Flexing closed hyperbolic manifolds.

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We show that for certain closed hyperbolic manifolds, one can nontrivially deform the real hyperbolic structure when it is considered as a real projective structure. It is also shown that in the presence of a mild smoothness hypothesis, the existence of such real projective deformations is equivalent to the question of whether one can nontrivially deform the canonical representation of the real hyperbolic structure when it is considered as a group of complex hyperbolic isometries. The set of closed hyperbolic manifolds for which one can do this seems mysterious.

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1 Introduction

As remarked by Schwartz [20], it is a basic problem to understand how discrete faithful representations $\rho\colon \Gamma \longrightarrow G_0$ can be deformed if we extend the Lie group G_0 to a larger Lie group G_1. The best understood example of this is the case of *quasifuchsian deformation*, where Γ is the fundamental group of a closed orientable surface and the Lie group pair (G_0, G_1) is $(PSL(2, \mathbb{R}), PSL(2, \mathbb{C}))$. However, there has been an enormous amount of interesting work, we mention the exploration of bending constructions and related issues, which concern the case $(SO_0(3, 1), SO_0(4, 1))$, see for example Johnson and Millson [13], as well as Goldman and Parker [10] and Schwartz [21], which prove results about triangle groups for the pair $(SO_0(2, 1), PU(2, 1))$.

By considering the Klein model one sees that a hyperbolic structure is in particular a projective structure and this paper is devoted to results originally inspired by the question: Can the hyperbolic structure on a closed hyperbolic 3–manifold be deformed to a nontrivial real projective structure? In the framework described above, this is the case $(SO_0(3, 1), PGL(4, \mathbb{R}))$; see below. Our first result is that with a mild smoothness hypothesis, the existence of such real projective deformations is in some sense equivalent to the existence of a deformation of the representation into the isometry group of complex hyperbolic space:

Theorem 1.1 *Suppose that $\rho\colon \pi_1(M) \longrightarrow SO_0(n, 1)$ is a representation of a closed hyperbolic n–manifold which is a smooth point of the representation variety $V = \mathrm{Hom}(\pi_1(M), PGL(n+1, \mathbb{R}))$. Then ρ is a smooth point of $\mathrm{Hom}(\pi_1(M), PU(n, 1))$, and further,*

$$\dim_{\mathbb{R}}(\mathrm{Hom}(\pi_1(M), PU(n, 1))) = \dim_{\mathbb{R}}(\mathrm{Hom}(\pi_1(M), PGL(n+1, \mathbb{R})))$$

near ρ.

In fact this is a result about groups (see Theorem 2.2) and so can be applied to orbifolds and in particular the question of deforming triangle groups mentioned above. For, results of Goldman and Choi have shown in [7] that the deformation space for real projective structures on a closed hyperbolic 2–orbifold of negative Euler characteristic is a union of cells of dimension $-8\chi(S)+6k-2b$, where S is the underlying closed surface and k is the number of cone points, not of order two, and b is the number of cone points of order two. Thurston has shown that the dimension of the space of real hyperbolic deformations is $-3\chi(S) + 2k$, so that provided $-8\chi(S) + 6k - 2b > -3\chi(S) + 2k$ there are hyperbolic structures which can be deformed to nonhyperbolic real projective structures. Theorem 1.1 now implies that these hyperbolic structures can also be deformed to (genuine) complex hyperbolic structures. The most interesting and intensively studied case is that of a triangle group; the above computations show that these are rigid for deformations into $PSL(2, \mathbb{C})$, however, at any point which corresponds to a smooth point of the space of real projective structures, since $-8 \cdot 2 + 6 \cdot 3 > 0$, Theorem 1.1 proves the existence of complex hyperbolic deformations. Goldman–Parker [10] and Schwartz [21] have extensively investigated deformations of this type (see also related references).

Moreover, it turns out that if the original representation is the discrete faithful representation, then sufficiently close to ρ in $\mathrm{Hom}(\pi_1(M), PU(n, 1))$, the deformed representations are also discrete and faithful. We are indebted to the referee for pointing out that this result (with more or less the same proof) is originally due to Guichard [11, Theorem 2].

Theorem 1.2 *In the notation above, there is a small neighbourhood of the discrete faithful representation $\rho\colon \pi_1(M) \longrightarrow SO_0(n, 1)$ so that the deformed representations which have image in $\mathrm{Isom}(\mathbb{CH}^n) \cong PU(n, 1)$ are discrete and faithful.*

In particular, the deformations predicted above for triangle groups are discrete and faithful near to the canonical representation into $SO_0(2, 1)$. This special case $n = 2$ was first proved by Parker and Platis [18] although in this case by a somewhat different method.

Using this technique we can actually go further: we are also able to give the first examples which deform the discrete faithful representation of a closed hyperbolic 3–manifold from $SO_0(3, 1)$ to a discrete faithful representation into $PU(3, 1)$. (Examples of *discrete* representations of certain hyperbolic 3–manifold groups into $PU(2, 1)$ are already known, see Falbel [9] and Schwartz [22].)

Our examples are constructed by exhibiting closed hyperbolic manifolds where the dimension of the space of cocycles $\dim_{\mathbb{R}}(Z^1(\pi_1(M); \mathfrak{pgl}(n+1, \mathbb{R})))$ at the discrete faithful representation is strictly bigger than $\dim(PGL(n+1, \mathbb{R})) = (n+1)^2 - 1$ and then by exhibiting enough deformed representations (corresponding to nontrivial real projective structures) to guarantee that we have a smooth point of the representation variety. In the closed case, nearby structures provided by Theorem 1.1 are necessarily not real hyperbolic by Mostow rigidity and therefore we obtain deformations of the discrete faithful representation into $PU(3, 1)$.

We note that one simple way to produce such a deformation is to take a manifold M which contains an embedded totally geodesic surface and use the bending construction, for example as described by Johnson and Millson [13]. However, (most of) the examples of Appendix A contain no immersed totally geodesic surface (and therefore for arithmetic reasons no nonelementary Fuchsian subgroup) and indeed, some are known to contain no closed incompressible surface at all. It is an interesting and intriguing open question to understand how the deformation happens in these examples. Since there seems to be nothing obvious along which these examples are bending, we refer to this type of deformation as *flexing* and such manifolds we call *flexible*.

A fundamental but as yet unresolved issue is the question of which closed hyperbolic manifolds are flexible in this sense. Noting that $\dim_{\mathbb{R}}(PGL(n+1, \mathbb{R})) = \dim_{\mathbb{R}}(PU(n, 1))$ and since the proof of Theorem 1.1 holds with $PGL(n+1, \mathbb{R})$ and $PU(n, 1)$ exchanged, this theorem shows in particular that in the presence of the smoothness hypothesis, nontrivial real projective structures near to ρ are somehow equivalent to complex hyperbolic flexings near ρ. However the question of which hyperbolic manifolds admit a nonstandard real projective structure seems mysterious. We note that of the 4500 two generator manifolds in the census, there are 61 which are infinitesimally deformable, of these 25 have been rigorously proven to be flexible and there is compelling evidence that 27 of the remaining ones are flexible. (Three of the other nine have been proven rigid.) Thus it would seem that flexing is fairly rare for the census 3–manifolds. However, we should point out that flexing is in fact a good deal more common than deformations into $SO_0(4, 1)$ - only *one* manifold in the first two thousand or so in the census seems to deform into $SO_0(4, 1)$. In this sense, the complex hyperbolic deformation problem (and indeed closely related question of the existence of a nontrivial real projective structure) is perhaps more natural than its $SO_0(4, 1)$ counterpart.

