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Highly connected manifolds with positive Ricci curvature

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We prove the existence of Sasakian metrics with positive Ricci curvature on certain highly connected odd dimensional manifolds. In particular, we show that manifolds homeomorphic to the 2k–fold connected sum of $S^{2n-1}\times S^{2n}$ admit Sasakian metrics with positive Ricci curvature for all k. Furthermore, a formula for computing the diffeomorphism types is given and tables are presented for dimensions 7 and 11.

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Introduction

An important problem in global Riemannian geometry is that of describing the class of manifolds that admit metrics of positive Ricci curvature. The only known obstructions for obtaining such metrics come from either the classical Myers theorem or the obstructions to the existence of positive scalar curvature which is fairly well understood (see the recent review article by Rosenberg and Stolz [36] for discussion and references). Given the lack of obstructions it seems most natural to develop techniques for proving the existence of positive Ricci curvature metrics. Over the years several methods for doing so have appeared. These include symmetry methods, bundle constructions, surgery theory, algebro-geometric techniques. We refer the reader to recent papers (Boyer, Galicki and Nakamaye [13; 15], Grove and Ziller [23] and Schwachhöfer and Tuschmann [37]) for a discussion of the history and pertinent references. In [13; 15], with M Nakamaye, we introduced a method for proving the existence of positive Ricci curvature on odd dimensional manifolds which relies on a transverse version of Yau's famous proof of the Calabi conjecture. The odd dimensional manifolds to which this method has been applied are hypersurfaces of isolated singularities coming from weighted homogeneous polynomials. All such manifolds are what are sometimes called "highly connected". So far, with collaborators, we have applied our methods successfully mainly to rational homology spheres [4; 11; 7], homotopy spheres [15; 9; 10], and connected sums of $S^2\times S^3$ [14; 12; 5]; (see also Kollár [28]). A recent review of our method can be found in [6].

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The purpose of this note is to prove the existence of Sasakian metrics of positive Ricci curvature on certain odd dimensional highly connected smooth manifolds. Manifolds of dimension $2n$ or $2n+1$ that are $n-1$ connected are often referred to as *highly connected* manifolds. They are relatively tractable and there is a classification of such manifolds by C T C Wall [41; 42] and his students (Barden [1] and Wilkens [43]) as well as Crowley [18]. Our first result concerns dimension $4n-1$, where we prove the following:

Theorem 1 *Let $n \geqslant 2$ be an integer, then for each positive integer k there exist Sasakian metrics with positive Ricci curvature in $D_n(k)$ of the $|bP_{4n}|$ oriented diffeomorphism classes of the $(4n-1)$–manifolds $2k\#(S^{2n-1} \times S^{2n})$ that bound a parallelizable manifold, where the number $D_n(k)$ is determined by the explicit formula* (A–4) *given in the Appendix. In particular, $D_n(1) = |bP_{4n}|$ for all $n \geqslant 2$, so $2\#(S^{2n-1} \times S^{2n})$ admits Sasakian metrics of positive Ricci curvature in every oriented diffeomorphism class.*

Here $k\#(M_1 \times M_2)$ denotes the k–fold connected sum of the manifold $M_1 \times M_2$. In the case of the connected sums of products of standard spheres, metrics of positive Ricci curvature have been constructed previously by Sha and Yang [39]. However, the existence of such metrics for the exotic differential structures appears to be new. In dimension $4n+1$ we prove a somewhat weaker result:

Theorem 2 *For each pair of positive integers (n,k) there exists an oriented $(2n-1)$–connected $(4n+1)$–manifold K with $H_{2n}(K,\mathbb{Z})$ free of rank k which admits a Sasakian metric of positive Ricci curvature. Furthermore, K is diffeomorphic to one of the manifolds*

$$\#k(S^{2n} \times S^{2n+1}), \quad \#(k-1)(S^{2n} \times S^{2n+1})\#T, \quad \#k(S^{2n} \times S^{2n+1})\#\Sigma^{4n+1},$$

where $T = T_1(S^{2n+1})$ is the unit tangent bundle of S^{2n+1}, and Σ^{4n+1} is the Kervaire sphere. For $k=1$ the manifolds

$$S^{2n} \times S^{2n+1}, \qquad (S^{2n} \times S^{2n+1})\#\Sigma^{4n+1}, \qquad T$$

all admit Sasakian metrics with positive Ricci curvature. If $n = 1,3$ then $\#k(S^{2n} \times S^{2n+1})$ admits a Sasakian metric with positive Ricci curvature for all k.

The result here for $n=1$ was given previously in [13]. For highly connected rational homology spheres we have:

Theorem 3 *Every $(2n-2)$–connected oriented $(4n-1)$–manifold that is the boundary of a parallelizable manifold whose homology group $H_{2n-1}(K,\mathbb{Z})$ is isomorphic to $\mathbb{Z}_3$ admits Sasakian metrics with positive Ricci curvature. There are precisely $2|bP_{4n}|$ such smooth oriented manifolds.*

There are two distinct oriented topological manifolds in this theorem and they are distinguished by their linking form in $H_{2n-1}(K,\mathbb{Z})\approx\mathbb{Z}_3$. However, they are equivalent as non-oriented manifolds. Moreover, each oriented manifold is comprised of $|bP_{4n}|$ distinct oriented diffeomorphism types.

