Holomorphic generating functions for invariants counting coherent sheaves on Calabi–Yau 3–folds

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Let X be a Calabi–Yau 3–fold, $\mathcal{T} = D^b(\mathrm{coh}(X))$ the derived category of coherent sheaves on X, and $\mathrm{Stab}(\mathcal{T})$ the complex manifold of Bridgeland stability conditions on $\mathcal{T}$. It is conjectured that one can define invariants $J^\alpha(Z, \mathcal{P}) \in \mathbb{Q}$ for $(Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{T})$ and $\alpha \in K(\mathcal{T})$ generalizing Donaldson–Thomas invariants, which "count" $(Z, \mathcal{P})$–semistable (complexes of) coherent sheaves on X, and whose transformation law under change of $(Z, \mathcal{P})$ is known.

This paper explains how to combine such invariants $J^\alpha(Z, \mathcal{P})$, if they exist, into a family of holomorphic generating functions $F^\alpha \colon \mathrm{Stab}(\mathcal{T}) \to \mathbb{C}$ for $\alpha \in K(\mathcal{T})$. Surprisingly, requiring the F^α to be continuous and holomorphic determines them essentially uniquely, and implies they satisfy a p.d.e., which can be interpreted as the flatness of a connection over $\mathrm{Stab}(\mathcal{T})$ with values in an infinite-dimensional Lie algebra $\mathcal{L}$.

The author believes that underlying this mathematics there should be some new physics, in String Theory and Mirror Symmetry. String Theorists are invited to work out and explain this new physics.

14J32; 14D20, 18E30

1 Introduction

To set the scene we start with an analogy, explained by McDuff and Salamon [17]. If (M, ω) is a compact symplectic manifold, one can define the *Gromov–Witten invariants* $\Phi_A(\alpha, \beta, \gamma)$ of M. For $A \in H_2(M, \mathbb{Z})$ and $\alpha, \beta, \gamma \in H_{\mathrm{even}}(M, \mathbb{Z})$, to define $\Phi_A(\alpha, \beta, \gamma) \in \mathbb{Q}$ we choose an almost complex structure J on M compatible with ω and cycles $C_\alpha, C_\beta, C_\gamma$ in M representing α, β, γ, and then $\Phi_A(\alpha, \beta, \gamma)$ is roughly speaking the "number" of J–holomorphic rational curves Σ in M intersecting $C_\alpha, C_\beta, C_\gamma$, with $[\Sigma] = A$ in $H_2(M, \mathbb{Z})$. It is independent of the choices of J and $C_\alpha, C_\beta, C_\gamma$.

It is natural to encode the Gromov–Witten invariants in a *holomorphic generating function* $\mathcal{S} \colon H^{\mathrm{even}}(M, \mathbb{C}) \to \mathbb{C}$ called the *Gromov–Witten potential*, given by a (formal)

power series with coefficients the $\Phi_A(\alpha, \beta, \gamma)$. Identities on the $\Phi_A(\alpha, \beta, \gamma)$ imply that $\mathcal{S}$ satisfies a p.d.e., the *WDVV equation*. This p.d.e. can be interpreted as the flatness of a 1–parameter family of connections defined using $\mathcal{S}$, which make $H^{\mathrm{even}}(M, \mathbb{C})$ into a *Frobenius manifold*.

The goal of this paper (which we do not achieve) is to tell a story with many similar features. Let X be a Calabi–Yau 3–fold, $\mathrm{coh}(X)$ the abelian category of coherent sheaves on X, and $\mathcal{T} = D^b(\mathrm{coh}(X))$ its bounded derived category. Let $K(\mathcal{T})$ be the image of the Chern character map $K_0(\mathcal{T}) \to H^{\mathrm{even}}(X, \mathbb{Q})$, a lattice of finite rank. Define the *Euler form* $\bar{\chi}\colon K(\mathcal{T}) \times K(\mathcal{T}) \to \mathbb{Z}$ by

$$\sum_{k \in \mathbb{Z}}(-1)^k \dim \mathrm{Hom}_{\mathcal{T}}(U, V[k]) = \bar{\chi}([U], [V]) \quad \text{for all } U, V \in \mathcal{T}.$$

Then $\bar{\chi}$ is biadditive, antisymmetric, and nondegenerate.

Following Bridgeland [4] one can define *stability conditions* $(Z, \mathcal{P})$ on the triangulated category $\mathcal{T}$, consisting of a group homomorphism $Z\colon K(\mathcal{T}) \to \mathbb{C}$ called the *central charge* and extra data $\mathcal{P}$ encoding the $(Z, \mathcal{P})$-*semistable objects* in $\mathcal{T}$. The family of stability conditions $\mathrm{Stab}(\mathcal{T})$ is a finite-dimensional complex manifold, with the map $(Z, \mathcal{P}) \mapsto Z$ a local biholomorphism $\mathrm{Stab}(\mathcal{T}) \to \mathrm{Hom}(K(\mathcal{T}), \mathbb{C})$. In String Theory terms, the "stringy Kähler moduli space" of X should be thought of as a complex Lagrangian submanifold of $\mathrm{Stab}(\mathcal{T})$, the subset of stability conditions represented by Super Conformal Field Theories.

We would like to define invariants $J^\alpha(Z, \mathcal{P}) \in \mathbb{Q}$ "counting" $(Z, \mathcal{P})$–semistable objects in each class $\alpha \in K(\mathcal{T}) \setminus \{0\}$, so roughly counting semistable sheaves. In the final version of the theory these should be extensions of *Donaldson–Thomas invariants* [6; 20] and invariant under deformations of X, but for the present we may make do with the author's "motivic" invariants defined in [15] for the abelian category case, which are not invariant under deformations of X.

The important thing about the invariants $J^\alpha(Z, \mathcal{P})$ is that their transformation laws under change of stability condition are known completely, and described in the abelian case in [15]. Basically $J^\alpha(Z, \mathcal{P})$ is a locally constant function of $(Z, \mathcal{P})$, except that when $\alpha = \alpha_1 + \cdots + \alpha_n$ for $\alpha_k \in K(\mathcal{T})$ and $(Z, \mathcal{P})$ crosses a locus in $\mathrm{Stab}(\mathcal{T})$ where $Z(\alpha_1), \ldots, Z(\alpha_n) \in \mathbb{C} \setminus \{0\}$ all have the same phase, then $J^\alpha(Z, \mathcal{P})$ jumps by a multiple of $J^{\alpha_1}(Z, \mathcal{P}) \cdots J^{\alpha_n}(Z, \mathcal{P})$.

This paper studies the problem of how best to combine such invariants $J^\alpha(Z, \mathcal{P})$ into generating functions which should be continuous, holomorphic functions of $(Z, \mathcal{P})$ on $\mathrm{Stab}(\mathcal{T})$, a bit like the Gromov–Witten potential. In fact we shall define a function

$F^\alpha\colon \mathrm{Stab}(\mathcal{T}) \to \mathbb{C}$ for each $\alpha \in K(\mathcal{T}) \setminus \{0\}$, given by

(1)
$$
F^\alpha(Z, \mathcal{P}) = \sum_{\substack{n \geqslant 1,\, \alpha_1, \ldots, \alpha_n \in K(\mathcal{T}) \setminus \{0\}: \\ \alpha_1 + \cdots + \alpha_n = \alpha,\, Z(\alpha_k) \neq 0 \text{ all } k}} F_n\big(Z(\alpha_1), \ldots, Z(\alpha_n)\big) \prod_{i=1}^{n} J^{\alpha_i}(Z, \mathcal{P}) \cdot
$$
$$
\left[\frac{1}{2^{n-1}} \sum_{\substack{\text{connected, simply connected digraphs } \Gamma: \\ \text{vertices } \{1, \ldots, n\},\, \text{edge } \bullet^i \to \bullet^j \text{ implies } i < j}} \prod_{\substack{\text{edges} \\ \bullet^i \to \bullet^j \\ \text{in } \Gamma}} \bar{\chi}(\alpha_i, \alpha_j) \right],
$$

where $F_n\colon (\mathbb{C}^\times)^n \to \mathbb{C}$ are some functions to be determined, and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Here the sum over graphs comes from the transformation laws (25) below for the $J^\alpha(Z, \mathcal{P})$, determined in the abelian case in [15, Section 6.5].

Let us admit at once that there are two very major issues about (1) that this paper does not even attempt to solve, which is why the goals of the paper are not achieved. The first is that we do not define the invariants $J^\alpha(Z, \mathcal{P})$. In the abelian category case $\mathcal{A} = \mathrm{coh}(X)$, for Gieseker type stability conditions $(\tau, T, \leqslant)$, we do define and study such invariants $J^\alpha(\tau)$ in [15]. But the extension to Bridgeland stability conditions on $D^b(\mathrm{coh}(X))$ still requires a lot of work.

The second issue is the *convergence* of the infinite sum (1) and of other infinite sums below. I am not at all confident about this: it may be that (1) does not converge at all, or does so only in special limiting corners of $\mathrm{Stab}(\mathcal{T})$, and I am not going to conjecture that (1) or other sums converge. Instead, we shall simply treat our sums as convergent. This means that the results of this paper are rigorous and the sums known to converge only in rather restricted situations: working with abelian categories $\mathcal{A}$ rather than triangulated categories $\mathcal{T}$, and imposing finiteness conditions on $\mathcal{A}$ that do not hold for coherent sheaves $\mathcal{A} = \mathrm{coh}(X)$, but do work for categories of quiver representations $\mathcal{A} = \mathrm{mod}\text{–}\mathbb{K}Q$.

The question we do actually answer in this paper is the following. Suppose for the moment that (1) converges in as strong a sense as necessary. What are the conditions on the functions $F_n(z_1, \ldots, z_n)$ for F^α to be both continuous and holomorphic? Since the $J^\alpha(Z, \mathcal{P})$ are not continuous in $(Z, \mathcal{P})$, to make F^α continuous the F_n must have discontinuities chosen so that the jumps in $J^\alpha(Z, \mathcal{P})$ and F_n exactly cancel. The simplest example of this is that across the real hypersurface $z_{l+1}/z_l \in (0, \infty)$ in $(\mathbb{C}^\times)^n$, the function $F_n(z_1, \ldots, z_n)$ must jump by $F_{n-1}(z_1, \ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2}, \ldots, z_n)$.

We shall show that the condition that F^α be holomorphic and continuous, plus a few extra assumptions on the symmetry and growth of the F_n and the normalization $F_1 \equiv (2\pi i)^{-1}$, actually determine the F_n *uniquely*. Furthermore, on the open subset

of $(\mathbb{C}^\times)^n$ where F_n is continuous it satisfies the p.d.e.

$$(2) \qquad dF_n(z_1, \ldots, z_n) = \sum_{k=1}^{n-1} F_k(z_1, \ldots, z_k) F_{n-k}(z_{k+1}, \ldots, z_n) \cdot \left[\frac{dz_{k+1} + \cdots + dz_n}{z_{k+1} + \cdots + z_n} - \frac{dz_1 + \cdots + dz_k}{z_1 + \cdots + z_k} \right].$$

This in turn implies that the generating functions F^α satisfy the p.d.e.

$$(3) \quad dF^\alpha(Z, \mathcal{P}) = - \sum_{\beta, \gamma \in K(\mathcal{T}) \setminus \{0\} : \alpha = \beta + \gamma} \overline{\chi}(\beta, \gamma) F^\beta(Z, \mathcal{P}) F^\gamma(Z, \mathcal{P}) \frac{d(Z(\beta))}{Z(\beta)}.$$

It seems remarkable that simply requiring the F^α to be holomorphic and continuous implies they must satisfy the p.d.e. (3), which has appeared more-or-less out of nowhere. In the Gromov–Witten case the generating function $\mathcal{S}$ also satisfies a p.d.e., the WDVV equation. Note however that the WDVV equation holds because of identities upon Gromov–Witten invariants, but in our case (3) holds because of any identities not on the $J^\alpha(Z, \mathcal{P})$ for fixed $(Z, \mathcal{P})$, but rather because of identities on how the $J^\alpha(Z, \mathcal{P})$ transform as $(Z, \mathcal{P})$ changes.

Just as the WDVV equation implies the flatness of a connection constructed using the Gromov–Witten potential, so we can interpret (3) in terms of flat connections. Define $\mathcal{L}$ to be the $\mathbb{C}$–Lie algebra with basis formal symbols c^α for $\alpha \in K(\mathcal{T})$ and Lie bracket $[c^\alpha, c^\beta] = \overline{\chi}(\alpha, \beta) c^{\alpha + \beta}$. Ignoring questions of convergence, define an $\mathcal{L}$–valued connection matrix Γ on $\mathrm{Stab}(\mathcal{T})$ by

$$\Gamma(Z, \mathcal{P}) = \sum_{\alpha \in K(\mathcal{T}) \setminus \{0\}} F^\alpha(Z, \mathcal{P}) \, c^\alpha \otimes \frac{d(Z(\alpha))}{Z(\alpha)}.$$

Then (3) implies that Γ is *flat*, that is, the curvature $R_\Gamma = d\Gamma + \frac{1}{2}\Gamma \wedge \Gamma \equiv 0$. But we do not expect that $d\Gamma \equiv 0$ and $\Gamma \wedge \Gamma \equiv 0$ as happens in the Gromov–Witten case, so we do not have a 1–parameter family of flat connections and a Frobenius manifold type structure.

All this cries out for an explanation, but I do not have one. However, I am convinced that the explanation should be sought in String Theory, and that underlying this is some new piece of physics to do with Mirror Symmetry, just as the context of the derived category $D^b(\mathrm{coh}(X))$ of coherent sheaves on X is the core of the Homological Mirror Symmetry programme of Kontsevich [16]. I invite any physicists with ideas on its interpretation to please let me know.

Two possible pointers towards an interpretation are discussed in Section 6. Firstly, ignoring convergence issues, we show that in the Calabi–Yau 3–fold triangulated

category case the connection Γ above induces a flat connection on $T\,\mathrm{Stab}(\mathcal{T})$, which is in fact the Levi-Civita connection of a flat holomorphic metric $g_{\mathbb{C}}$ on $\mathrm{Stab}(\mathcal{T})$, provided $g_{\mathbb{C}}$ is nondegenerate. Secondly, again ignoring convergence issues, for $\lambda \in \mathbb{C}^{\times}$ and fixed $a, b \in \mathbb{Z}$ define a $(0, 1)$–form on $\mathrm{Stab}(\mathcal{T})$ by

$$\Phi_{\lambda}(Z, \mathcal{P}) = \sum_{\alpha \in K(\mathcal{T}) \setminus \{0\}} \lambda^a e^{\lambda^b Z(\alpha)} \, \overline{F^{\alpha}(Z, \mathcal{P})} \, \frac{\overline{\mathrm{d}(Z(\alpha))}}{\overline{Z(\alpha)}}.$$

Then (3) implies an equation in $(0, 2)$–forms on $\mathrm{Stab}(\mathcal{T})$:

$$\left(\overline{\partial}\Phi_{\lambda}(Z, \mathcal{P})\right)_{\overline{i}\,\overline{j}} = -\tfrac{1}{2}\lambda^{-a-2b}(\overline{\chi})^{ij}\left(\partial\Phi_{\lambda}(Z, \mathcal{P})\right)_{i\overline{i}}\left(\partial\Phi_{\lambda}(Z, \mathcal{P})\right)_{j\overline{j}},$$

using complex tensor index notation, where $(\overline{\chi})^{ij}$ is the $(2, 0)$ part of $\overline{\chi}$. This is a little similar to the *holomorphic anomaly equation* of Bershadsky et al [1; 2].

Here is a brief description of the paper. Despite this introduction we mostly work neither with Calabi–Yau 3–folds, nor with triangulated categories. Instead, we work with abelian categories $\mathcal{A}$ such as quiver representations mod–$\mathbb{K}Q$, and slope stability conditions $(\mu, \mathbb{R}, \leqslant)$ determined by a morphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$. Then we can use the author's series [12; 13; 14; 15] on invariants counting μ–semistable objects in abelian categories; the facts we need are summarized in Section 2. Section 3 studies generating functions f^{α} generalizing F^{α} in (1), in the abelian category setting, and expressed in terms of Lie algebras $\mathcal{L}$ following [12; 13; 14; 15].

In Section 3.1 we find conditions on F_n for these f^{α} to be holomorphic and continuous, including some conditions from the triangulated category case, and show that with a few extra assumptions any such functions F_n are unique. In Section 3.2 we guess a p.d.e. generalizing (3) for the f^{α} to satisfy, deduce that it implies (2), and use (2) to construct a family of functions F_n by induction on n. Then Section 3.3 shows that these F_n constructed using (2) satisfy all the conditions of Section 3.1, and so are unique. Section 4 discusses $\mathcal{L}$–valued flat connections Γ as above, and Section 5 the extension to triangulated categories. Finally, Section 6 explains how the ideas work out for Calabi–Yau 3–folds.

2 Background material

The author has written six long, complicated papers [10; 11; 12; 13; 14; 15] developing a framework for studying stability conditions $(\tau, T, \leqslant)$ on an abelian category $\mathcal{A}$, and

interesting invariants counting τ–semistable objects in $\mathcal{A}$, and the transformation laws of these invariants under change of stability condition. Section 2.1–Section 2.2 explain only the minimum necessary for this paper; for more detail, see [10; 11; 12; 13; 14; 15]. Section 2.3 discusses the extension to triangulated categories.

2.1 The general set-up of [12; 13; 14; 15]

We start with a very brief summary of parts of the author's series [12; 13; 14; 15]. Here [12, Assumptions 7.1 and 8.1] is the data we require.

Assumption 2.1 Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{A}$ a $\mathbb{K}$–linear noetherian abelian category with $\text{Ext}^i(A, B)$ finite-dimensional $\mathbb{K}$–vector spaces for all $A, B \in \mathcal{A}$ and $i \geqslant 0$. Let $K(\mathcal{A})$ be the quotient of the Grothendieck group $K_0(\mathcal{A})$ by some fixed subgroup. Suppose that if $A \in \mathcal{A}$ with $[A] = 0$ in $K(\mathcal{A})$ then $A \cong 0$.

To define moduli stacks of objects or configurations in $\mathcal{A}$, we need some *extra data*, to tell us about algebraic families of objects and morphisms in $\mathcal{A}$, parametrized by a base scheme U. We encode this extra data as a *stack in exact categories* $\mathfrak{F}_\mathcal{A}$ on the *category of $\mathbb{K}$–schemes* $\text{Sch}_\mathbb{K}$, made into a *site* with the *étale topology*. The $\mathbb{K}, \mathcal{A}, K(\mathcal{A}), \mathfrak{F}_\mathcal{A}$ must satisfy some complex additional conditions [12, Assumptions 7.1 and 8.1], which we do not give.

In [12, Section 9–Section 10] we define data $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_\mathcal{A}$ satisfying Assumption 2.1 in some large classes of examples, including the abelian category $\text{coh}(X)$ of coherent sheaves on a projective $\mathbb{K}$–scheme X, and the following:

Example 2.2 A *quiver* Q is a finite directed graph. That is, Q is a quadruple (Q_0, Q_1, b, e), where Q_0 is a finite set of *vertices*, Q_1 is a finite set of *arrows*, and $b, e: Q_1 \to Q_0$ are maps giving the *beginning* and *end* of each arrow.

A *representation* (V, ρ) of Q consists of finite-dimensional $\mathbb{K}$–vector spaces V_v for each $v \in Q_0$, and linear maps $\rho_a: V_{b(a)} \to V_{e(a)}$ for each $a \in Q_1$. A *morphism* of representations $\phi: (V, \rho) \to (W, \sigma)$ consists of $\mathbb{K}$–linear maps $\phi_v: V_v \to W_v$ for all $v \in Q_0$ with $\phi_{e(a)} \circ \rho_a = \sigma_a \circ \phi_{b(a)}$ for all $a \in Q_1$. Write mod–$\mathbb{K}Q$ for the *abelian category of representations* of Q. It is of finite length.

Write $\mathbb{N}^{Q_0}$ and $\mathbb{Z}^{Q_0}$ for the sets of maps $Q_0 \to \mathbb{N}$ and $Q_0 \to \mathbb{Z}$, where $\mathbb{N} = \{0, 1, 2, \ldots\} \subset \mathbb{Z}$. Define the *dimension vector* $\dim(V, \rho) \in \mathbb{N}^{Q_0} \subset \mathbb{Z}^{Q_0}$ of a representation $(V, \rho) \in \text{mod–}\mathbb{K}Q$ by $\dim(V, \rho): v \mapsto \dim_\mathbb{K} V_v$. This induces a surjective group homomorphism dim: $K_0(\text{mod–}\mathbb{K}Q) \to \mathbb{Z}^{Q_0}$. Define $K(\text{mod–}\mathbb{K}Q)$ to be the quotient of $K_0(\text{mod–}\mathbb{K}Q)$ by the kernel of dim. Then $K(\text{mod–}\mathbb{K}Q) \cong \mathbb{Z}^{Q_0}$, and

for simplicity we identify $K(\text{mod}-\mathbb{K}Q)$ and $\mathbb{Z}^{Q_0}$, so that for $(V, \rho) \in \text{mod}-\mathbb{K}Q$ the class $[(V, \rho)]$ in $K(\text{mod}-\mathbb{K}Q)$ is $\dim(V, \rho)$. As in [12, Example 10.5] we can define a stack in exact categories $\mathfrak{F}_{\text{mod}-\mathbb{K}Q}$ so that $\mathcal{A} = \text{mod}-\mathbb{K}Q$, $K(\text{mod}-\mathbb{K}Q)$, $\mathfrak{F}_{\text{mod}-\mathbb{K}Q}$ satisfy Assumption 2.1.

We will need the following notation [12, Definition 7.3], [14, Definition 3.8]:

Definition 2.3 We work in the situation of Assumption 2.1. Define

(4) $$C(\mathcal{A}) = \{[U] \in K(\mathcal{A}) : U \in \mathcal{A}, \; U \ncong 0\} \subset K(\mathcal{A}),$$

and $\bar{C}(\mathcal{A}) = C(\mathcal{A}) \cup \{0\}$. That is, $C(\mathcal{A})$ is the set of classes in $K(\mathcal{A})$ of nonzero objects $U \in \mathcal{A}$, and $\bar{C}(\mathcal{A})$ the set of classes of objects in $\mathcal{A}$. We think of $C(\mathcal{A})$ as the "positive cone" and $\bar{C}(\mathcal{A})$ as the "closed positive cone" in $K(\mathcal{A})$. In Example 2.2 we have $\bar{C}(\mathcal{A}) = \mathbb{N}^{Q_0}$ and $C(\mathcal{A}) = \mathbb{N}^{Q_0} \setminus \{0\}$.

A set of $\mathcal{A}$-*data* is a triple $(I, \preceq, \kappa)$ such that $(I, \preceq)$ is a finite partially ordered set (poset) and $\kappa: I \to C(\mathcal{A})$ a map. In this paper we will be interested only in the case when $\preceq$ is a total order, so that $(I, \preceq)$ is uniquely isomorphic to $(\{1, \ldots, n\}, \leqslant)$ for $n = |I|$. We *extend κ to the set of subsets of I* by defining $\kappa(J) = \sum_{j \in J} \kappa(j)$. Then $\kappa(J) \in C(\mathcal{A})$ for all $\varnothing \neq J \subseteq I$, as $C(\mathcal{A})$ is closed under addition.

Then [12, Section 7] defines *moduli stacks* $\mathfrak{Obj}_{\mathcal{A}}$ of objects in $\mathcal{A}$, and $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ of objects in $\mathcal{A}$ with class α in $K(\mathcal{A})$, for each $\alpha \in \bar{C}(\mathcal{A})$. They are Artin $\mathbb{K}$–stacks, locally of finite type, with $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ an open and closed $\mathbb{K}$–substack of $\mathfrak{Obj}_{\mathcal{A}}$. The underlying geometric spaces $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ are the sets of isomorphism classes of objects U in $\mathcal{A}$, with $[U] = \alpha$ for $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$.

In [14, Section 4] we study (*weak*) *stability conditions* on $\mathcal{A}$, generalizing Rudakov [19]. The next three definitions are taken from [14, Definitions 4.1–4.3, 4.6 and 4.7].

Definition 2.4 Let Assumption 2.1 hold and $C(\mathcal{A})$ be as in (4). Suppose $(T, \leqslant)$ is a totally ordered set, and $\tau: C(\mathcal{A}) \to T$ a map. We call $(\tau, T, \leqslant)$ a *stability condition* on $\mathcal{A}$ if whenever $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$, or $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$, or $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$. We call $(\tau, T, \leqslant)$ a *weak stability condition* on $\mathcal{A}$ if whenever $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) \leqslant \tau(\beta) \leqslant \tau(\gamma)$, or $\tau(\alpha) \geqslant \tau(\beta) \geqslant \tau(\gamma)$.

Definition 2.5 Let $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$, $K(\mathcal{A})$ as above. Then we say that a nonzero object U in $\mathcal{A}$ is

(i) τ-*semistable* if for all $S \subset U$ with $S \ncong 0, U$ we have $\tau([S]) \leqslant \tau([U/S])$;

(ii) τ-*stable* if for all $S \subset U$ with $S \ncong 0, U$ we have $\tau([S]) < \tau([U/S])$;

(iii) τ-*unstable* if it is not τ–semistable.

Definition 2.6 Let Assumption 2.1 hold and $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$. For $\alpha \in C(\mathcal{A})$ define

$$\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau) = \big\{[U] \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K}) : U \text{ is } \tau\text{–semistable}\big\} \subset \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}).$$

Write $\delta_{\mathrm{ss}}^{\alpha}(\tau)\colon \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \to \{0, 1\}$ for its characteristic function.

We call $(\tau, T, \leqslant)$ a *permissible* weak stability condition if

 (i) $\mathcal{A}$ is τ-*artinian*, that is, there are no chains of subobjects $\cdots \subset A_2 \subset A_1 \subset U$ in $\mathcal{A}$ with $A_{n+1} \neq A_n$ and $\tau([A_{n+1}]) \geqslant \tau([A_n/A_{n+1}])$ for all n;

(ii) $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ is a *constructible set* in $\mathfrak{Obj}_{\mathcal{A}}$ for all $\alpha \in C(\mathcal{A})$, using the theory of constructible sets and functions on Artin $\mathbb{K}$–stacks developed in [10].

Examples of (weak) stability conditions on $\mathcal{A} = \mathrm{mod}\text{–}\mathbb{K}Q$ and $\mathcal{A} = \mathrm{coh}(X)$ are given in [14, Section 4.3–Section 4.4]. Most of them are permissible. Here is [14, Example 4.14].

Example 2.7 Let Assumption 2.1 hold, and $c, r\colon K(\mathcal{A}) \to \mathbb{R}$ be group homomorphisms with $r(\alpha) > 0$ for all $\alpha \in C(\mathcal{A})$. Define $\mu\colon C(\mathcal{A}) \to \mathbb{R}$ by $\mu(\alpha) = c(\alpha)/r(\alpha)$ for $\alpha \in C(\mathcal{A})$. Then μ is called a *slope function* on $K(\mathcal{A})$, and $(\mu, \mathbb{R}, \leqslant)$ is a *stability condition* on $\mathcal{A}$.

It will be useful later to re-express this as follows. Define the *central charge* $Z\colon K(\mathcal{A}) \to \mathbb{C}$ by $Z(\alpha) = -c(\alpha) + i\,r(\alpha)$. The name will be explained in Section 2.3. Then $Z \in \mathrm{Hom}\big(K(\mathcal{A}), \mathbb{C}\big)$ is a group homomorphism, and maps $C(\mathcal{A})$ to the upper half plane $H = \{x + iy : x \in \mathbb{R}, \ y > 0\}$ in $\mathbb{C}$.

For $\alpha \in C(\mathcal{A})$, the argument $\arg \circ Z(\alpha)$ lies in $(0, \pi)$, and $\mu(\alpha) = -\cot \circ \arg \circ Z(\alpha)$, where cot is the cotangent function. So $(\mu, \mathbb{R}, \leqslant)$ can be recovered from Z. Since $-\cot\colon (0, \pi) \to \mathbb{R}$ is strictly increasing, it fixes orders in $\mathbb{R}$. Thus $(\arg \circ Z, \mathbb{R}, \leqslant)$ is an equivalent stability condition to $(\mu, \mathbb{R}, \leqslant)$, that is, $U \in \mathcal{A}$ is μ-(semi)stable if and only if it is $\arg \circ Z$-(semi)stable. Write

$$(5) \qquad \begin{aligned} \mathrm{Stab}(\mathcal{A}) = \big\{ &Z \in \mathrm{Hom}(K(\mathcal{A}), \mathbb{C}) : Z(C(\mathcal{A})) \subset H, \text{ and the stability} \\ &\text{condition } (\mu, \mathbb{R}, \leqslant) \text{ defined by } Z \text{ is permissible}\big\}. \end{aligned}$$

In the cases we are interested in $\mathrm{Stab}(\mathcal{A})$ is an *open* subset of the complex vector space $\mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$, and so is a *complex manifold*.

Such stability conditions can be defined on all the quiver examples of [12, Section 10], and they are automatically *permissible* by [14, Corollary 4.13]. In Example 2.2, as $K(\mathcal{A}) = \mathbb{Z}^{Q_0}$ and $C(\mathcal{A}) = \mathbb{N}^{Q_0} \setminus \{0\}$ we may write c, r as

$$c(\alpha) = \sum_{v \in Q_0} c_v \, (\dim \alpha)(v) \quad \text{and} \quad r(\alpha) = \sum_{v \in Q_0} r_v \, (\dim \alpha)(v),$$

where $c_v \in \mathbb{R}$ and $r_v \in (0, \infty)$ for all $v \in Q_0$. Thus $\text{Stab}(\mathcal{A}) = H^{Q_0} \subset \mathbb{C}^{Q_0}$.

