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Virtually Haken fillings and semi-bundles

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Suppose that M is a fibered three-manifold whose fiber is a surface of positive genus with one boundary component. Assume that M is not a semi-bundle. We show that infinitely many fillings of M along ∂M are virtually Haken. It follows that infinitely many Dehn-surgeries of any non-trivial knot in the three-sphere are virtually Haken.

57M10; 57M25

1 Introduction

In this paper *manifold* will always mean a compact, connected, orientable, possibly bounded, three-manifold. A *bundle* means a manifold which fibers over the circle. A *semi-bundle* is a manifold which is the union of two twisted I–bundles (over connected surfaces) whose intersection is the corresponding ∂I–bundle. An irreducible, ∂–irreducible manifold that contains a properly embedded incompressible surface is called *Haken*. A manifold is *virtually Haken* if has a finite cover that is Haken.

Waldhausen's *virtually Haken conjecture* is that every irreducible closed manifold with infinite fundamental group is virtually Haken. It was shown by Cooper and Long [1] that *most* Dehn-fillings of an atoroidal Haken manifold with torus boundary are virtually Haken provided the manifold is not a bundle.

Theorem 1 *Suppose that M is a bundle with fiber a compact surface F and that F has exactly one boundary component. Also suppose that M is not a semi-bundle and not $S^1 \times D^2$. Then infinitely many Dehn-fillings of M along ∂M are virtually Haken.*

Corollary 2 *Let k be a knot in a homology three-sphere N. Suppose that $N-k$ is irreducible and that k does not bound a disk in N. Then infinitely many Dehn-surgeries along k are virtually Haken.*

The main idea is to construct a surface of *invariant slope* (see Section 3) in a particular finite cover of M. Such surfaces are studied in arbitrary covers using representation theory in a sequel [2]. While writing this paper we noticed that Thurston's theory

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of bundles extends to semi-bundles, and in particular there are manifolds which are semi-bundles in infinitely many ways. We discuss this in the next section.

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2 Bundles and semi-bundles

Various authors have studied semi-bundles, in particular Hempel and Jaco [6] and Zulli [10; 11]. Suppose a manifold has a regular cover which is a surface bundle. We wish to know when a particular fibration in the cover corresponds to a bundle or semi-bundle structure on the quotient. The following has the same flavor as some results of Hass [5].

Theorem 3 *Let M be a compact, connected, orientable, irreducible three-manifold, $p\colon \widetilde{M} \to M$ a finite regular cover, and G the group of covering automorphisms. Suppose that $\phi\colon \widetilde{M} \to S^1$ is a fibration of $\widetilde{M}$ over the circle. Suppose that the cyclic subgroup V of $H^1(\widetilde{M};\mathbb{Z})$ generated by $[\phi]$ is invariant under the action of G. Then one of the following occurs:*

(1) *The action of G on V is trivial. Then M also fibers over the circle. Moreover there is a fibering of M which is covered by a fibering of $\widetilde{M}$ that is isotopic to the original fibering.*

(2) *The action of G on V is non-trivial. Then M is a semi-bundle. Moreover there is a semi-fibering of M which is covered by a fibering of $\widetilde{M}$ that is isotopic to the original fibering.*

Proof Define $N = \ker[\phi_*\colon \pi_1\widetilde{M} \to \pi_1 S^1]$. Since ϕ is a fibration N is finitely generated. If N is cyclic then the fiber is a disc or annulus. In these cases the result is easy. Thus we may assume N is not cyclic. Because V is G–invariant, it follows that N is a normal subgroup of $\pi_1 M$ and $Q = \pi_1 M/N$ is infinite. Using [6, Theorem 3] it follows that M is a bundle or semi-bundle (depending on case 1 or 2) with fiber a compact surface F and N has finite index in $\pi_1 F$. The pull-back of this (semi)fibration of M gives a fibration of $\widetilde{M}$ in the cohomology class of ϕ and is therefore isotopic to the given fibration. □

Suppose that $G \cong (\mathbb{Z}_2)^n$ acts on a real vector space V and let $X = Hom(G,\mathbb{C})$ denote the set of characters on G. Then $X \cong Hom(G,\mathbb{Z}_2)$. For each $\epsilon \in X$ there is a G–invariant *generalized ϵ–eigenspace*

$$V_\epsilon = \{\ v \in V\ :\ \ \forall g \in G\ \ g\cdot v = \epsilon(g)v\ \}.$$

Then V is the direct sum of these subspaces V_ϵ.