Examples exist of manifolds which are infinitesimally rigid and therefore cannot flex, but have finite sheeted coverings which do flex. (See Section A.1.2) Indeed, there are examples which have, in a natural sense, one dimension's worth of flexes, but are finitely covered by manifolds with two dimensions' worth of flexes. And while most of the small volume flexible examples have rational Chern–Simons invariant, there are deformable examples for which the Chern–Simons invariant is apparently irrational. (See Section A.1.3). The absence of any obvious common thread, as well as the virtual behaviour noted above leads us to conjecture (weakly) that perhaps this is the general situation:

Conjecture 1.3 *Every closed hyperbolic 3–manifold is virtually flexible.*

This is equivalent to the statement that every closed hyperbolic 3–manifold has a finite sheeted covering in which the canonical representation can be deformed to a nonhyperbolic real projective structure.

This paper is organized as follows. In Section 2, we prove Theorem 1.1 and prove the fact that the nearby representations into $PU(3, 1)$ are discrete and faithful. We then touch briefly upon the difficult question of degenerations. Using Chuckrow's theorem in the complex hyperbolic setting it is not difficult to show the following.

Theorem 1.4 *Let $\rho_n \colon \pi_1(M) \longrightarrow \mathrm{Isom}(\mathbb{CH}^n)$ be any family of discrete faithful representations of a closed real hyperbolic 3–manifold. Then one can find a sequence of $\mathrm{Isom}(\mathbb{CH}^n)$ conjugacies so that the sequence $g_n.\rho_n.g_n^{-1}$ subconverges algebraically to a discrete faithful representation $\rho \colon \pi_1(M) \longrightarrow \mathrm{Isom}(\mathbb{CH}^n)$.*

There are several theorems in this spirit in the literature, dating back to the original ideas of Morgan–Shalen, through Bestvina and Paulin. A more general compactness result is proved by Belegradek [2].

While the theorem shows that the limit representation is discrete and faithful, determining exactly what can happen seems to be a hard problem. Even for the simplest example *vol3*, although there is a conjectural value for the limit point, a conjectural description of the degeneration, and a beautiful limit representation for $\pi_1(vol3)$, it appears to be extremely hard to verify this picture and in particular to rule out the possibility of degeneration before this conjectural limit point. This is hardly surprising given the extreme difficulty even in dimension two (see Schwartz [20]).

We go on to describe one example in some detail, and sketch how one can perform some of the necessary computations to prove flexibility as well as rigidity. A zoo of known examples is included as Appendix A. Somewhat surprisingly, amongst small volume hyperbolic manifolds, the vanishing of the cocycle obstruction usually ensures flexibility.

2 Flexing closed hyperbolic 3–manifolds

We recall that a *projective structure* on a closed n–manifold is an atlas of charts $U \to \mathbb{RP}^n$ so that the transition functions are projective maps, which is to say the restrictions of maps of $\mathrm{Aut}(\mathbb{RP}^n) = PGL(n+1, \mathbb{R})$. In the language of (G, X) structures, it is a $(PGL(n+1, \mathbb{R}), \mathbb{RP}^n)$ structure.

For some n, let M be a closed hyperbolic n–manifold and let ϕ_0 be the discrete faithful representation into $SO_0(n, 1)$. (We recall that $SO_0(n, 1)$ is the subgroup of $O(n, 1)$ of matrices of determinant 1 and with the additional property that they do not permute the two sheets of the sphere of radius -1 in Minkowski space.) If $n \geq 3$, then by Mostow rigidity, ϕ_0 is determined up to $O(n, 1)$ conjugacy. From consideration of the Klein model for hyperbolic space, we see that a hyperbolic structure is a special case of a real projective structure. As remarked in the introduction, it seems natural in this setting to ask whether one can nontrivially deform the complete hyperbolic structure to a real projective structure. The following theorem, whose original argument is due to Thurston (see Choi [6] for the extension to orbifolds) shows that in the closed case one can understand such deformations in terms of deformations of the holonomy representation into $PGL(n+1, \mathbb{R})$.

Theorem 2.1 *Let M be a closed real projective n–manifold with associated holonomy representation $\rho\colon \pi_1(M) \to \mathrm{Aut}(\mathbb{RP}^n)$. Then representations sufficiently close to ρ also give a real projective structure.*

Sketch proof (Thurston) The projective structure gives rise to a submersive developing map $D\colon \widetilde{M} \to \mathbb{RP}^n$ which satisfies the equivariance condition $D(g \cdot m) = \rho(g) \cdot D(m)$ with respect to the holonomy. A convenient way to code these structures simultaneously is to let $\pi_1(M)$ act on $\widetilde{M} \times \mathbb{RP}^n$ by $g \cdot (m, x) \to (g \cdot m, \rho(g)(x))$ and to form the flat projective bundle

$$\mathbb{RP}^n \to E(\rho) \cong (\widetilde{M} \times \mathbb{RP}^n)/\pi_1(M) \to M$$

equipped with a section denoted $\sigma\colon m \to [(\widetilde{m}, D(\widetilde{m}))]$. Here $\widetilde{m}$ is any choice of a lift of m to $\widetilde{M}$; the image equivalence class is well defined in $E(\rho)$.

The manifold $\widetilde{M} \times \mathbb{RP}^n$ admits two $\pi_1(M)$–equivariant foliations, and therefore the bundle $E(\rho)$ admits two foliations, namely a vertical foliation, coming from projecting $\{m\} \times \mathbb{RP}^n$ and a horizontal foliation $\mathcal{F}$ which comes by projecting $\widetilde{M} \times \{p\}$. The fact that D is a submersion means that the image of the section $\sigma(M)$ is transverse to $\mathcal{F}$. Locally, the projection along $\mathcal{F}$ of $\sigma(M)$ to the vertical foliation gives the charts of the real projective structure.

