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Deforming Euclidean cone 3–manifolds

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Given a closed orientable Euclidean cone 3–manifold C with cone angles $\leq \pi$ and which is not almost product, we describe the space of constant curvature cone structures on C with cone angles $< \pi$. We establish a regeneration result for such Euclidean cone manifolds into spherical or hyperbolic ones and we also deduce global rigidity for Euclidean cone structures.

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1 Introduction

Let C be a closed orientable Euclidean cone 3–manifold with cone angles $\leq \pi$. Its singular locus $\mathrm{Sing}(C)$ is a trivalent graph consisting of q circles and edges, that we enumerate from $i = 1$ to q. To those edges one can associate a multiangle $\overline{\alpha} = (\alpha_1, \dots, \alpha_q)$, where α_i is the angle of the ith circle or edge of $\mathrm{Sing}(C)$. Let further $\overline{l} = (l_1, \dots, l_q)$ denote the vector of lengths of singular circles and edges. We are interested in the space of cone structures on C obtained by changing the multiangle.

Definition 1.1 A *hyperbolic cone structure* on C is a finite volume hyperbolic cone manifold X, possibly with totally geodesic boundary, with an embedding given by $i\colon (X, \mathrm{Sing}(X)) \to (C, \mathrm{Sing}(C))$ such that $i(X^{\mathrm{smooth}})$ is a retract of C^{smooth}.

Notice that this definition is adapted to small cone angles, because when three singular edges meet at one vertex, the sum of their cone angles is $> 2\pi$. When cone angles decrease so that this sum becomes $= 2\pi$, the vertex goes to infinity, becoming a cusp with horospherical section a Euclidean cone surface. When cone angles decrease further, this horospherical section becomes a totally geodesic boundary component. The cone angle of a singular edge or circle may also go to zero, corresponding to a cusp, see Boileau et al [2] or Weiss [15] for more details.

Definition 1.2 A Euclidean cone 3–manifold C is *almost product* if $C = (S^1 \times E^2)/G$, where E^2 is a two-dimensional cone manifold and $G < \mathrm{Isom}(S^1) \times \mathrm{Isom}(E^2)$ is a finite subgroup.

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Theorem 1.3 *Let C be a closed orientable Euclidean cone 3-manifold with cone angles $\leq \pi$. If C is not almost product, then for every multiangle $\overline{\alpha} \in (0,\pi)^q$ there exists a unique cone structure of constant curvature in $\{-1,0,1\}$ on C with those cone angles.*

If all cone angles of C are π, then every point in $(0,\pi)^q$ is the multiangle of a hyperbolic cone structure on C.

If at least one of the cone angles is $< \pi$, then the subset $E \subseteq (0,\pi)^q$ of multiangles of Euclidean cone structures is a smooth, properly embedded hypersurface that splits $(0,\pi)^q$ into two connected components S and H, corresponding to multiangles of spherical and hyperbolic cone structures respectively. Furthermore, for each $\overline{\alpha} \in E$ the tangent space of E at $\overline{\alpha}$ is orthogonal to the vector of singular lengths $\overline{l}$.

This theorem describes the structures corresponding to multiangles in the open cube $(0,\pi)^q$. We can also describe the structures corresponding to multiangles contained in some parts of the boundary of the closed cube $[0,\pi]^q$. For instance, we show that the multiangles contained in $\partial[0,\pi]^q \cap [0,\pi)^q$ (ie none of the angles is π and at least one of them is zero) are angles of hyperbolic cone structures. However, we do not give a description of structures in all points of $\partial[0,\pi]^q$, as this would involve studying collapses at cone angle π and deformations and regenerations of Seifert fibered geometries.

Corollary 1.4 *Let $\mathcal{O}$ be an irreducible closed orientable 3–orbifold. Assume there exists a Euclidean cone structure C on $\mathcal{O}$ with cone angles less than or equal to the orbifold angles of $\mathcal{O}$. If one of the cone angles of C is strictly less than the orbifold angle, then $\mathcal{O}$ is spherical.*

Proof To prove this corollary, we show in Lemma 2.5 that the Euclidean cone structure is not almost product, using the irreducibility of $\mathcal{O}$. By Theorem 1.3 we can obtain a spherical cone structure on C by increasing any of the cone angles. We are using here that E is orthogonal to the vector of singular lengths $\overline{l}$. By Proposition 8.2 the orbifold angles can be realized by a spherical cone structure. □

This corollary gives an alternative argument to the last step in the proof of the orbifold theorem in [2], which is more natural from the point of view of cone manifolds. D Cooper and S P Kerckhoff have announced a different approach to the spherical uniformization.

A special case of this theorem was proved in Porti [11], assuming that the singular locus was a knot and a technical hypothesis involving cohomology. Even if some of the techniques of [11] are used, this paper does not rely on it.

Now we describe the organization of the paper. We are interested in the rotational part of the holonomy of the Euclidean cone manifold in $SO(3)$ (in fact its lift to $\mathrm{Spin}(3)$), that we denote by ρ_0. Some basic properties of this representation are studied in Section 2. In order to deform the structure, we view $SO(3)$ as the stabilizer of a point in $\mathbb{H}^3$ or $\mathbb{S}^3$, thus we study the local properties of the varieties of representations in the isometries of $\mathbb{H}^3$ and $\mathbb{S}^3$ around ρ_0. This is done in Section 4, using the cohomological results of Section 3. In Section 5 we give conditions for a deformation of the representation ρ_0 to correspond to a path of hyperbolic or spherical cone manifolds. This is applied in Section 6 to construct paths of hyperbolic and spherical structures by deforming one of the cone angles. The analysis of the local parameterization of the variety of representations is completed in Section 7, where all deformations of the structure are constructed. The global results are established in Section 8. Finally, in Section 9 we illustrate the main theorem for cone structures on the 3–sphere with singular locus given by the Whitehead link.

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2 Almost product structures

Given a Euclidean cone manifold C, its smooth part is denoted by $M = C \setminus \mathrm{Sing}(C)$. Then M has a non-complete and non-singular flat Riemannian metric. In particular it has a developing map

$$\mathrm{dev}\colon \widetilde{M} \to \mathbb{R}^3$$

and a holonomy representation

$$\mathrm{hol}\colon \pi_1 M \to \mathrm{Isom}^+ \mathbb{R}^3 = SO(3) \ltimes \mathbb{R}^3.$$

We consider the composition of hol with the projection $\mathrm{ROT}\colon \mathrm{Isom}^+ \mathbb{R}^3 \to SO(3)$.

Lemma 2.1 *Let C be a closed Euclidean cone manifold. Then C is almost product if and only if the image of $\mathrm{ROT} \circ \mathrm{hol}\colon \pi_1 M \to SO(3)$ is contained in $O(2) \subset SO(3)$.*

Notice that the singular locus can be assumed to be empty.

Proof If C is almost product then the lemma is straightforward. To prove the converse, we assume that the image of $\mathrm{ROT} \circ \mathrm{hol}$ is contained in $O(2) \subset SO(3)$. Since $O(2)$ has two components, composition with $O(2) \to \pi_0 O(2)$ defines a morphism $\pi_1 M \to \mathbf{Z}/2\mathbf{Z}$ and, if it is non-trivial, we take the corresponding covering. This induces a branched covering of C, with branching locus contained in the edges and circles of $\mathrm{Sing}(C)$ that have cone angle π. Hence we may assume that the image of $\mathrm{ROT} \circ \mathrm{hol}$ is contained in $SO(2)$. The vertical foliation of $\mathbb{R}^3$ invariant under $SO(2)$ induces an isometric flow of M. It extends to a flow on the whole of C, since this restriction of the holonomy implies that the components of $\mathrm{Sing}(C)$ are parallel to the leaves of the flow.

If $\mathrm{Sing}(C) = \varnothing$, then C is a smooth flat manifold and the lemma is well known. Hence we assume that $\mathrm{Sing}(C) \neq \varnothing$. Since the components of $\mathrm{Sing}(C)$ are leaves of the flow, either all leaves are closed or they accumulate on tori at constant distance from the components of $\mathrm{Sing}(C)$. The latter is not possible, since the concentric tori of accumulation of fibers have principal curvatures of constant sign, the convex side pointing to the singular locus, and C is closed. Hence the leaves of the flow are closed, and they are leaves of a Seifert fibration, whose basis has a natural structure of a Euclidean cone 2–manifold.

One can construct a horizontal surface H perpendicular to the flow, and we can enlarge it as much as possible to be complete. Since H has cone angles $\leq \pi$, the singular locus of H is finite. Thus H is compact, because if H was accumulating somewhere, then by parallel transport it would intersect infinitely many times a singularity. Therefore the Seifert fibration is virtually a product. □

Definition 2.2 The Euclidean cone manifold C is *Seifert fibered* if it admits a Seifert fibration such that $\mathrm{Sing}(C)$ is a union of fibers. It is *almost Seifert fibered* if it is the quotient of a Seifert fibered cone manifold by a finite subgroup of isometries that preserve the fibration.

A cone manifold which is almost Seifert fibered but not Seifert fibered admits a partition by circles and intervals, so that the end-points of the intervals lie in singular edges with cone angle π. This corresponds to an orbifold Seifert fibration. In particular, if cone angles are $< \pi$, then almost Seifert fibered implies Seifert fibered.

Corollary 2.3 *The Euclidean cone manifold C is almost product iff it is almost Seifert fibered. In particular, if all cone angles of C are $< \pi$ and its smooth part is not Seifert fibered, then C is not almost product.*

Proof Almost product implies trivially almost Seifert fibered. Assume now that C is almost Seifert fibered. Up to a covering, possibly branched at singular points with cone angle π, we may assume that C is orientable, it is Seifert fibered with orientable basis and $\mathrm{Sing}(C)$ consists of fibers. Then, as both C and the basis of the fibration are orientable, the smooth part of C is a product. Let $f \in \pi_1(C \setminus \mathrm{Sing}(C))$ represent the fiber of this fibration. Since the components of $\mathrm{Sing}(C)$ are also fibers, the holonomy of f is either a nontrivial translation or a screw motion with non-trivial translation length. Since f is central, the direction of this vector must be preserved. Thus the image of $\mathrm{ROT} \circ \mathrm{hol}$ is contained in $SO(2)$. It follows that when C is almost Seifert fibered, then the image of $\mathrm{ROT} \circ \mathrm{hol}$ is contained in $O(2)$, and by Lemma 2.1 C is almost product. $\square$

We consider the action by conjugation of $SO(3)$ on $M_{3\times 3}(\mathbf{R}) = \mathrm{End}(\mathbb{R}^3)$, the vector space of 3×3 matrices with real entries.

Lemma 2.4 *C is not almost product if and only if the only subspace of $M_{3\times 3}(\mathbb{R})$ invariant under* $\mathrm{ROT} \circ \mathrm{hol}$ *is the space of multiples of the identity matrix.*

Proof As $SO(3)$–module acting by conjugation, we have the splitting:

$$M_{3\times 3}(\mathbb{R}) = \mathbb{R} \oplus \mathfrak{so}(3) \oplus Z^5,$$

where $\mathbb{R}$ is the subspace of multiples of the identity matrix, $\mathfrak{so}(3) \cong \mathbb{R}^3$ is the subspace of antisymmetric matrices and $Z^5 \cong \mathbb{R}^5$ is the subspace of traceless symmetric matrices.

