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Finite group extensions and the Baum–Connes conjecture

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In this note, we exhibit a method to prove the Baum–Connes conjecture (with coefficients) for extensions with finite quotients of certain groups which already satisfy the Baum–Connes conjecture. Interesting examples to which this method applies are torsion-free finite extensions of the pure braid groups, eg the full braid groups, and fundamental groups of certain link complements in S^3.

The Baum–Connes conjecture for a group G states that the Baum–Connes map

$$\mu_r\colon K_*^G(\underline{E}G, A) \to K_*(C^*_{\text{red}}(G, A)) \tag{1}$$

is an isomorphism for every C^*–algebra A with an action of G by C^*–algebra homomorphisms. In this note the term "Baum–Connes conjecture" will always mean "Baum–Connes conjecture with coefficients", and all group are assumed to be discrete and countable.

Here, the left hand side is the equivariant K–homology with coefficients in A of the universal space $\underline{E}G$ for proper G–actions, which is homological in nature. The right hand side, the K–theory of the reduced crossed product of A and G, belongs to the world of C^*–algebras and—to some extent—representations of groups. If $A = \mathbb{C}$ with the trivial action, the right hand side becomes the K–theory of the reduced C^*–algebra of G. If, in addition, G is torsion-free, the left hand side is the K–homology of the classifying space of G.

The Baum–Connes conjecture has many important connections to other questions and areas of mathematics. The injectivity of μ_r implies the Novikov conjecture about homotopy invariance of higher signatures. It also implies the stable Gromov–Lawson–Rosenberg conjecture about the existence of metrics with positive scalar curvature on spin-manifolds. The surjectivity, on the other hand, gives information in particular about $C^*_{\text{red}}G$. If G is torsion-free, it implies eg that this C^*–algebra contains no idempotents different from zero and one. Since we are only considering the Baum–Connes conjecture with coefficients, all these properties follow for all subgroups of G, as well.

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We do not want to repeat the construction of the K–groups and the map in the Baum–Connes conjecture (1), instead, the reader is referred to Baum, Connes and Higson [1] and Julg [8]. Higson, Lafforgue and Skandalis [7], using groups constructed by Gromov [4], have produced counterexamples to the conjecture (with non-trivial, commutative coefficients). However, we will concentrate on groups for which the Baum–Connes conjecture is known to be true and will prove it for new examples. A well-known fact is that the Baum–Connes conjecture is inherited by arbitrary subgroups, and there are rather precise results describing its behavior under group extensions (we will recall this below). However, it is almost completely unknown what is happening for an extension with finite quotient of a group which satisfies the Baum–Connes conjecture. The most prominent open examples are probably the full braid groups, which contain the pure braid groups as subgroups of finite index. For the latter, the Baum–Connes conjecture is well known to be true.

The main goal of this note is to prove Baum–Connes for the full braid groups, and for other classes of groups which arise as (finite) extensions of groups for which Baum–Connes is known. We use the following results.

Theorem 1 *Assume G is a group which satisfies the Baum–Connes conjecture with coefficients. If H is a subgroup of G, then H satisfies the Baum–Connes conjecture with coefficients, too.*

Theorem 2 (Oyono-Oyono [15]) *Assume G is a group acting on a tree. The Baum–Connes conjecture is true for G if and only if it is true for every isotropy subgroup of the action on the vertices.*

Note that, from a logical point of view, Theorem 1 is a consequence of Theorem 2.

Theorem 3 (Chabert–Echterhoff [2, Section 3] and Oyono-Oyono [16, Theorem 3.1]) *Assume we have an extension of groups*

$$1 \to H \to G \xrightarrow{\pi} Q \to 1.$$

For every finite subgroup E of Q let $H_E < G$ be the inverse image under π of E in G. Assume Q and all groups H_E satisfy the Baum–Connes conjecture with coefficients.

Then the Baum–Connes conjecture with coefficients is also true for G.