Now M is compact, so if we make a sufficiently small perturbation of the holonomy map to a new map ρ', the projection of the section $\sigma(M)$ will continue to be everywhere transverse to the horizontal foliation of the bundle $E(\rho')$. Hence we continue to have local projection maps and hence a real projective structure. $\qquad\square$

It follows from this theorem that if we can find $GL(n+1, \mathbb{R})$ representations near to the complete structure of a closed manifold then (potentially) we have deformed the hyperbolic structure to an inequivalent projective structure. However, there are certain deformations which do not give rise to a different projective structure. For example, the deformations resulting from conjugation by any small matrix in $GL(n+1, \mathbb{R}))$ give rise to a $((n+1)^2-1)$–dimensional family of deformations where the structure continues to be hyperbolic. Also, if there exists an epimorphism $\psi\colon \pi_1(M) \longrightarrow \mathbb{Z}$, there will be deformations of the shape $g \to \alpha_t(\psi(g)).\phi_0(g)$ where $\alpha_t\colon \mathbb{Z} \to GL(n+1, \mathbb{R})$ is some path of homomorphisms into the centre running through the identity; this is another description of the same real projective structure. We shall refer to such deformations of ϕ_0 as *trivial* and concentrate on deformations which do not arise in either of these fashions.

One can begin by looking for infinitesimal obstructions to small deformations; we shall briefly recall how this is done.

Let V denote the irreducible component of $\mathrm{Hom}(\pi_1(M), SL(n+1, \mathbb{R}))$ containing ϕ_0. The first obstruction to the existence of deformed representations is obtained by computation of the so-called Zariski tangent space. We briefly recall that if, for example, the group $\pi_1(M)$ is generated by g_1 and g_2 and they satisfy a relation $w(g_1, g_2) = I$, then we can ask if there is a path of nearby representations through the representation ϕ_0. Such a path is conveniently coded as $(v_t \cdot \phi_0(g_1), w_t \cdot \phi_0(g_2))$ where v_t and w_t are matrices close to the identity matrix. If one regards the condition $w(v_t \cdot \phi_0(g_1), w_t \cdot \phi_0(g_2)) = I$ as a function of t and takes the derivative, one obtains a linear condition providing an obstruction to the existence of such a path, namely that the vector $(\mathbf{v}, \mathbf{w}) = \frac{d}{dt}(v_t, w_t)|_{t=0}$ must lie in the kernel of a certain linear map which is easily computable, using for example the method of group cocycles.

The kernel of this linear map is usually referred to as the *Zariski tangent space* at ϕ_0 and its dimension is an upper bound for the actual dimension of deformations. We shall identify this subspace with the space of group 1–cocycles, and denote it by $Z^1(\pi_1(M); \mathfrak{sl}(n, \mathbb{R}))$ since the representation in question will usually be clear from the context. There is no guarantee that such vectors are integrable, that is, they correspond to actual deformed representations; indeed, there is in general an infinite sequence of obstructions that have to be surmounted, see for example Artin [1]. (In Section 2 we describe a reasonably straightforward method for testing the order 3

obstruction, which has successfully showed that certain manifolds admit non-integrable infinitesimal deformations.) If V is a smooth manifold of dimension r at ϕ_0, the Zariski tangent space at ϕ_0 has dimension at least r; conversely, it is well known that if $\dim(Z^1(\pi_1(M); \mathfrak{sl}(n, \mathbb{R}))) = r$, and if one can exhibit a smooth r–dimensional family of deformations near ϕ_0, then in fact ϕ_0 is a smooth point of V.

Since, as noted above, there are certain deformations which need to be factored out, our computations will be directed towards examples where these considerations are ignored. For example, in most of the cases that we consider the group $H_1(M; \mathbb{Z})$ is finite, and there the only trivial deformations come from conjugacy. It follows that we may restrict attention to those cases where $\dim(Z^1(\pi_1(M); \mathfrak{sl}(n+1, \mathbb{R}))) > (n+1)^2 - 1$; in all the cases computed in this paper $n = 3$.

Computations of the Zariski tangent space are essentially a matter of linear algebra. The first twenty manifolds in the census for which this obstruction vanishes are listed in Section A.1. Somewhat surprisingly, most of these infinitesimal deformations turn out to be integrable.

2.1 The first main result

The next result shows that in the presence of a smoothness hypothesis, nontrivial projective deformations and nontrivial deformations into isometries of complex hyperbolic space imply each other. In this way we can exhibit the first examples of discrete faithful representations of closed hyperbolic 3–manifold groups flexing into $\mathrm{Isom}(\mathbb{C}\mathbb{H}^3)$ where there is no nonelementary Fuchsian subgroup.

We note that for odd n, $PGL(n, \mathbb{R}) = PSL(n, \mathbb{R}) \approx SL(n, \mathbb{R})$, whereas for even n the Lie group $PGL(n, \mathbb{R})$ has two components, of which the component subgroup $PSL(n, \mathbb{R})$ containing the identity is doubly covered by $SL(n, \mathbb{R})$. On the other hand, the natural map $SU(n, 1) \to PU(n, 1)$ is an $(n+1)$–sheeted covering map for each n, as the kernel consists of matrices of form $\mathrm{diag}(\zeta, \dots, \zeta)$, where $\zeta^{n+1} = 1$.

Theorem 1.1 follows easily from the following result, which exploits the fact that $SL(n+1, \mathbb{R})$ and $SU(n, 1)$ are both fixed point sets of antiholomorphic involutions on $SL(n+1, \mathbb{C})$.

Theorem 2.2 *Let Γ be a finitely generated group, and let $\rho\colon \Gamma \to SO_0(n, 1)$ be a smooth point of the representation variety $\mathrm{Hom}(\Gamma, SL(n+1, \mathbb{R}))$.*

Then ρ is also a smooth point of $\mathrm{Hom}(\Gamma, SU(n, 1))$. Further, near ρ the real dimensions of $\mathrm{Hom}(\Gamma, SL(n+1, \mathbb{R}))$ and $\mathrm{Hom}(\Gamma, SU(n, 1))$ are equal.

Proof Let V be the irreducible component of $\mathrm{Hom}(\Gamma, SL(n+1, \mathbb{R}))$ containing ρ, and let $V_{\mathbb{C}} \subset \mathrm{Hom}(\Gamma, SL(n+1, \mathbb{C}))$ be its complexification. The variety V is the zero set in real affine space of the ideal generated by a collection of polynomials $f_1, \ldots, f_m$ in variables $X_1, \ldots, X_n$ with real coefficients, and we may define $V_{\mathbb{C}}$ as the zero set in complex affine space of the same polynomials. The inclusion $\mathbb{R} \subset \mathbb{C}$ induces an embedding of V in $V_{\mathbb{C}}$, and the real dimension of $V_{\mathbb{C}}$ is twice that of V. It is a standard fact that a smooth point p of V is also a smooth point of $V_{\mathbb{C}}$; to see this, observe that the rank of the Jacobian of $f_1, \ldots, f_m$ at p is independent of whether we consider $f_1, \ldots, f_m$ to be functions of real variables or functions of complex variables.

The Lie group $SL(n+1, \mathbb{C})$ admits two antiholomorphic involutions that are of importance in the present context. The first of these, $\sigma \colon A \mapsto \overline{A}$, has as fixed point set $SL(n+1, \mathbb{R})$, and may be regarded as the defining involution for $SL(n+1, \mathbb{R})$. The second involution is that which defines $SU(n, 1)$:

$$\tau \colon A \mapsto J \cdot (A^*)^{-1} \cdot J,$$

where $J = \mathrm{diag}(-1, 1, 1, \ldots, 1)$ is the matrix defining the form of signature $(n, 1)$, and where * denotes conjugate transpose.