The action on $\mathbb{R}$ is trivial and this space is always invariant. The action on $\mathfrak{so}(3)$ is equivalent to the usual action of $SO(3)$ on $\mathbb{R}^3$, and having a nonzero invariant vector in $\mathbb{R}^3$ is equivalent for $\mathrm{ROT} \circ \mathrm{hol}$ to be contained in $SO(2)$.

Since symmetric matrices diagonalize orthogonally, matrices in Z^5 have an orthogonal basis of eigenvectors. Furthermore the sum of eigenvalues is zero, and therefore every non-trivial element has at least a one dimensional eigenspace. Thus having a non-trivial invariant element of Z^5 is equivalent for $\mathrm{ROT} \circ \mathrm{hol}$ to have an invariant line, ie to be contained in $O(2)$. $\square$

Lemma 2.5 *Let C and $\mathcal{O}$ be as in the situation of Corollary 1.4. Then C is not almost product.*

Proof By hypothesis the cone angles of C are $\leq \pi$ and at least one of them is $< \pi$. Therefore if C is almost product, then C is a quotient of $E^2 \times S^1$ where $E^2 \cong S^2(\alpha, \beta, \gamma)$ is a Euclidean turnover (a cone structure on S^2 with three cone points of cone angles satisfying $\alpha + \beta + \gamma = 2\pi$). This cone manifold E^2 embeds in C and induces an essential spherical 2–orbifold in $\mathcal{O}$, contradicting the irreducibility of $\mathcal{O}$. $\square$

3 Cohomology of the tangent bundle

We assume from now on that M is the smooth part of a closed Euclidean cone manifold C which is not almost product. The latter assumption may be viewed as a nondegeneracy condition and is in that respect similar to the assumption of not being Seifert fibered in the deformation theory of spherical cone manifolds, cf Boileau et al [2] and Weiss [14].

In the Euclidean case the flat bundle of infinitesimal isometries $\mathcal{E} = \mathfrak{so}(TM) \oplus TM$ contains the bundle of infinitesimal translations $\mathcal{E}_{\mathrm{trans}} = \{0\} \oplus TM$ as a parallel subbundle. The inclusion $\mathcal{E}_{\mathrm{trans}} \subset \mathcal{E}$ fits into a short exact sequence of flat vectorbundles and connection-preserving maps

$$0 \to \mathcal{E}_{\mathrm{trans}} \to \mathcal{E} \to \mathcal{E}_{\mathrm{rot}} \to 0,$$

where $\mathcal{E}_{\mathrm{rot}} = \mathfrak{so}(TM) \oplus \{0\}$ carries the *quotient* connection ($\mathcal{E}_{\mathrm{rot}} \subset \mathcal{E}$ is *not* parallel!). In fact $\mathcal{E}_{\mathrm{rot}}$ is again isomorphic to TM as a flat bundle. This can for instance be seen if we look at the corresponding short exact sequence of $\mathrm{Isom}^+ \mathbb{R}^3$–modules

$$0 \longrightarrow \mathbb{R}^3_{\mathrm{Ad}(\cdot)|_{\mathbb{R}^3}} \longrightarrow \mathfrak{isom}^+\mathbb{R}^3_{\mathrm{Ad}} \xrightarrow{\mathrm{ROT}_*} so(3)_{\mathrm{Ad}\circ\mathrm{ROT}} \longrightarrow 0.$$

Now we have isomorphisms of $\mathrm{Isom}^+ \mathbb{R}^3$–modules $\mathfrak{so}(3)_{\mathrm{Ad}\circ\mathrm{ROT}} \cong \mathbb{R}^3_{\mathrm{Ad}(\cdot)|_{\mathbb{R}^3}} \cong \mathbb{R}^3_{\mathrm{ROT}}$, which in turn yield isomorphisms of flat bundles $\mathcal{E}_{\mathrm{rot}} \cong \mathcal{E}_{\mathrm{trans}} \cong TM$. Note that the first isomorphism is particular to our 3–dimensional situation. We will freely identify TM with either $\mathcal{E}_{\mathrm{trans}}$ or $\mathcal{E}_{\mathrm{rot}}$ from now on.

The translational part of a Euclidean isometry ϕ w.r.t. a base point $p \in \mathbb{R}^3$

$$\mathrm{trans}_p(\phi) = \phi(p) - p$$

gives rise to a cocycle $\mathrm{trans}_p\colon \mathrm{Isom}^+ \mathbb{R}^3 \to \mathbb{R}^3$ twisted by ROT, ie

$$\mathrm{trans}_p(\phi_1\phi_2) = \mathrm{trans}_p(\phi_1) + \mathrm{ROT}(\phi_1)(\mathrm{trans}_p(\phi_2))$$

for $\phi_1, \phi_2 \in \mathrm{Isom}^+ \mathbb{R}^3$.

Definition 3.1 The *holonomy cocycle* is the composition

$$\mathrm{trans}_p \circ \mathrm{hol}\colon \pi_1 M \longrightarrow \mathbb{R}^3$$

for some fixed $p \in \mathbb{R}^3$.

This is a cocycle twisted by $\mathrm{ROT} \circ \mathrm{hol}$, ie

$$\mathrm{trans}_p \circ \mathrm{hol}(\gamma_1\gamma_2) = \mathrm{trans}_p \circ \mathrm{hol}(\gamma_1) + (\mathrm{ROT} \circ \mathrm{hol}(\gamma_1))(\mathrm{trans}_p \circ \mathrm{hol}(\gamma_2)).$$

for $\gamma_1, \gamma_2 \in \pi_1 M$. The cohomology class of this cocycle is independent of p and is denoted by

$$v = [\mathrm{trans}_p \circ \mathrm{hol}] \in H^1(\pi_1 M; \mathbb{R}^3_{\mathrm{ROT} \circ \mathrm{hol}}).$$

Let $\omega \in \Omega^1(M; TM)$ be closed and γ a loop based at $x \in M$. We define the integral of ω along γ as

$$\int_\gamma \omega = \int_0^1 \tau^{-1}_{\gamma(t)} \omega(\dot\gamma(t)) dt \in \mathbb{R}^3,$$

where $\tau_{\gamma(t)}$ denotes parallel transport along γ from $x = \gamma(0)$ to $\gamma(t)$ and we identify $T_x M$ with $\mathbb{R}^3$ by developing M on $\mathbb{R}^3$. For $\omega \in \Omega^1(M; TM)$ closed the assignment $\gamma \mapsto \int_\gamma \omega$ defines a group cocycle $z_\omega \in Z^1(\pi_1 M; \mathbb{R}^3_{\mathrm{ROT} \circ \mathrm{hol}})$, the so-called period cocycle of ω. The period map $\omega \mapsto z_\omega$ descends to an isomorphism in cohomology

$$\begin{aligned} P\colon H^1(M; TM) &\xrightarrow{\cong} H^1(\pi_1 M; \mathbb{R}^3_{\mathrm{ROT} \circ \mathrm{hol}}) \\ [w] &\longmapsto [z_\omega]. \end{aligned}$$

Observe that on each side there is a canonically defined cohomology class. On the left hand side we have $\mathrm{id} \in \Omega^1(M; TM) = \Gamma(M; T^*M \otimes TM)$. The form id is parallel and gives rise to a class $[\mathrm{id}] \in H^1(M; TM)$. On the right hand side we have the class of the holonomy cocycle v. In fact, these classes coincide via the period map and correspond geometrically to a global rescaling of M:

Lemma 3.2 *$P([\mathrm{id}]) = v$ and the image of v in $H^1(\pi_1 M; \mathfrak{isom}^+\mathbb{R}^3_{Ad \circ \mathrm{hol}})$ is tangent to a deformation, which globally rescales the Euclidean metric on M.*

Proof Let γ be a loop based at $x \in M$ and $\tilde\gamma$ a lift of γ to $\widetilde{M}$. We compute

$$\int_\gamma \mathrm{id} = \mathrm{trans}_{\mathrm{dev}(\tilde\gamma(0))} \circ \mathrm{hol}(\gamma),$$

which shows that the periods of id reproduce the holonomy cocycle defined with respect to $\mathrm{dev}(\tilde\gamma(0)) \in \mathbb{R}^3$. Furthermore for $\gamma \in \pi_1 M$ let

$$\rho_t(\gamma) = (\mathrm{ROT} \circ \mathrm{hol}(\gamma), (t+1)\, \mathrm{trans}_0 \circ \mathrm{hol}(\gamma)) \in \mathrm{Isom}^+ \mathbb{R}^3.$$

Then $\rho_0 = \mathrm{hol}$ and ρ_t is the holonomy representation of M with rescaled Euclidean metric $(t+1)^2 g$. We obtain $\rho_t(\gamma)\, \mathrm{hol}(\gamma)^{-1} = (1, t \cdot \mathrm{trans}_0 \circ \mathrm{hol}(\gamma))$ and consequently $\frac{d}{dt}\big|_{t=0} \rho_t(\gamma)\, \mathrm{hol}(\gamma)^{-1} = (0, \mathrm{trans}_0 \circ \mathrm{hol}(\gamma))$. This means the corresponding path of characters is precisely tangent to v, cf the later discussion in Section 4. □

Let $U_\varepsilon(\mathrm{Sing}(C))$ denote the ε–tube around $\mathrm{Sing}(C)$ in C minus the singular locus itself. Let $\overline{M}$ denote a compact core of M. $\partial\overline{M}$ is a disjoint union of tori and surfaces of higher genus. Note that the pair $(M, U_\varepsilon(\mathrm{Sing}(C)))$ is homotopy equivalent to the pair $(\overline{M}, \partial \overline{M})$, by homotopy invariance we may identify the corresponding cohomology groups.

If (r_i, θ_i, z_i) are cylindrical coordinates around the i-th edge and $\varphi_i = \varphi_i(z_i)$ is a function with $\varphi_i \equiv 0$ near $z_i = 0$ and $\varphi_i \equiv l_i$ near $z_i = l_i$, we define as in [14]

$$\omega^i_{\mathrm{len}} = d\varphi_i \otimes \tfrac{\partial}{\partial z_i} \in \Omega^1(U_\varepsilon(\mathrm{Sing}(C)); TM).$$

As in [14] we obtain the following result.

Lemma 3.3 *The forms ω^i_{len} are bounded on $U_\varepsilon(\mathrm{Sing}(C))$, in particular L^2.*

Geometrically, the classes $[\omega^i_{\mathrm{len}}]$ correspond to deformations of $U_\varepsilon(\mathrm{Sing}(C))$ changing the length of the i th edge. These deformations are independent. Therefore the following Lemma does not come as a surprise.

Lemma 3.4 *The classes $[\omega^i_{\mathrm{len}}]$ are linearly independent in $H^1(\partial \overline{M}; TM)$, furthermore the relation $[\mathrm{id}]|_{\partial \overline{M}} = \sum_i [\omega^i_{\mathrm{len}}]$ holds.*

Proof The first assertion follows as in [14]. The periods of both forms id and $\sum_i \omega^i_{\mathrm{len}}$ reproduce the holonomy cocycle restricted to $\pi_1 \partial \overline{M}$. Since the period map descends to an isomorphism in cohomology, the result follows. □

Lemma 3.5 $H^0(M; TM) = 0$.