1 Highly connected manifolds

The most obvious subclass of highly connected manifolds are the homotopy spheres which were studied in detail in the seminal paper of Kervaire and Milnor [27]. We briefly summarize their results. Kervaire and Milnor defined an Abelian group Θ_n which consists of equivalence classes of homotopy spheres of dimension n that are equivalent under oriented h–cobordism. By Smale's famous h–cobordism theorem [40] (see also Milnor [33]) this implies equivalence under oriented diffeomorphism. The group operation on Θ_n is connected sum. Now Θ_n has an important subgroup bP_{n+1} which consists of equivalence classes of those oriented homotopy spheres which are the boundary of a parallelizable manifold. It is the subgroup bP_{2n} that is important for us in the present work. Kervaire and Milnor proved:

(i) $bP_{2m+1}=0$.

(ii) bP_{4m} $(m\geqslant 2)$ is cyclic of order $2^{2m-2}(2^{2m-1}-1)$ times the numerator of $\left(\frac{4B_m}{m}\right)$, where B_m is the m-th Bernoulli number. Thus, for example $|bP_8|=28$, $|bP_{12}|=992$, $|bP_{16}|=8128$, $|bP_{20}|=130,816$.

(iii) bP_{4m+2} is either 0 or $\mathbb{Z}_2$.

Determining which bP_{4m+2} is $\{0\}$ and which is $\mathbb{Z}_2$ has proven to be difficult in general, and is still not completely understood. If $m\neq 2^i-1$ for any $i\geqslant 3$, then Browder [17] proved that $bP_{4m+2}=\mathbb{Z}_2$. However, bP_{4m+2} is the identity for $m=1,3,7,15$ (Mahowald and Tangora [32] and Barratt, Jones and Mahowald [2]). See Lance [31] for a recent survey of results in this area and complete references. The answer is still unknown in the remaining cases. Using surgery Kervaire was the first to show that there is an exotic sphere in dimension 9. His construction works in all dimensions of the form $4m+1$, but as just discussed they are not always exotic.

In analogy with the Kervaire–Milnor group bP_{2n}, Durfee [20] defined the group BP_{2n}. Actually, he first defined this as a semigroup in [19], and later in [20] with the same notation denoted the corresponding Grothendieck group. Thus, we have:

Definition 1.1 For $n \geqslant 3$ let SBP_{2n} denote the semigroup of oriented diffeomorphism classes of closed oriented $(n-2)$–connected $(2n-1)$–manifolds that bound parallelizable manifolds, and let BP_{2n} denote its Grothendieck completion.

As with bP_{2n} mulitiplication in SBP_{2n} is the connected sum operation, and the standard sphere is a two-sided identity. Thus, SBP_{2n} is a monoid. Furthermore, the Kervaire and Milnor group bP_{2n} is a subgroup of BP_{2n} again by Smale's h–cobordism theorem. Unless otherwise stated we shall heretofore assume that $n \geqslant 3$, and that "manifold" will mean oriented manifold. We are mainly interested in those highly connected manifolds that can be realized as links of isolated hypersurface singularities defined by weighted homogeneous polynomials, so we have:

Definition 1.2 We denote by WHP_{2n-1} the set of oriented diffeomorphism classes of smooth closed oriented $(2n-1)$ dimensional manifolds that can be realized as the link of an isolated hypersurface singularity of a weighted homogeneous polynomial in $\mathbb{C}^{n+1}$.

It is easy to see that elements of WHP_{2n-1} enjoy some nice properties.

Theorem 1.3 *Let $M \in WHP_{2n-1}$. Then*

(i) *M is highly connected, that is, it is $(n-2)$–connected.*
(ii) *M is the boundary of a compact $(n-1)$–connected parallelizable manifold V of dimension $2n$ with $H_n(V,\mathbb{Z})$ free.*
(iii) *If n is even the $(n-1)$st Betti number $b_{n-1}(M)$ is even.*
(iv) *M is a spin manifold.*

Proof Parts (i), (ii) and (iv) are well known. By [3] M admits a Sasakian structure, so its odd Betti numbers are even up to the middle dimension by a well known result of Fujitani and Blair–Goldberg (cf [8]) which proves (iii). □

From (i) and (ii) of Theorem 1.3 one sees that there is a map

$$\Phi\colon WHP_{2n-1} \longrightarrow BP_{2n}$$

which is the composition of the inclusion $WHP_{2n-1} \hookrightarrow SBP_{2n}$ with the natural semigroup homomorphism $SBP_{2n} \longrightarrow BP_{2n}$. The image $\Phi(WHP_{2n-1})$ is a subset of BP_{2n}, and by (iii) of Theorem 1.3 it is a proper subset at least when n is even. This can be contrasted with bP_{2n} which by a result of Brieskorn [16] satisfies $bP_{2n} \cap$

$\Phi(WHP_{2n-1}) = bP_{2n}$ if $n \geqslant 3$. Notice, however, that $\Phi(WHP_{2n-1})$ is not generally a submonoid.