Suppose that M is an atoroidal irreducible manifold with boundary consisting of incompressible tori. According to Thurston there is a finite collection (possibly empty), $\mathcal{C} = \{C_1, \cdots, C_k\}$, called *fibered faces*. Each fibered face is the interior of a certain top-dimensional face of the unit ball of the Thurston norm on $H_2(M, \partial M; \mathbb{R})$. It is an open convex set with the property that fibrations of M correspond to rational points in the projectivized space $\mathbb{P}(\cup_i C_i) \subset \mathbb{P}(H_2(M, \partial M; \mathbb{R}))$.

Let $G = H_1(M; \mathbb{Z}/2)$. The regular cover $\widetilde{M}_s$ of M with covering group G is called the $\mathbb{Z}_2$–*universal cover*. Let $\mathcal{D} = \{D_1, \cdots, D_l\}$ be the fibered faces for this cover. For each $\epsilon \in H^1(M; \mathbb{Z}_2)$ there is an ϵ–eigenspace $H_{2,\epsilon}$ of $H_2(\widetilde{M}_s, \partial \widetilde{M}_s; \mathbb{R})$. For each $1 \le i \le l$ and $\epsilon \in H^1(M; \mathbb{Z}_2)$ we call $S_{i,\epsilon} = D_i \cap H_{2,\epsilon}$ a *semi-fibered face* if it is not empty. It is the interior of a compact convex polyhedron whose interior is in the interior of some fibered face for $\widetilde{M}_s$. Let S_i be the union of the $S_{i,\epsilon}$ where ϵ is non-trivial.

Theorem 4 *With the above notation there is a bijection between isotopy classes of semi-fiberings of M and rational points in $\mathbb{P}(\cup_i S_i)$.*

Proof A semi-fibration of M gives such a rational point by considering the induced fibration on $\widetilde{M}_s$. The converse follows from Theorem 3. We leave it as an exercise to check uniqueness up to isotopy. □

We believe that all points in $\mathbb{P}(\cup_i S_i)$ correspond to isotopy classes of non-transversally-orientable, transversally-measured, product-covered 2–dimensional foliations of M. This is true for rational points and therefore holds on a dense open set (using the fact that the set of non-degenerate twisted 1–forms is open). However, since we have no use for this fact, we have not tried very hard to prove it.

Definition A manifold is a *sesqui-bundle* if it is both a bundle and a semi-bundle.

An example is the torus bundle M with monodromy $-\mathrm{Id}$. This is the quotient of Euclidean three-space by the group $\mathcal{G}_2$ (Wolf [8, Theorem 3.5.5]). M has infinitely many semi-fibrations with generic fiber a torus and two Klein-bottle fibers. In addition, M is a bundle thus a sesqui-bundle.

A hyperbolic example may be obtained from M as follows. Let C be a 1–submanifold in M which is a small C^1–perturbation of a finite set of disjoint, immersed, closed geodesics in M chosen so that:

(1) No two components of C cobound an annulus and no component bounds a Mobius strip.

(2) C intersects every flat torus and flat Klein bottle.

(3) Each component of C is transverse to both a chosen fibration and semi-fibration.

Let N be M with a regular neighborhood of C removed. Then the interior of N admits a complete hyperbolic metric. By (3) it is a sesqui-bundle. This answers a question of Zulli who asked in [11] if there are non-Seifert 3–manifolds which are sesqui-bundles.

3 Virtually Haken fillings

The following is well-known, but we include it here for ease of reference.

Lemma 5 *Suppose M is Seifert fibered and has one boundary component . Then one of the following holds:*

(1) *M is $D^2 \times S^1$ or a twisted I–bundle over the Klein bottle.*

(2) *Infinitely many Dehn-fillings are virtually Haken.*

Proof The base orbifold Q has one boundary component and no corners. If $\chi^{\text{orb}} Q > 0$ then Q is a disc with at most one cone point thus $M = D^2 \times S^1$. If $\chi^{\text{orb}} Q = 0$ then Q is a Mobius band or a disc with two cone points labeled 2 and in either case Q has a 2–fold orbifold-cover that is an annulus A. But then M is 2–fold covered by a circle bundle over A. Since M is orientable it follows that this bundle is $S^1 \times A$ and hence M is a twisted I–bundle over the Klein bottle.

