Order in the concordance group and Heegaard Floer homology

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We use the Heegaard–Floer homology correction terms defined by Ozsváth–Szabó to formulate a new obstruction for a knot to be of finite order in the smooth concordance group. This obstruction bears a formal resemblance to that of Casson and Gordon but is sensitive to the difference between the smooth versus topological category. As an application we obtain new lower bounds for the concordance order of small crossing knots.

57M25; 57R58

1 Introduction

A knot K in S^3 is called *slice* if $(S^3, K) = \partial(B^4, D^2)$ where D^2 is a 2–disk smoothly and properly embedded in the 4–ball B^4. Knots K_1 and K_2 are called *concordant* if $K_1 \# \overline{K}_2$ is slice where $\overline{K}$ represents the mirror image of K with reversed string orientation. The set of concordance classes of knots forms an Abelian group under the connected sum operation called the smooth concordance group and is denoted by $\mathcal{C}_1$. The order of K in this group is the least positive n for which the connected sum of n copies of K is slice.

In this paper we use the correction terms for 3–manifolds stemming from Heegaard–Floer homology to obstruct torsion in $\mathcal{C}_1$. Specifically, we focus our attention on knots with 10 or fewer crossings. Among these there are, as of this writing, 26 knots with unknown concordance order. Table 1 below, courtesy of *KnotInfo*[1], lists these knots along with lower bounds on their orders. The existing lower bounds have been determined by A Tamulis [21]. The structure of $\mathcal{C}_1$ is still rather poorly understood and virtually nothing is known about torsion in $\mathcal{C}_1$. We briefly summarize the current state of understanding of $\mathcal{C}_1$ and point out connections to the knots from Table 1.

[1] *KnotInfo* is an online atlas of knots maintained by Charles Livingston. It can be found at http://www.indiana.edu/∼knotinfo/.

Published: 30 May 2007 DOI: 10.2140/gt.2007.11.979

Knot K	Order of K	Knot K	Order of K	Knot K	Order of K
8_{13}	≥ 4	10_{26}	≥ 4	10_{102}	≥ 4
8_{17}	≥ 4	10_{28}	≥ 4	10_{109}	≥ 4
9_{14}	≥ 4	10_{34}	≥ 4	10_{115}	≥ 4
9_{19}	≥ 4	10_{58}	≥ 4	10_{118}	≥ 4
9_{30}	≥ 4	10_{60}	≥ 4	10_{119}	≥ 4
9_{33}	≥ 4	10_{79}	≥ 4	10_{135}	≥ 4
9_{44}	≥ 4	10_{81}	≥ 4	10_{158}	≥ 2
10_{10}	≥ 4	10_{88}	≥ 4	10_{164}	≥ 4
10_{13}	≥ 4	10_{91}	≥ 4		

Table 1

There is a surjective homomorphism $\Theta\colon \mathcal{C}_1 \to \mathcal{G}$ (Levine, [9; 10]) from $\mathcal{C}_1$ onto the *algebraic concordance group* $\mathcal{G}$, which consists of Witt classes of Seifert forms under orthogonal sums. It is known that $\mathcal{G}$ is isomorphic to the infinite direct sum

$$\mathcal{G} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty.$$

The analogous homomorphism from odd dimensional concordance groups $\mathcal{C}_{2n+1}$ (concordance classes of embeddings of S^{2n+1} into S^{2n+3}) is an isomorphism for $n > 1$ and it is injective onto an index 2 subgroup of $\mathcal{G}$ when $n = 1$. In the case of $\mathcal{C}_1$ the kernel is nontrivial as first proved by Casson and Gordon [1; 2]. In fact, the kernel of Θ, referred to as the subgroup of *algebraically slice knots*, is known to contain a subgroup isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$ by work of Jiang [7] and Livingston [11]. All the knots in Table 1 map to order two elements in $\mathcal{G}$ and are therefore of either infinite order or finite and even order in $\mathcal{C}_1$.

Remark 1.1 *Given the isomorphism of the higher dimensional concordance groups $\mathcal{C}_{2n+1}$ with the group $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, it is a reasonable guess to expect $\mathcal{C}_1$ to exhibit 4–torsion elements (besides existing 2–torsion, see below) and perhaps no other finite torsion. In view of this, obstructing 4–torsion in $\mathcal{C}_1$ is of particular importance.*

The obstruction we use (elucidated in Section 3 in detail) for a knot to be of order n in $\mathcal{C}_1$ applies in principle to all $n \geq 2$. However, computational complexity prevents us from checking the obstruction for $n > 4$. Nonetheless, for $n = 4$ the algorithm gives the following improvement on the above table (compare to Remark 1.1):

Theorem 1.2 *The concordance order of any knot K from the set of 14 knots*

$$\{8_{13}, 9_{14}, 9_{19}, 9_{33}, 9_{44}, 10_{13}, 10_{26}, 10_{28}, 10_{34}, 10_{58}, 10_{60}, 10_{102}, 10_{119}, 10_{135}\}$$

is at least 6.

While the correction term obstruction (see Section 3) bears a formal resemblance to that of Casson and Gordon [1; 2], it is nonetheless bound to be substantially different. While the Casson–Gordon obstruction does not differentiate between the subtle distinction of smooth versus topological sliceness, our methods are indeed sensitive to it. For example, the pretzel knot $P(7,-3,5)$ is topologically slice (and so all of its Casson–Gordon obstructions vanish) but our methods can be applied to show that its order in $\mathcal{C}_1$ is infinite. Further examples of this type can be found in Owens–Manolescu [13].