The usual notion of slope stability on $\mathcal{A} = \text{coh}(X)$ for X a smooth projective curve is a slight generalization of the above. We take $c([U])$ to be the *degree* and $r([U])$ the *rank* of $U \in \text{coh}(X)$. But then for $\alpha \in C(\mathcal{A})$ coming from a torsion sheaf U we have $r(\alpha) = 0$ and $c(\alpha) > 0$, so we must allow μ to take values in $(-\infty, +\infty]$, with $\mu(\alpha) = +\infty$ if $r(\alpha) = 0$.

Here [14, Theorem 4.4] is a useful property of weak stability conditions. We call $0 = A_0 \subset \cdots \subset A_n = U$ in Theorem 2.8 the *Harder–Narasimhan filtration* of U.

Theorem 2.8 *Let $(\tau, T, \leqslant)$ be a weak stability condition on an abelian category $\mathcal{A}$. Suppose $\mathcal{A}$ is noetherian and τ–artinian. Then each $U \in \mathcal{A}$ admits a unique filtration $0 = A_0 \subset \cdots \subset A_n = U$ for $n \geqslant 0$, such that $S_k = A_k / A_{k-1}$ is τ–semistable for $k = 1, \ldots, n$, and $\tau([S_1]) > \tau([S_2]) > \cdots > \tau([S_n])$.*

2.2 A framework for discussing counting invariants

Given $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$ satisfying Assumption 2.1 and weak stability conditions $(\tau, T, \leqslant)$, $(\tilde{\tau}, \tilde{T}, \leqslant)$ on $\mathcal{A}$, the final paper [15] in the series was mostly concerned with defining interesting *invariants* $I_{\text{ss}}^{\alpha}(\tau), J^{\alpha}(\tau), \ldots$ which "count" τ–semistable objects in class α for all $\alpha \in C(\mathcal{A})$, and computing the *transformation laws* which these invariants satisfy under changing from $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$.

These different invariants all share a common structure, involving an algebra and a Lie algebra. We will now abstract this structure (which was not done in [15]) and express the various invariants of [15] as examples of this structure. We first explain, using an example, how transformation between stability conditions can be written in terms of identities in an algebra.

Example 2.9 Let Assumption 2.1 hold with $\mathbb{K}$ of characteristic zero. Write $\text{CF}(\mathfrak{Obj}_{\mathcal{A}})$ for the vector space of $\mathbb{Q}$–valued *constructible functions* on $\mathfrak{Obj}_{\mathcal{A}}$. In [13] we defined an associative, noncommutative multiplication $*$ on $\text{CF}(\mathfrak{Obj}_{\mathcal{A}})$, such that for $f, g \in \text{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $V \in \mathcal{A}$, $(f * g)([V])$ is the "integral" over all short exact sequences $0 \to U \to V \to W \to 0$ in $\mathcal{A}$ of $f([U])g([W])$, with respect to a measure defined using the Euler characteristic of constructible subsets of $\mathbb{K}$–stacks. The

identity in $CF(\mathfrak{Obj}_{\mathcal{A}})$ is $1 = \delta_{[0]}$, the characteristic function of $[0] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Thus, $CF(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-*algebra*.

Suppose $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is of *finite type* for all $\alpha \in C(\mathcal{A})$. This holds in Example 2.2, for instance. Then the set of $\mathbb{K}$–points $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ of $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is a constructible set in $\mathfrak{Obj}_{\mathcal{A}}$. Write $\delta_{\mathrm{all}}^{\alpha} \in CF(\mathfrak{Obj}_{\mathcal{A}})$ for the characteristic function of $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$, that is, $\delta_{\mathrm{all}}^{\alpha}$ is the characteristic function of all objects in class α in $\mathcal{A}$.

Let $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$, and as in Definition 2.6 write $\delta_{\mathrm{ss}}^{\alpha}(\tau)$ for the characteristic function of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$, that is, the characteristic function of all τ–semistable objects in class α in $\mathcal{A}$. As $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is of finite type and $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ is open, $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ is constructible, and so $\delta_{\mathrm{ss}}^{\alpha}(\tau) \in CF(\mathfrak{Obj}_{\mathcal{A}})$.

We can rewrite Theorem 2.8 as an identity in $CF(\mathfrak{Obj}_{\mathcal{A}})$ for $\alpha \in C(\mathcal{A})$:

$$(6) \qquad \delta_{\mathrm{all}}^{\alpha} = \sum_{\substack{n \geqslant 1,\, \alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\\ \alpha = \alpha_1 + \cdots + \alpha_n,\ \tau(\alpha_1) > \tau(\alpha_2) > \cdots > \tau(\alpha_n)}} \delta_{\mathrm{ss}}^{\alpha_1}(\tau) * \delta_{\mathrm{ss}}^{\alpha_2}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\alpha_n}(\tau).$$

Here if $U \in \mathcal{A}$ then both sides of (6) are zero at U if $[U] \neq \alpha$ in $C(\mathcal{A})$, and if $[U] = \alpha$ then the left hand side of (6) is 1 at U, and exactly one term on the right hand side is 1, coming from the Harder–Narasimhan filtration of U with $\alpha_i = [S_i]$ in $C(\mathcal{A})$, with all other terms zero.

Suppose now that $(\tau, T, \leqslant)$ and $(\tilde{\tau}, \tilde{T}, \leqslant)$ are two different weak stability conditions on $\mathcal{A}$. To understand the transformation from $(\tau, T, \leqslant)$ and $(\tilde{\tau}, \tilde{T}, \leqslant)$ we would like to characterize $\tilde{\tau}$–semistability in terms of τ–semistability, that is, we would like to write the $\delta_{\mathrm{ss}}^{\alpha}(\tilde{\tau})$ in terms of the $\delta_{\mathrm{ss}}^{\beta}(\tau)$. Equation (6) suggests an algorithm for doing this. Following an idea of Markus Reineke, we can recursively solve (6) in the algebra $CF(\mathfrak{Obj}_{\mathcal{A}})$ to write $\delta_{\mathrm{ss}}^{\alpha}(\tau)$ in terms of the $\delta_{\mathrm{all}}^{\beta}$, giving

$$(7) \qquad \delta_{\mathrm{ss}}^{\alpha}(\tau) = \sum_{\substack{n \geqslant 1,\, \alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\, \alpha = \alpha_1 + \cdots + \alpha_n,\\ \tau(\alpha_1 + \cdots + \alpha_i) > \tau(\alpha_{i+1} + \cdots + \alpha_n),\ 1 \leqslant i < n}} (-1)^{n-1} \delta_{\mathrm{all}}^{\alpha_1} * \delta_{\mathrm{all}}^{\alpha_2} * \cdots * \delta_{\mathrm{all}}^{\alpha_n}.$$

Then we substitute (6) into (7) with $\tilde{\tau}$ in place of τ to get an identity

$$(8) \qquad \delta_{\mathrm{ss}}^{\alpha}(\tilde{\tau}) = \sum_{\substack{n \geqslant 1,\, \alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\\ \alpha = \alpha_1 + \cdots + \alpha_n}} S(\alpha_1,\ldots,\alpha_n; \tau, \tilde{\tau})\, \delta_{\mathrm{ss}}^{\alpha_1}(\tau) * \delta_{\mathrm{ss}}^{\alpha_2}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\alpha_n}(\tau),$$

where $S(\alpha_1,\ldots,\alpha_n; \tau, \tilde{\tau}) \in \mathbb{Z}$ are combinatorial coefficients which depend on the orders of $\tau(\alpha_1),\ldots,\tau(\alpha_n)$ in T and of $\tilde{\tau}(\alpha_1 + \cdots + \alpha_i)$ and $\tilde{\tau}(\alpha_{i+1} + \cdots + \alpha_n)$ for $1 \leqslant i < n$ in $\tilde{T}$. In [15] we show that (8) is still valid in many situations in which $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is not of finite type, so that (6) and (7) no longer make sense in $CF(\mathfrak{Obj}_{\mathcal{A}})$ since $\delta_{\mathrm{all}}^{\alpha}$ is not a constructible function.

Here [15, Definition 4.2] is a definition of the coefficients $S(\alpha_1, \ldots, \alpha_n; \tau, \tilde{\tau})$ appearing in (8). But following [15] we instead write $S(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde{\tau})$, where $\kappa \colon \{1, \ldots, n\} \to C(\mathcal{A})$ is defined by $\kappa(i) = \alpha_i$. This is because [15] also deals with a more general situation in which $(\{1, \ldots, n\}, \leqslant, \kappa)$ is replaced by $\mathcal{A}$-*data* $(I, \preceq, \kappa)$, in the sense of Definition 2.3.

Definition 2.10 Let Assumption 2.1 hold, $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ be weak stability conditions on $\mathcal{A}$, and $(\{1, \ldots, n\}, \leqslant, \kappa)$ be $\mathcal{A}$–data. If for all $i = 1, \ldots, n-1$ we have either

   (a) $\tau \circ \kappa(i) \leqslant \tau \circ \kappa(i+1)$ and $\tilde{\tau} \circ \kappa(\{1, \ldots, i\}) > \tilde{\tau} \circ \kappa(\{i+1, \ldots, n\})$ or

   (b) $\tau \circ \kappa(i) > \tau \circ \kappa(i+1)$ and $\tilde{\tau} \circ \kappa(\{1, \ldots, i\}) \leqslant \tilde{\tau} \circ \kappa(\{i+1, \ldots, n\})$,

then define $S(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde{\tau}) = (-1)^r$, where r is the number of $i = 1, \ldots, n-1$ satisfying (a). Otherwise define $S(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde{\tau}) = 0$.

If $(I, \preceq, \kappa)$ is $\mathcal{A}$–data with $\preceq$ a *total order*, there is a unique bijection $\phi \colon \{1, \ldots, n\} \to I$ with $n = |I|$ and $\phi_*(\leqslant) = \preceq$, and $(\{1, \ldots, n\}, \leqslant, \kappa \circ \phi)$ is $\mathcal{A}$–data. Define $S(I, \preceq, \kappa, \tau, \tilde{\tau}) = S(\{1, \ldots, n\}, \leqslant, \kappa \circ \phi, \tau, \tilde{\tau})$.

Next we explain, using an example, how the algebra in our problem may contain a much smaller, interesting Lie algebra, and the algebra elements $\delta^\alpha_{\mathrm{ss}}(\tau)$ may be expressed in terms of Lie algebra elements $\epsilon^\alpha_{\mathrm{ss}}(\tau)$ by a universal formula.

Example 2.11 Continue in the situation of Example 2.9. Write $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ for the vector subspace of constructible functions $f \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *supported on indecomposables*, that is, $f([X]) = 0$ unless $X \in \mathcal{A}$ is indecomposable. Then [13, Section 4.4] shows that $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ is a *Lie subalgebra* of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, under the natural Lie bracket $[f, g] = f * g - g * f$. If for all fixed $X, Z \in \mathcal{A}$ there are only finitely many isomorphism classes of $Y \in \mathcal{A}$ fitting into an exact sequence $0 \to X \to Y \to Z \to 0$, then [13, Section 4.6] shows that $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is isomorphic to the *universal enveloping algebra* of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$.

Let $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$. If $0 \ncong X \in \mathcal{A}$ then we may write X uniquely up to order and isomorphism as $X \cong X_1 \oplus \cdots \oplus X_n$, where $0 \ncong X_i \in \mathcal{A}$ is indecomposable for $i = 1, \ldots, n$. Furthermore X is τ–semistable if and only if X_i is τ–semistable with $\tau([X_i]) = \tau([X])$ for all $i = 1, \ldots, n$. Using this we show in [14, Section 7.3] that if we define

$$(9) \qquad \epsilon^\alpha_{\mathrm{ss}}(\tau) = \sum_{\substack{n \geqslant 1, \, \alpha_1, \ldots, \alpha_n \in C(\mathcal{A}): \\ \alpha = \alpha_1 + \cdots + \alpha_n, \\ \tau(\alpha_i) = \tau(\alpha), \, i = 1, \ldots, n}} \frac{(-1)^{n-1}}{n} \, \delta^{\alpha_1}_{\mathrm{ss}}(\tau) * \delta^{\alpha_2}_{\mathrm{ss}}(\tau) * \cdots * \delta^{\alpha_n}_{\mathrm{ss}}(\tau),$$

then $\epsilon_{\mathrm{ss}}^{\alpha}(\tau)$ is *supported on indecomposables*, that is, $\epsilon_{\mathrm{ss}}^{\alpha}(\tau)$ lies in the Lie algebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Here $\epsilon_{\mathrm{ss}}^{\alpha}(\tau)(X) = 1$ if X is τ–stable and lies in class α, and $\epsilon_{\mathrm{ss}}^{\alpha}(\tau)(X) = 0$ if X is τ–unstable or decomposable or does not lie in class α, and $\epsilon_{\mathrm{ss}}^{\alpha}(\tau)(X) \in \mathbb{Q}$ otherwise.

We can also invert (9) to obtain

$$(10) \qquad \delta_{\mathrm{ss}}^{\alpha}(\tau) = \sum_{\substack{n \geqslant 1, \alpha_1, \ldots, \alpha_n \in C(\mathcal{A}): \\ \alpha = \alpha_1 + \cdots + \alpha_n, \\ \tau(\alpha_i) = \tau(\alpha), i = 1, \ldots, n}} \frac{1}{n!} \, \epsilon_{\mathrm{ss}}^{\alpha_1}(\tau) * \epsilon_{\mathrm{ss}}^{\alpha_2}(\tau) * \cdots * \epsilon_{\mathrm{ss}}^{\alpha_n}(\tau).$$

Thus, the $\epsilon_{\mathrm{ss}}^{\alpha}(\tau)$ are an alternative set of generators for the subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\delta_{\mathrm{ss}}^{\beta}(\tau)$. Now let $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ are two different weak stability conditions on $\mathcal{A}$. By substituting (10) into (8) into (9) with $\tilde{\tau}$ in place of τ we obtain a transformation law for the $\epsilon_{\mathrm{ss}}^{\alpha}(\tau)$, of the form

$$(11) \qquad \epsilon_{\mathrm{ss}}^{\alpha}(\tilde{\tau}) = \sum_{\substack{n \geqslant 1, \alpha_1, \ldots, \alpha_n \in C(\mathcal{A}): \\ \alpha = \alpha_1 + \cdots + \alpha_n}} U(\alpha_1, \ldots, \alpha_n; \tau, \tilde{\tau}) \, \epsilon_{\mathrm{ss}}^{\alpha_1}(\tau) * \epsilon_{\mathrm{ss}}^{\alpha_2}(\tau) * \cdots * \epsilon_{\mathrm{ss}}^{\alpha_n}(\tau),$$

where $U(\alpha_1, \ldots, \alpha_n; \tau, \tilde{\tau}) \in \mathbb{Q}$ are combinatorial coefficients which depend on the orders of $\tau(\alpha_1), \ldots, \tau(\alpha_n)$ in T and of $\tilde{\tau}(\alpha_i), \tilde{\tau}(\alpha_1 + \cdots + \alpha_i)$ and $\tilde{\tau}(\alpha_{i+1} + \cdots + \alpha_n)$ for $1 \leqslant i < n$ in $\tilde{T}$.

Here [15, Definition 4.4] is a definition of the coefficients $U(\alpha_1, \ldots, \alpha_n; \tau, \tilde{\tau})$ appearing in (11), but following [15] we instead write $U(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde{\tau})$, where $\kappa: \{1, \ldots, n\} \to C(\mathcal{A})$ is defined by $\kappa(i) = \alpha_i$.

Definition 2.12 Let Assumption 2.1 hold, $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ be weak stability conditions on $\mathcal{A}$, and $(\{1, \ldots, n\}, \leqslant, \kappa)$ be $\mathcal{A}$–data. Define

$$U(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde{\tau}) =$$

$$(12) \quad \sum_{1 \leqslant l \leqslant m \leqslant n} \quad \sum_{\substack{\text{surjective } \psi: \{1, \ldots, n\} \to \{1, \ldots, m\} \\ \text{and } \xi: \{1, \ldots, m\} \to \{1, \ldots, l\}: \\ i \leqslant j \text{ implies } \psi(i) \leqslant \psi(j), \, i \leqslant j \text{ implies } \xi(i) \leqslant \xi(j). \\ \tau \circ \kappa \equiv \tau \circ \lambda \circ \mu: I \to T \text{ and } \tilde{\tau} \circ \mu \equiv \tilde{\tau}(\alpha) \\ \text{where } \lambda: \{1, \ldots, m\} \to C(\mathcal{A}) \text{ is } \lambda(b) = \kappa(\psi^{-1}(b)) \\ \text{and } \mu: \{1, \ldots, l\} \to C(\mathcal{A}) \text{ is } \mu(a) = \lambda(\xi^{-1}(a)).}} \prod_{a=1}^{l} S(\xi^{-1}(\{a\}), \leqslant, \lambda, \tau, \tilde{\tau}) \cdot \frac{(-1)^{l-1}}{l} \cdot \prod_{b=1}^{m} \frac{1}{|\psi^{-1}(b)|!} \cdot$$

If $(I, \preceq, \kappa)$ is $\mathcal{A}$–data with $\preceq$ a *total order*, there is a unique bijection $\phi: \{1, \ldots, n\} \to I$ with $n = |I|$ and $\phi_*(\leqslant) = \preceq$, and $(\{1, \ldots, n\}, \leqslant, \kappa \circ \phi)$ is $\mathcal{A}$–data. Define $U(I, \preceq, \kappa, \tau, \tilde{\tau}) = U(\{1, \ldots, n\}, \leqslant, \kappa \circ \phi, \tau, \tilde{\tau})$.

In Definition 2.10 and Definition 2.12 we call $S, U(*, \tau, \tilde{\tau})$ *transformation coefficients*, as they are combinatorial factors appearing in transformation laws from $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$. Here [15, Definition 5.1] are some finiteness conditions we will need on changes between stability conditions.

Definition 2.13 Let Assumption 2.1 hold and $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ be weak stability conditions on $\mathcal{A}$. We say *the change from $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$ is locally finite* if for all constructible $C \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, there are only finitely many sets of $\mathcal{A}$–data $(\{1, \ldots, n\}, \leqslant, \kappa)$ for which $S(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde{\tau}) \neq 0$ and

$$C \cap \boldsymbol{\sigma}(\{1, \ldots, n\})_* \big(\mathcal{M}_{\mathrm{ss}}(\{1, \ldots, n\}, \leqslant, \kappa, \tau)_{\mathcal{A}} \big) \neq \varnothing.$$

We say *the change from $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$ is globally finite* if this holds for $C = \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ (which is not constructible, in general) for all $\alpha \in C(\mathcal{A})$. Since any constructible $C \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ is contained in a finite union of $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$, globally finite implies locally finite.

The following assumption encapsulates the structure common to Example 2.9 and Example 2.11, and to most of the invariants studied in [15], with some oversimplifications we discuss in Remark 2.15.

Assumption 2.14 Let Assumption 2.1 hold. Suppose we are given a $\mathbb{C}$–algebra $\mathcal{H}$ with identity 1 and multiplication $*$ (which is associative, but not in general commutative), with a decomposition into $\mathbb{C}$–vector subspaces $\mathcal{H} = \bigoplus_{\alpha \in \bar{C}(\mathcal{A})} \mathcal{H}^{\alpha}$, such that $1 \in \mathcal{H}^0$ and $\mathcal{H}^{\alpha} * \mathcal{H}^{\beta} \subseteq \mathcal{H}^{\alpha + \beta}$ for all $\alpha, \beta \in \bar{C}(\mathcal{A})$.

Suppose we are given a $\mathbb{C}$–Lie subalgebra $\mathcal{L}$ of $\mathcal{H}$ with Lie bracket $[f, g] = f * g - g * f$, with a decomposition into $\mathbb{C}$–vector subspaces $\mathcal{L} = \bigoplus_{\alpha \in C(\mathcal{A})} \mathcal{L}^{\alpha}$ such that $\mathcal{L}^{\alpha} \subseteq \mathcal{H}^{\alpha}$ and $[\mathcal{L}^{\alpha}, \mathcal{L}^{\beta}] \subseteq \mathcal{L}^{\alpha + \beta}$ for all $\alpha, \beta \in C(\mathcal{A})$.

Whenever $(\tau, T, \leqslant)$ is a permissible weak stability condition on $\mathcal{A}$, let there be given elements $\delta^{\alpha}(\tau) \in \mathcal{H}^{\alpha}$ and $\epsilon^{\alpha}(\tau) \in \mathcal{L}^{\alpha}$ for all $\alpha \in C(\mathcal{A})$. These satisfy

$$(13) \qquad \epsilon^{\alpha}(\tau) = \sum_{\substack{\mathcal{A}\text{–data } (\{1, \ldots, n\}, \leqslant, \kappa) : \\ \kappa(\{1, \ldots, n\}) = \alpha, \, \tau \circ \kappa \equiv \tau(\alpha)}} \frac{(-1)^{n-1}}{n} \delta^{\kappa(1)}(\tau) * \delta^{\kappa(2)}(\tau) * \cdots * \delta^{\kappa(n)}(\tau),$$

$$(14) \qquad \delta^{\alpha}(\tau) = \sum_{\substack{\mathcal{A}\text{–data } (\{1, \ldots, n\}, \leqslant, \kappa) : \\ \kappa(\{1, \ldots, n\}) = \alpha, \, \tau \circ \kappa \equiv \tau(\alpha)}} \frac{1}{n!} \epsilon^{\kappa(1)}(\tau) * \epsilon^{\kappa(2)}(\tau) * \cdots * \epsilon^{\kappa(n)}(\tau),$$

where there are only finitely many nonzero terms in each sum.

If $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ are permissible weak stability conditions on $\mathcal{A}$ and the change from $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$ is globally finite, for all $\alpha \in C(\mathcal{A})$ we have

$$(15) \quad \sum_{\substack{\mathcal{A}\text{–data }(\{1,\ldots,n\},\leqslant,\kappa): \\ \kappa(\{1,\ldots,n\})=\alpha}} S(\{1,\ldots,n\}, \leqslant, \kappa, \tau, \tilde{\tau}) \cdot \delta^{\kappa(1)}(\tau) * \delta^{\kappa(2)}(\tau) * \cdots * \delta^{\kappa(n)}(\tau) = \delta^{\alpha}(\tilde{\tau}),$$

$$(16) \quad \sum_{\substack{\mathcal{A}\text{–data }(\{1,\ldots,n\},\leqslant,\kappa): \\ \kappa(\{1,\ldots,n\})=\alpha}} U(\{1,\ldots,n\}, \leqslant, \kappa, \tau, \tilde{\tau}) \cdot \epsilon^{\kappa(1)}(\tau) * \epsilon^{\kappa(2)}(\tau) * \cdots * \epsilon^{\kappa(n)}(\tau) = \epsilon^{\alpha}(\tilde{\tau}),$$

where there are only finitely many nonzero terms in each sum.

Equation (16) may be rewritten:

$$(17) \quad \epsilon^{\alpha}(\tilde{\tau}) = \sum_{\substack{\text{iso classes} \\ \text{of finite} \\ \text{sets } I}} \frac{1}{|I|!} \sum_{\substack{\kappa: I \to C(\mathcal{A}): \\ \kappa(I)=\alpha}} \left[\sum_{\substack{\text{total orders } \preceq \text{ on } I. \\ \text{Write } I = \{i_1,\ldots,i_n\}, \\ i_1 \preceq i_2 \preceq \cdots \preceq i_n}} U(I, \preceq, \kappa, \tau, \tilde{\tau}) \cdot \epsilon^{\kappa(i_1)}(\tau) * \cdots * \epsilon^{\kappa(i_n)}(\tau) \right].$$

The term $[\cdots]$ in (17) is a finite $\mathbb{Q}$–linear combination of multiple commutators of $\epsilon^{\kappa(i)}$ for $i \in I$, and so it lies in the Lie algebra $\mathcal{L}$, not just the algebra $\mathcal{H}$. Thus (16) and (17) can be regarded as identities in $\mathcal{L}$ rather than $\mathcal{H}$.

Remark 2.15 (a) $\delta^{\alpha}(\tau)$ is an invariant of the moduli space $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ of τ–semistable objects in class α, which "counts" such τ–semistable objects. Often it is of the form $\delta^{\alpha}(\tau) = \Phi(\delta^{\alpha}_{\mathrm{ss}}(\tau))$, where $\delta^{\alpha}_{\mathrm{ss}}(\tau) \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is the characteristic function of $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$, $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is the vector space of constructible functions on the Artin $\mathbb{K}$–stack $\mathfrak{Obj}_{\mathcal{A}}$ as in [10], and $\Phi: \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathcal{H}$ is a linear map with special multiplicative properties.

(b) The $\epsilon^{\alpha}(\tau)$ are an alternative set of generators to the $\delta^{\alpha}(\tau)$. Here (14) is the inverse of (13), and given (12)–(14), equations (15) and (16) are equivalent. Thus, the main nontrivial claim about the $\epsilon^{\alpha}(\tau)$ is that they lie in the Lie algebra $\mathcal{L}$, which may be much smaller than $\mathcal{H}$. Roughly speaking, the $\epsilon^{\alpha}(\tau)$ count τ–semistable objects S in class α weighted by a rational number depending on the factorization of S into τ–stables, which is 1 if U is τ–stable. If S is decomposable this weight is 0, so $\epsilon^{\alpha}(\tau)$ counts only indecomposable τ–semistables. The Lie algebra $\mathcal{L}$ is the part of $\mathcal{H}$ "supported on indecomposables".

(c) In [15] we worked with (Lie) algebras over $\mathbb{Q}$, not $\mathbb{C}$. But here we complexify, as we shall be discussing holomorphic functions into $\mathcal{H}, \mathcal{L}$.

(d) In parts of [15], equations (15)–(17) are only proved under an extra assumption, the existence of a third weak stability condition $(\hat{\tau}, \hat{T}, \leqslant)$ compatible with $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ in certain ways. But we will not worry about this.

(e)   In parts of [15] we relax the assumption that $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ are *permissible* (taking them instead to be τ-*artinian*, or *essentially permissible*), and we allow the change from $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$ to be *locally finite* rather than globally finite. Then equations (13)–(17) need no longer have only finitely many nonzero terms, and they are interpreted using a notion of convergence in $\mathcal{H}$.

(f)   The $\delta^{\alpha}(\tau), \epsilon^{\alpha}(\tau)$ are only the simplest of the invariants studied in [15]—we could call them "one point invariants", as they depend on only one class $\alpha \in C(\mathcal{A})$. We also considered systems of "n point invariants" depending on n classes $\alpha \in C(\mathcal{A})$, which will not enter this paper. One thing that makes the one point invariants special is that their transformation laws (15)–(16) depend only on other one point invariants, not on n point invariants for all $n \geqslant 1$.

The next six examples explain how various results in [15] fit into the framework of Assumption 2.14. The first continues Example 2.9 and Example 2.11, but without supposing the $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ are of finite type.

Example 2.16   Let Assumption 2.1 hold with $\mathbb{K}$ of characteristic zero. Take $\mathcal{H} = \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \mathbb{C}$, the vector space of $\mathbb{C}$–valued *constructible functions* on $\mathfrak{Obj}_{\mathcal{A}}$, and $\mathcal{H}^{\alpha}$ the subspace of functions supported on $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$. The multiplication $*$ on $\mathcal{H}$, studied at length in [13], has the following approximate form: for $V \in \mathcal{A}$, $(f * g)([V])$ is the "integral" over all short exact sequences $0 \to U \to V \to W \to 0$ in $\mathcal{A}$ of $f([U])g([W])$, with respect to a measure defined using the Euler characteristic of constructible subsets of $\mathbb{K}$–stacks.

The identity is $1 = \delta_{[0]}$, the characteristic function of $[0] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. The Lie subalgebra $\mathcal{L}$ is $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \mathbb{C}$, functions supported on points $[U]$ for $U \in \mathcal{A}$ indecomposable, and $\delta^{\alpha}(\tau) = \delta_{\mathrm{ss}}^{\alpha}(\tau)$, the characteristic function of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$. Then [13; 14; 15] show Assumption 2.14 holds, except that (15)–(17) are only proved under extra conditions as in Remark 2.15(d) above.

We can also replace $\mathcal{H} = \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \mathbb{C}$ and $\mathcal{L} = \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \mathbb{C}$ by the much smaller (Lie) subalgebras $\mathcal{H}_{\tau}^{\mathrm{to}} \otimes_{\mathbb{Q}} \mathbb{C}, \mathcal{L}_{\tau}^{\mathrm{to}} \otimes_{\mathbb{Q}} \mathbb{C}$ of [14, Section 7] generated by the $\delta_{\mathrm{ss}}^{\alpha}(\tau)$ and $\epsilon_{\mathrm{ss}}^{\alpha}(\tau)$, since by [15, Section 5] these are very often independent of the choice of permissible weak stability condition $(\tau, T, \leqslant)$ used to define them.

Example 2.17   Let Assumption 2.1 hold. Take $\mathcal{H} = \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \mathbb{C}$, the algebra of stack functions on $\mathfrak{Obj}_{\mathcal{A}}$ with algebra stabilizers defined in [13, Section 5], using the theory of *stack functions* from [11], a universal generalization of constructible functions. Let $\mathcal{L} = \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \mathbb{C}$, the subspace of $\mathcal{H}$ supported on "virtual indecomposables', and let $\mathcal{H}^{\alpha}, \mathcal{L}^{\alpha}$ be the subspaces of $\mathcal{H}, \mathcal{L}$ supported on $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$. Set

$\delta^\alpha(\tau) = \bar{\delta}^\alpha_{\mathrm{ss}}(\tau)$, in the notation of [14]. Then [13; 14; 15] show Assumption 2.14 holds, but with (15)–(17) only proved under extra conditions.

This also works with $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ replaced by one of the "twisted stack function" spaces $\overline{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, $\overline{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$, $\overline{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$ of [13].