Theorem 4 (Higson–Kasparov [6; 5]; compare also Julg [8]) *Let G be an amenable group. Then the Baum–Connes conjecture with coefficients holds for G.*

Theorem 3 and Theorem 4 together immediately imply the following corollary.

Corollary 5 *Assume we have an extension $1 \to H \to G \to A \to 1$ of groups, A is torsion-free and amenable, and the Baum–Connes conjecture with coefficients holds for H. Then it also holds for G.*

Our (naive) idea to deal with finite group extensions is to exhibit them as extension with torsion-free quotient. More precisely, assume we have an extension $1 \to H \to G \to Q \to 1$ where H satisfies the Baum–Connes conjecture and Q is a finite group. Our goal is, to find a normal subgroup U of G which is contained in H and such that G/U is torsion-free and amenable (eg virtually nilpotent or virtually solvable). If H is torsion-free, a necessary condition for the existence of such a factorization is that G is torsion-free, too. We will observe that there are large classes of torsion-free groups such that every torsion-free finite extension admits a factorization of the type we are looking for, including all pure braid groups.

We need the following notation.

Definition 6 Let H be a group and p a prime number. Let $\widehat{H}^p$ denote the pro-p completion of H, ie the inverse limit of the system of finite p–group quotients of H. There is a natural homomorphism $H \to \widehat{H}^p$ (not necessarily injective). The Galois cohomology $H^*(\widehat{H}^p, \mathbb{Z}/p)$ is defined to be the direct limit of the cohomology of the finite p-group quotients of H (with $\mathbb{Z}/p$–coefficients). For the theory of profinite groups and Galois cohomology compare eg Wilson [17].

The group H is called *cohomologically complete* if the natural homomorphisms $H^*(\widehat{H}^p, \mathbb{Z}/p) \to H^*(H, \mathbb{Z}/p)$ are isomorphisms for every prime p.

The second property of H we are going to use is the existence of many quotients which are torsion-free and amenable.

Definition 7 Let H be a group. We say H has *enough amenable torsion-free quotients* if for every normal subgroup U of H of finite p–power index (for some prime p) another normal subgroup V of H exists which is contained in U and such that H/V is torsion-free and elementary amenable. We say that H has *enough nilpotent torsion-free quotients* if we find V such that H/V is nilpotent.

Remark 8 Recall that the class of elementary amenable groups is the smallest class of groups which contains all finite and all abelian groups and which is closed under extensions and directed unions. It contains in particular all nilpotent and all solvable-by-finite groups.

For the purpose of our paper, we could weaken the condition to H/V being amenable. However, this would deviate from the notation used in Linnel–Schick [11]. Moreover, all examples relevant to us satisfy the condition that H/V is solvable-by-finite.

These two properties are important because of the following result.

Theorem 9 (Linnel–Schick [11, Theorem 3.46]) *Let H be a group which is cohomologically complete and which has enough amenable torsion-free quotients. Assume that there is a finite model for the classifying space of H.*

Let $1 \to H \to G \to Q \to 1$ be an extension with finite quotient Q and such that G is torsion-free. Then there is a normal subgroup U of G, contained in H, such that G/U is torsion-free and elementary amenable.

The proof of this theorem uses Sylow's theorems to reduce to the case where Q is a p–group. One then uses the pro–p completions as an intermediate step to show that if no such torsion-free quotient exists, then the projection $G \to Q$ induces a split injective map in cohomology (we might have to replace Q with a non-trivial subgroup first). Using an Atiyah–Hirzebruch spectral sequence argument, we show that the same is true for stable cohomotopy. But then a fixed-point theorem of Jackowski implies that the map $G \to Q$ itself splits, which contradicts the assumption that G is torsion-free.

We now obtain our main technical result.