The involutions σ, τ induce antiholomorphic involutions on $V_{\mathbb{C}}$, which we also denote σ, τ respectively. The fixed point set (that is, the real form) of σ is precisely V, whereas the fixed point set W of τ consists of representations $\Gamma \to SU(n, 1)$. We note that since $SO_0(n, 1)$ is simultaneously in $SL(n+1, \mathbb{R})$ and in $SU(n, 1)$, the representation ρ is simultaneously in V and in W. It remains for us to show that $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} V_{\mathbb{C}}$, and that ρ is a smooth point of W. This is a consequence of Onishchik and Vinberg [17, Theorem 6, Section 2.3], which shows that by suitably re-embedding $V_{\mathbb{C}}$ in complex affine space the involution τ becomes coordinatewise complex conjugation, whereby one deduces that W is a real form for τ; however, we may establish this very simply and directly by means of the following transversality argument.

A simple computation shows that the map $d\tau_\rho$ acts (again component wise) on the complex tangent space $T_\rho(V_{\mathbb{C}})$ as

$$d\tau_\rho(\mathbf{v}) = -J \cdot \mathbf{v}^* \cdot J$$

and the condition that a vector corresponds to eigenvalue 1 is $\mathbf{v} = -J \cdot \mathbf{v}^* \cdot J$, which is if and only if $\mathbf{v} \in \mathfrak{pu}(n, 1)$. Similarly, eigenvalue -1 corresponds to $\mathbf{v} \in i \cdot \mathfrak{pu}(n, 1)$.

Suppose that $V_{\mathbb{C}}$ is smooth of complex dimension k at ρ. Note that the map $d\tau_\rho$ is an $\mathbb{R}$–linear involution of $T_\rho(V_{\mathbb{C}})$, so that as real vector spaces we have a decomposition

$$V_{\mathbb{C}} \cong V_+ \oplus V_-$$

where $V_{\pm}$ denotes the ± 1–eigenspace for $d\tau_\rho$. The computation above shows that the map $\mathbf{v} \to i \cdot \mathbf{v}$ is an isomorphism of V_+ with V_-; whence each of these subspaces has dimension k.

Now we may choose a Riemannian metric on $V_{\mathbb{C}}$ and average it so that τ is an isometry. Using this metric to define the exponential map at ρ we see that the fixed set of τ near ρ is given as the exponential of a small neighbourhood of the 1 eigenspace for $d\tau_\rho$ acting on $T_\rho(V)$. So $\mathrm{Hom}(\pi_1(M), SU(n, 1))$, which is given as the fixed set of τ near ρ in $V_{\mathbb{C}}$, has dimension k and is smooth as required. $\qquad\square$

Remarks The proof of Theorem 2.2 remains valid if we interchange the roles of $SL(n+1, \mathbb{R})$ and $SU(n, 1)$; this justifies the statement made at the beginning of this section.

Of course, to be interesting, one needs to exhibit examples to which this theorem applies. We defer this issue to Section 2.4 and A.1, where examples are given of deformations of this type. These examples are interesting in that they are not accounted for by the presence of totally geodesic subobjects.

An important consequence for such deformations is the following result. As remarked in the introduction, this was independently proved, using more or less the same ideas, by Guichard [11].

Theorem 2.3 *In the notation above, there is a small neighbourhood of the discrete faithful representation $\rho\colon \pi_1(M) \longrightarrow SO_0(n, 1)$ for which the deformed representations which have image in $\mathrm{Isom}(\mathbb{CH}^n)$ are discrete and faithful.*

Proof Let $G = \pi_1(M)$ and fix some set of generators for G. Form the abstract Cayley graph $\Gamma(G)$ with respect to this set of generators, regarded as usual as a metric space in which all the edges have length 1.

Fix some real vector $\mathbf{v}$ in $\mathbb{R}^{n+1}$, say with norm -1 for the form J of Theorem 2.2. Using the orbit of the projective class of this vector and the same set of generators we can construct an immersed copy of $\Gamma(G)$ by joining $\Gamma(G)$–connected vertices by a geodesic in the J version of real hyperbolic space. Using the fact that M is a closed manifold, we may apply a standard argument of Cannon [5] to show that the obvious map of the abstract metric Cayley graph $\Gamma(G)$ to the copy of $\Gamma(\rho(G))$ lying in real hyperbolic space is a quasi-isometry.

Now regarding the form J as a Hermitian form on $\mathbb{C}^{n+1}$ we can regard the construction of the above paragraph as a map of $\Gamma(G)$ into $\mathbb{CH}^n$. Complex conjugation on $\mathbb{C}^{n+1}$

induces a map on $\mathbb{C}P^n$ which restricted to $\mathbb{C}H^n$ is an isometry leaving precisely the real points fixed. The real hyperbolic space containing the image graph $\Gamma(\rho(G))$ is therefore totally geodesic and it follows that the map

$$\Gamma(G) \longrightarrow \Gamma(\rho(G)) \subset \mathbb{C}H^n$$

is still a quasi-isometry onto its image, in particular, geodesics in the Cayley graph are carried to quasi-geodesics in complex hyperbolic space.

We denote a smooth family of deformations of ρ which have image in $\mathrm{Isom}(\mathbb{C}H^n)$ (for example as prescribed by Theorem 2.2) by ρ_v, where v is some real parameter. We claim that for sufficiently small deformations, the action of the image group $\rho_v(G)$ on $\mathbb{C}H^n$ is discrete and faithful.

Before proving this, we recall some general facts. In a geodesic metric space, an arc α is a (λ, ϵ) quasi-geodesic if the constants $\lambda > 1$ and $\epsilon \geq 0$ have the property that for any two points x and y on α we have

$$d_\alpha(x, y) \leq \lambda d(x, y) + \epsilon$$

Here d_α is the distance measured along the arc α (assumed rectifiable, say) and d is the distance measured in the ambient space. It is a standard fact that in a space with bounded negative curvatures (in particular in a negatively curved group), there is a constant L so that any path which is locally quasi-geodesic over all its subpaths of length $\leq L$ is actually a global quasi-geodesic (with different and worse constants of course). (See Cannon [5].)

Since the complex hyperbolic space $\mathbb{C}H^n$ formed by using J has bounded negative sectional curvatures, we may apply the above observations; we fix once and for all some L as in the above paragraph. (Standard models of $\mathbb{C}H^n$ have curvatures lying in the interval $[-4, -1]$ or $[-1, -1/4]$ depending on convention.)

