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On the virtual Betti numbers of arithmetic hyperbolic 3–manifolds

D COOPER
D D LONG
A W REID

We show that closed arithmetic hyperbolic 3–manifolds with virtually positive first Betti number have infinite virtual first Betti number. As a consequence, such manifolds have large fundamental group.

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1 Introduction

The main result of this paper is the following.

Theorem 1.1 *Suppose that M is a closed arithmetic hyperbolic 3–manifold which virtually has positive first Betti number.*

Then M has infinite virtual Betti number.

An interesting feature of our argument is that although it uses arithmetic in an essential way, it is largely geometric; in particular there is no use of Borel's theorem [2]. This makes Theorem 1.1 strictly stronger than [2] in this setting, since no congruence assumptions are made.

We recall that a group is said to be *large* if it has a subgroup of finite index which maps onto a free group of rank two. We remark that large is known in the setting of bounded hyperbolic 3–manifolds arithmetic or not, see Cooper–Long–Reid [3], so that throughout this paper we restrict our attention to closed manifolds.

Using entirely different ideas, it is shown in Lackenby–Long–Reid [6, Theorem 6.1], that once an arithmetic hyperbolic 3–manifold has Betti number at least four, it is large, so that taken in conjunction with this result, Theorem 1.1 implies a somewhat stronger theorem:

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Theorem 1.2 *Suppose that M is a closed arithmetic hyperbolic 3–manifold which virtually has positive first Betti number.*

Then $\pi_1(M)$ is large.

Combining with classical results of Bonahon and Thurston we have the following corollary.

Corollary 1.3 *If M is closed arithmetic and contains a geometrically infinite surface group, then $\pi_1(M)$ is large.*

Remark Since the writing of this paper, two other papers (Venkataramana [11] and Agol [1]) have appeared which give different proofs of a more general result than Theorem 1.1. In both of these cases, the methods are more in the spirit of [2].

2 Main results

2.1 Preliminaries

For convenience we recall the definition of an arithmetic Kleinian group and arithmetic hyperbolic 3–orbifold (see Maclachlan–Reid [7] for further details).

Arithmetic Kleinian groups are obtained as follows. Let k be a number field having exactly one complex place, and B a quaternion algebra over k which ramifies at all real places of k. Let $\rho\colon B \to M(2,\mathbf{C})$ be an embedding, $\mathcal{O}$ an order of B, and $\mathcal{O}^1$ the elements of norm one in $\mathcal{O}$. Then $\mathrm{P}\rho(\mathcal{O}^1) < \mathrm{PSL}(2,\mathbf{C})$ is a finite co-volume Kleinian group, which is co-compact if and only if B is not isomorphic to $M(2,\mathbf{Q}(\sqrt{-d}))$, where d is a square free positive integer. An *arithmetic Kleinian group* Γ is a subgroup of $\mathrm{PSL}(2,\mathbf{C})$ commensurable with a group of the type $\mathrm{P}\rho(\mathcal{O}^1)$. We call $Q = \mathbf{H}^3/\Gamma$ *arithmetic* if Γ is arithmetic.

An arithmetic Kleinian group Γ is called *derived from a quaternion algebra* if Γ is a subgroup of some group of the form $\mathrm{P}\rho(\mathcal{O}^1)$. It follows from the characterisation theorem for arithmetic Kleinian groups (see Maclachlan–Reid [7, Corollary 8.3.5]), that if Γ is a finite co-volume Kleinian group, then Γ is arithmetic if and only if the group $\Gamma^{(2)}$ (the subgroup of Γ generated by the squares of elements in Γ) is derived from a quaternion algebra.

The following lemma is important in what follows; this is the crucial property of arithmeticity which is used. This is implicit in [6] but we give the proof for completeness.

Lemma 2.1 *Let $M = \mathbf{H}^3/\Gamma$ be an arithmetic hyperbolic 3–manifold. Let $\gamma \subset M$ be a closed geodesic.*

Then there exists a finite sheeted cover $M_\gamma \to M$ such that M_γ admits an orientation-preserving involution τ for which the fixed point set of τ contains a component of the preimage of γ.