Proof This group is isomorphic to the subspace of $\mathbb{R}^3$ invariant under $\mathrm{ROT} \circ \mathrm{hol}$, which is trivial since we assume that C is not almost product (see the proof of Lemma 2.4). □

From the long exact cohomology sequence we obtain the following consequence.

Corollary 3.6 *$H^0(M; \mathcal{E}) = 0$ and the map $H^1(M; TM) \to H^1(M; \mathcal{E})$ is an injection.*

To compute $H^1(M; TM)$ we use the L^2–cohomology: $H^1_{L^2}(M; TM)$. Our starting point is the following theorem of [14], where the hypothesis about cone angles is used:

Theorem 3.7 *If M is the smooth part of a closed Euclidean cone manifold with cone angles $\leq \pi$, then*

$$H^1_{L^2}(M; TM) \cong \{\omega \in \Omega^1(M; TM) | \nabla\omega = 0\}.$$

Corollary 3.8 $H^1_{L^2}(M;TM)\cong\mathbb{R}$.

Proof Since every element of $H^1_{L^2}(M;TM)$ is represented by a parallel form, this cohomology group is isomorphic to the space of elements in $(\mathbb{R}^3)^*\otimes\mathbb{R}^3=\mathrm{End}(\mathbb{R}^3)\cong M_{3\times 3}(\mathbb{R}^3)$ which are invariant under the action of $\mathrm{ROT}\circ\mathrm{hol}$ by conjugation. By Lemma 2.4, this subspace is one dimensional. □

Let $\overline{M}$ denote a compact core of M. By homotopy invariance $H^*(M;TM)\cong H^*(\overline{M};TM)$.

Lemma 3.9 *The map $H^1(M;TM)\to H^1(\partial\overline{M};TM)$ is an injection.*

Proof Looking at the exact sequence of the pair $(\overline{M},\partial\overline{M})$, it suffices to show that the image of the map $j\colon H^1(\overline{M},\partial\overline{M};TM)\to H^1(\overline{M};TM)$ is zero. To prove it, we observe that j factors through $H^1_{L^2}(M;TM)$. By Corollary 3.8, $H^1_{L^2}(M;TM)$ is one-dimensional and its generator is represented by the identity of TM. Therefore the image of j is contained in the span of [id]. By Lemma 3.4, [id] restricts to a nontrivial class in $H^1(\partial\overline{M};TM)$. On the other hand, by exactness the image of j is contained in the kernel of $H^1(M;TM)\to H^1(\partial\overline{M};TM)$, hence it is zero. □

As a corollary of the proof above we notice the following.

Corollary 3.10 *The map $H^1_{L^2}(M;TM)\to H^1(M;TM)$ is an injection.*

We may thus view L^2–cohomology as a subspace of ordinary cohomology in degree 1.

Lemma 3.11 $\dim H^1(M;TM)=q$.

Proof By Lemma 3.9, the long exact sequence of the pair and Poincaré duality, we have a short exact sequence:

$$0\to H^1(M;TM)\to H^1(\partial\overline{M};TM)\to H^2(\overline{M},\partial\overline{M};TM)\to 0$$

Since $H^1(M;TM)$ and $H^2(\overline{M},\partial\overline{M};TM)$ are Poincaré dual,

$$\dim H^1(M;TM)=\frac{1}{2}\dim H^1(\partial\overline{M};TM).$$

In addition, $\dim H^1(\partial\overline{M};TM)=2q$ (see [2] or [14]). □

Continuing the long exact sequence we get the following isomorphism.

Corollary 3.12 *The inclusion induces an isomorphism* $H^2(M;TM)\cong H^2(\partial\overline{M};TM)$.

Let $H^*(\pi_1 M;\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}})$ denote the cohomology of the $\pi_1 M$–module $\mathbb{R}^3$ with the action induced by $\mathrm{ROT}\circ\mathrm{hol}$. Notice that the induced flat bundle over M is isomorphic to TM.

Lemma 3.13 *There is a natural map*

$$H^i(\pi_1 M;\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}})\to H^i(M;TM)$$

which is an isomorphism for $i=0,1$ *and an injection for* $i=2$.

Remark 3.14 *When* $i=1$, *this map is the inverse of the isomorphism* P *in Lemma 3.2.*

Proof There is a natural isomorphism

$$H^i(\pi_1 M;\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}})\cong H^i(K(\pi_1 M,1);\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}}),$$

where $K(\pi_1 M,1)$ denotes any aspherical CW–complex with the same fundamental group as M, and $\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}}$ also denotes the flat bundle over $K(\pi_1 M,1)$ with fiber $\mathbb{R}^3$ and holonomy $\mathrm{ROT}\circ\mathrm{hol}$. The space $K(\pi_1 M,1)$ can be constructed from M itself by adding cells of dimension ≥ 3. Thus the cohomology of the pair $(K(\pi_1 M,1),M)$ is trivial in dimension ≤ 2, and the lemma follows easily from the long exact sequence in cohomology for the pair $(K(\pi_1 M,1),M)$. □

Corollary 3.15 *Let* $\partial_1\overline{M},\dots,\partial_k\overline{M}$ *denote the components of* $\partial\overline{M}$. *The restriction induces an injection:*

$$0\to H^2(\pi_1 M;\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}})\to\bigoplus_{i=1}^{k} H^2(\pi_1\partial_i\overline{M};\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}}).$$

Proof The result follows from Corollary 3.12, Lemma 3.13 and the asphericity of the surfaces $\partial_i\overline{M}$. □

4 Varieties of representations

Since M is three dimensional by a theorem of Culler, cf [5, Proposition 2.1], hol lifts to a representation

$$\rho=\widetilde{\mathrm{hol}}\colon \pi_1 M\to\widetilde{\mathrm{Isom}^+\mathbb{R}^3}\cong SU(2)\ltimes\mathbb{R}^3,$$

and similarly $\mathrm{ROT} \circ \mathrm{hol}$ lifts to

$$\rho_0 = \widetilde{\mathrm{ROT}} \circ \widetilde{\mathrm{hol}}\colon \pi_1 M \to \widetilde{SO(3)} \cong SU(2),$$

where $\widetilde{ROT}\colon \widetilde{\mathrm{Isom}^+ \mathbb{R}^3} \to SU(2)$ is a lift of ROT. Notice that Culler's theorem applies because M is parallelizable, and therefore the frame bundle of M has a section. We will drop the notational distinction between $\widetilde{\mathrm{hol}}$ and hol, resp. between $\widetilde{\mathrm{ROT}}$ and ROT from now on.

Let $R(M, SU(2)) = R(\pi_1 M, SU(2))$ and $R(M, SL_2(\mathbb{C})) = R(\pi_1 M, SL_2(\mathbb{C}))$ denote the varieties of representations in $SU(2)$ and $SL_2(\mathbb{C})$ respectively, and let $X(M, SU(2))$ and $X(M, SL_2(\mathbb{C}))$ be the varieties of characters. In addition, let $\chi_0 \in X(M, SU(2)) \subset X(M, SL_2(\mathbb{C}))$ denote the character of ρ_0.

We recall some general facts about spaces of representations.

Let Γ be a finitely generated discrete group and G a Lie group. We equip $R(\Gamma, G)$ with the compact-open topology. Once and for all we fix a presentation $\langle \gamma_1, \dots, \gamma_n | (r_i)_{i \in I} \rangle$. Without loss of generality we may assume that the set of generators contains with any of its elements its inverse. The relations r_i determine functions $f_i\colon G^n \to G$ which identify $R(\Gamma, G)$ with the subset $\{f_i = 1\} \subset G^n$.

If G is a real algebraic group (as for example $G = SU(2)$ or $\widetilde{\mathrm{Isom}^+ \mathbb{R}^3}$), the f_i are polynomial functions and $R(\Gamma, G)$ acquires the structure of a real algebraic set. Similarly, if G is a complex algebraic group (for example $G = SL_2(\mathbb{C})$), then $R(\Gamma, G)$ is a complex algebraic set.

Let $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ be the semidirect product of G and $\mathfrak{g}$ defined via $\mathrm{Ad}\colon G \to Aut(\mathfrak{g})$. Let $\rho \in R(\Gamma, G)$ be given. A simple but important observation is that $z \in Z^1(\Gamma; \mathfrak{g}_{\mathrm{Ad} \circ \rho})$ if and only if $(\rho, z) \in R(\Gamma, G \ltimes_{\mathrm{Ad}} \mathfrak{g})$. The relations therefore determine functions $g_i\colon G^n \times \mathfrak{g}^n \to \mathfrak{g}$ such that $R(\Gamma, G \ltimes_{\mathrm{Ad}} \mathfrak{g}) = \{(f_i, g_i) = (1, 0)\}$ and $Z^1(\Gamma; \mathfrak{g}_{\mathrm{Ad} \circ \rho}) = \{g_i(\rho, \cdot) = 0\}$. Note that for ρ fixed the functions $g_i(\rho, \cdot)\colon \mathfrak{g}^n \to \mathfrak{g}$ are linear. If we identify $T_A G$ with $\mathfrak{g}$ via right translation, then we have the following Lemma.

Lemma 4.1 $g_i(\rho, \cdot) = df_i(\rho) \quad \forall i \in I$.

Proof The differential of the group multiplication $f\colon G \times G \to G$ at $(A_1, A_2) \in G \times G$ considered as a map $df(A_1, A_2)\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ via right translation is given by

$$df(A_1, A_2)(b_1, b_2) = b_1 + \mathrm{Ad}(A_1) b_2.$$

On the other hand the group multiplication law in $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ reads

$$(A_1, b_1)(A_2, b_2) = (A_1 A_2, b_1 + \mathrm{Ad}(A_1) b_2),$$

which proves the claim in this case. The general case is easily reduced to this one. □

In particular if the f_i cut out $R(\Gamma, G)$ transversely, then $T_\rho R(\Gamma, G)$ is identified with $Z^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\circ\rho})$. In general, if $\rho \in R(\Gamma, G)$ is a smooth point, there is a linear injection

$$\begin{aligned} T_\rho R(\Gamma, G) &\longrightarrow Z^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \\ \dot\rho = \tfrac{d}{dt}\big|_{t=0}\rho_t &\longmapsto \dot\rho\rho^{-1} = \{\gamma \mapsto \tfrac{d}{dt}\big|_{t=0}\rho_t(\gamma)\rho(\gamma)^{-1}\} \end{aligned} \tag{1}$$

which identifies $T_\rho R(\Gamma, G)$ with a subspace of $Z^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\circ\rho})$. This injection is known as *Weil's construction*, and it extends naturally to the non-smooth case by using the Zariski tangent space. A cocycle z is a coboundary, ie $z \in B^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\circ\rho})$, if and only if z is tangent to the orbit of the action of G on $R(\Gamma, G)$ via conjugation.

Lemma 4.2 *$\rho_0 \in R(M, SU(2))$ is a smooth point with tangent space $Z^1(\pi_1 M; \mathbb{R}^3_{\rho_0})$. In a similar way, $\rho_0 \in R(M, SL_2(\mathbb{C}))$ is also a smooth point with the tangent space $Z^1(\pi_1 M; \mathfrak{sl}_2(\mathbb{C})_{\mathrm{Ad}\circ\rho_0})$.*

Proof There is an infinite sequence of obstructions for a cocycle $z \in Z^1(\pi_1 M; \mathbb{R}^3_{\rho_0})$ to be integrated into an actual path of representations tangent to z, cf Goldman [6]. These obstructions live in $H^2(\pi_1 M; \mathbb{R}^3_{\rho_0})$. Using the injection of Corollary 3.15 and the fact that the obstructions are natural, we conclude that they vanish, because they vanish in $H^2(\pi_1 \partial \overline{M}; \mathbb{R}^3_{\rho_0})$ (cf Goldman [7]). Then using Artin's theorem [1] they are in fact integrable. Thus every element in $Z^1(\pi_1 M; \mathbb{R}^3_{\rho_0})$ is integrable, and therefore ρ_0 is a smooth point of $R(M, SU(2))$ with tangent space $Z^1(\pi_1 M; \mathbb{R}^3_{\rho_0})$.

