it is of the form

$$f = c + \sum_{i=1}^{n} (b_i z_i + \overline{b}_i \overline{z}_i) + \sum_{i,j=1}^{n} a_{ij} z_i \overline{z}_j \tag{2–6}$$

for some $c \in \mathbb{R}$, b_i, $a_{ij} \in \mathbb{C}$ with $a_{ij} = \overline{a}_{ji}$ and $\sum_{i=1}^n a_{ii} = 0$. A harmonic Hermitian quadratic with $c = 0$, $a_{ij} = 0$ for all i and j corresponds to the moment map of a translation. A harmonic Hermitian quadratic with $c = 0$, $b_i = 0$ for all i corresponds to the moment map of the element $A = \sqrt{-1}\,(a_{ij}) \in \mathfrak{su}(n)$. Moreover, these functions satisfy conditions (i) and (ii) above, and thus give rise to a moment map for the whole group action.

Since this group action also preserves the condition to be special Lagrangian, the restriction to special Lagrangian submanifolds of the moment map of any $\mathfrak{su}(\mathfrak{n}) \ltimes \mathbb{C}^n$ vector field enjoys some special properties.

Lemma 2.4 (Joyce [32, Lemma 3.4] and Fu [14, Theorem 3.2]) *Let $\mu\colon \mathbb{C}^n \to \mathbb{R}$ be a moment map for a vector field x in $\mathfrak{su}(n) \ltimes \mathbb{C}^n$. Then the restriction of μ to any special Lagrangian n–fold in $\mathbb{C}^n$ is a harmonic function on L with respect to the metric induced on L by $\mathbb{C}^n$.*

Conversely, a function f on $\mathbb{C}^n$ is harmonic on every SL n–fold in $\mathbb{C}^n$ if and only if

$$d(\iota_{X_f} \operatorname{Im}(\Omega)) = 0 \tag{2–7}$$

where $X_f = -J\nabla f$ is the Hamiltonian vector field associated with f. For $n \geq 3$, f satisfies (2–7) if and only if f is a harmonic Hermitian quadratic.

In other words, any special Lagrangian n–fold L in $\mathbb{C}^n$ automatically has certain distinguished harmonic functions. We can use Lemma 2.4 together with the Maximum Principle to conclude that a compact special Lagrangian submanifold with boundary inherits all the symmetries of the boundary.

Proposition 2.5 (Haskins [23, Proposition 3.6]) *Let L^n be a compact connected special Lagrangian submanifold of $\mathbb{C}^n$ with boundary Σ. Let G be the identity component of the subgroup of $SU(n) \ltimes \mathbb{C}^n$ which preserves Σ. Then G admits a moment map $\mu\colon \mathbb{C}^n \to \mathfrak{g}^*$, both Σ and L are contained in $\mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$ and G also preserves L.*

Here is an asymptotically conical analogue of this result:

Proposition 2.6 (Haskins [23, Propostion B.1]) *Let L be an ACSL submanifold L with rate λ and regular special Lagrangian cone C. Let G be the identity component of the subgroup of $SU(n)$ preserving C. Then*

(i) *G admits a moment map μ and $C \subset \mu^{-1}(0)$;*

(ii) *if $\lambda < 0$, then $L \subset \mu^{-1}(0)$ and G also preserves L;*

(iii) *if $\lambda = 0$ and L has one end, then $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$ and G preserves L.*

The following lemma shows that associated to any SL cone C there is a natural 1–parameter family of ACSL submanifolds L_t, asymptotic with rate $\lambda = 2-n$ to the union of the two SL cones $C \cup e^{i\pi/n}C$. Hence there always exist natural "two-ended" desingularizations of any SL cone C. It may be useful to emphasize that this construction does not provide a solution to the ACSL smoothing problem for C, unless C is $(e^{i\pi/n})$–invariant. Notice that this lemma is compatible with the fact that the disjoint union of any oriented manifold Σ together with $-\Sigma$ is always nullcobordant.

Lemma 2.7 [23, Lemma B.3] *Let C be any regular SL cone in $\mathbb{C}^n$ with link Σ. For any $t \neq 0 \in \mathbb{R}$ define L_t as*

$$L_t = \left\{\sigma \in \mathbb{C}^n : \sigma \in \Sigma, z \in \mathbb{C}, \text{ with } \operatorname{Im} z^n = t, \ \arg z \in (0, \tfrac{\pi}{n})\right\}.$$

Then L_t is an ACSL n–fold with rate $\lambda = 2-n$, and cone $C \cup e^{i\pi/n}C$. Moreover, L_t has the same symmetry group as C.

2.7 Nonsmoothable SL singularities II

Proposition 2.6 and Lemma 2.7 show that any ACSL smoothing (with appropriate decay) of a G–invariant SL cone C is also G–invariant, and that C has a canonical 1–parameter family of G–invariant two-ended ACSL smoothings. For certain group actions G we will see that the only G–invariant SLG n–folds are the SL cones C themselves and their canonical two-ended ACSL smoothings. Hence we will be able to prove that certain G–invariant SL cones C admit no (one-ended) ACSL smoothings (with decay).

The following three infinite families of very symmetric SL cones have been studied previously by a number of authors [10; 11; 42; 43].

Example 2.8 For $n \geq 3$, let $\mathfrak{gl}(n,\mathbb{C})$ denote the space of all $n \times n$ complex matrices equipped with the hermitian inner product $\langle\langle A, B\rangle\rangle := \operatorname{Tr} AB^*$, and let SU(n) act on $\mathfrak{gl}(n,\mathbb{C})$ by $(A, B) \mapsto AB$. Then the map ϕ: SU(n) $\to \mathfrak{gl}(n,\mathbb{C})$ given by

$$\phi(A) := \frac{1}{\sqrt{n}} A$$

gives an SU(n)–invariant special Legendrian submanifold of the unit sphere in $\mathfrak{gl}(n,\mathbb{C})$ diffeomorphic to SU(n).

Example 2.9 For $n \geq 3$, let Sym$(n,\mathbb{C})$ denote the space of all symmetric $n \times n$ complex matrices equipped with the hermitian inner product induced from $\mathfrak{gl}(n,\mathbb{C})$. Let SU(n) act on $\mathrm{Sym}(n,\mathbb{C})$ by $(A,B)\mapsto ABA^T$. Then the map ϕ: SU(n) $\to \mathrm{Sym}(n,\mathbb{C})$ given by

$$\phi(A) := \frac{1}{\sqrt{n}} AA^T$$

induces an SU(n)–invariant special Legendrian submanifold of the unit sphere in $\mathrm{Sym}(n,\mathbb{C})$ diffeomorphic to $\mathrm{SU(n)}/\mathrm{SO(n)}$. Note, these are the same examples that already appeared in Theorem 2.2.

Example 2.10 For $n \geq 2$, let $\mathfrak{so}(2n,\mathbb{C})$ denote the space of all skew-symmetric $2n\times 2n$ complex matrices equipped with the hermitian inner product induced from $\mathfrak{gl}(2n,\mathbb{C})$. Let SU(2n) act on $\mathfrak{so}(2n,\mathbb{C})$ by $(A,B)\mapsto ABA^T$. Then the map ϕ from SU(2n) to $\mathfrak{so}(2n,\mathbb{C})$ given by

$$\phi(A) := \frac{1}{\sqrt{n}} AJ_nA^T, \quad \text{where } J_n = \left(\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array}\right),$$

induces an SU(2n)–invariant special Legendrian submanifold of the unit sphere in $\mathfrak{so}(2n,\mathbb{C})$ diffeomorphic to $\mathrm{SU(2n)}/\mathrm{Sp}(n)$.

Theorem 2.11 *Let C be any of the SL cones described in Example 2.8, Example 2.9 and Example 2.10. Then the ACSL smoothing problem with decay is not solvable for the cone C.*

Proof For Example 2.8–Example 2.10, Castro–Urbano [10, Section 2] proved that the only G–invariant SL submanifolds (with the group G and its action as described in the examples) are the cones themselves and their canonical two-ended ACSL smoothings. However, by Proposition 2.6(iii) any ACSL smoothing of C with decay rate $\lambda \leq 0$ is also G–invariant. Hence no such ACSL smoothing of C with decay rate $\lambda \leq 0$ exists. □

Here is a way to think about why we should expect a result like Theorem 2.11 to hold. In the examples above, the common feature is that we have a low cohomogeneity action of the nonabelian group G for which $Z(\mathfrak{g}^*) = (0)$. Let μ denote the associated moment map. Then we know that all G–invariant Lagrangian submanifolds are contained in $\mu^{-1}(0)$. Thus there are rather few G–invariant Lagrangian submanifolds for these groups G.

If C is a G–invariant SL cone and G has cohomogeneity one (as do the above examples) then any ACSL smoothing with decay will also be G–invariant with cohomogeneity one. Thus to obtain (one-ended) ACSL smoothings of C it is necessary that there be ω–isotropic orbits of G whose dimension is neither maximal (corresponding to the link of the cone) nor minimal (when the orbit is just the origin of $\mathbb{C}^n$). If isotropic orbits cannot collapse in this intermediate way then there is no way to obtain G–invariant one-ended ACSL n–folds. For instance, in Example 2.8 it is easy to see that SU(n) acts freely on $\mu^{-1}(0)$, except at the origin. Hence all the G–orbits in $\mu^{-1}(0)$ except the origin are diffeomorphic to SU(n), and thus there are no "intermediate" isotropic orbits.

However, if instead G is abelian, then any level set of the corresponding moment map μ is G–invariant, and not just the zero level set $\mu^{-1}(0)$. Thus in the abelian case there are many more G–invariant Lagrangian submanifolds and many more isotropic G–orbits. In particular, if G is the maximal abelian subgroup $\mathbb{T}^{n-1} \subset \mathrm{SU(n)}$ acting in the obvious way on $\mathbb{C}^n$, then there are "intermediate" isotropic G–orbits. As a result the $\mathbb{T}^{n-1}$–invariant SL cone C over the Legendrian Clifford torus T^{n-1} does admit one-ended ACSL smoothings. In fact, one can classify all the one-ended ACSL smoothings with decay of the Legendrian Clifford torus $\mathbb{T}^{n-1} \subset \mathbb{S}^{2n-1}$ this way; see [29, Section 10] for a detailed description of the $n = 3$ case.

One can use the same idea for other group actions G. Provided G is "large enough" then we can expect to classify *all* G–invariant SL n–folds and hence to classify all ACSL smoothings with decay of any such G–invariant SL cone.

In Example 2.8 the link of the SL cone is diffeomorphic to SU(n). Since any Lie group is parallelizable, all its Pontrjagin and Stiefel–Whitney classes vanish and hence it is nullcobordant. Hence we have infinitely many SL cones where there is no differential-topological obstruction to smoothing the singularity, but which nevertheless admit no ACSL smoothing (with decay).

In this case a natural weaker question to ask is: do the nullcobordant special Legendrian links Σ above admit Lagrangian or Maslov-zero Lagrangian smoothings? We will see that we can answer the immersed version of this question using a relative version of the Gromov–Lees h–principle for Lagrangian immersions. We will see that in general there are obstructions to Lagrangian smoothing that are of an algebro-topological nature which go beyond the cobordism type of the link.

In order to describe these obstructions we need to review the h–principle for Lagrangian immersions, beginning with the version for closed manifolds. Although much of this material is well-known to symplectic topologists, we expect that many readers whose background is in minimal submanifolds or special Lagrangian geometry will be unfamiliar with it.

3 The h–principle for Lagrangian immersions of closed manifolds

Let L^n be a compact connected (not necessarily orientable) manifold without boundary. A natural question in symplectic geometry is:

When does L admit a Lagrangian immersion into $\mathbb{R}^{2n}$?

Gromov and Lees showed that Lagrangian immersions of closed manifolds satisfy the so-called h–principle, where h stands for homotopy. The idea is that for a Lagrangian immersion to exist certain conditions of an algebro-topological nature must be satisfied, and saying that the h–principle holds for Lagrangian immersions means that these necessary topological conditions are in fact sufficient.

Hence the previous, apparently geometric question about the existence of Lagrangian immersions reduces to a topological problem which in many cases can be solved.

3.1 The h–principle for immersions of closed manifolds

We begin with a description of the h–principle for immersions of a closed manifold, ie for the moment we drop the Lagrangian condition. Let V and W be smooth manifolds of dimension n and q respectively. Suppose that $n \leq q$ and that there exists an immersion $f\colon V \to W$. Then the differential $df\colon TV \to TW$ is a smooth bundle map (ie a smooth map which fibrewise is linear) which is injective on each fibre. This motivates the following definition:

Definition 3.1 A *monomorphism* $F\colon TV \to TW$ is a smooth bundle morphism which is fibrewise injective. We will usually denote by $f = \mathrm{bs}(F)$ the underlying "base map" $V \to W$. Notice that f is a smooth map, but it need not be an immersion.

Eliashberg and Mishachev [13] call such a monomorphism a *formal solution* of the immersion problem. Clearly, the existence of a monomorphism $F\colon TV \to TW$ (a formal solution) is a necessary condition for the existence of an immersion $f\colon V \to W$ (a genuine solution). A monomorphism of the form $F = df$ is also sometimes called *holonomic*. One says that a differential relation $\mathcal{R}$ (eg the immersion relation) satisfies the h–principle if every formal solution of $\mathcal{R}$ (a monomorphism $TV \to TW$ in our case) is homotopic in the space of formal solutions to a genuine solution of $\mathcal{R}$ (an immersion $V \to W$). Hirsch [27] extending work of Smale [46] proved that the h–principle holds for the differential relation associated with immersions of a closed n–manifold V into a q–manifold W provided $q > n$. In the case where V is an n–manifold and $W = \mathbb{R}^q$ we know from the results of Whitney [52] that any smooth

n–manifold ($n > 1$) can be immersed in $\mathbb{R}^q$ for any $q \geq 2n-1$. In particular, this gives a very indirect argument that we must always be able to construct a monomorphism from TV to $T\mathbb{R}^q$, for $q \geq 2n-1$ whatever the n–manifold V. Perhaps more importantly it shows that the Smale–Hirsch h–principle is (in the case of immersions into Euclidean space) of greatest interest in the case of relatively low codimension.

3.2 The h–principle for Lagrangian immersions

To understand the statement of the h–principle for Lagrangian immersions first we have to understand what the appropriate notion of a formal solution to the Lagrangian immersion problem is.

Suppose as in the previous section that V and W are smooth manifolds of dimension n and q respectively and that $q > n$. Any monomorphism $F\colon TV \to TW$ induces a natural map $GF\colon V \to \mathrm{Gr}_n(W)$, where $\mathrm{Gr}_n(W)$ denotes the Grassmannian bundle of n–planes in W. In particular, for any immersion $f\colon V \to W$ we have an associated (tangential) Gauss map written Gdf. If (W,ω) is a symplectic manifold then one can define a number of natural subbundles of the Grassmannian n–plane bundles. In particular, if $q = 2n$ we can define the subset $\mathrm{Gr}_{\mathrm{lag}}(W)$ of all Lagrangian n–planes in (W,ω). Clearly, if $f\colon V^n \to (W^{2n},\omega)$ is a Lagrangian immersion then the image of the associated (tangential) Gauss map Gdf is contained in the subbundle $\mathrm{Gr}_{\mathrm{lag}}(W)$ of $\mathrm{Gr}_n(W)$.

Similarly, one can also define a subbundle $\mathrm{Gr}^+_{\mathrm{lag}}(W) \subset \mathrm{Gr}^+_n(W)$ corresponding to the set of all oriented Lagrangian planes inside the set of all oriented n–planes. If $f\colon V^n \to (W^{2n},\omega)$ is a Lagrangian immersion of an oriented manifold V then the associated Gauss map Gdf is contained in $\mathrm{Gr}^+_{\mathrm{lag}}(W)$. Similarly, if (W,J) is an almost-complex manifold then there are other natural subbundles of the Grassmannian n–plane bundles. For instance, a linear subspace S of a complex vector space (W,J) is *totally real* if $JS \cap S = (0)$. Hence, in an almost complex manifold (W^{2n},J) there is a subbundle $\mathrm{Gr}_{\mathrm{real}}(W) \subset \mathrm{Gr}_n(W)$ such that any totally real immersion $f\colon V^n \to (W^{2n},J)$ has tangential (Gauss) map Gdf contained in $\mathrm{Gr}_{\mathrm{real}}(W)$. If J and ω are compatible then $\mathrm{Gr}_{\mathrm{lag}}(W) \subset \mathrm{Gr}_{\mathrm{real}}(W)$, so that any Lagrangian immersion is totally real. The converse is false. There is also an oriented version of this totally real Grassmannian bundle.

Definition 3.2 Let V be a smooth (not necessarily orientable) n–manifold and let (W,ω) be a symplectic $2n$–manifold. A *Lagrangian monomorphism* F is a monomorphism $F\colon TV \to TW$ such that $GF(V) \subset \mathrm{Gr}_{\mathrm{lag}}(W)$. Similarly, one can define a *totally real monomorphism* in an almost-complex manifold (W,J) by replacing $\mathrm{Gr}_{\mathrm{lag}}(W)$ with $\mathrm{Gr}_{\mathrm{real}}(W)$.

A Lagrangian monomorphism is the correct notion of a formal solution of the Lagrangian immersion problem. For the case of Lagrangian immersions of closed manifolds into $(\mathbb{R}^{2n}, \omega)$ with its standard symplectic structure the following version of the h–principle, due to Gromov [17] and Lees [38], holds.

Theorem 3.3 (Lagrangian h–principle for closed manifolds) *Let L be a smooth closed n–manifold. Suppose there is a Lagrangian monomorphism $F\colon TL \to T\mathbb{R}^{2n}$. Then there exists a family of Lagrangian monomorphisms $F_t\colon [0,1] \times TL \to T\mathbb{R}^{2n}$ such that $F_0 = F$ and F_1 is holonomic; ie $F_1 = d\widetilde{f}$. In particular, the base map $bs(F_1) = \widetilde{f}\colon L \to \mathbb{C}^n$ is a Lagrangian immersion. Furthermore, $\widetilde{f}$ is exact.*

Remark 3.4 (Exactness of the resulting Lagrangian immersion) Let $f\colon \Sigma \to \mathbb{R}^{2n}$ be an immersion of a manifold Σ^k, of dimension $k \le n$. Recall that f is said to be an *exact* immersion if the 1–form $f^*\lambda$ is exact, ie $f^*\lambda = dz$, for some $z \in C^\infty(\Sigma, \mathbb{R})$. Equivalently f is exact if $\int_\gamma f^*\lambda = 0$ for all closed curves γ in Σ. Since $\omega = d\lambda$, any exact immersion is automatically ω–isotropic.

If f is ω–isotropic, $f^*\lambda$ is a closed 1–form on Σ and the class $[f^*\lambda] \in H^1(L;\mathbb{R})$ is called the *symplectic area class* of the isotropic immersion f. The isotropic immersion f is exact if and only if the symplectic area class of f vanishes. Hence any Lagrangian immersion of a simply connected manifold is exact. Also, any Lagrangian submanifold is "locally exact" near any smooth point, because any closed 1–form is locally exact. In Section 6 we will see that this is also true near certain types of isolated singularities.

There is an important reformulation of the condition that an immersion $f\colon \Sigma \to \mathbb{R}^{2n}$ be exact; f is exact if and only if it can be lifted to an immersion in $\mathbb{R}^{2n+1}$ which is isotropic with respect to its standard contact structure $\alpha = dz - \lambda$. In particular, (up to constants) there is a one-to-one correspondence between exact Lagrangian immersions in $(\mathbb{R}^{2n}, \omega)$ and Legendrian immersions in $(\mathbb{R}^{2n+1}, \alpha)$. The proof of Theorem 3.3 rests on the fact that any Lagrangian monomorphism into $\mathbb{R}^{2n}$ lifts to a Legendrian monomorphism into $\mathbb{R}^{2n+1}$; one can then apply the corresponding result for Legendrian monomorphisms; see [13, Section 16.1.3]. Hence the previous observation explains why the Lagrangian immersion that we obtain from the h–principle in Theorem 3.3 must be exact.

Recall that two Lagrangian immersions are said to be *regularly homotopic* if there exists a homotopy through Lagrangian immersions. We will also need the following parametric version of the Lagrangian h–principle (also due to Gromov and Lees).