A negative amphicheiral knot, that is a knot which is isotopic to its mirror image with reversed orientation, is clearly of concordance order 2. Other than this nothing is currently known about torsion in $\mathcal{C}_1$. In higher dimensions there are order 2 concordance classes not represented by negative amphicheiral knots (Coray–Michel [6]). In dimension three it is unknown whether or not the corresponding order 2 algebraic concordance classes have any concordance order 2 representatives.

Levine's set of invariants of algebraic concordance includes the Tristram–Levine signatures, which being additive integral invariants, vanish for any knot representing a finite order algebraic concordance class. Accordingly all of these invariants vanish for the knots from Table 1.

In [3] Cochran, Orr and Teichner gave a geometric filtration

$$0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_{n.0} \subset \cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_{1.0} \subset \mathcal{F}_{0.5} \subset \mathcal{C}_{\mathrm{top}}$$

which produces an infinite sequence of obstructions that a (topologically) slice knot needs to obey. In their subsequent work [4; 5] it was shown that the quotient groups $\mathcal{F}_{n.5}/\mathcal{F}_{n.0}$ are nontrivial at each stage, $\mathcal{C}/\mathcal{F}_{1.0}$ is the algebraic concordance group, and Casson–Gordon invariants vanish for elements of $\mathcal{F}_{1.5}$. So far this filtration has not been used to provide information regarding torsion.

There are other, more subtle obstructions ("subtler" in the sense that they differentiate between the smooth and topological slice genus, a topic which we don't discuss here) to a knot representing a torsion class in $\mathcal{C}_1$:

- $\tau(K) =$ The Ozsváth–Szabó τ invariant from Heegaard Floer homology [16].
- $s(K) =$ The Rasmussen invariant defined using Khovanov homology [20].
- $\delta(K) =$ The δ–invariant of Manolescu and Owens also defined using Heegaard Floer homology [13].

If either of these is non-vanishing, the knot K is of infinite order in $\mathcal{C}_1$. For the knots from Table 1 all three of these invariants are either known or are readily calculated and are all vanishing.

Yet further information about the concordance order of knots comes from the following theorem proved by Livingston and Naik [12] using the Casson–Gordon obstructions to sliceness.

Theorem 1.3 *Let K be a knot in S^3 with 2–fold branched cover Y_K. If $H_1(Y_K;\mathbb{Z}) \cong \mathbb{Z}_{p^n} \oplus G$ with p a prime congruent to 3 mod 4, n odd and p not dividing the order of G, then K is of infinite order in $\mathcal{C}_1$.*

This theorem gives a rather strong obstruction to being a torsion element in $\mathcal{C}_1$, however, as is easy to check, none of the knots from Table 1 satisfy the hypothesis of Theorem 1.3.

Additional obstructions to sliceness were obtained in (Kirk–Livingston [8]) using the twisted Alexander polynomials which relate to determinants of Casson-Gordon invariants. Using these Tamulis showed in [21] that none of the knots from Table 1 have order 2 in $\mathcal{C}_1$.

In summary, the knots from Table 1 are rather resilient to most of the known concordance invariants. It is in this sense that the use of Heegaard Floer homology in the proof of Theorem 1.2 is a significant new method, one which we hope will bear more fruit in the near future.

The remainder of the article is organized as follows. Section 2 reviews relevant parts of Heegaard Floer homology and reminds the reader of basic properties of the 3–manifold correction terms $d(Y,\mathfrak{s})$. This section can safely be skipped by the "Heegaard Floer initiate", our hope is that it will provide an alternative exposition style for the "Heegaard Floer novice" complementing that found in the existing literature. Section 3 states the obstruction to being order n in $\mathcal{C}_1$ coming from the said correction terms. Section 4 explains how we calculated the correction terms for the double branched covers of the knots from Table 1. Finally, Section 5 explains how the results of Theorem 1.2 follow from our main obstruction. The MATHEMATICA script used in our computations can be downloaded from the first author's web page.

No originality is claimed on the material presented in sections 2–4. Our main obstruction Obstruction 3.1 has been first observed by Ozsváth and Szabó [15] and been successfully used by other authors (Owens–Strle [14], Owens–Manolescu [13]). Our contribution is the use of this obstruction to address long-standing questions about torsion in $\mathcal{C}_1$.

Acknowledgement

We would like to thank Peter Ozsváth, Brendan Owens and Sašo Strle for many helpful discussions and for so generously sharing their expertise. Special thanks are due to Charles Livingston for his feedback and for creating and maintaining the very helpful *KnotInfo*.

2 Heegaard Floer homology

This section serves as a reminder of some basic definitions and properties of the Heegaard Floer homology groups and the resulting correction terms for 3–manifolds.

2.1 The Heegaard Floer homology groups

In their seminal papers [17; 18] Peter Ozsváth and Zoltán Szabó introduced the Heegaard Floer homology groups $\widehat{HF}(Y,\mathfrak{s})$, $HF^{\pm}(Y,\mathfrak{s})$ and $HF^{\infty}(Y,\mathfrak{s})$ associated to a spin^c 3–manifold $(Y,\mathfrak{s})$. These Abelian groups come equipped with a relative $\mathbb{Z}_d$–grading gr where

$$d = \gcd\{\langle c_1(\mathfrak{s}), h\rangle | h \in H_2(Y;\mathbb{Z})\}.$$

In the case when $\mathfrak{s}$ is torsion (by which we mean that $c_1(\mathfrak{s})$ is torsion) the relative $\mathbb{Z}$–grading gr lifts to an absolute $\mathbb{Q}$–grading $\widetilde{\mathrm{gr}}$.