A small real deformation of v gives rise to only a small difference between the real hyperbolic version $\Gamma(\rho(G))$ (regarded as a subspace of a complex hyperbolic space) and the new complex hyperbolic version $\Gamma(\rho_v(G))$, at least for elements which are fairly close to the identity element. It follows that there is an obvious map from a large ball neighbourhood of the identity element in $\Gamma(\rho(G))$ to a large ball neighbourhood of the identity element in $\Gamma(\rho_v(G))$ which will be a graph isomorphism and a quasi-isometry for ρ_v sufficiently close to the complete structure, in the sense it will carry long quasi-geodesics in the given identity neighbourhood to quasi-geodesics (with a slightly worse constant).

In particular, this gives a map from a large ball neighbourhood of the identity in $\Gamma(G)$ to a large ball neighbourhood of the identity in $\Gamma(\rho_v(G))$ which is a graph isomorphism

and a quasi-isometry. By choosing a ρ_v sufficently close to the complete structure, and a large enough ball, we may arrange that a ball neighbourhood in the Cayley graph maps graph isomorpically and quasi-isometrically onto all the vertices of $\Gamma(\rho_v(G))$ which are (at least obviously) at distance $\leq 100L$ from the identity in complex hyperbolic space.

The action of the group $\rho_v(G)$ is by complex hyperbolic isometries and this local property is therefore global. It follows that every geodesic in the Cayley graph is mapped to a path which is L–locally a quasigeodesic, hence a quasi-geodesic. We deduce that all long geodesic paths in the Cayley graph are mapped so as to have endpoints which are very far apart in complex hyperbolic space, so that ρ_v is discrete and faithful as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remarks (i) The above proof made essential use of the fact that complex hyperbolic space is negatively curved. That this restriction is necessary may be seen by considering the Euclidean annulus $S^1 \times \mathbb{R}$ with holonomy consisting of a discrete group of translations in Isom($\mathbb{E}^2$). Arbitrarily close to this holonomy is a deformed holonomy where the generator maps to a rotation through a small angle fixing a point close to infinity. This deformed holonomy is not a discrete faithful representation of $S^1 \times \mathbb{R}$ and there is no complete Euclidean structure with this holonomy.

(ii) The work of Goldman and Parker [10] has shown that hyperbolic triangle groups, which are rigid in $SO_0(2, 1)$, can be deformed into Isom($\mathbb{CH}^2$). Theorem 2.2 accounts for this, since Goldman and Choi have shown that the deformation space of real projective structures in this setting has dimension $-8 \cdot 2 + 6 \cdot 3 > 0$. Theorem 2.3 now implies that sufficiently small deformations into Isom($\mathbb{CH}^2$) are discrete and faithful.

In fact the argument of Theorem 2.3 proves a little more:

Proposition 2.4 *Suppose that* $\rho\colon \pi_1(M) \longrightarrow$ Isom($\mathbb{CH}^n$) *is a (discrete and faithful) representation for which the Cayley graph formed from the orbit of* $\rho(\pi_1(M))$ *is quasi-isometric to the abstract Cayley graph of* $\pi_1(M)$.

Then there is a small neighbourhood of ρ *in* Hom($\pi_1(M)$, Isom($\mathbb{CH}^n$)) *which consists of discrete and faithful representations and for which the orbit continues to be quasi-isometric to the Cayley graph.*

To place these new deformations in a context, we recall that the most common bending construction in the context of $SO_0(4, 1)$ comes from totally geodesic surfaces; it appears to be not generally appreciated that this bending construction also gives deformations into Isom($\mathbb{CH}^n$).

Theorem 2.5 *Suppose that M is a real hyperbolic closed n–manifold containing a embedded totally geodesic hypersurface.*

Then there is a one (real) parameter family of discrete faithful deformations beginning at the real hyperbolic structure

$$\rho_t \colon \pi_1(M) \longrightarrow \mathrm{Isom}(\mathbb{CH}^n)$$

Proof For simplicity, suppose that F is an embedded separating totally geodesic hypersurface. We may conjugate inside $SO_0(n, 1)$ so that at the complete representation any element of the hypersurface group $\pi_1(F)$ represents as matrices $[1] \oplus K$ where $K \in SO_0(n-1, 1)$. Change basis so that the form on the n–dimensional subspace appears as the diagonal matrix $(1, \ldots, 1, -1)$ and extend orthogonally to the whole space. In these coordinates, the centralizer in $PGL(n+1, \mathbb{C})$ of $\pi_1(F)$ contains the diagonal matrix $(\exp(-ni\theta), \exp(i\theta), \ldots, \exp(i\theta))$ and this is a Hermitian isometry of the form $(1, \ldots, 1, -1)$. It follows that we can conjugate the representations on one side of the n–manifold to obtain the required bending deformation. That these deformations are discrete and faithful for small values of θ is proved as above. $\qquad \square$

A table of the first twenty examples in the census to which this machinery applies is included in Section A.1 (we also include some other provocative examples.) In most cases, the deformations cannot have arisen from a nonelementary totally geodesic subgroup, as there is none. For example, it follows from Maclachlan and Reid [16, Theorem 9.5.5] that the manifold *vol3* contains no immersed, closed, totally geodesic surface (indeed its fundamental group contains no nonelementary Fuchsian subgroup); moreover, N. Dunfield (unpublished) has shown that *vol3* is non-Haken. Therefore the deformations for *vol3* provided by Theorem 2.2 are distinct from (and more subtle than) those coming from Theorem 2.5.

The manifold *vol3* (and indeed several others in that table) contain nothing obvious along which one can "bend"; accordingly, we describe manifolds admitting a nontrivial deformation of this type as *flexible*. The question as to exactly which manifolds are flexible or virtually flexible currently seems very mysterious. Most of the small volume examples have rational Chern–Simons invariant, but there are examples whose Chern–Simons invariant is apparently irrational (see Section A.1.3).

2.2 Degenerations

As in Schwartz [20], once one has succeeded in flexing a representation, it is natural to attempt an understanding of the possible degenerations. This is already an impressively difficult problem in dimension two (see Schwartz [20; 21].) However, in Theorem 2.6

we are able to make a general observation about the nature of limit points, and this can be used to produce the first example of an (almost certainly faithful) nontotally-geodesic, discrete representation of the fundamental group of a closed hyperbolic 3–manifold into the fundamental group of finite covolume complex hyperbolic 3–manifold.

Theorem 2.6   *Let $\rho_n\colon \pi_1(M) \longrightarrow \mathrm{Isom}(\mathbb{C}\mathbb{H}^n)$ be any family of discrete faithful representations of a closed real hyperbolic 3–manifold. Then one can find a sequence of $\mathrm{Isom}(\mathbb{C}\mathbb{H}^n)$ conjugacies so that the sequence $g_n.\rho_n.g_n^{-1}$ subconverges algebraically to a discrete faithful representation $\rho\colon \pi_1(M) \longrightarrow \mathrm{Isom}(\mathbb{C}\mathbb{H}^n)$. Furthermore there is an embedding of M into $N = \mathrm{Isom}(\mathbb{C}\mathbb{H}^n)/\rho(\pi_1 M)$ which is a homotopy equivalence.*

This is proved by using Chuckrow's theorem for complex hyperbolic manifolds; we have not been able to find a proof of this in the literature so we include one for the convenience of the reader.