Theorem 3.5 (Parametric Lagrangian h–principle) *Let $f_0, f_1\colon L \to \mathbb{R}^{2n}$ be Lagrangian immersions. If the monomorphisms $F_0 := df_0$ and $F_1 := df_1$ are homotopic* *through Lagrangian monomorphisms, then they are also homotopic through holonomic Lagrangian monomorphisms; ie f_0 and f_1 are regularly homotopic.*

Remark 3.6 Notice that if f_0, f_1 are exact, a proof of Theorem 3.5 can be obtained along the same lines as before: we can lift the curve of Lagrangian monomorphisms F_s into $\mathbb{R}^{2n+1}$ and get a curve of Legendrian monomorphisms which are holonomic for $s = 0$ and $s = 1$. The parametric Legendrian h–principle now yields a curve of holonomic Legendrian monomorphisms which coincide with the given ones for $s = 0$ and $s = 1$; this curve projects back down to $\mathbb{R}^{2n}$ yielding the desired curve of holonomic Lagrangian monomorphisms.

If f_0 and/or f_1 are not exact, the proof is more complicated. One first needs to prove the equivalent of Theorem 3.3 for general exact ambient spaces $(N, \omega = d\alpha)$. Then one proves an analogue of Theorem 3.5 for exact $f_0, f_1\colon L \to N$ and s–dependent 1–forms α_s (this time we lift the given data into $N \times \mathbb{R}$ endowed with the appropriate s–dependent contact structure). It turns out, using the parametric h–principle for immersions, that the problem of Theorem 3.5 is equivalent to the following.

Let N^{2n} be a rank n vector bundle over L, endowed with a curve of exact symplectic forms $\omega_s = d\lambda_s$ for $s \in [0,1]$. Let $i\colon L \to N$ be the zero section and $\pi\colon N \to L$ be the projection. Assume we are given a curve of monomorphisms $F_s\colon TL \to TN$ such that F_s is Lagrangian with respect to ω_s and $F_0 = F_1 = di$. Then F_s can be homotoped, rel $s = 0$ and $s = 1$, to a curve of holonomic Lagrangian monomorphisms.

The proof of this last claim is rather simple. Let us choose a curve of closed 1–forms $\widetilde{\lambda}_s$ which coincide with $\pi^*(\lambda_s|_{i(L)})$ when $s = 0$ and $s = 1$. Let $\alpha_s := \lambda_s - \widetilde{\lambda}_s$. Notice that $d\alpha_s = \omega_s$ so the symplectic structure has not changed and the F_s are still Lagrangian. However $F_0 = di$ and $F_1 = di$ are now exact with respect to the 1–forms α_s, so we can apply the simpler parametric h–principle seen above.

Remark 3.7 (Regular homotopies vs Hamiltonian isotopies) Suppose $f\colon L^n \to \mathbb{R}^{2n}$ is a nonexact Lagrangian immersion. Since df is a Lagrangian monomorphism, we can first apply Theorem 3.3 to find an exact immersion $\widetilde{f}$; we can then apply Theorem 3.5 to prove that f is regularly homotopic to $\widetilde{f}$. There is however a finer classification of Lagrangian immersions than that given by the regular homotopy classes, namely the classification up to Hamiltonian isotopy. The condition that a Lagrangian immersion be exact or nonexact is invariant under Hamiltonian isotopy. This shows that using the h–principle to perturb a given Lagrangian submanifold will typically change the Hamiltonian isotopy class of that submanifold.

Remark 3.8 (Lagrangian and totally real embeddings) It is natural to ask whether the h–principle also holds for Lagrangian embeddings and not just immersions. By Whitney's hard embedding theorem any smooth n–manifold L admits an embedding into $\mathbb{R}^{2n}$. There is, however, a simple topological obstruction to a closed n–manifold admitting a Lagrangian embedding in $\mathbb{R}^{2n}$ which we now describe.

Suppose that L^n admits a Lagrangian embedding into $\mathbb{R}^{2n}$. The self-intersection number of any compact submanifold L^n of $\mathbb{R}^{2n}$ is zero since we may use translations to disjoin L from itself. Since L is embedded we can identify a neighbourhood of $L \subset \mathbb{R}^{2n}$ with a neighbourhood of L inside its normal bundle NL; thus the Euler number of the normal bundle of L is also zero. Since the Lagrangian condition allows us to identify NL with TL, this proves that the Euler number of the tangent bundle of L, ie the Euler characteristic of L, must also be zero. This shows, for example, that the only orientable surface which could admit a Lagrangian embedding into $\mathbb{R}^4$ is the torus T^2, and the product $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C} \times \mathbb{C}$ gives such a Lagrangian embedding. Notice that we cannot conclude anything from this Euler characteristic information about Lagrangian embeddings of closed orientable 3–manifolds in $\mathbb{R}^6$, and in particular of S^3.

In fact, Gromov proved that Lagrangian embeddings do not satisfy the h–principle, as an application of his theory of J–holomorphic curves in symplectic manifolds [16]. In particular, he proved that there are no exact Lagrangian embeddings of closed n–manifolds in $(\mathbb{R}^{2n}, \omega)$. For example, there is no Lagrangian embedding of S^3 in $\mathbb{R}^6$.

This also shows that if we apply Theorem 3.3 to perturb a (nonexact) embedded Lagrangian submanifold, the resulting exact Lagrangian submanifold will typically only be immersed. On the other hand, there are totally real embeddings of S^3 in $\mathbb{R}^6$. In fact both totally real immersions and totally real embeddings satisfy versions of the h–principle (see Gromov [17] or Eliashberg and Mishachev [13, Section 19.3.1, Section 19.4.5]).

3.3 Existence and classification of Lagrangian immersions

As we have just seen, the h–principle reduces questions about the existence of Lagrangian immersions to questions about the existence of Lagrangian monomorphisms. In this section we show that it suffices to prove the existence of totally real monomorphisms; this last problem has a straightforward solution which leads to a classification of Lagrangian immersions in terms of topological data.

Since $T\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, we can think of any monomorphism $F \colon TL \to T\mathbb{R}^{2n}$ as the data (f, ϕ), where $f \colon L \to \mathbb{R}^{2n}$ is the underlying base map and $\phi \colon TL \to \mathbb{R}^{2n}$ is a fibrewise injective homomorphism. In what follows we will often use ϕ to pull back the standard metric g on $\mathbb{R}^{2n}$, obtaining a metric $\phi^* g$ on TL.

Let us start by showing that totally real monomorphisms are closely related to the *complexified tangent bundle* $TL^{\mathbb{C}}$; as usual, we will say that $TL^{\mathbb{C}}$ is trivial if there exists a complex vector bundle isomorphism (a *complex trivialization*) $TL^{\mathbb{C}} \simeq L \times \mathbb{C}^n$. Any map ϕ as above has a natural *complexification* $\phi^{\mathbb{C}}\colon TL^{\mathbb{C}} \to \mathbb{R}^{2n}$, defined by

$$\phi^{\mathbb{C}}(v + iw) := \phi(v) + J\phi(w)$$

where J is the standard complex structure on $\mathbb{R}^{2n}$.

Lemma 3.9 *There is a one-to-one correspondence between totally real injections $TL \to \mathbb{R}^{2n}$ and trivializations of $TL^{\mathbb{C}}$.*

Proof Let $\phi\colon TL \to \mathbb{R}^{2n}$ be a fibrewise linear map. We will first show that ϕ is totally real if and only if $\phi^{\mathbb{C}}\colon TL^{\mathbb{C}} \to \mathbb{R}^{2n}$ is a complex isomorphism (with respect to J). Since $\phi^{\mathbb{C}}$ is a complex linear map between complex spaces of the same dimension, it suffices to show surjectivity of $\phi^{\mathbb{C}}$. But $\phi^{\mathbb{C}}$ is surjective at $p \in L$ if and only if the two real n–planes $\phi(T_pL)$ and $J\phi(T_pL)$ in $\mathbb{R}^{2n}$ intersect transversally, ie if and only if ϕ is a totally real monomorphism. Hence any totally real injection naturally induces a complex trivialization of $TL^{\mathbb{C}}$.

Conversely, any complex trivialization $\phi^{\mathbb{C}}\colon TL^{\mathbb{C}} \to L \times \mathbb{C}^n$ determines a unique totally real injection $\phi : TL \to \mathbb{C}^n$ defined by $\phi := \phi^{\mathbb{C}} \circ i$ where $i\colon TL \to TL^{\mathbb{C}}$ denotes the natural (totally real) inclusion of TL in $TL^{\mathbb{C}}$. □

Lemma 3.9 indicates the existence of a distinguished subset of trivializations of $TL^{\mathbb{C}}$: the ones obtained by complexifying Lagrangian injections $TL \to \mathbb{C}^n$. In Lemma 3.15 we will refer to these as the *Lagrangian trivializations* of $TL^{\mathbb{C}}$.

Even though Lagrangian monomorphisms form a proper subset of the set of all totally real monomorphisms (analogously, the Lagrangian trivializations are a proper subset of the set of all trivializations of $TL^{\mathbb{C}}$), Corollary 3.12 below will show that the difference between these two categories disappears when one works in terms of homotopy classes. This is a direct consequence of the following result.

Lemma 3.10 (Polar decomposition) *There exists a strong deformation retraction $\rho\colon GL(n,\mathbb{C}) \times I \to GL(n,\mathbb{C})$ onto $U(n)$ which is equivariant with respect to $U(n)$–multiplication on the left or the right; ie*

$$\rho_t(AM) = A\rho_t(M),\ \rho_t(MA) = \rho_t(M)A,\ \text{for all } A \in U(n),\ M \in GL(n,\mathbb{C}),\ t \in [0,1].$$

Proof The standard polar decomposition theorem proves the existence of a unique decomposition $M = PU$ with $U \in \mathrm{U}(n)$ and $P = \sqrt{MM^*} \in \mathcal{P}^+$, where $\mathcal{P}^+$ denotes the set of positive self-adjoint matrices. Note if $P \in \mathcal{P}^+$ then the matrix $P_t := t\,\mathrm{Id} + (1-t)P \in \mathcal{P}^+$. Let us define $\rho_t(M) := P_t U$. Clearly $\rho_0(M) = M$ and $\rho_1(M) = U$, so ρ_t is a strong retraction as claimed. Now suppose that A is in $\mathrm{U}(n)$. The equivariance of ρ_t with respect to the right action of U(n) is easy to see. To show the left-equivariance, notice that $AM = APU = APA^{-1}AU = \widetilde{P}AU$, where $\widetilde{P} := APA^{-1}$ is positive self-adjoint and $AU \in \mathrm{U}(n)$. Uniqueness of the decomposition implies that the polar decomposition of AM is $\widetilde{P}(AU)$. Notice also that $\widetilde{P}_t = AP_tA^{-1}$. This shows that $\rho_t(AM) = \widetilde{P}_t AU = AP_tU = A\rho_t(M)$. □

Remark 3.11 If we restrict ρ to the set of real matrices $\mathrm{GL(n,}\mathbb{R}) \subset \mathrm{GL(n,}\mathbb{C})$, we obtain corresponding statements for the *real* polar decomposition theorem: for any $M \in \mathrm{GL(n,}\mathbb{R})$, $M = PO$ where P is positive symmetric and $O \in \mathrm{O(n)}$; furthermore, $\mathrm{GL(n,}\mathbb{R})$ retracts O(n)–equivariantly onto O(n).

Corollary 3.12 *A totally real monomorphism $F\colon TL \to T\mathbb{R}^{2n}$ is homotopic through a path of totally real monomorphisms to a Lagrangian monomorphism $\widetilde{F}\colon TL \to T\mathbb{R}^{2n}$. Moreover, two totally real monomorphisms $F_1, F_2\colon TL \to T\mathbb{R}^{2n}$ are homotopic via totally real monomorphisms if and only if the corresponding Lagrangian monomorphisms $\widetilde{F}_1, \widetilde{F}_2$ are homotopic through Lagrangian monomorphisms.*

Proof The proof is basically a direct consequence of the Lemma 3.10. For completeness, we provide here the details.

Let $F = (f, \phi)$. For any $p \in L$, let $\{v_i\}_{i=1}^n$ be an orthonormal basis of T_pL (with respect to the pullback metric on L induced by F). Let $M = M(p)$ denote the matrix expressing the homomorphism $\phi^{\mathbb{C}}\colon T_pL^{\mathbb{C}} \to \mathbb{C}^n$ in terms of the basis $\{v_i\}$ of $T_pL^{\mathbb{C}}$ and the standard basis of $\mathbb{C}^n$. Since F is totally real, $M \in \mathrm{GL(n,}\mathbb{C})$. Define $M_t := \rho_t(M)$ where ρ is the retraction defined in the Lemma 3.10. M_t gives a homotopy through $\mathrm{GL(n,}\mathbb{C})$ matrixes between M and the matrix $L := \rho_1(M) \in \mathrm{U(n)}$. We now define $\phi_t\colon T_pL \to \mathbb{C}^n$ to be the homomorphism corresponding to the matrix M_t, and hence obtain a one-parameter family of monomorphisms $F_t := (f, \phi_t)$. Since $M_t \in \mathrm{GL(n,}\mathbb{C})$ and $L \in \mathrm{U(n)}$, F_t (for $t \in [0,1)$) are totally real monomorphisms and $\widetilde{F} := F_1$ is a Lagrangian monomorphism (furthermore, the pullback metrics induced by F, F_1 coincide). Notice that if we begin with a different orthonormal basis $\{w_1, \dots, w_n\}$ then $w_i = v_j a_{ji}$ for some $A = (a_{ij}) \in \mathrm{O}(n)$; now ϕ gives rise to the matrix MA and hence, by the right-equivariance of ρ, ϕ_1 corresponds to LA. Hence the construction is independent of the choice of basis of T_pL, so that $\widetilde{F}$ is well-defined. Since all this also works parametrically, it also proves the second statement. □

Remark 3.13 The same methods apply to show that polar decomposition induces a natural retraction from $\mathrm{Gr}_{\mathrm{real}}(\mathbb{R}^{2n})$ to $\mathrm{Gr}_{\mathrm{lag}}(\mathbb{R}^{2n})$. Using the U(n)–invariance of polar decomposition mentioned in Lemma 3.10 these results can be extended to totally real monomorphisms from TL into any symplectic ambient manifold (W, ω) endowed with a compatible almost-complex structure J and metric g. The proof is as above: we choose an orthonormal basis v_i for T_pL and a unitary basis e_i for $T_{f(p)}W$ and reduce the question to matrices, then prove that the result is independent of the choice of both sets of bases using the left and right equivariance of polar decomposition.

Hence we have the following result:

Corollary 3.14 *Let L be a closed smooth (not necessarily orientable) n–manifold. Then the following conditions are equivalent.*

(1) *L admits a Lagrangian immersion $f\colon L \to \mathbb{R}^{2n}$.*

(2) *$TL^{\mathbb{C}}$ is trivial.*

(3) *L admits a totally real monomorphism $F\colon TL \to T\mathbb{R}^{2n}$.*

(4) *L admits a Lagrangian monomorphism $F\colon TL \to T\mathbb{R}^{2n}$.*

Furthermore, there is a one-to-one correspondence between the homotopy classes of the above objects.

Proof Notice that there is an obvious map from the set $\mathcal{I}$ of regular homotopy classes of Lagrangian immersions to the set $\mathcal{M}$ of homotopy classes of Lagrangian monomorphisms, namely,

$$\mathcal{I} \to \mathcal{M}, \quad [f] \mapsto [df].$$

Theorem 3.3 shows that this map is surjective; this also proves the equivalence between statements (1) and (4). Theorem 3.5 shows that this map is injective. Corollary 3.12 proves the equivalence of (3) and (4) and of their homotopy classes. To prove the equivalence between (2) and (3), let (f, ϕ) be a totally real monomorphism. Lemma 3.9 then proves that ϕ defines a complex trivialization of $TL^{\mathbb{C}}$. Conversely, given ϕ we can use any smooth map $f\colon L \to \mathbb{R}^{2n}$ (eg a constant map) to obtain a totally real monomorphism (f, ϕ). Since $\mathbb{R}^{2n}$ is topologically trivial all such f are homotopic, so the correspondence between homotopy classes is simple. □

Corollary 3.14 part (2) thus completely solves the "Lagrangian immersion in $\mathbb{R}^{2n}$ problem" for closed manifolds in terms of a topological condition on L. There is one last thing we can do, which is to classify these homotopy classes. This is based on the following simple fact.

Lemma 3.15 *Any given trivialization ϕ_0 of $TL^{\mathbb{C}}$ induces a bijective correspondence between the set of all trivializations of $TL^{\mathbb{C}}$ and the set of all maps $L \to GL(n, \mathbb{C})$. Two trivializations are homotopic if and only if the corresponding maps are homotopic.*

Furthermore we have the following:

(1) *Let h and g denote the standard Hermitian and Riemannian metrics on $\mathbb{C}^n$. Then the above correspondence, restricted to the set of maps $L \to U(n)$, yields the set of all other trivializations whose pullback Hermitian metric $\phi^* h$ on $TL^{\mathbb{C}}$ coincides with $\phi_0^* h$.*

(2) *Now assume ϕ_0 is a Lagrangian trivialization of $TL^{\mathbb{C}}$ (as defined above). Then the above correspondence, restricted to the set of maps $L \to U(n)$, yields the set of all other Lagrangian trivializations whose pullback Riemannian metric $\phi^* g$ on TL coincides with $\phi_0^* g$.*

Proof For $i = 0, 1$ let $\phi_i \colon TL^{\mathbb{C}} \to L \times \mathbb{C}^n$ be complex trivializations of $TL^{\mathbb{C}}$. Then $\phi_1 \circ \phi_0^{-1} : L \times \mathbb{C}^n \to L \times \mathbb{C}^n$ is a fibrewise complex isomorphism, or equivalently, a map from L into $\mathrm{Aut}(\mathbb{C}^n)$. We identify $\mathrm{Aut}(\mathbb{C}^n)$ with GL(n, $\mathbb{C}$) by choosing the standard basis $e_1, \dots, e_n$ of $\mathbb{C}^n$. We write the map $L \to$ GL(n, $\mathbb{C}$) determined by ϕ_0 and ϕ_1 (and the choice of our standard basis for $\mathbb{C}^n$) as M_{ϕ_1, ϕ_0}. Hence if we choose a reference trivialization ϕ_0 of $TL^{\mathbb{C}}$, then any other trivialization ϕ of $TL^{\mathbb{C}}$ determines a map $L \to$ GL(n, $\mathbb{C}$), namely M_{ϕ, ϕ_0}. Conversely, a map $M \colon L \to$ GL(n, $\mathbb{C}$) determines a fibrewise complex-linear automorphism $A(M)$ of $L \times \mathbb{C}^n$, and hence another complex trivialization of $TL^{\mathbb{C}}$, denoted ϕ_0^M and given by $\phi_0^M := A(M) \circ \phi_0$.

Now let $v_i := \phi_0^{-1} e_i$. Notice that $\phi^* h = \phi_0^* h$ if and only if $\phi^* h(v_i, v_j) = \phi_0^* h(v_i, v_j)$. However, the left hand side of this last expression is $(\phi \circ \phi_0^{-1})^* h(e_i, e_j)$ while the right hand side is $h(e_i, e_j)$, so we conclude that $\phi^* h = \phi_0^* h$ if and only if $\phi \circ \phi_0^{-1}$ is unitary, ie $M_{\phi, \phi_0} \in$ U(n). This proves (1). Now recall that $h = g + i\omega$, so if ϕ_0 is Lagrangian then $\phi_0^* h = \phi_0^* g$. In this case $\phi^* h = \phi_0^* h$ if and only if $\phi^* h = \phi_0^* g$; in particular this implies that $\phi^* \omega = 0$, ie that $\phi(TL)$ is Lagrangian in $\mathbb{C}^n$, and that $\phi^* g = \phi_0^* g$. This proves (2). □

We can now classify the homotopy classes of Lagrangian immersions as follows (cf [5, page 274]).