Finally, if $\chi^{\text{orb}}(Q) < 0$ then all but one filling of M is Seifert fibered. There are infinitely many fillings of M which give a Seifert fibered space, P, with base orbifold Q' and $\chi^{\text{orb}}(Q') < 0$. There is an orbifold-covering of Q' which is a closed surface of negative Euler characteristic. The induced covering of P contains an essential vertical torus and is therefore virtually Haken. □

Definitions A *slope* on a torus T is the isotopy class of an essential simple closed curve on T. We say that a slope *lifts* to a covering of T if it is represented by a loop which lifts. The following is immediate:

Lemma 6 *Suppose $\widetilde{T} \to T$ is a finite covering. Then the following are equivalent:*

(1) *Some slope on T lifts to $\widetilde{T}$.*

(2) *The covering is finite cyclic.*

(3) *Infinitely many slopes on T lift to $\widetilde{T}$.*

The *distance,* $\Delta(\alpha,\beta)$, between slopes α,β on T is the minimum number of intersection points between representative loops. If α is a slope on a torus boundary component of M then $M(\alpha)$ denotes the manifold obtained by Dehn-filling M using α. A surface S in a manifold M is *essential* if it is compact, connected, orientable, incompressible, properly-embedded, and not boundary-parallel. Let M be a manifold with boundary a torus and $\alpha \subset \partial M$ a slope. Suppose that N is a finite cover of M. An essential surface $S \subset N$ has *invariant slope* α if $\partial S \neq \phi$ and every component of ∂S projects to a loop homotopic to a non-zero multiple of α. We call a finite cover $p\colon N \to M$ a ∂*–cover* if there is an integer $d > 0$ and a homomorphism $\theta\colon \pi_1(\partial M) \to \mathbb{Z}_d$ such that for every boundary component T of N we have $p_*(\pi_1 T) = \ker\theta$. The existence of θ ensures each component of ∂N is the same cyclic cover of ∂M.

The following lemma reduces the proof of the main theorem to constructing an essential non-fiber surface of invariant slope in a ∂-cover of M.

Lemma 7 *Suppose that M is a compact, connected, orientable irreducible 3–manifold with one torus boundary component. Suppose that there is a ∂–cover N of M and an essential non-separating surface $S \subset N$ of invariant slope. Assume that S is not a fiber of a fibration of N. Then M has infinitely many virtually-Haken Dehn-fillings.*

Proof We first remark that the particular case that concerns us in this paper is that M is a bundle with boundary and thus M is irreducible. Since M is irreducible at most 3 fillings give reducible manifolds (Gordon and Luecke [4]). A cover of an irreducible manifold is irreducible (Meeks and Yau [7]). Therefore it suffices to show there are infinitely many fillings of M which have a finite cover containing an essential surface.

If M contains an essential torus then this torus remains incompressible for infinitely many Dehn-fillings by Culler–Gordon–Luecke–Shalen [3, Theorem 2.4.2]. If M is Seifert fibered then by Lemma 5 either the result holds or $M = S^1 \times D^2$ or is a twisted I–bundle over the Klein bottle. The latter two possibilities do not contain a surface S as in the hypotheses. By Thurston's hyperbolization theorem we are reduced to case that M is hyperbolic.

Since $p\colon N \to M$ is a ∂–cover there is $d > 0$ such that every component of ∂N is a d–fold cover of ∂M. Let k be a positive integer coprime to d. Let $p_k\colon \widetilde{N}_k \to N$ be the k–fold cyclic cover dual to S. We claim that there is a homomorphism $\theta_k\colon \pi_1 M \to \mathbb{Z}_{kd}$ such that every slope in $\ker\theta_k$ lifts to every component of $\partial\widetilde{N}_k$.

Assuming this, the filling $M(\gamma)$ of M is covered by a filling, $\widetilde{N}_k(\gamma)$, of $\widetilde{N}_k$ if and only if the slope $\gamma \subset \partial M$ lifts to each component of $\partial\widetilde{N}_k$. Since S is non-separating, by Wu [9, Theorem 5.7], there is $K>0$ such that if $k\geq K$ then there is an essential closed surface $F_k\subset \widetilde{N}_k$ obtained by Freedman tubing two lifts of S. We choose such k coprime to d. By [9, Theorem 5.3], there is a finite set of slopes $\beta_1,\cdots,\beta_n$ on ∂M and $L>0$ so that if $\gamma\subset\partial M$ is a slope and $\Delta(\gamma,\beta_i)\geq L$ for all i then the projection of F_k into $M(\gamma)$ is π_1–injective. Assuming the claim, there are infinitely many slopes $\gamma\in\ker\theta_k$ satisfying these inequalities. For such γ the cover $\widetilde{N}_k(\gamma)\to M(\gamma)$ contains the essential surface F_k.