For $SL_2(\mathbb{C})$ the same argument applies since $\mathfrak{sl}_2(\mathbb{C})_{\mathrm{Ad}\circ\rho_0}$ is the complexification of the module $\mathfrak{su}(2)_{\mathrm{Ad}\circ\rho_0} \cong \mathbb{R}^3_{\rho_0}$. □

Proposition 4.3 *$\chi_0 \in X(M, SU(2))$ is a smooth point of local real dimension q and tangent space isomorphic to $H^1(\pi_1 M; \mathbb{R}^3_{\rho_0})$. Similarly, $\chi_0 \in X(M, SL_2(\mathbb{C}))$ is a smooth point of local complex dimension q and tangent space isomorphic to $H^1(\pi_1 M; \mathfrak{sl}_2(\mathbb{C})_{\mathrm{Ad}\circ\rho_0})$.*

Proof Since ρ_0 is irreducible, $Z(\rho_0(\pi_1 M)) = \{\pm 1\}$ in each case. Since $SU(2)$ is compact (resp. the action of $SL_2(\mathbb{C})$ is proper on the irreducible part of $R(M, SL_2(\mathbb{C}))$, cf [14, Lemma 6.24]) and the tangent space to the orbit is given by $Z^1(\pi_1 M; \mathbb{R}^3_{\rho_0})$ (resp. by $Z^1(\pi_1 M; \mathfrak{sl}_2(\mathbb{C})_{\mathrm{Ad}\circ\rho_0})$), the result follows with Lemma 4.2. □

The fiber of the map

$$\mathrm{ROT}\colon\ R(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3}) \to R(M, SU(2))$$

at $\rho \in R(M, SU(2))$ is the the space of cocycles $Z^1(\pi_1 M; \mathbb{R}^3_\rho)$, which is given by a system of linear equations whose coefficients depend continuously on ρ. Therefore its dimension is an upper semi-continuous function of ρ in general.

If $\rho_0 \in R(M, SU(2))$ is a smooth point with $T_{\rho_0} R(M, SU(2)) = Z^1(\pi_1 M; \mathbb{R}^3_{\rho_0})$ (which is the case if the f_i cut out $R(M, SU(2))$ transversely or for $\rho_0 = \mathrm{ROT} \circ \mathrm{hol}$ according to Lemma 4.2), then $T_\rho R(M, SU(2))$ injects into $Z^1(\pi_1 M; \mathbb{R}^3_\rho)$ for ρ in a neighborhood of ρ_0 via Weil's construction (1):

$$\begin{aligned} T_\rho R(M, SU(2)) &\longrightarrow Z^1(\pi_1 M; \mathbb{R}^3_\rho) \\ \dot\rho &\longmapsto \dot\rho\rho^{-1} \end{aligned}$$

and hence ROT: $R(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3}) \to R(M, SU(2))$ is locally the projection of a vector bundle. More precisely we have the following Lemma.

Lemma 4.4 ROT: $R(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3}) \to R(M, SU(2))$ *is isomorphic to the tangent bundle* $TR(M, SU(2))$ *near* ρ_0.

Proof The map

$$\begin{aligned} TR(M, SU(2)) &\longrightarrow R(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3}) \\ (\rho, \dot\rho) &\longmapsto (\rho, \dot\rho\rho^{-1}) \end{aligned}$$

is a vector bundle isomorphism near ρ_0. □

Corollary 4.5 $\mathrm{hol} \in R(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3})$ *is a smooth point.*

Let $\chi \in X(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3})$ denote the character of $\mathrm{hol} \in R(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3})$.

Proposition 4.6 $\chi \in X(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3})$ *is a smooth point of local real dimension* $2q$. *The induced map* ROT: $X(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3}) \to X(M, SU(2))$ *is locally the projection of a vector bundle isomorphic to the tangent bundle* $TX(M, SU(2))$ *near* ρ_0.

Proof Let us digress into a more general situation first.

Let G be a compact Lie group, M a smooth manifold (not necessarily compact) and $G \times M \to M$ a smooth free action. The associated infinitesimal action is the Lie algebra homomorphism $\mathfrak{g} \to \Gamma(M, TM)$ defined by $a \mapsto \{p \mapsto \frac{d}{dt}\big|_{t=0} \exp(ta)p\}$. Since the original action was free, this homomorphism is injective and we will identify $\mathfrak{g}$ with its image in $\Gamma(M, TM)$.

Let $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ be the semidirect product of G and $\mathfrak{g}$ defined via Ad: $G \to Aut(\mathfrak{g})$. We extend the natural action of G on TM via the differential to an action of the group $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ in the following way:

$$\begin{aligned}(G \ltimes_{\mathrm{Ad}} \mathfrak{g}) \times TM &\longrightarrow TM \\ ((g,a),v) &\longmapsto dg(v) + a(g(\pi(v))).\end{aligned}$$

The relation $dg(a(p)) = (\mathrm{Ad}(g)a)(gp)$ ensures that this defines a group action. This action is free since the original action was free, it is clearly proper.

Lemma 4.7 $T(M/G) = TM/(G \ltimes_{\mathrm{Ad}} \mathfrak{g})$.

Proof The fibers of the map $d\pi\colon TM \to T(M/G)$ are precisely the orbits of the action of the group $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ on TM. □

We return to the situation of Proposition 4.6.

Clearly $SU(2) \ltimes_{\mathrm{Ad}} \mathfrak{su}(2) \cong \widetilde{\mathrm{Isom}^+ \mathbb{R}^3}$ via $\mathfrak{su}(2)_{\mathrm{Ad}} \cong \mathbb{R}^3$ as $SU(2)$–modules. For a representation $(\rho, z) \in R(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3})$ and $(A,b) \in \widetilde{\mathrm{Isom}^+ \mathbb{R}^3}$ we have

$$(A,b)(\rho,z)(A,b)^{-1} = (A\rho A^{-1}, Az + b - A\rho A^{-1} b).$$

Therefore the map $TR(M, SU(2)) \to R(M, \widetilde{\mathrm{Isom}^+ \mathbb{R}^3})$ is $\widetilde{\mathrm{Isom}^+ \mathbb{R}^3}$–equivariant and Lemma 4.7 yields the proposition. □

From the long exact cohomology sequence and Proposition 4.6 we obtain the following Corollary.

Corollary 4.8 *There is a short exact sequence*

$$0 \to H^1(M; TM) \to H^1(M; \mathcal{E}) \to H^1(M; TM) \to 0.$$

Remark 4.9 *Modulo the choice of a splitting of the exact sequence in Corollary 4.8, an infinitesimal deformation of the holonomy of the Euclidean structure on M is therefore determined by the infinitesimal deformation of its rotational part and of its translational part, and both can be independently prescribed.*

Let $m_1, \dots, m_q \in \pi_1 M$ a system of meridians for $\pi_1 M$ (ie one for each component of $\mathrm{Sing}(C)$). We define the angle function $\mu_j\colon U \subseteq X(M, SU(2)) \to \mathbb{R}$ in a neighborhood U of χ_0, so that $\mu_j(\chi_0) = \alpha_j$. It is related to the trace by the equality:

$$\mathrm{trace}(\rho(m_j)) = \pm 2 \cos \frac{\mu_j(\chi_\rho)}{2}.$$

In particular μ_j is analytic. We extend μ_j to a neighborhood in $X(M;SL_2(\mathbb{C}))$ as a complex analytic function:

$$\mu_j\colon V\subset X(M,SL_2(\mathbb{C}))\to\mathbb{C},$$

so that the complex length of $\rho(m_j)$ is $i\ \mu_j$ (ie a translation of length $\mathrm{Re}\,(i\ \mu_j)$ plus a rotation of angle $\mathrm{Im}\,(i\ \mu_j)$).

The differentials $d\mu_j$ live in the cotangent space to the varieties of characters, thus

$$d\mu_j\in H^1(M;TM)^*\cong H_1(M;TM).$$

Proposition 4.10 $\ker\langle d\mu_1,\dots,d\mu_q\rangle=\langle v\rangle$*, where* v *is the class of the holonomy cocycle.*

Proof To check that $d\mu_j(v)=0$, we observe that

$$H^1(\pi_1M;\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}})\to H^1(\langle m_j\rangle;\mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}})$$

maps v to zero, because $\mathrm{hol}(m_j)$ is a rotation with a fixed axis. In particular $d\mu_j$ evaluated at v is zero. To see the other inclusion, we use that

$$\ker\langle d\mu_1,\dots,d\mu_q\rangle=H^1(M;TM)\cap\left\langle[\omega^1_{\mathrm{len}}],\dots,[\omega^q_{\mathrm{len}}]\right\rangle,$$

where we view $H^1(M;TM)$ as a subspace of $H^1(\partial\overline{M};TM)$ via Lemma 3.9. Since the forms ω^j_{len} are L^2 near the singular locus according to Lemma 3.3, we conclude that $\ker\langle d\mu_1,\dots,d\mu_q\rangle\subseteq H^1_{L^2}(M;TM)$, which we view as a subspace of $H^1(M;TM)$ via Corollary 3.10. Since $H^1_{L^2}(M;TM)$ is spanned by [id] the result follows. □

Corollary 4.11 $\dim\langle d\mu_1,\dots,d\mu_q\rangle=q-1$.

5 Regeneration of structures

Along this section $\mathbb{X}^3$ denotes either $\mathbb{S}^3$ or $\mathbb{H}^3$. The stabilizer of a point $p\in\mathbb{X}^3$ is isomorphic to $SU(2)\cong\mathrm{Spin}(3)$. Thus we view ρ_0 as a representation of π_1M in this stabilizer.

We consider a path of characters

$$\begin{array}{rcl}[0,\varepsilon)&\to&X(M,\widetilde{\mathrm{Isom}^+\mathbb{X}^3})\\ t&\mapsto&\chi_t\end{array}$$

with $\chi_0 = \chi_{\rho_0}$. Assume that this path is differentiable to the right at 0. The derivative $\frac{\partial \chi_t}{\partial t}(0)$ is an element of $H^1(\pi_1 M; \mathfrak{g}_{\mathrm{Ad}\circ\mathrm{hol}})$, where $\mathfrak{g}$ is the Lie algebra of $\widetilde{\mathrm{Isom}^+ \mathbb{X}^3}$ (using Weil's construction).