Proposition 3.16 *Given a Lagrangian immersion $L \to \mathbb{C}^n$, the set of regular homotopy classes of all Lagrangian immersions $L \to \mathbb{C}^n$ (or of homotopy classes of Lagrangian monomorphisms) is in bijective correspondence with the set $[L; U(n)]$ of homotopy classes of maps from L into U(n) (or equivalently into GL(n, $\mathbb{C}$)).*

Proof Corollary 3.14 allows us to rephrase the statement in terms of Lagrangian monomorphisms; the proof of Corollary 3.14 shows that all base maps are homotopic, allowing us to further reduce to Lagrangian trivializations of $TL^{\mathbb{C}}$. We can now prove the statement as follows. The given Lagrangian immersion defines a reference Lagrangian trivialization ϕ_0 and a pullback metric $\phi_0^* g$ on TL. Choose another Lagrangian trivialization ϕ; for any point $p \in L$, let π_p denote the image Lagrangian plane in $\mathbb{C}^n$, with the metric induced from g. Then $\phi(p)\colon T_pL \to \pi_p$ is an isomorphism, and by the real polar decomposition lemma (see Remark 3.11) it can be homotoped to an isometry $\widetilde{\phi}$ (as in Corollary 3.12) without changing the image plane π_p. Since $\widetilde{\phi}$ has the same image π_p, $\widetilde{\phi}$ still defines a Lagrangian trivialization of $TL^{\mathbb{C}}$; thus part (2) of Lemma 3.15 shows that $\widetilde{\phi}$ corresponds to a map $L \to \mathrm{U(n)}$. □

3.4 Gauss maps and the Maslov class

To simplify the discussion, in this section we will assume that L is oriented; at the end of the section we will add a few comments on the nonorientable case.

Since $\mathbb{R}^{2n}$ is contractible, $\mathrm{Gr}^+_{\mathrm{lag}}(\mathbb{R}^{2n})$ is a trivial bundle over $\mathbb{R}^{2n}$ which has fibre $\mathrm{U}(n)/\mathrm{SO}(n)$. Hence given any Lagrangian monomorphism $F\colon TL \to T\mathbb{R}^{2n}$ the tangential Gauss map GF gives us a map

$$GF\colon L \to \mathrm{U}(n)/\mathrm{SO}(n).$$

More explicitly, fix $x \in L$ and consider the homomorphism $F_x\colon T_xL \to \mathbb{C}^n$. We can use any positive orthonormal basis $\{v_i\}_{i=1}^n$ (with respect to the pullback metric) of T_xL and the standard basis of $\mathbb{C}^n$ to represent F_x via a matrix in $\mathrm{U(n)}$. This matrix depends on the choice of $\{v_i\}$ but its equivalence class in $\mathrm{U(n)/SO(n)}$ does not; this is the Gauss map $GF(x)$. Composition with the complex determinant yields an associated *determinant map*

$$\det_{\mathbb{C}} \circ\, GF\colon L \to \mathbb{S}^1.$$

This construction induces a well-defined map μ on the set of homotopy classes $\mathcal{M}$ of Lagrangian monomorphisms, which we call the *Maslov map* $\mu\colon \mathcal{M} \to [L;\mathbb{S}^1]$ and is given by

$$\mu_F := [\det_{\mathbb{C}} \circ\, GF].$$

When the Lagrangian monomorphism F arises from an oriented Lagrangian immersion then the determinant map defined above coincides with the Lagrangian phase function $e^{i\theta}$ defined in Section 2.2.

We now recall some well-known facts about homotopy classes of maps to $\mathbb{S}^1$. Since $\mathbb{S}^1$ is an abelian Lie group the homotopy classes of maps into $\mathbb{S}^1$ inherit the structure of an abelian group. We will let $d\theta$ denote the standard harmonic 1–form on $\mathbb{S}^1$, normalized so that $[d\theta]$ is the generator of $H^1(\mathbb{S}^1,\mathbb{Z}) \simeq \mathbb{Z}$.

Lemma 3.17 *For any CW complex K the map $[K;\mathbb{S}^1] \to H^1(K,\mathbb{Z})$ induced by*

$$f \mapsto f^*[d\theta]$$

is an isomorphism of abelian groups. Also, there is an isomorphism between $H^1(K,\mathbb{Z})$ and $\mathrm{Hom}(\pi_1(K),\mathbb{Z})$. In particular, a map $f\colon K \to \mathbb{S}^1$ is homotopic to a constant if and only if $f|_\gamma$ is homotopic to a constant, for all closed loops $\gamma \subset K$.

Proof If K is any CW complex and Y is an Eilenberg–Mac Lane space of type $K(\pi,n)$, there is a natural isomorphism between the homotopy classes of maps $[K;Y]$ and $H^n(K;\pi)$ [7, Theorem VII.12.1]. The first part now follows since $\mathbb{S}^1$ is an Eilenberg–Mac Lane space $K(\mathbb{Z},1)$. From the Universal Coefficient Theorem we have $H^1(K,\mathbb{Z}) \simeq \mathrm{Hom}(H_1(K),\mathbb{Z}) + \mathrm{Ext}(H_0(K),\mathbb{Z})$ [7, page 282]. Since $H_0(K)$ is always free, we have $H^1(K,\mathbb{Z}) \simeq \mathrm{Hom}(H_1(K),\mathbb{Z}) \simeq \mathrm{Hom}(\pi_1(K),\mathbb{Z})$ since $\mathbb{Z}$ is abelian. □

Using Lemma 3.17 we can identify μ_F with the element $(\det_{\mathbb{C}} \circ GF)^*[d\theta] \in H^1(L,\mathbb{Z})$; this is the *Maslov class* of F. Furthermore, μ_F is equivalent to a homomorphism $m_F\colon \pi_1(L) \to \mathbb{Z}$. Given a loop $\gamma\colon \mathbb{S}^1 \to L$, $m_F([\gamma]) \in \mathbb{Z}$ simply calculates the degree of the map $\det_{\mathbb{C}} \circ GF \circ \gamma\colon \mathbb{S}^1 \to \mathbb{S}^1$; we call this integer the *Maslov index* of the curve γ. Notice that according to Corollary 3.14 we can always represent μ_F by a Lagrangian immersion f in the same homotopy class; we will then use the notation μ_f. We now want to describe a second way to compute μ_F, along the lines of Proposition 3.16. Let us fix a Lagrangian monomorphism $F_0 = (f_0,\phi_0)$. As explained in Proposition 3.16, any other Lagrangian monomorphism $F = (f,\phi)$ can be homotoped so that $f = f_0$ (see the proof of Corollary 3.14) and the induced pullback metrics $\phi^* g$, $\phi_0^* g$ on TL coincide, so F will correspond to a map $M_{F,F_0}\colon L \to \mathrm{U(n)}$.

Lemma 3.18 *In the situation just described, $\mu_F = [\det_{\mathbb{C}} M_{F,F_0}] \cdot \mu_{F_0}$, where $\cdot$ denotes group multiplication in the abelian group $[L;\mathbb{S}^1]$. In particular, if F_0 has zero Maslov class then $\mu_F = [\det_{\mathbb{C}} M_{F,F_0}]$.*

Proof Let $F_0 = (f,\phi_0)$ and $F = (f,\phi)$. Fix $p \in L$; let $\{v_i\}_{i=1}^n$ be an orthonormal basis of T_pL with respect to $\phi_0^* g$. Let $M(\phi_0)$, $M(\phi)$ be the matrices expressing $\phi_0^{\mathbb{C}}$ and $\phi^{\mathbb{C}}$ with respect to the basis $\{v_i\}$ of $T_pL^{\mathbb{C}}$ and the standard basis of $\mathbb{C}^n$. Notice that $\phi^{\mathbb{C}} = \phi^{\mathbb{C}} \circ (\phi_0^{\mathbb{C}})^{-1} \circ \phi_0^{\mathbb{C}}$. Thus $\det_{\mathbb{C}} \circ GF = \det_{\mathbb{C}} M(\phi) = \det_{\mathbb{C}} M_{F,F_0} \cdot (\det_{\mathbb{C}} \circ GF_0)$ so $\mu_F = [\det_{\mathbb{C}} M_{F,F_0}] \cdot \mu_{F_0}$. □

Proposition 3.19 *Let L be a closed orientable n–manifold L with $TL^{\mathbb{C}}$ trivial, and let μ be an arbitrary element of $H^1(L,\mathbb{Z})$.*

(1) *L admits a Lagrangian monomorphism $F\colon TL \to T\mathbb{C}^n$ with $\mu_F = \mu$.*

(2) *L admits an exact Lagrangian immersion $f\colon L \to \mathbb{R}^{2n}$ with $\mu_f = \mu$.*

Proof Since $TL^{\mathbb{C}}$ is trivial there exists at least one Lagrangian monomorphism $F_0\colon TL \to T\mathbb{R}^{2n}$ which we can fix as our reference monomorphism. As in Lemma 3.15, all other such monomorphisms are determined (up to homotopy) by maps from L to U(n) and the corresponding Maslov classes can be calculated as in Lemma 3.18. The inclusion $\mathbb{S}^1 \hookrightarrow$ U(n), given by

$$e^{i\theta} \mapsto \mathrm{diag}\,(e^{i\theta}, 1, \dots, 1)$$

shows that any map $L \to \mathbb{S}^1$ is the determinant of some map $L \to$ U(n); in other words, the map

$$[L, \mathrm{U(n)}] \to [L, \mathbb{S}^1] \simeq H^1(L,\mathbb{Z})$$

induced by $\det_{\mathbb{C}}$ is surjective. Thus for any $\mu \in H^1(L,\mathbb{Z})$ we can find $M\colon L \to$ U(n) such that $\mu = [\det_{\mathbb{C}} M]\cdot\mu_{F_0}$. Let F be the Lagrangian monomorphism determined by F_0 and M. It follows from Lemma 3.18 that $\mu_F = \mu$ as required.

The second statement is a direct consequence of the Lagrangian h–principle. □

It follows from Proposition 3.19 that in the orientable case there are no extra obstructions to finding Maslov-zero monomorphisms (or immersions).

It is an interesting issue how one can use information on the Gauss map to study a given Lagrangian monomorphism. To investigate this, let us introduce a new way of thinking about Lagrangian trivializations. Let $\phi\colon TL^{\mathbb{C}} \to L \times \mathbb{C}^n$ be a Lagrangian trivialization. Let $SO(L)$ denote the bundle of positive orthonormal frames on L induced by the pullback metric. Recall that $SO(L)$ is a SO(n)–principal fibre bundle over L with respect to the right action of SO(n) defined as follows: if $(v_1, \dots, v_n) \in SO(L)$ and $A \in$ SO(n), then $(v_1, \dots, v_n)\cdot A := (w_1, \dots, w_n)$ where $w_i := v_j a_{ji}$. We will denote by $p\colon SO(L) \to L$ the obvious projection.

There is a natural map

$$\Phi\colon SO(L) \to \mathrm{U(n)}, \quad (v_1, \dots, v_n) \in p^{-1}(x) \mapsto \Phi(v_1, \dots, v_n)$$

where $\Phi(v_1, \dots, v_n)$ is the matrix which represents ϕ_x with respect to the basis $(v_1, \dots, v_n)$ and the standard basis of $\mathbb{C}^n$. This map is SO(n)–equivariant with respect to the action of SO(n) on U(n) determined by right multiplication. Denote the natural projection by $q\colon \mathrm{U(n)} \to \mathrm{U}(n)/\mathrm{SO}(n)$. Notice that the Gauss map satisfies the relation

$p \circ G\phi = \Phi \circ q$. In other words, Φ is a map between the principal fibre bundles $SO(L)$ and U(n) which covers the corresponding Gauss map on the base spaces L and U(n)/SO(n).

More generally, any Lagrangian trivialization inducing the same metric defines an equivariant map $SO(L) \to$ U(n). Furthermore, it is clear that this procedure defines a one-to-one correspondence between all such trivializations and all such maps. This correspondence respects homotopy classes in the following sense: two such trivializations are homotopic if and only if the corresponding maps are homotopic.

Notice that a given map $L \to \mathrm{U}(n)/\mathrm{SO}(n)$ is a Gauss map if and only if it admits a lift to an equivariant map $SO(L) \to$ U(n). It follows from the general theory of principal fibre bundles that the existence of such lifts is a homotopically invariant property: if $g\colon L \to \mathrm{U}(n)/\mathrm{SO}(n)$ is a Gauss map, ie $g = G\phi$, and g is homotopic to g' then g' is the Gauss map of some ϕ' and the corresponding Φ, Φ' are themselves homotopic through equivariant maps $SO(L) \to$ U(n). We can use this as follows.

Definition 3.20 Let $\mathrm{Gr}_{\mathrm{SL}}(\mathbb{C}^n)$ denote the Grassmannian of special Lagrangian planes in $\mathbb{C}^n$; it is a trivial subbundle of $\mathrm{Gr}^+_{\mathrm{lag}}(\mathbb{R}^{2n})$, with fibre SU(n)/SO(n). A Lagrangian monomorphism F is *special Lagrangian* if its Gauss map takes values in $\mathrm{Gr}_{\mathrm{SL}}(\mathbb{C}^n)$.

SL monomorphisms are the obvious formal analogue of SL immersions.

Proposition 3.21 *Let L be an oriented manifold. Then any Maslov-zero monomorphism F is homotopic through Lagrangian monomorphisms to an SL monomorphism F'.*

Proof Write $F = (f, \phi)$ and let Φ denote the corresponding equivariant map from $SO(L)$ to U(n) (with respect to the induced metric). Recall that $\mathrm{U(n)} \simeq \mathbb{S}^1 \times \mathrm{SU(n)}$ and that $\mathrm{U(n)/SO(n)} \simeq \mathbb{S}^1 \times \mathrm{SU(n)/SO(n)}$. Thinking of $G\phi$ as a map from L to $\mathbb{S}^1 \times$ SU(n)/SO(n), we see that F is Maslov-zero if and only if $G\phi$ is homotopic to a map $g'\colon L \to \mathrm{SU(n)/SO(n)}$. By the homotopic invariance of the lifting property, g' is the Gauss map of some Φ' homotopic to Φ. By construction, Φ' is a map $SO(L) \to \mathrm{SU(n)}$; the corresponding monomorphism $F' = (f, \phi')$ is a SL monomorphism homotopic to F. □

From the viewpoint of SL geometry, this proposition justifies our interest in Maslov-zero monomorphisms and immersions. It also shows that there are no extra obstructions to finding SL monomorphisms. However there do exist strong obstructions to finding SL immersions. For example, since $\mathbb{C}^n$ contains no closed minimal submanifolds, under our assumptions (L compact) there cannot exist SL immersions $f\colon L \to \mathbb{C}^n$. In particular this shows that SL monomorphisms do not satisfy an h–principle.

We conclude this section with a few comments on the nonorientable case. The nonoriented Lagrangian Grassmannian has fibre U(n)/O(n), so to get a well-defined determinant map it is necessary to replace $\det_{\mathbb{C}}$ with $\det^2_{\mathbb{C}}$. Unoriented analogues of the Maslov map, the Maslov class and the Maslov index of a curve can then be defined as before. The following lemma shows that if the unoriented Maslov data of a Lagrangian monomorphism $F\colon TL^{\mathbb{C}} \to \mathbb{C}^n$ satisfies certain conditions then in fact L must be orientable.

Lemma 3.22 *Suppose that L (not assumed to be orientable) admits a Lagrangian monomorphism $F\colon TL^{\mathbb{C}} \to \mathbb{C}^n$ such that the (unoriented) Maslov index*

$$m_F(\gamma) := \deg(\det^2_{\mathbb{C}} \circ GF \circ \gamma)$$

of every loop γ in L is even. Then L is orientable.

Proof Consider the two-to-one covering map $p\colon \mathrm{U(n)/SO(n)} \to \mathrm{U(n)/O(n)}$ obtained by forgetting the orientation of an oriented Lagrangian n–plane in $\mathbb{R}^{2n}$. One can show that $p_*(\pi_1(\mathrm{U(n)/SO(n)})$ is an index two subgroup of $\pi_1(\mathrm{U(n)/O(n)})$ isomorphic to $2\mathbb{Z} \subset \mathbb{Z}$. By standard covering space theory the map $GF\colon L \to \mathrm{U(n)/O(n)}$ lifts to the two-fold cover U(n)/SO(n) if and only if $GF_*\pi_1(L) \subseteq p_*\pi_1(\mathrm{U(n)/SO(n)})$. But this is equivalent to the fact that the Maslov index of every loop in L is even. Hence GF lifts to a map $GF^+\colon L \to \mathrm{U(n)/SO(n)}$, and this gives an orientation of each tangent space of L as needed. □

An immediate corollary of the previous Lemma is the following:

Corollary 3.23 *Suppose L admits a Lagrangian monomorphism $F\colon TL^{\mathbb{C}} \to \mathbb{C}^n$ such that $\det^2_{\mathbb{C}} GF\colon L \to \mathbb{S}^1$ is homotopic to a constant. Then L is orientable, and the corresponding oriented Gauss map is Maslov-zero.*

The above shows that Proposition 3.19 cannot hold for closed nonorientable manifolds L; in particular, $0 \in H^1(L, \mathbb{Z})$ cannot be realized as a Maslov class of any nonorientable manifold. The analogous result, which can be proved via the same methods, is as follows.

Proposition 3.24 *Let L be a closed, nonorientable manifold. Assume $TL^{\mathbb{C}}$ is trivial, so that there exists a Lagrangian monomorphism F with (unoriented) Maslov class $\mu_F \in H^1(L, \mathbb{Z})$. Then any other Lagrangian monomorphism from L has Maslov class in the set $2H^1(L, \mathbb{Z}) + \mu_F \subset H^1(L, \mathbb{Z})$, and any such cohomology class can be realized this way.*

3.5 Examples

We conclude this section by using Corollary 3.14 to give examples of manifolds which do and do not admit Lagrangian immersions into $\mathbb{R}^{2n}$.

3.5.1 Low-dimensional cases For $n \leq 3$, one can use the previous results to give a very good description of which closed n–manifolds admit Lagrangian immersions into $\mathbb{C}^n$ and to describe the regular homotopy classes of Lagrangian immersions.

$n = 1$ In this case $L = S^1$ and any immersion of L into $\mathbb{C}$ is Lagrangian. Let us fix one immersion $f_0\colon S^1 \to \mathbb{C}$. As explained in Lemma 3.15, any other immersion f now defines an element in $[S^1; \mathrm{U}(1)] = \pi_1(\mathrm{U}(1)) = \mathbb{Z}$, ie an integer. Notice that the Gauss maps take values in the Grassmannian $\mathrm{U}(1)/\mathrm{SO}(1) \simeq \mathbb{S}^1$ and that $\det_{\mathbb{C}} = \mathrm{Id}$. Choosing f_0 to have zero-Maslov class is equivalent to making Gdf_0 homotopically trivial; for example, we could choose f_0 to be a "figure eight curve" inside $\mathbb{C}$. With such a choice of f_0 the above integer is the turning number of f in $\mathbb{C}$. Hence the classification of regular homotopy classes of Lagrangian immersions of S^1 in $\mathbb{C}$ given by Proposition 3.16 reduces to Whitney's classification of immersions of S^1 in $\mathbb{R}^2$ according to their turning number.

$n = 2$ Let L be a closed surface, not necessarily orientable. In this case one can show that the complexified tangent bundle $TL^{\mathbb{C}}$ is trivial if and only if the Euler characteristic $\chi(L)$ is even [5, page 274]. Hence by Corollary 3.14 every orientable surface admits Lagrangian immersions in $\mathbb{C}^2$, while if L is the connected sum of k copies of $\mathbb{RP}^2$ then it admits Lagrangian immersions into $\mathbb{C}^2$ if and only if k is even. For example, $\mathbb{RP}^2$ has no Lagrangian immersions into $\mathbb{C}^2$, while the Klein bottle K does.

In the oriented case, let us now fix one such immersion. Recall from the proof of Proposition 3.19 that the map $[L; \mathrm{U}(2)] \to [L; \mathbb{S}^1]$ induced by $\det_{\mathbb{C}}$ is surjective. Since the map $\det_{\mathbb{C}}\colon \mathrm{U(n)} \to \mathbb{S}^1$ induces isomorphisms on π_1 and π_2, it follows using techniques similar to those of Section 4.4 that in dimension 2 this map is also injective [7, Corollary VII.11.13]. Hence the map $M \mapsto (\det_{\mathbb{C}} M)^*[d\theta]$ induces a bijection from $[L; \mathrm{U}(2)]$ to $H^1(L, \mathbb{Z})$. In other words, for any closed oriented surface we can strengthen Proposition 3.19 as follows: the regular homotopy class of the immersion f is completely determined by μ.

$n = 3$ We shall assume that L is a closed orientable 3–manifold. In this case, it is a theorem originally stated by Stiefel that L is parallelizable [41, page 148]. Complexification gives an obvious complex parallelization of L. Hence any closed orientable 3–manifold admits exact Lagrangian immersions into $\mathbb{C}^3$. In the special case where $L = S^3$ then the regular homotopy classes of Lagrangian immersions are in one-to-one correspondence with $\pi_3(\mathrm{U}(3)) = \mathbb{Z}$. In particular, unlike the case of curves and surfaces, the Maslov class of a Lagrangian immersion f of L^3 no longer determines its regular homotopy class. However, one can prove that there is a bijection between $[L; \mathrm{U}(3)]$ and $H^1(L,\mathbb{Z}) \times H^3(L,\mathbb{Z})$ [2, Proposition 1].