Proof We can assume without loss of generality that Γ is derived from a quaternion algebra B/k. Let $a \in \Gamma$ be a hyperbolic element whose axis A_a projects to γ. Let $b \in \Gamma$ be chosen so that its axis A_b is disjoint from A_a. Now the Lie product $ab - ba$ defines an involution $\tau_{a,b}$ for which the axis of rotation is the perpendicular bisector of A_a and A_b in $\mathbf{H}^3$. Denote this geodesic by δ. As shown in [6, Proposition 2.4], there is an order $\mathcal{O}$ of B for which $\tau_{a,b}$ lies in the image in $\mathrm{PSL}(2,\mathbf{C})$ of the normalizer of $\mathcal{O}$ in B. This is an arithmetic Kleinian group commensurable with Γ (see [6] or [7, Chapter 6]). Hence there is a hyperbolic element $g \in \Gamma$ whose axis A_g is the geodesic δ.

It follows that A_a is now the perpendicular bisector of the axes A_g and aA_g. Repeating the argument of the previous paragraph provides an involution fixing A_a (namely arising from the Lie product of the elements g and aga^{-1}) which lies in an arithmetic Kleinian group Δ commensurable with Γ. To complete the proof, take the core of $\Gamma \cap \Delta$ in Δ, and let M_γ be the corresponding cover of M. □

Remark It is a famous result of Margulis that a closed hyperbolic manifold is *not* arithmetic if and only if there is a unique minimal element in the commensurability class. From this theorem, it is not difficult to show that the property described in Lemma 2.1 is actually a characterisation of arithmeticity. See Reid [8].

We will also use some results from Hass [5] concerning least area surfaces. There it is shown in particular that if M is a closed orientable hyperbolic 3–manifold, then given any non-zero class in $H_2(M)$, (not necessarily primitive), there is a smooth immersion with embedded image achieving the minimal area over all immersed surfaces representing that class (see [5, Lemma 2.1]). A key fact for us will be the following.

Theorem 2.2 [5, Theorem 2.3] *Let M be a closed hyperbolic 3–manifold and suppose that G_1 and G_2 are embedded closed oriented surfaces, homologous up to sign and least area for the class that they represent.*

Let F_1 and F_2 be connected components of G_1 and G_2 respectively, then either $F_1 \cap F_2$ is empty or F_1 and F_2 coincide.

2.2 Increasing the homology of an arithmetic hyperbolic 3–manifold.

In this section we prove Theorem 1.1 modulo Theorem 2.5, which will be proved in Section 2.3.

We begin with a simple lemma.

Lemma 2.3 *Let $p\colon \widetilde{M} \longrightarrow M$ be a finite sheeted covering and suppose that $H_2(\widetilde{M})$ has the same rank as $H_2(M)$.*

Fix a connected embedded oriented closed surface F in M representing some nonzero class in $H_2(M)$ and let $\widetilde{F}_1, \dots, \widetilde{F}_k$ be the components of $p^{-1}(F)$.

Then for every i, j, $[\widetilde{F}_i] = \pm[\widetilde{F}_j]$ in $H_2(\widetilde{M})$.

Proof Observe that $p_*[\widetilde{F}_i]$ and $p_*[\widetilde{F}_j]$ are both integral multiples of $[F]$, and since the ranks of $H_2(\widetilde{M})$ and $H_2(M)$ are the same, the map p_* is a rational isomorphism. It follows that

$$b[\widetilde{F}_i] = a[\widetilde{F}_j]$$

for integers a and b. Now $[F]$ is nonzero, whence so are all the $[\widetilde{F}_i]$ classes, and since they are all connected embedded surfaces, they represent primitive classes in $H_2(\widetilde{M})$. It follows that $a = \pm b$ as required. □

Lemma 2.4 *In the notation of Lemma 2.3, suppose in addition that F is least area for $[F]$.*

Then every $\widetilde{F}_i$ has the same area and this is least area for the class $[\widetilde{F}_i]$.