We can also replace $\mathcal{H} = \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \mathbb{C}$ and $\mathcal{L} = \mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \mathbb{C}$ by the much smaller (Lie) subalgebras $\bar{\mathcal{H}}^{\mathrm{to}}_\tau \otimes_{\mathbb{Q}} \mathbb{C}$, $\bar{\mathcal{L}}^{\mathrm{to}}_\tau \otimes_{\mathbb{Q}} \mathbb{C}$ of [14, Section 8], since by [15, Section 5] these are very often independent of the choice of permissible weak stability condition $(\tau, T, \leqslant)$ used to define them.

Example 2.18 Let Assumption 2.1 hold and $\chi \colon K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be biadditive and satisfy

$$(18) \qquad \dim_{\mathbb{K}} \mathrm{Hom}(U, V) - \dim_{\mathbb{K}} \mathrm{Ext}^1(U, V) = \chi\big([U], [V]\big) \quad \text{for all } U, V \in \mathcal{A}.$$

This happens when $\mathcal{A} = \mathrm{coh}(X)$ with X a smooth projective curve, and for $\mathcal{A} = \mathrm{mod}\text{–}\mathbb{K}Q$ in Example 2.2 with χ given by the *Ringel form*

$$\chi(\alpha, \beta) = \sum_{v \in Q_0} \alpha(v)\beta(v) - \sum_{a \in Q_1} \alpha(b(a))\beta(e(a)) \ \text{ for } \alpha, \beta \in \mathbb{Z}^{Q_0}.$$

Define $\Lambda = \mathbb{C}(z)$, the algebra of rational functions $p(z)/q(z)$ for polynomials p, q with coefficients in $\mathbb{C}$ and $q \neq 0$, and define a special element $\ell = z^2$ in Λ. Define Λ° to be the subalgebra of $p(z)/q(z)$ in Λ for which $z \pm 1$ do not divide q. The facts we need about Λ, Λ° are that the *virtual Poincaré polynomial* $P(X; z)$ of a $\mathbb{K}$–variety X takes values in $\Lambda^\circ \subset \Lambda$, and $\ell = P(\mathbb{K}; z)$ for $\mathbb{K}$ the affine line, and ℓ and $\ell^k + \ell^{k-1} + \cdots + 1$ are invertible in Λ°, and $\ell - 1$ is invertible in Λ.

Let a^α for $\alpha \in \bar{C}(\mathcal{A})$ be formal symbols, and define $\mathcal{H} = A(\mathcal{A}, \Lambda, \chi)$ to be the Λ–module with basis $\{a^\alpha : \alpha \in \bar{C}(\mathcal{A})\}$ as in [13, Section 6.2], with the obvious notions of addition and multiplication by $\mathbb{C}$. Define a *multiplication* $*$ on $\mathcal{H}$ by

$$\big(\textstyle\sum_{i \in I} \lambda_i \, a^{\alpha_i}\big) * \big(\textstyle\sum_{j \in J} \mu_j \, a^{\beta_j}\big) = \textstyle\sum_{i \in I} \sum_{j \in J} \lambda_i \mu_j \, \ell^{-\chi(\beta_j, \alpha_i)} \, a^{\alpha_i + \beta_j}.$$

Then $\mathcal{H}$ is a $\mathbb{C}$–algebra, with identity a^0. Define $\mathcal{H}^\alpha = \Lambda \cdot a^\alpha$ for $\alpha \in \bar{C}(\mathcal{A})$. Define $\mathcal{L}^\alpha = (\ell - 1)^{-1} \Lambda^\circ \cdot a^\alpha \subset \mathcal{H}^\alpha$ for $\alpha \in C(\mathcal{A})$, and $\mathcal{L} = \bigoplus_{\alpha \in C(\mathcal{A})} \mathcal{L}^\alpha$. Then $\mathcal{L}$ is a Lie subalgebra of $\mathcal{H}$, as $(\ell^{-\chi(\beta, \alpha)} - \ell^{-\chi(\alpha, \beta)})/(\ell - 1) \in \Lambda^\circ$.

For $(\tau, T, \leqslant)$ a permissible weak stability condition on $\mathcal{A}$ and $\alpha \in C(\mathcal{A})$, define $\delta^\alpha(\tau) = I^\alpha_{\mathrm{ss}}(\tau) a^\alpha$, where $I^\alpha_{\mathrm{ss}}(\tau)$ is the *virtual Poincaré function* of $\mathrm{Obj}^\alpha_{\mathrm{ss}}(\tau)$, as defined in [11, Section 4.2], where we regard $\mathrm{Obj}^\alpha_{\mathrm{ss}}(\tau)$ as a finite type open $\mathbb{K}$–substack with affine geometric stabilizers in the Artin $\mathbb{K}$–stack $\mathfrak{Obj}^\alpha_{\mathcal{A}}$. Define $\epsilon^\alpha(\tau)$ by (13). Then $\epsilon^\alpha(\tau) \in \mathcal{H}^\alpha$, so we can write $\epsilon^\alpha(\tau) = (\ell - 1)^{-1} J^\alpha(\tau) a^\alpha$ for $J^\alpha(\tau) \in \Lambda$. We show in [15, Theorem 6.8] that $J^\alpha(\tau) \in \Lambda^\circ$, so $\epsilon^\alpha(\tau) \in \mathcal{L}^\alpha$.

Then [13, Section 6.2] and [15, Section 6.2] show that Assumption 2.14 holds in its entirety when $\mathcal{A} = \text{mod–}\mathbb{K}Q$, and with extra conditions as in Remark 2.15(d) above in general. It also holds with Λ replaced by other commutative $\mathbb{C}$–algebras, and virtual Poincaré polynomials replaced by other Λ–valued "motivic invariants" Υ of $\mathbb{K}$–varieties with $\ell = \Upsilon(\mathbb{K})$; for details see [11; 13; 15].

Example 2.19 Let $\mathbb{K}$ be an algebraically closed field and X a smooth projective surface over $\mathbb{K}$ with K_X^{-1} *numerically effective* (*nef*). Take $\mathcal{A} = \text{coh}(X)$ with data $K(\mathcal{A}), \mathfrak{F}_\mathcal{A}$ satisfying Assumption 2.1 as in [12, Example 9.1]. Then there is a biadditive $\chi\colon K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ such that for all $U, V \in \mathcal{A}$ we have

$$\dim_\mathbb{K} \text{Hom}(U, V) - \dim_\mathbb{K} \text{Ext}^1(U, V) + \dim_\mathbb{K} \text{Ext}^2(U, V) = \chi([U], [V]).$$

Define $\Lambda, \mathcal{H}, *$ and $\mathcal{H}^\alpha$ as in Example 2.18, but set $\mathcal{L}^\alpha = \mathcal{H}^\alpha$ for $\alpha \in C(\mathcal{A})$ and $\mathcal{L} = \bigoplus_{\alpha \in C(\mathcal{A})} \mathcal{L}^\alpha$. Then in [15, Section 6.4], for a class of weak stability conditions $(\tau, T, \leqslant)$ on $\mathcal{A}$ based on Gieseker stability, we define invariants $I_{\text{ss}}^\alpha(\tau), \bar{J}^\alpha(\tau) \in \Lambda$ such that Assumption 2.14 holds with $\delta^\alpha(\tau) = I_{\text{ss}}^\alpha(\tau)a^\alpha$ and $\epsilon^\alpha(\tau) = (\ell - 1)^{-1}\bar{J}^\alpha(\tau)a^\alpha$. But we do not prove that $\bar{J}^\alpha(\tau) \in \Lambda^\circ$, which is why we modify the definitions of $\mathcal{L}^\alpha, \mathcal{L}$.

Example 2.18 and Example 2.19 illustrate the relationship between "invariants" $I_{\text{ss}}^\alpha(\tau)$, $J^\alpha(\tau), \bar{J}^\alpha(\tau)$ which "count" τ–semistables in class α, and our (Lie) algebra approach. In this case, the transformation laws (15)–(16) for $\delta^\alpha(\tau), \epsilon^\alpha(\tau)$ are equivalent to the following laws for $I_{\text{ss}}^\alpha(\tau), J^\alpha(\tau)$, from [15, Theorem 6.8]:

$$(19) \qquad I_{\text{ss}}^\alpha(\tilde{\tau}) = \sum_{\substack{\mathcal{A}\text{–data}(\{1,\ldots,n\}, \leqslant, \kappa): \\ \kappa(\{1,\ldots,n\}) = \alpha}} S(\{1,\ldots,n\}, \leqslant, \kappa, \tau, \tilde{\tau}) \cdot \ell^{-\sum_{1 \leqslant i < j \leqslant n} \chi(\kappa(j), \kappa(i))} \cdot \prod_{i=1}^n I_{\text{ss}}^{\kappa(i)}(\tau),$$

$$(20) \qquad J^\alpha(\tilde{\tau}) = \sum_{\substack{\mathcal{A}\text{–data}(\{1,\ldots,n\}, \leqslant, \kappa): \\ \kappa(\{1,\ldots,n\}) = \alpha}} U(\{1,\ldots,n\}, \leqslant, \kappa, \tau, \tilde{\tau}) \cdot \ell^{-\sum_{1 \leqslant i < j \leqslant n} \chi(\kappa(j), \kappa(i))} \cdot (\ell - 1)^{1-n} \prod_{i=1}^n J^{\kappa(i)}(\tau).$$

Observe that (15)–(16) are simpler than (19)–(20), since the powers of ℓ in (19)–(20) are packaged in the multiplication $*$ in $\mathcal{H}$. This is more pronounced in our next two examples, where the formulae for $*$ are much more complicated, so the transformation laws for invariants are too. One moral is that working in the framework of Assumption 2.14 is simpler than working with systems of invariants, which is why we have adopted it.

Example 2.20 Let Assumption 2.1 hold and $\chi\colon K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be biadditive and satisfy (18), and let $\Lambda, \Lambda^\circ, \ell$ be as in Example 2.18. Consider pairs (I, κ) with I a finite set and $\kappa\colon I \to C(\mathcal{A})$ a map. Define an equivalence relation $\approx$ on such (I, κ)

by $(I, \kappa) \approx (I', \kappa')$ if there exists a bijection $i \colon I \to I'$ with $\kappa' \circ i = \kappa$. Write $[I, \kappa]$ for the $\approx$–equivalence class of (I, κ). Introduce formal symbols $b_{[I,\kappa]}$ for all such equivalence classes $[I, \kappa]$.

As in [13, Section 6.3], let $\mathcal{H} = B(\mathcal{A}, \Lambda, \chi)$ be the Λ–module with basis the $b_{[I,\kappa]}$. Define $\mathcal{H}^\alpha = \bigoplus_{[I,\kappa] : \kappa(I) = \alpha} \Lambda \cdot b_{[I,\kappa]}$. Define a *multiplication* $*$ on $\mathcal{H}$ by

$$
b_{[I,\kappa]} * b_{[J,\lambda]} = \sum_{\text{eq. classes } [K, \mu]} b_{[K,\mu]} \cdot \frac{(\ell - 1)^{|K| - |I| - |J|}}{|\operatorname{Aut}(K, \mu)|} \cdot
$$

$$
(21) \quad \left[\sum_{\substack{\text{iso.} \\ \text{classes} \\ \text{of finite} \\ \text{sets } L}} \frac{(-1)^{|L| - |K|}}{|L|!} \sum_{\substack{\phi \colon I \to L, \, \psi \colon J \to L \text{ and} \\ \theta \colon L \to K \colon \phi \amalg \psi \text{ surjective}, \\ \mu(k) = \kappa((\theta \circ \phi)^{-1}(k)) + \\ \lambda((\theta \circ \psi)^{-1}(k)), \, k \in K}} \prod_{k \in K} (|\theta^{-1}(k)| - 1)! \prod_{\substack{i \in I, \, j \in J : \\ \phi(i) = \psi(j)}} \ell^{-\chi(\lambda(j), \kappa(i))} \right]
$$

extended Λ–bilinearly. Then $\mathcal{H}$ is a $\mathbb{C}$–algebra with identity $b_{[\varnothing, \varnothing]}$.

For $\alpha \in C(\mathcal{A})$ define $b^\alpha = b_{[\{1\}, \alpha']}$ where $\alpha'(1) = \alpha$, define $\mathcal{L}^\alpha = \Lambda^\circ \cdot b^\alpha$ and $\mathcal{L} = \bigoplus_{\alpha \in C(\mathcal{A})} \mathcal{L}^\alpha$. Equation (21) yields

$$
[b^\alpha, b^\beta] = \frac{\ell^{-\chi(\beta, \alpha)} - \ell^{-\chi(\alpha, \beta)}}{\ell - 1} b^{\alpha + \beta},
$$

and $(\ell^{-\chi(\beta, \alpha)} - \ell^{-\chi(\alpha, \beta)}) / (\ell - 1) \in \Lambda^\circ$, so $\mathcal{L}$ is a Lie subalgebra of $\mathcal{H}$.

If $(\tau, T, \leqslant)$ is a permissible weak stability condition we put $\epsilon^\alpha(\tau) = J^\alpha(\tau) b^\alpha$ for the same $J^\alpha(\tau) \in \Lambda^\circ$ as in Example 2.18. We then define $\delta^\alpha(\tau)$ by (14), giving a much more complicated answer than in Example 2.18. From [13, Section 6.3] and [15, Section 6] it follows that Assumption 2.14 holds in its entirety when $\mathcal{A} = \text{mod–}\mathbb{K}Q$, and with extra conditions as in Remark 2.15(d) above in general.

In Example 2.18 and Example 2.20 the algebras Λ, Λ°, the Lie algebra $\mathcal{L}$, and the Lie algebra elements $\epsilon^\alpha(\tau)$, are the same, provided we identify the generator $(\ell - 1)^{-1} a^\alpha$ of $\mathcal{L}^\alpha$ in Example 2.18 with the generator b^α of $\mathcal{L}^\alpha$ in Example 2.20. But the algebra $\mathcal{H}$ in Example 2.20 is much larger than that in Example 2.18. If we set $\Lambda^\circ = \Lambda$, then in Example 2.18 we would have $\mathcal{L} = \mathcal{H}$, or at least $\mathcal{H} = \Lambda \cdot a^0 \oplus \mathcal{L}$, but in Example 2.20, $\mathcal{H}$ would be the Λ–universal enveloping algebra of $\mathcal{L}$.

One difference between Example 2.18 and Example 2.20 is that $\delta^\alpha(\tau)$ records more information about the moduli space $\operatorname{Obj}_{\text{ss}}^\alpha(\tau)$ in Example 2.20. In Example 2.18, $\delta^\alpha(\tau)$ is basically the virtual Poincaré polynomial of $\operatorname{Obj}_{\text{ss}}^\alpha(\tau)$, multiplied by b^α. But in Example 2.20, $\delta^\alpha(\tau)$ is a sum of terms $\beta_{[I,\kappa]} b_{[I,\kappa]}$, where roughly speaking $\beta_{[I,\kappa]}$ is the virtual Poincaré polynomial of the subspace of $X \in \operatorname{Obj}_{\text{ss}}^\alpha(\tau)$ admitting a

decomposition $X \cong \bigoplus_{i \in I} X_i$ with X_i indecomposable and $[X_i] = \kappa(i) \in C(\mathcal{A})$ for all $i \in I$. However, since by (13)–(14) knowing all the $\epsilon^\alpha(\tau)$ is equivalent to knowing all the $\delta^\alpha(\tau)$, and the $\epsilon^\alpha(\tau)$ are the same in Example 2.18 and Example 2.20, the information in the family of all $\delta^\alpha(\tau)$ is the same in Example 2.18 and Example 2.20.

Example 2.21 Let Assumption 2.1 hold and $\bar{\chi} \colon K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be antisymmetric and biadditive and satisfy

$$
\begin{aligned}
(22) \quad & \big(\dim_{\mathbb{K}} \mathrm{Hom}(U, V) - \dim_{\mathbb{K}} \mathrm{Ext}^1(U, V)\big) - \\
& \big(\dim_{\mathbb{K}} \mathrm{Hom}(V, U) - \dim_{\mathbb{K}} \mathrm{Ext}^1(V, U)\big) = \bar{\chi}\big([U], [V]\big) \quad \text{for all } U, V \in \mathcal{A}.
\end{aligned}
$$

Note that (18) implies (22) with $\bar{\chi}(\alpha, \beta) = \chi(\alpha, \beta) - \chi(\beta, \alpha)$, so this holds for $\mathcal{A} = \mathrm{mod}\text{–}\mathbb{K}Q$ and $\mathcal{A} = \mathrm{coh}(X)$ for X a smooth projective curve. But we also show in [13, Section 6.6] using Serre duality that (22) holds when $\mathcal{A} = \mathrm{coh}(X)$ for X a *Calabi–Yau* 3–*fold* over $\mathbb{K}$.

As in Example 2.20, introduce symbols $c_{[I,\kappa]}$ for all equivalence classes $[I, \kappa]$, and let $\mathcal{H} = C(\mathcal{A}, \Omega, \frac{1}{2}\bar{\chi})$ be the $\mathbb{C}$–vector space with basis the $c_{[I,\kappa]}$. Furthermore, define $\mathcal{H}^\alpha = \bigoplus_{[I,\kappa] \colon \kappa(I) = \alpha} \mathbb{C} \cdot c_{[I,\kappa]}$. Define a *multiplication* $*$ on $\mathcal{H}$ by

$$
\begin{aligned}
(23) \quad c_{[I,\kappa]} * c_{[J,\lambda]} = & \sum_{\text{eq. classes } [K, \mu]} c_{[K,\mu]} \cdot \frac{1}{|\mathrm{Aut}(K,\mu)|} \sum_{\substack{\eta \colon I \to K,\, \xi \colon J \to K: \\ \mu(k) = \kappa(\eta^{-1}(k)) + \lambda(\xi^{-1}(k))}} \\
& \left[\sum_{\substack{\text{simply connected directed graphs } \Gamma: \\ \text{vertices } I \amalg J,\, \text{edges } \bullet^i \to \bullet^j,\, i \in I,\, j \in J, \\ \text{conn. components } \eta^{-1}(k) \amalg \xi^{-1}(k),\, k \in K}} \quad \prod_{\substack{\text{edges} \\ \bullet^i \to \bullet^j \\ \text{in } \Gamma}} \tfrac{1}{2}\bar{\chi}(\kappa(i), \lambda(j)) \right]
\end{aligned}
$$

extended $\mathbb{C}$–bilinearly. Then $\mathcal{H}$ is a $\mathbb{C}$–algebra with identity $c_{[\varnothing, \varnothing]}$. For $\alpha \in C(\mathcal{A})$ define $c^\alpha = c_{[\{1\},\alpha']}$ where $\alpha'(1) = \alpha$, define $\mathcal{L}^\alpha = \mathbb{C} \cdot c^\alpha$ and $\mathcal{L} = \bigoplus_{\alpha \in C(\mathcal{A})} \mathcal{L}^\alpha$. Equation (23) yields $[c^\alpha, c^\beta] = \bar{\chi}(\alpha, \beta)\, c^{\alpha+\beta}$, so $\mathcal{L}$ is a Lie subalgebra.

Then [15, Section 6.5] defines invariants $J^\alpha(\tau) \in \mathbb{Q}$ for $\alpha \in C(\mathcal{A})$, such that if we set $\epsilon^\alpha(\tau) = J^\alpha(\tau)\, c^\alpha$ and define $\delta^\alpha(\tau)$ by (14) then Assumption 2.14 holds, with extra conditions as in Remark 2.15(d) above. These invariants $J^\alpha(\tau)$ are defined using the *Euler characteristic* of constructible sets in Artin $\mathbb{K}$–stacks, in a rather subtle way. As Euler characteristic and virtual Poincaré polynomials are related by $\chi(X) = P(X; -1)$, these are specializations of the virtual Poincaré polynomial invariants of Example 2.18 and Example 2.20.

Note that we cannot define the $\delta^\alpha(\tau)$ directly, but only reconstruct them from the $\epsilon^\alpha(\tau)$. In the notation of Example 2.17, this is because $\epsilon^\alpha(\tau)$ is defined using a Lie

algebra morphism $\Psi: \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_\mathcal{A}) \to \mathcal{L}$ which does *not* extend to an algebra morphism $\Psi: \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_\mathcal{A}) \to \mathcal{H}$, so we cannot define $\delta^\alpha(\tau) = \Psi(\bar{\delta}_{\mathrm{ss}}^\alpha(\tau))$ as we might hope. The above also holds with $\mathbb{C}$ replaced by other commutative $\mathbb{C}$–algebras Ω, and Euler characteristics replaced by other Ω–valued "motivic invariants" Θ of $\mathbb{K}$–varieties with $\Theta(\mathbb{K}) = 1$; for details see [11; 13; 15].

To rewrite (16) as a transformation law for the $J^\alpha(\tau)$ we need to compute $c^{\alpha_1} * \cdots * c^{\alpha_n}$ in $\mathcal{H}$. Actually it is enough to know the projection of this to $\mathcal{L}$. As in [15, Section 6.5], calculation shows this is given by:

(24)

$$c^{\alpha_1} * \cdots * c^{\alpha_n} = \text{ terms in } c_{[I,\kappa]}, \ |I| > 1,$$

$$+ \left[\frac{1}{2^{n-1}} \sum_{\substack{\text{connected, simply connected digraphs } \Gamma: \\ \text{vertices } \{1,\ldots,n\}, \text{ edge } \bullet^i \to \bullet^j \text{ implies } i < j}} \prod_{\substack{\text{edges} \\ \bullet^i \to \bullet^j \\ \text{in } \Gamma}} \bar{\chi}(\alpha_i, \alpha_j) \right] c^{\alpha_1 + \cdots + \alpha_n}.$$

Here a *digraph* is a directed graph.

Let $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ be weak stability conditions on $\mathcal{A}$, Γ be a connected, simply connected digraph with finite vertex set I, and $\kappa: I \to C(\mathcal{A})$. Define $V(I, \Gamma, \kappa, \tau, \tilde{\tau}) \in \mathbb{Q}$ by

$$V(I, \Gamma, \kappa, \tau, \tilde{\tau}) = \frac{1}{2^{|I|-1}|I|!} \sum_{\substack{\text{total orders } \preceq \text{ on } I: \\ \text{edge } \bullet^i \to \bullet^j \text{ in } \Gamma \text{ implies } i \preceq j}} U(I, \preceq, \kappa, \tau, \tilde{\tau}).$$

Then using $\epsilon^\alpha(\tau) = J^\alpha(\tau) \, c^\alpha$ and (24), it turns out [15, Theorem 6.28] that (16) is equivalent to

$$(25) \qquad J^\alpha(\tilde{\tau}) = \sum_{\substack{\text{iso.} \\ \text{classes} \\ \text{of finite} \\ \text{sets } I}} \sum_{\substack{\kappa: I \to C(\mathcal{A}): \\ \kappa(I) = \alpha}} \sum_{\substack{\text{connected,} \\ \text{simply connected} \\ \text{digraphs } \Gamma, \\ \text{vertices } I}} V(I, \Gamma, \kappa, \tau, \tilde{\tau}) \cdot \prod_{\text{edges } \bullet^i \to \bullet^j \text{ in } \Gamma} \bar{\chi}(\kappa(i), \kappa(j)) \\ \cdot \prod_{i \in I} J^{\kappa(i)}(\tau).$$

Example 2.21 is the reason why the title of the paper involves Calabi–Yau 3–folds, why we believe that the ideas of this paper have to do with Mirror Symmetry and String Theory, and why we want to bring them to the attention of String Theorists in particular so that they may explain them in physical terms. In brief, the point is this.

In [15, Section 6.5], as the culmination of a great deal of work in [10; 11; 12; 13; 14; 15], the author defined invariants $J^\alpha(\tau) \in \mathbb{Q}$ "counting" τ–semistable sheaves in class $\alpha \in K(\mathcal{A})$ on a Calabi–Yau 3–fold X, which transform according to a complicated

transformation law (25) under change of weak stability condition, reminiscent of Feynman diagrams.

The author expects that some related invariants which extend Donaldson–Thomas invariants and transform according to the same law (25) should be important in String Theory, perhaps counting numbers of branes or BPS states. For conjectures on this see [15, Section 6.5]. This paper will study natural ways of combining these invariants in holomorphic generating functions; the author expects that these generating functions, and the equations they satisfy, should also be significant in String Theory.

2.3 Comments on the extension to triangulated categories

The series [12; 13; 14; 15] studied only abelian categories, such as the coherent sheaves $\mathrm{coh}(X)$ on a projective $\mathbb{K}$–scheme X. But for applications to String Theory and Mirror Symmetry, the whole programme should be extended to *triangulated categories*, such as the *bounded derived category* $D^b(\mathrm{coh}(X))$ of coherent sheaves on X. The issues involved in this are discussed in [15, Section 7]. For a recent survey on derived categories of coherent sheaves on Calabi–Yau m–folds, see Bridgeland [5].

One justification for this is Kontsevich's *Homological Mirror Symmetry* proposal [16], which explains Mirror Symmetry of Calabi–Yau 3–folds $X, \widehat{X}$ as an equivalence between $D^b(\mathrm{coh}(X))$ and the derived Fukaya category $D^b(F(\widehat{X}))$ of $\widehat{X}$. This relates the complex algebraic geometry of X, encoded in $D^b(\mathrm{coh}(X))$, to the symplectic geometry of $\widehat{X}$, encoded in $D^b(F(\widehat{X}))$. Building on Kontsevich's ideas, triangulated categories of branes have appeared in String Theory in the work of Douglas, Aspinwall, Diaconescu, Lazaroiu and others.
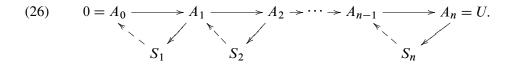
The following notion of stability condition on a triangulated category, due to Bridgeland [4, Section 1.1], will be important in this programme. For background on triangulated categories, see Gelfand and Manin [7].

Definition 2.22 Let $\mathcal{T}$ be a triangulated category, and $K(\mathcal{T})$ the quotient of its Grothendieck group $K_0(\mathcal{T})$ by some fixed subgroup. For instance, if $\mathcal{T}$ is of *finite type* over a field $\mathbb{K}$ we can take $K(\mathcal{T})$ to be the *numerical Grothendieck group* $K^{\mathrm{num}}(\mathcal{T})$ as in [4, Section 1.3], and then Bridgeland calls the resulting stability conditions *numerical stability conditions*.

A *stability condition* $(Z, \mathcal{P})$ on $\mathcal{T}$ consists of a group homomorphism $Z: K(\mathcal{T}) \to \mathbb{C}$ called the *central charge*, and full subcategories $\mathcal{P}(\phi) \subset \mathcal{T}$ for each $\phi \in \mathbb{R}$ of *semistable objects with phase* ϕ, satisfying the following:

(a) If $S \in \mathcal{P}(\phi)$ then $Z([S]) = m([S])e^{i\pi\phi}$ for some $m([S]) \in (0, \infty)$.

(b) For all $t \in \mathbb{R}$, $\mathcal{P}(t+1) = \mathcal{P}(t)[1]$.

(c) If $t_1 > t_2$ and $S_j \in \mathcal{P}(t_j)$ for $j = 1, 2$ then $\mathrm{Hom}_{\mathcal{T}}(S_1, S_2) = 0$.

(d) for $0 \neq U \in \mathcal{T}$ there is a finite sequence $t_1 > t_2 > \cdots > t_n$ in $\mathbb{R}$ and a collection of distinguished triangles with $S_j \in \mathcal{P}(t_j)$ for all j:

$$(26) \qquad 0 = A_0 \longrightarrow A_1 \longrightarrow A_2 \to \cdots \to A_{n-1} \longrightarrow A_n = U.$$

This is the generalization to triangulated categories of the *slope function* stability conditions of Example 2.7. In both cases we have a central charge homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ or $Z\colon K(\mathcal{T}) \to \mathbb{C}$, and semistability can be expressed in terms of $\arg \circ Z$. In the abelian case $\arg \circ Z$ takes a unique value in $(0, \pi)$, but in the triangulated case one has to choose a value of $\arg \circ Z$ and lift phases from $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$. This need to choose phases is why in the abelian case Z determines the stability condition, but in the triangulated case we also need extra data $\mathcal{P}$. Equation (26) is the analogue of Theorem 2.8, since both decompose an arbitrary object $U \in \mathcal{A}$ or $\mathcal{T}$ into semistable objects $S_1, \ldots, S_n$ with phases satisfying $\mu(S_1) > \cdots > \mu(S_n)$ or $t_1 > \cdots > t_n$.

There is also a generalized notion of stability condition on $\mathcal{T}$ due to Gorodentscev et al [9], not involving a central charge, which is closer in spirit to Definition 2.4 above. But we will not use it. Here is Bridgeland's main result [4, Theorem 1.2], slightly rewritten:

Theorem 2.23 *Let $\mathcal{T}$ be a triangulated category and $K(\mathcal{T})$ as in Definition 2.22. Write $\mathrm{Stab}(\mathcal{T})$ for the set of stability conditions $(Z, \mathcal{P})$ on $\mathcal{T}$. Then $\mathrm{Stab}(\mathcal{T})$ has a natural, Hausdorff topology. Let Σ be a connected component of $\mathrm{Stab}(\mathcal{T})$. Then there is a complex vector subspace V_Σ in $\mathrm{Hom}(K(\mathcal{T}), \mathbb{C})$ with a well-defined linear topology such that the map $\Sigma \to \mathrm{Hom}(K(\mathcal{T}), \mathbb{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeomorphism $\Sigma \to V_\Sigma$.*

When V_Σ is finite-dimensional, which happens automatically when $K(\mathcal{T})$ has finite rank, Σ can be given the structure of a complex manifold uniquely so that $(Z, \mathcal{P}) \mapsto Z$ is a local biholomorphism $\Sigma \to V_\Sigma$.