The various Heegaard Floer groups are related by means of long exact sequences. For example $HF^{\pm}(Y,\mathfrak{s})$ and $HF^{\infty}(Y,\mathfrak{s})$ fit into the sequence

$$(1) \qquad \cdots \to HF^-(Y,\mathfrak{s}) \to HF^{\infty}(Y,\mathfrak{s}) \xrightarrow{\pi} HF^+(Y,\mathfrak{s}) \to HF^-(Y,\mathfrak{s}) \to \cdots .$$

If $\mathfrak{s}$ is torsion then the maps in the above sequence preserve the absolute grading $\widetilde{\mathrm{gr}}$ except the map $HF^+(Y,\mathfrak{s}) \to HF^-(Y,\mathfrak{s})$ which drops degree by 1.

2.2 Cobordism induced maps

The Heegaard Floer homology groups fit into a TQFT framework in the following sense: given a spin^c 4–manifold $(W,\mathfrak{t})$ with $\partial W = -Y_1 \sqcup Y_2$ (where $-Y$ is Y with its orientation reversed) there are induced group homomorphisms

$$F^{\circ}_{W,\mathfrak{t}}\colon HF^{\circ}(Y_1,\mathfrak{t}|_{Y_1}) \to HF^{\circ}(Y_2,\mathfrak{t}|_{Y_2})$$

where $\circ$ stands for any of $\widehat{\ }$, $+$, $-$, ∞. When $\mathfrak{t}|_{Y_1}$ and $\mathfrak{t}|_{Y_2}$ are both torsion the degree shift of the map $F^{\circ}_{W,\mathfrak{t}}$ is

$$(2) \qquad \deg F^{\circ}_{W,\mathfrak{t}} := \widetilde{\mathrm{gr}}(F^{\circ}_{W,\mathfrak{t}}(x)) - \widetilde{\mathrm{gr}}(x) = \frac{(c_1(\mathfrak{t}))^2 - 2e_W - 3\sigma_W}{4}$$

where e_W and σ_W are the Euler number and signature of W respectively and $x \in HF^\circ(Y_1, \mathfrak{t}|_{Y_1})$ is any homogeneous element. Said differently, $F^\circ_{W,\mathfrak{t}}$ is a homogeneous map of degree $((c_1(\mathfrak{t}))^2 - 2e_W - 3\sigma_W)/4$.

Proposition 2.1 (Ozsváth-Szabó, [15]) *When $b_2^+(W) = 0$ the homomorphism $F^\infty_{W,\mathfrak{t}}$ is an isomorphism for all spin^c–structures $\mathfrak{t}$ on W.*

The exact sequence (1) is functorial under cobordism induced maps in the sense that one obtains the commutative diagram (with exact rows):

$$\begin{array}{ccccccccc}
\longrightarrow & HF^-(Y_1,\mathfrak{s}_1) & \longrightarrow & HF^\infty(Y_1,\mathfrak{s}_1) & \xrightarrow{\pi} & HF^+(Y_1,\mathfrak{s}_1) & \longrightarrow \\
 & {\scriptstyle F^-_{W,\mathfrak{t}}}\big\downarrow & & {\scriptstyle F^\infty_{W,\mathfrak{t}}}\big\downarrow & & {\scriptstyle F^+_{W,\mathfrak{t}}}\big\downarrow & \\
\longrightarrow & HF^-(Y_2,\mathfrak{s}_2) & \longrightarrow & HF^\infty(Y_2,\mathfrak{s}_2) & \xrightarrow{\pi} & HF^+(Y_2,\mathfrak{s}_2) & \longrightarrow
\end{array} \tag{3}$$

In the above diagram $\mathfrak{s}_i$ stands for $\mathfrak{t}|_{Y_i}$.

2.3 The correction terms for 3–manifolds

Let Y be a rational homology sphere and let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ be a spin^c–structure on Y. The *correction term* $d(Y,\mathfrak{s})$ is defined to be

$$d(Y,\mathfrak{s}) = \min\{\widetilde{\mathrm{gr}}(\pi(x)) | x \in HF^\infty(Y,\mathfrak{s})\}$$

where $\pi\colon HF^\infty(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s})$ is the map from the exact sequence (1).