Theorem 10 *Let H be a group with finite classifying space which is cohomologically complete and which has enough amenable torsion-free quotients. Assume H satisfies the Baum–Connes conjecture. Let $1 \to H \to G \to Q \to 1$ be an extension of groups such that G is torsion-free and such that Q satisfies the Baum–Connes conjecture. Then the same is true for G.*

Proof Because of Theorem 3 it suffices to prove the result if Q is finite (and still G torsion-free). In this case, by Theorem 9 we can replace the extension $1 \to H \to G \to Q \to 1$ by another extension $1 \to U \to G \to A \to 1$ where U is a subgroup of H and A is torsion-free and elementary amenable. Theorem 1 implies that U fulfills the Baum–Connes conjecture. Because of Corollary 5 the same is then true for G. □

In the last part of this note, we discuss examples of groups H, to which Theorem 10 applies.

Definition 11 Let $\mathcal{F}$ be the class of groups G which fulfill the following properties: G has a finite classifying space, G is cohomologically complete and G has enough nilpotent torsion-free quotients.

Let $\mathbf{LH}\mathcal{ETH}$ be the class of groups defined in Mislin–Valette [14, Definition 5.22].

Set $\mathcal{F}^+ := \mathcal{F} \cap \mathbf{LH}\mathcal{ETH}$.

Remark 12 We do not recall its complicated definition, but remark that it has the following important properties: The class $\mathbf{LH}\mathcal{ETH}$ is closed under passing to subgroups, under extensions with torsion free quotients and under finite products and finite free products. More generally, it is closed under passing to fundamental groups of graphs of groups. $\mathbf{LH}\mathcal{ETH}$ contains in particular all one-relator groups and all Haken 3–manifold groups (and hence all knot or link groups), as well as all a-T-menable groups (in particular all amenable groups). All these facts of the class $\mathbf{LH}\mathcal{ETH}$ and more information can be found in Mislin–Valette [14], in particular [14, Theorem 5.23].

The following result is stated as Mislin–Valette [14, Theorem 5.23] or Lück–Reich [12, Theorem 5.2].

Proposition 13 *If $H \in \mathbf{LH}\mathcal{ETH}$ then the Baum–Connes conjecture with coefficients is true for H.*

Corollary 14 *Assume that $H \in \mathcal{F}^+$ and we have an extension*

$$1 \to H \to G \to Q \to 1$$

where Q fulfills the Baum–Connes conjecture with coefficients and G is torsion-free. Then the Baum–Connes conjecture with coefficients is true for G.

Proof Because $H \in \mathcal{F}^+ \subset \mathbf{LH}\mathcal{ETH}$, by Proposition 13 H satisfies the Baum–Connes conjecture. The definition of $\mathcal{F}^+$ implies that we can apply Theorem 10 in our situation, which implies the assertion. □

Definition 15 A semidirect product $H \rtimes Q$ of two groups H and Q is called *H_1–trivial* if the induced action of Q on $H_1(H, \mathbb{Z})$ is trivial.

Definition 16 A one-relator group G is called *primitive* if it is finitely generated and if it has a presentation $G = \langle x_1, \dots, x_d \mid r \rangle$ such that the element r in the free group F generated by $x_1, \dots, x_d$ is contained in the lower central series subgroup $\gamma_n(F)$ but not in $\gamma_{n+1}(F)$ and the image of r in $\gamma_n(F)/\gamma_{n+1}(F)$ is not a proper power.

Example 17 Fundamental groups of orientable two-dimensional surfaces are primitive one-relator groups, as well as one-relator groups where the least common multiple of the exponent-sums for the different generators $x_1, \dots, x_n$ in the relator r is one.

Proof For the convenience of the reader, we sketch proofs of these well known facts.

In the second case, the image of r in the abelianization $\gamma_1(F)/\gamma_2(F) = F/[F, F]$ of the free group is a product of multiples $a_1[x_1], \dots, a_n[x_n]$ of the generators $[x_1], \dots, [x_n]$

of the free abelian group $F/[F,F]$ with generators $[x_1],\dots,[x_n]$ with least common multiple of $(a_1,\dots,a_n)$ equal to 1, so is non-zero and not a proper power.