We now discuss invariants that distinguish elements of BP_{2n}. First, by Poincaré duality the only non-vanishing homology groups occur in dimension $0, n-1, n$ and $2n-1$. Moreover, $H_n(K,\mathbb{Z})$ is free and rank $H_n(K,\mathbb{Z}) =$ rank $H_{n-1}(K,\mathbb{Z})$. Thus, our first invariant is the rank of $H_{n-1}(K,\mathbb{Z})$, so we define

$$BP_{2n}(k) = \{K \in BP_{2n} \mid \text{rank } H_{n-1}(K,\mathbb{Z}) = k\}. \tag{1–1}$$

This provides BP_{2n} with a grading, namely

$$BP_{2n} = \bigoplus_k BP_{2n}(k) \tag{1–2}$$

which is compatible with multiplication in BP_{2n} in the sense that

$$\times\colon\ BP_{2n}(k_1) \times BP_{2n}(k_2) \longrightarrow BP_{2n}(k_1 + k_2). \tag{1–3}$$

Note that $BP_{2n}(0)$ is the submonoid of highly connected rational homology spheres, and that $BP_{2n}(k)$ is a bP_{2n}–module.

The remaining known invariants (Wall [42], Durfee [19; 20]) are a linking form on the torsion subgroup of $H_{n-1}(K)$ and a quadratic invariant on the $2n$–manifold whose boundary is K. The precise nature of these invariants depends on whether n is even or odd. For the case n even Durfee [20] shows that for $n \geqslant 3$ and $n \neq 4, 8$ there is an exact sequence

$$0 \longrightarrow bP_{2n} \longrightarrow BP_{2n} \xrightarrow{\Psi} \mathbb{Z} \oplus KQ(\mathbb{Z}) \longrightarrow 0, \tag{1–4}$$

where $KQ(\mathbb{Z})$ denotes the Grothendieck group of regular bilinear form modules over $\mathbb{Z}$. Let us describe the map Ψ. The projection onto the first factor is just the rank of $H_{n-1}(K)$ while the projection onto the second factor is Wall's quadratic form [42] which is essentially the classical linking form b on the torsion subgroup of $H_{n-1}(K)$. Any two manifolds $K_1, K_2 \in BP_{2n}$ such that $\Psi(K_1) = \Psi(K_2)$ differ by a homotopy sphere, ie, there is $\Sigma \in bP_{2n}$ such that $K_2 \approx K_1 \# \Sigma$ where $\approx$ means diffeomorphic. It is well known (Brieskorn [16]) that for n even the elements $\Sigma \in bP_{2n}$ are determined by the signature of V. This completes the diffeomorphism classification for $n \neq 4, 8$ even. The cases $n = 4, 8$ are more complicated (Wilkens [43], Crowley [18]). Now, in addition to the group $H_{n-1}(K)$ and the linking form b, there is an obstruction cocycle $\widehat{\beta} \in H^n(K, \pi_{n-1}(SO)) \approx H^n(K,\mathbb{Z})$. The tangent bundle of K restricted to the $(n-1)$–skeleton is trivial and $\widehat{\beta}$ gives the obstruction to triviality on the n–skeleton. If the torsion subgroup of $H_{n-1}(K)$ has odd order, then up to decomposability these are all the invariants. However, if the torsion subgroup of $H_{n-1}(K)$ has even order, things are even more complicated, and the analysis in [43] was not complete. It was recently completed in [18]. The important point for us is that if the torsion subgroup of $H_{n-1}(K)$ vanishes, K is determined completely up to diffeomorphism by the rank of $H_{n-1}(K)$. Summarizing we have:

Theorem 1.4 *Let M be a highly connected manifold in BP_{4n} such that $H_{2n-1}(M,\mathbb{Z}) = \mathbb{Z}^k$. Then M is diffeomorphic to $k\#(S^{2n-1}\times S^{2n})\#\Sigma^{4n-1}$ for some $\Sigma^{4n-1}\in bP_{4n}$.*

Notice that by a well-known result of Fujitani and Blair–Goldberg (cf [8]) $k\#(S^{2n-1}\times S^{2n})\#\Sigma^{4n-1}$ can admit a Sasakian structure only if k is even.