It only remains to prove the claim. Let T be a component of ∂N and $\beta\subset T$ be the slope given by $S\cap T$. Let $\widetilde{T}$ be a component of $\partial\widetilde{N}_k$ which covers T. The cover $p_k|\colon \widetilde{T}\to T$ is cyclic of degree k' some divisor of k (depending only on $|S\cap T|$). Also β lifts to this cover. Suppose that a slope $\gamma\subset\partial M$ lifts to a slope $\widetilde{\gamma}\subset T$. It follows that $\widetilde{\gamma}$ lifts to $\widetilde{T}$ if k' divides $\Delta(\widetilde{\gamma},\beta)$. If this condition is satisfied by some lift, $\widetilde{\gamma}$, of γ then, since S has invariant slope and $N\to M$ is a ∂–cover, it is satisfied by every such lift.

Let $\widetilde{T}\to T$ be the k'–fold cyclic cover dual to β. Since k' and d are coprime the composite of this cover and the cyclic d–fold cover $T\to\partial M$ is a cyclic cover of degree dk'. By Lemma 6 there are infinitely many slopes on ∂M which lift to $\widetilde{T}$. Every slope on ∂M which lifts to $\widetilde{T}$ also lifts to every component of $\partial\widetilde{N}_k$. This proves the claim. □

Proof of Theorem 1 We attempt to construct S and N as in Lemma 7. The action of the monodromy on $H_1(F;\mathbb{Z}_2)$ has some finite order m. Therefore there is a finite cyclic m–fold cover $W\to M$ such that W is a bundle with fiber F and the action of the monodromy for W on $H_1(F;\mathbb{Z}_2)$ is trivial. We then have

$$H^1(W;\mathbb{Z}_2)\cong H^1(F;\mathbb{Z}_2)\oplus H^1(S^1;\mathbb{Z}_2).$$

Since F has boundary and $F\neq D^2$ we may choose a non-zero element $\phi=(b,0)\in H^1(F;\mathbb{Z}_2)\oplus H^1(S^1;\mathbb{Z}_2)$. This determines a two-fold cover $\widetilde{W}$ of W. Since F has one boundary component, ϕ vanishes on $H_1(\partial W;\mathbb{Z}_2)$, and since W has one boundary component, $\widetilde{W}$ has exactly two boundary components T_1 and T_2. The action of the covering involution, τ, swaps these tori. In particular $\widetilde{W}\to M$ is a ∂–cover.

We claim that there is an essential surface S in $\widetilde{W}$ such that

$$\tau_*[S]=-[S]\neq 0\in H_2(\widetilde{W},\partial\widetilde{W};\mathbb{Z}).$$

Using real coefficients, all cohomology groups have direct-sum decomposition into ± 1 eigenspaces for τ^*; thus $H^1(\partial\widetilde{W};\mathbb{R})=V_+\oplus V_-$. Since τ swaps T_1 and T_2 then, with obvious notation, it swaps μ_1 with μ_2 and λ_1 with λ_2. If $\epsilon = \pm 1$ then V_ϵ has basis $\{\mu_1 + \epsilon\mu_2, \lambda_1 + \epsilon\lambda_2\}$ and thus has dimension 2. Let

$$K = \mathrm{Im}\left[\mathrm{incl}^*\colon\ H^1(\widetilde{W};\mathbb{R}) \to H^1(\partial\widetilde{W};\mathbb{R})\right].$$

Decompose $K = K_+ \oplus K_-$. We claim that $\dim(K_+) = \dim(K_-) = 1$. Since $\dim(K) = 2$ the only other possibilities are that $K_+ = V_+$ or $K_- = V_-$. The intersection pairing on $\partial\widetilde{W}$ is dual to the pairing on $H^1(\partial\widetilde{W},\mathbb{R})$ given by $<\phi,\psi> = (\phi\cup\psi)\cap[\partial\widetilde{W}]$. This pairing vanishes on K. Since $<\mu_1+\epsilon\mu_2,\lambda_1+\epsilon\lambda_2> = 2<\mu_1,\lambda_1> = \pm 2$, the restriction of $<,>$ to each of $V_\pm$ is non-degenerate. This contradicts $K = V_\pm$.

Choose a primitive class $\phi \in H^1(\widetilde{W};\mathbb{Z})$ with $\mathrm{incl}^*\,\phi \in K_-$. Let S be an essential oriented surface in $\widetilde{W}$ representing the class Poincaré dual to ϕ. Then $\tau_*[S] = -[S]$ as required.