On the other hand, it is easy to find nonorientable closed 3–manifolds which do not admit Lagrangian immersions; $\mathbb{RP}^2 \times S^1$ is probably the simplest example.

3.5.2 Manifolds with trivial or stably trivial tangent bundles In the previous example we saw that if there exists a trivialization of TL then complexification gives an obvious trivialization of $TL^{\mathbb{C}}$. Hence any closed parallelizable n–manifold admits Lagrangian immersions into $\mathbb{C}^n$. In particular any compact Lie group admits Lagrangian immersions into $\mathbb{C}^n$ and so do the spheres and real projective spaces in dimensions 1, 3 and 7 (which are well-known to be the only parallelizable spheres or projective spaces). More generally, one can show that if TL is only stably parallelizable (ie the direct sum of TL with some trivial bundle is itself trivial) then $TL^{\mathbb{C}}$ is still trivial [5, page 273]. As we have already mentioned, any embedded hypersurface in $\mathbb{R}^{n+1}$ has this property. This gives an alternative proof that any compact orientable surface admits Lagrangian immersions into $\mathbb{R}^4$. It also proves that S^n admits Lagrangian immersions into $\mathbb{R}^{2n}$ for any n (on the other hand we will see that for most n, $\mathbb{RP}^n$ does not admit Lagrangian immersions). Furthermore, the regular homotopy classes of Lagrangian immersions are in one-to-one correspondence with $[S^n, \mathrm{U(n)}] = \pi_n(\mathrm{U(n)})$, which equals 0 or $\mathbb{Z}$ depending on whether n is even or odd respectively.

There are many more stably parallelizable manifolds than parallelizable manifolds. For example, the class of stably parallelizable manifolds is closed under taking products and connected sums [35, page 187] and also contains all manifolds which are homotopy spheres [35, page 191].

3.5.3 Nonexistence of Lagrangian immersions via characteristic classes One can sometimes give simple proofs of Lagrangian nonimmersion results by characteristic class arguments. Below we give representative examples of this type of argument using the Stiefel–Whitney and Pontrjagin classes of TL.

Given any real vector bundle $E^r \to L$ over any closed manifold L^n, one can define Stiefel–Whitney classes $w_i(E) \in H^i(L;\mathbb{Z}/2)$, with $w_0(E) = 1$ [41, Section 4]. $w(E) := w_0(E) + \cdots + w_r(E)$ is called the total Stiefel–Whitney class of the bundle. Recall that $w(E)$ has the following fundamental properties: $w(E \oplus F) = w(E) \cdot w(F)$ and, if E is trivial, then $w(E) = 1$. The total Stiefel–Whitney class of $T\mathbb{RP}^n$ is well-known to be

$$w(\mathbb{RP}^n) = (1+a)^{n+1}, \tag{3–1}$$

where a denotes the nonzero element of $H^1(\mathbb{RP}^n, \mathbb{Z}/2)$ which under cup product generates the full $\mathbb{Z}/2$ cohomology ring of $\mathbb{RP}^n$ [41, Theorem 4.5]. As unoriented real vector bundles $TL^{\mathbb{C}} \simeq TL \oplus TL$. Hence if $TL^{\mathbb{C}}$ is trivial, then $w(TL)^2 = 1$. Using (3–1) it is straightforward to show that $w(\mathbb{RP}^n)^2 = 1$ if and only if $n+1$ is a power of 2. Hence there are no Lagrangian immersions of $\mathbb{RP}^n$ in $\mathbb{C}^n$ unless $n = 2^k - 1$ for some $k \in \mathbb{Z}$. For $k = 1, 2, 3$ we already saw in Section 3.5.2 that such Lagrangian immersions do exist.

The Pontrjagin classes of a real vector bundle $E^r \to L$ may be obtained from the Chern classes of its complexification $E \otimes_{\mathbb{R}} \mathbb{C}$. The standard definition of the i–th Pontrjagin class is $p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(L, \mathbb{Z})$ [41, page 174]. The total Pontrjagin class $p(E)$ is defined to be

$$p(E) = 1 + p_1(E) + \ldots + p_{[n/2]}(E) \in H^*(L). \tag{3–2}$$

The class $p(E)$ equals the total Chern class of $E \otimes \mathbb{C}$, ignoring the odd Chern classes $c_{2i+1}(E \otimes \mathbb{C})$ which vanish in de Rham cohomology and are order 2 in the integral cohomology (Bott and Tu give a different definition of $p(E)$ which does not ignore these odd Chern classes [6, page 289]). If the complex vector bundle $E \otimes \mathbb{C}$ is trivial, then from the basic properties of Chern classes we have $p(E) = 1$. In particular, if L admits a Lagrangian immersion in $\mathbb{R}^{2n}$ then $TL^{\mathbb{C}}$ is trivial and hence $p(TL) = 1$. For instance, the total Pontrjagin class of $\mathbb{CP}^n$ is well-known [41, page 177] to be

$$p(\mathbb{CP}^n) = (1+a^2)^{n+1} = 1 + (n+1)a^2 + \ldots \tag{3–3}$$

where $a \in H^2(\mathbb{CP}^n; \mathbb{Z})$ generates $H^*(\mathbb{CP}^n; \mathbb{Z})$ under cup product. In particular, $p(\mathbb{CP}^n) \neq 1$ for $n > 1$, and hence admits no Lagrangian immersions in $\mathbb{C}^{2n}$. For an oriented 4–manifold M there is only one nontrivial Pontrjagin class $p_1(M) \in H^4(M, \mathbb{Z})$ and hence only one Pontrjagin number $p_1[M] := \langle p_1(TM), [M] \rangle \in \mathbb{Z}$. If Σ_d is a nonsingular algebraic hypersurface of degree $d > 0$ in $\mathbb{CP}^3$, then a standard characteristic class computation (see Donaldson and Kronheimer [12, Section 1.1.7]) shows that the Pontrjagin number of Σ_d is given by

$$p_1[\Sigma_d] = (4 - d^2)d. \tag{3–4}$$

In particular, Σ_d does not admit Lagrangian immersions into $\mathbb{C}^4$ unless $d = 2$, in which case $\Sigma_2 \simeq S^2 \times S^2$ which is stably parallelizable and hence admits a Lagrangian immersion.

4 Prescribed Boundary Problem

We now turn to Lagrangian immersions of compact manifolds with boundary. Specifically, we will prescribe an immersion along the boundary Σ^{n-1} and try to find a "Lagrangian filling" of this data. In Section 4.2 we will describe this "Prescribed Boundary Problem" more precisely. In Section 4.1 we begin with a preliminary, related question. Throughout this section, our focus will be on the orientable case.

4.1 Preliminary considerations

Let Σ^{n-1} be a compact oriented (not necessarily connected) manifold without boundary, and $i\colon \Sigma \to \mathbb{R}^{2n}$ be an immersion. Consider the following question.

Lagrangian Cobordism Problem Does there exist a compact oriented n–manifold L bounding Σ and a Lagrangian immersion $f\colon L \to \mathbb{R}^{2n}$ extending $i\colon \Sigma \to \mathbb{R}^{2n}$?

Our main interest is actually in a stronger version of this question, which we call the Prescribed Boundary Problem. Our goal now is thus not to solve the Lagrangian Cobordism Problem, but simply to highlight some of the restrictions that it imposes on Σ and i.

Topological restrictions The first question is whether Σ bounds any compact oriented smooth manifold L. As we have already discussed in Section 2.5 this is equivalent to the vanishing of all the Pontrjagin and Stiefel–Whitney numbers of Σ.

The second topological issue is that L must admit Lagrangian immersions. As in Section 3, it is easy to prove that this implies that $TL^{\mathbb{C}}$ must be trivial. The existence of such an L is an additional constraint on Σ.

Geometric restrictions Suppose that we have overcome the topological restrictions described above, ie Σ bounds some compact n–manifold L with $TL^{\mathbb{C}}$ trivial. If there exists a Lagrangian immersion $f: L^n \to \mathbb{R}^{2n}$ which extends $i\colon \Sigma \to \mathbb{R}^{2n}$ then i must be isotropic, ie $i^*\omega = 0$. Thus, from now on we assume that i is an isotropic immersion.

A second necessary condition on i comes from cohomological considerations. First consider the following:

Example 4.1 For dimensional reasons, any immersion $i\colon S^1 \to \mathbb{R}^4$ is isotropic. Assume i admits an oriented Lagrangian extension $f\colon L \to \mathbb{R}^4$. Then by Stokes' Theorem,

$$\int_{S^1} i^*\lambda = \int_{S^1} f^*\lambda = \int_L f^*\omega = 0,$$

so i is exact. For instance, this shows the isotropic immersion $i\colon \mathbb{S}^1 \to \mathbb{C}^2$ given by

$$i(e^{i\theta}) = (e^{i\theta}, 0)$$

does not admit any oriented Lagrangian filling.

More generally, suppose i admits an oriented Lagrangian extension $f\colon L \to \mathbb{R}^{2n}$. Then, for any closed curve $\gamma \subset \Sigma$ which is homologically trivial in L, Stokes' Theorem shows that $\int_\gamma \lambda = 0$. Another way of saying this is as follows. If $\gamma \subset \Sigma$ is homologically trivial in L, there exists $\alpha \in H_2(L,\Sigma)$ such that $\partial\alpha = \gamma$. On the other hand, since i is isotropic any extension f defines an element $[f^*\omega]$ in the relative de Rham cohomology $H^2(L,\Sigma)$. If f is Lagrangian then $[f^*\omega] = 0$. By duality this occurs if and only if $\langle f^*\omega, \alpha\rangle = 0$ for all $\alpha \in H_2(L,\Sigma)$. Now, using the particular α defined above, we see that

$$\int_\gamma \lambda = \int_\alpha f^*\omega = \langle f^*\omega, \alpha\rangle = 0.$$

This condition is not automatic from the fact that i is an isotropic immersion.

However if we assume that the immersion i is not merely isotropic but *exact*, then we have $\int_\gamma \lambda = 0$ for any closed curve γ in Σ. Example 4.1 shows that when $n = 2$ (and Σ is connected) exactness is actually a necessary condition. Moreover, we will see that (as in the closed case) Lagrangian submanifolds with boundary produced using the relative version of the Lagrangian h–principle are automatically exact and hence so are their boundaries. Thus, if we want to make use of the relative h–principle then the exactness of Σ is a necessary assumption and not a mere convenience. Finally, when we come to apply our results on the Prescribed Boundary Problem to Lagrangian desingularization problems we will find that the local geometry near the singular points often forces Σ to be exact and not just isotropic.

Hence from now on we make the following assumptions about Σ.

Assumptions about Σ

A. Σ bounds a compact oriented manifold L.

B. $TL^{\mathbb{C}}$ is trivial.

C. The immersion $i\colon \Sigma \to \mathbb{R}^{2n}$ is exact.

4.2 Lagrangian thickenings and the Prescribed Boundary Problem

In Section 6 we investigate the existence of Lagrangian smoothings of a singular Lagrangian object with only isolated singular points. We attempt to find our Lagrangian smoothings by removing a small neighbourhood U of a singular point and gluing in a smooth Lagrangian immersion of some compact manifold L with boundary $\Sigma := \partial U$. To ensure smoothness of the new submanifold along the boundary ∂U we need to modify the Lagrangian Cobordism Problem by assigning as initial data, instead of just Σ, a neighbourhood of Σ to be "filled" by L.

Following [13; 17] let us introduce the notation $\mathcal{O}p\,\Sigma$ to denote some open neighbourhood of $\Sigma \subset L$, which can be varied as appropriate by restriction to a smaller neighbourhood of Σ. In our applications we will usually choose $\mathcal{O}p\,\Sigma$ to be topologically $[0,\epsilon)\times\Sigma$.

Definition 4.2 Let L^n be a compact connected oriented manifold with oriented boundary Σ, such that $TL^{\mathbb{C}}$ is trivial. Let $f\colon \mathcal{O}p\,\Sigma \to \mathbb{R}^{2n}$ be an exact Lagrangian immersion of an open neighbourhood of $\Sigma \subset L$ into $\mathbb{R}^{2n}$. We say that the triple (Σ, L, f) is *(exact) initial data* for the Prescribed Boundary Problem. A *solution to the Prescribed Boundary Problem with initial data* (Σ, L, f) is any exact Lagrangian immersion $\widehat{f}: L \to \mathbb{R}^{2n}$ which agrees with f on some (possibly smaller) open neighbourhood of Σ.

Remark 4.3 As in Remark 3.7, it will follow from (the parametric version of) Theorem 4.6 below that any Lagrangian immersion $\widehat{f}\colon L \to \mathbb{C}^n$ which agrees with the initial data (Σ, L, f) on some neighbourhood of Σ is regularly homotopic to an exact Lagrangian immersion with the same initial data. For this reason there is no loss of generality in incorporating exactness directly into the definition of "solution".

Any solution of the Prescribed Boundary Problem yields a solution of the Lagrangian Cobordism Problem, defined by setting $i = f|_{\Sigma}$. The prescription of a neighbourhood of Σ adds however an important cohomological constraint to the problem, as follows.

Assume for simplicity that $\mathcal{O}p\,\Sigma \simeq [0,\epsilon)\times\Sigma$. Notice that the initial data (let us call it a "Lagrangian thickening" of i) determines a Maslov class $\mu_f \in [\Sigma\times[0,\epsilon), \mathbb{S}^1] \simeq [\Sigma, \mathbb{S}^1] \simeq H^1(\Sigma,\mathbb{Z})$ defined, as before, via the corresponding Gauss map Gdf. Now suppose that $(L, \widehat{f})$ is a solution to the induced Lagrangian Cobordism Problem. If $(L, \widehat{f})$ also solves the Prescribed Boundary Problem, then in particular the Maslov classes μ_f and $\mu_{\widehat{f}}$ must satisfy

$$i^*\mu_{\widehat{f}} = \mu_f,$$

where $i^*\colon H^1(L,\mathbb{Z}) \to H^1(\Sigma,\mathbb{Z})$ is the map induced by the inclusion $\Sigma \times [0,\epsilon) \subset L$. Hence, the choice of initial data f with a different Maslov class μ_f will impose a different condition on $\mu_{\hat{f}}$ but will not change the induced Lagrangian Cobordism Problem.

In particular, we see that if a solution to the Prescribed Boundary Problem does exist then μ_f must belong to the image of i^*. Thus for example if $\mu_f \neq 0$ and $H^1(L,\mathbb{Z}) = 0$, the Prescribed Boundary Problem will not admit solutions even though by hypothesis the Lagrangian Cobordism Problem does (eg we could use $\Sigma = S^1$ and $L = D^2$).

The above argument supposes that the same Σ can support many different Lagrangian thickenings, with different Maslov classes. One can think of several different constructions to show that this is indeed true. Two of the simplest are outlined in the examples below. Notice that since Lagrangian thickenings have the same topology as Σ, they are exact if and only if Σ is.

Example 4.4 If the isotropic immersion $i\colon \Sigma^{n-1} \to \mathbb{R}^{2n}$ is not just smooth but real analytic, we can produce Lagrangian thickenings of i as follows. Let $L := (-\epsilon,\epsilon) \times \Sigma$. Then, for ϵ sufficiently small, and any $e^{i\theta} \in \mathbb{S}^1$, an application of Cartan–Kähler theory proves that there exists a unique θ–special Lagrangian immersion of L extending i [22, Theorem III.5.5].

Example 4.5 Suppose i is actually a Lagrangian immersion of Σ into $\mathbb{C}^{n-1}$ so that it defines a Maslov class $\mu_i \in [\Sigma, \mathbb{S}^1]$. Let f be a Lagrangian thickening of i in $\mathbb{C}^n$ and $\mu_f \in [\Sigma,\mathbb{S}^1]$ be its Maslov class. For example, if $f\colon \Sigma \times [0,\epsilon) \to \mathbb{C}^n$ is the "cylindrical thickening" of i given by

$$(x,t) \mapsto (i(x),t)$$

then $\mu_f = \mu_i$. On the other hand, if i is real analytic then Example 4.4 shows how to build Maslov-zero thickenings of i. This shows that the same i may admit Lagrangian thickenings with different Maslov classes.

4.3 The Prescribed Boundary Problem and the relative h–principle

Our main tool to solve the Prescribed Boundary Problem described in Definition 4.2 will again be the Gromov–Lees h–principle, but this time in its relative form.

Theorem 4.6 (Lagrangian h–principle, relative version, Eliashberg and Mishachev [13, Section 6.2.C, Section 16.3.1]) *Suppose $\Sigma = \partial L$ and that there exists a Lagrangian monomorphism $F\colon TL \to T\mathbb{R}^{2n}$ which is holonomic over an open neighbourhood $\mathcal{O}p\,\Sigma$ of $\Sigma \subset L$; ie $F = (f,\phi)$, where ϕ satisfies $\phi|_{\mathcal{O}p\,\Sigma} = df|_{\mathcal{O}p\,\Sigma}$.*

Assume furthermore that $f|_{\mathcal{O}p\,\Sigma}$ is exact. Then there exists a family of Lagrangian monomorphisms $F_t : [0,1] \times TL \to T\mathbb{R}^{2n}$ such that $F_0 = F$, $F_t|_{\mathcal{O}p\,\Sigma} = F_0|_{\mathcal{O}p\,\Sigma}$ and F_1 is holonomic; ie $F_1 = d\widetilde{f}$. In particular, the base map $bs(F_1) = \widetilde{f}\colon L \to \mathbb{R}^{2n}$ is a Lagrangian immersion with the prescribed boundary data $f|_{\mathcal{O}p\,\Sigma}$. Furthermore, $\widetilde{f}$ is exact.

Remark 4.7 Theorem 4.6 still holds without assuming orientability of Σ or L.

Remark 4.8 As in the closed case, the proof of the relative h–principle for Lagrangian immersions relies on lifting the Lagrangian monomorphism to a Legendrian one in $\mathbb{R}^{2n+1}$ and applying the relative h–principle for Legendrian immersions; again this explains why the resulting Lagrangian immersion is exact. However, in order to apply the Legendrian h–principle we need this lift to be holonomic on $\mathcal{O}p\,\Sigma$: this explains the extra assumption in the statement.

Theorem 4.6 shows that to solve the Prescribed Boundary Problem with exact initial data, the only obstruction one needs to overcome is the construction of a Lagrangian monomorphism with the prescribed initial data. The triviality condition on $TL^{\mathbb{C}}$ is not sufficient to guarantee that such monomorphisms exist: the boundary data imposes an additional constraint, which we formalize as follows.

Definition 4.9 Let (Σ, L, f) be initial data for the Prescribed Boundary Problem. A trivialization $TL^{\mathbb{C}} \to L \times \mathbb{C}^n$ is *compatible* with the initial data if it extends the trivialization of $T(\mathcal{O}p\,\Sigma)^{\mathbb{C}}$ induced by $df^{\mathbb{C}}$.

The theory now proceeds exactly as in Section 3. In particular, the following statement can be proved using those same methods.

Corollary 4.10 *Let (Σ, L, f) be exact initial data for the Prescribed Boundary Problem. Then the following conditions are equivalent.*

1. *The Prescribed Boundary Problem with initial data (Σ, L, f) admits a solution.*
2. *$TL^{\mathbb{C}}$ admits a compatible trivialization.*
3. *L admits a totally real monomorphism $F\colon TL \to T\mathbb{R}^{2n}$ which extends df.*
4. *L admits a Lagrangian monomorphism $F\colon TL \to T\mathbb{R}^{2n}$ which extends df.*

Furthermore, there is a one-to-one correspondence between the homotopy classes of the above objects. In particular, if a solution to the Prescribed Boundary Problem exists, it induces a bijective correspondence between the set of all compatible trivializations of $TL^{\mathbb{C}}$ and the set of all maps $L \to GL(n, \mathbb{C})$ which map a neighbourhood of Σ to the identity matrix. Two trivializations are homotopic if and only if the corresponding maps are homotopic.

The theory is not yet completely satisfactory: we would still like a constructive procedure to verify the existence of compatible trivializations. To this end, let (Σ, L, f) be initial data for the Prescribed Boundary Problem. Notice that since $\mathbb{R}^{2n}$ is contractible there is no obstruction to extending f to a smooth map $\widehat{f}\colon L \to \mathbb{R}^{2n}$; we can thus fix one such extension $\widehat{f}$ and use it as the base map for all our monomorphisms.