The Lie algebra $\mathfrak{g}$ decomposes into rotational and translational part with respect to p:

$$0 \to \mathfrak{k} \to \mathfrak{g} \xrightarrow{\text{trans}} T_p\mathbb{X}^3 \to 0$$

where $\mathfrak{k} \cong \mathfrak{su}(2)$ is the Lie algebra of the stabilizer of $p \in \mathbb{X}^3$. Thus

$$\operatorname{trans}\left(\frac{\partial \chi_t}{\partial t}(0)\right) \in H^1(\pi_1 M; \mathbb{R}^3_{\mathrm{ROT}\circ\mathrm{hol}}).$$

Theorem 5.1 *Let $\{\chi_t\}_{t\in[0,\varepsilon)}$ be a path in $X(M, \widetilde{\mathrm{Isom}^+ \mathbb{X}^3})$ with $\chi_0 = \chi_{\rho_0}$ and differentiable to the right at 0. If $\operatorname{trans}(\frac{\partial \chi_t}{\partial t}(0)) = v$, then χ_t is the holonomy character of a $\mathbb{X}^3$–structure on M, for $t \in (0, \delta)$ and some $0 < \delta < \varepsilon$.*

If in addition $\chi_t(m_j)$ is a rotation for each meridian m_j, then the structure on M completes to a cone structure.

Proof Consider $\mathbb{X}_t$, the space of constant sectional curvature ϵt^2, where $\epsilon = \pm 1$ is the curvature of $\mathbb{X}^3$, for $t \in [0, \varepsilon)$. Equivalently, $\mathbb{X}_t$ is $\mathbb{X}^3$ with the metric tensor scaled by t^{-2}. We fix a base point p in $\mathbb{X}_t$ independently of t, so that the pointed Euclidean space $(\mathbb{X}_0, p)$ is the limit of $(\mathbb{X}_t, p)$ when $t \to 0$, and consider the union

$$\overline{\mathbb{X}} = \bigcup_{t\in[0,\varepsilon]} \mathbb{X}_t.$$

We equip $\overline{\mathbb{X}}$ with a manifold structure, using the local charts given by the exponential maps at p and the parameter $t \in [0, \varepsilon)$, after identifying isometrically $T_p\mathbb{X}_t \cong \mathbb{R}^3$ for every $t \in [0, \varepsilon)$. Notice that the choice of p is relevant for the topology of $\overline{\mathbb{X}}$.

Choose a smooth path of representations ρ_t with character χ_t so that $\rho_0 = \mathrm{ROT}\circ\mathrm{hol}$ and $\operatorname{trans}(\frac{\partial \rho_t}{\partial t}(0)) = \operatorname{trans}(\mathrm{hol})$.

Lemma 5.2 *For every $\gamma \in \pi_1 M$, the action of $\rho_t(\gamma)$ on $\mathbb{X}_t$ for $t > 0$ extends continuously to the action of $\mathrm{hol}(\gamma)$ on $\mathbb{X}_0$ for the $\mathcal{C}^1$–topology.*

Notice that ρ_t acts on $\mathbb{X}_t$ isometrically, since rescaling the metric does not change the isometry group. The previous lemma provides an action on $\overline{\mathbb{X}}$, which is isometric on each $\mathbb{X}_t$.

Proof We start describing the local coordinates. Let $\exp_p^{(t)}\colon T_p\mathbb{X}_t \to \mathbb{X}_t$ denote the Riemannian exponential. Using the isometric identification $\mathbb{R}^3 \cong T_p\mathbb{X}_t$, we have

$$\exp_p^{(t)}(v) = \exp_p(t\,v) \quad \forall v \in \mathbb{R}^3,$$

where $\exp_p = \exp_p^{(1)}\colon T_p\mathbb{X}^3 \to \mathbb{X}^3$. Thus, the (inverse of) the local charts is given as follows. Given an open set $V \subset \mathbb{R}^3$, $V \times [0,\varepsilon)$ parametrizes a subset of $\overline{\mathbb{X}}$ via the map:

$$\begin{array}{rcl} V\times\{0\} & \to & \mathbb{X}_0 \\ (v,0) & \mapsto & v \end{array} \quad \begin{array}{rcl} V\times(0,\varepsilon) & \to & \overline{\mathbb{X}} \\ (v,t) & \mapsto & \exp_p^{(t)}(v) = \exp_p(t\,v) \in \mathbb{X}_t. \end{array}$$

To prove the lemma it suffices to show that

$$\lim_{t\to 0^+} \frac{1}{t}\exp_p^{-1}(\rho_t(\gamma)\exp_p(t\,v)) = \mathrm{hol}(\gamma)(v)$$

uniformly for $v \in \mathbb{R}^3$ in a compact set for the $\mathcal{C}^1$–topology.

The Lie algebra of $\operatorname{Isom}\mathbb{X}_1$ decomposes as a sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the subalgebra of infinitesimal rotations around p and $\mathfrak{p}$ the subspace of infinitesimal translations with respect to p. There is an isometric identification $\mathfrak{p} \cong T_p\mathbb{X}_1 \cong \mathbb{R}^3$, so that for any $v \in \mathbb{R}^3$:

$$\exp_p(v) = \exp(v)(p),$$

here we view $v \in T_p\mathbb{X}_1 \cong \mathbb{R}^3$ when we write $\exp_p(v)$ and $v \in \mathfrak{p}$ for $\exp(v)(p)$.

According to global Cartan's decomposition, we write $\rho_t(\gamma) = \exp(b_t)\,a_t$, where $b_t \in \mathfrak{p}$ and a_t belongs to the stabilizer of p in $\mathbb{X}_1$. Notice that $\frac{b_t}{t} \to \mathrm{trans}(\mathrm{hol}(\gamma))$ as $t \to 0$ by hypothesis.

Using this notation:

$$\begin{aligned} \rho_t(\gamma)(\exp_p(tv)) &= (\exp(b_t)a_t\exp(t\,v))(p) = (\exp(b_t)a_t\exp(tv)a_t^{-1})(p) \\ &= \exp(b_t)\exp(ta_t(v))(p) = \exp(b_t + ta_t(v) + t^2\,C)(p) \end{aligned}$$

where $C = C(b_t/t, a_t, v, t)$ is an analytic function. Here we use that $b_t = O(t)$ together with the Campbell–Hausdorff formula. Using again this formula and the fact that $a_t(v), b_t \in \mathfrak{p}$, we get:

$$\rho_t(\gamma)(\exp_p(t\,v)) = \exp_p(b_t + t\,a_t(v) + t^2\,C'),$$

where $C' = C'(b_t/t, a_t, v, t)$ is also analytic. Hence

$$\frac{1}{t}\exp_p^{-1}(\rho_t(\gamma)\exp_p(t\,v)) = a_t(v) + \frac{b_t}{t} + tC'(b_t/t, a_t, v, t).$$

When $t \to 0$ this converges to $\mathrm{hol}(\gamma)(v)$ uniformly for v in a compact subset of $\mathbb{R}^3$, because $\frac{b_t}{t} \to \mathrm{trans}(\mathrm{hol}(\gamma))$ and $a_0 = \rho_0(\gamma)$ is a lift of $\mathrm{ROT}(\mathrm{hol}(\gamma))$, by hypothesis.

To prove convergence in the $\mathcal{C}^1$–topology, we write $v + \varepsilon w$ with $v, w \in \mathbb{R}^3$, and $\varepsilon > 0$ small. The previous calculation yields easily:

$$\frac{1}{t}\exp_p^{-1}(\rho_t(\gamma)\exp_p(t(v+\varepsilon w))) = a_t(v+\varepsilon w) + \frac{b_t}{t} + tC'(b_t/t, a_t, v+\varepsilon w, t).$$

We can compute the derivative with respect to ε when $\varepsilon = 0$ and we get $\mathcal{C}^1$–convergence. □

We modify slightly Goldman's construction to deform the structure. Let $\overline{M}$ be a compact core of $M = C^{\mathrm{smooth}}$. Consider $E = \widetilde{\overline{M}} \times_{\pi_1 M} \overline{\mathbb{X}}$, which is a bundle over $\overline{M}$ with fiber $\overline{\mathbb{X}}$. It is in fact a union of bundles $E_t = \widetilde{\overline{M}} \times_{\pi_1 M} \mathbb{X}_t$ with fiber $\mathbb{X}_t$. Each E_t has a natural flat connection that varies continuously with t. The developing map of the Euclidean structure induces a section $s\colon M \to E$ with values in $\mathbb{X}_0$ transverse to the flat connection. Since the image of s is compact, we compose it with the flow Φ_t of the vector field tangent to the direction of t (this is defined globally when $\epsilon = -1$ but not when $\epsilon = 1$, because $\mathbb{X}_0$ is not homeomorphic to $\mathbb{X}_t$). For small values of $t > 0$, $\Phi_t \circ s\colon \overline{M} \to E_t$ is a section, still transverse to the flat connection by $\mathcal{C}^1$–continuity, hence inducing a developing map.

This provides a structure on $\overline{M}$, that can be completed by controlling its behavior on $\partial\overline{M}$, using the hypothesis about the meridians. □

The volume of these cone manifolds is increasing with t, for the unscaled metric of constant curvature ± 1, because the starting Euclidean structure is viewed as totally degenerate with volume zero. In fact we get more precisely the following formula.

Proposition 5.3 (Schläfli's formula) *Let C_t be the family of cone manifolds constructed in Theorem 5.1 of constant curvature $K \in \mathbb{R}$. Assume moreover that the path of characters is analytic. Then*

$$K d \operatorname{vol} C_t = \frac{1}{2}\sum_e \operatorname{length}(e) d\alpha_e$$

where the sum runs over the edges and circles e of $\mathrm{Sing}(C)$.

Sketch of proof The construction of developing maps in the proof of Theorem 5.1 can be made so that we have a set of points $Z = \{z_1, \ldots, z_k\}$ such that the balls $B_{r_i}(z_i)$ cover C, where r_i is much smaller that the injectivity radius, and $\mathcal{D}_t(\widetilde{z}_i)$

varies analytically (we assume that the path of characters is analytic). Let C_t the cone structure with holonomy χ_t and define

$$P_i(t) = \{x \in C_t \mid d(x, z_i) \leq (x, z_j) \quad \forall j = 1, \dots, k\}$$

By construction, this P_i is an analytic family of polyhedra in the space $\mathbb{X}^3$. Analyticity implies that the topological type of P_i changes in a discrete subset of times t. Thus we may apply Schläfli's formula to them. Adding all the terms, we get the formula of the proposition, because the contribution of nonsingular edges is trivial. See [11, Proposition 4.2] for further details. □

Remark 5.4 *In the hyperbolic case,* $\widetilde{\mathrm{Isom}^+ \mathbb{H}^3} \cong SL_2(\mathbb{C})$ *and* $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$. *Therefore in this case* trans: $H^1(\pi_1 M; \mathfrak{g}_{\mathrm{Ad} \circ \mathrm{hol}}) \to H^1(\pi_1 M; \mathbb{R}^3_{\mathrm{ROT} \circ \mathrm{hol}})$ *can be viewed as the imaginary part.*

In the spherical case, $\widetilde{\mathrm{Isom}^+ \mathbb{S}^3} \cong SU(2) \times SU(2)$ *and the Lie algebra of the stabilizer of a point* $p \in \mathbb{S}^3$ *is conjugate to the diagonal subalgebra of* $\mathfrak{su}(2) \times \mathfrak{su}(2)$. *Therefore in this case* trans: $H^1(\pi_1 M; \mathfrak{g}_{\mathrm{Ad} \circ \mathrm{hol}}) \to H^1(\pi_1 M; \mathbb{R}^3_{\mathrm{ROT} \circ \mathrm{hol}})$ *can be viewed as the difference between factors.*

Remark 5.5 *For every* ρ *with character* $\chi_\rho \in V \subset X(M, SL_2(\mathbb{C}))$,

$$\rho(m_i) \text{ is a rotation if and only if } \mu_i(\chi_\rho) \in \mathbb{R}.$$

For every (ρ_1, ρ_2) *with character* $(\chi_{\rho_1}, \chi_{\rho_2}) \in U \times U \subset X(M, SU(2)) \times X(M, SU(2))$,

$$(\rho_1(m_i), \rho_2(m_i)) \text{ is a rotation if and only if } \mu_i(\chi_{\rho_1}) = \mu_i(\chi_{\rho_2}).$$

6 Constructing paths of hyperbolic and spherical structures

Up to changing the indices, by Corollary 4.11 we may assume that $d\mu_2, \dots, d\mu_q$ are linearly independent. Thus the set

$$\mathcal{C} = \{\chi \in U \subset X(M, SL_2(\mathbb{C})) \mid \mu_i(\chi) = \mu_i(\chi_0),\ i \geq 2\}$$

is a smooth complex curve in a neighborhood U of χ_0.