For the case n odd the diffeomorphism classification was obtained by Wall [42], but for our purposes, the presentation in [19] is more convenient. Let $K\in BP_{2n}$ with $K=\partial V$, where V can be taken as $(n-1)$–connected and parallelizable. In this case the key invariant is a $\mathbb{Z}_2$–quadratic form

$$\psi\colon H_n(V,\mathbb{Z})/2H_n(V,\mathbb{Z})\longrightarrow\mathbb{Z}_2$$

defined as follows: Let X be an embedded n–sphere in V that represents a non-trivial homology class in $H_n(V,\mathbb{Z})$, and let $[X]$ denote its image in $H_n(V,\mathbb{Z})/2H_n(V,\mathbb{Z})$. Then $\psi([X])$ is the characteristic class in the kernel $\ker(\pi_{n-1}(SO(n)\longrightarrow\pi_{n-1}(SO))\approx\mathbb{Z}_2$ of the normal bundle of X. Let rad ψ be the radical of ψ, ie, the subspace of the $\mathbb{Z}_2$–vector space $H_n(V,\mathbb{Z})/2H_n(V,\mathbb{Z})$ where ψ is singular. Then Durfee [19] (see also [21]) proves:

Theorem 1.5 *Let $K_i\in BP_{2n}$ for $i=1,2$ with $n\geqslant 3$ odd be boundaries of parallelizable $(n-1)$–connected $2n$ manifolds V_i with $\mathbb{Z}_2$ quadratic forms ψ_i. Suppose that $H_{n-1}(K_1,\mathbb{Z})\approx H_{n-1}(K_2,\mathbb{Z})$, then*

(i) *if $n=3$ or 7, then K_1 and K_2 are diffeomorphic;*

(ii) *if the torsion subgroups of $H_{n-1}(K_i,\mathbb{Z})$ have odd order and $\psi_i|\mathrm{rad}\,\psi_i\equiv 0$ for $i=1,2$, then $K_1\approx K_2\#(c(\psi_1)+c(\psi_2))\Sigma$, where c is the Arf invariant and Σ is the Kervaire sphere, ie, the generator of bP_{2n};*

(iii) *if the torsion subgroups of $H_{n-1}(K_i,\mathbb{Z})$ have odd order and $\psi_i|\mathrm{rad}\,\psi_i\not\equiv 0$ for $i=1,2$, then $K_1\approx K_2\approx K_2\#\Sigma$.*

It is convenient to define $WHP_{2n-1}(k)$ to be the subset of WHP_{2n-1} such that H_{n-1} has rank k. Then (iii) of Theorem 1.3 implies $WHP_{4n-1}(2k+1)=\varnothing$, whereas we shall see that $WHP_{4n-1}(2k)\neq\varnothing$ as well as $WHP_{4n+1}(k)\neq\varnothing$ for all k. Recently, in the case $n=3$, Kollár [29; 30] has discovered strong restrictions on the torsion subgroups of $H_2(K,\mathbb{Z})$ in order that K admit a Sasakian structure which implies that $\Phi(WHP_6(0))$ is a proper subset of $BP_6(0)$. One certainly expects these types of restrictions to persist in higher dimension as well.

2 Branched covers and periodicity

In this section we discuss some results of Durfee and Kauffman [21] concerning the periodicity of branched covers. Let $K \subset S^{2n+1}$ be a simple fibered knot or link ($n \geqslant 1$), by which we mean an $(n-2)$ connected $(2n-1)$ embedded submanifold of S^{2n+1} for which the Milnor fibration theorem holds. If F is the Milnor fiber of the fibration $\phi\colon S^{2n+1} - K \longrightarrow S^1$ then the *monodromy map* $h\colon H_n(F) \longrightarrow H_n(F)$ is a fundamental invariant of the link K. Let K_k be a k–fold cyclic branched cover of S^{2n+1} branched along K. Then Durfee and Kauffman [21] show that there is an exact sequence

$$H_n(F) \xrightarrow{\mathbb{1}+h+\cdots+h^{k-1}} H_n(F) \longrightarrow H_n(K_k) \longrightarrow 0. \tag{2–1}$$

So homologically K_k is determined by the cokernel of the map $\mathbb{1}+h+\cdots+h^{k-1}$. Now suppose that K is a rational homology sphere and that the monodromy map h of K has period d. Then since $\mathbb{1}-h$ is invertible, $\mathbb{1}+h+\cdots+h^{d-1}$ is the zero map in (2–1), and this determines the homology of K_d. Summarizing we have:

Lemma 2.1 (Durfee–Kauffman) *Let K be a fibered knot in S^{2n+1} which is a rational homology sphere such that the monodromy map has period d. Suppose further that K_k is a k–fold cyclic cover of S^{2n+1} branched along K. Then*

(i) $H_n(K_d) \approx H_n(F) \approx \mathbb{Z}^\mu$ *where μ is the Milnor number of K.*

(ii) $H_*(K_{k+d}) \approx H_*(K_k)$ *for all $k > 0$.*

(iii) $H_*(K_{d-k}) \approx H_*(K_k)$ *for all $0 < k < d$.*

Notice that (i) determines a large class of $n-1$ connected $2n+1$–manifolds whose middle homology group H_n is free, and in certain cases this determines the manifold up to homeomorphism. Items (ii) and (iii) give a homological periodicity.