Let us now fix a reference trivialization $F_0 = (\widehat{f}, \phi_0)$ of $TL^{\mathbb{C}}$ (not necessarily compatible with the initial data). As seen in Lemma 3.15 this choice determines a one-to-one correspondence between all other trivializations of $TL^{\mathbb{C}}$ and maps $L \to \mathrm{GL}(n, \mathbb{C})$. In particular, $df^{\mathbb{C}}$ determines a map $M_{df^{\mathbb{C}}, F_0}\colon \mathcal{O}p\,\Sigma \to \mathrm{GL}(n, \mathbb{C})$. The proof of the following result is a direct consequence of Definition 4.9.

Lemma 4.11 (Existence of compatible trivializations) *There exists a trivialization compatible with the initial data (Σ, L, f) if and only if the map $M := M_{df^{\mathbb{C}}, F_0}$ from $\mathcal{O}p\,\Sigma$ to GL(n, $\mathbb{C}$) determined by $df^{\mathbb{C}}$ and by the choice of reference trivialization F_0 can be extended to a map $\widehat{M}\colon L \to$ GL(n, $\mathbb{C}$).*

Remark 4.12 The existence of a compatible trivialization should be independent of any choices made in the construction of the matrix-valued map M; in particular if we change the reference trivialization we will obtain a different map M', and M should be extensible if and only if M' is. Notice that the change of trivialization will be described by a matrix-valued map defined on the whole L; using this it is simple to see that the above indeed holds.

In other words, the fact that $TL^{\mathbb{C}}$ is trivial allows us to translate the Prescribed Boundary Problem into an extension problem for maps into $\mathrm{GL}(n, \mathbb{C})$. In Section 4.4 we will describe a standard obstruction theory framework for dealing with such extension problems using algebraic topology, following the treatments in Bredon [7] and Hatcher [26]. In the meantime we extend the definitions and results of Section 3.4 to manifolds with boundary.

Given a manifold with boundary L and a Lagrangian monomorphism $F\colon TL \to T\mathbb{C}^n$, we define its Gauss map $GF\colon L \to \mathrm{U(n)/SO(n)}$ and determinant map $\det_{\mathbb{C}} \circ GF\colon L \to \mathbb{S}^1$ exactly as before. This allows us to continue to use the same definition also for the Maslov map $\mu\colon \mathcal{M} \to [L, \mathbb{S}^1] \simeq H^1(L, \mathbb{Z})$, where again $\mathcal{M}$ denotes the set of homotopy classes of Lagrangian monomorphisms.

Lemma 4.13 *Let L be a compact oriented n–manifold with boundary. Suppose $TL^{\mathbb{C}}$ is trivial and fix a reference Lagrangian monomorphism F_0. Then, for any other Lagrangian monomorphism F (homotoped so that the corresponding matrix map*

M_{F,F_0} takes values in U(n)), $\mu_F = [\det_{\mathbb{C}} M_{F,F_0}] \cdot \mu_{F_0}$. In particular, if F_0 has zero Maslov class then $\mu_F = [\det_{\mathbb{C}} M_{F,F_0}]$.

Furthermore, let μ be an arbitrary element of $H^1(L,\mathbb{Z})$. Then L admits a Lagrangian monomorphism $F\colon TL \to T\mathbb{C}^n$ with $\mu_F = \mu$. Finally, any Maslov-zero Lagrangian monomorphism can be homotoped to a SL monomorphism.

Proof The proofs are exactly the same as those of Lemma 3.18, Proposition 3.19 and Proposition 3.21. □

Notice that the second part of Proposition 3.19 does not extend so easily because the h–principle for manifolds with boundary requires additional assumptions about the monomorphism near the boundary. Fortunately the above Lemma will be sufficient for our purposes.

Example 4.14 Concerning the nonorientable case, suppose for example that L is the Möbius strip. Recall that L is a subset of the Klein bottle K, which from Section 3.5 has $TK^{\mathbb{C}}$ trivial; thus $TL^{\mathbb{C}}$ is trivial. The analogue of Proposition 3.24 for nonorientable manifolds with boundary shows that the set of unoriented Maslov classes of L must be either $2\mathbb{Z}$ or $2\mathbb{Z}+1$ inside $H^1(L,\mathbb{Z}) \simeq \mathbb{Z}$. However, since L is nonorientable, 0 cannot be a Maslov class so the subset realized by Maslov classes is $2\mathbb{Z}+1 \subset \mathbb{Z}$.

4.4 The Extension Problem and obstruction theory

Let (Σ, L, f) be initial data for the Prescribed Boundary Problem. By Lemma 4.11 the existence of a trivialization compatible with the initial data (in the sense of Definition 4.9) is equivalent to the extensibility to L of a map $M\colon \mathcal{O}p\,\Sigma \to \mathrm{GL}(n,\mathbb{C})$. In this section first we recall a standard approach to the abstract extension problem for continuous maps of topological spaces. Then we apply these results to prove various existence and nonexistence results for compatible trivializations.

Let A be a subspace of the topological space X and let $M : A \to Y$ be a continuous map. The Extension Problem asks:

Extension Problem When can M be extended to a continuous map $\widehat{M}\colon X \to Y$?

It is not always possible to give a continuous extension of M. For example, take $(X,A) = (D^{n+1}, S^n)$ and $Y = S^n$. Then M extends to D^{n+1} if and only if the class $[M] = 0 \in \pi_n(S^n) \equiv \mathbb{Z}$. In particular, $M\colon S^n \to S^n$ extends to D^{n+1} if and only if $\deg(M) = 0$. Thus there is a homotopy-theoretic obstruction to extending M. If we keep $(X,A) = (D^n, S^{n-1})$ and use $Y = \mathrm{GL}(n,\mathbb{C})$, then this is simplest case of interest in Lemma 4.11. Once again $M\colon S^{n-1} \to \mathrm{GL}(n,\mathbb{C})$ extends to D^n if and only if $[M] = 0 \in \pi_{n-1}(\mathrm{GL}(n,\mathbb{C}))$. In particular, if n is odd then this homotopy group vanishes (see below) and hence from Corollary 4.10 and Lemma 4.11 we obtain:

Corollary 4.15 *If n is odd, then the Prescribed Boundary Problem is solvable for any initial data of the form (D^n, S^{n-1}, f).*

On the other hand, if n is even, then $\pi_{n-1}(\mathrm{GL}(n,\mathbb{C})) = \mathbb{Z}$. For the case $n=2$, we will show in the next section that the Prescribed Boundary Problem is solvable if and only if the Maslov class of the initial data (D^2, S^1, f) is zero; see Theorem 5.4 for this result in a slightly more general context. This is again a homotopy-theoretic obstruction to solving the Prescribed Boundary Problem. Furthermore, since S^{n-1} is simply connected for $n \geq 3$, we see that these obstructions will be sensitive to more than the Maslov class of the initial data.

Note For the remainder of this section a map between two topological spaces means a continuous map even if we do not say so explicitly.

Now we would like to understand the Extension Problem for more general pairs (X,A). In the simple extension problems considered above the extensibility of M depended only on the homotopy class of M. It is natural to ask if this is always the case.

An equivalent reformulation of this question is: given two homotopic maps M_0 and M_1 from A to Y and an extension $\widehat{M}_0\colon X \to Y$ of M_0, is there always an extension $\widehat{M}_1\colon X \to Y$ of M_1 which is homotopic to $\widehat{M}_0$? If for fixed (X,A) and Y we can always find such a homotopic extension $\widehat{M}_1$ then (X,A) is said to have the *homotopy extension property* with respect to Y.

In other words, if (X,A) has the homotopy extension property with respect to Y then the extensibility of maps $M\colon A \to Y$ depends only on the homotopy class of M; in particular in this case, any map $M\colon A \to Y$ which is homotopic to a constant map has an extension $\widehat{M}\colon X \to Y$ which is also homotopic to a constant map (M may also have other homotopically distinct extensions).

Not all spaces satisfy the homotopy extension property [7, page 430]. However, under rather mild assumptions on (X,A) we get the much stronger conclusion that (X,A) satisfies the homotopy extension property with respect to any space Y [7, VII.1.1]. For example, if (X,A) is a CW-pair, ie A is a subcomplex of a CW-complex X, then the homotopy extension property holds for any space Y and hence the Extension Problem is a homotopy-theoretic one [7, Corollary VII.1.4].

If, in addition, the space Y is path-connected and simple (that is, $\pi_1(Y, y_0)$ acts trivially on $\pi_n(Y, y_0)$ for all $y_0 \in Y$ and $n \geq 1$) then one can use a Postnikov decomposition of Y [7, page 501] to solve the Extension Problem as follows:

Theorem 4.16 (Extension/Obstruction Theorem [7, page 507]) *Let (X, A) be a CW-pair and Y be path-connected and simple and let $M\colon A \to Y$ be a (continuous) map. Then there exists a sequence of obstructions*

$$c_M^{n+1} \in H^{n+1}(X, A; \pi_n(Y)), \quad n \geq 1,$$

(where c_M^{n+1} is defined only when all the previous obstructions vanish and depends on the previous liftings made) such that there is a solution of the Extension Problem for M if and only if there is a complete sequence of obstructions c_M^{n+1} all of which are zero. Furthermore, if $\widehat{M}_1$ and $\widehat{M}_2$ are two different extensions of M then there exists a sequence of obstructions

$$d^n(\widehat{M}_1, \widehat{M}_2) \in H^n(X, A; \pi_n(Y))$$

to the existence of a homotopy rel A between $\widehat{M}_1$ and $\widehat{M}_2$.

We will discuss later how to construct these obstruction cocycles c_M^i in some cases.

In our applications $(X, A) = (L, \partial L)$, where L is a smooth compact oriented manifold with oriented boundary ∂L. It is a classical result of J H C Whitehead [51] (see also Milnor and Stasheff [41, page 240]) that a smooth compact manifold with boundary can be triangulated as a simplicial complex with the boundary as a subcomplex. In particular, $(L, \partial L)$ is a CW-pair. Furthermore, the space Y will be a connected Lie group, in which case Y is known to be simple [47, pages 88–89], and hence Theorem 4.16 applies. For the application of obstruction theory to Lemma 4.11 and the construction of compatible trivializations we are interested in the case where $Y = \mathrm{GL(n, \mathbb{C})}$; the retraction $\mathrm{GL(n, \mathbb{C})} \to \mathrm{U(n)}$ allows us, from now on, to concentrate on the case $Y = \mathrm{U(n)}$.

For our applications of Theorem 4.16 we are only interested in $\pi_i(\mathrm{U(n)})$ for $i \leq n$. These fall within the "stable range" where $[(i+2)/2] \leq n$ and hence by Bott Periodicity are equal to 0 when n is even and $\mathbb{Z}$ when n is odd [7, pages 467–8]. Hence we have the following immediate corollary of Theorem 4.16:

Corollary 4.17 *A map $M\colon \Sigma^{n-1} \to U(n)$ extends to L if and only there is a sequence of $[n/2]$ obstructions*

$$c_M^{2i} \in H^{2i}(L, \Sigma; \mathbb{Z}) \quad \text{for } i \in \{1, \ldots, [n/2]\}$$

which all vanish (where c_M^{2i} is defined only when all the previous obstructions vanish and depends on the previous extensions made) . Furthermore if $\widehat{M}_1$ and $\widehat{M}_2$ are two different extensions of M there is a sequence of obstructions

$$d^{2i+1}(\widehat{M}_1, \widehat{M}_2) \in H^{2i+1}(L, \Sigma; \mathbb{Z}), \quad \text{for } i = 0, \dots, [(n-1)/2],$$

to the existence of a homotopy rel A between $\widehat{M}_1$ and $\widehat{M}_2$.

To analyze further the extensibility of maps into U(n) it is convenient first to break the problem up into two separate extension problems. There are two advantages of splitting the problem into these separate extension problems. The first advantage is that it shows that the higher obstructions cocycles c_M^{2i} $(i > 1)$ for a map into U(n) can be studied independently of the first obstruction cocycle c_M^2. The second advantage is that it will allow us in Corollary 4.25 to identify the first obstruction cocycle c_M^2 in a very concrete manner. The splitting of the U(n) extension problem is achieved as follows:

Given $U \in \mathrm{U(n)}$ define a map $\pi_{\mathbb{S}^1} : \mathrm{U(n)} \to \mathbb{S}^1$ by $\pi_{\mathbb{S}^1}(U) = \det_{\mathbb{C}} U$ and a map $\pi_{\mathrm{SU(n)}}\colon \mathrm{U(n)} \to \mathrm{SU(n)}$ by $\pi_{\mathrm{SU(n)}}(U) = \mathrm{diag}(\pi_{\mathbb{S}^1(U)}, 1, \dots, 1)^{-1} U$. Since any U in U(n) may be written uniquely as a product $U = \mathrm{diag}(\pi_{\mathbb{S}^1}(U), 1, \dots, 1) \cdot \pi_{\mathrm{SU(n)}}(M)$, we see that U(n) is diffeomorphic to $\mathbb{S}^1 \times \mathrm{SU(n)}$. In particular, any map M from any topological space X into U(n) induces two composition maps $M_{\mathbb{S}^1} := \pi_{\mathbb{S}^1} \circ M$ and $M_{\mathrm{SU(n)}} := \pi_{\mathrm{SU(n)}} \circ M$ from X into $\mathbb{S}^1$ and SU(n) respectively. Conversely any pair of maps from X into $\mathbb{S}^1$ and SU(n) determines a unique map from X into U(n). In particular, a map $M\colon \Sigma \to \mathrm{U(n)}$ extends to $\widehat{M}\colon L \to \mathrm{U(n)}$ if and only if the maps $M_{\mathbb{S}^1}\colon \Sigma \to \mathbb{S}^1$ and $M_{\mathrm{SU(n)}}\colon \Sigma \to \mathrm{SU(n)}$ both extend to L.

Remark 4.18 Using the observation above it follows that for any CW-complex K the homotopy classes of maps $[K; \mathrm{U(n)}]$ can be identified (as a set but not necessarily as a group) with $[K; \mathbb{S}^1] \times [K; \mathrm{SU(n)}] \simeq H^1(K, \mathbb{Z}) \times [K; \mathrm{SU(n)}]$ (using Lemma 3.17 for the final identification).

We must now apply the Theorem 4.16 to maps with target SU(n) or $\mathbb{S}^1$ and compare the results obtained to those of Corollary 4.17, where the target is U(n). In particular, we want to see how the $[n/2]$ obstructions which occur in the U(n) case split into two parts according to the decomposition of the map into its $\mathbb{S}^1$ and SU(n) parts.

Since $\mathrm{U(n)} \simeq \mathbb{S}^1 \times \mathrm{SU(n)}$ as manifolds (but not as Lie groups), $\pi_i(\mathrm{U(n)}) = \pi_i(\mathrm{SU(n)})$ for $i \geq 2$. We also have $\pi_1(\mathrm{SU(n)}) = 0$. Therefore, given a map $M\colon \Sigma \to \mathrm{SU(n)}$, by Theorem 4.16 there is a sequence of $[n/2]-1$ obstructions $c_M^{2i} \in H^{2i}(L, \Sigma; \mathbb{Z})$ for $i \in \{2, \dots, [n/2]\}$ to extending M to $\widehat{M}\colon L \to \mathrm{SU(n)}$. In particular, the first obstruction to extending a map M into U(n) disappears if instead M maps into SU(n). In particular, if $n=2$ or $n=3$ then there are no obstructions to extending M. In other words, we have the following extension result for maps into SU(2) or SU(3).

Corollary 4.19 *Let L be a compact smooth n–manifold with boundary Σ and let $M\colon \Sigma \to SU(n)$ be any map. Then for $n=2,3$, M extends to a map $\widehat{M}\colon L \to SU(n)$. Furthermore, the homotopy classes of maps $L \to SU(n)$ rel Σ which equal M on Σ are in one-to-one correspondence with 0 and $\mathbb{Z}$ for $n=2$ and $n=3$ respectively.*

Remark 4.20 Corollary 4.19 is already sufficient for a proof of Corollary 5.6 (see the second method of proof given there) and Corollary 5.9, and (via Remark 5.7) for the applications of Section 6.

The difference between the extensibility of maps $M\colon \Sigma \to \mathrm{U(n)}$ and $M_{SU}\colon \Sigma \to \mathrm{SU(n)}$ will be captured by the extensibility of the map $M_{\mathbb{S}^1}\colon \Sigma \to \mathbb{S}^1$. Since $\mathbb{S}^1$ has trivial homotopy groups for $i>1$, given a map $M\colon \Sigma \to \mathbb{S}^1$ then Theorem 4.16 gives us precisely one obstruction cocycle $c^2_M \in H^2(L,\Sigma;\mathbb{Z})$ to extending it to L.

Hence as expected in total we have the same number of obstruction cocycles (namely $[n/2]$ of them) which measure the nonextensibility for a map into U(n) or for a pair of maps into SU(n) and $\mathbb{S}^1$; one of these cocycles encodes the extensibility of the map into $\mathbb{S}^1$, while the remaining $[n/2]-1$ encode the extensibility of the map into SU(n).

To proceed further we need to understand better the obstruction cocycles. For the construction of these cocycles c^{i+1}_M in the general case we refer the reader to Bredon [7, VII.13]. However, in the case that the only possible nonzero obstruction is the so-called primary one it is simple to describe this obstruction. This will allow us to understand very concretely the obstruction to extending a map into $\mathbb{S}^1$.

Theorem 4.21 [7, Corollary VII.13.13] *Let (X,A) be a CW-pair and Y be simple and $(k-1)$–connected. Suppose that $H^{i+1}(X,A;\pi_i(Y))=0$ for all $i>k$. Then a map $M\colon A \to Y$ can be extended to a map $\widehat{M}\colon X \to Y$ if and only if the homomorphism*

$$\delta^* M^*\colon H^k(Y;\pi) \to H^{k+1}(X,A;\pi)$$

is trivial, where $\pi=\pi_k(Y)$ and δ^* is the usual map $\delta^*: H^k(A;\pi) \to H^{k+1}(X,A;\pi)$ which appears in the exact cohomology sequence of the pair (X,A).

Remark 4.22 There is an easy way to see that the nontriviality of the map $\delta^* M^*$ of Theorem 4.21 is an obstruction to extending M. Suppose $\widehat{M}\colon X \to Y$ is an extension of M. Then $\widehat{M}\circ i = M$ and hence $\delta^* M^* = \delta^* i^* \widehat{M}^* = 0$ since $\delta^* i^* = 0$ by the exactness of the cohomology sequence of any pair (X,A).

The following elementary lemma about δ^* is useful for applications of Theorem 4.21.

Lemma 4.23 *Let L be a compact connected n–manifold M with boundary $\partial L = \Sigma$. The map $\delta^*\colon H^{n-1}(\partial L; G) \to H^n(L, \partial L; G)$ is surjective for any coefficient group G. If additionally, L is orientable and ∂L is connected, then δ^* is an isomorphism.*

Proof Any compact n–manifold with nonempty boundary has the homotopy type of a CW complex of dimension at most $n-1$ [35, page 170]. In particular, $H^n(L) = 0$ for any coefficient group G. Surjectivity of $\delta^*\colon H^{n-1}(\partial L) \to H^n(L, \partial L)$ now follows immediately from the exactness of the cohomology sequence of the pair $(L, \partial L)$. If L is orientable, then we may use Poincaré–Lefschetz duality [7, page 357] to identify $\ker \delta^*$ with $\ker i_*$, where $i_*\colon H_0(\partial L) \to H_0(L)$ is the map induced by the inclusion $i\colon \partial L \to L$. If ∂L is connected then i_* is an isomorphism and hence so is δ^*. □

Remark 4.24 If L is orientable but Σ is disconnected with $k \geq 2$ components, then $\delta^*\colon H^{n-1}(\Sigma) \to H^n(L, \Sigma)$ fails to be an isomorphism (since $H^n(L, \Sigma; G) = G$ while $H^{n-1}(\Sigma; G) \simeq H_0(\Sigma; G) = G^k$).

It remains to use Theorem 4.21 to study the obstructions to extending maps into $\mathbb{S}^1$ and SU(n). We begin with maps into $\mathbb{S}^1$.