Proof Write $p_*[\widetilde{F}_1] = a[F]$ where a is the degree of the restriction of the covering map p to $\widetilde{F}_1$. It follows that the area of $\widetilde{F}_1$ is $a \cdot \mathrm{Area}(F)$. However, Lemma 2.3 implies that each component of $p^{-1}(F)$ represents the same homology class, so each projects under p_* to $a[F]$, and so they all have the same area.

Next we observe that a least area representative for $a[F]$ has area $a \cdot \mathrm{Area}(F)$. The reason is this. Clearly the least area representative has area at most $a \cdot \mathrm{Area}(F)$. Now by [5], the actual least area surface is represented by an embedded oriented possibly disconnected surface S (where we allow multiplicities). We recall (see Thurston [10, Lemma 1]) that if such an embedded surface represents the class $a[F]$, then S can be written as a union of a subsurfaces each of which represents $[F]$. Each of these subsurfaces must have area at least $\mathrm{Area}(F)$, so that the area of S is at least $a \cdot \mathrm{Area}(F)$.

The argument may now be concluded as follows. If F' in $\widetilde{M}$ is any (necessarily embedded) least area surface representing the primitive class $[\widetilde{F}_1]$, we see that $a[F] = p_*[F'] = [p(F')]$, so that the immersion $p(F')$ represents the class $a[F]$ and therefore has area at least $a \cdot \mathrm{Area}(F)$. Whence this is the minimum for the class $[\widetilde{F}_1]$. $\square$

Remark As pointed out in [5], a least area representative for a homology class may not be connected, even if the class in question is primitive. However each of the above lemmas applies to the components of the preimages of each component of a least area representative. This means that if $G = \cup G_j$ is least area with components G_j, then each component is least area in its homology class and we may therefore apply the above to the decomposition of $p^{-1}(G_j)$ into components.

We also remark that it is shown in the course of the proof of Lemma 2.4 that the least area function is linear on rays.

In Section 2.3 we shall prove the following theorem.

Theorem 2.5 *Let M be a closed hyperbolic 3–manifold. Then there is a closed geodesic η with the property that it has at least one non-right angle transverse intersection with every least area surface in M.*

Proof of Theorem 1.1 assuming Theorem 2.5

Let M be an arithmetic hyperbolic 3–manifold which we can assume has positive first Betti number. If $\pi_1(M)$ is large, then we are done, so we may assume that this is not the case.

Let η be a geodesic in M of the type promised by Theorem 2.5. By Lemma 2.1, we may take a finite sheeted covering $\widetilde{M} \longrightarrow M$ so that some component $\tilde{\eta}$ of the preimage of η is a closed geodesic which is part of the fixed point set of some involution $\tau\colon \widetilde{M} \longrightarrow \widetilde{M}$.

We claim that the rank of $H_2(\widetilde{M})$ must be strictly larger than that of $H_2(M)$. Suppose to the contrary they have the same rank, so that $p_*\colon H_2(\widetilde{M}) \longrightarrow H_2(M)$ is a rational isomorphism. Pick a connected embedded surface F, whose homology class represents an eigenvalue ± 1 for the action of τ_* on $H_2(\widetilde{M})$. Consider the class $p_*[F] \in H_2(M)$, this might not be primitive, so take a least area embedded surface G in M representing the primitive class. Hence we may write $p_*[F] = a[G]$ for some integer a.

As observed above, in general a least area representative of a class might not be connected even if the class is primitive. However in our setting we may suppose that G is connected. For if it were not connected, write $G = \cup G_j$ a union of components.

If G_i and G_j are independent classes in rational homology, then we have exhibited two surfaces whose union doesn't separate, whence $\pi_1(M)$ maps to a free group of rank two and we had already supposed that this did not happen. Thus all the G_j's lie in the same one-dimensional rational subspace, whence G only has one component by primitivity.

Notice that $p^{-1}(G)$ consists of components each of which is an embedded surface and therefore a primitive class in $H_2(\widetilde{M})$. Furthermore, since p_* is a rational isomorphism, $[F] = \pm[G^*]$, where G^* is any choice of a component of $p^{-1}(G)$. It follows we have that $\tau_*[G^*] = \pm[G^*]$. Moreover, by Lemma 2.4, any such G^* is a least area surface in the homology class $[G^*] = \pm[F]$.