Bridgeland's stability conditions were motivated by Douglas' work on Pi-stability, and are natural objects in String Theory. Suppose we wish to define some kind of generating function f^α encoding invariants "counting" $(Z, \mathcal{P})$–semistable objects in class α in $\mathcal{T}$. These invariants will depend on $(Z, \mathcal{P})$, so the generating function

f^α should be a function on $\mathrm{Stab}(\mathcal{T})$ (and perhaps in other variables as well). Now Theorem 2.23 shows $\mathrm{Stab}(\mathcal{T})$ is a *complex manifold*, so it makes sense to require f^α to be a *holomorphic function* on $\mathrm{Stab}(\mathcal{T})$. We can also try to make f^α *continuous*, despite the fact that the invariants it encodes will change discontinuously over real hypersurfaces in $\mathrm{Stab}(\mathcal{T})$.

This problem also makes sense in the abelian setting of Example 2.7, where we can try to define a generating function f^α which is a continuous, holomorphic function on the complex manifold $\mathrm{Stab}(\mathcal{A})$ of (5). In fact most of the rigorous part of the paper is about Example 2.7, but we have done it in a way that the author expects will generalize to the triangulated case when (if ever) the extension of [12; 13; 14; 15] to triangulated categories has been worked out.

3   Holomorphic generating functions

Consider the following situation. Let Assumption 2.1 and Assumption 2.14 hold for $\mathcal{A}$, with $K(\mathcal{A})$ of finite rank, and suppose $\mathrm{Stab}(\mathcal{A})$ in Example 2.7 is a nonempty open subset of $\mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$, and so a complex manifold. This works for all the quiver examples $\mathcal{A} = \mathrm{mod}\text{–}\mathbb{K}Q, \mathrm{nil}\text{–}\mathbb{K}Q, \mathrm{mod}\text{–}\mathbb{K}Q/I, \mathrm{nil}\text{–}\mathbb{K}Q/I, \mathrm{mod}\text{–}A$ of [12, Section 10], with $\mathcal{H}, \mathcal{L}, \ldots$ chosen as in one of Example 2.16–Example 2.21.

Then we have a complex manifold $\mathrm{Stab}(\mathcal{A})$ of central charges Z, each of which defines a permissible stability condition $(\mu, \mathbb{R}, \leqslant)$. For this μ we have invariants $\delta^\alpha(\mu) \in \mathcal{H}^\alpha$ and $\epsilon^\alpha(\mu) \in \mathcal{L}^\alpha$ for all $\alpha \in C(\mathcal{A})$. Regarded as functions of Z, these $\delta^\alpha(\mu), \epsilon^\alpha(\mu)$ change *discontinuously* across real hypersurfaces in $\mathrm{Stab}(\mathcal{A})$ where $\arg \circ Z(\beta) = \arg \circ Z(\gamma)$ for $\beta, \gamma \in C(\mathcal{A})$ according to the transformation laws (15)–(16), and away from such hypersurfaces are locally constant.

For $\alpha \in C(\mathcal{A})$ we shall consider a generating function $f^\alpha \colon \mathrm{Stab}(\mathcal{A}) \to \mathcal{H}^\alpha$ of the following form, where $(\mu, \mathbb{R}, \leqslant)$ is the stability condition induced by Z:

$$(27) \quad f^\alpha(Z) = \sum_{\substack{n \geqslant 1,\, \alpha_1, \ldots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \cdots + \alpha_n = \alpha}} F_n\big(Z(\alpha_1), \ldots, Z(\alpha_n)\big) \epsilon^{\alpha_1}(\mu) * \epsilon^{\alpha_2}(\mu) * \cdots * \epsilon^{\alpha_n}(\mu).$$

We explain why we chose this form, and what conditions the F_n must satisfy:

Remark 3.1 (a)   The general form of (27) is modelled on (13)–(16) above. The functions F_n should map $(\mathbb{C}^\times)^n \to \mathbb{C}$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. For the abelian category case of Example 2.7 we have $Z(\alpha) \in H = \{x + iy : x \in \mathbb{R},\ y > 0\}$ for $\alpha \in C(\mathcal{A})$, so it would be enough to define F_n only on H^n. However, for the extension to the

triangulated category case discussed in Section 2.3 we must allow $Z(\alpha_k) \in \mathbb{C}^\times$, which is why we chose the domain $(\mathbb{C}^\times)^n$.

(b) We require that the functions F_n satisfy

$$(28) \qquad F_n(\lambda z_1, \ldots, \lambda z_n) = F_n(z_1, \ldots, z_n) \quad \text{for all } \lambda, z_1, \ldots, z_n \in \mathbb{C}^\times.$$

The reason is easiest to explain in the triangulated category case. Let $\mathcal{T}, K(\mathcal{T})$ and $(Z, \mathcal{P})$ be as in Definition 2.22, and let $r > 0$ and $\psi \in \mathbb{R}$. Define a new stability condition $(Z', \mathcal{P}')$ on $\mathcal{T}$ by $Z' = r e^{i\psi} Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(\phi - \psi/\pi)$.

This gives an action of $(0, \infty) \times \mathbb{R}$ on $\mathrm{Stab}(\mathcal{T})$, which does not change the sets of $(Z, \mathcal{P})$–semistable objects, but only their phases ϕ. So we expect that in an appropriate extension of Assumption 2.14 to the triangulated case, the invariants $\delta^\alpha(Z, \mathcal{P}), \epsilon^\alpha(Z, \mathcal{P})$ "counting" $(Z, \mathcal{P})$–semistable objects in class α should be also unchanged by this action. Therefore we can try and make f^α and each term in (27) invariant under $Z \mapsto r e^{i\psi} Z$, which is equivalent to (28).

We can make a similar argument in the abelian case Example 2.7, but we have to restrict to $r e^{i\psi}$ such that $r e^{i\psi} Z[C(\mathcal{A})] \subset H$, which makes the argument less persuasive. Requiring f^α and F_n instead to be *homogeneous of degree* $d \in \mathbb{Z}$, so that $F_n(\lambda z_1, \ldots, \lambda z_n) = \lambda^d F_n(z_1, \ldots, z_n)$ for all λ, z_k, is equivalent to replacing $f^\alpha(Z)$ by $Z(\alpha)^d f^\alpha(Z)$. So we lose nothing by restricting to $d = 0$.

(c) Equation (28) implies that F_1 is constant, say $F_1 \equiv c$. For $\lambda \in \mathbb{C}^\times$ we may replace f^α, F_n, c by $\lambda f^\alpha, \lambda F_n, \lambda c$ without changing whether f^α is holomorphic or continuous, so all nonzero choices of c are equivalent. We shall take

$$(29) \qquad\qquad F_1 \equiv (2\pi i)^{-1},$$

as this simplifies formulae in Section 3.2 and the rest of the paper.

Think of (27) as saying $f^\alpha(Z) = c\, \epsilon^\alpha(\mu) +$ "higher order terms". If $\epsilon^\alpha(\mu)$ is an invariant "counting" μ–semistables in class α, then so is $f^\alpha(Z)$, to leading order. But $\epsilon^\alpha(\mu)$ changes discontinuously with Z, whereas $f^\alpha(Z)$ includes higher order correction terms which smooth out these changes and make f^α continuous.

(d) Following equations (16) and (17), we may rewrite (27) as:

$$f^\alpha(Z) =$$

$$(30) \qquad \sum_{\substack{\text{iso classes} \\ \text{of finite} \\ \text{sets } I}} \frac{1}{|I|!} \sum_{\substack{\kappa: I \to C(\mathcal{A}): \\ \kappa(I) = \alpha}} \left[\sum_{\substack{\text{total orders } \preceq \text{ on } I. \\ \text{Write } I = \{i_1, \ldots, i_n\}, \\ i_1 \preceq i_2 \preceq \cdots \preceq i_n}} F_{|I|}(Z \circ \kappa(i_1), \ldots, Z \circ \kappa(i_n)) \cdot \right. \\ \left. \epsilon^{\kappa(i_1)}(\mu) * \cdots * \epsilon^{\kappa(i_n)}(\mu) \right].$$

As for (17), we shall require the functions F_n to have the property that the term $[\cdots]$ in (30) is a finite $\mathbb{C}$–linear combination of multiple commutators of $\epsilon^{\kappa(i)}$ for $i \in I$, and so it lies in the Lie algebra $\mathcal{L}$, not just the algebra $\mathcal{H}$. Thus (27) and (30) make sense in $\mathcal{L}$, and f^α actually maps $\mathrm{Stab}(\mathcal{A}) \to \mathcal{L}^\alpha$.

This is why we choose to write (27) in terms of the $\epsilon^\alpha(\mu)$ rather than the $\delta^\alpha(\mu)$. By substituting (13) into (27) we get another equation of the same form for f^α, but with $\delta^{\alpha_i}(\mu)$ instead of $\epsilon^{\alpha_i}(\mu)$, and different functions F_n. But using the $\epsilon^{\alpha_i}(\mu)$ means we can work in $\mathcal{L}$ rather than $\mathcal{H}$, which is a great simplification if $\mathcal{L}$ is much smaller than $\mathcal{H}$. This happens in Example 2.21, our motivating Calabi–Yau 3–fold example, when $\mathcal{L}^\alpha = \mathbb{C} \cdot c^\alpha$ so f^α is really just a holomorphic function, but $\mathcal{H}^\alpha$ is in general infinite-dimensional.

Now for $|I| > 1$, if a $\mathbb{C}$–linear combination of products of $\epsilon^{\kappa(i)}(\mu)$ for $i \in I$ is a sum of multiple commutators, it is easy to see that the sum of the coefficients of the products must be zero in $\mathbb{C}$. Thus, a necessary condition for $[\cdots]$ in (30) to be a linear combination of multiple commutators is that

$$(31) \qquad \sum_{\sigma \in S_n} F_n(z_{\sigma(1)}, \dots, z_{\sigma(n)}) = 0 \quad \text{for all } n > 1 \text{ and } (z_1, \dots, z_n) \in (\mathbb{C}^\times)^n,$$

where S_n is the *symmetric group* of permutations σ of $\{1, \dots, n\}$.

(e) We require that f^α be a *continuous* and *holomorphic* function on $\mathrm{Stab}(\mathcal{A})$. These translate to conditions on the functions F_n. In Section 3.1 we will compute the conditions on F_n for f^α to be continuous; it turns out that across real hypersurfaces $\arg z_l = \arg z_{l+1}$, F_n must jump by expressions in F_k for $k < n$. For f^α to be holomorphic, it is enough that the F_n be holomorphic wherever they are continuous.

Thus, F_n is a branch of a multivalued holomorphic function, except along $\arg z_l = \arg z_{l+1}$ where it jumps discontinuously from one branch to another; but the discontinuities in $\epsilon^\alpha(\mu)$ and $F_n(\cdots)$ cancel out to make f^α continuous. A simple comparison is a branch of $\log z$ on $\mathbb{C}^\times$, cut along $(0, \infty)$.

(f) We shall ensure uniqueness of the F_n by imposing a *growth condition*:

$$(32) \qquad |F_n(z_1, \dots, z_n)| = o(|z_k|^{-1}) \quad \text{as } z_k \to 0 \text{ with } z_l \text{ fixed}, l \neq k, \text{ for all } k.$$

This may assist the convergence of (27) in situations when the sum is infinite.

(g) Equation (27) is an example of a transformation of the general form

$$(33) \qquad b^\alpha = \sum_{\substack{n \geqslant 1, \, \alpha_1, \dots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \cdots + \alpha_n = \alpha}} P_n(\alpha_1, \dots, \alpha_n) \, a^{\alpha_1} * a^{\alpha_2} * \cdots * a^{\alpha_n},$$

where $\{a^\alpha : \alpha \in C(\mathcal{A})\}$ and $\{b^\alpha : \alpha \in C(\mathcal{A})\}$ are generating sets for some subalgebra of $\mathcal{H}$, and $P_n(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}$. Equations (13)–(16) of Assumption 2.14 are also of this form. Transformations of type (33) are closed under composition, and invertible provided $P_1(\alpha) \neq 0$ for all $\alpha \in C(\mathcal{A})$.

For the moment we impose the following extra condition. It implies there are only finitely many possibilities for n and $\alpha_1, \ldots, \alpha_n$ in (27), and so avoids problems with infinite sums and convergence. It holds for all the quiver examples $\mathcal{A} = \mathrm{mod}\text{–}\mathbb{K}Q, \ldots$ of [12, Section 10], but not for $\mathcal{A} = \mathrm{coh}(X)$ when $\dim X > 0$.

Assumption 3.2 In the situation of Assumption 2.1, for each $\alpha \in C(\mathcal{A})$ there are only finitely pairs $\beta, \gamma \in C(\mathcal{A})$ with $\alpha = \beta + \gamma$.

In the rest of the section we construct functions F_n satisfying the requirements of Remark 3.1, and show that they are *unique*, and satisfy interesting *partial differential equations*.

3.1 Conditions on the functions F_n for f^α to be continuous

Let Assumption 2.1, Assumption 2.14 and Assumption 3.2 hold for $\mathcal{A}$, with $K(\mathcal{A})$ of finite rank, and suppose $\mathrm{Stab}(\mathcal{A})$ in Example 2.7 is a nonempty open subset of $\mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$, and so a complex manifold. Let $Z, \widetilde{Z} \in \mathrm{Stab}(\mathcal{A})$, with associated stability conditions $(\mu, \mathbb{R}, \leqslant)$ and $(\widetilde{\mu}, \mathbb{R}, \leqslant)$. We think of Z as varying in $\mathrm{Stab}(\mathcal{A})$ and $\widetilde{Z}$ as a fixed base point.

We need some notation for the coefficients $S, U(\{1, \ldots, n\}, \leqslant, \kappa, \mu, \widetilde{\mu})$ of Section 2.1. They depend on the $2n$ complex numbers $Z \circ \kappa(k), \widetilde{Z} \circ \kappa(k)$ for $k = 1, \ldots, n$ in $H = \{x + iy : x \in \mathbb{R}, \ y > 0\}$, and the definition makes sense for any $2n$ elements of H. Thus there are unique functions $s_n, u_n \colon H^{2n} \to \mathbb{Q}$ written $s_n, u_n(z_1, \ldots, z_n; \widetilde{z}_1, \ldots, \widetilde{z}_n)$ such that

$$
\begin{aligned}
(34) \quad & S(\{1, \ldots, n\}, \leqslant, \kappa, \mu, \widetilde{\mu}) = s_n(Z \circ \kappa(1), \ldots, Z \circ \kappa(n); \widetilde{Z} \circ \kappa(1), \ldots, \widetilde{Z} \circ \kappa(n)), \\
& U(\{1, \ldots, n\}, \leqslant, \kappa, \mu, \widetilde{\mu}) = u_n(Z \circ \kappa(1), \ldots, Z \circ \kappa(n); \widetilde{Z} \circ \kappa(1), \ldots, \widetilde{Z} \circ \kappa(n)).
\end{aligned}
$$

Then using (16) with $\alpha_i, \widetilde{\mu}, \mu$ in place of $\alpha, \tau, \widetilde{\tau}$ respectively to express $\epsilon^{\alpha_i}(\mu)$ in (27) in terms of $\epsilon^{\kappa(j)}(\widetilde{\mu})$, using (34) and rewriting, we find that:

$$(35) \quad f^{\alpha}(Z) = \sum_{\substack{n \geqslant 1, \, \alpha_1, \ldots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \cdots + \alpha_n = \alpha}} \epsilon^{\alpha_1}(\widetilde{\mu}) * \epsilon^{\alpha_2}(\widetilde{\mu}) * \cdots * \epsilon^{\alpha_n}(\widetilde{\mu}) \cdot$$

$$\left[\sum_{\substack{m=1,\ldots,n, \\ 0=a_0<a_1< \\ \cdots<a_m=n}} F_m\bigl(Z(\alpha_{a_0+1}+\cdots+\alpha_{a_1}),\ldots, Z(\alpha_{a_{m-1}+1}+\cdots+\alpha_{a_m})\bigr) \atop \prod_{k=1}^{m} u_{a_k-a_{k-1}}\bigl(\widetilde{Z}(\alpha_{a_{k-1}+1}),\ldots,\widetilde{Z}(\alpha_{a_k}); Z(\alpha_{a_{k-1}+1}),\ldots, Z(\alpha_{a_k})\bigr) \right].$$

We get this by decomposing $\alpha = \beta_1 + \cdots + \beta_m$ in (27), and then decomposing $\beta_k = \alpha_{a_{k-1}+1} + \cdots + \alpha_{a_k}$ as part of an expression (16) for $\epsilon^{\beta_k}(\mu)$, for $k = 1, \ldots, m$.

We rewrite the bottom line $[\cdots]$ of (35) as

$$(36) \quad f^{\alpha}(Z) = \sum_{\substack{n \geqslant 1, \, \alpha_1, \ldots, \alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \cdots + \alpha_n = \alpha}} G_n\bigl(Z(\alpha_1),\ldots,Z(\alpha_n); \widetilde{Z}(\alpha_1),\ldots,\widetilde{Z}(\alpha_n)\bigr) \cdot \epsilon^{\alpha_1}(\widetilde{\mu}) * \epsilon^{\alpha_2}(\widetilde{\mu}) * \cdots * \epsilon^{\alpha_n}(\widetilde{\mu}),$$

using a function $G_n \colon H^{2n} \to \mathbb{C}$ given by

$$G_n(z_1,\ldots,z_n;\widetilde{z}_1,\ldots,\widetilde{z}_n) =$$

$$(37) \quad \sum_{\substack{m=1,\ldots,n, \\ 0=a_0<a_1<\cdots<a_m=n}} F_m(z_{a_0+1}+\cdots+z_{a_1},\ldots,z_{a_{m-1}+1}+\cdots+z_{a_m}) \atop \prod_{k=1}^{m} u_{a_k-a_{k-1}}(\widetilde{z}_{a_{k-1}+1},\ldots,\widetilde{z}_{a_k}; z_{a_{k-1}+1},\ldots,z_{a_k}).$$

In (36), the terms $\epsilon^{\alpha_1}(\widetilde{\mu}) * \cdots * \epsilon^{\alpha_n}(\widetilde{\mu})$ and $\widetilde{Z}(\alpha_1),\ldots,\widetilde{Z}(\alpha_n)$ are constants independent of Z. Thus it is clear that f^{α} is continuous, or holomorphic, provided the function $(z_1,\ldots,z_n) \mapsto G_n(z_1,\ldots,z_n;\widetilde{z}_1,\ldots,\widetilde{z}_n)$ is continuous, or holomorphic, for each fixed $(\widetilde{z}_1,\ldots,\widetilde{z}_n) \in H^n$.

We can now substitute (16) with $\alpha_i, \mu, \widetilde{\mu}$ in place of $\alpha, \tau, \widetilde{\tau}$ respectively to express $\epsilon^{\alpha_i}(\widetilde{\mu})$ in (36) in terms of $\epsilon^{\kappa(j)}(\mu)$. Rewriting gives an expression for $f^{\alpha}(Z)$ as a linear combination of $\epsilon^{\alpha_1}(\mu) * \cdots * \epsilon^{\alpha_n}(\mu)$, as in (27). In fact the coefficients of $\epsilon^{\alpha_1}(\mu) * \cdots * \epsilon^{\alpha_n}(\mu)$ in the two expressions must agree; we can prove this either by using [14, Example 7.10], in which the $\epsilon^{\alpha_1}(\mu) * \cdots * \epsilon^{\alpha_n}(\mu)$ are linearly independent in $\mathcal{H}$ for all $\alpha_1, \ldots, \alpha_n$ satisfying some conditions, or by using combinatorial properties of the coefficients $U(\cdots)$ from [15, Theorem 4.8].

Equating the two writes $F_n\bigl(Z(\alpha_1),\ldots,Z(\alpha_n)\bigr)$ in terms of the functions G_m and u_k. Since $Z(\alpha_k), \widetilde{Z}(\alpha_k)$ can take arbitrary values in H, we deduce an expression for

$F_n(z_1, \ldots, z_n)$ when $z_k, \tilde{z}_k \in H$, the inverse of (37):

$$F_n(z_1, \ldots, z_n) =$$

$$(38) \quad \sum_{\substack{m=1,\ldots,n, \\ 0=a_0<a_1<\cdots<a_m=n}} G_m(z_{a_0+1}+\cdots+z_{a_1}, \ldots, z_{a_{m-1}+1}+\cdots+z_{a_m}; \\ \tilde{z}_{a_0+1}+\cdots+\tilde{z}_{a_1}, \ldots, \tilde{z}_{a_{m-1}+1}+\cdots+\tilde{z}_{a_m}) \cdot \\ \prod_{k=1}^m u_{a_k-a_{k-1}}(z_{a_{k-1}+1}, \ldots, z_{a_k}; \tilde{z}_{a_{k-1}+1}, \ldots, \tilde{z}_{a_k}).$$

Note that although we want F_n to map $(\mathbb{C}^\times)^n \to \mathbb{C}$, for the moment (38) is defined only when $z_k, \tilde{z}_k \in H$, since we have so far defined u_n and G_n only on H^{2n}, not on $(\mathbb{C}^\times)^{2n}$. Note too that (38) holds for arbitrary $\tilde{z}_1, \ldots, \tilde{z}_n \in H$.

Here are some conclusions so far.

Proposition 3.3 *Suppose Assumption 2.1, Assumption 2.14 and Assumption 3.2 hold for $\mathcal{A}$ with $K(\mathcal{A})$ of finite rank and $\mathrm{Stab}(\mathcal{A})$ is a nonempty open subset of $\mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$, and let some functions $F_n \colon (\mathbb{C}^\times)^n \to \mathbb{C}$ be given. Then a sufficient condition for the function f^α of (27) to be continuous, or holomorphic, is that for fixed $(\tilde{z}_1, \ldots, \tilde{z}_n) \in H^n$ the function $(z_1, \ldots, z_n) \mapsto G_n(z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n)$ is continuous, or holomorphic. This holds for some $(\tilde{z}_1, \ldots, \tilde{z}_n) \in H^n$ if and only if it holds for all $(\tilde{z}_1, \ldots, \tilde{z}_n)$.*

This condition is also necessary, for all values of n occurring in (27), if the terms $\epsilon^{\alpha_1}(\mu) * \epsilon^{\alpha_2}(\mu) * \cdots * \epsilon^{\alpha_n}(\mu)$ occurring in (27) are linearly independent in $\mathcal{H}$. This happens in the examples of [14, Example 7.10], for arbitrarily large n.

To go further we must understand the functions $s_n, u_n(z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n)$ better. From Example 2.7, Definition 2.10 and Definition 2.12, we see that these depend on whether the inequalities $\arg(z_a + \cdots + z_b) > \arg(z_{b+1} + \cdots + z_c)$ and $\arg(\tilde{z}_a + \cdots + \tilde{z}_b) > \arg(\tilde{z}_{b+1} + \cdots + \tilde{z}_c)$ hold for each choice of $1 \leqslant a \leqslant b < c \leqslant n$, choosing $\arg(\cdots)$ uniquely in $(0, \pi)$ as "$\cdots$" lies in H.

For each $(\tilde{z}_1, \ldots, \tilde{z}_n) \in H^n$, define

$$(39) \quad N_{(\tilde{z}_1,\ldots,\tilde{z}_n)} = \big\{ (z_1, \ldots, z_n) \in H^n : \text{if } \arg(\tilde{z}_a + \cdots + \tilde{z}_b) > \arg(\tilde{z}_{b+1} + \cdots + \tilde{z}_c) \\ \text{then } \arg(z_a + \cdots + z_b) > \arg(z_{b+1} + \cdots + z_c), \text{ for all } 1 \leqslant a \leqslant b < c \leqslant n \big\}.$$

Then $N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$ is an *open* subset of H^n, as it is defined by strict inequalities, and $(\tilde{z}_1, \ldots, \tilde{z}_n) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$. As in [14, Definition 4.10] we say that $(\tilde{\mu}, \mathbb{R}, \leqslant)$ *dominates* $(\mu, \mathbb{R}, \leqslant)$ if $\tilde{\mu}(\alpha) > \tilde{\mu}(\beta)$ implies $\mu(\alpha) > \mu(\beta)$ for all $\alpha, \beta \in C(\mathcal{A})$. From [15, Section 5.2], this implies that

$$(40) \quad S(\{1, \ldots, n\}, \leqslant, \kappa, \mu, \tilde{\mu}) = \begin{cases} 1, & \mu \circ \kappa(1) > \cdots > \mu \circ \kappa(n), \ \tilde{\mu} \circ \kappa \equiv \tilde{\mu}(\alpha), \\ 0, & \text{otherwise,} \end{cases}$$

for all $\mathcal{A}$–data $(\{1,\ldots,n\},\leqslant,\kappa)$ with $\kappa(\{1,\ldots,n\})=\alpha$.

If $(z_1,\ldots,z_n)\in N_{(\widetilde{z}_1,\ldots,\widetilde{z}_n)}$ then the same argument shows that

$$s_n(z_1,\ldots,z_n;\widetilde{z}_1,\ldots,\widetilde{z}_n) = \begin{cases} 1, & \begin{array}{l}\arg(z_1)>\cdots>\arg(z_n) \text{ and} \\ \arg(\widetilde{z}_k)=\arg(\widetilde{z}_1+\cdots+\widetilde{z}_n), \text{ all } k, \end{array} \\ 0, & \text{otherwise,} \end{cases}$$

since the conditions in (39) play the same role as $\widetilde{\mu}(\alpha) > \widetilde{\mu}(\beta)$ implies $\mu(\alpha) > \mu(\beta)$ does in (40). From (12) we deduce that if $(z_1,\ldots,z_n)\in N_{(\widetilde{z}_1,\ldots,\widetilde{z}_n)}$ then:

(41) $u_n(z_1,\ldots,z_n;\widetilde{z}_1,\ldots,\widetilde{z}_n) =$

$$\sum_{1\leqslant l\leqslant m\leqslant n} \sum_{\substack{\text{surjective } \psi:\{1,\ldots,n\}\to\{1,\ldots,m\} \\ \text{and } \xi:\{1,\ldots,m\}\to\{1,\ldots,l\}: \\ a\leqslant b \text{ implies } \psi(a)\leqslant\psi(b),\, a\leqslant b \text{ implies } \xi(a)\leqslant\xi(b), \\ \arg(\widetilde{z}_a)=\arg(\widetilde{z}_1+\cdots+\widetilde{z}_n), \text{ all } a, \\ \psi(a)=\psi(b) \text{ implies } \arg(z_a)=\arg(z_b), \\ \psi(a)<\psi(b) \text{ and } \xi\circ\psi(a)=\xi\circ\psi(b) \text{ imply } \arg(z_a)>\arg(z_b)}} \frac{(-1)^{l-1}}{l}\cdot\prod_{c=1}^{m}\frac{1}{|\psi^{-1}(c)|!},$$

so that $u_n(z_1,\ldots,z_n;\widetilde{z}_1,\ldots,\widetilde{z}_n) = 0$ if $\arg(\widetilde{z}_k)\neq\arg(\widetilde{z}_1+\cdots+\widetilde{z}_n)$, some k

We have been working with $z_k,\widetilde{z}_k\in H$, since s_n,u_n,G_n are, so far, defined only on H^{2n}. We shall now restate the conditions of Proposition 3.3 for f^α to be continuous, or holomorphic, in a way which makes sense for $z_k,\widetilde{z}_k\in\mathbb{C}^\times$.

Condition 3.4 Let some functions $F_n\colon(\mathbb{C}^\times)^n\to\mathbb{C}$ be given for $n\geqslant 1$. For all $n\geqslant 1$ and $(\widetilde{z}_1,\ldots,\widetilde{z}_n)\in(\mathbb{C}^\times)^n$ there should exist an open neighbourhood $N_{(\widetilde{z}_1,\ldots,\widetilde{z}_n)}$ of $(\widetilde{z}_1,\ldots,\widetilde{z}_n)$ in $(\mathbb{C}^\times)^n$, such that if $(z_1,\ldots,z_n)\in N_{(\widetilde{z}_1,\ldots,\widetilde{z}_n)}$ then $\mathrm{Re}(z_k\widetilde{z}_k^{-1})>0$ for $k=1,\ldots,n$. For $(z_1,\ldots,z_n)\in N_{(\widetilde{z}_1,\ldots,\widetilde{z}_n)}$ we must have

$$F_n(z_1,\ldots,z_n) =$$

$$\sum_{\substack{m=1,\ldots,n,\ 0=a_0<a_1<\cdots<a_m=n \\ \text{and } c_1,\ldots,c_m\in[0,2\pi): \\ \widetilde{z}_a\in e^{ic_k}(0,\infty),\, a_{k-1}<a\leqslant a_k,\, k=1,\ldots,m}} G_m(z_{a_0+1}+\cdots+z_{a_1},\ldots,z_{a_{m-1}+1}+\cdots+z_{a_m};$$
$$\widetilde{z}_{a_0+1}+\cdots+\widetilde{z}_{a_1},\ldots,\widetilde{z}_{a_{m-1}+1}+\cdots+\widetilde{z}_{a_m})\cdot$$

(42)
$$\prod_{k=1}^{m}\ \sum_{\substack{1\leqslant l_k\leqslant m_k\leqslant a_k-a_{k-1} \\ \text{surjective } \psi_k:\{a_{k-1}+1,\ldots,a_k\}\to\{1,\ldots,m_k\} \\ \text{and } \xi_k:\{1,\ldots,m_k\}\to\{1,\ldots,l_k\}: \\ a\leqslant b \text{ implies } \psi_k(a)\leqslant\psi_k(b),\, a\leqslant b \text{ implies } \xi_k(a)\leqslant\xi_k(b), \\ \psi_k(a)=\psi_k(b) \text{ implies } \arg(z_a)=\arg(z_b), \\ \psi_k(a)<\psi_k(b) \text{ and } \xi_k\circ\psi_k(a)=\xi_k\circ\psi_k(b) \text{ imply } \arg(z_a)>\arg(z_b), \\ \text{taking } \arg(z_a),\arg(z_b)\in(c_k-\pi/2,c_k+\pi/2)}} \frac{(-1)^{l_k-1}}{l_k}\cdot\prod_{c=1}^{m_k}\frac{1}{|\psi_k^{-1}(c)|!},$$

where $G_m(\cdots)$ are some functions defined on the subsets of $(\mathbb{C}^\times)^{2m}$ required by (42), such that the maps $(z_1,\ldots,z_m) \mapsto G_m(z_1,\ldots,z_m;\tilde{z}_1,\ldots,\tilde{z}_m)$ are continuous (for f^α to be continuous), and holomorphic (for f^α to be holomorphic), in their domains.