Theorem 2.7   (Chuckrow's theorem for complex hyperbolic manifolds)   *Suppose that*

$$\rho_k\colon \Gamma \longrightarrow \mathrm{Isom}(\mathbb{C}\mathbb{H}^n)$$

is an algebraically convergent sequence of discrete faithful representations of a group Γ that is not virtually nilpotent. Then the limit representation ρ_∞ is discrete and faithful.

Proof   By the Margulis lemma (see for example, Thurston's book Theorem 4.1.16), there is an $\epsilon > 0$ so that any discrete group generated by elements that are all within a distance ϵ of the identity in $PU(n, 1)$ is virtually nilpotent. This implies:

Lemma 2.8   *Given any $K > 0$, there is an $\eta > 0$ so that if $G = \langle g_1, \ldots, g_p \rangle$ is a discrete group in $PU(n, 1)$ and*

(i) $1 \neq g \in G$ *is distance less than η from the identity in $PU(n, 1)$ and*

(ii) g_i *is distance less than K from the identity in $PU(n, 1)$ for each $1 \leq i \leq p$*

then G is virtually nilpotent.

Proof of Lemma 2.8   We can suppose that we have chosen $\eta < \epsilon$; then by shrinking further if necessary, we can arrange that the elements $g_i g g_i^{-1}$ are all a distance less than ϵ from the identity in $PU(n, 1)$. It follows from the Margulis lemma that the group $\langle g_1 g g_1^{-1}, \ldots, g_p g g_p^{-1} \rangle$ is virtually nilpotent.

It now follows from the classification of complex hyperbolic isometries that this forces $G = \langle g_1, \ldots, g_p \rangle$ to be virtually nilpotent.   □

To prove Chuckrow's theorem, we proceed as follows. Suppose that $\gamma_1, \ldots, \gamma_p$ generate Γ. Then by definition of algebraic convergence, there is $K > 0$ such that for all m and all $1 \le i \le p$ then $\rho_m(\gamma_i)$ is within a distance K of the identity in $PU(n, 1)$. With this K let η be as in the lemma. If the limit is not discrete and faithful then there is m and $1 \ne \gamma \in \Gamma$ such that $\rho_m(\gamma)$ is within a distance η of the identity in $PU(n, 1)$. Since each representation is faithful $\rho_m(\gamma) \ne 1$. Since $\rho_m(\Gamma)$ is discrete, by the Lemma $\rho_m(\Gamma)$ is virtually nilpotent. Since ρ_m is injective, Γ is virtually nilpotent, and this is a contradiction as required. $\qquad\square$

Proof of Theorem 2.6 The manifold M contains no π_1–injective surface of nonnegative Euler characteristic, so by the work of Morgan and Shalen (or see Bestvina [3, Theorem B]), $\pi_1(M)$ cannot act on an $\mathbb{R}$–tree as a group of isometries with small edge stabilisers. However, the space of discrete faithful representations of $\pi_1(M)$ into Isom(X) up to conjugacy (where X is a δ–hyperbolic space) is naturally compactified by such actions (See Bestvina [3] or Paulin [19]) and it follows that the space of such representations is compact. The limit representation must be discrete and faithful by Chuckrow's theorem.

The last conclusion is proved as follows. First observe that M and N are both a $K(\pi_1(M), 1)$ and so there is $f \colon M \to N$ which is a homotopy equivalence. Since $\dim(M) = 3$ and $\dim(N) = 6$ using the Whitney immersion theorem we may homotop f to be an immersion. Since $\dim(M)$ is odd, the self intersection number of M in N is zero, (e.g. for cup product reasons) and since $\dim(W) \ge 6$ using the Whitney trick we can homotop this immersion to a smooth embedding. $\qquad\square$

2.3 Local rigidity

There are actually quite small examples for which the Zariski tangent space obstruction vanishes, but which we can prove fail to be deformable; the first such example is $m149(-4, 1)$. This section sketches the method used to prove:

Theorem 2.9 *The manifolds* $m149(-4, 1)$, $m159(2, 3)$ *and* $m293(4, 1)$ *have nontrivial Zariski tangent space, but are rigid in* $SL(4, \mathbb{R})$ *and* $PU(3, 1)$.

There is a standard cohomological obstruction theory which implies that a Zariski tangent vector is tangent to a smooth path in the representation variety iff each of an infinite sequence of obstructions vanishes.

In order to explain how to compute these obstructions we will take a very elementary approach. The problem is the following. We are given a finitely generated group Γ,

for example the fundamental group of a 3–manifold. The set $V = \mathrm{Hom}(\Gamma, SL(4, \mathbb{R}))$ is then a real algebraic variety defined as the set of zeroes of some finite collection of real polynomials. We are given a representation $\rho \in V$ and wish to show there is no smooth curve γ in V tangent at ρ to a certain tangent vector.

This is a special case of a standard problem in computational real-algebraic geometry. We are given a polynomial map $f \colon \mathbb{R}^m \to \mathbb{R}^n$ and a point, p, in the real affine variety $V = f^{-1}(0)$, and a vector $\vec{v} \in \ker(df_p)$. We wish to show how one might prove that there is no smooth curve γ in V tangent at p to $\vec{v}$.

From the point of view of smooth topology the problem is that f need not be transverse to 0. Typically, even if V is smooth, one may have redundant equations in which case $\dim(V) > m - n$. The function $x \mapsto \mathrm{rank}(df_x)$ in general is only lower semi-continuous: at a singular point, p, one in general only has $\mathrm{rank}(df_x) \geq \mathrm{rank}(df_p)$ for x near p. The idea for proving $\vec{v}$ is not tangent to a smooth curve in V is to think of the curve $\gamma(t)$ as given by a power series in t. One can attempt to compute this power series term-by-term. At each stage, to compute the next term, one must solve a certain finite system of linear equations to obtain the next coefficient in the power series. The obstruction is then whether or not this system of linear equations has a solution.

Our treatment is based on the following elementary result which says that, given a polynomial curve which approximately satisfies some polynomial equations up to degree n, there is an analytic curve which exactly satisfies the equations and equals the given approximation up to degree n. The hypothesis on the equations is a transversality condition. Although we do not need this, the term *polynomial* can be replaced by *real analytic* in the following statement.