Notice that the class of the holonomy cocycle v is the tangent vector to $\mathcal{C}$ at χ_0, by Proposition 4.10. Thus, using Remark 5.4 we have the following.

Remark 6.1 *To every path* $\gamma\colon [0, \varepsilon) \to \mathcal{C}$ *with* $\gamma(0) = \chi_0$ *and* $\mathrm{Im}(\gamma'(0)) \neq 0$, *Theorem 5.1 applies.*

Lemma 6.2 *The restricted map $\mu_1|_{\mathcal{C}}\colon \mathcal{C} \to \mathbb{C}$ is a branched covering of degree two.*

Proof By Proposition 4.10, $d\mu_1|_{\mathcal{C}}(\chi_0) = 0$, thus $\mu_1|_{\mathcal{C}}$ is either constant or a branched covering of degree $d \geq 2$. Seeking a contradiction, assume that $\mu_1|_{\mathcal{C}}$ is constant, then we have a path of characters in the curve $\mathcal{C}$ to which we apply Theorem 5.1. Since each μ_i stays constant on this curve, this path corresponds to hyperbolic cone manifolds with constant cone angle. Thus by Schläfli's formula (Proposition 5.3) they have constant volume, contradicting the fact that the volume increases from zero. Thus $\mu_1|_{\mathcal{C}}$ is a branched covering of degree $d \geq 2$.

To prove that the degree d is precisely 2, we assume that $d > 2$ and seek again a contradiction. We look at the inverse image or real points $(\mu_1|_{\mathcal{C}})^{-1}(\mathbb{R})$, because for characters here, the image of μ_1 is a rotation. This inverse image is a graph with $2d$ branches starting at χ_0, and the angle between the branches is π/d. Thus two of the branches are real and the remaining $2d-2$ have nontrivial imaginary part. Hence Theorem 5.1 applies to those $2d-2$ branches. In addition, if $d > 2$, then there are branches for which μ_1 is strictly larger than $\mu_1(\chi_0)$ and branches for which μ_1 is strictly less than $\mu_1(\chi_0)$. Since μ_1 is the cone angle, we have constructed regenerating families of hyperbolic cone manifolds with both increasing and decreasing cone angles, contradicting Schläfli's formula. □

Corollary 6.3 *There is a family of hyperbolic cone structures obtained by decreasing α_1.*

Proof Since $\mu_1|_{\mathcal{C}}$ is a branched covering of degree 2, $(\mu_1|_{\mathcal{C}})^{-1}(\mathbb{R})$ has 4 branches starting at χ_0. Two of them are real, and to the other two one applies Theorem 5.1. The cone angle α_1 of those branches must decrease by Schläfli's formula. These two branches correspond to complex conjugate representations, ie with opposite orientations, because changing the sign of v corresponds to changing the orientation. □

Proposition 6.4 *There is a family of spherical cone structures obtained by increasing α_1.*

Proof To construct the spherical structure we consider

$$\mathcal{D} = \mathcal{C} \cap X(M, SU(2)).$$

This is a real analytic curve in a neighborhood V of χ_0. It is the set of real points of $\mathcal{C}$.

By Lemma 6.2, the map $\mu_1|_{\mathcal{D}}\colon \mathcal{D} \to \mathbb{R}$ is locally equivalent to the map $x \mapsto x^2$ in a neighborhood of $0 \in \mathbb{R}$. Thus the inverse fiber of the map $\mu_1|_{\mathcal{D}}$ consists of pairs of points, except for $\mu_1(\chi_0)$, which consists of a single point. Let $\mathcal{D}^{\pm} \subset \mathcal{D}$ be two subintervals of $\mathcal{D}$ such that $\mathcal{D}^{+} \cup \mathcal{D}^{-} = \mathcal{D}$ and $\mathcal{D}^{+} \cap \mathcal{D}^{-} = \{\chi_0\}$. We consider the set

$$S = \{(\chi_+, \chi_-) \in \mathcal{D}^{+} \times \mathcal{D}^{-} \mid \mu_1(\chi_+) = \mu_1(\chi_-)\}$$

because $\mu_1(\chi_+) = \mu_1(\chi_-)$ is the condition that guarantees that (χ_+, χ_-) is the character of a representation ρ in Spin(4) so that $\rho(\mu_1)$ is a rotation, by Remark 5.5. Theorem 5.1 applies to the latter two branches, because the difference of tangent vectors gives the translational part, that is necessarily a non-zero multiple of v (see Remark 5.4). □

If we replace $\mathcal{D}^{+} \times \mathcal{D}^{-}$ by $\mathcal{D}^{-} \times \mathcal{D}^{+}$ (ie we change the order of the factors) then we get the same structures with different orientation.

Remark 6.5 *We have constructed regenerations by deforming the cone angle α_1 because we assumed that the forms*

$$d\mu_2, \dots, d\mu_q$$

are linearly independent. Below in Corollary 7.8 we prove that the linear relation satisfied by these forms is:

$$\sum_{i=1}^{q} l_i d\mu_i = 0 \quad \text{for some } l_i > 0.$$

Thus we can regenerate by deforming any of the cone angles.

7 Fold locus and Euclidean structures

Now we analyze the behavior of the real analytic map

$$\mu = (\mu_1, \dots, \mu_q)\colon X(M, SU(2)) \to \mathbb{R}^q$$

in a neighborhood of χ_0.

Lemma 7.1 *In a neighborhood of χ_0, μ is equivalent to the map*

$$(x_1, x_2, \dots, x_q) \mapsto (x_1^2, x_2, \dots, x_q)$$

in a neighborhood of the origin via a real analytic change of coordinates at both the source and at the target.

Proof By Corollary 4.11 and up to permuting coordinates, we may assume that the forms $d\mu_2, \dots, d\mu_q \in T^*_{\chi_0} X(M, SU(2))$ are linearly independent. Thus the set

$$\mathcal{D} = \{\chi \in U \subset X(M, SU(2)) \mid \mu_i(\chi) = \mu_i(\chi_0), i \geq 2\}$$

is a smooth (real analytic) curve in a neighborhood U of χ_0. By Lemma 6.2 we can choose real analytic coordinates $(x_1, \dots, x_q)$ in a neighborhood of χ_0 with $x_i(\chi_0) = 0$ such that $\mu_1|_{\mathcal{D}} = \mu_1(\chi_0) + x_1^2$ and $\mu_i = \mu_i(\chi_0) + x_i$ for $i \geq 2$. Note that the curve $\mathcal{D}$ corresponds to the x_1–axis in these coordinates.

Since $\partial\mu_1/\partial x_1(0) = 0$ and $\partial^2\mu_1/\partial x_1^2(0) = 1$, the implicit function theorem yields that the set $\{\partial\mu_1/\partial x_1 = 0\}$ is locally around 0 the graph of a real analytic function $f = f(x_2, \dots, x_q)$. Let $g = g(x_2, \dots, x_q)$ be the real analytic function defined by

$$g(x_2, \dots, x_q) = \mu_1(f(x_2, \dots, x_q), x_2, \dots, x_q).$$

Let further h be the unique real analytic function satisfying

$$h^2(x_1, \dots, x_q) = \mu_1(x_1, \dots, x_q) - g(x_2, \dots, x_q)$$

and $\partial h/\partial x_1(0) = 1$. Then the map ϕ defined by

$$\phi(x_1, \dots, x_q) = (h(x_1, \dots, x_q), x_2, \dots, x_q)$$

is a local diffeomorphism. We obtain $\mu_1 \circ \phi^{-1} = x_1^2 + g(x_2, \dots, x_q)$ and $\mu_i \circ \phi^{-1} = x_i + \mu_i(\chi_0)$ for $i \geq 2$. With the local diffeomorphism ψ defined by

$$\psi(x_1, \dots, x_q) = (x_1 - g(x_2, \dots, x_q), x_2 - \mu_2(\chi_0), \dots, x_q - \mu_q(\chi_0))$$

we obtain $\psi \circ \mu \circ \phi^{-1}(x_1, \dots, x_q) = (x_1^2, x_2, \dots, x_q)$. □

Remark 7.2 *The same result holds true for the complex analytic extension*

$$\mu = (\mu_1, \dots, \mu_q)\colon X(M, SL_2(\mathbb{C})) \to \mathbb{C}^q$$

just by composing with the complex analytic extensions of the coordinate changes.

Definition 7.3 The *fold locus* $\mathcal{F} \subset X(M, SU(2))$ is the set of points where μ is not a local diffeomorphism.

In the coordinate system of Lemma 7.1 the fold locus is the set $\{x_1 = 0\}$, hence the following Corollary.

Corollary 7.4 *There exists a neighborhood U of χ_0 such that both $\mathcal{F} \cap U$ and $\mu(\mathcal{F} \cap U)$ are codimension one submanifolds. In addition, for each $\chi \in \mathcal{F}$,*

$$\dim \ker\langle d\mu_1, \dots, d\mu_k\rangle = 1.$$

Proposition 7.5 *Each $\chi \in \mathcal{F} \cap U$ is the rotational part of the holonomy of a Euclidean cone structure. The translational part lies in* $\ker\langle d\mu_1, \dots, d\mu_k\rangle$.

Proof When we deform a character χ in $\mathcal{F}$ we can also deform continuously the vector v in $\ker\langle d\mu_1, \dots, d\mu_k\rangle \subset T_\chi X(M, SU(2))$, which is a one dimensional subspace that varies continuously, by Lemma 7.1. This corresponds to deforming continuously the holonomy hol by a family of representations of $\pi_1 M$ in $\widetilde{\mathrm{Isom}^+ \mathbb{R}^3}$. The condition $v \in \ker\langle d\mu_1, \dots, d\mu_k\rangle$ ensures that the image of meridians are rotations (cf [11, Proposition 9.6]). Since those representations map the meridians to rotations, it follows that they correspond to holonomy representations of Euclidean cone manifolds. $\square$

Corollary 7.6 *Points of $E = \mu(\mathcal{F})$ are angles of Euclidean cone structures on C.*

Let $\overline{l} = (l_1, \dots, l_q)$ denote the lengths of the singular circles and components of C. Those are unique up to homothety.