Here are some remarks on this:

- From (41), for $(z_1,\ldots,z_n)$ in an open neighbourhood of $(\tilde{z}_1,\ldots,\tilde{z}_n)$ we see that the term $u_{a_k-a_{k-1}}(z_{a_{k-1}+1},\ldots,z_{a_k};\tilde{z}_{a_{k-1}+1},\ldots,\tilde{z}_{a_k})$ in (38) is zero unless $\arg(\tilde{z}_a) = c_k$ for all $a_{k-1} < a \leqslant a_k$ and some c_k. We have put this in as a restriction in the first line of (42), expressing it as $\tilde{z}_a \in e^{ic_k}(0,\infty)$ rather than $\arg(\tilde{z}_a) = c_k$ because of the multivalued nature of arg.

- We put in an extra condition $\mathrm{Re}(z_k\tilde{z}_k^{-1}) > 0$ for all k when $(z_1,\ldots,z_n) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$. The main point of this is in (42) we have that $\mathrm{Re}(e^{-ic_k}z_a) > 0$ for $a_{k-1} < a \leqslant a_k$, so $\mathrm{Re}(e^{-ic_k}(z_{a_{k-1}+1} + \cdots + z_{a_k})) > 0$, and thus the argument $z_{a_{k-1}+1} + \cdots + z_{a_k}$ in $G_m(\cdots)$ in (42) is nonzero. That is, (42) only needs G_m to be defined on a subset of $(\mathbb{C}^\times)^{2m}$.
  We also use this condition in the second line, as when $a_{k-1} < a, b \leqslant a_k$ we can choose $\arg(z_a), \arg(z_b)$ uniquely in $(c_k - \pi/2, c_k + \pi/2)$.

- When restricted to H^n, Condition 3.4 is equivalent to the conditions of Proposition 3.3 for f^α to be continuous, or holomorphic, as the arguments above show. But Condition 3.4 also makes sense on $(\mathbb{C}^\times)^n$, where we want F_n to be defined for the extension to the triangulated category case. Calculations by the author indicate that Condition 3.4 is the correct extension to the triangulated case.

- The point of restricting to neighbourhoods $N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$ is partly because there we can use the formula (41), but mostly because we do not have a meaningful extension of u_n from H^{2n} to all of $(\mathbb{C}^\times)^{2n}$, so that (37) and (38) do not make sense. But for any fixed $(\tilde{z}_1,\ldots,\tilde{z}_n)$ we can use (41) to define $u_n(z_1,\ldots,z_n;\tilde{z}_1,\ldots,\tilde{z}_n)$ for $(z_1,\ldots,z_n)$ sufficiently close to $(\tilde{z}_1,\ldots,\tilde{z}_n)$, and this is the basis of Condition 3.4.

Now suppose that $(\tilde{z}_1,\ldots,\tilde{z}_n) \in (\mathbb{C}^\times)^n$ with $\tilde{z}_{k+1}/\tilde{z}_k \notin (0,\infty)$ for all $1 \leqslant k < n$, and let $(z_1,\ldots,z_n) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$. Then in the first sum in (42) we cannot have $a_{k-1} \leqslant a_k - 2$ for any k, as then $\tilde{z}_{a_k}, \tilde{z}_{a_k-1} \in e^{ic_k}(0,\infty)$, contradicting $\tilde{z}_{a_k}/\tilde{z}_{a_k-1} \notin (0,\infty)$. Thus the only term in the first sum is $m = n$ and $a_k = k$ for $0 \leqslant k \leqslant n$, so the only terms in the second line are $l_k = m_k = 1$ and $\{a_k\} \xrightarrow{\psi_k} \{1\} \xrightarrow{\xi_k} \{1\}$, and (38) reduces to

$$(43) \quad F_n(z_1,\ldots,z_n) = G_n(z_1,\ldots,z_n;\tilde{z}_1,\ldots,\tilde{z}_n)$$

$$\text{if } \tilde{z}_{k+1}/\tilde{z}_k \notin (0,\infty) \text{ for all } k \text{ and } (z_1,\ldots,z_n) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}.$$

Thus Condition 3.4 requires F_n to be continuous, or holomorphic, on $N_{(\tilde{z}_1,\dots,\tilde{z}_n)}$, an open neighbourhood of $(\tilde{z}_1,\dots,\tilde{z}_n)$. So we deduce:

Proposition 3.5 *Condition 3.4 implies that the function F_n must be continuous, and holomorphic, on the set*

$$(44) \qquad \{(z_1,\dots,z_n) \in (\mathbb{C}^\times)^n : z_{k+1}/z_k \notin (0,\infty) \text{ for all } 1 \leqslant k < n\}.$$

Similarly, suppose $(\tilde{z}_1,\dots,\tilde{z}_n) \in (\mathbb{C}^\times)^n$ with $\tilde{z}_{l+1}/\tilde{z}_l \in (0,\infty)$ for some $1 \leqslant l < n$, and $\tilde{z}_{k+1}/\tilde{z}_k \notin (0,\infty)$ for all $1 \leqslant k < n$, $k \neq l$. Then in the first sum there are two terms, $m = n$ and $a_k = k$ for $0 \leqslant k \leqslant n$ as before, and $m = n-1$ and $a_k = k$ for $0 \leqslant k < l$, $a_k = k+1$ for $l \leqslant k < n$. Rewriting $\arg(z_l) > \arg(z_{l+1})$ as $\mathrm{Im}(z_{l+1}/z_l) < 0$, and so on, we find (42) reduces to

$$F_n(z_1,\dots,z_n) = G_n(z_1,\dots,z_n;\tilde{z}_1,\dots,\tilde{z}_n)$$

$$+ G_{n-1}(z_1,\dots,z_{l-1},z_l+z_{l+1},z_{l+2},\dots,z_n; \\ \tilde{z}_1,\dots,\tilde{z}_{l-1},\tilde{z}_l+\tilde{z}_{l+1},\tilde{z}_{l+2},\dots,\tilde{z}_n) \cdot \begin{cases} \frac{1}{2}, & \mathrm{Im}(z_{l+1}/z_l) < 0, \\ 0, & \mathrm{Im}(z_{l+1}/z_l) = 0, \\ -\frac{1}{2}, & \mathrm{Im}(z_{l+1}/z_l) > 0. \end{cases}$$

By (43) this $G_{n-1}(\cdots)$ is $F_{n-1}(z_1,\dots,z_{l-1},z_l+z_{l+1},z_{l+2},\dots,z_n)$, giving:

Proposition 3.6 *Condition 3.4 implies that if $(\tilde{z}_1,\dots,\tilde{z}_n) \in (\mathbb{C}^\times)^n$ with $\tilde{z}_{l+1}/\tilde{z}_l \in (0,\infty)$ for some $1 \leqslant l < n$, and $\tilde{z}_{k+1}/\tilde{z}_k \notin (0,\infty)$ for all $1 \leqslant k < n$ with $k \neq l$, then for $(z_1,\dots,z_n)$ in an open neighbourhood of $(\tilde{z}_1,\dots,\tilde{z}_n)$ in $(\mathbb{C}^\times)^n$, the following function is continuous, and holomorphic:*

$$\begin{aligned} &F_n(z_1,\dots,z_n) - \\ (45) \qquad &F_{n-1}(z_1,\dots,z_{l-1},z_l+z_{l+1},z_{l+2},\dots,z_n) \cdot \begin{cases} \frac{1}{2}, & \mathrm{Im}(z_{l+1}/z_l) < 0, \\ 0, & \mathrm{Im}(z_{l+1}/z_l) = 0, \\ -\frac{1}{2}, & \mathrm{Im}(z_{l+1}/z_l) > 0. \end{cases} \end{aligned}$$

To summarize: away from the real hypersurfaces $z_{l+1}/z_l \in (0,\infty)$ in $(\mathbb{C}^\times)^n$ for $1 \leqslant l < n$, the F_n must be continuous and holomorphic. As we cross the hypersurface $z_{l+1}/z_l \in (0,\infty)$ at a generic point, the function $F_n(z_1,\dots,z_n)$ jumps by $F_{n-1}(z_1,\dots,z_{l-1},z_l+z_{l+1},z_{l+2},\dots,z_n)$, with the value on the hypersurface being the average of the limiting values from either side.

Where several of the hypersurfaces $z_{l+1}/z_l \in (0,\infty)$ intersect, $F_n(z_1,\dots,z_n)$ satisfies a more complicated condition. Roughly speaking, this says that several different sectors

of (44) come together where the hypersurfaces intersect, and on the intersection F_n should be a weighted average of the limiting values in each of these sectors. We now show these conditions determine the functions F_n uniquely, provided they exist at all.

Theorem 3.7 *There exists at most one family of functions $F_n \colon (\mathbb{C}^\times)^n \to \mathbb{C}$ satisfying Condition 3.4 and equations* (28), (29), (31), (32) *of Remark 3.1.*

Proof Suppose F_n and F'_n for $n \geqslant 1$ are two families of functions satisfying all the conditions, using functions G_m, G'_m respectively in Condition 3.4. We shall show that $F_n \equiv F'_n$ for all n, by induction on n. We have $F_1 \equiv F'_1 \equiv (2\pi i)^{-1}$ by (29). So let $n \geqslant 2$, and suppose by induction that $F_k \equiv F'_k$ for all $k < n$. By Condition 3.4 and induction on k this implies that $G_k = G'_k$ for $k < n$. So taking the difference of (38) for F_n and F'_n gives

$$
\begin{aligned}
f(z_1, \dots, z_n) &= F_n(z_1, \dots, z_n) - F'_n(z_1, \dots, z_n) \\
&= G_n(z_1, \dots, z_n; \tilde{z}_1, \dots, \tilde{z}_n) - G'_n(z_1, \dots, z_n; \tilde{z}_1, \dots, \tilde{z}_n)
\end{aligned}
$$

in an open neighbourhood of $(\tilde{z}_1, \dots, \tilde{z}_n)$.

As G_n, G'_n are continuous and holomorphic in the z_k, we see $f \colon (\mathbb{C}^\times)^n \to \mathbb{C}$ is holomorphic. By (28), f is the pullback of a holomorphic function $\tilde{f} \colon \{[z_1, \dots, z_n] \in \mathbb{CP}^{n-1} : z_k \neq 0, \ k = 1, \dots, n\} \to \mathbb{C}$. Taking the difference of (32) for F_n, F'_n gives $|\tilde{f}| = o(|z_k|^{-1})$ near points in just one hypersurface $z_k = 0$ in $\mathbb{CP}^{n-1}$. So by standard results in complex analysis, $\tilde{f}$ extends holomorphically over these parts of $\mathbb{CP}^{n-1}$, and so is defined except on intersections of two or more hypersurfaces $z_k = 0$ in $\mathbb{CP}^{n-1}$. By Hartog's theorem $\tilde{f}$ extends holomorphically to all of $\mathbb{CP}^{n-1}$, and so is constant. Since $n > 1$, equation (31) gives $\sum_{\sigma \in S_n} \tilde{f}([z_{\sigma(1)}, \dots, z_{\sigma(n)}]) = 0$ for $z_k \in \mathbb{C}^\times$, forcing $\tilde{f} \equiv 0$. Hence $f \equiv 0$ and $F_n \equiv F'_n$. The theorem follows by induction. $\qquad\square$

Note that we actually prove slightly more than the theorem says: any functions $F_1, \dots, F_n$ satisfying the conditions up to n are unique.

3.2 Partial differential equations satisfied by f^α and F_n

We wish to construct a family of holomorphic generating functions $f^\alpha \colon \mathrm{Stab}(\mathcal{A}) \to \mathcal{L}^\alpha$ for $\alpha \in C(\mathcal{A})$. Clearly, it would be interesting if this family satisfied some nontrivial partial differential equations. We are now going to *guess* a p.d.e. for the f^α to satisfy, and deduce a p.d.e. for the F_n. We will then use this p.d.e. to *construct* the functions F_n that we want by induction on n. In Section 3.3 we will prove they satisfy Remark 3.1 and Condition 3.4. Theorem 3.7 then shows these F_n are unique.

This means that the p.d.e.s that we shall guess for the f^α and F_n are actually *implied* by the general assumptions Remark 3.1 and Condition 3.4, which seems very surprising, as these imposed no differential equations other than being holomorphic. One possible conclusion is that our p.d.e.s are not simply something the author made up, but are really present in the underlying geometry and combinatorics, and have some meaning of their own.

To guess the p.d.e. we start by determining the function F_2. Equation (28) implies we may write $F_2(z_1, z_2) = f(z_2/z_1)$ for some $f: \mathbb{C}^\times \to \mathbb{C}$, and then Proposition 3.5, Proposition 3.6 and (29) imply that f is holomorphic in $\mathbb{C} \setminus [0, \infty)$ with the following continuous over $(0, \infty)$:

$$f(z) - \frac{1}{2\pi i} \cdot \begin{cases} \frac{1}{2}, & \operatorname{Im}(z) < 0, \\ 0, & \operatorname{Im}(z) = 0, \\ -\frac{1}{2}, & \operatorname{Im}(z) > 0. \end{cases}$$

Since $\log z$ cut along $(0, \infty)$ jumps by $2\pi i$ across $(0, \infty)$, the obvious answer is $f(z) = (2\pi i)^{-2} \log z + C$ for some constant C, where we define $\log z$ on $\mathbb{C} \setminus [0, \infty)$ such that $\operatorname{Im} \log z \in (0, 2\pi)$. But equation (31) reduces to $f(z) + f(z^{-1}) \equiv 0$, which holds provided $C = -\pi i/(2\pi i)^2$. This suggests that

$$(46) \qquad F_2(z_1, z_2) = \begin{cases} \frac{1}{(2\pi i)^2}\big(\log(z_2/z_1) - \pi i\big), & z_2/z_1 \notin (0, \infty), \\ \frac{1}{(2\pi i)^2} \log(z_2/z_1), & z_2/z_1 \in (0, \infty), \end{cases}$$

where $\log z$ is defined so that $\operatorname{Im} \log z \in [0, 2\pi)$. It is now easy to check that these F_1, F_2 satisfy Condition 3.4 and (28), (29), (31), (32) up to $n = 2$, so Theorem 3.7 shows (46) is the *unique* function F_2 which does this.

Let us now consider a simple situation in which we are interested only in classes $\beta, \gamma, \beta + \gamma$ in $C(\mathcal{A})$, and β, γ cannot be written as $\delta + \epsilon$ for $\delta, \epsilon \in C(\mathcal{A})$, and the only ways to write $\beta + \gamma = \delta + \epsilon$ for $\delta, \epsilon \in C(\mathcal{A})$ are $\delta, \epsilon = \beta, \gamma$ or $\delta, \epsilon = \gamma, \beta$. In this case, from (27), (29) and (46) we see that when $Z \in \operatorname{Stab}(\mathcal{A})$ with $Z(\gamma)/Z(\beta) \notin (0, \infty)$ we have

$$f^\beta(Z) = \tfrac{1}{2\pi i}\, \epsilon^\beta(\mu), \qquad f^\gamma(Z) = \tfrac{1}{2\pi i}\, \epsilon^\gamma(\mu),$$

$$f^{\beta+\gamma}(Z) = \tfrac{1}{2\pi i} \epsilon^{\beta+\gamma}(\mu) + \tfrac{1}{(2\pi i)^2}\left(\log\left(\tfrac{Z(\gamma)}{Z(\beta)}\right) - \pi i\right)\big(\epsilon^\beta(\mu) * \epsilon^\gamma(\mu) - \epsilon^\gamma(\mu) * \epsilon^\beta(\mu)\big).$$

These satisfy the p.d.e. on $\mathrm{Stab}(\mathcal{A})$, at least for $Z(\gamma)/Z(\beta) \notin (0, \infty)$:

$$(47) \quad \mathrm{d}f^{\beta+\gamma}(Z) = \left(f^{\beta}(Z) * f^{\gamma}(Z) - f^{\gamma}(Z) * f^{\beta}(Z) \right) \otimes \left(\tfrac{\mathrm{d}(Z(\gamma))}{Z(\gamma)} - \tfrac{\mathrm{d}(Z(\beta))}{Z(\beta)} \right).$$

Here $f^{\beta+\gamma}$ is an $\mathcal{L}$–valued function on $\mathrm{Stab}(\mathcal{A})$, so $\mathrm{d}f^{\beta+\gamma}$ is an $\mathcal{L}$–valued 1–form, that is, a section of $\mathcal{L} \otimes_{\mathbb{C}} T_{\mathbb{C}}^* \mathrm{Stab}(\mathcal{A})$. Also $Z(\gamma), Z(\beta)$ are complex functions on $\mathrm{Stab}(\mathcal{A})$, so $\mathrm{d}(Z(\gamma))/Z(\gamma)$, $\mathrm{d}(Z(\beta))/Z(\beta)$ are complex 1–forms on $\mathrm{Stab}(\mathcal{A})$, and tensoring over $\mathbb{C}$ with $f^{\beta}(Z) * f^{\gamma}(Z) - f^{\gamma}(Z) * f^{\beta}(Z)$ also gives an $\mathcal{L}$–valued 1–form on $\mathrm{Stab}(\mathcal{A})$. Note that $\epsilon^{\beta}(\mu), \epsilon^{\gamma}(\mu), \epsilon^{\beta+\gamma}(\mu)$ are locally constant in Z away from $Z(\gamma)/Z(\beta) \in (0, \infty)$, so there are no terms from differentiating them. Also, by construction $f^{\beta}, f^{\gamma}, f^{\beta+\gamma}$ are continuous and holomorphic over the hypersurface $Z(\gamma)/Z(\beta) \in (0, \infty)$, so by continuity (47) holds there too.

We now guess that the generating functions f^{α} of (27) should satisfy the p.d.e., for all $\alpha \in C(\mathcal{A})$:

$$
\begin{aligned}
\mathrm{d}f^{\alpha}(Z) &= \sum_{\beta,\gamma \in C(\mathcal{A}):\alpha=\beta+\gamma} \left(f^{\beta}(Z) * f^{\gamma}(Z) \right) \otimes \left(\frac{\mathrm{d}(Z(\gamma))}{Z(\gamma)} - \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} \right) \\
&= \sum_{\beta,\gamma \in C(\mathcal{A}):\alpha=\beta+\gamma} \left(\tfrac{1}{2}[f^{\beta}(Z), f^{\gamma}(Z)] \right) \otimes \left(\frac{\mathrm{d}(Z(\gamma))}{Z(\gamma)} - \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} \right) \\
&= - \sum_{\beta,\gamma \in C(\mathcal{A}):\alpha=\beta+\gamma} \left([f^{\beta}(Z), f^{\gamma}(Z)] \right) \otimes \frac{\mathrm{d}(Z(\beta))}{Z(\beta)},
\end{aligned}
$$

(48)

where the three lines are equivalent, noting that we may exchange β, γ. In the simple case above this reduces to (47) when $\alpha = \beta + \gamma$.

We can now explain our choice of constant $F_1 \equiv (2\pi i)^{-1}$ in (29). The $2\pi i$ comes from the jumping of $\log z$ over $(0, \infty)$ as above. If we had instead set $F_1 \equiv c$ for some $c \in \mathbb{C}$, then f^{α}, F_n and the right hand side of (46) would be multiplied by $2\pi i\, c$, and the right hand sides of (47)–(48), and (65) below, would be multiplied by $(2\pi i\, c)^{-1}$. We picked $c = (2\pi i)^{-1}$ to eliminate constant factors in the p.d.e. (48) and flat connection (65), and so simplify our equations.

For (48) to hold, it is clearly necessary that the right hand side should be closed. We check this by applying d to it and using (48), giving:

$$
d\left[\sum_{\beta,\gamma\in C(\mathcal{A}):\alpha=\beta+\gamma}(f^\beta(Z) * f^\gamma(Z))\otimes\left(\tfrac{d(Z(\gamma))}{Z(\gamma)}-\tfrac{d(Z(\beta))}{Z(\beta)}\right)\right]
$$

(49)
$$
=\sum_{\epsilon,\delta\in C(\mathcal{A}):\alpha=\epsilon+\delta}\left(df^\epsilon(Z) * f^\delta(Z)\right)\wedge\left(\tfrac{d(Z(\delta))}{Z(\delta)}-\tfrac{d(Z(\epsilon))}{Z(\epsilon)}\right)
$$
$$
+\sum_{\beta,\epsilon\in C(\mathcal{A}):\alpha=\beta+\epsilon}\left(f^\beta(Z) * df^\epsilon(Z)\right)\wedge\left(\tfrac{d(Z(\epsilon))}{Z(\epsilon)}-\tfrac{d(Z(\beta))}{Z(\beta)}\right)
$$
$$
=\sum_{\beta,\gamma,\delta\in C(\mathcal{A}):\alpha=\beta+\gamma+\delta}\left(f^\beta(Z) * f^\gamma(Z) * f^\delta(Z)\right)\otimes
$$
$$
\left[\left(\tfrac{d(Z(\gamma))}{Z(\gamma)}-\tfrac{d(Z(\beta))}{Z(\beta)}\right)\wedge\left(\tfrac{d(Z(\delta))}{Z(\delta)}-\tfrac{d(Z(\beta+\gamma))}{Z(\beta+\gamma)}\right)\right.
$$
$$
\left.+\left(\tfrac{d(Z(\delta))}{Z(\delta)}-\tfrac{d(Z(\gamma))}{Z(\gamma)}\right)\wedge\left(\tfrac{d(Z(\gamma+\delta))}{Z(\gamma+\delta)}-\tfrac{d(Z(\beta))}{Z(\beta)}\right)\right]=0.
$$

Here expanding the first line gives $\sum_{\beta,\gamma}df^\beta * f^\gamma\wedge(\cdots)+\sum_{\beta,\gamma}f^\beta * df^\gamma\wedge(\cdots)$, as the final 1–form is closed. These two terms appear in the second and third lines, with β,γ relabelled as ϵ,δ in the second line and β,ϵ in the third. The fourth, fifth and sixth lines substitute (48) into the second and third lines, with ϵ in place of α for the second line and ϵ,γ,δ in place of α,β,γ for the third line. The final step holds as the 2–form $[\cdots]$ on the fourth and fifth lines is zero.

Thus equation (48) has the attractive property that *it implies its own consistency condition*; that is, the condition for (48) to be locally solvable for f^α is equation (48) for β,γ. We express (48) in terms of the functions F_n and G_n.

Proposition 3.8 *Suppose Assumption 2.1, Assumption 2.14 and Assumption 3.2 hold for $\mathcal{A}$ with $K(\mathcal{A})$ of finite rank and $\mathrm{Stab}(\mathcal{A})$ is a nonempty open subset of $\mathrm{Hom}(K(\mathcal{A}),\mathbb{C})$, and let some functions $F_n\colon(\mathbb{C}^\times)^n\to\mathbb{C}$ be given. Then a sufficient condition for the functions f^α of (27) to satisfy (48), is that for fixed $(\tilde{z}_1,\dots,\tilde{z}_n)\in H^n$ the functions $(z_1,\dots,z_n)\mapsto G_n(z_1,\dots,z_n;\tilde{z}_1,\dots,\tilde{z}_n)$ of Section 3.1 should satisfy*

(50)
$$
dG_n(z_1,\dots,z_n;\tilde{z}_1,\dots,\tilde{z}_n)=\sum_{k=1}^{n-1}G_k(z_1,\dots,z_k;\tilde{z}_1,\dots,\tilde{z}_k)\cdot
$$
$$
G_{n-k}(z_{k+1},\dots,z_n;\tilde{z}_{k+1},\dots,\tilde{z}_n)\left(\frac{dz_{k+1}+\cdots+dz_n}{z_{k+1}+\cdots+z_n}-\frac{dz_1+\cdots+dz_k}{z_1+\cdots+z_k}\right)
$$

in H^n. This holds for some $(\tilde{z}_1,\dots,\tilde{z}_n)$ in H^n if and only if it holds for all $(\tilde{z}_1,\dots,\tilde{z}_n)$ in H^n.

This condition is also necessary, for all values of n occurring in (27), if the terms $\epsilon^{\alpha_1}(\mu) * \epsilon^{\alpha_2}(\mu) * \cdots * \epsilon^{\alpha_n}(\mu)$ occurring in (27) are linearly independent in $\mathcal{H}$. This happens in the examples of [14, Example 7.10], for arbitrarily large n.

Now suppose Condition 3.4 holds. Then equation (50) holds for $(z_1, \ldots, z_n)$ in $N_{(\widetilde{z}_1, \ldots, \widetilde{z}_n)}$ for all $n \geq 1$ and all fixed $(\widetilde{z}_1, \ldots, \widetilde{z}_n) \in (\mathbb{C}^\times)^n$ if and only if the following p.d.e. holds on the domain (44) for all $n \geq 1$:

$$
(51) \qquad \mathrm{d}F_n(z_1, \ldots, z_n) = \sum_{k=1}^{n-1} F_k(z_1, \ldots, z_k) F_{n-k}(z_{k+1}, \ldots, z_n) \cdot \left[\frac{\mathrm{d}z_{k+1} + \cdots + \mathrm{d}z_n}{z_{k+1} + \cdots + z_n} - \frac{\mathrm{d}z_1 + \cdots + \mathrm{d}z_k}{z_1 + \cdots + z_k} \right].
$$

Proof For the first part, substitute (36) in for f^α, f^β and f^γ in the top line of (48). Then both sides can be rewritten as a sum over $\alpha_1, \ldots, \alpha_n \in C(\mathcal{A})$ with $\alpha_1 + \cdots + \alpha_n = \alpha$ of $\epsilon^{\alpha_1}(\widetilde{\mu}) * \cdots * \epsilon^{\alpha_n}(\widetilde{\mu})$ tensored with complex 1–forms. Equating the complex 1–form coefficients of $\epsilon^{\alpha_1}(\widetilde{\mu}) * \cdots * \epsilon^{\alpha_n}(\widetilde{\mu})$ gives (50), evaluated at $z_a = Z(\alpha_a)$ and $\widetilde{z}_a = \widetilde{Z}(\alpha_a)$, where the G_k term comes from f^β in (48) with $\beta = \alpha_1 + \cdots + \alpha_k$, and the G_{n-k} term comes from f^γ in (48) with $\gamma = \alpha_{k+1} + \cdots + \alpha_n$. The first two paragraphs follow, by the same arguments used to prove Proposition 3.3.

For the final part, let Condition 3.4 hold. If $(\widetilde{z}_1, \ldots, \widetilde{z}_n) \in (\mathbb{C}^\times)^n$ with $\widetilde{z}_{k+1}/\widetilde{z}_k \notin (0, \infty)$ for $1 \leq k < n$ and $(z_1, \ldots, z_n) \in N_{(\widetilde{z}_1, \ldots, \widetilde{z}_n)}$, then the proof of Proposition 3.5 shows that:

$$
\begin{aligned}
F_n(z_1, \ldots, z_n) &= G_n(z_1, \ldots, z_n; \widetilde{z}_1, \ldots, \widetilde{z}_n), \\
F_k(z_1, \ldots, z_k) &= G_k(z_1, \ldots, z_k; \widetilde{z}_1, \ldots, \widetilde{z}_k), \\
F_{n-k}(z_{k+1}, \ldots, z_n) &= G_{n-k}(z_{k+1}, \ldots, z_n; \widetilde{z}_{k+1}, \ldots, \widetilde{z}_n).
\end{aligned}
$$

Thus (50) is equivalent to (51) in $N_{(\widetilde{z}_1, \ldots, \widetilde{z}_n)}$, so (50) implies (51) in the domain (44).

For the reverse implication, suppose Condition 3.4 holds and (51) holds in (44). Then the argument above shows (50) holds for $(z_1, \ldots, z_n)$, $(\widetilde{z}_1, \ldots, \widetilde{z}_n)$ in (44) with $(z_1, \ldots, z_n) \in N_{(\widetilde{z}_1, \ldots, \widetilde{z}_n)}$. But whether (50) holds or not is unaffected by small changes in $(\widetilde{z}_1, \ldots, \widetilde{z}_n)$, and (44) is dense in $(\mathbb{C}^\times)^n$. Thus, (50) holds for all $(\widetilde{z}_1, \ldots, \widetilde{z}_n) \in (\mathbb{C}^\times)^n$ and $(z_1, \ldots, z_n) \in N_{(\widetilde{z}_1, \ldots, \widetilde{z}_n)}$ with $(z_1, \ldots, z_n)$ in (44). Now by Condition 3.4 the functions $(z_1, \ldots, z_m) \mapsto G_m(z_1, \ldots, z_m; \widetilde{z}_1, \ldots, \widetilde{z}_m)$ are continuous and holomorphic, so as (44) is open and dense we see that (50) must hold for all $(z_1, \ldots, z_n) \in N_{(\widetilde{z}_1, \ldots, \widetilde{z}_n)}$, by continuity. $\qquad\square$

The proof of Proposition 3.8 conceals a subtlety. One might think that for generic $Z \in \text{Stab}(\mathcal{A})$, all terms $(Z(\alpha_1), \ldots, Z(\alpha_n))$ occurring in (27) will lie in the open dense domain (44), so that (51) on (44) implies (48) for generic Z in the obvious way, and so (48) must hold for all Z by continuity. However, this is false. For example, if $\alpha_1 = \alpha_2$ then $Z(\alpha_2)/Z(\alpha_1) \equiv 1 \in (0, \infty)$, so $(Z(\alpha_1), \ldots, Z(\alpha_n))$ does not lie in (44) for any $Z \in \text{Stab}(\mathcal{A})$. So assuming (51) on (44) apparently tells us nothing about how such terms contribute to (48).