Example 2.2 *Consider S^3 with its unique spin-structure $\mathfrak{s}_0$. Recall from [18] that $HF^\infty(S^3,\mathfrak{s}_0) \cong \mathbb{Z}[U,U^{-1}]$ and $HF^+(S^3,\mathfrak{s}_0) \cong \mathbb{Z}[U^{-1}]$. The absolute grading on both groups is specified by $\widetilde{\mathrm{gr}}(U^k) = -2k$ and the map $\pi\colon HF^\infty(S^3,\mathfrak{s}_0) \to HF^+(S^3,\mathfrak{s}_0)$ is the obvious quotient map*

$$\mathbb{Z}[U,U^{-1}] \to \frac{\mathbb{Z}[U,U^{-1}]}{U\mathbb{Z}[U]} \cong \mathbb{Z}[U^{-1}].$$

Thus π is surjective and therefore $d(S^3,\mathfrak{s}_0)$ is the lowest grading in $HF^+(S^3,\mathfrak{s}_0)$ which in turn is given by

$$d(S^3,\mathfrak{s}_0) = \widetilde{\mathrm{gr}}(U^0) = 0. \tag{4}$$

The correction terms enjoy a number of nice properties. Given $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ let $\overline{\mathfrak{s}}$ be the conjugate spin^c–structure. Then

$$\begin{aligned} d(Y,\overline{\mathfrak{s}}) &= d(Y,\mathfrak{s}) \\ d(-Y,\mathfrak{s}) &= -d(Y,\mathfrak{s}) \\ d(Y_1\#Y_2,\mathfrak{s}_1\#\mathfrak{s}_2) &= d(Y_1,\mathfrak{s}_1)+d(Y_2,\mathfrak{s}_2). \end{aligned} \tag{5}$$

2.4 Correction terms for 3–manifolds bounding rational homology 4–balls

Consider now two rational homology 3–spheres Y_1 and Y_2 equipped with spin^c–structures $\mathfrak{s}_i \in \mathrm{Spin}^c(Y_i)$. Consider furthermore a negative definite cobordism $(W,\mathfrak{t})$ from $(Y_1,\mathfrak{s}_1)$ to $(Y_2,\mathfrak{s}_2)$ (ie a 4–manifold W with $\partial W = -Y_1 \sqcup Y_2$, $\mathfrak{t}|_{Y_i} = \mathfrak{s}_i$ and $b_2^+(W)=0$). Let $x_2 \in HF^\infty(Y_2,\mathfrak{s}_2)$ be an element with $\widetilde{\mathrm{gr}}(\pi(x_2)) = d(Y_2,\mathfrak{s}_2)$ where π is the map from (1). According to Proposition 2.1 the homomorphism $F^\infty_{W,\mathfrak{t}}\colon HF^\infty(Y_1,\mathfrak{s}_1) \to HF^\infty(Y_2,\mathfrak{s}_2)$ is an isomorphism. Let $x_1 \in HF^\infty(Y_1,\mathfrak{s}_1)$ be the unique preimage of x_2 under this map. The degree-shift formula (2) and the commutative diagram (3) show that

$$\widetilde{\mathrm{gr}}(\pi(x_2)) - \widetilde{\mathrm{gr}}(\pi(x_1)) = \frac{(c_1(\mathfrak{t}))^2 - 2e_W - 3\sigma_W}{4}.$$

Since $d(Y_1,\mathfrak{s}_1) \le \widetilde{\mathrm{gr}}(\pi(x_1))$ by definition and $d(Y_2,\mathfrak{s}_2) = \widetilde{\mathrm{gr}}(\pi(x_2))$ by choice of x_2, the above equality becomes the inequality

$$d(Y_1,\mathfrak{s}_1) \le d(Y_2,\mathfrak{s}_2) - \frac{(c_1(\mathfrak{t}))^2 - 2e_W - 3\sigma_W}{4}. \tag{6}$$

Let us now turn to the special case when $Y_2 = S^3$ and W has the rational homology of a punctured 4–ball. Then $e_W = 0 = \sigma_W$ and $(c_1(\mathfrak{t}))^2 = 0$ for all $\mathfrak{t} \in \mathrm{Spin}^c(W)$. The above inequality along with Example 2.2 then shows that $d(Y_1,\mathfrak{s}_1) \le 0$ for all spin^c–structures $\mathfrak{s}_1 \in \mathrm{Spin}^c(Y_1)$ which lie in the image of the restriction map $\mathrm{Spin}^c(W) \to \mathrm{Spin}^c(Y_1)$. Reversing the orientation on W and applying (6) once more shows that $d(Y_1,\mathfrak{s}_1) \ge 0$ also. Therefore

$$d(Y_1,\mathfrak{s}_1) = 0 \quad \forall \mathfrak{s}_1 \in \mathrm{Im}[\mathrm{Spin}^c(W) \to \mathrm{Spin}^c(Y_1)].$$

If we fill in the S^3 boundary component of W with a 4–ball, we see that $Y = Y_1$ bounds a rational homology ball $X = W \cup_{S^3} B^4$. It is well known (and follows easily from the universal coefficient theorem and the exact sequence of the pair (X,Y)) that such a 3–manifold has second cohomology of square order, say $|H^2(Y;\mathbb{Z})| = n^2$, and that the order of the image $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is n. After suitable affine identifications of $\mathrm{Spin}^c(X) \cong H^2(X;\mathbb{Z})$ and $\mathrm{Spin}^c(Y) \cong H^2(Y;\mathbb{Z})$ the restriction map $\mathrm{Spin}^c(X) \to \mathrm{Spin}^c(Y)$ corresponds precisely to the restriction induced map $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$. We summarize our discussion in the following Theorem.

Theorem 2.3 *Let Y be a rational homology 3–sphere which bounds a rational homology 4–ball X. Then $|H^2(Y;\mathbb{Z})| = n^2$ for some n and there is a subgroup $\mathcal{P}$ of $H^2(Y;\mathbb{Z})$ of order n such that*

$$d(Y,\mathfrak{s}) = 0 \quad \forall \mathfrak{s} \in \mathcal{P}$$

under a suitable identification $\mathrm{Spin}^c(Y) \cong H^2(Y;\mathbb{Z})$.