Proposition 2.10 *Suppose that $f \colon \mathbb{R}^a \to \mathbb{R}^b$ is a polynomial map and $V = f^{-1}(0)$. Suppose that p is a point in V which has a neighborhood in V which is a smooth manifold and such that $T_p V = \ker(df_p)$. Suppose we are given a polynomial curve $\gamma \colon (-\epsilon, \epsilon) \to \mathbb{R}^a$ through $p = \gamma(0)$ such that $f \circ \gamma(t) = O(t^{n+1})$. More formally this means there is a polynomial map $q \colon (-\epsilon, \epsilon) \to \mathbb{R}^b$ and $\kappa \in \mathbb{R}^b$ such that $f \circ \gamma(t) = t^{n+1} \cdot \kappa + t^{n+2} \cdot q(t)$. Then, after possibly decreasing ϵ, there is a real analytic curve $\eta \colon (-\epsilon, \epsilon) \to V \subset \mathbb{R}^b$ which equals γ up to order t^{n+1}. More formally there is a real analytic $\delta \colon (-\epsilon, \epsilon) \to \mathbb{R}^a$ and $\lambda \in \mathbb{R}^a$ such that $\eta(t) = \gamma(t) + t^{n+1} \cdot \lambda + t^{n+2} \cdot \delta(t)$ and $f \circ \eta \equiv 0$. Furthermore $\kappa = -df_p(\lambda)$.*

Proof There is a neighborhood U of p in $\mathbb{R}^a$ such that the nearest point retraction $\pi \colon U \to V$ is real analytic. We may assume ϵ is small enough that the image of γ is in U and then define $\eta = \pi \circ \gamma$. It remains to check that the first n derivatives of $\eta(t) - \gamma(t)$ vanish at $t = 0$.

Since $T_p V = \ker(df_p)$ there is $K > 0$ such that for all x in U

$$d(x, V) \leq K \cdot \|f(x)\|.$$

The hypothesis on γ implies there is $L > 0$ such that for all t

$$f \circ \gamma(t) \leq L \cdot |t|^{n+1}.$$

Putting these together gives

$$\|\eta(t) - \gamma(t)\| = d(\gamma(t), V) \leq K \cdot L \cdot |t|^{n+1}.$$

This gives the first conclusion. Using the Taylor expansion for f around $\gamma(t)$

$$\begin{aligned}
f(\eta(t)) &= f(\gamma(t) + t^{n+1} \cdot \lambda + t^{n+2} \cdot \delta(t)) \\
&= f(\gamma(t)) + t^{n+1} df_{\gamma(t)}(\lambda) + O(t^{n+2}) \\
&= t^{n+1} \cdot \kappa + t^{n+2} \cdot q(t) + t^{n+1} df_{\gamma(t)}(\lambda) + O(t^{n+2}) \\
&= t^{n+1}(\kappa + df_{\gamma(t)}(\lambda)) + O(t^{n+2}).
\end{aligned}$$

Now we use that $df_{\gamma(t)}(\lambda) = df_{\gamma(0)}(\lambda) + O(t)$ to get

$$f(\eta(t)) = t^{n+1}(\kappa + df_p(\lambda)) + O(t^{n+2}).$$

Since $f \circ \eta(t) \equiv 0$ we get $\kappa + df_p(\lambda) = 0$ as asserted. $\qquad \square$

We will now apply this to prove that certain representations which deform infinitesimally are in fact rigid. In what follows $G = SL(4, \mathbb{R}) \subset \mathbb{R}^{16}$. By choosing a presentation of $\pi_1 M$ with generators $g_1, \ldots, g_m$ and relator words $w_1, \ldots, w_n$ we obtain a polynomial map $f \colon \mathbb{R}^{16m} \to \mathbb{R}^{16n} \times \mathbb{R}^m$ as follows. A point in $\mathbb{R}^{16m}$ may be thought of as a set $(A_1, A_2, \ldots, A_m)$ of matrices in $M_4(\mathbb{R})$ where A_i is the image of g_i. Then $f = (f_1 - I, f_2 - I, \ldots, f_n - I, det_1 - 1, det_2 - 1, \det_m - 1)$ where $f_i = (A_1, \ldots, A_m)$, and det_i is the determinant of A_i, and I is the identity matrix in $SL(4, \mathbb{R})$. Then V may be identified with $f^{-1}(0)$.

Suppose that ρ is an irreducible representation in $V = \mathrm{Hom}(\pi_1 M, G)$. Let U be a small neighborhood of ρ in V. Since $\dim(G) = 15$ and ρ is irreducible, the subspace U_c of U consisting of representations conjugate to ρ is a manifold of dimension 15. *Suppose that the nullity of $df_\rho = 16$.* Then there are two possibilities for U.

Case 1 U is a smooth manifold of dimension 16.

Case 2 $U = U_c$ has dimension 15.

In order to show Case 2 holds we proceed as follows. We assume (for contradiction) that Case 1 holds. This implies that there is a curve in U containing ρ consisting of

representations no two of which are conjugate. Since nullity$(df_\rho) = 16$ the hypotheses of the proposition are satisified by V and f with $p = \rho$.

We choose a vector $v \in \ker(df_\rho)$ that is not tangent to the smooth manifold U_c. If Case 1 holds then $T_\rho = \ker(df_\rho)$ and so there is a smooth curve η in U tangent to v at ρ. We show how a certain computation can contradict this.

Inductively suppose that for some integer $n \geq 1$ we construct a polynomial curve γ_n in $\mathbb{R}^{16m}$ such that $\gamma_n(0) = \rho$ and $\gamma_n'(0) = v$ and $f \circ \gamma_n(t) = t^{n+1} \cdot \kappa + O(t^{n+2})$. We start with $\gamma_1(t) = \rho + t \cdot v$. From the proposition we see that a necessary condition for η to exist is that $\kappa \in \ker(df_\rho)$. This gives a sequence of possible obstructions which can be computed. If any sequence of choices in this construction lead to some γ_n for which $\kappa \notin \ker(df_\rho)$ then Case 2 holds.

Remark Although we will not use this, if all the obstructions vanish then we can construct a formal curve θ given by a power series such that $f \circ \theta$ is formally zero. It then follows from a deep theorem of M Artin [1] that for each integer n there is a real analytic curve η with $f \circ \eta \equiv 0$ and the first n terms in the power series for η and θ are equal. In particular, applying this with $n = 1$, there is a smooth curve in U tangent to v at ρ, and thus case (1) holds.

We will now describe in more detail how the computations can be performed. To avoid ungainly proliferation of notation, we shall make the following simplification. Let us suppose that $\pi_1(M)$ has a two generator, two relator presentation, and to get the necessary smoothness we make the assumption that $\dim_\mathbb{R}(H^1(\pi_1(M); \mathfrak{sl}(4, \mathbb{R}))) = 1$ at ρ. Notice that proving rigidity for $SL(4, \mathbb{R})$ in this setting also proves it for $PU(3, 1)$; since if the representation is nontrivially deformable into $PU(3, 1)$ then the $SL(4, \mathbb{C})$ representation variety is nontrivial at ρ (that is, not entirely accounted for by conjugacy) and therefore smooth at ρ since $1 = \dim_\mathbb{R}(H^1(\pi_1(M); \mathfrak{sl}(4, \mathbb{R}))) = \dim_\mathbb{C}(H^1(\pi_1(M); \mathfrak{sl}(4, \mathbb{C})))$. We may now apply the arguments of Section 2 with the involution being complex conjugacy to deduce that ρ admits smooth $SL(4, \mathbb{R})$ deformations.