Durfee and Kauffman also show that there are both homeomorphism and diffeomorphism periodicities in the case that n is odd and $n \neq 1, 3, 7$. In particular in this case, when the link K is a rational homology sphere whose monodromy map has period d, K_{k+d} is homeomorphic to K_k. To obtain the diffeomorphism periodicity let σ_k denote the signature of the intersection form on the Milnor fiber F_k. Again assuming that K is a rational homology sphere and h has periodicity d, one finds that K_{k+d} is diffeomorphic to $\frac{\sigma_{d+1}}{8}\Sigma \# K_k$ where $\frac{\sigma_{d+1}}{8}\Sigma$ denotes $\frac{\sigma_{d+1}}{8}$ copies of the Milnor sphere Σ. Here we state the slightly more general theorem of Durfee [20, Theorem 6.4]:

Theorem 2.2 *For even $n \neq 2,4,8$ let K_i be $(n-2)$–connected manifolds that bound parallelizable manifolds V_i, with $i=1,2$. Suppose that the quadratic forms of K_i are isomorphic and $H_{n-1}(K_1,\mathbb{Z}) \approx H_{n-1}(K_2,\mathbb{Z})$. Then $\sigma(V_2)-\sigma(V_1)$ is divisible by 8, and K_2 is diffeomorphic to $K_1\#\frac{1}{8}(\sigma(V_2)-\sigma(V_1))\Sigma$ where $\sigma(V)$ is the Hirzebruch signature of V.*

Remark 2.1 [20, Theorem 6.4] as well as [21, Theorem 5.3] exclude the cases $n=4$ and 8. However, it follows from [43] and [18] that the diffeomorphism classification still holds in these cases since the links we are considering here have no element of even order in the torsion subgroup of H_{n-1} (In fact the torsion subgroup vanishes in the case above). This remark also pertains to the discussion for Theorem 3 below.

3 Positive Ricci curvature on links

Recall [13] that a Sasakian structure (ξ,η,Φ,g) is *positive* if the basic Chern class $c_1(\mathcal{F}_\xi)$ of the characteristic foliation $\mathcal{F}_\xi$ is positive. The importance of positive Sasakian structures comes from Theorem A of [13] which states that they give rise to Sasakian metrics with positive Ricci curvature. An important ingredient in the proof of this result is the 'transverse Yau theorem' of El Kacimi-Alaoui [22], or equivalently for the cases at hand, the orbifold version of Yau's theorem. Now there is a natural induced Sasakian structure on the link of a hypersurface singularity of a weighted homogeneous polynomial [3]. Combining this with 'orbifold adjunction theory' [8] we obtain:

Theorem 3.1 *Let L_f be the link of an isolated hypersurface singularity of a weighted homogeneous polynomial f of degree d and weight vector $\mathbf{w}$. Suppose further that $|\mathbf{w}|-d>0$. Then L_f admits a Sasakian metric with positive Ricci curvature.*

It is a simple task to construct positive Sasakian structures on links by increasing the dimension.

Proposition 3.2 *Let $L_{f'}$ be the link of a weighted homogeneous polynomial $f'(z_2,\cdots,z_n)$ in $n-1$ variables with weight vector $\mathbf{w}'$ and degree d'. Assume that the origin in $\mathbb{C}^{n-1}$ is the only singularity so that $L_{f'}$ is smooth. Consider the weighted homogeneous polynomial*

$$f = z_0^2 + z_1^2 + f'$$

of degree $d=\mathrm{lcm}(2,d')$. Then the link L_f admits a Sasakian structure with positive Ricci curvature and $b_{n-1}(L_f)=b_{n-3}(L_{f'})$.

Proof There are two cases. If d' is odd then the weight vector of f is $\mathbf{w} = (d', d', 2\mathbf{w}')$, whereas, if d' is even, then $\mathbf{w} = (\frac{d'}{2}, \frac{d'}{2}, \mathbf{w}')$. In the first case we have $|\mathbf{w}| - d = d' + d' + 2|\mathbf{w}'| - 2d' = 2|\mathbf{w}'| > 0$, while in the second case $|\mathbf{w}| - d = \frac{d'}{2} + \frac{d'}{2} + |\mathbf{w}'| - d' = |\mathbf{w}'| > 0$. In either case L_f admits a Sasakian metric with positive Ricci curvature by Theorem 3.1. The equality of Betti numbers is well known and follows from a theorem of Sebastiani and Thom [38; 26]. □

We note that it is easy to see that the appearance of the two 2's in f implies that the klt conditions used to imply the existence of Sasakian-Einstein metrics [9; 7] cannot be satisfied. So we can say nothing at present about the existence of Sasakian-Einstein metrics on these links.