The 1–manifold $\alpha_i = T_i \cap \partial S$ with the induced orientation is a 1–cycle in $\partial\widetilde{W}$. Then $[\partial S] = [\alpha_1]+[\alpha_2] \in H_1(\partial\widetilde{W})$. Since T_i is a torus all the components of α_i are parallel. Since $\tau(T_1) = T_2$ all components of ∂S project to isotopic loops in ∂W thus S has invariant slope for the cover $\widetilde{W} \to M$. This gives:

Case (i) If S is not the fiber of a fibration of $\widetilde{W}$ then the result follows from Lemma 7.

Thus we are left with the case that S is the fiber of a fibration of $\widetilde{W}$. Let N be the $\mathbb{Z}_2$–universal covering of W. This is a regular covering and each component of ∂N is a two-fold cover of ∂W. We claim that the composition of coverings $N \to W \to M$ is regular.

Recall that a subgroup $H < G$ is *characteristic* if it is preserved by $Aut(G)$. The $\mathbb{Z}_2$–universal covering $N \to W$ corresponds to the characteristic subgroup $\pi_1 N < \pi_1 W$. The cover $W \to M$ is cyclic and so $\pi_1 W$ is normal in $\pi_1 M$. A characteristic subgroup of a normal subgroup is normal. Hence $\pi_1 N$ is also normal in $\pi_1 M$. This proves the claim. It follows that $N \to M$ is a ∂–cover. A pre-image, $\widetilde{S}$, of S in N is a fiber of a fibration.

Case (ii) Suppose the one-dimensional vector space of $H_2(N,\partial N;\mathbb{R})$ spanned by $[\widetilde{S}]$ is invariant under the group of covering transformations of $N \to M$.

Then, by Theorem 3, M is semi-fibered which contradicts our hypothesis. This completes case (ii). Therefore there is some covering transformation, σ, such that $\sigma_*[\widetilde{S}] \neq \pm[\widetilde{S}]$.

Because $\widetilde{S}$ and $\sigma\widetilde{S}$ are fibers, they both meet every boundary component of N. Since S has invariant slope for the cover $N \to M$ it follows that $\widetilde{S}$ and $\sigma\widetilde{S}$ have the same invariant slope for this cover.

Case (iii) Suppose S is a fiber and $[\partial\widetilde{S}] \neq \pm\sigma_*[\partial\widetilde{S}] \in H_1(\partial N)$.

Given a boundary component of N, there are integers a and b such that the class $a[\widetilde{S}] + b\cdot\sigma_*[\widetilde{S}] \in H_2(N, \partial N)$ is non-zero and represented by an essential surface G that misses this boundary component. Thus G is not a fiber of a fibration. Clearly G has invariant slope. The result now follows from Lemma 7 applied to the surface G in the ∂–cover N. This completes case (iii). The remaining case is:

Case (iv) S is a fiber and there is $\epsilon \in \{\pm 1\}$ with $\sigma_*[\partial\widetilde{S}] = \epsilon\cdot[\partial\widetilde{S}] \in H_1(\partial N)$.

Consideration of the homology exact sequence for the pair $(N, \partial N)$ shows $x = \sigma_*[\widetilde{S}] - \epsilon\cdot[\widetilde{S}] \in H_2(N, \partial N)$ is the image of some $y \in H_2(N)$. Using exactness of the sequence again it follows that $y + i_*H_2(\partial N)$ is not zero in $H_2(N)/i_*H_2(\partial N)$. Hence every filling of N produces a closed manifold with $\beta_2 > 0$. Infinitely many slopes on ∂M *lift* to slopes on ∂N. The result follows. This completes the proof of case (iv) and thus of the Theorem 1. □

Proof of Corollary 2 Let $\eta(K)$ be an open tubular neighborhood of k. By hypothesis the knot exterior $M = N\setminus\eta(K)$ is irreducible. Every semibundle contains two disjoint compact surfaces whose union is non-separating, thus the first Betti number with mod-2 coefficients of a semi-bundle is at least 2. Because N is a homology sphere $H_1(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, therefore M is not a semi-bundle. Since N is a homology sphere it, and therefore M, are orientable.

If M is a bundle with fiber F then, since N is a homology sphere, F has exactly one boundary component. Since k does not bound a disk in N it follows that $M \neq D^2\times S^1$. The result now follows from Theorem 1. If M contains a closed essential surface then infinitely many fillings are Haken, [3, Theorem 2.4.2]. The remaining possibilities are that M is hyperbolic and not a bundle, or else Seifert fibered. The hyperbolic non-bundle case follows from [1].

This leaves the case that M is Seifert fibered. The manifold M is not a twisted I–bundle over the Klein bottle because the latter has mod-2 Betti number 2. The result now follows from Lemma 5. □

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