Proposition 7.7 *The vector $\overline{l}$ is normal to $E = \mu(\mathcal{F})$ at $\mu(\chi_0)$.*

Proof This is a consequence of Schläfli's formula (Proposition 5.3). $\square$

Corollary 7.8 *On $T_{\chi_0}(X(M, SU(2)))$ holds $\sum_i l_i \, d\mu_i = 0$.*

Proof It holds on $T_{\chi_0}\mathcal{F}$ by Proposition 7.7. In addition, it also holds on the span of $v = [\mathrm{trans} \circ \mathrm{hol}]$. By Lemma 7.1 and Proposition 4.10, those spaces generate the whole $T_{\chi_0}(X(M, SU(2))$. $\square$

It follows from this corollary that we can regenerate into hyperbolic or spherical cone structures by decreasing or increasing any of the cone angles, the geometry depends on the sign of $\sum l_i \alpha_i'$. (See Remark 6.5.)

Corollary 7.9 (Local Rigidity for Euclidean cone manifolds) *Let C be a closed Euclidean cone manifold with cone angles $\leq \pi$ which is not almost product. Then deformations up to dilations of C into Euclidean cone structures are parameterized by the q cone angles in a manifold E of dimension $q-1$ and transverse to the vector of singular lengths $\overline{l} = (l_1, \dots, l_q)$.*

Proof The previous analysis determines all representations of $\pi_1 M$ into $\widetilde{\mathrm{Isom}^+ \mathbb{R}^3}$ up to conjugation that map meridians to rotations, ie pairs (v, χ_ρ) such that $\chi_\rho \in X(M, SU(2))$ and further $v \in \ker\langle d\mu_1, \dots, d\mu_k\rangle \subset T_\chi X(M, SU(2))$, see Figure 1. $\square$

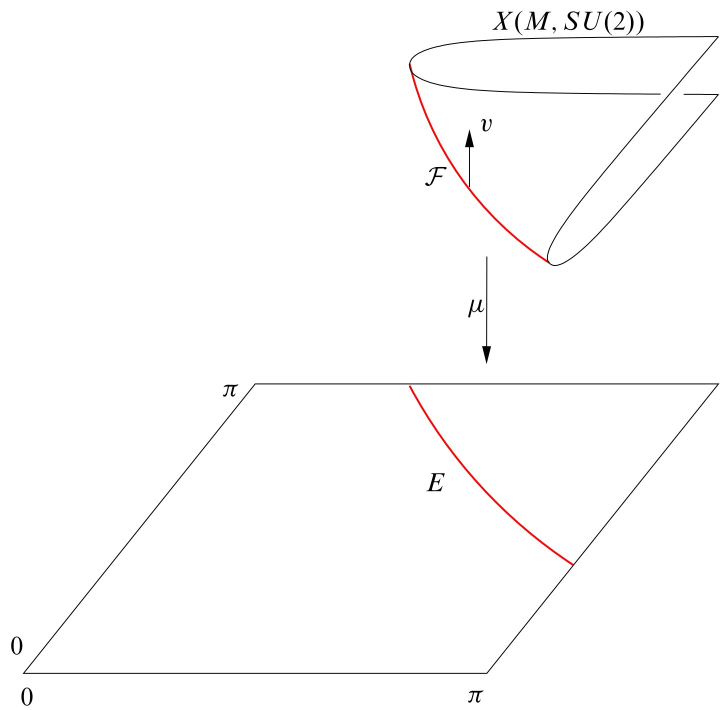

Figure 1: Geometry of the map μ and the fold locus.

Corollary 7.10 (Local Rigidity for regenerations) *Let C be as above. Then deformations of C into constant curvature structures up to dilations are parameterized by the q cone angles, and the type of geometry depends on the side of E.*

Proof The analysis of this section and previous ones determine the structure of the spaces of characters of $\pi_1 M$ in $\widetilde{\mathrm{Isom}^+ \mathbb{X}^3}$. Namely, spherical regenerations correspond to pairs of $SU(2)$–characters (χ_+, χ_-) such that $\mu(\chi_+) = \mu(\chi_-)$. According to Lemma 7.1, these pairs are determined by the value of μ, which is the multiangle of the structure, cf Figure 1. Hyperbolic regenerations correspond to $SL_2(\mathbb{C})$–characters χ with $\mu(\chi) \in \mathbb{R}^q$, which are not $SU(2)$–characters, cf Figure 2. Again such a character is determined up to complex conjugation by the value of μ.

Next, we shall show that the deformations of the structures are uniquely determined by the deformations of the characters, adapting an argument from Goldman [8].

We choose a compact core $\overline{M}$ of $M = C^{\text{smooth}}$ as follows. We fix $\varepsilon > 0$ less than one third the normal radius of $\mathrm{Sing}(C)$, so that the tubular neighborhood $U_{3\varepsilon}(\mathrm{Sing}(C))$ of

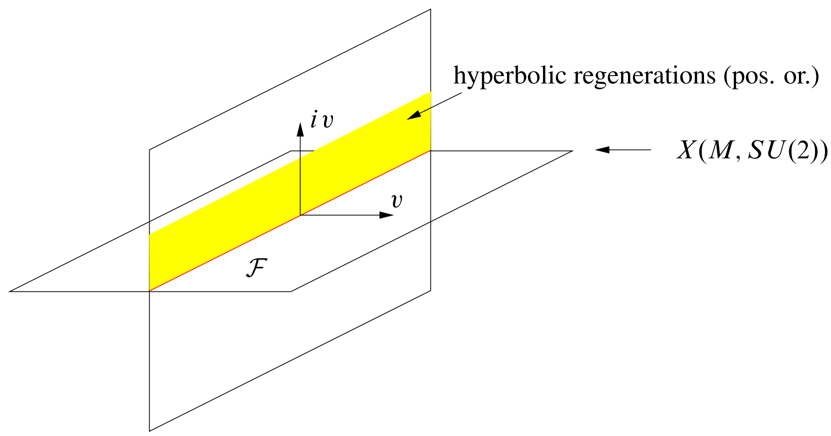

Figure 2: The locus of $SL_2(\mathbb{C})$–characters having real traces.

radius 3ε is well embedded, and choose $\overline{M} = C \setminus U_{2\varepsilon}(\mathrm{Sing}(C))$. We will consider deformations of structures on $\overline{M}$, knowing that $C \setminus U_{\varepsilon}(\mathrm{Sing}(C))$ is a thickening of $\overline{M}$, as in Canary et al [4], and that deformations on $U_{3\varepsilon}(\mathrm{Sing}(C))$ are determined by the holonomy representation that maps meridian curves of the singular locus to rotations.

Using the notation of Section 5, $E = \cup_{t \in [0,t)} E_t$ is the union of flat bundles E_t with fiber $\mathbb{X}_t$ the space of constant curvature $\pm t^2$, and holonomy the deformation of the Euclidean holonomy representation. Since E_t is flat, there is a horizontal foliation, which is also used to define locally a projection onto the fibers $\mathbb{X}_t$.

The Euclidean structure is induced by a section $s_0\colon \overline{M} \to E_0$ transverse to the horizontal foliation. In Section 5, s_0 is deformed to $s_t\colon \overline{M} \to E_t$, following the flow in the time direction on E. Let $\sigma_t\colon \overline{M} \to E_t$ be another deformation, ie a section of class $\mathcal{C}^1$ such that $\sigma_0 = s_0$ and $\sigma_t|_{\partial \overline{M}} = s_t|_{\partial \overline{M}}$. We claim that, for $t \geq 0$ sufficiently small, there exist a diffeomorphism h_t of M isotopic to the identity so that the developing maps corresponding to σ_t and $s_t \circ h_t$ are equivalent (namely, the sections σ_t and $s_t \circ h_t$ composed with the local projections to $\mathbb{X}_t$ differ by isometries of $\mathbb{X}_t$). Following [8], we take a foliated neighborhood N of $s_0(\overline{M})$ in E_0, so that the intersection of leaves with N are balls, and consider $N \times [0,t)$ in E, via the flow as in Section 5. For t sufficiently small, closeness and transversality imply that the image of both s_t and σ_t meet each horizontal leave in $N \times \{t\}$ precisely once. The intersection with the horizontal leaves can be used to define the map h_t as in [8]. $\square$

8 Global results

In this section we finish the proof of Theorem 1.3. We first establish some properties of the set of Euclidean angles and recall some results about the set of hyperbolic and spherical ones.

Let E denote the set of multiangles with a Euclidean cone structure.

Proposition 8.1 *E is a properly embedded hypersurface of $(0,\pi)^q$.*

Proof First we show that every Euclidean cone structure on C with cone angles in $(0,\pi)$ is not almost product. This will imply that E is a $q-1$ submanifold, by Corollary 7.6. By contradiction, we assume that there exists such a Euclidean cone structure on C which is almost product. Since all cone angles are in $(0,\pi)$, this implies that C is Seifert fibered and $\mathrm{Sing}(C)$ is a union of fibers. By Corollary 2.3, in this case C is almost product for any cone angle, contradicting the hypothesis of Theorem 1.3.

To prove properness, we show that if x_n is a sequence in E converging to $x_\infty \in (0,\pi)^q$, then $x_\infty \in E$. Let C_n be the corresponding sequence of cone structures on C with multiangles x_n. We can rescale them so that $\operatorname{diam} C_n = 1$. Since the cone angles are uniformly $< \pi$, by Boileau et al [2, Theorem 5.3 and Corollary 5.4] the cone manifolds C_n are uniformly thick: they have a base point $x_n \in C_n$ with $inj(x_n) \geq c > 0$. Thus they have a convergent subsequence to a Euclidean cone manifold of diameter 1, hence a cone structure on C with cone angles x_∞. □

In [15, Chapter 3] the following Proposition is proved, which applies to C under the assumptions of the paper.

Proposition 8.2 *If $\overline{\alpha} \in (0,\pi]^q$ is a multiangle of a spherical cone structure on C which is not Seifert fibered, then every $\overline{\beta} = (\beta_1,\dots,\beta_q) \in (0,\pi]^q$ with $\alpha_i \leq \beta_i$ is the multiangle of a spherical cone structure with cone angles $\overline{\beta}$.*

If $\overline{\alpha} \in (0,\pi]^q$ is a multiangle of a hyperbolic cone structure on C, then every $\overline{\beta} = (\beta_1,\dots,\beta_q) \in (0,\pi]^q$ with $0 < \beta_i \leq \alpha_i$ is the multiangle of a hyperbolic cone structure with cone angles $\overline{\beta}$.

Remark 8.3 *When all cone angles are π, the orbifold is spherical and hence small (without essential 2–suborbifolds). In particular all turnovers in C are compressible or boundary parallel.*

Before proving Theorem 1.3, we need to establish incompatibility between structures of different sign. We normalize the constant curvature to be -1, 0 or 1.

Lemma 8.4 *Given $\overline{\alpha} = (\alpha_1, \dots, \alpha_q) \in (0, \pi]^q$, $\overline{\alpha}$ cannot be the multiangle of two cone structures with different normalized constant curvatures (-1, 0 or 1).*

Proof Seeking a contradiction, assume for instance that $\overline{\alpha} \in (0, \pi]^q$ is the multiangle of a hyperbolic and Euclidean cone structure simultaneously. Using Corollary 6.3, the Euclidean cone structure can be regenerated to hyperbolic ones, which, by construction, have arbitrarily small diameter, and also cone multiangle arbitrarily close to $\overline{\alpha}$. On the other hand, when we perturb the hyperbolic cone structure with cone multiangle $\overline{\alpha}$, we find hyperbolic cone manifolds with the same multiangles but diameter bounded below away from zero. This contradicts the global rigidity of hyperbolic cone manifolds proved in [15, Theorem 1.4]. The same argument works for spherical and Euclidean (global rigidity in the spherical case is the content of [15, Theorem 1.7]).