3 The sliceness obstruction

Let K be a knot in S^3 and let Y_K be the double branched cover of S^3 branched along K. The order of the second cohomology of Y_K is given by

$$|H^2(Y_K;\mathbb{Z})| = |\det(K)| = |\Delta_K(-1)|$$

where $\Delta_K(t)$ is the Alexander polynomial of K.

If K is slice with slice disk $D^2 \hookrightarrow B^4$ we let X_K be the double branched cover of B^4 branched along D^2. The manifold X_K is a rational homology ball with boundary $\partial X_K = Y_K$. Thus according to Theorem 2.3 we must have $|\det(K)| = n^2$ for some integer n and $d(Y_K,\mathfrak{s}) = 0$ for all $\mathfrak{s}$ in some subgroup $\mathcal{P}$ of $H^2(Y_K;\mathbb{Z})$ of order n. As sample calculations show, this turns out to be a rather strong obstruction to the sliceness of K.

To apply this algorithm to the question of the order of a knot K in $\mathcal{C}_1$ consider the knot $K' = \#^{2m} K$, the $2m$–fold connected sum of K with itself.

If K is of order $2m$ then K' is slice and the above algorithm asserts the vanishing of $d(Y_{K'},\mathfrak{s}')$ for spin^c–structures $\mathfrak{s}'$ from some (affine) subgroup $\mathcal{O}$ of $\mathrm{Spin}^c(Y_{K'})$ of order $|\det(K)|^m$. Recall that

$$Y_{K_1\#K_2} \cong Y_{K_1}\#Y_{K_2} \quad \mathrm{Spin}^c(Y_1\#Y_2) \cong \mathrm{Spin}^c(Y_1)\times\mathrm{Spin}^c(Y_2).$$

Thus a spin^c–structure $\mathfrak{s}' \in \mathrm{Spin}^c(Y_{K'})$ corresponds to a collection of $2m$ spin^c–structures $\mathfrak{s}' = (\mathfrak{s}_1,\dots,\mathfrak{s}_{2m})$ with $\mathfrak{s}_i \in \mathrm{Spin}^c(Y_K)$. Furthermore (5) implies that for such an $\mathfrak{s}'$ the correction term $d(Y_{K'},\mathfrak{s}')$ is given by

$$d(Y_{K'},\mathfrak{s}') = d(Y_K,\mathfrak{s}_1) + \cdots + d(Y_K,\mathfrak{s}_{2m}).$$

To summarize we obtain the following.

Obstruction 3.1 *If K is of order $2m$ in the smooth knot concordance group $\mathcal{C}_1$ there exists a subgroup $\mathcal{O}$ of $\left(H^2(Y;\mathbb{Z})\right)^{2m} \cong (\mathrm{Spin}^c(Y_K))^{\times 2m}$ of order $|\det(K)|^m$ with*

$$d(Y_K,\mathfrak{s}_1)+\cdots+d(Y_K,\mathfrak{s}_{2m})=0 \quad \forall(\mathfrak{s}_1,\ldots,\mathfrak{s}_{2m})\in\mathcal{O}. \tag{7}$$

In the above Y_K is the double branched cover of S^3 branched along K.

One drawback of this obstruction algorithm is that there is a priori no way of knowing what the group $\mathcal{O}$ might be in the case of a concrete knot K. We are thus forced to consider all subgroups $\mathcal{O}$ of $H^2(Y_K;\mathbb{Z})^{\times 2m}$ of order $|\det(K)|^m$ and hope that for none of them relation (7) holds. If this is the case, K cannot be of order $2m$.

To use this obstruction for the knots K from Table 1 one needs to calculate the correction terms $d(Y_K,\mathfrak{s})$ for all $\mathfrak{s}\in\mathrm{Spin}^c(Y_K)$. In the next section we do this by distinguishing a number of cases.

4 Calculating obstruction terms

4.1 2–bridge knots

Some of the knots from Table 1 are 2–bridge knots and so their double branched covers are lens spaces. The correction terms for lens spaces have been calculated by Ozsváth and Szabó in [15] and follow the recursive formula

$$d(-L(p,q),i)=\left(\frac{pq-(2i+1-p-q)^2}{4pq}\right)-d(-L(q,r),j)$$

where r and j are the mod q reductions of p and i respectively. Here i is an integer $0\le i<p+q$ whose mod p reduction represents the spin^c–structure $[i]\in\mathbb{Z}_p\cong\mathrm{Spin}^c(-L(p,q))$.