The standard presentation $\langle x_1,y_1,\dots,x_n,y_n \mid [x_1,y_1],\dots,[x_n,y_n]\rangle$ of a the fundamental group of an orientable surface of genus n shows that the relation r, the product of the commutators $[x_i,y_i]$ is contained in $\gamma_2(F)$, but is a product of certain free generators of the free abelian group $\gamma_2(F)/\gamma_3(F)$, (this abelian group is freely generated by the images of all non-trivial commutators of the generators $x_1,\dots,x_n$, as follows from Magnus characterization of $\gamma_n(F)$, compare Magnus [13]). Consequently, the image of r in $\gamma_2(F)/\gamma_3(F)$ is non-zero and not a proper power. □

Definition 18 We denote the fundamental group G of the complement of a tame link with d components in S^3 a *link group with d components*. We define the *linking diagram* to be the edge-labeled graph whose vertices are the components of the link, and such that any pair of vertices is joined by exactly one edge. Each edge is labeled with the linking number of the two link components involved.

We say the link group G is *primitive* if for each prime p there is a spanning subtree of the linking diagram such that none of the labels of the edges of this subtree is congruent to 0 modulo p.

Observe that in particular every *knot group* (ie a link group with only one component) is primitive, as the linking diagram has one vertex and no edges, ie a tree where we don't have to worry about any label of any edge.

Theorem 19 *The class $\mathcal{F}^+$ is closed under H_1–trivial semidirect products. It contains*

- *all primitive link groups*
- *all primitive one-relator groups*
- *the fundamental groups of all fiber-type arrangements (as defined in Falk–Randell [3]) and*
- *Artin's pure braid groups P_n.*

Proof It follows from Linnell–Schick [11, Theorems 5.26 and 5.40, Corollary 5.27, Propositions 5.30 and 5.34], using Linnell–Schick [10] and Kümpel [9] that the groups which are mentioned belong to $\mathcal{F}$. By Remark 12 every link-group and every 1–relator group belongs to $\mathbf{LH}\mathcal{ETH}$. Moreover, Artin's pure braid groups are iterated extensions of free groups. Since free groups belong to $\mathbf{LH}\mathcal{ETH}$ and $\mathbf{LH}\mathcal{ETH}$ is closed under extensions with torsion-free quotients, the pure braid groups belong to $\mathbf{LH}\mathcal{ETH}$. Exactly the same argument applis to general fiber-type arrangements as defined in Falk–Randell [3]. □

Corollary 20 *All the torsion-free finite extensions of the groups listed in Theorem 19, in particular Artin's full braid groups B_n, satisfy the Baum–Connes conjecture.*

Proof It is a classical result that the full braid group is torsion-free, and it is a finite extension of the pure braid group. Theorem 19 implies that we can apply Corollary 14 to the extension

$$1 \to P_n \to B_n \to B_n/P_n \to 1,$$

or the corresponding general exact sequence, and the assertion follows. □

In Corollary 14 we can relax the condition that H belongs to $\mathcal{F}^+$ a little bit.

Proposition 21 *Assume $G_1, \dots, G_N \in \mathcal{F}^+$, and $Q \in \mathbf{LH}\mathcal{ETH}$ has a finite classifying space, is cohomologically complete, and has enough amenable torsion-free quotient. For example, Q could be a free product $Q = Q_1 * \cdots * Q_m$ of finitely many groups $Q_i \in \mathcal{F}^+$. Define*

$$G := G_1 \rtimes \big(G_2 \rtimes (\cdots (G_N \rtimes Q))\big),$$

where each semidirect product is H_1–trivial. Then G has a finite classifying space, enough amenable torsion-free quotients, and is cohomologically complete. If there is an exact sequence

$$1 \to G \to H \to A \to 1$$

with H torsion-free such that A satisfies the Baum–Connes conjecture, then the same is true for H.

Proof In Linnell–Schick [11, Proposition 4.30] we prove that G has the desired properties. Because $\mathbf{LH}\mathcal{ETH}$ is closed under extension, $G \in \mathbf{LH}\mathcal{ETH}$, in particular G fulfills the Baum–Connes conjecture. The last statement now follows from Theorem 10. □

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