By choice of η, there is at least one component of the preimage of G which has a non-right angle transverse intersection with $\widetilde{\eta}$, make this choice for G^*. However, using this surface contradicts Theorem 2.2, since the surfaces G^* and $\tau(G^*)$ are homologous up to orientation, least area, yet they meet without coinciding for angle reasons. The proof of Theorem 1.1 is now completed by repeated application of this argument. □

2.3 Finding the geodesic η.

This section is devoted to proving Theorem 2.5. Here is the outline of the proof. First, we show that it follows from a result of R Schoen that a least area surface looks like a totally geodesic surface on a very small scale in the sense that there is a $\delta > 0$, independent of the surface, so that discs of radius δ in the surface are very close to totally geodesic discs of radius δ. We then argue that we can find a very large collection of geodesic arcs with the property that given a point on any least area surface, there is at least one arc in the collection which punctures the surface near to this point, transversely at an angle close to $\pi/4$. Finally, we argue that there is a closed geodesic which runs sufficiently close to this family of arcs, that it must meet every least area surface transversely at an angle fairly close to $\pi/4$.

Suppose S is a surface smoothly immersed in a 3–manifold M. Let $\nu(p)$ be the unit normal vector at a point p in S. Choose an orthonormal basis e_1, e_2 of T_pS. The second fundamental form on S at p is given in this basis by the 2×2 matrix A with entries

$$a_{ij} = \langle \nabla_{e_i}\nu, e_j\rangle.$$

Here ∇ is the covariant derivative on M. The *length squared* of the second fundamental form is

$$|A|^2 = \sum_{i,j=1}^{2} a_{ij}^2.$$

We shall be concerned with the case that M is hyperbolic 3-space. Let B denote the unit ball $\{\|x\| < 1 : x \in \mathbb{R}^3\}$. The Poincaré metric on B is given by $ds^2/(1-\|x\|^2)^2$ where ds is the Euclidean metric on $\mathbb{R}^3$. In fact this metric has constant curvature -4, but for simplicity we will ignore this. Let P denote a plane in $\mathbb{R}^3$ containing the origin. Given $p \in B$ and $v \in T_pB$ we use $\|v\|_E$ for the Euclidean norm and $\|v\|_H$ for the hyperbolic norm. Recall that the Euclidean and hyperbolic angle between vectors in T_pB coincide.

The following says that surfaces in hyperbolic space with everywhere bounded second fundamental form are uniformly almost flat.

Lemma 2.6 *Given $K, \epsilon > 0$ there is $\delta > 0$ with the following property. Suppose that S is a smooth surface immersed in B, that the induced metric on S is complete, and that the length squared of the second fundamental form on S (with respect to the hyperbolic metric) is everywhere less than K. Finally suppose that S is tangent to P at the origin. Let Δ denote the disc in P centered on the origin and of radius δ in the Euclidean metric. Let e denote the constant vector field in $\mathbb{R}^3$ with $\|e\|_E = 1$ and which is orthogonal to P. Let $\nu(p)$ denote the unit normal vector to S at p.*

Then there is a neighborhood, U, of the origin in S which projects along e onto Δ and at every point in U the unit normal vector to S lies within ϵ of e.

Proof The Euclidean coordinate system on $\mathbb{R}^3$ coincides with Riemann normal coordinates for the Poincaré metric on the unit ball at the origin. These are the coordinates given by the exponential map at the origin. Let U and V be smooth vector fields defined on an open subset of B. We may write these vector fields in the usual coordinates on $\mathbb{R}^3$ with components U^i and V^j. We wish to relate the covariant derivatives of V with respect to U in the hyperbolic metric and the Euclidean metric. In the Euclidean metric we have

$$\left(\nabla^E_U V\right)^i = \sum_{j=1}^{3} U^j \frac{\partial V^i}{\partial x^j}.$$

In the hyperbolic metric we get

$$\left(\nabla^H_U V\right)^i = \sum_{j=1}^{3} U^j \frac{\partial V^i}{\partial x^j} + \sum_{j,k=1}^{3} \Gamma^i_{jk} U^j V^k.$$