Corollary 4.25 *Let L be a compact connected smooth n–manifold with boundary Σ and $i\colon \Sigma \to L$ denote the natural inclusion map. A map $M\colon \Sigma^{n-1} \to \mathbb{S}^1$ extends to a map $\widehat{M}\colon L^n \to \mathbb{S}^1$ if and only if*

$$M^*[d\theta] \in \mathrm{Im}(i^*\colon H^1(L, \mathbb{Z}) \to H^1(\Sigma, \mathbb{Z})).$$

Hence if M is homotopic to a constant map then M extends to L. Moreover, in this case there is an extension $\widehat{M}$ which is homotopic to a constant map from L, and the possible homotopy classes of different extensions of M are parameterized by $H^1(L, \Sigma; \mathbb{Z})$.

Conversely, if either

(i) *L is an orientable surface with connected boundary Σ, or*

(ii) *$H^1(L; \mathbb{Z}) = 0$,*

then M extends to L if and only if M is homotopic to a constant.

Proof Since $\mathbb{S}^1$ is 0–connected and its higher homotopy groups π_i for $i \geq 2$ all vanish then M extends if and only if the map $\delta^* M^*\colon H^1(\mathbb{S}^1;\mathbb{Z}) \to H^2(L,\Sigma;\mathbb{Z})$ is zero by Theorem 4.21. Since $H^1(\mathbb{S}^1;\mathbb{Z}) = \mathbb{Z}$ and is generated by $[d\theta]$, this is true if and only if $\delta^* M^*[d\theta] = 0$. But by the exactness of the long exact sequence in cohomology of the pair (L,Σ), $M^*[d\theta] \in \ker \delta^*$ if and only if $M^*[d\theta] \in \operatorname{Im} i^*$. Clearly, if $H^1(L,\mathbb{Z}) = (0)$ then $M^*[d\theta] \in \operatorname{Im} i^*$ if and only if $M^*[d\theta] = 0$, which by Lemma 3.17 is equivalent to M being homotopic to a constant. In case (i), it follows from Lemma 4.23 that $\delta^*\colon H^1(\Sigma;\mathbb{Z}) \to H^2(L,\Sigma;\mathbb{Z})$ is an isomorphism. Hence $\delta^* M^* = 0$ if and only if $M^*[d\theta] = 0$, as in case (ii). If M is homotopic to a constant then since the pair (L,Σ) satisfies the homotopy extension property for any space Y then there is an extension $\widehat{M}\colon L \to \mathbb{S}^1$ which is homotopic to a constant map. The statement about the different homotopy classes of extensions follows from the second part of Theorem 4.16 since the only nonzero obstruction space $H^i(L,\Sigma,\pi_i(\mathbb{S}^1))$ occurs when $i = 1$. □

Remark 4.26 It is interesting to check what Corollary 4.25 says in the nonorientable case. For example, suppose L is the Möbius strip with boundary $\Sigma = S^1$. Then $H^1(L,\mathbb{Z}) \simeq \mathbb{Z} \simeq H^1(\Sigma,\mathbb{Z})$ and i^* is the map $n \mapsto 2n$, so it has image $2\mathbb{Z} \subset H^1(\Sigma,\mathbb{Z})$. Thus a map $M\colon \Sigma \to \mathbb{S}^1$ extends to L if and only if $M^*[d\theta] \in 2\mathbb{Z}$. From Example 4.14, the image under i^* of the possible Maslov classes of $L \subset \mathbb{C}^2$ is the set $4\mathbb{Z}+2 \subset H^1(\Sigma,\mathbb{Z})$. On the other hand, the standard immersion of Σ in $\mathbb{C}$ has Maslov class $2 \in H^1(\Sigma,\mathbb{Z})$ (when calculated via $\det^2_{\mathbb{C}}$); this is also the Maslov class of any exact perturbation of this immersion, and of the corresponding "cylindrical thickening" f in $\mathbb{C}^2$ (see Example 4.5). One could use these facts as a basis for proving that the Prescribed Boundary Problem determined by this (Σ, L, f) is solvable. The solution is actually known explicitly [3].

Similarly, if we apply Theorem 4.21 with $Y = \mathrm{SU(n)}$ then we obtain the following extension of Corollary 4.19.

Corollary 4.27 *Let L be a compact connected smooth n–manifold with boundary Σ and $i\colon \Sigma \to L$ denote the natural inclusion map. Let $M\colon \Sigma \to SU(n)$ be a map. Then for $2 \leq n \leq 5$, M extends to a map of L if and only if*

$$M^*\Theta \in \operatorname{Im}(i^*\colon H^3(L,\mathbb{Z}) \to H^3(\Sigma,\mathbb{Z}))$$

where Θ is the generator of $H^3(SU(n),\mathbb{Z}) = \mathbb{Z}$.

Furthermore, if $n = 4$ and L is orientable with connected boundary Σ then M extends if and only if M is homotopic to a constant map.

Proof The proof of the first part is entirely analogous to the proof of the previous Corollary, the only difference being that SU(n) is 2–connected. If $n = 4$, L is orientable and Σ is connected, then by Lemma 4.23 $\delta^*\colon H^3(\Sigma;\mathbb{Z}) \to H^4(L;\Sigma,\mathbb{Z})$ is an isomorphism. Hence $\delta^* M^* = 0$ if and only if $M^*\Theta = 0$. But since once can prove that the map $M \mapsto M^*\Theta$ induces an isomorphism between the groups $[\Sigma^3, \mathrm{SU}(4)]$ and $H^3(\Sigma^3;\mathbb{Z})$, $M^*\Theta = 0$ if and only if M is homotopic to a constant map. □

If $n > 5$ the situation is not as simple since there are further obstructions to extending the map $M_{\mathrm{SU(n)}}$, eg a second obstruction $c^6_M \in H^6(L,\Sigma,\pi_5(SU(n))) = H^6(L,\Sigma,\mathbb{Z})$ can now be nonzero.

We now summarize the main results of this section in terms of the set-up introduced in the context of Lemma 4.11.

Theorem 4.28 *Let (Σ^{n-1}, L^n, f) be initial data for the Prescribed Boundary Problem and assume $2 \le n \le 5$. Fix a reference Maslov-zero Lagrangian monomorphism F_0 (this exists by Lemma 4.13); let $M := M_{df^{\mathbb{C}},F_0} : \mathcal{O}p\,\Sigma \to GL(n,\mathbb{C})$ be the map corresponding to $df^{\mathbb{C}}$ via F_0. Using the retraction $GL(n,\mathbb{C}) \to U(n)$, M is homotopic to a map which takes values in U(n); we will continue to denote this map M. Let $M_{\mathbb{S}^1}$ and $M_{SU(n)}$ denote the associated maps into $\mathbb{S}^1$ and SU(n) respectively. Then there exists a compatible trivialization if and only if*

(i) $M^*_{\mathbb{S}^1}[d\theta] \in \mathrm{Im}(i^*\colon H^1(L,\mathbb{Z}) \to H^1(\Sigma,\mathbb{Z}))$, *and*

(ii) $M^*_{SU(n)}\Theta \in \mathrm{Im}(i^*\colon H^3(L,\mathbb{Z}) \to H^3(\Sigma,\mathbb{Z}))$,

where Θ is the generator of $H^3(SU(n),\mathbb{Z})$, i is the natural immersion $\Sigma \to L$, and the pullback operation is performed using the restrictions of $M_{\mathbb{S}^1}$ and $M_{SU(n)}$ to Σ.

Furthermore, there is a natural identification of $M^*_{\mathbb{S}^1}[d\theta]$ with the Maslov map μ_f of the initial data.

Proof Notice that by the homotopy extension property (setting $Y := \mathrm{GL(n,\mathbb{C})}$ and thinking of $\mathrm{U(n)} \subset \mathrm{GL(n,\mathbb{C})}$) the map into $\mathrm{GL(n,\mathbb{C})}$ is extensible if and only if the map into U(n) is. The idea of the proof is now of course to apply the previous obstruction theory results to the maps $M_{\mathbb{S}^1}$ and $M_{\mathrm{SU(n)}}$. Notice however that, as required in Lemma 4.11, we want to extend not only the values of these maps on Σ but also on a neighbourhood of Σ. Furthermore we want the extension to be smooth. To achieve this, let us choose a (closed, sufficiently small) tubular neighbourhood $N \simeq \Sigma \times [0,\epsilon] \subseteq \mathcal{O}p\,\Sigma$; let Σ_ϵ denote the "inner boundary" $\Sigma \times \{\epsilon\} \subset N$. We can apply the obstruction theory results to $M_{\mathbb{S}^1}$ and $M_{\mathrm{SU(n)}}$ restricted to Σ_ϵ, obtaining obstructions $M^*_{\mathbb{S}^1}[d\theta]$ and $M^*_{\mathrm{SU(n)}}\Theta$.

Note To be precise, these cocycles live in $H^1(\Sigma_\epsilon, \mathbb{Z})$ and $H^3(\Sigma_\epsilon, \mathbb{Z})$ but we can identify these with $H^1(\Sigma, \mathbb{Z})$ and $H^3(\Sigma, \mathbb{Z})$, respectively.

Conditions (i) and (ii) now follow from Corollary 4.25 and Corollary 4.27. When the conditions are satisfied we obtain continuous extensions defined on the complement $L \setminus N$ of the tubular neighbourhood; together with the given values of $M_{\mathbb{S}^1}$ and $M_{\mathrm{SU(n)}}$ on N, we now have continuous extensions defined on the whole of L. Standard perturbation results show that we can assume that these extensions are smooth.

It remains only to show that $M^*_{\mathbb{S}^1}[d\theta]$ can be identified with the Maslov class of the Lagrangian map $f\colon \mathcal{O}p\,\Sigma \to \mathbb{C}^n$. By considering a sufficiently small neighbourhood of Σ, we can assume that $\mathcal{O}p\,\Sigma$ is contractible onto N. The homotopy invariance of μ_f then allows us to restrict f to N. Recall from Lemma 4.13 that, since F_0 is Maslov-zero, μ_f is the homotopy class of $\det_{\mathbb{C}} M : N \to \mathbb{S}^1$. The claim then follows from the fact that $\det_{\mathbb{C}} M = M_{\mathbb{S}^1}$ and the identifications $[N, \mathbb{S}^1] \simeq [\Sigma, \mathbb{S}^1] \simeq H^1(\Sigma, \mathbb{Z})$ (where the last isomorphism is via pullback of $[d\theta]$; see Lemma 3.17). □

5 The Prescribed Boundary Problem in low dimensions

Corollary 4.10 showed that solving the Prescribed Boundary Problem is equivalent to finding a compatible trivialization, which by Lemma 4.11 is equivalent to solving a certain extension problem. Theorem 4.28 shows that in low dimensions the obstructions to solving this extension problem can be written down explicitly. We will now examine these cases one by one.

Note To simplify the exposition, throughout this section we will assume that $\mathcal{O}p\,\Sigma$ is a tubular neighbourhood of $\Sigma \subset L$ so that $\mathcal{O}p\,\Sigma \simeq \Sigma \times [0,\epsilon)$. As in the proof of Theorem 4.28, we will use this to identify $\mu_f \in H^1(\mathcal{O}p\,\Sigma, \mathbb{Z})$ with the corresponding element in $H^1(\Sigma, \mathbb{Z})$.

5.1 The Prescribed Boundary Problem in $\mathbb{R}^4$

Suppose we are given an immersion $i\colon \Sigma \to \mathbb{R}^4$ where Σ is a finite union of oriented circles. Our first task is to check whether Σ and i satisfy Assumptions A–C.

It is clear that Σ always admits orientable fillings L. If Σ is connected then $\Sigma = \partial L$ for any orientable surface L with connected boundary; if Σ is not connected, the claim now follows via oriented connect sums. Furthermore, (unlike the case of closed surfaces) any orientable surface with nonempty boundary is parallelizable. (A proof of this fact can be seen as follows: any n–manifold with nonempty boundary has the homotopy type of a CW complex of dimension at most $n-1$. Since any orientable surface with boundary can be embedded in $\mathbb{R}^3$ it is stably parallelizable. But any stably trivial bundle of rank $k > d$ is actually trivial over any space homotopy equivalent to a CW complex of dimension at most d [35, Corollary IX.1.5]. Hence the tangent bundle of L is trivial.) Thus assumptions A and B are both satisfied.

For dimensional reasons i is always isotropic. In general however it will not satisfy the exactness condition C, so we must make this additional assumption on i. Example 4.1 shows that if Σ is connected then exactness of Σ is in fact a necessary condition for the existence of a smooth Lagrangian extension $f\colon L \to \mathbb{R}^4$.

Before applying the obstruction theory results to this set-up, we begin by making the following simple observation.

Lemma 5.1 *Let γ be the boundary of an immersed oriented Lagrangian surface $f\colon L \to \mathbb{R}^4$ and suppose that γ is connected. Then the Maslov index of γ, $\mu_f([\gamma])$, must be zero.*

Proof This is immediate since $\mu_f([\gamma])$ depends only on the homology class of the loop γ. □

Remark 5.2 It follows from Lemma 5.1 that any connected separating curve in a closed oriented Lagrangian surface has zero Maslov index.

Lemma 5.1 shows that when Σ is connected a smooth oriented solution to the Prescribed Boundary Problem exists only if the initial data (γ, L^2, f) is Maslov-zero. One can give concrete examples of exact initial data which are not Maslov zero as follows.

Example 5.3 Given any pair of relatively prime positive integers p, q define a curve $\gamma \subset \mathbb{S}^3$ by

$$\gamma_{p,q}(\theta) := \frac{1}{\sqrt{p+q}}\left(\sqrt{q}e^{ip\theta}, i\sqrt{p}e^{-iq\theta}\right), \quad \theta \in [0, 2\pi].$$

Each curve $\gamma_{p,q}$ is a Legendrian curve in $\mathbb{S}^3$, and hence is exact. Since $\gamma_{p,q}$ is Legendrian the cone over $\gamma_{p,q}$, denoted $C_{p,q}$ is Lagrangian. It is not difficult to show that the Maslov index of the curve $\gamma_{p,q}$ in the Lagrangian cone $C_{p,q}$ is $p-q$ [45, Remark 4.4]. (The cones $C_{p,q}$ were studied by Schoen–Wolfson and are important because up to unitary transformations they account for all the Hamiltonian stationary cones in $\mathbb{C}^2$.) Now take γ to be $\gamma_{p,q}$ and take the Lagrangian thickening of $\gamma_{p,q}$ to be a truncation of the cone over γ, eg we could take $f\colon \gamma \times [0,1) \to \mathbb{R}^4$ to be

$f(\theta,t) = (1+t)\gamma_{p,q}(\theta))$. Then for any oriented filling L the exact initial data (Σ, L^2, f) has nonzero Maslov class provided $p \neq q$, and hence there is no smooth solution to the Lagrangian Prescribed Boundary Problem with such initial data.

The main result of this section, Theorem 5.4, provides a converse to Lemma 5.1. The reader may also like to compare Lemma 5.1 with the following result of Schoen–Wolfson: given any exact smooth Jordan curve γ in $\mathbb{R}^4$ and $m \in \mathbb{Z}$, there exists a *piecewise* smooth Lagrangian disk D bounding γ so that γ has Maslov index equal to m [45, Proposition 4.1].

We now state the main result of this section.

Theorem 5.4 *Let L be a connected oriented surface with boundary Σ and suppose (Σ, L, f) is exact initial data in the sense of Definition 4.2.*

(1) *If Σ is connected then the Prescribed Boundary Problem is solvable if and only if f has zero Maslov class, ie $\mu_f = 0$.*

(2) *In general, the Prescribed Boundary Problem is solvable if and only if*

$$\mu_f \in \operatorname{Im}(i^*\colon H^1(L,\mathbb{Z}) \to H^1(\Sigma,\mathbb{Z})).$$

Furthermore, if $\mu_f = 0$ then one can always find a solution with zero Maslov class.

Proof Part (2) is a direct consequence of Theorem 4.28; in dimension 2 the second condition there is vacuous, ie the matrix-valued map $M_{SU(2)}$ corresponding to $df^{\mathbb{C}}$ is always extensible (see also Corollary 4.19). The existence of a Maslov-zero solution follows from the fact that if μ_f is zero then the map $M_{\mathbb{S}^1}$ is homotopic to a constant map by Lemma 3.17. Corollary 4.25 then shows that this map admits a homotopically constant extension.

Part (1) now follows from case (i) of Corollary 4.25. $\square$

Example 5.5 Let L be any closed n–manifold with $TL^{\mathbb{C}}$ trivial. By Proposition 3.19, given any $\mu \in H^1(L,\mathbb{Z})$, there exists an exact Lagrangian immersion $f\colon L \to \mathbb{C}^n$ with $\mu_f = \mu$. Hence if $H^1(L,\mathbb{Z}) \neq 0$, then we have Maslov nonzero exact Lagrangian immersions of L.

Let $f\colon L \to \mathbb{C}^n$ be any exact Lagrangian immersion with nonzero Maslov class. The immersion $C(f) : L \times [0,1] \subset \mathbb{C}^n \times \mathbb{C}$ defined by $C(f)(p,t) = (f(p),t)$ defines an exact Lagrangian “generalized cylinder” in $\mathbb{C}^{n+1}$ and gives a solution to the Prescribed Boundary Problem determined by the data $C(f) : L \times [0,\epsilon) \cup L \times (1-\epsilon, 1] \to \mathbb{C}^{n+1}$. It is clear that this initial data has $\mu_{C(f)} \neq 0$, and thus shows that part (1) of Theorem 5.4 is false if we do not assume that Σ is connected.

An immediate corollary of Theorem 5.4 is the following result:

Corollary 5.6 *Let (Σ, L^2, f) be exact SL initial data. Then the Prescribed Boundary Problem admits a Maslov-zero solution.*

In particular, let Σ be an exact real-analytic curve of $\mathbb{R}^4$ (eg any real-analytic Legendrian curve in $\mathbb{S}^3$). Choose any oriented surface L bounding Σ. Then for any $\theta \in [0, 2\pi)$ there exists an exact Lagrangian immersion $f_\theta\colon L \to \mathbb{R}^4$ with zero Maslov class which coincides in a neighbourhood of Σ with the θ-SL extension of Σ.

Proof If f is SL it is Maslov-zero, so we can apply the last statement of Theorem 5.4. The statement regarding real analytic curves is based on Example 4.4. □

For applications in Section 6 it will be useful to give a second proof of the previous result.

Second proof of Corollary 5.6 According to Lemma 4.13, in Theorem 4.28 we can choose the reference trivialization to be SL. If we start off with SL initial data, the map $M_{\mathbb{S}^1}$ in Theorem 4.28 will have constant value 1, so we can extend it to a map on L with constant value 1. Corollary 4.19 shows that there is no obstruction to extending $M_{\mathrm{SU}(2)}$. We thus obtain a SL monomorphism $\widehat{F}$ extending df; in particular $\widehat{F}$ is Maslov-zero. We can homotope $\widehat{F}$ using the h–principle to prove the existence of an exact Maslov-zero Lagrangian extension.

Remark 5.7 More generally, the idea in the second proof of Corollary 5.6 can also be used to prove part of Theorem 5.4 without relying on the full set of results presented in Section 4.4. Assume the initial data f is (exact and) Maslov-zero. Choose a reference SL trivialization ϕ_0 of $TL^{\mathbb{C}}$. Again using Lemma 4.13, it is possible to homotope $df^{\mathbb{C}}$ to a SL monomorphism of $\mathcal{O}p\,\Sigma$ which, via matrices and Corollary 4.19, admits an extension to L. By the homotopy property of extensions, this means that $df^{\mathbb{C}}$ is extensible to a Lagrangian monomorphism of L. Applying the h–principle to this monomorphism allows us to conclude that the Prescribed Boundary Problem admits a Maslov-zero solution for such initial data. Clearly, this works also for $n = 3$.

5.2 The Prescribed Boundary Problem in $\mathbb{R}^6$

Let Σ be any compact oriented surface not necessarily connected. If Σ is connected with genus g then it admits one obvious "topological filling" L^3, namely, the standard genus g handlebody. One can then construct more complicated fillings of Σ using the standard topological surgeries. Using oriented connect sums one can show that

Σ admits (connected) fillings even if it is disconnected. Recall also that any compact orientable 3–manifold L is parallelizable. Finally, Example 5.10 below shows that any compact oriented surface Σ admits exact immersions i into $\mathbb{R}^6$. Hence Assumptions A–C can be satisfied for any compact orientable surface Σ.

We now apply our obstruction theory results to this case; once again, the second obstruction identified in Theorem 4.28 vanishes for dimensional reasons so the results (and proofs) are largely analogous to those obtained in dimension 2.

Theorem 5.8 *Let L be a connected oriented 3–manifold with boundary Σ and let (Σ, L, f) be exact initial data in the sense of Definition 4.2. Then the Prescribed Boundary Problem is solvable if and only if $\mu_f \in \mathrm{Im}(i^*\colon H^1(L,\mathbb{Z}) \to H^1(\Sigma,\mathbb{Z}))$.*

Furthermore, if $\mu_f = 0$ then one can always find a solution with zero Maslov class.