The knots from Table 1 whose double branched covers Y_K are lens spaces are given in Table 2. For example, the correction terms of $Y_{8_{13}}$ thus obtained are

$$\left\{-\tfrac{2}{29},-\tfrac{18}{29},\tfrac{8}{29},\tfrac{18}{29},\tfrac{12}{29},-\tfrac{10}{29},\tfrac{10}{29},\tfrac{14}{29},\tfrac{2}{29},-\tfrac{26}{29},-\tfrac{12}{29},-\tfrac{14}{29},-\tfrac{32}{29},-\tfrac{8}{29},0,\right.$$
$$\left.-\tfrac{8}{29},-\tfrac{32}{29},-\tfrac{14}{29},-\tfrac{12}{29},-\tfrac{26}{29},\tfrac{2}{29},\tfrac{14}{29},\tfrac{10}{29},-\tfrac{10}{29},\tfrac{12}{29},\tfrac{18}{29},\tfrac{8}{29},-\tfrac{18}{29},-\tfrac{2}{29}\right\}.$$

4.2 Alternating knots

When K is a knot which possesses an alternating projection D the correction terms of Y_K can be calculated from the Goeritz matrix G associated to D. The details of this

Knot K	Y_K	Knot K	Y_K
8_{13}	$L(29,11)$	10_{13}	$L(53,22)$
9_{14}	$L(37,14)$	10_{26}	$L(61,17)$
9_{19}	$L(41,16)$	10_{28}	$L(53,19)$
10_{10}	$L(45,17)$	10_{34}	$L(37,13)$

Table 2

have been worked out by Ozsváth and Szabó in [19] and we summarize them here for the benefit of the reader.

Let D be an alternating projection of a knot K. We color the regions of D black and white according to the convention from Figure 1, to obtain a checkerboard pattern.

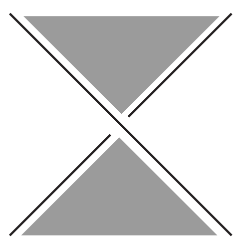

Figure 1: The coloring conventions near a crossing.

From such a pattern we extract a graph in the following way: The vertices of the graph are in bijection with the white regions in the diagram (including the unbounded region if it happens to be white). There is an edge between two vertices for each touching point of their corresponding white regions. Figure 2 shows the checkerboard diagram and the associated graph for the knot 8_{17}.

From the graph we now extract a matrix - the Goeritz matrix of the projection D. Pick and discard one of the vertices of the graph (the vertex enclosed in a dotted circle in Figure 2) while retaining all of its edges. Give the remaining vertices an arbitrary ordering. The Goeritz matrix $G=[g_{ij}]$ has the entries

$$g_{ij} = \begin{cases} \text{Number of edges between the } i\text{th and } j\text{th vertex} & i \neq j \\ -1 \cdot \text{Valence of the } i\text{th vertex} & i = j. \end{cases}$$

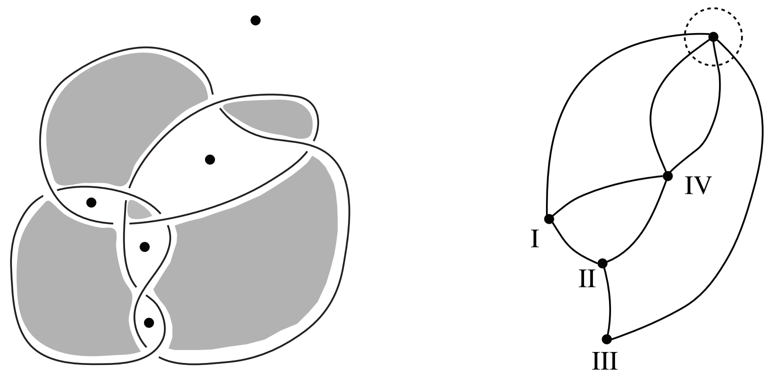

Figure 2: The checkerboard diagram and graph associated to the knot 8_{17}. In the construction of the Goeritz matrix we drop the vertex enclosed by the dotted circle.

For example, the Goeritz matrix associated to the projection of 8_{17} from Figure 2 with the ordering of the vertices as indicated is

$$G_{8_{17}} = \begin{bmatrix} -3 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & 1 & 0 & -4 \end{bmatrix}.$$

Finally, from the Goeritz matrix it is now a matter of arithmetic to extract the correction terms $d(Y_K, \mathfrak{s})$: Consider $G\colon V \otimes V \to \mathbb{Z}$ as a negative definite bilinear quadratic form where $V = \mathbb{Z}^\ell$ if G is of dimension $\ell \times \ell$. Let $g\colon V \to V^*$ and $G^*\colon V^* \otimes V^* \to \mathbb{Q}$ be the obvious maps induced by G, namely

$$g(v) = G(v, \cdot) \text{ and } G^*(G(v,\cdot), G(w,\cdot)) = G(v,w).$$

Let $M_g\colon \mathrm{Coker}(g) \to \mathbb{Q}$ be

$$M_g(\xi) = \frac{1}{4}\left(\max_{\{v \in V^* | [v] = \xi\}} G^*(v_0 + 2v, v_0 + 2v) + \mathrm{rk}(V) \right)$$

where v_0 is any characteristic vector of G (ie any vector $v_0 \in V^*$ with $v_0(w) \equiv G(w,w) (\mathrm{mod}\ 2)$ for all $w \in V$). It is shown in [19] that there is an isomorphism $\varphi\colon \mathrm{Coker}(g) \to H^2(Y_K; \mathbb{Z})$ such that

$$d(Y_K, \varphi(\xi)) = M_g(\xi)$$

for some affine identification of $H^2(Y_K; \mathbb{Z})$ with $\mathrm{Spin}^c(Y_K)$.