Because of this, for (48) to hold when f^α in (27) includes terms with dependencies such as $\alpha_1 = \alpha_2$, we need F_n to satisfy not just (51) on (44), but other more complicated conditions on the real hypersurfaces $z_{k+1}/z_k \in (0, \infty)$ as well. The point of the proof is that these other conditions are implied by (51) on (44) and Condition 3.4, as we can express the conditions in terms of the $G_n(z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n)$ and use the fact that they are continuous and holomorphic in $(z_1, \ldots, z_n)$ over the hypersurfaces $z_{k+1}/z_k \in (0, \infty)$.

Equations (50) and (51) apparently have poles on the hypersurfaces $z_1 + \cdots + z_k = 0$ and $z_{k+1} + \cdots + z_n = 0$. So we would expect solutions G_n, F_n to have log-type singularities along these hypersurfaces; in particular, this suggests that there should not be single-valued solutions on the domain (44). In fact this is false, and single-valued, nonsingular solutions can exist across these hypersurfaces. The next proposition explains why.

Proposition 3.9 *For $n \geqslant 2$ the following is a nonempty, connected set in $\mathbb{C}^n$:*

(52)
$$\{(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n : z_{k+1}/z_k \notin (0, \infty) \text{ for } k = 1, \ldots, n-1$$
$$\text{and } z_1 + \cdots + z_n = 0\}.$$

If F_n satisfies (51) on the domain (44) then $F_n \equiv C_n$ on (52) for some $C_n \in \mathbb{C}$. If F_n also satisfies (31) as in Remark 3.1 then $C_n = 0$. In this case we have $F_n(z_1, \ldots, z_n) = (z_1 + \cdots + z_n) H_n(z_1, \ldots, z_n)$ for a holomorphic function H_n defined on the domain (44), including where $z_1 + \cdots + z_n \equiv 0$. Using these we rewrite (51) as

(53)
$$dF_n(z_1, \ldots, z_n) = \sum_{k=1}^{n-1} H_k(z_1, \ldots, z_k) H_{n-k}(z_{k+1}, \ldots, z_n) \cdot$$
$$\big((z_1 + \cdots + z_k)(dz_{k+1} + \cdots + dz_n) - (z_{k+1} + \cdots + z_n)(dz_1 + \cdots + dz_k)\big).$$

Note that (53) has no poles on (44).

Proof Let $(z_1, \ldots, z_n), (z'_1, \ldots, z'_n)$ lie in (52). We shall construct a path between them in (52), showing (52) is connected. It is easy to see that

$$\begin{aligned}
(54) \quad &\{w \in \mathbb{C} : (z_1, \ldots, z_{n-2}, w, z_{n-1} + z_n - w) \text{ lies in (52)}\} = \\
&\qquad \mathbb{C} \setminus \big(\{x(z_{n-1} + z_n) : x \in [0,1]\} \cup \{xz_{n-2} : x \in [0, \infty)\}\big),
\end{aligned}$$

$$\begin{aligned}
(55) \quad &\{w \in \mathbb{C} : (z'_1, \ldots, z'_{n-2}, w, z'_{n-1} + z'_n - w) \text{ lies in (52)}\} = \\
&\qquad \mathbb{C} \setminus \big(\{x(z'_{n-1} + z'_n) : x \in [0,1]\} \cup \{xz'_{n-2} : x \in [0, \infty)\}\big),
\end{aligned}$$

which are both *connected* subsets of $\mathbb{C}$, containing z_{n-1} and z'_{n-1} respectively. Choose some w_0 in both (54) and (55) with $|w_0| \gg |z_k|, |z'_k|$ for all $k = 1, \ldots, n$. Choose paths between z_{n-1} and w_0 in (54), and between z'_{n-1} and w_0 in (55). These induce paths in (52) between $(z_1, \ldots, z_n)$ and $(z_1, \ldots, z_{n-2}, w_0, z_{n-1} + z_n - w_0)$, and between $(z'_1, \ldots, z'_n)$ and $(z'_1, \ldots, z'_{n-2}, w_0, z'_{n-1} + z'_n - w_0)$.

It remains to find a path in (52) between $(z_1, \ldots, z_{n-2}, w_0, z_{n-1} + z_n - w_0)$ and $(z'_1, \ldots, z'_{n-2}, w_0, z'_{n-1} + z'_n - w_0)$. To do this, choose a path $(x_1(t), \ldots, x_{n-2}(t))$ for $t \in [0, 1]$ between $(z_1, \ldots, z_{n-2})$ and $(z'_1, \ldots, z'_{n-2})$ in

$$\begin{aligned}
\{(y_1, \ldots, y_{n-2}) \in (\mathbb{C}^\times)^{n-2} : &\; y_{k+1}/y_k \notin (0, \infty) \text{ for } k = 1, \ldots, n-3 \\
&\text{and } y_{n-2}/w_0 \notin (0, \infty)\},
\end{aligned}$$

which is possible as using y_{k+1}/y_k, y_{n-2}/w_0 as coordinates we see this domain is homeomorphic to $(\mathbb{C} \setminus [0, \infty))^{n-2}$, and thus is connected. Making w_0 larger if necessary, we can also assume that $|w_0| \gg |x_k(t)|$ for all $k = 1, \ldots, n-2$ and $t \in [0, 1]$. Then it is easy to see that the path $(x_1(t), \ldots, x_{n-2}(t), w_0, -x_1(t) - \cdots - x_{n-2}(t) - w_0)$ for $t \in [0, 1]$ links $(z_1, \ldots, z_{n-2}, w_0, z_{n-1} + z_n - w_0)$ and $(z'_1, \ldots, z'_{n-2}, w_0, z'_{n-1} + z'_n - w_0)$ in (52). Therefore (52) is connected.

For the second part, observe that on the hypersurface $z_1 + \cdots + z_n = 0$ we have $z_{k+1} + \cdots + z_n \equiv -(z_1 + \cdots + z_k)$, so the 1–form $[\cdots]$ in (51) restricts to zero on $z_1 + \cdots + z_n = 0$. Thus the restriction of dF_n to (52) is zero, so $F_n \equiv C_n$ on (52) for some $C_n \in \mathbb{C}$, as (52) is connected. For generic $(z_1, \ldots, z_n)$ in (52) all the permutations $(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ for $\sigma \in S_n$ lie in (52) as well, so (31) becomes $n!C_n = 0$, giving $C_n = 0$. Thus, the holomorphic function F_n is zero along the nonsingular hypersurface $z_1 + \cdots + z_n = 0$ in its domain (44). Properties of holomorphic functions imply that $F_n(z_1, \ldots, z_n) = (z_1 + \cdots + z_n) H_n(z_1, \ldots, z_n)$ for a unique holomorphic function H_n on (44). Equation (53) is immediate. $\square$

Suppose we are given some holomorphic functions F_n on the domains (44) satisfying (51). Analytically continue the F_n to *multivalued*, *singular* holomorphic functions $\widetilde{F}_n$ on $(\mathbb{C}^\times)^n$, still satisfying (51). The argument above shows that $\widetilde{F}_n$ is locally constant

along $z_1 + \cdots + z_n = 0$, but it can take a different value on each sheet. So (51) can have genuine poles along $z_1 + \cdots + z_k = 0$ and $z_{k+1} + \cdots + z_n = 0$ when $\widetilde{F}_k, \widetilde{F}_{n-k}$ are nonzero constants.

Thus $\widetilde{F}_n$ will have log-like singularities along $z_1 + \cdots + z_k = 0$ and $z_{k+1} + \cdots + z_n = 0$, and more generally singularities along $z_a + \cdots + z_b = 0$ for $1 \leqslant a \leqslant b \leqslant n$ with $(a, b) \neq (1, n)$. One moral is that our functions F_n manage to be single-valued and nonsingular on (44) for very special reasons, and their analytic continuations have much worse singularities and branching behaviour.

We now construct functions F_n for $n \geqslant 1$ satisfying (51), by induction on n.

Proposition 3.10 *There exists a unique series of holomorphic functions F_n for $n \geqslant 1$ defined on the domain (44) with $F_1 \equiv (2\pi i)^{-1}$, such that F_n satisfies (51) and is zero on (52). Also $F_n(z_1, \ldots, z_n) \equiv (z_1 + \cdots + z_n) H_n(z_1, \ldots, z_n)$ for a unique holomorphic function H_n defined on (44), and (53) holds.*

Proof Suppose by induction that for some $m \geqslant 2$ we have constructed unique holomorphic functions F_n, H_n on the domains (44) for $n = 1, \ldots, m - 1$ satisfying all the conditions of the proposition for $n < m$. For $m = 2$ this is trivial, as we must have $F_1(z) = (2\pi i)^{-1}$ and $H_1(z) = (2\pi i z)^{-1}$. We will construct F_m, H_m, and show they are unique.

Equations (51) and (53) for $n = m$ give equivalent expressions for dF_m on (44), with (53) being manifestly holomorphic on all of (44). Write α_m for the right hand side of (51) or (53), so that α_m is a holomorphic 1–form on (44), and we need to construct F_m with $dF_m = \alpha_m$. Following (49), we can compute $d\alpha_m$ by applying d to the r.h.s. of (51) for $n = m$, and using (51) for $n < m$ (which holds by induction) to substitute in for dF_k and dF_{n-k}. Then everything cancels giving $d\alpha_m = 0$, so α_m is a *closed* 1–form.

Although (44) is not simply connected, it is the pullback to $\mathbb{C}^m \setminus \{0\}$ of

$$(56) \qquad \{[z_1, \ldots, z_m] \in \mathbb{CP}^{m-1} : z_k \neq 0 \text{ and } z_{k+1}/z_k \notin (0, \infty) \text{ for all } k\},$$

which is homeomorphic to $(\mathbb{C} \setminus [0, \infty))^{m-1}$, and so is simply connected. Now α_m is the pullback of a 1–form on (56), which is closed as α_m is closed, and so is *exact* as (56) is simply connected.

Therefore α_m is an exact holomorphic 1–form on (44), so there exists a holomorphic function F_m on (44) with $dF_m = \alpha_m$, which is unique up to addition of a constant, as (44) is connected. To choose the constant, note that the restriction of α_m to the connected set (52) is zero as in Proposition 3.9, so requiring F_m to be zero on (52) fixes

F_m uniquely. Since F_m is zero along the nonsingular hypersurface $z_1 + \cdots + z_m = 0$, by properties of holomorphic functions there is a unique holomorphic function H_m on (44) with $F_m(z_1, \ldots, z_m) \equiv (z_1 + \cdots + z_m) H_m(z_1, \ldots, z_m)$. These F_m, H_m satisfy all the conditions for $n = m$, and the proposition follows by induction. $\qquad\square$

3.3 Reconciling the approaches of Section 3.1 and Section 3.2

So far we have given two quite different approaches to the functions F_n used to define f^α in (27). In Section 3.1 we found conditions on the F_n on $(\mathbb{C}^\times)^n$ for the f^α to be continuous and holomorphic, and showed such F_n would be unique if they existed. In Section 3.2 we found different conditions on the F_n on a subdomain (44) of $(\mathbb{C}^\times)^n$ for the f^α to satisfy the p.d.e. (51), neglecting the question of whether f^α would be continuous for these F_n, and constructed unique F_n satisfying these second conditions. There seems no *a priori* reason why these two sets of conditions on F_n should be compatible, but we now prove that they are. That is, we show that the F_n on (44) constructed in Proposition 3.10 extend uniquely to $(\mathbb{C}^\times)^n$ so as to satisfy Remark 3.1 and Condition 3.4.

First we show, in effect, that Proposition 3.6 holds for the F_n of Proposition 3.10. For F_n as in Proposition 3.10, define a function $D_{l,n}$ for $1 \leqslant l < n$ by:

$$D_{l,n} : \left\{ (\widetilde{z}_1, \ldots, \widetilde{z}_n) \in (\mathbb{C}^\times)^n : \widetilde{z}_{l+1}/\widetilde{z}_l \in (0, \infty), \right.$$

$$\left. \widetilde{z}_{k+1}/\widetilde{z}_k \notin (0, \infty) \text{ for } 1 \leqslant k < n, \, k \neq l \right\} \longrightarrow \mathbb{C},$$

$$(57) \qquad D_{l,n}(\widetilde{z}_1, \ldots, \widetilde{z}_n) = \lim_{\substack{(z_1, \ldots, z_n) \to (\widetilde{z}_1, \ldots, \widetilde{z}_n): \\ (z_1, \ldots, z_n) \text{ lies in } (44), \\ \mathrm{Im}(z_{l+1}/z_l) < 0}} F_n(z_1, \ldots, z_n) - \lim_{\substack{(z_1, \ldots, z_n) \to (\widetilde{z}_1, \ldots, \widetilde{z}_n): \\ (z_1, \ldots, z_n) \text{ lies in } (44), \\ \mathrm{Im}(z_{l+1}/z_l) > 0}} F_n(z_1, \ldots, z_n).$$

These limits exist and give a continuous function $D_{l,n}$, since the proof in Proposition 3.10 that the F_n are continuous and holomorphic in their domains extends locally from either side over the hypersurface $z_{l+1}/z_l \in (0, \infty)$. The next result would follow from (45) if we knew Proposition 3.6 applied.

Proposition 3.11 *We have*

$$D_{l,n}(z_1, \ldots, z_n) \equiv F_{n-1}(z_1, \ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2}, \ldots, z_n)$$

on the domain of $D_{l,n}$, where F_{n-1} is as in Proposition 3.10.

Proof Taking the difference of the limits of (51) from both sides of the hypersurface $z_{l+1}/z_l \in (0, \infty)$ gives an equation in 1–forms on the domain of $D_{l,n}$:

$$(58) \quad \mathrm{d}D_{l,n}(z_1, \ldots, z_n) =$$

$$\sum_{k=1}^{l-1} F_k(z_1, \ldots, z_k) D_{l-k,n-k}(z_{k+1}, \ldots, z_n) \cdot \left[\frac{\mathrm{d}z_{k+1} + \cdots + \mathrm{d}z_n}{z_{k+1} + \cdots + z_n} - \frac{\mathrm{d}z_1 + \cdots + \mathrm{d}z_k}{z_1 + \cdots + z_k} \right] +$$

$$\sum_{k=l+1}^{n-1} D_{l,k}(z_1, \ldots, z_k) F_{n-k}(z_{k+1}, \ldots, z_n) \cdot \left[\frac{\mathrm{d}z_{k+1} + \cdots + \mathrm{d}z_n}{z_{k+1} + \cdots + z_n} - \frac{\mathrm{d}z_1 + \cdots + \mathrm{d}z_k}{z_1 + \cdots + z_k} \right].$$

Here if $(z_1, \ldots, z_n)$ lies in the domain of $D_{l,n}$ and $k < l$ then F_k is defined and continuous at $(z_1, \ldots, z_k)$ but F_{n-k} is not defined (nor continuous) at $(z_{k+1}, \ldots, z_n)$, so the difference in limits of $F_k(z_1, \ldots, z_k) F_{n-k}(z_{k+1}, \ldots, z_n)$ in (51) is equal to $F_k(z_1, \ldots, z_k) D_{l-k,n-k}(z_{k+1}, \ldots, z_n)$, giving the first term in (58). Similarly, $k > l$ gives the second term. There is no term $k = l$ in (58), since F_l and F_{n-l} are both defined and continuous at $(z_1, \ldots, z_l)$ and $(z_{l+1}, \ldots, z_n)$ respectively, so the limits from each side of $z_{l+1}/z_l \in (0, \infty)$ cancel.

As $F_n(z_1, \ldots, z_n) = 0$ when $z_1 + \cdots + z_n = 0$, if $(\tilde{z}_1, \ldots, \tilde{z}_n)$ lies in the domain of $D_{l,n}$ with $\tilde{z}_1 + \cdots + \tilde{z}_n = 0$ then both limits in (57) are zero, as $(\tilde{z}_1, \ldots, \tilde{z}_n)$ is the limit of points $(z_1, \ldots, z_n)$ with $z_1 + \cdots + z_n = 0$ from both sides of $z_{l+1}/z_l \in (0, \infty)$. Thus

$$(59) \qquad\qquad D_{l,n}(z_1, \ldots, z_n) = 0 \quad \text{if } z_1 + \cdots + z_n = 0.$$

Also, it is easy to verify from Proposition 3.10 that F_2 is given by (46) in its domain, so from properties of logs we see that

$$(60) \qquad\qquad D_{1,2}(z_1, z_2) \equiv (2\pi i)^{-1} \equiv F_1(z_1 + z_2).$$

By induction, suppose $D_{l,n}(z_1, \ldots, z_n) \equiv F_{n-1}(z_1, \ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2}, \ldots, z_n)$ whenever $1 \leqslant l < n \leqslant m$, for some $m \geqslant 2$. The first case $m = 2$ is (60). Let $n = m+1$ and $1 \leqslant l < n$. Then comparing (51) and (58) and using the inductive hypothesis shows that

$$\mathrm{d}D_{l,n}(z_1, \ldots, z_n) \equiv \mathrm{d}F_{n-1}(z_1, \ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2}, \ldots, z_n).$$

Thus $D_{l,n}(z_1, \ldots, z_n) - F_{n-1}(z_1, \ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2}, \ldots, z_n)$ is constant on the domain of $D_{l,n}$. But this domain is connected and contains $(z_1, \ldots, z_n)$ with $z_1 + \cdots + z_n = 0$ as $n \geqslant 3$, and both $D_{l,n}(\cdots)$ and $F_{n-1}(\cdots)$ are zero at such points by Proposition 3.10 and (59). So the constant is zero, proving the inductive step and the proposition. $\qquad\square$

Using this we extend the F_n of Proposition 3.10 so that Condition 3.4 holds.

Theorem 3.12 *The functions F_n of Proposition 3.10, defined on the domain* (44), *can be extended uniquely to $F_n\colon (\mathbb{C}^\times)^n \to \mathbb{C}$ satisfying Condition 3.4.*

Proof The idea of the proof is that by induction on n we shall construct functions $G_n(z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n)$ that for each fixed $(\tilde{z}_1, \ldots, \tilde{z}_n)$ are continuous and holomorphic and satisfy (50) in $(z_1, \ldots, z_n)$ on $N_{(\tilde{z}_1, \ldots, \tilde{z}_n)}$, such that (42) holds with F_n as in Proposition 3.10 whenever $(z_1, \ldots, z_n)$ lies in the intersection of (44) and $N_{(\tilde{z}_1, \ldots, \tilde{z}_n)}$. We then extend F_n uniquely from (44) to $(\mathbb{C}^\times)^n$ by requiring (42) to hold on all of $N_{(\tilde{z}_1, \ldots, \tilde{z}_n)}$, for all $(\tilde{z}_1, \ldots, \tilde{z}_n)$.

Suppose by induction that for some $p \geqslant 2$ and for all $n < p$ we have found open neighbourhoods $N_{(\tilde{z}_1, \ldots, \tilde{z}_n)}$ of $(\tilde{z}_1, \ldots, \tilde{z}_n)$ in $(\mathbb{C}^\times)^n$ for all $(\tilde{z}_1, \ldots, \tilde{z}_n) \in (\mathbb{C}^\times)^n$, and functions G_n, and extensions of F_n in Proposition 3.10 to $(\mathbb{C}^\times)^n$, such that Condition 3.4 holds for $n < p$ and the G_n satisfy (50), and $G_n(z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n) = 0$ if $z_1 + \cdots + z_n = 0$. The first case $p = 2$ is trivial, taking $F_1 \equiv (2\pi i)^{-1} \equiv G_1$ and $N_{(\tilde{z}_1)} = \mathbb{C}^\times$. We shall now construct open neighbourhoods $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$, the function G_p, and an extension of F_p, satisfying all the conditions.

Choose a connected, simply connected open neighbourhood $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ of each point $(\tilde{z}_1, \ldots, \tilde{z}_p)$ in $(\mathbb{C}^\times)^p$, such that $(z_1, \ldots, z_p) \in N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ implies that (a) if $1 \leqslant m < p$, $0 = a_0 < a_1 < \cdots < a_m = p$ and $c_1, \ldots, c_m \in [0, 2\pi)$ with $\tilde{z}_a \in e^{ic_k}(0, \infty)$ for $a_{k-1} < a \leqslant a_k$, then

$$(z_{a_0+1} + \cdots + z_{a_1}, \ldots, z_{a_{m-1}+1} + \cdots + z_{a_m}) \in N_{(\tilde{z}_{a_0+1} + \cdots + \tilde{z}_{a_1}, \ldots, \tilde{z}_{a_{m-1}+1} + \cdots + \tilde{z}_{a_m})},$$

and (b) if $1 \leqslant k < p$ then $(z_1, \ldots, z_k) \in N_{(\tilde{z}_1, \ldots, \tilde{z}_k)}$ and $(z_{k+1}, \ldots, z_p) \in N_{(\tilde{z}_{k+1}, \ldots, \tilde{z}_p)}$. This is satisfied if $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ is a small enough open ball about $(\tilde{z}_1, \ldots, \tilde{z}_p)$. The point is that (a) ensures that all the terms in (42) with $n = p$ and $m < p$ are well-defined when $(z_1, \ldots, z_p) \in N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$, and (b) ensures that the right hand side of (50) for $n = p$ is well-defined when $(z_1, \ldots, z_p) \in N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$.

Now regard $(\tilde{z}_1, \ldots, \tilde{z}_p)$ as fixed, and consider equation (50) with $n = p$ for $(z_1, \ldots, z_p)$ in $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$. The left hand side $dG_p(\cdots)$ has not yet been defined. The right hand side involves G_k for $k < p$, which by induction are defined on their domains and satisfy (50). The choice of $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ implies the r.h.s. is a 1–form defined on $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$, and taking d and using (50) for $n < p$ we find this 1–form is *closed*, as for dF_m in the proof of Proposition 3.10. Also, as for (49) and for F_n in Proposition 3.9, the inductive assumption $G_n(z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n) = 0$ if $z_1 + \cdots + z_n = 0$ ensures that the terms $(z_1 + \cdots + z_k)^{-1}$, $(z_{k+1} + \cdots + z_n)^{-1}$ in (50) do not induce singularities.

This proves that the right hand side of (50) for $n = p$ is a well-defined, closed, holomorphic, nonsingular 1–form on the connected, simply connected domain $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$.

Hence there exists a holomorphic function $(z_1, \ldots, z_p) \mapsto G_p(z_1, \ldots, z_p; \tilde{z}_1, \ldots, \tilde{z}_p)$ on $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$, unique up to addition of a constant, such that (50) holds. Here is how we fix the constant. Recall that so far F_p has been defined on the open dense domain (44) in Proposition 3.10, and F_n for $n < p$ has been defined on all of $(\mathbb{C}^\times)^n$. Thus, every term in (42) with $n = p$ is now defined on the intersection of (44) and $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$; note that the only term on the r.h.s. of (42) with $m = n = p$ is $G_p(z_1, \ldots, z_p; \tilde{z}_1, \ldots, \tilde{z}_p)$. This intersection is also open and nonempty, as $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ is nonempty and (44) is dense.

We claim that there is a unique function G_p satisfying (50) such that (42) holds on the intersection of (44) and $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$. To see this, note that by Proposition 3.8, equations (50) and (51) are equivalent when (42) holds. Thus, for any choice of G_p satisfying (50), applying d to both sides of (42) gives the same thing, so the difference between the left and right hand sides of (42) is locally constant on the intersection of $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ and (44). Fix a connected component C of this intersection. Then we can choose G_p uniquely such that (42) holds on this connected component, and on every other connected component the difference between the left and right hand sides of (42) is constant.

Suppose C', C'' are connected components of the intersection of $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ and (44) which meet along the real hypersurface $z_{l+1}/z_l \in (0, \infty)$ for $1 \leqslant l < p$. (That is, the closures of C', C'' must contain a nonempty open subset of this hypersurface). Then Proposition 3.11 computes how much F_p jumps across this hypersurface, which by Proposition 3.6 follows from the condition for G_p to be continuous across the hypersurface. It is not difficult to deduce that the difference between the left and right hand sides of (42) must take the same constant value on C' and C''. Since this value is 0 on one component C, and as $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ is open and connected we can get from C to any other component C' by crossing hypersurfaces $z_{l+1}/z_l \in (0, \infty)$ one after the other, the constant is zero for every C'. This proves the claim.

We have now defined the functions G_p. If $(z_1, \ldots, z_p)$ lies in the intersection of (44) and $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ with $z_1 + \cdots + z_p = 0$ then (42) holds at $(z_1, \ldots, z_p)$. There is a term $G_p(z_1, \ldots, z_p; \tilde{z}_1, \ldots, \tilde{z}_p)$ on the right hand side, and every other term is zero by $F_p(z_1, \ldots, z_p) = 0$ when $z_1 + \cdots + z_p = 0$ and the inductive hypothesis. Hence $G_p(z_1, \ldots, z_p; \tilde{z}_1, \ldots, \tilde{z}_p) = 0$. By continuity this extends to all $(z_1, \ldots, z_p)$ in $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ with $z_1 + \cdots + z_p = 0$, as we have to prove.

By construction, (42) holds on the intersection of $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ and the subset (44) where F_p is already defined by Proposition 3.10. We now extend F_p to $(\mathbb{C}^\times)^p$ by requiring F_p to satisfy (44) with $n = p$ on each domain $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$. Since the $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)}$ cover $(\mathbb{C}^\times)^p$ this defines F_p uniquely, but we must check that given $(\tilde{z}_1, \ldots, \tilde{z}_p)$ and

$(\hat{z}_1, \ldots, \hat{z}_p)$, equation (44) for $n = p$ gives the same answer for $F_p(z_1, \ldots, z_p)$ with the $\tilde{z}_k$ and $\hat{z}_k$ on the intersection $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)} \cap N_{(\hat{z}_1, \ldots, \hat{z}_p)}$.

This holds for the same reason that the conditions of Proposition 3.3 hold for some $(\tilde{z}_1, \ldots, \tilde{z}_n)$ if and only if they hold for all $(\tilde{z}_1, \ldots, \tilde{z}_n)$. The point is that the condition for f^α to be continuous is that we can write F_p in the form (44) near $(\tilde{z}_1, \ldots, \tilde{z}_p)$ for G_k continuous in $(z_1, \ldots, z_k)$, and these continuity conditions for the points $(\tilde{z}_1, \ldots, \tilde{z}_p), (\hat{z}_1, \ldots, \hat{z}_p)$ must be equivalent in the overlap $N_{(\tilde{z}_1, \ldots, \tilde{z}_p)} \cap N_{(\hat{z}_1, \ldots, \hat{z}_p)}$. We are using (42) to determine how to extend F_p from (44) to $(\mathbb{C}^\times)^p$ in a way that makes the f^α continuous, and these continuity conditions are independent of the choice of $(\tilde{z}_1, \ldots, \tilde{z}_p)$ or $(\hat{z}_1, \ldots, \hat{z}_p)$. Thus F_p is well defined and satisfies (42). This completes the inductive step, and the proof of Theorem 3.12. $\square$

Our next three results verify the remaining conditions of Remark 3.1.

Theorem 3.13 *For $n \geq 1$, define A_n to be the free $\mathbb{C}$–algebra with generators $e_1, \ldots, e_n$ and multiplication $*$, and L_n to be the free Lie subalgebra of A_n generated by $e_1, \ldots, e_n$ under the Lie bracket $[f, g] = f * g - g * f$. Then for any $(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n$ the following expression lies in L_n:*

(61) $$\sum_{\sigma \in S_n} F_n(z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)}) \, e_{\sigma(1)} * e_{\sigma(2)} * \cdots * e_{\sigma(n)},$$

where the F_n are as in Theorem 3.12 and S_n is the symmetric group. Also (31) holds, and f^α in (27) maps $\mathrm{Stab}(\mathcal{A}) \to \mathcal{L}^\alpha$, as in Remark 3.1(d).

Proof We shall first prove the first part of the theorem on the domain

(62) $$\left\{ (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n : z_k/z_l \notin (0, \infty) \text{ for all } 1 \leq k < l \leq n \right\}.$$

The point of this is that if $(z_1, \ldots, z_n)$ lies in (62) then $(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ lies in the domain (44) where F_n is holomorphic and satisfies (51) for all $\sigma \in S_n$.