Here Γ^i_{jk} are the Christoffel symbols defined in terms of the metric tensor g_{ij} by

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{m=1}^{3} g^{im} \left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right).$$

One may easily compute this using that $g_{ij} = \delta_{ij}(1 - \|x\|^2)^{-2}$ however we do not need to do this. Thus

$$\left(\nabla^H_U\ V\right)^i - \left(\nabla^E_U\ V\right)^i = \sum_{j,k=1}^{3} \Gamma^i_{jk} U^j V^k.$$

In what follows we make various estimates in the case x is very close to the origin so that $\|x\|_E$ is very small. In normal coordinates $\Gamma^i_{jk}(0) = 0$ thus $\Gamma^i_{jk}(x) = O(\|x\|_E)$ and

$$\left(\nabla^H_U\ V\right)^i - \left(\nabla^E_U\ V\right)^i = O(\|x\|_E) \cdot \|U\|_E \cdot \|V\|_E. \tag{1}$$

We will now apply this to the computation of the second fundamental form. Take $V = \nu(x)$ to be normal vector (in the hyperbolic metric) to S at x with $\|\nu\|_H = 1$. Let $e_1, e_2 \in T_x S$ be an orthonormal basis in the hyperbolic norm. The given bound on the second fundamental form implies

$$|\langle \nabla^H_{e_i} \nu, e_j \rangle_H| \le \sqrt{K}. \tag{2}$$

We now translate this into a bound for the Euclidean metric. In normal coordinates $g_{ij}(x) = \delta_{ij} + O(\|x\|^2_E)$ thus if $v \in T_x B$ then

$$\|v\|_H = (1 + O(\|x\|^2_E)) \cdot \|v\|_E.$$

The vectors $f_i = e_i \cdot \|e_i\|^{-1}_E$ are a Euclidean orthonormal basis of $T_p S$ and $\nu \cdot \|\nu\|^{-1}_E$ is the Euclidean normal to S of Euclidean length 1. We wish to bound the second fundamental form on S with respect to the Euclidean metric on $\mathbb{R}^3$. Thus we need to bound

$$\langle \nabla^E_{f_i}(\nu \cdot \|\nu\|^{-1}_E), f_j \rangle_E.$$

The covariant derivative $\nabla_a b$ is a linear function of a thus

$$\begin{aligned} \langle \nabla^E_{f_i}(\nu \cdot \|\nu\|^{-1}_E), f_j \rangle_E &= \left(\frac{\|f_i\|_E}{\|e_i\|_E} \frac{\|f_j\|_E}{\|e_j\|_E} \right) \langle \nabla^E_{e_i}(\nu \cdot \|\nu\|^{-1}_E), e_j \rangle_E \\ &= (1 + O(\|x\|^2_E))\ \langle \nabla^E_{e_i}(\nu \cdot \|\nu\|^{-1}_E), e_j \rangle_E. \end{aligned} \tag{3}$$

Using the product rule

$$\begin{aligned} \nabla^E_{e_i}(\nu \cdot \|\nu\|^{-1}_E) &= \left(\|\nu\|^{-1}_E\right) \nabla^E_{e_i} \nu + \left(\nabla^E_{e_i}\left(\|\nu\|^{-1}_E\right)\right) \nu \\ &= (1 + O(\|x\|^2_E)) \nabla^E_{e_i} \nu + O(\|x\|_E) \nu. \end{aligned} \tag{4}$$

From (1) and using that $\|e_i\|_E = 1 + O(\|x\|^2_E) = \|\nu\|_E$ gives

$$\nabla^E_{e_i} \nu = \nabla^H_{e_i} \nu + O(\|x\|_E).$$

Substituting this into (4) gives

$$\nabla^E_{e_i}(\nu \cdot \|\nu\|_E^{-1}) = \nabla^H_{e_i}\nu + O(\|x\|_E).$$

Substituting into (3) yields

$$\langle \nabla^E_{f_i}(\nu \cdot \|\nu\|_E^{-1}), f_j\rangle_E = \langle \nabla^H_{e_i}\nu, e_j\rangle_E + O(\|x\|_E).$$