As in the previous subsection, this result has an immediate corollary:

Corollary 5.9 *Let (Σ, L^3, f) be exact SL initial data. Then the Prescribed Boundary Problem admits a Maslov-zero solution $\widetilde{f}\colon L \to \mathbb{R}^6$.*

In particular, let Σ be a real-analytic exact surface in $\mathbb{R}^6$ (eg any real-analytic Legendrian submanifold of $\mathbb{S}^5$). Choose any oriented 3–manifold L bounding Σ. Then for any $\theta \in [0, 2\pi)$ there exists an exact Lagrangian immersion $f_\theta\colon L \to \mathbb{R}^6$ with zero Maslov class which coincides in a neighbourhood of Σ with the θ-SL extension of Σ.

Example 5.10 Since the standard contact structure on $\mathbb{S}^5$ with one point removed is contactomorphic to the standard contact structure on $\mathbb{R}^5$ [15], a closed surface Σ admits Legendrian immersions into $\mathbb{S}^5$ if and only if it admits a Legendrian immersion into $\mathbb{R}^5$ with its standard contact structure. Hence by previous remarks Σ admits Legendrian immersions into $\mathbb{S}^5$ if and only if it admits exact Lagrangian immersions into $\mathbb{R}^4$. But we saw in Section 3.5.1 that any compact orientable surface Σ admits such immersions. Thus, it is possible to find an exact immersion of Σ in $\mathbb{R}^6$ for any orientable surface Σ.

Another special case for Theorem 5.8 occurs when $H^1(L,\mathbb{Z}) = 0$: in this case the condition on μ_f simplifies to $\mu_f = 0$. However, for $n \geq 3$ there are constraints on the topological complexity of any L which bounds Σ.

Lemma 5.11 *Let L be a smooth oriented $(2i+1)$–dimensional manifold with connected boundary Σ. Then $\dim H^i(L,\mathbb{R}) \geq \frac{1}{2} b^i(\Sigma)$.*

The proof of Lemma 5.11 follows from Poincaré–Lefschetz duality for compact orientable manifolds with boundary; see eg Bredon [7, page 360]. In particular, when L is 3–dimensional the special case $H^1(L,\mathbb{Z}) = 0$ can occur only if $\Sigma = S^2$.

5.3 The prescribed boundary problem in $\mathbb{R}^8$ and $\mathbb{R}^{10}$

We saw in Section 4, that when n equals 4 or 5, not every compact oriented n–manifold Σ admits an oriented filling L. Furthermore, even if Σ bounds it is not clear that the bounding manifold L will have $TL^{\mathbb{C}}$ trivial or that exact immersions of Σ in $\mathbb{R}^{2n}$ exist. Rather than try to analyze further conditions on Σ under which Assumptions A–C can be verified we shall simply assume we are given exact initial data (Σ, L, f). Then we can apply Theorem 4.28 to obtain the following results.

Theorem 5.12 *Let L be a connected oriented n–manifold with boundary Σ and let (Σ, L, f) be exact initial data in the sense of Definition 4.2.*

(1) *If $n = 4$ and Σ is connected then the Prescribed Boundary Problem is solvable if and only if the following two conditions hold:*
$$\begin{aligned} \mu_f &\in \mathrm{Im}(i^*\colon H^1(L,\mathbb{Z}) \to H^1(\Sigma,\mathbb{Z})); \\ M^*_{SU}\Theta &= 0. \end{aligned}$$

(2) *In general, if $n = 4$ or $n = 5$ then the Prescribed Boundary Problem is solvable if and only if the following two conditions hold:*
$$\begin{aligned} \mu_f &\in \mathrm{Im}(i^*\colon H^1(L,\mathbb{Z}) \to H^1(\Sigma,\mathbb{Z})); \\ M^*_{SU}\Theta &\in \mathrm{Im}(i^*\colon H^3(L,\mathbb{Z}) \to H^3(\Sigma,\mathbb{Z})). \end{aligned}$$

If the Prescribed Boundary Problem is solvable and moreover $\mu_f = 0$, then one can always find a solution with zero Maslov class.

Proof The proof is similar to that of Theorem 5.4; when Σ is connected and $n = 4$ we can prove the stronger condition using Corollary 4.27. □

Once again, it is clearly useful to know if Σ admits simply connected fillings. Using Surgery Theory one can prove the following simplification result for the fundamental group.

Theorem 5.13 [40] *Let X be any oriented compact manifold (with or without boundary) of dimension n with $n \geq 4$. Then there exists a simply connected oriented manifold X' (obtained by a finite sequence of surgeries of type $(1, n-2)$) with $\partial X = \partial X'$. In particular, if L is an oriented n–manifold which bounds Σ and $n \geq 4$ then there is a simply connected manifold L' which bounds Σ.*

The basic point of the proof of Theorem 5.13 is that any nontrivial loop in $\pi_1(X)$ can be represented by an embedded S^1 which has a trivial normal bundle. Then by performing surgery on these embedded loops one can systematically kill the fundamental group of X. In order to try to simplify the higher-dimensional homotopy or homology groups one needs to find higher-dimensional embedded spheres with trivial normal bundles representing elements of these groups. Hence with further assumptions about the bounding manifold L one can use Surgery Theory to simplify its higher homotopy groups; see Kosinski [35] for results in this direction.

6 Lagrangian desingularizations

We now apply the results of Section 5 to answer the three questions posed in the Introduction to this paper.

6.1 Lagrangian submanifolds with exact isolated singularities in $\mathbb{C}^n$

Definition 6.1 Let X be a connected topological space with a finite number of points $\{x_1, \dots, x_m\}$ such that the (not necessarily connected) space

$$X' := X \setminus \{x_1, \dots, x_m\}$$

is an oriented smooth n–manifold. We say that X is an *oriented manifold with isolated singularities* if, in addition, each x_i has a connected open neighbourhood U_i such that $U_i \setminus \{x_i\} \simeq (0,1) \times \Sigma_i$, for some compact (not necessarily connected) $(n-1)$-manifold Σ_i, and so that $U_i \cap U_j = \varnothing$, if $i \neq j$.

Let X be an oriented n–manifold with isolated singularities. We say that $f\colon X \to \mathbb{R}^{2n}$ is an *oriented Lagrangian submanifold with isolated singularities* if f is continuous and $f|_{X'}$ is a Lagrangian immersion.

We say that X has *exact singularities* if each restriction $f|_{U_i}$ is exact. This "local exactness" condition is satisfied automatically near any smooth point of X. We say that X is exact or that it has zero Maslov class if this is true for X'.

Example 6.2 Let X be any oriented Lagrangian cone in $\mathbb{R}^{2n}$ with an isolated singularity at the origin, and let Σ denote the intersection of X with $\mathbb{S}^{2n-1}$. Σ is an oriented Legendrian submanifold of $\mathbb{S}^{2n-1}$ which is not a totally geodesic sphere. The origin is then an isolated singular point of X which has a neighbourhood in X whose boundary is diffeomorphic to Σ; X' has the same number of connected components as Σ.

We now give two ways to see that any Lagrangian cone X is exact. Since Σ is Legendrian and the standard contact form α on $\mathbb{S}^{2n-1}$ is just the restriction of λ to $\mathbb{S}^{2n-1}$ it follows immediately that Σ and hence X is exact. Alternatively, given any closed curve $\gamma \subset X$ and $r \in \mathbb{R}^+$, we can "slide" γ to the curve $r \cdot \gamma$. As $r \to 0$ the length of $r \cdot \gamma$ goes to 0. Hence, by Stokes' Theorem,

$$\int_\gamma \lambda = \int_{r\cdot\gamma} \lambda \to 0 \quad \text{as } r \to 0.$$

Since $\int_\gamma \lambda$ is however independent of r, it must be zero.

Example 6.3 Let X be a smooth oriented n–manifold and $f\colon X \to \mathbb{R}^{2n}$ be a continuous map which is a Lagrangian immersion except at a finite number of points $\{x_1, \dots, x_m\}$. Then f defines a Lagrangian submanifold with isolated singularities. In this example the singularities arise from f rather than from X, so each Σ_i is diffeomorphic to $\mathbb{S}^{n-1}$.

It is important to find conditions on f ensuring that its singularities are exact. This can be done fairly easily, as shown by the following example.

Example 6.4 Suppose $f\colon X \to \mathbb{C}^n$ is a Lagrangian submanifold with isolated singularities $\{x_1, \dots, x_m\}$. Let g denote the pullback metric on $U_i \setminus \{x_i\} \simeq (0,1) \times \Sigma_i$. Let r denote the variable on $(0,1)$ and p any point on Σ_i.

Suppose that g is a "$O(r^\beta)$–approximation" of a metric $g' = dr^2 + r^{2\alpha} g_i$, for some $\alpha, \beta > 0$ and some smooth metric g_i on the cross-section Σ_i. More precisely, we ask that $g = g' + h$, for some symmetric tensor $h = h(r,p)$ whose norm (calculated with respect to g') is bounded by some r^β. Then the singularities of f are locally exact. The proof is similar to that given for Lagrangian cones in Example 6.2, but we now use closed curves $\gamma \subset \Sigma_i$. The condition on the metric ensures that the length of the curves $r \cdot \gamma$ tends to zero as $r \to 0$.

In particular, if X is a SL n–fold with isolated conical singularities in the sense of Joyce [31, Definition 3.6] then X is an oriented Lagrangian submanifold with exact isolated singularities in the sense of Definition 6.1.

Definition 6.5 Let $f\colon X \to \mathbb{R}^{2n}$ be a Lagrangian submanifold with isolated singularities $x_1, \dots, x_m$, and mutually disjoint connected neighbourhoods $U_1, \dots, U_m$ of these singular points as in Definition 6.1. Let Y be a smooth oriented n–manifold (not necessarily connected) and let $V_1, \dots, V_m$ be mutually disjoint (not necessarily connected) open subsets of Y so that $Y \setminus \bigcup_{i=1}^m V_i$ is diffeomorphic to $X \setminus \bigcup_{j=1}^m U_j$. We call a Lagrangian immersion $\widehat{f}\colon Y \to \mathbb{R}^{2n}$ an *oriented Lagrangian desingularization of* X if $\widehat{f} = f$ on the sets $Y \setminus \bigcup_{i=1}^m V_i = X \setminus \bigcup_{j=1}^m U_j$.

Remark 6.6 As a very simple example we can consider the connected 2–dimensional singular space X with one singular point x_1, formed by taking a 2–sphere and identifying two distinct points. Then there is a neighbourhood U_1 of x_1 that has two boundary components each diffeomorphic to S^1. If we take V_1 to be the disjoint union of two discs bounded by these two circles, then the resulting manifold Y will be diffeomorphic to S^2. If instead we take V_1 to be a cylinder bounding these two circles, the resulting manifold will be a 2–torus.

More generally, even though we start with a connected space X, when we remove the singular points the resulting nonsingular manifold X' will in general be disconnected. The manifold Y may take different components of X' and reconnect them into fewer components, but not necessarily into one component only. In particular, different components of the deleted neighbourhood of one singular point $U_i \setminus \{x_i\}$ can end up in different connected components of Y.

Given a Lagrangian submanifold X with isolated singularities, it is natural to ask if it admits any Lagrangian desingularization Y. Furthermore, if desingularizations do exist then we would like to know what topology Y can have, and whether the desingularization process preserves extra properties such as exactness or zero Maslov class. In the oriented case we can now prove the following results.

Theorem 6.7 *For $n = 2$ or $n = 3$, let $f\colon X \to \mathbb{C}^n$ be an oriented Maslov-zero Lagrangian submanifold with exact isolated singularities $\{x_1, \dots, x_m\}$ in the sense of Definition 6.1. Then X admits oriented Lagrangian desingularizations in the sense of Definition 6.5 with arbitrarily complicated topology.*

Furthermore, let U_i be a connected neighbourhood of x_i with smooth (not necessarily connected) oriented boundary Σ_i. If either

(i) *each Σ_i is connected, or*

(ii) *X is SL,*

then X also admits Maslov-zero oriented desingularizations with arbitrarily complicated topology. Finally, if each Σ_i is connected and X is also exact, then all these desingularizations are exact.

Proof Recall from Definition 6.1 that $U_i \setminus \{x_i\} \simeq (0,1) \times \Sigma_i$. Choose any oriented filling L_i of Σ_i. By assumption X is locally exact, so we can apply the results of Section 5 to f restricted to $\mathcal{O}p\,\Sigma_i \simeq \Sigma_i \times (1-\epsilon, 1]$; this proves that the Prescribed Boundary Problem defined by the initial data (Σ_i, L_i, f) admits an exact Maslov-zero solution $\widehat{f_i}$. Iterating this construction near each singular point produces an oriented Lagrangian desingularization $\widehat{f}\colon Y \to \mathbb{C}^n$ of X in the sense of Definition 6.1. Y is smooth because we chose, as initial data, an open neighbourhood of Σ_i rather than Σ_i itself. The topology of Y is "arbitrarily complicated" in the sense already seen in Section 5: we are allowed to perform any number of topological surgeries on each L_i so as to increase its topological complexity.

Suppose now that each Σ_i is connected. Let γ be a closed curve in Y. Since each Σ_i is connected, we can homologically split γ into the sum of r closed curves γ_k such that each γ_k is completely contained either in $X \setminus \bigcup_{j=1}^m U_j$ or in some L_k. Then, since on each of these sets $\widehat{f}$ is Maslov-zero,

$$\mu_{\widehat{f}}([\gamma]) = \sum_{k=1}^{r} \mu_{\widehat{f}}([\gamma_k]) = 0.$$

Hence Y is Maslov-zero. The same method proves the corresponding statement for exact desingularizations.

Suppose instead that X is SL. The second proof of Corollary 5.6 (which works also for $n=3$) shows that we can extend the initial data df on each $\mathcal{O}p\,\Sigma_i$ to a SL monomorphism $\widehat{F}_i$ defined on L_i. The corresponding Lagrangian immersion of L_i, obtained via the h–principle, is clearly Maslov-zero but has the additional virtue of being SL on $\mathcal{O}p\,\Sigma_i$. Let us now choose any closed curve γ in Y. Let $G(\gamma)$ denote the corresponding curve of tangent planes, defined as $G(\gamma)(t) := T_{\gamma(t)}Y$. This is a closed curve in the Lagrangian Grassmannian bundle $\mathrm{Gr}_{\mathrm{lag}}(\mathbb{R}^{2n})$. According to Lemma 3.17, to show that Y has zero Maslov class it is equivalent to show that for any such γ the corresponding curve $\det_{\mathbb{C}} \circ G(\gamma)$ is homotopically trivial in $\mathbb{S}^1$.

If γ is completely contained in $X \setminus \bigcup_{j=1}^m U_j$, then $G(\gamma)$ is by hypothesis a curve of SL planes so $\det_{\mathbb{C}} \circ G(\gamma) \equiv 1$; in particular, it is homotopically trivial. The same is true if γ is completely contained in some L_i. Finally, assume γ enters some L_i at a point $p_i \in \Sigma_i$; it is then forced to exit at a point $q_i \in \Sigma_i$. Consider the portion of γ contained in L_i. The h–principle gives us a homotopy between the distribution of SL planes determined by the SL monomorphism $\widehat{F}_i$ and the tangent planes of L_i. Restricting this homotopy to the corresponding portion of $G(\gamma)$, shows that our curve can be homotoped rel $\{p_i, q_i\}$ to a curve of SL planes. We can repeat this procedure for each L_i; the remaining portion of $G(\gamma)$ is already special Lagrangian. This proves that also in this case $\det_{\mathbb{C}} \circ G(\gamma)$ is homotopically trivial. □

Remark 6.8 Our desingularization method starts with a Maslov-zero X and replaces a neighbourhood of each singular point with a Maslov-zero L_i. We should not expect *any* such desingularization to be Maslov-zero. After all, this procedure is purely local and any Lagrangian submanifold is locally Maslov-zero, though not necessarily globally so. The same is true for the exactness condition.

Corollary 6.9 *Let X be a SL cone in $\mathbb{C}^3$. Then X admits connected oriented exact Maslov-zero desingularizations of arbitrarily complicated topology.*

Proof Recall from Example 6.2 or Example 6.4 that X is exact. Let Σ denote the (not necessarily connected) link of X, ie the intersection of X with the sphere $\mathbb{S}^5 \subset \mathbb{C}^3$. Choose any connected filling L of Σ; then the methods of Theorem 6.7 prove the existence of a Maslov-zero Lagrangian desingularization Y. Since any curve in Y is homotopic to a curve in L and L is exact by construction, then Y is exact even if Σ is not connected. □

Example 6.10 The result of Corollary 6.9 also holds for $n = 2$, but in this case it is well-known that SL desingularizations can be found explicitly as follows. Any 2–dimensional minimal cone (and in particular any SL cone) is the union of 2–planes. The "hyperkähler rotation" trick shows that a submanifold of $\mathbb{R}^4$ is SL if and only if it is a complex submanifold with respect to one particular (nonstandard) complex structure on $\mathbb{R}^4$. Accordingly, the i–th plane π_i can be described by some complex linear equation $a_{i1}z_1 + a_{i2}z_2 = 0$, so our cone will be given by the equation $\prod_{i=1}^n (a_{i1}z_1 + a_{i2}z_2) = 0$. For any nonzero $\epsilon \in \mathbb{C}$ the set $\prod_{i=1}^n (a_{i1}z_1 + a_{i2}z_2) = \epsilon$ describes a smooth complex surface in $\mathbb{R}^4$. Notice that these surfaces do not coincide with the original cone outside a compact set, but do become asymptotic to the cone at infinity. In this sense, these surfaces are SL desingularizations of the original SL cone.

6.2 Desingularization of Lagrangian singularities in almost Calabi–Yau manifolds

The goal of this section is to extend the results of Section 6.1 to more general ambient manifolds. Let (M^{2n}, ω) be a smooth symplectic manifold. By Darboux's Theorem M is locally symplectomorphic to $\mathbb{R}^{2n}$; notice that Definition 6.1 relies on purely local properties of f and X, so it is simple to extend this definition so that it includes the notion of Lagrangian submanifolds $f\colon X \to M$ with isolated singularities. Unless

M is *exact*, ie $\omega = d\lambda$ for some 1–form λ, it is not possible to define a global exactness condition for X (unless X happens to be contained inside one Darboux chart). However, it still makes sense to discuss whether X is locally exact near each of its singular points.

The generalization of the Maslov class to Lagrangian submanifolds of general symplectic manifolds is more complicated but is well-known. For completeness we describe this generalization briefly and indicate the simplifications which occur when the symplectic manifold is an *almost Calabi–Yau manifold*.

Let L^n be an oriented Lagrangian submanifold of (M,ω). As in Section 3 denote by $\mathrm{Gr}^+_{\mathrm{lag}}(M)$ the oriented Lagrangian Grassmannian bundle of M. The main issue is that in general $\mathrm{Gr}^+_{\mathrm{lag}}(M)$ is now a nontrivial $\mathrm{U(n)}/\mathrm{SO(n)}$ fibre bundle over M.

Let us fix a compatible almost complex structure J on M and let g be the metric defined by ω and J. Let $\widehat{\gamma}\colon (D^2, \partial D) \to (M, L)$ be a disk in M with boundary $\gamma := \widehat{\gamma}(\partial D) \subset L$. Since D is contractible there exists a complex volume form Ω defined on a neighbourhood of D in M. Any choice of Ω defines an evaluation map

$$\Omega\colon \mathrm{Gr}^+_{\mathrm{lag}}(M)|_D \to \mathbb{C}^*,$$

defined as follows. Let $\{e_i\}_{i=1}^n$ be a g–orthonormal basis for T_pL. Since T_pL is Lagrangian $\{e_i\}_{i=1}^n$ is a basis for the complex vector space (T_pM, J). In particular, $\{e_i\}_{i=1}^n$ is linearly independent over $\mathbb{C}$ and hence defines an element $e_1 \wedge \ldots \wedge e_n \neq 0$ in $\Lambda^{n,0}(T_pM, J)$. Hence we obtain a map from T_pL to $\mathbb{C}^*$ by evaluating Ω on this nonzero n–vector. It is easy to check that this map does not depend on the choice of the orthonormal basis of T_pL.