For example, the correction terms for $Y_{8_{17}}$ calculated this way are

$$\Big\{-\tfrac{20}{37},-\tfrac{32}{37},\tfrac{18}{37},-\tfrac{18}{37},\tfrac{8}{37},\tfrac{22}{37},\tfrac{24}{37},\tfrac{14}{37},-\tfrac{8}{37},\tfrac{32}{37},-\tfrac{14}{37},\tfrac{2}{37},\tfrac{6}{37},$$
$$-\tfrac{2}{37},-\tfrac{22}{37},\tfrac{20}{37},-\tfrac{24}{37},-\tfrac{6}{37},0,-\tfrac{6}{37},-\tfrac{24}{37},\tfrac{20}{37},-\tfrac{22}{37},-\tfrac{2}{37},\tfrac{6}{37},$$
$$\tfrac{2}{37},-\tfrac{14}{37},\tfrac{32}{37},-\tfrac{8}{37},\tfrac{14}{37},\tfrac{24}{37},\tfrac{22}{37},\tfrac{8}{37},-\tfrac{18}{37},\tfrac{18}{37},-\tfrac{32}{37},-\tfrac{20}{37}\Big\}.$$

Of the knots from Table 1 which do not appear in Table 2, the ones which have alternating projections are listed in Table 3.

Knots having alternating projections
8_{17}, 9_{30}, 9_{33}, 10_{58}, 10_{60}, 10_{79}, 10_{81}
10_{88}, 10_{91}, 10_{102}, 10_{109}, 10_{115}, 10_{118}, 10_{119}

Table 3

4.3 The remaining cases

Section 4.1 and Section 4.2 allow for a calculation of the correction terms of Y_K for most knots K from Table 1. The knots from that table which do not fall into either category are

$$9_{44} \quad 10_{135} \quad 10_{158} \quad 10_{164} \tag{8}$$

and these require special attention. All four of these knots however "resemble" alternating knots sufficiently so that a calculation of their correction terms can be done by using the Goeritz matrix again.

The following algorithm has been described in [19], see also [13]. Suppose that K is a knot with a knot projection D which outside some region R is alternating and inside R consists of k left-handed half-twists of two parallel strands, see Figure 3. Such projections can be found for all four knots from (8).

Let L be the 2–component link obtained from K by replacing R with R'. For example, Figure 4 shows a knot projection of 10_{158} with the marked region R, Figure 5 depicts the corresponding link L.

There is a restriction we impose: the vertex from the checkerboard pattern for L that we drop in the computation of the Goeritz matrix of L, should always be one of the vertices from the region R. In Figure 5, two such vertices are indicated.

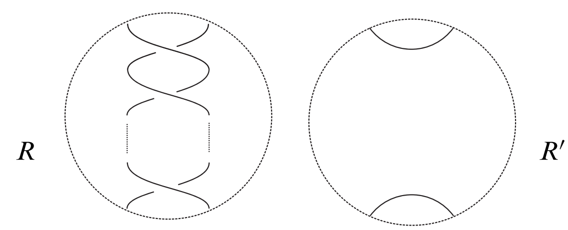

Figure 3: The left-handed orientation of the twists in the region R is shown on the left. The region R' on the right is used to replace R in forming L from K.

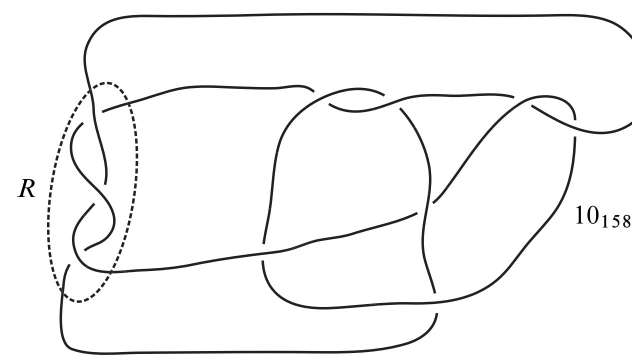

Figure 4: The region R is the portion of this projection of 10_{158} inside the dotted oval.

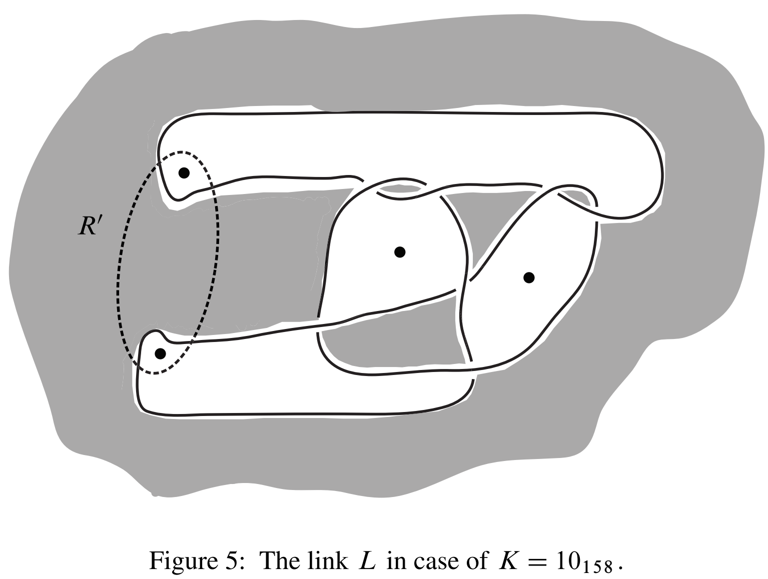

Figure 5: The link L in case of $K = 10_{158}$.