4 Proofs of Theorems 1, 2 and 3

The links that we need to prove Theorems 1–3 involve Brieskorn–Pham polynomials of the form

$$f_{p,q} = z_0^p + z_1^q + z_2^2 + \cdots + z_n^2. \tag{4–1}$$

The link associated with $f_{p,q}$ is

$$L_{p,q} = \{f_{p,q} = 0\} \cap S^{2n+1}.$$

By Proposition 3.2 all such links admit Sasakian metrics with positive Ricci curvature. One can view $L_{p,q}$ as a p–fold branched cover of S^{2n-1} branched over the link L_q defined by the polynomial

$$f_q = z_1^q + z_2^2 + \cdots + z_n^2.$$

Proof of Theorem 1 Here we need the link $L_{2(2k+1),2k+1}$, ie $p = 2(2k+1), q = 2k+1$, with n even (here n corresponds to $2n$ in the statement of the theorem). In this case the degree of $L_{2(2k+1),2k+1}$ is $d = 2(2k+1)$ which is the period of the monodromy map of the link L_{2k+1}. Furthermore, L_{2k+1} is a homotopy sphere by the Brieskorn Graph Theorem [16] or [8]. Now the link $L_{2(2k+1),2k+1}$ is a $2(2k+1)$ branched cover of S^{2n-1} branched over L_{2k+1}, so by item (i) of Lemma 2.1, we have

$$H_{n-1}(L_{2(2k+1),2k+1}, \mathbb{Z}) \approx H_n(L_{2(2k+1),2k+1}, \mathbb{Z}) \approx \mathbb{Z}^{\mu} = \mathbb{Z}^{2k}. \tag{4–2}$$

Here μ is the Milnor number [34] of the link L_{2k+1} which is easily computed by the formula for Brieskorn polynomials, namely

$$\mu = \prod_{i=1}^{n}(a_i - 1) = (2k+1-1)\cdot 1\cdots 1 = 2k.$$

Remark 4.1 Notice that the link $L_{2(2k+1),2k+1}$ can be obtained by iterating Proposition 3.2 beginning with the Brieskorn manifold $M(2(2k+1), 2k+1, 2)$ which is described in [35, Example 1 page 320]. As discussed there it is the total space of the circle bundle with Chern number -1 over a Riemann surface of genus k.

It now follows from Theorem 1.4 that $L_{2(2k+1),2k+1}$ is diffeomorphic to $2k\#(S^{n-1}\times S^n)\#\Sigma^{4n-1}$ for some $\Sigma^{4n-1}\in bP_{4n}$. (Here n is as in the statement of the theorem.) We now use the periodicity results of Durfee and Kauffman to determine the diffeomorphism type. First we notice that Theorem 1.4 together with [21, Theorem 4.5] imply that for every positive integer i and every positive integer k, the link $L_{2i(2k+1),2k+1}$ is homeomorphic to the connected sum $2k\#(S^{n-1}\times S^n)$. The diffeomorphism types are determined by Theorem 2.2 ([20, Theorem 6.4], see also [21, Theorem 5.3]) together with Remark 2.1. Let $F_{i,k}$ denote the Milnor fibre of the link $L_{2i(2k+1),2k+1}$ and $\sigma(F_{i,k})$ its Hirzebruch signature. Then Theorem 2.2 says that for each pair of positive integers i,j there is a diffeomorphism

$$L_{2i(2k+1),2k+1} \approx \Big(\frac{\sigma(F_{i,k})-\sigma(F_{j,k})}{8}\Sigma\Big)\# L_{2j(2k+1),2k+1}, \tag{4–3}$$

where $l\Sigma$ denotes the connected sum of l copies of the Milnor sphere, and a minus sign corresponds to reversing orientation. Actually this formula follows from a signature periodicity result of Neumann as stated in [21, Theorem 5.2]. From Durfee's theorem the difference in signatures is always divisible by 8, so this expression makes sense. Equation (4–3) can be iterated; so it is enough to consider the case $i=2$ and $j=1$. In order to determine how many distinct diffeomorphism types occur in (4–3), we need to compute the signature of the Milnor fibres. This is done in Appendix A. It is interesting to note that not all diffeomorphism types can be attained. This ends the proof of Theorem 1. □

Proof of Theorem 2 Now we have n odd (corresponding to $2n+1$ in the statement of the theorem) and there are several cases. First we take $p=2(2k+1), q=2k+1$ as in the proof of Theorem 1. Again this leads to the link $L_{2(2k+1),2k+1}$ with free homology satisfying Equation (4–2) except now n is odd. Next we consider $q=2k$ in Equation (4–1). The link L_{2k} of the Brieskorn–Pham polynomial $f_{2k}=z_1^{2k}+z_2^2+\cdots+z_n^2$

is a rational homology sphere by the Brieskorn Graph Theorem. Furthermore, its monodromy map has period $2k$. Then choosing $p = 2k$ in Equation (4–1) the link $L_{2k,2k}$ is $2k$–fold branched cover over S^{2n+1} branched over the rational homology sphere L_{2k}, so by item (i) of Lemma 2.1, we have

$$H_{n-1}(L_{2k,2k}, \mathbb{Z}) \approx H_n(L_{2k,2k}, \mathbb{Z}) \approx \mathbb{Z}^{\mu} = \mathbb{Z}^{2k-1}.$$