Using the bound from (2) gives

$$|\langle \nabla^E_{f_i}(\nu \cdot \|\nu\|_E^{-1}), f_j\rangle_E| \le \sqrt{K} + O(\|x\|_E) \le \sqrt{2K}.$$

The length squared of the second fundamental form, A_E, for S in the Euclidean metric is then

$$|A_E|^2 \le 8K.$$

This holds on the part of S lying within some fixed distance of the origin, independent of S. For a surface in Euclidean space, the rate of turning of the normal vector at a moving point p in the surface is bounded above by the velocity of p times $|A_E|$. Thus if p is a point on S which can be joined to the origin by a path in S of length L then the angle between $\nu(p)$ and $\nu(0)$ is at most $L \cdot |A_E|$. This easily implies the lemma. □

In [9], Schoen proved the following theorem.

Theorem 2.7 [9, Theorem 3, first part] *Let S be an immersed stable surface in M^3. Given $r_0 \in (0,1]$ and a point $P_0 \in S$ such that $B_{r_0}(P_0)$ has compact closure in S then there is a constant c_{20} depending only on K_{P_0,r_0} so that*

$$|A|^2(P_0) \ \le \ c_{20} r_0^{-2}.$$

We may make certain simplifications since we are dealing with a compact least area surface S immersed in a hyperbolic 3–manifold M. We recall that a least area surface is stable, and since S is compact every ball in S has compact closure. Thus we may use $r_0 = 1$ in the theorem. The quantity $K_{p,r}$ is defined as

$$K_{p,r} = \sup_{\mathrm{B}^3_\mathrm{r}(\mathrm{p})}\{|\operatorname{curv}|, |\mathrm{D}\operatorname{curv}|\}.$$

Here $B^3_r(p)$ is a ball in M and curv refers to the curvature tensor on M and D is the covariant derivative. Since $\mathbf{H}^3$ is homogeneous $K_{p,r}$ is a universal constant independent of p and r. Thus c_{20} is therefore a universal constant for least area surfaces in hyperbolic 3–manifolds. We thus obtain a universal bound on the second fundamental form of a least area surface in hyperbolic 3–space.

Corollary 2.8 ([9]) *There is a universal constant c_{20} such that if S is a least area surface in a hyperbolic 3–manifold M and p is a point on S then $|A|^2(p) \leq c_{20}$.*

Combining this with our lemma gives the closeness theorem announced at the beginning of this section.

Theorem 2.9 *Given $\epsilon > 0$ there is $\delta > 0$ with the following property. Suppose that S is a closed least area surface immersed in a complete hyperbolic 3–manifold M. Let $\pi\colon \widetilde{M} \to M$ denote the universal cover of M and $\widetilde{S}$ a component of $\pi^{-1}(S)$. Given $p \in \widetilde{S}$ identify $\widetilde{M}$ with the Poincaré ball model of hyperbolic space in such way that p is identified to the origin and $T_p\widetilde{S}$ is the xy–plane. Let Δ denote the disc in the xy–plane centered on the origin and of radius δ in the Euclidean metric.*

Then there is a neighborhood U of p in S which projects onto Δ under the Euclidean projection $\pi(x,y,z) = (x,y)$. Furthermore at every point q of U the normal vector $\nu(q)$ to S at q satisfies $\|\nu - (0,0,1)\| < \epsilon$.

We now show that there is a family of arcs which meet any least area surface transversely at an angle bounded away from $\pi/2$. To this end, we make the following definition.

Definition Fix an $\epsilon > 0$. A *geodesic arc field* in M is a collection of geodesic arcs each of length 2ϵ.

It follows from Theorem 2.9 that there is a $\delta > 0$ which can be chosen independently of the least area surface F, so that at any point $x \in F$, the surface near to F is given by a uniformly controlled graph over the totally geodesic δ–ball centred at x and tangent to F at x.