Let $\widetilde{G}$ be the Goeritz matrix of L and let G be the matrix obtained from $\widetilde{G}$ as

$$G = \left[\begin{array}{ccc|c} & & & 0 \\ & \widetilde{G} & & \vdots \\ & & & 0 \\ & & & 1 \\ \hline 0 & \dots & 0 \; 1 & -k \end{array}\right] \tag{9}$$

where k is the number of negative half-twists in the region R.

When K is any of 9_{44}, 10_{135}, 10_{158} or 10_{164} the correction terms of Y_K are calculated from G in the way described in Section 4.2 for alternating knots.

For example, the Goeritz matrix $\widetilde{G}$ of the link L (Figure 5) and its associated matrix G for the knot $K = 10_{158}$ (Figure 4) are

$$\widetilde{G} = \begin{bmatrix} -4 & 1 & 2 \\ 1 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \quad G = \begin{bmatrix} -4 & 1 & 2 & 0 \\ 1 & -4 & 2 & 0 \\ 2 & 2 & -4 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

leading to the correction terms

$$\begin{aligned} \Big\{ & -\tfrac{2}{45}, -\tfrac{2}{5}, \tfrac{8}{9}, -\tfrac{8}{45}, \tfrac{2}{5}, \tfrac{28}{45}, \tfrac{22}{45}, 0, -\tfrac{38}{45}, -\tfrac{2}{45}, \tfrac{2}{5}, \tfrac{22}{45}, \tfrac{2}{9}, -\tfrac{2}{5}, \tfrac{28}{45}, \\ & -\tfrac{32}{45}, -\tfrac{2}{5}, -\tfrac{4}{9}, -\tfrac{38}{45}, \tfrac{2}{5}, -\tfrac{32}{45}, -\tfrac{8}{45}, 0, -\tfrac{8}{45}, -\tfrac{32}{45}, \tfrac{2}{5}, -\tfrac{38}{45}, -\tfrac{4}{9}, -\tfrac{2}{5}, -\tfrac{32}{45}, \\ & \tfrac{28}{45}, -\tfrac{2}{5}, \tfrac{2}{9}, \tfrac{22}{45}, \tfrac{2}{5}, -\tfrac{2}{45}, -\tfrac{38}{45}, 0, \tfrac{22}{45}, \tfrac{28}{45}, \tfrac{2}{5}, -\tfrac{8}{45}, \tfrac{8}{9}, -\tfrac{2}{5}, -\tfrac{2}{45} \Big\}. \end{aligned}$$

5 Applying the obstruction

Given a knot K, Obstruction 3.1 implies that if K has order $2m$ in $\mathcal{C}_1$ then there is a subgroup $\mathcal{O}$ of $H^2(Y_K;\mathbb{Z})^{\times 2m}$ of order $|\det(K)|^m$ for which all corresponding correction terms vanish. To check the obstruction for a concrete knot K one needs to:

(1) Calculate all correction terms of Y_K.

(2) Find all subgroups $\mathcal{O}$ of $H^2(Y_K;\mathbb{Z})^{\times 2m}$ of order $|\det(K)|^m$.

(3) Check that

$$d(Y_K,\mathfrak{s}_1) + \cdots + d(Y_K,\mathfrak{s}_m) = 0 \quad \forall (\mathfrak{s}_1,\dots,\mathfrak{s}_m) \in \mathcal{O}.$$

We have written a MATHEMATICA script which performs each of the 3 steps above. Computationally the most demanding part by far is step 2. In fact, our computational resources only allowed us to use $m = 2$ (and thus test for 4–torsion in $\mathcal{C}_1$) and even in that case we were forced to use a weaker version of obstruction Obstruction 3.1.

Obstruction 5.1 *If $|\det(K)| = p$ or $|\det(K)| = p \cdot q$ where $p \neq q$ are primes, then if K is of order 4 there exists a subgroup $\widetilde{\mathcal{O}}$ of $H^2(Y_{K'};\mathbb{Z})$ isomorphic to $\mathbb{Z}_{|\det(K)|}$ with*

$$d(Y_K, \mathfrak{s}_1) + d(Y_K, \mathfrak{s}_2) + d(Y_K, \mathfrak{s}_3) + d(Y_K, \mathfrak{s}_4) = 0$$

for all $(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4) \in \widetilde{\mathcal{O}}$. Here K' denotes $\#^4 K$.

This is a direct consequence of Obstruction 3.1. The results of Theorem 1.2 follow from our MATHEMATICA implementation of Obstruction 5.1.

An easy check reveals that all knots from Table 1 satisfy the hypothesis of Obstruction 5.1 except 10_{10}, 10_{158} and 10_{164}. Each of these 3 knots has determinant 45. If either of them were of order 4, it would still follow from Obstruction 3.1 that there is a subgroup $\widetilde{\mathcal{O}}$ of $H^2(Y_{K'};\mathbb{Z})$ of order 45 whose associated correction terms vanish. However, unlike in Obstruction 5.1, there are now 2 possibilities for the isomorphism type of $\widetilde{\mathcal{O}}$, namely $\mathbb{Z}_{45}$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_{15}$. While for each of the knots 10_{10}, 10_{158} and 10_{164} no group of the former type (with vanishing correction terms) exists, there are groups of the latter type and so no conclusions can be drawn.

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Proposed: Ron Stern Received: 20 November 2006
Seconded: Peter Teichner and Peter Ozsváth Revised: 6 February 2007