Finally we deal with the case that $\overline{\alpha}$ is simultaneously the multiangle of a hyperbolic and spherical structure on C. Consider $n \in \mathbb{N}$ sufficiently large so that $2\pi/n < \alpha_i$ for $i = 1, \dots, q$. By Proposition 8.2, $(\frac{2\pi}{n}, \dots, \frac{2\pi}{n})$ is the multiangle of a hyperbolic cone manifold, hence a hyperbolic orbifold. Notice that this orbifold cannot be spherical, because its fundamental group is infinite. Consider now a path of cone multiangles between $\overline{\alpha}$ and $(\frac{2\pi}{n}, \dots, \frac{2\pi}{n})$ which is decreasing along each component. By Proposition 8.2 every multiangle in this path corresponds to a hyperbolic cone structure. However, sphericity must fail at some multiangle of this path, because $(\frac{2\pi}{n}, \dots, \frac{2\pi}{n})$ is not the multiangle of a spherical structure. By the arguments in the proof of Proposition 8.1, when sphericity fails we find precisely a Euclidean structure, and we have reduced to the first case of incompatibility between hyperbolic and Euclidean structures. $\square$

The following Proposition is also used in the proof of Theorem 1.3.

Proposition 8.5 *Every multiangle in $[0, \pi)^q$ with at least one angle 0 is the multiangle of a hyperbolic cone structure.*

Proof By [15, Corollary 1.5], the smooth part of C is hyperbolic. Applying Thurston's hyperbolic Dehn filling theorem, we have the proposition for a neighborhood of the origin. To cover the rest of the multiangles, we just have to enlarge some of the angles, keeping the other ones equal to zero (ie complete cusps). Thus we have a lower bound on the cone angles, and in addition the diameter is infinite. Applying the results of [2], we can enlarge each one of the cone angles up to $< \pi$. Notice that some convergence results of [2] apply to cone manifolds without essential turnovers. In our case there are no turnovers by Remark 8.3 except if we allow cusped turnovers (ie turnovers with some cone angle 0). However cusped turnovers are not a problem in those arguments, because all cone angles are $< \pi$, and therefore they cannot converge to a Euclidean cone 2–manifold. □

Remark 8.6 *The hyperbolic structures of the previous proposition can be deformed in a neighborhood in* $[0,\pi)^q$.

This remark can be easily proved adapting the arguments of the proof of hyperbolic Dehn filling for orbifolds in Boileau–Porti [3, Appendix B] and using the infinitesimal rigidity results established in Weiss [14].

Proof of Theorem 1.3 Assume first that C is a Euclidean cone manifold as in the statement and all cone angles are π. Then using the regeneration results of Section 6 and Proposition 8.2, every point in $(0,\pi)^q$ is the multiangle of a hyperbolic cone structure on C.

Assuming that at least one of the angles of C is $< \pi$, then there exists a spherical cone structure on C with all cone angles π, again by Section 6 and Proposition 8.2. We consider all segments in $[0,\pi)^q$ starting at $(\pi,\dots,\pi)$ and ending at some point with at least one coordinate zero. The first point of the segment is the multiangle of a spherical cone structure and the last one is the multiangle of a hyperbolic one by Proposition 8.5. By Remark 8.6, we can assume that the multiangle of the hyperbolic structure lies in the open cube $(0,\pi)^q$. Starting at $(\pi,\cdots,\pi)$, we move along the segment by decreasing the cone angles and obtaining a family of spherical cone manifolds. This family cannot be spherical all the time because the endpoint of the segment corresponds to a hyperbolic structure and they are incompatible by Lemma 8.4. Since all cone angles are $< \pi$, the end of the spherical subsegment is the multiangle of a Euclidean structure by the same argument as in Proposition 8.1. By the regeneration results of Section 6 and Proposition 8.2, we connect the multiangle of this Euclidean structure to the endpoint of the segment by multiangles of hyperbolic cone structures.

The previous argument shows that every point in $(0,\pi)^q$ is the multiangle of a constant curvature cone structure on C, and that E is a hypersurface that divides $(0,\pi)^q$ in two components H and S corresponding respectively to hyperbolic and spherical multiangles.

For the uniqueness, notice first that in Lemma 8.4 we establish incompatibility between hyperbolic, spherical and Euclidean for a given multiangle. Global rigidity for hyperbolic and spherical structures is proved in Weiss [15]. Using local rigidity of the regenerations (Corollary 7.10) and global rigidity for the hyperbolic structures, we get global rigidity for the Euclidean ones. □

9 The Whitehead link

In this section we illustrate the main theorem for cone structures on the 3–sphere with singular locus given by the Whitehead link, which is the 2–component link depicted in Figure 3. Furthermore we discuss geometric structures corresponding to multiangles contained in the boundary of $[0, \pi]^2$ to some extent.

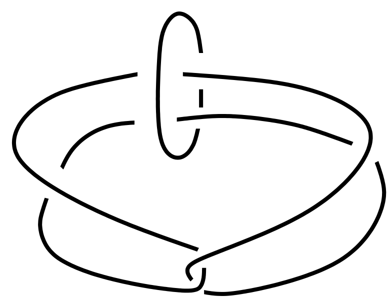

Figure 3: The Whitehead link.

R N Shmatkov has computed the curve of multiangles of Euclidean structures by constructing the corresponding cone manifolds explicitly, cf [12]. These structures are not almost product. The hyperbolic region had earlier been computed by A Mednykh in a similar way, cf [10] and references therein.

Alternatively, one can proceed as follows: Let M denote the complement of the Whitehead link L in S^3. If $a, b \in \pi_1 M$ are meridians around the two components of L, the fundamental group of M has the following presentation, cf Hilden et al [9]:

$$\pi_1 M = \langle a, b | awa^{-1}w^{-1} \rangle$$

with $w = bab^{-1}a^{-1}b^{-1}ab$. The $SL_2(\mathbb{C})$-character variety of M has been computed in [9]. Namely, after identifying $X(M, SL_2(\mathbb{C}))$ with the image of the map

$$\begin{aligned} (t_a, t_b, t_{ab})\colon\ R(M, SL_2(\mathbb{C})) &\to \mathbb{C}^3 \\ \rho &\mapsto (\operatorname{tr}\rho(a), \operatorname{tr}\rho(b), \operatorname{tr}\rho(ab)) \end{aligned}$$

in $\mathbb{C}^3$, it is given by

$$X(M, SL_2(\mathbb{C})) = \{(x, y, z) \in \mathbb{C}^3 : p(x, y, z) \cdot q(x, y, z) = 0\}$$

with

$$p(x, y, z) = xy - (x^2 + y^2 - 2)z + xyz^2 - z^3$$

and

$$q(x,y,z) = x^2 + y^2 + z^2 - xyz - 4.$$

Again by [9], the irreducible part of the character variety, ie those characters which correspond to irreducible representations, is given by

$$X^{irr}(M, SL_2(\mathbb{C})) = \{p(x,y,z) = 0\} \setminus \{q(x,y,z) = 0\};$$

moreover

$$\{p(x,y,z) = 0\} \cap \{q(x,y,z) = 0)\} = \{x = \pm 2, z = \pm y\} \cup \{y = \pm 2, z = \pm x\}.$$

Holonomy representations of hyperbolic cone structures lift to irreducible representations ρ with $t_a(\rho) \in (-2,2)$ and $t_b(\rho) \in (-2,2)$. We may write

$$x = \pm 2\cos(\alpha/2)$$

and

$$y = \pm 2\cos(\beta/2)$$

with α and β the cone angles around the two components of L.

Rotational parts of Euclidean holonomies correspond to representations as above where in addition t_a and t_b fail to be local coordinates. This is precisely the locus where the discriminant of p computed with respect to the variable z vanishes. The discriminant is given by the following polynomial:

$$\begin{aligned} f(x,y) = x^6y^2 - 2x^4y^4 + 2x^4y^2 + x^2y^6 + 2x^2y^4 - 11x^2y^2 + 32 \\ - 48x^2 - 48y^2 + 24y^4 + 24x^4 - 4x^6 - 4y^6. \end{aligned}$$

For the resulting curve of multiangles of Euclidean cone structures see Figure 4.

For multiangles contained in some parts of the boundary of $[0,\pi]^2$ we can construct cone structures as well using our main theorem: For multiangles of the form $(0,\beta)$, $0 \le \beta < \pi$, resp. $(\alpha, 0)$, $0 \le \alpha < \pi$, we obtain hyperbolic cone structures, whereas for multiangles of the form (π, β), $\pi/2 < \beta \le \pi$, resp. (α, π), $\pi/2 < \alpha \le \pi$, we obtain spherical ones.

A *Nil*–orbifold structure with branching indices $(4,2)$, resp. $(2,4)$, ie corresponding to multiangles $(\pi/2, \pi)$, resp. $(\pi, \pi/2)$, has been constructed by E Suárez, cf [13].

For the remaining part of the boundary we do not give a complete description, we rather prove the following statement.

Lemma 9.1 *There is no hyperbolic cone structure corresponding to multiangles contained in* $[0,\pi/2) \times \{\pi\} \cup \{\pi\} \times [0,\pi/2)$.

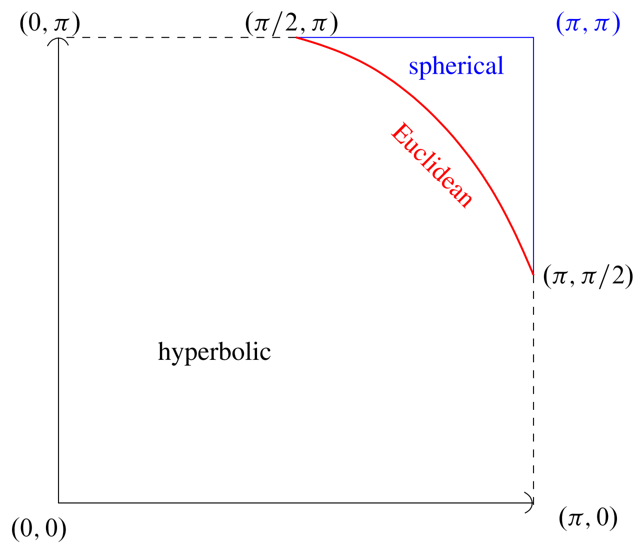

Figure 4: The curve of Euclidean multiangles and the hyperbolic and spherical regions.

Proof Let us suppose there existed such a structure. Then we consider the double branched cover of S^3 branched along the component of the Whitehead link with cone angle π. This branched cover is again S^3 since the components of the Whitehead link are unknotted. The other component of the Whitehead link lifts to the torus link $T(4,2)$, whose complement is known to be non-hyperbolic. On the other hand, the lift of the initial hyperbolic cone metric will be a hyperbolic cone metric on S^3 with singular locus $T(4,2)$, which is a contradiction in view of the results in [15]. □

Remark 9.2 *Multiangles of the form $(2\pi/n,\pi)$, resp. $(\pi,2\pi/n)$, with $n \geq 5$ correspond to orbifold structures modelled on $\widetilde{PSL_2(\mathbb{R})}$–geometry.*

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