The Gauss map associates to γ a loop $G(\gamma)$ of Lagrangian tangent spaces defined by $G(\gamma)(t) := T|_{\gamma(t)}L$; we can thus define the *Maslov index* $\mu_L(\widehat{\gamma})$ of $\widehat{\gamma}$ to be the degree of the map

$$\frac{\Omega}{|\Omega|}\colon G(\gamma) \subset \mathrm{Gr}^+_{\mathrm{lag}}(M) \to \mathbb{S}^1.$$

This number is independent of the choice of J and Ω: this follows from the contractibility of D and of the space of compatible almost complex structures. For the same reason it depends only on the relative homotopy class of $\widehat{\gamma}$; we have thus defined a map on the relative homotopy group

$$\mu_L\colon \pi_2(M, L) \to \mathbb{Z}.$$

Let us now assume that (M, ω, g, J) is a Kähler manifold with trivial canonical bundle K_M and fix a global never-vanishing holomorphic section Ω of K_M. In the special Lagrangian literature the data $(M, \omega, g, J, \Omega)$ often goes under the name *almost*

Calabi–Yau manifold; the manifold is *Calabi–Yau* if Ω is covariantly constant and satisfies a normalization condition. Almost Calabi–Yau manifolds provide a particularly convenient framework for discussing Maslov indices, because the disk D is no longer needed to guarantee the existence of the complex volume form in a neighbourhood of a loop $\gamma \subset L$. Hence the Maslov index can be thought of as a map

$$\mu_L\colon \pi_1(L) \to \mathbb{Z}.$$

Alternatively, we could proceed as in Section 3.4 using $\Omega/|\Omega|$ instead of $\det_{\mathbb{C}}$ to define a *Maslov class* $\mu_L \in H^1(L;\mathbb{Z})$. Once again, homotopy considerations show that the Maslov class is locally independent of the particular Ω, J used in this construction. In particular, suppose we are given a Lagrangian submanifold of M with isolated singularities. Then, in a Darboux neighbourhood of a singularity, it is Maslov-zero according to the definition just given if and only if it is Maslov-zero with respect to $\det_{\mathbb{C}}$, ie according to the definition of Section 3.4.

We say that a Lagrangian submanifold L of an almost Calabi–Yau manifold is *special Lagrangian* if

$$\operatorname{Im}\Omega|_L = 0 \quad \text{and} \quad \operatorname{Re}\Omega|_L > 0.$$

Equivalently, notice that the data (ω, g, J, Ω) defines a $\mathrm{SU}(n)$ principal fibre bundle over M, given by the set of all (g, J)–unitary frames $\{e_1, \dots, e_n\}$ such that $\Omega(e_1, \dots, e_n) \in \mathbb{R}^+$; L is special Lagrangian if, for each $p \in L$, we can find an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pL which, as a unitary frame of T_pM, belongs to this fibre bundle. Any SL submanifold is automatically oriented by the volume form $\operatorname{Re}(\Omega)|_L$, and clearly has zero Maslov class.

The above considerations show that in Darboux coordinates any SL submanifold X with isolated singularities is locally Maslov-zero in the sense of Section 3.4. As seen in Example 6.4, it is easy to use either the ambient metric g or the local metric induced by Darboux coordinates to find conditions ensuring that X is locally exact. Finally, the C^0*–dense* version of the h–principle [13, Section 6.2.D] proves that the methods used in the previous Sections are completely local; in other words the desingularizations that we build there, near any singular point $x_i \in X$, can be made to live in any small neighbourhood of x_i. In particular, the construction can take place completely inside any Darboux coordinate chart. Our methods thus apply verbatim, proving the following result.

Corollary 6.11 *Let M be an almost Calabi–Yau manifold of dimension 2 or 3. Let $X \subset M$ be a SL submanifold with isolated exact singularities. Then X admits oriented Lagrangian desingularizations with zero Maslov class and arbitrarily complicated topology.*

6.3 Soft asymptotically conical smoothings of SL cones

Finally, in this section we address the third question posed in the Introduction to this paper. We begin with the following definition, adapted from Joyce [31, Definition 7.1].

Definition 6.12 Let C^n be a regular oriented cone in $\mathbb{C}^n$ with oriented (not necessarily connected) link Σ^{n-1}. Let $i : \Sigma \times (0,\infty) \to \mathbb{C}^n$ denote the corresponding immersion. We say that an immersion $f\colon \Sigma \times (1-\epsilon,\infty) \to \mathbb{C}^n$ is an *asymptotically conical (AC) end with decay* λ *and cone* C if

$$|\nabla^k (f - i)| = O(r^{\lambda-1-k}) \text{ as } r \to \infty \text{ for } k = 0, 1.$$

Here, ∇ and $|\cdot|$ are defined using the natural metric g' on C and we assume $\lambda < 2$.

Now suppose both i and f are Lagrangian. Even though i is automatically exact, in general there is no reason for f to be exact. However, it is simple to find additional assumptions ensuring that f is also exact. One such assumption was described by Joyce [31, Proposition 7.3]: if $\lambda < 0$ then f is exact. Notice also that, under this assumption, if C is Maslov-zero then this will be true also for f because the two manifolds are C^1–asymptotically close at infinity. Another possible assumption is as follows. Suppose ψ is a Hamiltonian diffeomorphism of $\mathbb{C}^n$ and that $f := \psi \circ i$ (where we restrict i to $\Sigma \times [1-\epsilon,\infty)$ and assume that ψ decays appropriately at infinity (so as to ensure the AC condition on f). Then f is exact and it is Maslov-zero if and only if C is.

Suppose now that C is SL and that Σ satisfies Assumptions A–C; as we have already remarked, these assumptions can be satisfied in low dimensions via any compact oriented choice of filling L. We can then try to solve the Prescribed Boundary Problem for the exact initial data (Σ, L, f) where now f is restricted to $\Sigma \times (1-\epsilon, 1]$. Since C is Maslov-zero, under either of the above assumptions f will be Maslov-zero also. We can thus conclude that in dimensions 2 and 3 the Prescribed Boundary Problem is solvable. The resulting submanifold Y still has the AC ends given by f, and Y is thus an exact Maslov-zero Lagrangian submanifold asymptotic to the SL cone C.

Appendix A Comparisons with other results in the literature

The goal of this Appendix is to fit our results into a broader context by describing some similarities, differences and relationships with other constructions in the Lagrangian and SL literature.

A.1 Lagrangian cobordism groups

Arnold [1] defined various types of Lagrangian and Legendrian cobordism groups. In [3; 4], Audin provides a detailed study of the oriented and nonoriented *exact Lagrangian cobordism groups*.

The abstract set-up is by now standard for cobordism-type theories: given a specific category of geometric objects, one defines an equivalence relationship such that, under disjoint union, the equivalence classes inherit a group structure. Following Thom, one then looks for an algebraic formulation of these groups in terms of homotopy groups of certain universal classifying spaces. The calculation of these homotopy groups is often possible but relies on rather sophisticated methods in homotopy theory.

In Audin's case, the geometric objects in question are pairs (L, f) where L is a closed (not necessarily connected) smooth manifold with $TL^{\mathbb{C}}$ trivial and $f\colon L \to \mathbb{C}^n$ is an exact Lagrangian immersion. Two such pairs (L_0, f_0) and (L_1, f_1) are considered equivalent if

(1) L_0, L_1 are cobordant in the usual sense; ie there exists a manifold Y such that $\partial Y = L_0 \cup L_1$ (in the oriented category, we also require that the orientations be compatible in the sense that $\partial Y = -L_0 \cup L_1$;

(2) Y has $TY^{\mathbb{C}}$ trivial and admits an exact Lagrangian immersion $F\colon Y \to \mathbb{C}^{n+1}$ which, on the boundary, restricts to the isotropic immersions $(f_0, 0, 0)$ and $(f_1, 1, 0)$;

(3) let n denote the inward unit conormal of $L_i \subset Y$. Then, using the standard notation for complex variables $z_j = x_j + iy_j$, we require that $dF(n) = \partial x_{n+1}$ along L_0 and $dF(n) = -\partial x_{n+1}$ along L_1.

The third condition should be thought of as a transversality condition which implies a canonical form for F near the boundary similar to our "cylindrical thickenings" in Example 4.5. A more thorough discussion of these groups can be found in [4, page 21].

[4] provides an algebraic reformulation of these Lagrangian cobordism groups (both oriented and nonoriented) in terms of a Thom-type set-up. Using this reformulation Audin computes the oriented cobordism groups in dimensions up to 10, often identifying explicit generators. Along the way, Audin defines and studies the Maslov-zero analogues of these groups ("SL cobordism groups") as a tool to compute the full oriented cobordism groups [4, page XII]. In the unoriented case matters simplify considerably: if L_0 and L_1 are cobordant as smooth unoriented manifolds and both admit Lagrangian immersions f_0, f_1 in $\mathbb{C}^n$, then (L_0, f_0) and (L_1, f_1) are exact Lagrangian cobordant [3, Corollary 2.2]

Audin's results apply to the Lagrangian Cobordism Problem stated in Section 4.1 as follows. Using the notation of that section, suppose that the exact isotropic immersion $i:\Sigma\to\mathbb{R}^{2n}$ is in fact an exact Lagrangian immersion into $\mathbb{R}^{2n-2}$. If the pair (Σ, i) is zero in the exact Lagrangian cobordism group then our Lagrangian Cobordism Problem with boundary data (Σ, i) is solvable. A priori, this does not allow us to prescribe the topology of the filling L nor the data of the Lagrangian thickening. A deeper understanding of the relationship between the Lagrangian cobordism groups and our Prescribed Boundary Problem requires many of the tools described in Section 3 and Section 4.

We expect that the more direct methods used in this paper and the explicit results of Section 5 and Section 6 are closer to the needs of differential geometers interested in SL singularities.

A.2 Lagrangian surgeries

The fillings and desingularizations obtained in Section 5 and Section 6 are in general only immersed, not embedded. Moreover, as already mentioned in Remark 3.8, Lagrangian embeddings of a fixed manifold L do not satisfy any type of h–principle. However, a generic Lagrangian immersion of L will contain only isolated transverse double points and there exists a standard Lagrangian surgery procedure which replaces a neighborhood of any such self-intersection with an embedded Lagrangian 1–handle [44]. If the number of double points is finite, an iteration of this procedure will thus generate an embedded Lagrangian submanifold $\widetilde{L}$. Clearly, this procedure changes the topology of L: L and $\widetilde{L}$ have different cohomology groups in dimensions 1 and $n-1$. It will thus affect the Maslov class of L and usually also its orientability. However, when n is odd-dimensional one can always choose this surgery so that $\widetilde{L}$ is orientable provided L is. Thus, for example, in the $n=3$ case the only problem introduced by these surgeries concerns the Maslov class. However it is not clear how to deal with this issue. In particular, it is not clear if it is possible to apply the Lagrangian surgery procedure to the immersed Maslov-zero desingularizations obtained in Section 6 and get embedded Maslov-zero desingularizations. It seems very likely that there would be further topological obstructions to being able to do this.

Polterovich's surgery procedure relies on the existence of an explicit Lagrangian model for desingularizing the union of any two Lagrangian planes. It is precisely the lack of analogous models for other types of singularities, such as most Lagrangian cones, that motivates our use of nonexplicit h–principle techniques to prove the existence of (immersed) desingularizations.

A.3 Fu's SL moment conditions

It is well-known that in a Calabi–Yau manifold $(M, \Omega, g, J, \omega)$, an orientable submanifold L is SL (with the correct choice of orientation) if and only if ω and $\beta := \operatorname{Im}\Omega$ restrict to zero on L. Thus if a compact Lagrangian submanifold L^n (with or without boundary) is SL, or is even only homologous to a SL, it must satisfy the condition $\int_L \beta = 0$.

One can immediately derive other constraints that any SL submanifold of a Calabi–Yau manifold must satisfy by considering the *differential ideal* $\mathcal{I}$ generated by ω and β. Let $\Omega(M)$ denote the algebra of real differential forms on M. Recall that a differential ideal on a smooth manifold M is an ideal $\mathcal{I}$ of $\Omega(M)$ which is also d–closed; in other words, $\mathcal{I}$ is a differential ideal if $\alpha \in \mathcal{I}$ and $\beta \in \Omega(M)$ implies that $\alpha \wedge \beta \in \mathcal{I}$ and $d\alpha \in \mathcal{I}$. Using the standard properties of forms under pullback and exterior differentiation we see that if f is a SL-immersion, then $f^*\sigma = 0$ for any $\sigma \in \mathcal{I}(\omega, \beta)$, the differential ideal generated by ω and β. In particular, if L is SL then $\int_L \alpha = 0$ for *any* $\alpha \in \mathcal{I}^n$, where $\mathcal{I}^n$ is the degree n part of $\mathcal{I}$.

Fu [14] used these constraints to determine necessary conditions for a compact orientable $(n-1)$–dimensional isotropic submanifold $\Sigma \subset \mathbb{C}^n$ to bound a SL n–fold. In many ways this is the SL analogue of the Lagrangian Cobordism Problem, so we now briefly summarize Fu's construction. For notational simplicity we will write Ω^k for $\Omega^k(M)$.

Consider the following complex:

$$(\Omega^0, \mathcal{I}^0) \xrightarrow{d} \ldots \to (\Omega^{k-1}, \mathcal{I}^{k-1}) \xrightarrow{d} (\Omega^k, \mathcal{I}^k) \to \cdots \xrightarrow{d} (\Omega^m, \mathcal{I}^m).$$

Let $\mathcal{H}^k$ denote the corresponding (relative) cohomology groups. That is, we define

$$\mathcal{Z}^k := \{\alpha \in \Omega^k : d\alpha \in \mathcal{I}^{k+1}\} \quad \text{and} \quad \mathcal{B}^k := d\Omega^{k-1} + \mathcal{I}^k,$$

and then $\mathcal{H}^k$ is defined as the quotient space $\mathcal{H}^k := \mathcal{Z}^k / \mathcal{B}^k$. Using the facts that Σ is isotropic and that elements γ of $\mathcal{I}^{n-1}$ are of the form $\gamma = \omega \wedge \gamma'$, it follows that $\int_\Sigma [\alpha]$ is well-defined for all $[\alpha] \in \mathcal{H}^{n-1}$. If Σ admits a smooth SL filling, then Σ satisfies the *SL moment conditions*

$$\int_\Sigma [\alpha] = 0, \quad \text{for all} \quad [\alpha] \in \mathcal{H}^{n-1}.$$

In fact, even if Σ is only the boundary of a (possibly singular) n–dimensional SL-rectifiable current then Σ still satisfies these moments conditions. Furthermore, if Σ admits a (possibly singular) SL filling, then *any* Lagrangian filling L of Σ must satisfy the condition

$$\int_L d[\alpha] = 0, \quad \text{for any} \quad [\alpha] \in \mathcal{H}^{n-1}.$$

Hence the SL moment conditions are not sensitive enough to detect the difference between the existence of a smooth SL filling and a singular one. For example, the SL moment conditions will be satisfied even if Σ can only be filled by a singular SL cone. In this sense Fu's moment conditions are strictly weaker than what one would need to solve the "smooth SL Cobordism Problem". Furthermore, using an explicit description of the SL moment conditions in $\mathbb{C}^n$ (see below), Fu showed that there are isotropic manifolds Σ which satisfy all the SL moment conditions and yet do not bound any SL rectifiable current. This shows that the SL moment conditions are not sufficient for the "singular SL Cobordism Problem" either.

When $M=\mathbb{C}^n$, Fu was able to express the SL moment conditions very explicitly. In particular he showed that the space $\mathcal{H}^{n-1}$ is isomorphic to the space of functions f satisfying the condition $\mathcal{L}_{X_f}\beta=0$, where $\mathcal{L}$ denotes Lie differentiation and X_f is the Hamiltonian vector field generated by f. He then determined all such functions explicitly; for $n\geq 3$ they are exactly the hermitian harmonic quadratics defined following Lemma 2.4 and arising from the Hamiltonian action of $\mathrm{SU(n)}\ltimes\mathbb{C}^n$ on $\mathbb{C}^n$. Hence the SL moment conditions amount to exactly $n(n+2)$ independent conditions on any compact orientable isotropic $(n-1)$–manifold $\Sigma\subset\mathbb{C}^n$.

A.4 Joyce's SL gluing results

In the papers [33; 34] Joyce presents a series of results on the desingularization of certain kinds of singular SL submanifolds, which we have called SL conifolds. More specifically, these papers deal with a compact SL n–fold X in M with isolated singularities $x_1,\dots,x_m$ (as in our Definition 6.1), satisfying two additional constraints. First, he assumes that each singularity x_i is modelled metrically (in a precise sense) on a regular SL cone $C_i\subset\mathbb{C}^n$. Second, he assumes that there exists a smooth SL submanifold $L_i\subset\mathbb{C}^n$ which is asymptotic (again, in a precise sense) to the cone C_i. He then proves that, under certain conditions, it is possible to remove a neighbourhood of each singular point x_i in X and glue in a copy of L_i so that, after a global perturbation, the resulting submanifold Y is SL in M.

The simplest case in which Joyce obtains desingularization results is presented in [33, Theorem 6.13]. The first step in his process is to produce a Lagrangian gluing of each L_i to X as follows. Let Σ_i denote the link of the SL cone C_i. The use of Lagrangian neighbourhoods allows him to work inside the cotangent bundle $T^*(\Sigma_i\times(0,\infty))$, endowed with its standard symplectic structure. More precisely, the gluing happens inside a small region of this bundle, $T^*(\Sigma_i\times[t^\tau,2t^\tau])$ (depending on parameters t,τ). There, the relevant regions of L_i and X appear as the graphs of closed 1–forms χ_i and η_i respectively. Producing the required Lagrangian gluing reduces to proving that there exists a closed 1–form which interpolates between $\chi_i \in T^*(\Sigma_i \times \{t^\tau\})$ and $\eta_i \in T^*(\Sigma_i \times \{2t^\tau\})$. There is an obvious obstruction to doing this: the cone-like form of the metric on X close to each singular point x_i implies (as in our Example 6.4) that η_i is exact, so such an interpolation exists if and only if χ_i is also exact. The exactness of χ_i is an additional condition on L_i which is encoded in Joyce's cohomological invariant $Y(L_i)$ (see [31, Definition 7.2]). Thus the Lagrangian gluing procedure just described is possible if and only if $Y(L_i) = 0$.

There are some strong similarities between the Lagrangian gluing procedure outlined above and the methods we use, eg in Corollary 6.11. First of all, both are completely local in the sense that they perturb the original X only in a neighborhood of the singularity, leaving the rest untouched. Second, in both cases (and for the same reasons) the singularities are locally exact. The main difference is of course that Joyce relies on an *a priori* smoothing L_i of C_i while we create our own smoothings using h–principle techniques. Thus in our case the role played by exactness is different from in Joyce's work; in our setting it is a necessary condition for applying the h–principle.

There are SL smoothings L of SL cones C which do not satisfy $Y(L) = 0$ [30, Example 6.8]. For desingularizations based on these models, Joyce provides a second type of Lagrangian gluing process which perturbs X globally. Again, the gluing is possible only if a certain cohomological compatibility holds between the appropriate regions of L_i and X.

It may also be useful to point out that the reason why Joyce is not concerned with Maslov-class type problems is that, by being careful in the Lagrangian gluing process, the Lagrangian submanifolds he produces this way are almost SL (in a precise sense). Thus they are automatically Maslov-zero. In conclusion, it is the whole set-up Joyce begins with that allows him not to worry about the kind of topological questions we address here, and instead concentrate on the analytic aspects of the SL condition.

Remark A.1 Joyce also has results for a SL analogue of the Lagrangian surgery procedure explained in this Appendix, Section A.2. Once again, the starting point for this is provided by explicit local models due to Lawlor. These models furnish local SL desingularizations of certain pairs of SL planes, satisfying an angle condition. Again one has no Maslov class problems because one can create an almost SL smoothing which therefore has zero Maslov class. Joyce shows that any connected SL submanifold with transverse self-intersection points modelled by such pairs of planes can be desingularized, leading to an embedded SL submanifold. The angle condition is always verified in the $n = 3$ case. Joyce also provides results for a "SL connect sum" of smooth SL submanifolds which intersect one another transversely. Related results were obtained by A Butscher, Y I Lee and D Lee [8; 37; 36].

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Department of Mathematics, Imperial College London
South Kensington, London SW7 2AZ, England

Georgia Institute of Technology, Department of Mathematics
686 Cherry Street, Atlanta, GA 30332, USA

m.haskins@imperial.ac.uk, t.pacini@imperial.ac.uk

Proposed: Frances Kirwan
Seconded: Jim Bryan, Simon Donaldson

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