These two cases now give links whose middle homology groups are free of arbitrary positive rank. However, unlike the case for n even this does not determine the homeomorphism type unless $n = 3, 7$ in which case there is a unique diffeomorphism class. Indeed Theorem 1.5 implies we need to compute the quadratic form ψ, and this appears to be quite difficult in all but the simplest case. From Theorem 1.5 one can conclude [19] that if $M \in BP_{4n+2}$ with $H_{2n}(M, \mathbb{Z})$ free of rank one, then it is homeomorphic to $S^{2n} \times S^{2n+1}$ or the unit tangent bundle $T = T_1(S^{2n+1})$. (Now n is as in the statement of the theorem). So the diffeomorphism types at most differ by an exotic Kervaire sphere Σ^{4n+1}. Furthermore, $S^{2n} \times S^{2n+1}, T$ and $(S^{2n} \times S^{2n+1})\#\Sigma^{4n+1}$ generate the torsion-free submonoid of BP_{4n+2}, there being relations in the monoid, namely, $T\#T = 2\#(S^{2n} \times S^{2n+1})$ and $T\#\Sigma^{4n+1} = T$ (Some further relations may exist depending on n such as $T_1(S^3) \approx S^2 \times S^3$). This proves the first statement in Theorem 2.

To prove the second statement we follow Durfee and Kauffman and consider a slightly different Brieskorn–Pham polynomial, namely $z_0^{2k} + z_1^2 + \cdots + z_n^2$. For $k = 1$ we get as before a link $L_{2,2}$ whose middle homology group is free of rank one. Thus, it is diffeomorphic to one of the three generators above by (i) of Lemma 2.1. Now as k varies we have a homological periodicity by (ii) and (iii) of Lemma 2.1. Durfee and Kauffman show that there is an 8–fold diffeomorphism periodicity, and they compute the ψ invariant to show that

$$L_{2,2} \approx T, \quad L_{4,2} \approx (S^{2n} \times S^{2n+1})\#\Sigma^{4n+1},$$

$$L_{6,2} \approx T\#\Sigma^{4n+1} \approx T, \quad L_{8,2} \approx S^{2n} \times S^{2n+1}.$$

This proves Theorem 2. □

Proof of Theorem 3 This is essentially a corollary of [20, Proposition 7.2] where Durfee considers the link K_k of the Brieskorn–Pham polynomial $z_0^k + z_1^3 + z_2^2 + \cdots + z_n^2$ for even $n \geqslant 4$. He shows that $H_n(K_2, \mathbb{Z}) \approx H_n(K_4, \mathbb{Z}) \approx \mathbb{Z}_3$, but that K_2 and K_4 have inequivalent linking forms. Furthermore, K_{6l+2} is diffeomorphic to $K_2\#(-1)^{\frac{n}{2}} l\,\Sigma^{4n-1}$ and K_{6l+4} is diffeomorphic to $K_4\#(-1)^{\frac{n}{2}} l\,\Sigma^{4n-1}$ where $\Sigma^{4n-1} = K_5$ is the Milnor generator. □

Appendix A Computing the signature

There are several known methods for computing the signature of the Milnor fibre F of a Brieskorn manifold in the case when n is odd. This was first accomplished for homotopy spheres by Brieskorn [16] and developed further by Hirzebruch and Zagier [24; 25]. Our discussion follows that in [24]. Let $\mathbf{a} \in (\mathbb{Z}^+)^{n+1}$ and write $\mathbf{a} = (a_0, \cdots, a_n)$. Consider the Brieskorn manifold $M_{\mathbf{a}}$ defined by the link

$$\{z_0^{a_0} + \cdots + z_n^{a_n} = 0\} \cap S^{2n+1}.$$

The Milnor fibre $F_{\mathbf{a}}$ can be represented by the Brieskorn manifold

$$\{\mathbf{z} \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \cdots + z_n^{a_n} = 1\}.$$

For n even the Hirzebruch signature of $F_{\mathbf{a}}$ is given by the function

$$t(\mathbf{a}) = \#\{\mathbf{x} \in \mathbb{Z}^{n+1} \mid 0 < x_k < a_k \text{ and } 0 < \sum_{j=0}^{n} \frac{x_k}{a_k} < 1 \mod 2\}$$

$$-\#\{\mathbf{x} \in \mathbb{Z}^{n+1} \mid 0 < x_k < a_k \text{ and } 1 < \sum_{j=0}^{n} \frac{x_k}{a_k} < 2 \mod 2\}. \tag{A–1}$$

Using methods of Fourier analysis, Zagier has obtain the following formula for $t(\mathbf{a})$:

$$t(\mathbf{a}) = \frac{(-1)^{\frac{n}{2}}}{N} \sum_{j=0}^{N-1} \cot \frac{\pi(2j+1)}{2N} \cot \frac{\pi(2j+1)}{2a_0} \cdots \cot \frac{\pi(2j+1)}{2a_n}, \tag{A–2}$$

where N is any common multiple of the a_i's.