Suppose by induction that for some $m \geq 2$ and all $n < m$, the expression (61) lies in L_n for all $(z_1, \ldots, z_n)$ in (62). Write P_m for (61) with $n = m$, regarded as a holomorphic

function from (62) to H_m. Then we have

$$
\mathrm{d}P_m(z_1,\ldots,z_n)
$$

$$
= \sum_{\sigma \in S_m} \sum_{k=1}^{m-1} F_k(z_{\sigma(1)},\ldots,z_{\sigma(k)})\, e_{\sigma(1)}*\cdots*e_{\sigma(k)}*
$$
$$
F_{m-k}(z_{\sigma(k+1)},\ldots,z_{\sigma(m)})e_{\sigma(k+1)}*\cdots*e_{\sigma(m)}
$$
$$
\otimes \left[\frac{\mathrm{d}z_{\sigma(k+1)}+\cdots+\mathrm{d}z_{\sigma(m)}}{z_{\sigma(k+1)}+\cdots+z_{\sigma(m)}} - \frac{\mathrm{d}z_{\sigma(1)}+\cdots+\mathrm{d}z_{\sigma(k)}}{z_{\sigma(1)}+\cdots+z_{\sigma(k)}} \right]
$$

$$
= \tfrac{1}{2} \sum_{\sigma \in S_m} \sum_{k=1}^{m-1} \left[F_k(z_{\sigma(1)},\ldots,z_{\sigma(k)})\, e_{\sigma(1)}*\cdots*e_{\sigma(k)}, \right.
$$
(63)
$$
\left. F_{m-k}(z_{\sigma(k+1)},\ldots,z_{\sigma(m)})\, e_{\sigma(k+1)}*\cdots*e_{\sigma(m)} \right]
$$
$$
\otimes \left[\frac{\mathrm{d}z_{\sigma(k+1)}+\cdots+\mathrm{d}z_{\sigma(m)}}{z_{\sigma(k+1)}+\cdots+z_{\sigma(m)}} - \frac{\mathrm{d}z_{\sigma(1)}+\cdots+\mathrm{d}z_{\sigma(k)}}{z_{\sigma(1)}+\cdots+z_{\sigma(k)}} \right]
$$

$$
= \tfrac{1}{2} \sum_{\sigma \in S_m} \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left[\sum_{\tau \in S_k} F_k(z_{\sigma\circ\tau(1)},\ldots,z_{\sigma\circ\tau(k)})e_{\sigma\circ\tau(1)}*\cdots*e_{\sigma\circ\tau(k)}, \right.
$$
$$
\sum_{\upsilon \in S_{m-k}} F_{m-k}(z_{\sigma(k+\upsilon(1))},\ldots,z_{\sigma(k+\upsilon(m-k))})
$$
$$
\left. e_{\sigma(k+\upsilon(1))}*\cdots*e_{\sigma(k+\upsilon(m-k))} \right]
$$
$$
\otimes \left[\frac{\mathrm{d}z_{\sigma(k+1)}+\cdots+\mathrm{d}z_{\sigma(m)}}{z_{\sigma(k+1)}+\cdots+z_{\sigma(m)}} - \frac{\mathrm{d}z_{\sigma(1)}+\cdots+\mathrm{d}z_{\sigma(k)}}{z_{\sigma(1)}+\cdots+z_{\sigma(k)}} \right].
$$

Here the second step is immediate from (51). The third step is the average of two copies of the second, one copy as it stands, the other relabelled with $m-k$ in place of k and indices $\sigma(k+1),\ldots,\sigma(m),\sigma(1),\ldots,\sigma(k)$ in place of the indices $\sigma(1),\ldots,\sigma(k),\sigma(k+1),\ldots,\sigma(m)$ respectively; this is valid because of the sum over $\sigma \in S_m$. The fourth and final step uses the fact that symmetrizing over S_m on $1,\ldots,m$ is equivalent to first symmetrizing over S_k on $1,\ldots,k$ and S_{m-k} on $k+1,\ldots,m$, with factors $1/k!(m-k)!$, and then symmetrizing over S_m.

By the inductive hypothesis, as $k, m-k < m$, the terms $\sum_{\tau \in S_k} \cdots$ and $\sum_{\upsilon \in S_{m-k}} \cdots$ in the final line of (63) lie in L_k with generators $e_{\sigma(1)},\ldots,e_{\sigma(k)}$ and L_{m-k} with generators $e_{\sigma(k+1)},\ldots,e_{\sigma(m)}$ respectively, so they and their commutator in (63) lie in L_m. Hence $\mathrm{d}P_m$ is an L_m-*valued* 1–*form* on (62), not just an H_m–valued 1–form. As $m \geqslant 2$ it is easy to show that each connected component of (62) with $n = m$ contains a point $(z_1,\ldots,z_m)$ with $z_1+\cdots+z_m = 0$. At this point $F_m(z_{\sigma(1)},\ldots,z_{\sigma(m)}) = 0$ for all $\sigma \in S_m$, so $P_m(z_1,\ldots,z_m) = 0$, which lies in L_m. Thus $\mathrm{d}P_m$ is an L_m–valued 1–form and $P_m(z_1,\ldots,z_m)$ lies in L_m at one point in each connected component of (62), so $P_m(z_1,\ldots,z_m)$ lies in L_m at every point of (62), completing the inductive step.

It remains to extend this from (62) to $(\mathbb{C}^\times)^n$. We do this using an argument similar to Theorem 3.12, and facts about the coefficients $U(\cdots)$ from [15, Section 5]. The relationships between the functions F_n, G_n given in (37) and (38) were derived by using the change of stability condition formula (16) to transform between $\epsilon^\alpha(\mu)$ and $\epsilon^\beta(\tilde{\mu})$. By [15, Theorem 5.4], equation (16) can be rewritten as in (17) with the term $[\cdots]$ a sum of multiple commutators of $\epsilon^{\kappa(i)}(\tau)$ for $i \in I$, so that it lies in $\mathcal{L}^\alpha$ rather than just $\mathcal{H}^\alpha$.

Suppose the open neighbourhoods $N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$ in Theorem 3.12 are chosen so that $(z_1,\ldots,z_n) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$ if and only if $(z_{\sigma(1)},\ldots,z_{\sigma(n)}) \in N_{(\tilde{z}_{\sigma(1)},\ldots,\tilde{z}_{\sigma(n)})}$ for all $\sigma \in S_n$. As we can take the $N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$ to be sufficiently small open balls about $(\tilde{z}_1,\ldots,\tilde{z}_n)$, this is clearly possible. Then since the changes (37)–(38) between F_n, G_n come from Lie algebra transformations, one can show that (61) lies in L_n for all n and $(z_1,\ldots,z_n) \in (\mathbb{C}^\times)^n$ if and only if the expression

$$(64) \qquad \sum_{\sigma \in S_n} G_n(z_{\sigma(1)},\ldots,z_{\sigma(n)}; \tilde{z}_{\sigma(1)},\ldots,\tilde{z}_{\sigma(n)}) \, e_{\sigma(1)} * \cdots * e_{\sigma(n)}$$

lies in L_n for all $(\tilde{z}_1,\ldots,\tilde{z}_n) \in (\mathbb{C}^\times)^n$ and $(z_1,\ldots,z_n) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$.

In fact one can prove more than this. For $m \geqslant 1$, write:

$(*_m)$ Suppose (61) lies in L_n for all $n < m$ and $(z_1,\ldots,z_n) \in (\mathbb{C}^\times)^n$ and (64) lies in L_n for all $n < m$, $(\tilde{z}_1,\ldots,\tilde{z}_n) \in (\mathbb{C}^\times)^n$ and $(z_1,\ldots,z_n) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_n)}$.

One can show that if $(*_m)$ holds, $(\tilde{z}_1,\ldots,\tilde{z}_m) \in (\mathbb{C}^\times)^m$ and $(z_1,\ldots,z_m) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_m)}$, then (61) with $n = m$ and this $(z_1,\ldots,z_m)$ lies in L_m if and only if (64) with $n = m$ and these $(z_1,\ldots,z_m), (\tilde{z}_1,\ldots,\tilde{z}_m)$ lies in L_m. The point is that (64) is (61) plus sums of multiple commutators of terms we know lie in L_m by our assumptions for $n < m$, and vice versa.

Suppose by induction that $(*_m)$ holds for some $m \geqslant 1$. When $m = 1$ this is vacuous. Let $(\tilde{z}_1,\ldots,\tilde{z}_m) \in (\mathbb{C}^\times)^m$ and $(z_1,\ldots,z_m) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_m)}$ with $(z_1,\ldots,z_m)$ in (62) for $m = n$. Then (61) with $n = m$ and this $(z_1,\ldots,z_m)$ lies in L_m by the proof above, so (64) with $n = m$ and these $(z_1,\ldots,z_m), (\tilde{z}_1,\ldots,\tilde{z}_m)$ lies in L_m. As L_m is closed and $G_m(z_1,\ldots,z_m; \tilde{z}_1,\ldots,\tilde{z}_m)$ is continuous in $(z_1,\ldots,z_m)$ and the intersection of $N_{(\tilde{z}_1,\ldots,\tilde{z}_m)}$ with (62) for $m = n$ is dense in $N_{(\tilde{z}_1,\ldots,\tilde{z}_m)}$, taking limits shows (64) lies in L_m for any $(z_1,\ldots,z_m) \in N_{(\tilde{z}_1,\ldots,\tilde{z}_m)}$. As this holds for all $(\tilde{z}_1,\ldots,\tilde{z}_m) \in (\mathbb{C}^\times)^m$, equation (61) lies in L_m for all $(z_1,\ldots,z_m) \in (\mathbb{C}^\times)^m$. Hence by induction $(*_m)$ holds for all $m \geqslant 1$, which proves the first part of the theorem. The remaining two parts follow as in Remark 3.1(d).   □

Lemma 3.14 *If* $1 \leqslant k \leqslant n$ *and* $z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n$ *are fixed in* $\mathbb{C}^\times$, *the function* F_n *of Theorem 3.12 satisfies* $\left| F_n(z_1, \ldots, z_n) \right| \leqslant C(1 + |\log z_k|)^{n-1}$ *for all* $z_k \in \mathbb{C}^\times$, *for some* $C > 0$ *depending on* k, n *and* $z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n$.

Proof For $n = 1, 2$ the lemma follows from (29) and (46). On the domains (44), equation (51) gives an expression for $\partial F_n / \partial z_k$ in terms of F_l for $l < n$, and it is easy to use this and induction on n to prove the lemma on (44). To extend from (44) to $(\mathbb{C}^\times)^n$, we can observe that for $(z_1, \ldots, z_n)$ in the complement of (44) in $(\mathbb{C}^\times)^n$, $F_n(z_1, \ldots, z_n)$ is a weighted average of the limits of $F_n(z_1', \ldots, z_n')$ as $(z_1', \ldots, z_n') \to (z_1, \ldots, z_n)$, for $(z_1', \ldots, z_n')$ in the various sectors of (44) meeting at $(z_1, \ldots, z_n)$. In particular, $F_n(z_1, \ldots, z_n)$ lies in the convex hull in $\mathbb{C}$ of these limits, so estimates on $|F_n|$ on (44) imply the same estimates on $(\mathbb{C}^\times)^n$. □

Corollary 3.15 *The functions* F_n *of Theorem 3.12 satisfy Condition 3.4 and equations* (28), (29), (31) *and* (32) *of Remark 3.1. Thus by Theorem 3.7 they are the unique functions* F_n *satisfying the conditions of Section 3.1.*

Proof Condition 3.4 holds by Theorem 3.12. Given $\lambda \in \mathbb{C}^\times$ we note that all conditions on F_n, G_n are preserved by replacing $F_n(z_1, \ldots, z_n)$ by $F_n(\lambda z_1, \ldots, \lambda z_n)$ and $G_n(z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n)$ by $G_n(\lambda z_1, \ldots, \lambda z_n; \lambda \tilde{z}_1, \ldots, \lambda \tilde{z}_n)$. Thus, since these conditions determine F_n, G_n uniquely (28) must hold. Equation (29) holds by definition, and (31) and (32) follow from Theorem 3.13 and Lemma 3.14. □

Remark 3.16 It is an obvious question whether the functions F_n constructed above can be written in terms of known special functions. Tom Bridgeland has found a very nice answer to this, which will be published in [3]. It involves the *hyperlogarithms* of Goncharov [8, Section 2], a kind of *polylogarithm*, which are defined by iterated integrals and satisfy a p.d.e. reminiscent of (51).

Bridgeland shows that $F_n(z_1, \ldots, z_n)$ may be written on the domain (44) as an explicit sum over *rooted trees* with n leaves of a product over vertices of the tree of a hyperlogarithm whose arguments are various sums of $z_1, \ldots, z_n$, and a constant factor. This is interesting, as polylogarithms and hyperlogarithms have many links to other branches of mathematics such as number theory, Hodge theory and motives, and the author wonders whether the ideas of this paper will also have such links.

4   Flat connections

We now explain how to define a holomorphic $\mathcal{L}$–valued connection Γ on $\mathrm{Stab}(\mathcal{A})$ using the generating functions f^α, which the p.d.e. (48) implies is flat. Our formulae

involve infinite sums over all $\alpha \in C(\mathcal{A})$, so we need a notion of *convergence* of infinite sums in $\mathcal{L}$, that is, a topology on $\mathcal{L}$. This also clarifies the meaning of the infinite direct sum $\mathcal{L} = \bigoplus_{\alpha \in C(\mathcal{A})} \mathcal{L}^\alpha$ in Assumption 2.14, since we can take $\mathcal{L}$ to be the set of convergent sums $\sum_{\alpha \in C(\mathcal{A})} l^\alpha$ with $l^\alpha \in \mathcal{L}^\alpha$.

Here are simple definitions of convergence and the direct sum which go well with Assumption 3.2, and ensure the formulae below converge in this case. If Assumption 3.2 does not hold, choosing a topology on $\mathcal{L}$ to make the formulae below converge may be difficult or impossible; in this case the sum (27) defining f^α may not converge either. Also, we must consider whether the Lie bracket $[\,,\,]$ is defined on all of $\mathcal{L} \times \mathcal{L}$, and whether it commutes with limits.

Definition 4.1 In Assumption 2.14, by the *direct sum* $\mathcal{L} = \bigoplus_{\alpha \in C(\mathcal{A})} \mathcal{L}^\alpha$ we mean simply that $\mathcal{L}$ is the infinite Cartesian product of the spaces $\mathcal{L}^\alpha$. That is, elements of $\mathcal{L}$ are just arbitrary families $(l^\alpha)_{\alpha \in C(\mathcal{A})}$ with $l^\alpha \in \mathcal{L}^\alpha$, with no restriction on how many l^α are zero, and no other "smallness conditions" on the l^α. Write $\Pi^\alpha \colon \mathcal{L} \to \mathcal{L}^\alpha$ for the obvious projection.

A possibly infinite sum $\sum_{i \in I} l_i$ in $\mathcal{L}$ is called *convergent* if for each $\alpha \in C(\mathcal{A})$ there are only finitely many $i \in I$ with $\Pi^\alpha(l_i)$ nonzero. The *limit* $l = (l^\alpha)_{\alpha \in C(\mathcal{A})}$ in $\mathcal{L}$ is defined uniquely by taking l^α to be the sum of the nonzero $\Pi^\alpha(l_i)$. That is, $\sum_{i \in I} l_i = l$ if $\sum_{i \in I} \Pi^\alpha(l_i) = \Pi^\alpha(l)$ in $\mathcal{L}^\alpha$ for all $\alpha \in C(\mathcal{A})$, where the second sum is well-defined as it has only finitely many nonzero terms. The direct sum $\mathcal{H} = \bigoplus_{\bar{\alpha} \in C(\mathcal{A})} \mathcal{H}^\alpha$ and convergence of sums in $\mathcal{H}$ are defined in the same way.

If Assumption 3.2 holds, it is easy to see that the Lie brackets $[\,,\,]\colon \mathcal{L}^\alpha \times \mathcal{L}^\beta \to \mathcal{L}^{\alpha+\beta}$ extend to a unique Lie bracket $[\,,\,]\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ which commutes with limits. Otherwise, the Lie bracket of two convergent sums can be a nonconvergent sum, so $[\,,\,]$ can only be defined on a subspace of $\mathcal{L} \times \mathcal{L}$.

In the situation of Section 3, define a section Γ in $C^\infty\bigl(\mathcal{L} \otimes T^*_{\mathbb{C}} \operatorname{Stab}(\mathcal{A})\bigr)$ by

$$(65) \qquad \Gamma(Z) = \sum_{\alpha \in C(\mathcal{A})} f^\alpha(Z) \otimes \frac{\mathrm{d}(Z(\alpha))}{Z(\alpha)}.$$

Extended to $\mathcal{L} \otimes T^*_{\mathbb{C}} \operatorname{Stab}(\mathcal{A})$ in the obvious way, this infinite sum is convergent in the sense of Definition 4.1, since for each $\alpha \in C(\mathcal{A})$ there is only one term in the sum with $\Pi^\alpha(\cdots)$ nonzero. Then Γ is a *connection matrix* for a holomorphic connection on the trivial complex Lie algebra bundle $\mathcal{L} \times \operatorname{Stab}(\mathcal{A})$ over $\operatorname{Stab}(\mathcal{A})$.

By standard differential geometry, the *curvature* of this connection is the section R_Γ of the vector bundle $\mathcal{L} \otimes \Lambda^2 T^*_{\mathbb{C}} \operatorname{Stab}(\mathcal{A})$ over $\operatorname{Stab}(\mathcal{A})$ given by

$$(66) \qquad\qquad R_\Gamma = \mathrm{d}\Gamma + \tfrac{1}{2}\Gamma \wedge \Gamma.$$

To form $\Gamma \wedge \Gamma \in C^\infty\big(\mathcal{L} \otimes \Lambda^2 T_\mathbb{C}^* \operatorname{Stab}(\mathcal{A})\big)$ from $\Gamma \otimes \Gamma \in C^\infty\big((\mathcal{L} \otimes T_\mathbb{C}^* \operatorname{Stab}(\mathcal{A}))^2\big)$, project $\mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ using the Lie bracket $[\,,\,]$ on $\mathcal{L}$, and using the wedge product $\wedge$ to project $T_\mathbb{C}^* \operatorname{Stab}(\mathcal{A}) \otimes T_\mathbb{C}^* \operatorname{Stab}(\mathcal{A}) \to \Lambda^2 T_\mathbb{C}^* \operatorname{Stab}(\mathcal{A})$.

Combining (48), (65) and (66) we find that

$$
\begin{aligned}
R_\Gamma &= \sum_{\alpha \in C(\mathcal{A})} \mathrm{d}f^\alpha(Z) \wedge \frac{\mathrm{d}(Z(\alpha))}{Z(\alpha)} + \tfrac{1}{2} \sum_{\beta,\gamma \in C(\mathcal{A})} [f^\beta(Z), f^\gamma(Z)] \otimes \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} \wedge \frac{\mathrm{d}(Z(\gamma))}{Z(\gamma)} \\
&= \sum_{\substack{\alpha,\beta,\gamma \in C(\mathcal{A}): \\ \beta+\gamma=\alpha}} \big(\tfrac{1}{2}[f^\beta(Z), f^\gamma(Z)]\big) \otimes \left(\frac{\mathrm{d}(Z(\gamma))}{Z(\gamma)} - \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} \right) \wedge \frac{\mathrm{d}(Z(\alpha))}{Z(\alpha)} \\
&\quad + \sum_{\beta,\gamma \in C(\mathcal{A})} \big(\tfrac{1}{2}[f^\beta(Z), f^\gamma(Z)]\big) \otimes \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} \wedge \frac{\mathrm{d}(Z(\gamma))}{Z(\gamma)} \\
&= \sum_{\beta,\gamma \in C(\mathcal{A})} \big(\tfrac{1}{2}[f^\beta(Z), f^\gamma(Z)]\big) \otimes \\
&\qquad \left[\left(\frac{\mathrm{d}(Z(\gamma))}{Z(\gamma)} - \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} \right) \wedge \frac{\mathrm{d}(Z(\beta))+\mathrm{d}(Z(\gamma))}{Z(\beta)+Z(\gamma)} + \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} \wedge \frac{\mathrm{d}(Z(\gamma))}{Z(\gamma)} \right] = 0,
\end{aligned}
$$

as the term $[\cdots]$ in the last line is zero. Thus Γ is a *flat connection*. If Assumption 3.2 holds, these calculations are all valid as infinite convergent sums in the sense of Definition 4.1.

If $\rho\colon \mathcal{L} \to \operatorname{End}(V)$ is a representation of the Lie algebra $\mathcal{L}$ on a complex vector space V then Γ induces a flat connection $\nabla_{\rho(\Gamma)}$ on the trivial vector bundle $V \times \operatorname{Stab}(\mathcal{A})$ over $\operatorname{Stab}(\mathcal{A})$, with connection 1–form $\rho(\Gamma)$ in $C^\infty\big(\operatorname{End}(V) \otimes T_\mathbb{C}^* \operatorname{Stab}(\mathcal{A})\big)$. If $s\colon \operatorname{Stab}(\mathcal{A}) \to V$ is a smooth section of this bundle then $\nabla_{\rho(\Gamma)} s = \mathrm{d}s + \rho(\Gamma) \cdot s$ in $C^\infty\big(V \otimes T_\mathbb{C}^* \operatorname{Stab}(\mathcal{A})\big)$.

In particular, as the tangent bundle $T \operatorname{Stab}(\mathcal{A})$ is naturally isomorphic to the trivial vector bundle $\operatorname{Hom}(K(\mathcal{A}), \mathbb{C}) \times \operatorname{Stab}(\mathcal{A})$, if $\mathcal{L}$ has a representation ρ on $\operatorname{Hom}(K(\mathcal{A}), \mathbb{C})$ then $\nabla_{\rho(\Gamma)}$ is a flat connection on $T \operatorname{Stab}(\mathcal{A})$. We will see in Section 6 that this should happen in the triangulated category extension of the Calabi–Yau 3–fold invariants in Example 2.21.

Take V to be $\mathcal{L}$ and ρ the adjoint representation $\operatorname{ad}\colon \mathcal{L} \to \operatorname{End}(\mathcal{L})$. Define a section $s\colon \operatorname{Stab}(\mathcal{A}) \to \mathcal{L}$ by

$$
(67) \qquad\qquad s(Z) = \sum_{\alpha \in C(\mathcal{A})} f^\alpha(Z),
$$

which converges as in Definition 4.1. Then from (48) and (65) we see that

$$
\nabla_{\mathrm{ad}(\Gamma)} s = \sum_{\alpha \in C(\mathcal{A})} \mathrm{d} f^\alpha(Z) + \sum_{\beta \in C(\mathcal{A})} \Big[f^\beta(Z), \sum_{\gamma \in C(\mathcal{A})} f^\gamma(z) \Big] \otimes \frac{\mathrm{d}(Z(\beta))}{Z(\beta)}
$$

$$(68)$$

$$
= -\sum_{\substack{\alpha,\beta,\gamma \in C(\mathcal{A}): \\ \alpha = \beta + \gamma}} [f^\beta(Z), f^\gamma(Z)] \otimes \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} + \sum_{\beta,\gamma \in C(\mathcal{A})} [f^\beta(Z), f^\gamma(z)] \otimes \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} = 0,
$$

so that s in (67) is a *constant section* of $\mathcal{L} \times \mathrm{Stab}(\mathcal{A})$. If Assumption 3.2 holds then (68) is valid as infinite convergent sums in the sense of Definition 4.1.

Let $P\colon \mathcal{L} \to \mathbb{C}$ be smooth and invariant under $\mathrm{ad}(\mathcal{L})$, that is, $\mathrm{d}P(x)\cdot[x,y] = 0$ for all $x,y \in \mathcal{L}$. Then $\nabla^{\mathrm{ad}(\Gamma)} s = 0$ implies that $P(s)$ is constant on $\mathrm{Stab}(\mathcal{A})$. For example, if $\rho\colon \mathcal{L} \to \mathrm{End}(V)$ is a representation of $\mathcal{L}$ on a finite-dimensional $\mathbb{C}$–vector space V then $P(x) = \det\big(\rho(x) - \lambda\,\mathrm{id}_V\big)$ has these properties, so the characteristic polynomial of $\rho(s(Z))$ is constant on $\mathrm{Stab}(\mathcal{A})$.

In general, for s as in (67) the eigenvalues of $s(Z)$ in any representation of $\mathcal{L}$ should be constant on $\mathrm{Stab}(\mathcal{A})$. However, the author does not expect this construction to be useful with the topology on $\mathcal{L}$ in Definition 4.1, as it seems likely that the only finite-dimensional representations for such infinite-dimensional $\mathcal{L}$ will be nilpotent, and so have zero eigenvalues anyway.

The author feels that the topology on $\mathcal{L}$ given in Definition 4.1 is rather trivial, and that if the ideas of this section do have interesting applications in classes of examples it will be with a more complex topology on $\mathcal{L}$ appropriate to the examples. Then the convergence and validity of equations (65)–(68) would become conjectures to be (dis)proved in these examples, depending on asymptotic properties of the f^α for large α.

5 Extending all this to triangulated categories

Our programme cannot yet be rigorously extended from abelian categories $\mathcal{A}$ to triangulated categories $\mathcal{T}$, because the material of [12; 13; 14; 15] on which it rests has not yet been extended. Some remarks on the issues involved are given in [15, Section 7]. The work of Bertrand Toën [22; 21] is likely to be useful here. In particular, [22] defines a "derived Hall algebra" $\mathcal{DH}(\mathcal{T})$ under strong finiteness conditions on $\mathcal{T}$, and [21, Section 3.3.3] an "absolute Hall algebra" $\mathcal{H}_{\mathrm{abs}}(\mathcal{T})$ under weaker conditions.

It seems likely that the right way to construct examples of data satisfying a triangulated version of Assumption 2.14 is to use an algebra morphism $\Phi\colon \mathcal{DH}(\mathcal{T})$ or $\mathcal{H}_{\mathrm{abs}}(\mathcal{T}) \to \mathcal{H}$.

Also, [21] provides the tools needed to form moduli Artin ∞–stacks of objects and configurations in triangulated categories with dg-enhancement, which is the main ingredient needed to extend [12; 13; 14; 15] to the triangulated case. Here are some issues in extending the ideas of this paper to the triangulated case.

Lifting phases from $\mathbb{R}/2\pi i\mathbb{Z}$ **to** $\mathbb{R}$ The $\delta^{\alpha}(\tau), \epsilon^{\alpha}(\tau)$ of Assumption 2.14 are constructed from "characteristic functions" of τ–semistable objects in $\mathcal{A}$ in class $\alpha \in C(\mathcal{A})$. Now in Bridgeland's stability conditions $(Z, \mathcal{P})$ on a triangulated category $\mathcal{T}$, Definition 2.22, the $(Z, \mathcal{P})$–semistable objects in class $\alpha \in K(\mathcal{T})$ depend on a choice of *phase* for $Z(\alpha) \in \mathbb{C}^{\times}$.

That is, if we write $Z(\alpha) = re^{i\pi\phi}$ for $\phi \in \mathbb{R}$, then the $(Z, \mathcal{P})$–semistable objects in class α with phase ϕ are the objects U in $\mathcal{P}(\phi)$ with class $\alpha \in \mathbb{K}(\mathcal{T})$. Replacing ϕ by $\phi + 2n$ for $n \in \mathbb{Z}$ replaces $\mathcal{P}(\phi)$ by $\mathcal{P}(\phi + 2n) = \mathcal{P}(\phi)[2n]$, so replaces objects U by $U[2n]$, that is, applying the translation functor to the power $2n$. Note that replacing U by $U[2n]$ fixes the class α of U in $K(\mathcal{T})$.

It is natural to ask whether the triangulated analogues $\delta^{\alpha}(Z, \mathcal{P}), \epsilon^{\alpha}(Z, \mathcal{P})$ should also depend on a choice of phase ϕ for $Z(\alpha)$. The author's view is that for the purposes of this paper, they should *not* depend on choice of phase. Effectively this means working in a Hall-type algebra in which the translation squared operator $[+2]$ is the identity.

The reason is that if $\delta^{\alpha}(Z, \mathcal{P}), \epsilon^{\alpha}(Z, \mathcal{P})$ depended on phase then $f^{\alpha}(Z, \mathcal{P})$ should also depend on a choice of phase for $Z(\alpha)$, and $F_n(z_1, \ldots, z_n)$ on choices of phase for $z_1, \ldots, z_n$. That is, F_n should be a function of $(\log z_1, \ldots, \log z_n) \in \mathbb{C}^n$ rather than $(z_1, \ldots, z_n) \in (\mathbb{C}^{\times})^n$. Allowing this would invalidate nearly all of Section 3. In particular, the uniqueness result Theorem 3.7 would fail, and the p.d.e. (48) would no longer make sense, as for a given choice of phase for $Z(\alpha)$ there does not seem to be a natural way to choose phases for $Z(\beta), Z(\gamma)$ in the sum.

Replacing $C(\mathcal{A})$ **by** $\{\alpha \in K(\mathcal{T}) : Z(\alpha) \neq 0\}$ What should be the analogue of the *positive cone* $C(\mathcal{A})$ in a triangulated category $\mathcal{T}$? Replacing $\mathcal{A}$ by $\mathcal{T}$ in (4) will give $C(\mathcal{T}) = K(\mathcal{T})$, as for nonzero $\mathcal{T}$ every element of $K(\mathcal{T})$ will be represented by a nonzero object. However, the sums over $\alpha \in C(\mathcal{A})$ in Section 3 do not make sense when replaced by $\alpha \in K(\mathcal{T})$, because of problems when $Z(\alpha) = 0$. For instance, $F_n(Z(\alpha_1), \ldots, Z(\alpha_n))$ in (27) is undefined if any $Z(\alpha_k) = 0$, and (48) is undefined if any $Z(\beta) = 0$ or $Z(\gamma) = 0$.

The author proposes that the right answer is to replace sums over $\alpha \in C(\mathcal{A})$ in Section 3 involving $\epsilon^{\alpha}(\mu)$, such as (27), by sums over all $\alpha \in K(\mathcal{T})$ with $Z(\alpha) \neq 0$. For generic Z this amounts to summing over $\alpha \in K(\mathcal{T}) \setminus \{0\}$. Sums over $\alpha \in C(\mathcal{A})$ involving

$f^\alpha(Z)$ need a more subtle approach we describe below. We now explain two neat coincidences meaning that arguments in Section 3 still work with this replacement, although one might have expected them to fail.

First, note that if $(Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{T})$ and $\alpha \in K(\mathcal{T})$ with $Z(\alpha) = 0$ then we must have $\delta^\alpha(Z, \mathcal{P}) = \epsilon^\alpha(Z, \mathcal{P}) = 0$. This is because $\delta^\alpha(Z, \mathcal{P}), \epsilon^\alpha(Z, \mathcal{P})$ are constructed from $(Z, \mathcal{P})$–semistable objects in class α, but there are no such objects if $Z(\alpha) = 0$ by Definition 2.22. We also expect $\delta^\alpha(Z', \mathcal{P}') = \epsilon^\alpha(Z', \mathcal{P}') = 0$ for $(Z', \mathcal{P}')$ in a small open neighbourhood of $(Z, \mathcal{P})$ in $\mathrm{Stab}(\mathcal{T})$. This means that omitting terms in $\epsilon^{\alpha_i}(Z, \mathcal{P})$ in (27) when $Z(\alpha_i) = 0$ does not cause discontinuities on the hypersurface $Z(\alpha_i) = 0$ in $\mathrm{Stab}(\mathcal{T})$, since the omitted terms are zero near there anyway.