In particular the small variation of the normal vector implies the following. Fix, once and for all an $\epsilon > 0$ very small compared to the injectivity radius of M. Then there is a $\xi > 0$ with the following property. Choose any geodesic arc α centred at x whose length is 2ϵ and whose tangent vector at x makes an angle of $\pi/4$ to the vector in $T_x(M)$ which is orthogonal to $T_x(F)$. Then any geodesic arc of length 2ϵ which is within ξ of α in the Hausdorff metric meets F transversely near to x and at an angle near to $\pi/4$.

Denoting the unit tangent bundle of M by $UT(M)$, to each point $(x, \mathbf{v}) \in UT(M)$, we may associate a geodesic arc in M, namely the geodesic arc of length 2ϵ centred at x and tangent to $\mathbf{v}$ at x. Denote this arc by $\alpha(x,\mathbf{v})$. Since $UT(M)$ is compact, given the ξ of the previous paragraph, we may find a finite collection of points $\{(x_n, \mathbf{v}_n)\}$

with the property that if $(x, \mathbf{v})$ is any point in $UT(M)$, then $\alpha(x, \mathbf{v})$ is within $\xi/10$ of at least one arc $\alpha(x_n, \mathbf{v}_n)$ in the Hausdorff topology on M.

We claim that the geodesic arc field defined by the collection $\{\alpha(x_n, \mathbf{v}_n)\}$ has the property that given any least area surface F, there is an arc in the field which has transverse intersection with F making an angle close to $\pi/4$. The reason is this. Pick any $x \in F$ and let $\mathbf{v}$ be the vector in $T_x(M)$ normal to the tangent plane to F at x.

Consider the point $(x, \mathbf{v}^*)$ where $\mathbf{v}^*$ is any vector in $T_x(M)$ at angle $\pi/4$ to $\mathbf{v}$. As above, the geodesic arc $\alpha(x, \mathbf{v}^*)$ pierces the surface F at x at an angle $\pi/4$ to the normal vector. By construction there is an arc of our geodesic arc field $\alpha(x_k, \mathbf{v}_k)$ which is within $\xi/10$ of $\alpha(x, \mathbf{v}^*)$ in the Hausdorff topology and therefore pierces F very close to x and at an angle very close to $\pi/4$ as required.

Finally, we find a closed geodesic η which runs sufficiently close to every arc in the geodesic arc field. This threads a closed geodesic close to a geodesic arc field.

Theorem 2.10 *Given a geodesic arc field and $\xi > 0$ as above, we may find a closed geodesic η with the property that given any arc α in the geodesic arc field, it is within $\xi/10$ in the Hausdorff metric of least one subarc of η.*

Proof There is a geodesic which is dense in the unit tangent bundle of the manifold (Eberlein [4]), so given ξ we can find a geodesic arc a with the property that given any arc α in the geodesic arc field, there is a subarc of a which is within $\xi/100$ of α in the Hausdorff topology. It remains to show that we can improve a to a closed geodesic with this property with a slightly worse constant. A simple argument that shows this can be done is the following. Extend a some very long distance at both ends, (where our ξ determines how long we should extend) then some distance further if necessary so that the new (greatly extended) arc a^+ has its endpoints very close in the unit tangent bundle, again, the necessary closeness being determined by ξ. These lengths were chosen sufficiently large that we can join these ends of the arc together in M so that the geodesic curvature of the resulting curve C is very close to zero. Let η be the geodesic in the free homotopy class of C. The curves C and η cobound an annulus which by Gauss–Bonnet has very small area. It follows that all along the length of the arc $a \subset C$, a and η are extremely close, less than $\xi/100$ say, since otherwise too much area is contributed to the annulus. Thus each arc of the geodesic arc field is within at most $\xi/50$ of some subarc of η. $\square$

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Department of Mathematics, University of California,
Santa Barbara CA 93106, USA

Department of Mathematics, University of California,
Santa Barbara CA 93106, USA

Department of Mathematics, University of Texas,
Austin TX 78712, USA

cooper@math.ucsb.edu, long@math.ucsb.edu, areid@math.utexas.edu

Proposed: Dave Gabai Received: 13 December 2006
Seconded: Martin Bridson, Walter Neumann Accepted: 5 September 2007