We now adapt this formula to treat the link of the Brieskorn–Pham polynomial of Equation (4–1) with $N = 2(2k+1)$, namely, $\mathbf{a} = (2(2k+1), 2k+1), 2 \cdots, 2)$. Notice that we can always take the N in Zagier's formula (A–2) to be the same as the N in Equation (4–1). In this case we shall denote $t(\mathbf{a})$ by t_d since the degree $d = 2(2k+1)$ is the periodicity as well. Likewise, we denote by t_{2d} the signature $t(\mathbf{a})$ with $\mathbf{a} = (4(2k+1), 2k+1), 2 \cdots, 2)$. We find

$$t_d = \frac{(-1)^{\frac{n}{2}}}{4k+2} \sum_{j=0}^{4k+1} (-1)^j \cot^2 \frac{\pi(2j+1)}{8k+4} \cot \frac{\pi(2j+1)}{4k+2},$$

and

$$t_{2d} = \frac{(-1)^{\frac{n}{2}}}{8k+4} \sum_{j=0}^{8k+3} (-1)^j \cot^2 \frac{\pi(2j+1)}{16k+8} \cot \frac{\pi(2j+1)}{4k+2}.$$

Table 1 $2k\#(S^3 \times S^4)$			
k	τ_k	$D_2(k)$	$\frac{D_2(k)}{\lvert bP_8\rvert}$
1	1	28	1
2	3	28	1
3	6	14	$\frac{1}{2}$
4	10	14	$\frac{1}{2}$
5	15	28	1
6	21	4	$\frac{1}{7}$
7	28	1	$\frac{1}{28}$
8	36	7	$\frac{1}{4}$
9	45	28	1
10	55	28	1
20	210	2	$\frac{1}{14}$
48	1176	1	$\frac{1}{28}$
50	1275	28	1
100	5050	14	$\frac{1}{2}$
496	123256	1	$\frac{1}{28}$
500	125250	14	$\frac{1}{2}$

We want to compute $\tau_k = \frac{|t_{2d}-t_d|}{8}$. After some algebra we find that $(64k+32)\tau_k$ equals

$$\text{(A–3)} \qquad \sum_{j=0}^{8k+3} (-1)^j \cot\frac{\pi(2j+1)}{16k+8}\left(\cot\frac{\pi(2j+1)}{16k+8} - \cot\frac{\pi(2j+1)}{8k+4}\right)\cot\frac{\pi(2j+1)}{4k+2}.$$

Now τ_k is always an integer, and by (A–3) it is independent of n. We now define

$$\text{(A–4)} \qquad D_n(k) = \frac{|bP_{4n}|}{\gcd(\tau_k, |bP_{4n}|)}.$$

By Equation (4–3), $D_n(k)$ represents the number of distinct diffeomorphism types that can be represented by our construction. Using MAPLE we give two tables consisting of a list of τ_k and $D_n(k)$ together with the ratio

$$\frac{D_2(k)}{|bP_8|} = \frac{1}{\gcd(\tau_k, |bP_{4n}|)}$$

Table 2 $2k\#(S^5 \times S^6)$

k	τ_k	$D_3(k)$	$\frac{D_3(k)}{\lvert bP_{12}\rvert}$
1	1	992	1
2	3	992	1
3	6	496	$\frac{1}{2}$
4	10	496	$\frac{1}{2}$
5	15	992	1
6	21	992	1
7	28	248	$\frac{1}{4}$
8	36	248	$\frac{1}{4}$
9	45	992	1
10	55	992	1
31	496	2	$\frac{1}{496}$
48	1176	124	$\frac{1}{8}$
50	1275	992	1
62	1953	32	$\frac{1}{31}$
124	7750	16	$\frac{1}{62}$
248	30876	8	$\frac{1}{124}$
496	123256	4	$\frac{1}{248}$
500	125250	496	$\frac{1}{2}$
992	492528	2	$\frac{1}{496}$

for both the 7–manifolds $\#2k(S^3 \times S^4)$ and the 11–manifolds $\#2k(S^5 \times S^6)$ for various values of k.

Notice that the prime factorization of $|bP_{4n}|$ consists of high powers of two together with odd primes coming from the Bernoulli numbers. Since τ_k is independent of n, this gives rise to a bit of a pattern for the ratios $\frac{D_n(k)}{|bP_{4n}|}$. It is obvious that for $k=1$ all possible diffeomorphism types occur, but this seems also to hold for $k=2$. It is of course true whenever $|bP_{4n}|$ is relatively prime to 3. If we look at the next case namely, bP_{16}, we see that $|bP_{16}| = 8128 = 2^6 \cdot 127$. Comparing this with $|bP_{12}| = 992 = 2^5 \cdot 31$, we see that the same ratios will occur for the case $\#2k(S^7 \times S^8)$ as for $\#2k(S^5 \times S^6)$ for $k = 1, \cdots, 30$. It is interesting to contemplate whether the above gaps in the diffeomorphism types occur as a consequence of our method or whether they indicate an honest obstruction to the existence of positive Sasakian structures. At this stage we have no way of knowing.

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