Second, note that $f^\alpha(Z, \mathcal{P}) = 0$ when $Z(\alpha) = 0$, since (27) now involves terms in $\alpha_1, \ldots, \alpha_n$ with $Z(\alpha_k) \neq 0$ but $Z(\alpha_1) + \cdots + Z(\alpha_n) = Z(\alpha) = 0$. Hence $F_n(Z(\alpha_1), \ldots, Z(\alpha_n)) = 0$, and every term in (27) is zero. However, for $\alpha \neq 0$ we do not expect $f^\alpha(Z', \mathcal{P}') \equiv 0$ for $(Z', \mathcal{P}')$ near $(Z, \mathcal{P})$. So in sums such as (65) involving $f^\alpha(Z)/Z(\alpha)$, giving $0/0$ when $Z(\alpha) = 0$, it is not right to just omit α when $Z(\alpha) = 0$, for $\alpha \neq 0$. Instead, since $f^\alpha(Z, \mathcal{P})$ is holomorphic and zero when $Z(\alpha) = 0$, as for the functions H_n in Section 3.2, the holomorphic function $h^\alpha(Z, \mathcal{P}) = f^\alpha(Z, \mathcal{P})/Z(\alpha)$ on $Z(\alpha) \neq 0$ extends uniquely over $Z(\alpha) = 0$, so in (27), (29), (31) we replace terms $f^\alpha(Z, \mathcal{P})/Z(\alpha)$ by $h^\alpha(Z, \mathcal{P})$.

Convergence of sums Once we replace sums over $\alpha \in C(\mathcal{A})$ by sums over $\alpha \in K(\mathcal{T})$ with $Z(\alpha) \neq 0$, most of the equations in Section 3–Section 4 become infinite sums, and the question of whether they converge at all in any sense becomes acute. There seems to be no triangulated analogue of Assumption 3.2 that makes the sums finite, nor can the author find any way to make the sums converge in a formal power series sense. Here are two comments which may help.

Firstly, suppose the Lie algebra $\mathcal{L}$ is *nilpotent*. That is, define ideals $\mathcal{L} = \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots$ by $\mathcal{L}_1 = \mathcal{L}$, $\mathcal{L}_{n+1} = [\mathcal{L}, \mathcal{L}_n]$, and suppose $\bigcap_{n \geq 1} \mathcal{L}_n = \{0\}$. Then Theorem 3.13 implies that the sum of terms with fixed n in (27) lie in $\mathcal{L}_n$. Hence, projecting (27) to $\mathcal{L}/\mathcal{L}_k$ eliminates all terms with $n \geq k$. If we use a notion of convergence such that a sum converges in $\mathcal{L}$ if its projections to $\mathcal{L}/\mathcal{L}_k$ converge for all $k \geq 1$, then we only have to show the sum (27) for $n < k$ converges in $\mathcal{L}/\mathcal{L}_k$, which may be easier.

Secondly, even if the sum (27) defining f^α does not make sense, the p.d.e. (48) upon the f^α might still converge in the triangulated case, as it is a much simpler sum. For example, if $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Kac–Moody Lie algebra, it is known [13, Section 4.9] how to use Ringel–Hall algebras of abelian categories of quiver representations $\mathcal{A} = \mathrm{mod}\text{–}\mathbb{K}Q$ to realize $\mathcal{H} = U(\mathfrak{n}_+)$ and $\mathcal{L} = \mathfrak{n}_+$ in examples, and

people have hoped to use triangulated categories to obtain $\mathcal{H} = U(\mathfrak{g})$ and $\mathcal{L} = \mathfrak{g}$. If we could do this with $\mathfrak{g}$ a finite-dimensional semisimple Lie algebra, then α, β, γ in (48) would take values in the set of roots of $\mathfrak{g}$, with $\mathcal{L}^\alpha$ being the root space $\mathfrak{g}_\alpha$, and (48) would become a finite sum, so trivially convergent. However, (27) would still be an infinite sum.

Intuitively, what is going on is as follows. The functions F_n are related to certain finite-dimensional nilpotent Lie algebras N_n for $n \geqslant 1$. In a similar way to Ramakrishnan [18] for the higher logarithms $\ln_k$, one can use the F_k for $k \leqslant n$ to write down a nontrivial flat holomorphic N_n–valued connection on

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_a + \cdots + z_b \neq 0 \text{ for all } 1 \leqslant a \leqslant b \leqslant n, (a,b) \neq (1,n)\}.$$

In the Ringel–Hall case $\mathcal{H} = U(\mathfrak{n}_+)$, $\mathcal{L} = \mathfrak{n}_+$ above, equation (27) is about building the nilpotent Lie algebra $\mathfrak{n}_+$, and the flat $\mathfrak{n}_+$–valued connection Γ of Section 4, out of the standard family of nilpotent Lie algebras N_n, and standard flat N_n–valued connections.

However, it may not be possible to build semisimple Lie algebras $\mathfrak{g}$ and their flat connections from standard nilpotent building blocks N_n, which is why (27) may not converge. But (48) has to do with general Lie algebras, not just nilpotent Lie algebras, and so may make sense in a more general setting.

The remarks above suggest a shift in point of view, in which rather than starting with invariants $\delta^\alpha(Z)$ or $\epsilon^\alpha(Z)$ and constructing functions $f^\alpha(Z)$ as in Section 3, we instead regard the $f^\alpha(Z)$ or $f^\alpha(Z, \mathcal{P})$ and the p.d.e. (48) as primary, and first try to solve (48) to find the $f^\alpha(Z)$, and then reconstruct the $\delta^\alpha(Z)$ and $\epsilon^\alpha(Z)$ from the $f^\alpha(Z)$.

The most naive way to do this would be to attempt to solve (48) recursively, say in the abelian category case, by a form of induction on α, in which at the inductive step the r.h.s. of (48) is known, giving df^α, and we integrate this to obtain f^α. However, this only determines f^α up to an additive constant, which is basically $\epsilon^\alpha(Z_0)$ for some base stability condition Z_0.

Therefore the uncertainty in solving (48) inductively for all α is exactly $\epsilon^\alpha(Z_0)$ for all α, so we cannot hope to determine the $\delta^\alpha(Z), \epsilon^\alpha(Z)$ in this naïve way. But it is conceivable that by using more nontrivial global information (for instance, equivariance under monodromy transformations) about the moduli space $\mathrm{Stab}(\mathcal{T})$, say in the triangulated case, one might be able to determine solutions of (48); this would be similar to the method of Bershadsky et al [1; 2] for calculating higher genus Gromov–Witten invariants of Calabi–Yau 3–folds.

6 The Calabi–Yau 3–fold case

Finally we discuss and elaborate the ideas of Section 3–Section 5 in the Calabi–Yau 3–fold case of Example 2.21. We use the notation of this example and Section 3–Section 5 throughout.

6.1 Holomorphic functions F^α, H^α and their p.d.e.s

We begin with the abelian category case. Since by Theorem 3.13 f^α maps $\mathrm{Stab}(\mathcal{A})$ to $\mathcal{L}^\alpha$, and $\mathcal{L}^\alpha = \mathbb{C} \cdot c^\alpha$ we may write $f^\alpha = F^\alpha c^\alpha$ for a holomorphic function $F^\alpha \colon \mathrm{Stab}(\mathcal{A}) \to \mathbb{C}$, for $\alpha \in C(\mathcal{A})$. Also $\epsilon^\alpha(\mu) = J^\alpha(\mu)c^\alpha$ for $J^\alpha(\mu) \in \mathbb{Q}$, so combining (24) and (27) we find that

$$
\begin{aligned}
(69) \quad F^\alpha(Z) = &\sum_{\substack{n \geqslant 1,\, \alpha_1,\ldots,\alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \cdots + \alpha_n = \alpha}} F_n\big(Z(\alpha_1),\ldots,Z(\alpha_n)\big) \prod_{i=1}^{n} J^{\alpha_i}(\mu) \cdot \\
&\left[\frac{1}{2^{n-1}} \sum_{\substack{\text{connected, simply connected digraphs } \Gamma: \\ \text{vertices } \{1,\ldots,n\},\, \text{edge } \bullet^i \to \bullet^j \text{ implies } i < j}} \prod_{\substack{\text{edges} \\ \bullet^i \to \bullet^j \\ \text{in } \Gamma}} \bar{\chi}(\alpha_i,\alpha_j) \right],
\end{aligned}
$$

where μ is the slope function associated to Z. We also have $F^\alpha \equiv Z(\alpha)H^\alpha$ for a holomorphic function $H^\alpha \colon \mathrm{Stab}(\mathcal{A}) \to \mathbb{C}$ given by

$$
\begin{aligned}
(70) \quad H^\alpha(Z) = &\sum_{\substack{n \geqslant 1,\, \alpha_1,\ldots,\alpha_n \in C(\mathcal{A}): \\ \alpha_1 + \cdots + \alpha_n = \alpha}} H_n\big(Z(\alpha_1),\ldots,Z(\alpha_n)\big) \prod_{i=1}^{n} J^{\alpha_i}(\mu) \cdot \\
&\left[\frac{1}{2^{n-1}} \sum_{\substack{\text{connected, simply connected digraphs } \Gamma: \\ \text{vertices } \{1,\ldots,n\},\, \text{edge } \bullet^i \to \bullet^j \text{ implies } i < j}} \prod_{\substack{\text{edges} \\ \bullet^i \to \bullet^j \\ \text{in } \Gamma}} \bar{\chi}(\alpha_i,\alpha_j) \right].
\end{aligned}
$$

The p.d.e. (48) becomes

$$
\begin{aligned}
(71) \quad \mathrm{d}F^\alpha(Z) = &-\sum_{\beta,\gamma \in C(\mathcal{A}):\alpha=\beta+\gamma} \bar{\chi}(\beta,\gamma) F^\beta(Z) F^\gamma(Z) \frac{\mathrm{d}(Z(\beta))}{Z(\beta)} \\
= &-\sum_{\beta,\gamma \in C(\mathcal{A}):\alpha=\beta+\gamma} \bar{\chi}(\beta,\gamma) H^\beta(Z) H^\gamma(Z) Z(\gamma) \mathrm{d}(Z(\beta)),
\end{aligned}
$$

and the flat connection Γ of (65) is

$$
(72) \quad \Gamma(Z) = \sum_{\alpha \in C(\mathcal{A})} F^\alpha(Z)\, c^\alpha \otimes \frac{\mathrm{d}(Z(\alpha))}{Z(\alpha)} = \sum_{\alpha \in C(\mathcal{A})} H^\alpha(Z)\, c^\alpha \otimes \mathrm{d}(Z(\alpha)).
$$

In the triangulated category case we replace F^α, $H^\alpha(Z)$ and $J^\alpha(\mu)$ by F^α, H^α, $J^\alpha(Z, \mathcal{P})$, and replace sums over $C(\mathcal{A})$ by sums over $K(\mathcal{T}) \setminus \{0\}$ in (69)–(72), and also omit terms involving α_i with $Z(\alpha_i) = 0$ in (69)–(70).

In the triangulated case, the Lie algebra $\mathcal{L}$ is $\mathcal{L} = \langle c^\alpha : \alpha \in K(\mathcal{T}) \rangle_{\mathbb{C}}$, with $[c^\alpha, c^\beta] = \bar{\chi}(\alpha, \beta)c^{\alpha+\beta}$. Suppose $K(\mathcal{T})$ is a lattice of finite rank, and $\chi\colon K(\mathcal{T}) \times K(\mathcal{T}) \to \mathbb{Z}$ is nondegenerate. Then we can interpret $\mathcal{L}$ as a Lie algebra of complex functions on the real torus $T_{\mathcal{T}} = \mathrm{Hom}(K(\mathcal{T}), \mathbb{R})/\mathrm{Hom}(K(\mathcal{T}), \mathbb{Z})$ by identifying c^α with the function

$$C^\alpha\colon \mathrm{Hom}(K(\mathcal{T}), \mathbb{R})/\mathrm{Hom}(K(\mathcal{T}), \mathbb{Z}) \to \mathbb{C}, \quad C^\alpha\colon x + \mathrm{Hom}(K(\mathcal{T}), \mathbb{Z}) \mapsto e^{2\pi i x(\alpha)}.$$

Now $(2\pi i)^{-2}\bar{\chi}$ induces a section of $\Lambda^2 T(T_{\mathcal{T}}) \otimes_{\mathbb{R}} \mathbb{C}$ yielding a *Poisson bracket* $\{\,,\,\}$ on smooth complex functions on $T_{\mathcal{T}}$, with $\{C^\alpha, C^\beta\} = \bar{\chi}(\alpha, \beta)C^{\alpha+\beta}$.

Thus the map $c^\alpha \mapsto C^\alpha$ induces an injective Lie algebra morphism from $\mathcal{L}$ to a Lie algebra of complex functions on $T_{\mathcal{T}}$ with Poisson bracket $\{\,,\,\}$. It is not clear which class of functions on $T_{\mathcal{T}}$ we should consider. For instance, smooth functions $C^\infty(T_{\mathcal{T}})_{\mathbb{C}}$ or real analytic functions $C^\omega(T_{\mathcal{T}})_{\mathbb{C}}$ both give well behaved Lie algebras of functions on $T_{\mathcal{T}}$. These also come with natural topologies, and so yield notions of convergence of infinite sums in $\mathcal{L}$, as discussed in Section 4. However, the author expects that these notions of convergence will be too strict to make the sums of Section 3–Section 5 converge in interesting examples, and some much weaker convergence criterion than smoothness or real analyticity is required.

6.2 A flat connection on $T\,\mathrm{Stab}(\mathcal{T})$ in the triangulated case

In the triangulated category case, the invariants $J^\alpha(Z, \mathcal{P}) \in \mathbb{Q}$ "counting" $(Z, \mathcal{P})$–semistable objects in class $\alpha \in K(\mathcal{T})$ should satisfy $J^{-\alpha}(Z, \mathcal{P}) = J^\alpha(Z, \mathcal{P})$, since the translation operator $[+1]$ induces a bijection between $(Z, \mathcal{P})$–semistable objects in classes α and $-\alpha$. Thus we expect $F^{-\alpha} \equiv F^\alpha$ for all $\alpha \in K(\mathcal{T}) \setminus \{0\}$. Hence Γ in (72) is actually an $\mathcal{L}'$–valued connection, where $\mathcal{L}' = \langle c^\alpha + c^{-\alpha} : \alpha \in K(\mathcal{T}) \rangle_{\mathbb{C}}$ is a Lie subalgebra of $\mathcal{L}$ with

$$(73) \qquad [c^\alpha + c^{-\alpha}, c^\beta + c^{-\beta}] = \bar{\chi}(\alpha, \beta)\big((c^{\alpha+\beta} + c^{-\alpha-\beta}) - (c^{\alpha-\beta} + c^{-\alpha+\beta})\big).$$

Regarded as a Lie algebra of functions on $T_{\mathcal{T}}$, the functions in $\mathcal{L}'$ are invariant under $-1\colon T_{\mathcal{T}} \to T_{\mathcal{T}}$ acting by $x + \mathrm{Hom}(K(\mathcal{T}), \mathbb{Z}) \to -x + \mathrm{Hom}(K(\mathcal{T}), \mathbb{Z})$, so the Hamiltonian vector fields of functions in $\mathcal{L}'$ all vanish at $0 \in T_{\mathcal{T}}$. Therefore they have a Lie algebra action on $T_0 T_{\mathcal{T}} \cong \mathrm{Hom}(K(\mathcal{T}), \mathbb{C})$, and on its dual $K(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C}$. That is, we have found a Lie algebra representation $\rho\colon \mathcal{L}' \to \mathrm{End}\big(K(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C}\big)$, which is given explicitly on the generators $c^\alpha + c^{-\alpha}$ of $\mathcal{L}'$ by

$$(74) \qquad\qquad \rho(c^\alpha + c^{-\alpha})\colon \gamma \longmapsto 2\bar{\chi}(\alpha, \gamma)\alpha.$$

Comparing (73) and (74) shows ρ is a Lie algebra morphism. Note that ρ does *not* extend to a Lie algebra morphism $\mathcal{L} \to \mathrm{End}\big(K(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C}\big)$.

Now there is a natural isomorphism $K(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C} \cong T^* \mathrm{Stab}(\mathcal{T})$. Thus in the Calabi–Yau 3–fold triangulated category case, if all the relevant sums converge in $\mathrm{End}\big(K(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C}\big)$ (which seems rather unlikely), then applying ρ to the flat connection Γ of Section 4 induces a flat connection $\nabla_{\rho(\Gamma)}$ on the tangent bundle $T \mathrm{Stab}(\mathcal{T})$ and cotangent bundle $T^* \mathrm{Stab}(\mathcal{T})$ of $\mathrm{Stab}(\mathcal{T})$. This connection is easily seen to be *torsion-free*: the connection on $T \mathrm{Stab}(\mathcal{T})$ is a sum over $\alpha \in K(\mathcal{T}) \setminus \{0\}$ of a term linear in $\alpha \otimes \alpha \otimes \alpha$, and the torsion vanishes because of a symmetry in exchanging two copies of α. It also preserves the symplectic form on $\mathrm{Stab}(\mathcal{T})$ induced by $\bar{\chi}$. Integrating $\nabla_{\rho(\Gamma)}$ should give new, interesting flat local coordinate systems on $\mathrm{Stab}(\mathcal{T})$.

Ignoring convergence issues, define a section $g_{\mathbb{C}}$ of $S^2 T^* \mathrm{Stab}(\mathcal{T})$ by

$$(75) \qquad g_{\mathbb{C}}(Z, \mathcal{P}) = \sum_{\alpha \in K(\mathcal{T}) \setminus \{0\}} F^\alpha(Z, \mathcal{P}) \, dZ(\alpha) \otimes dZ(\alpha).$$

In a calculation related to (68), differentiating using $\nabla_{\rho(\Gamma)}$ gives

$$
\begin{aligned}
\nabla_{\rho(\Gamma)} g_{\mathbb{C}} = {}& \sum_{\alpha \in K(\mathcal{T}) \setminus \{0\}} dF^\alpha(Z, \mathcal{P}) \otimes dZ(\alpha) \otimes dZ(\alpha) + \\[2mm]
& \sum_{\beta, \gamma \in K(\mathcal{T}) \setminus \{0\}} \bar{\chi}(\beta, \gamma) F^\beta(Z, \mathcal{P}) F^\gamma(Z, \mathcal{P}) \frac{d(Z(\beta))}{Z(\beta)} \otimes \begin{bmatrix} dZ(\beta) \otimes dZ(\gamma) + \\ dZ(\gamma) \otimes dZ(\beta) \end{bmatrix} \\[2mm]
(76) \quad = {}& \sum_{\beta, \gamma \in K(\mathcal{T}) \setminus \{0\}} \bar{\chi}(\beta, \gamma) F^\beta(Z, \mathcal{P}) F^\gamma(Z, \mathcal{P}) \frac{d(Z(\beta))}{Z(\beta)} \otimes \\[2mm]
& \big[-\big(dZ(\beta) + dZ(\gamma)\big) \otimes \big(dZ(\beta) + dZ(\gamma)\big) + dZ(\beta) \otimes dZ(\gamma) + dZ(\gamma) \otimes dZ(\beta) \big] \\[2mm]
= {}& -\sum_{\beta, \gamma \in K(\mathcal{T}) \setminus \{0\}} \bar{\chi}(\beta, \gamma) F^\beta(Z, \mathcal{P}) F^\gamma(Z, \mathcal{P}) \frac{d(Z(\beta))}{Z(\beta)} \otimes \begin{bmatrix} dZ(\beta) \otimes dZ(\beta) + \\ dZ(\gamma) \otimes dZ(\gamma) \end{bmatrix} \\[2mm]
= {}& 0.
\end{aligned}
$$

Here the second line applies $\rho(\Gamma)$ to $g_{\mathbb{C}}$, where we replace α in the sum (72) defining Γ by β, and α in the sum (75) defining $g_{\mathbb{C}}$ by γ, and use the fact that

$$\rho(c^\beta + c^{-\beta})\big[dZ(\gamma) \otimes dZ(\gamma) \big] = 2\bar{\chi}(\beta, \gamma)\big[dZ(\beta) \otimes dZ(\gamma) + dZ(\gamma) \otimes dZ(\beta) \big].$$

The third and fourth lines of (76) substitute (71) into the first line and set $\alpha = \beta + \gamma$, and for the final step we note that as $F^{-\gamma}(Z, \mathcal{P}) = F^\gamma(Z, \mathcal{P})$, pairing terms in the fifth line with β, γ and $\beta, -\gamma$ shows that everything cancels.

Suppose now that $g_{\mathbb{C}}$ is a *nondegenerate* section of $S^2 T^* \mathrm{Stab}(\mathcal{T})$. (If it is nondegenerate at one point in $\mathrm{Stab}(\mathcal{T})$ it is degenerate everywhere in this connected component,

as it is constant under $\nabla_{\rho(\Gamma)}$ by (76).) Then $g_{\mathbb{C}}$ is a *holomorphic metric* on $\mathrm{Stab}(\mathcal{T})$. Since $\nabla_{\rho(\Gamma)}$ is torsion-free with $\nabla_{\rho(\Gamma)}g_{\mathbb{C}} = 0$, we see that $\nabla_{\rho(\Gamma)}$ is the *Levi-Civita connection* of $g_{\mathbb{C}}$, and thus $g_{\mathbb{C}}$ is *flat* as $\nabla_{\rho(\Gamma)}$ is flat. Note that Frobenius manifolds also have flat holomorphic metrics.

6.3 A variant of the holomorphic anomaly equation

Several people have commented to the author that the p.d.e. (51) on F_n resembles the *holomorphic anomaly equation* of Bershadsky, Cecotti, Ooguri and Vafa [1; 2], which is interpreted by Witten [23]. This equation is [2, Equation (3.6)]

$$(77) \qquad \bar{\partial}_{\bar{i}} F_g = \tfrac{1}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \big(\partial_j \partial_k F_{g-1} + \textstyle\sum_{r=1}^{g-1} \partial_j F_r \partial_k F_{g-r} \big),$$

which can be repackaged as a linear equation on $\exp\big(\sum_{g=1}^{\infty} \lambda^{2g-2} F_g\big)$. It is beyond the author's competence to properly explain (77). Very roughly, F_g is a complex-valued generating function which "counts" numbers of genus g holomorphic curves in a Calabi–Yau 3–fold X—just as our generating functions F^α "count" coherent sheaves on X. It is not holomorphic, but is nearly so, in that (77) expresses $\bar{\partial} F_g$ in terms of ∂F_r for $r < g$.

For $\lambda \in \mathbb{C}^\times$ and fixed $a, b \in \mathbb{Z}$ define a $(0,1)$–form on $\mathrm{Stab}(\mathcal{A})$ by

$$(78) \qquad \Phi_\lambda(Z) = \textstyle\sum_{\alpha \in C(\mathcal{A})} \lambda^a e^{\lambda^b Z(\alpha)} \, \overline{H^\alpha(Z)} \, \overline{\mathrm{d}(Z(\alpha))}.$$

The idea here is that we have taken the complex conjugate of (72), and then replaced the Lie algebra element $\overline{c^\alpha}$ by the holomorphic function $\lambda^a e^{\lambda^b Z(\alpha)}$. In the abelian category case, as $\mathrm{Im}\, Z(\alpha) > 0$ for $\alpha \in C(\mathcal{A})$, if $\mathrm{Im}(\lambda^b) \gg 0$ then $e^{\lambda^b Z(\alpha)}$ is small, and it seems plausible that (78) may actually converge. In the triangulated case, when $C(\mathcal{A})$ in (78) is replaced by $K(\mathcal{T}) \setminus \{0\}$, convergence seems less likely.

The $(1,1)$–form $\partial \Phi_\lambda$ and the $(0,2)$–form $\bar{\partial} \Phi_\lambda$ on $\mathrm{Stab}(\mathcal{A})$ are given by

$$\partial \Phi_\lambda(Z) = \lambda^{a+b} \sum_{\alpha \in C(\mathcal{A})} e^{\lambda^b Z(\alpha)} \, \overline{H^\alpha(Z)} \, \mathrm{d}(Z(\alpha)) \wedge \overline{\mathrm{d}(Z(\alpha))},$$

$$\bar{\partial} \Phi_\lambda(Z) = -\tfrac{1}{2} \lambda^a \sum_{\beta, \gamma \in C(\mathcal{A})} e^{\lambda^b Z(\beta)} \, \overline{H^\beta(Z)} \, e^{\lambda^b Z(\gamma)} \, \overline{H^\gamma(Z)} \cdot$$
$$\bar{\chi}(\beta, \gamma) \, \overline{\mathrm{d}(Z(\beta))} \wedge \overline{\mathrm{d}(Z(\gamma))},$$

where in the latter equation we have used $H^\alpha \equiv Z(\alpha)^{-1} F^\alpha$ and substituted in (71). Using index notation for complex tensors as in (77), so that i, j are type $(1,0)$ tensor indices and $\bar{i}, \bar{j}$ type $(0,1)$ tensor indices, we see these satisfy

$$(79) \qquad \big(\bar{\partial} \Phi_\lambda(Z) \big)_{\bar{i}\bar{j}} = -\tfrac{1}{2} \lambda^{-a-2b} (\bar{\chi})^{ij} \big(\partial \Phi_\lambda(Z) \big)_{i\bar{i}} \big(\partial \Phi_\lambda(Z) \big)_{j\bar{j}}.$$

Here $(\bar{\chi})^{ij}$ is the $(2,0)$ part of $\bar{\chi}$, regarded as a constant tensor in $\Lambda^2 T \operatorname{Stab}(\mathcal{A})$ $= \Lambda^2 \operatorname{Hom}(K(\mathcal{A}), \mathbb{C})$. Equation (79) is formally similar to the p.d.e. satisfied by $W_\lambda = \sum_{g=1}^{\infty} \lambda^{2g-2} F_g$ in the holomorphic anomaly case above, of the form

$$\bar{\partial} W_\lambda = \lambda^2 \big(\text{linear term in } \partial^2 W_\lambda + \partial W_\lambda \otimes \partial W_\lambda\big).$$

Note too that there are no convergence issues for (79), it always makes sense as an equation on $(0,1)$–forms Φ_λ on $\operatorname{Stab}(\mathcal{A})$ or $\operatorname{Stab}(\mathcal{T})$. The author has no idea whether all this is relevant to String Theory.

References

[1] **M Bershadsky**, **S Cecotti**, **H Ooguri**, **C Vafa**, *Holomorphic anomalies in topological field theories*, Nuclear Phys. B 405 (1993) 279–304   MR1240687

[2] **M Bershadsky**, **S Cecotti**, **H Ooguri**, **C Vafa**, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. 165 (1994) 311–427   MR1301851

[3] **T Bridgeland**, in preparation

[4] **T Bridgeland**, *Stability conditions on triangulated categories*, to appear in Ann. of Math. (2)   `arXiv:math.AG/0212237`

[5] **T Bridgeland**, *Derived categories of coherent sheaves*, from: "Proceedings of the International Congress of Mathematicians, Vol. 2 (Madrid, 2006)", (M Sanz-Solé, J Soria, J Varona, J Verdera, editors) (2007) 563–582   `arXiv:math.AG/0602129`

[6] **S K Donaldson**, **R P Thomas**, *Gauge theory in higher dimensions*, from: "The geometric universe (Oxford, 1996)", Oxford Univ. Press, Oxford (1998) 31–47   MR1634503

[7] **S I Gelfand**, **Y I Manin**, *Methods of homological algebra*, second edition, Springer Monographs in Mathematics, Springer, Berlin (2003)   MR1950475

[8] **A Goncharov**, *Multiple polylogarithms and mixed Tate motives*   `arXiv:math.AG/0103059`

[9] **A L Gorodentsev**, **S A Kuleshov**, **A N Rudakov**, *t-stabilities and t-structures on triangulated categories*, Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004) 117–150   MR2084563

[10] **D Joyce**, *Constructible functions on Artin stacks*, J. London Math. Soc. (2) 74 (2006) 583–606   MR2286434

[11] **D Joyce**, *Motivic invariants of Artin stacks and 'stack functions'*, to appear in Q. J. Math.   `arXiv:math.AG/0509722`

[12] **D Joyce**, *Configurations in abelian categories. I. Basic properties and moduli stacks*, Adv. Math. 203 (2006) 194–255   MR2231046

[13] **D Joyce**, *Configurations in abelian categories. II. Ringel–Hall algebras*, Adv. Math. 210 (2007) 635–706

[14] **D Joyce**, *Configurations in abelian categories. III. Stability conditions and identities*, to appear in Adv. Math. `arXiv:math.AG/0410267`

[15] **D Joyce**, *Configurations in abelian categories. IV. Invariants and changing stability conditions* `arXiv:math.AG/0410268`

[16] **M Kontsevich**, *Homological algebra of mirror symmetry*, from: "Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)", Birkhäuser, Basel (1995) 120–139 MR1403918

[17] **D McDuff**, **D Salamon**, *J -holomorphic curves and quantum cohomology*, University Lecture Series 6, American Mathematical Society, Providence, RI (1994) MR1286255

[18] **D Ramakrishnan**, *On the monodromy of higher logarithms*, Proc. Amer. Math. Soc. 85 (1982) 596–599 MR660611

[19] **A Rudakov**, *Stability for an abelian category*, J. Algebra 197 (1997) 231–245 MR1480783

[20] **R P Thomas**, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K*3 *fibrations*, J. Differential Geom. 54 (2000) 367–438 MR1818182

[21] **B Toën**, *Higher and derived stacks: a global overview* `arXiv:math.AG/0604504`

[22] **B Toën**, *Derived Hall algebras*, Duke Math. J. 135 (2006) 587–615 MR2272977

[23] **E Witten**, *Quantum background independence in String Theory* `arXiv: hep-th/9306122`

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