We write $\rho(x) = X$ and $\rho(y) = Y$ in $SO_0(3, 1)$ and we seek to examine whether the representation ρ has a nontrivial deformation into $SL(4, \mathbb{R})$; the cohomological simplification mentioned above ensures that if it does, ρ is a smooth point of the $SL(4, \mathbb{R})$ representation variety, which we denote by V.

Fix some vector $(\epsilon_1, \eta_1) \in Z^1(\pi_1(M), \mathfrak{sl}(4, \mathbb{R}))$; the fact that this cochain is a cocycle is the condition that when we substitute into the relators, we see the equation

$$R_i((I + t\epsilon_1)X, (I + t\eta_1)Y) = I + O(t^2) \qquad (i = 1, 2).$$

When these relations are expanded to order 2 we see expressions of the form $R_i((I + t\epsilon_1)X, (I + t\eta_1)Y) = I + t^2 e(2, i)$ where the expressions $e(2, i)$ are matrices formed from sums of matrices of the quadratic shape $W_1(X, Y) \cdot p \cdot W_2(X, Y) \cdot q \cdot W_3(X, Y)$; here $p, q \in \{\epsilon_1, \eta_1\}$ and the W_i's are appropriate subwords of the relevant relator.

Now we attempt to obtain an approximation accurate to second order. Consider the expression $R_i((I + t\epsilon_1 + 1/2t^2 v_2)X, (I + t\eta_1 + 1/2t^2 w_2)Y)$, where $v_2, w_2 \in \mathfrak{sl}(4, \mathbb{R})$. We expand this new expression in powers of t; some new terms of order two are created with the introduction of the vectors v_2 and w_2 and we need to choose these vectors suitably so that $R_i((I + t\epsilon_1 + 1/2t^2 v_2)X, (I + t\eta_1 + 1/2t^2 w_2)Y) = I + t^2(e(2, i) + c_{2,i}(v_2, w_2)) + O(t^3)$ has the zero vector in the coefficient of t^2.

To accomplish this we need to solve the linear system

$$(*) \qquad\qquad df_\rho(v_2, w_2) = -(e(2, 1), e(2, 2))$$

so that we can form a power series with higher order contact if and only if the vector $(e(2, 1), e(2, 2))$ lies in the image of df_ρ.

Assuming that this obstruction vanishes, that is to say, there is a solution to equation $(*)$, we may attempt to repeat this procedure: If we have constructed a power series which vanishes up to degree k, we compute the degree $k+1$ error term and ask whether there exists a degree $k+1$ power series which will cancel off this error term. For each k we wish to know whether a certain matrix is in the image of df_ρ. For the examples mentioned in the theorem the obstruction is non-trivial at the cubic stage, and therefore these examples do not deform.

Remark For a general group, there could be an obstruction at the second (quadratic) stage, if the vector $(e(2, 1), e(2, 2))$ does not lie in the image of df_ρ. However in the geometric context, other considerations can ensure the vanishing of this obstruction. For example, in the case of deformations in $PU(3, 1)$ (or $SO_0(4, 1)$), for the discrete faithful representation ρ, the second obstruction for deformations lies in $H^2(\pi_1(M); \mathfrak{pu}(3, 1))$ and this group is always zero as is shown below.

If $\rho\colon \Gamma \longrightarrow SO_0(3, 1)$ is locally rigid (for example Weil rigidity implies the discrete faithful representation is locally rigid), then the second obstruction to deformations into $PU(3, 1)$ vanishes. This is basically the argument of Kapovich and Millson [14, Theorem 11.1] (which in turn credits G Zuckermann): The inclusion $SO_0(3, 1) \rightarrow PU(3, 1)$ corresponds to the totally geodesic embedding $\mathbb{R}^4 \rightarrow \mathbb{C}^4$, let df_ρ be the map $\mathbb{C}^4 \rightarrow \mathbb{C}^4$ given by complex conjugation. Then df_ρ induces a map at the level of Lie algebras which has fixed set precisely $\mathfrak{so}(3, 1)$; clearly this algebra is fixed and given fixed vector, **v**, the one parameter subgroup $\exp(t\mathbf{v})$ is fixed by conjugation by df_ρ

and therefore lies in $SO_0(3,1)$. Therefore we have a splitting $\mathfrak{pu}(3,1) = \mathfrak{so}(3,1) \oplus m$ where df_ρ acts as $+1$ on $\mathfrak{so}(3,1)$ and -1 on m. It follows that if $\mathbf{v}$ and $\mathbf{w}$ are in m, then $[\mathbf{v}, \mathbf{w}] \in \mathfrak{so}(3,1)$.

The obstruction to integration of the cocycle $\mathbf{v} \in H^1(\Gamma; \mathfrak{pu}(3,1))$ is $[\mathbf{v}, \mathbf{v}]$ and the above splitting exhibits $H^1(\Gamma; \mathfrak{pu}(3,1))$ as the direct sum $H^1(\Gamma; \mathfrak{so}(3,1)) \oplus H^1(\Gamma; m)$. The first summand is zero by rigidity. It now follows that the cup product of any two classes in $H^1(\Gamma; \mathfrak{pu}(3,1))$ lies in $H^2(\Gamma; \mathfrak{so}(3,1))$ which is isomorphic by Poincaré duality to $H^1(\Gamma; \mathfrak{so}(3,1))$. This group is zero, since we have assumed that ρ is locally rigid.

2.4 The manifold *vol3*

The first flexible manifold in the census turns out to be the manifold usually referred to as *vol3*. We recall that *vol3* (it is the manifold with third-lowest volume in the census) is an arithmetic manifold with volume the same as that of a regular ideal simplex, that is, around 1.01494160640965, $H_1(vol3) = \mathbb{Z}_6 \oplus \mathbb{Z}_3$. Arithmetic methods show that this manifold contains no nonelementary Fuchsian subgroup and in fact it is non-Haken. The fundamental group has presentation

$$\langle a, b \mid aabbABAbb, aBaBabaaab \rangle$$

where $A = a^{-1}$ and $B = b^{-1}$.

Using the techniques described by the authors in [8] and sketched in Appendix A, one can show that there is a one complex parameter family of representations and our results apply. In fact we shall give two versions of this representation, each affording a certain utility. Despite the explicit nature of all the computations which follow, we are unable to offer much by way of geometrical insight.

Let $\alpha = \sqrt{v^2 - 4}$, $\beta = \sqrt{v^2 + 2}$, $\gamma = \sqrt{4v^2 + 5}$. Then the field generated by the matrix entries is $K = \mathbb{Q}(v)(\alpha, \beta, \gamma)$ and representing matrices are given by

$$\rho(a) = \begin{bmatrix} 0 & 0 & \frac{3v-\beta}{4\beta} & -\frac{3\gamma}{8\beta} \\ 0 & 0 & \frac{\gamma}{\beta} & \frac{-3v-\beta}{4\beta} \\ 1 & 0 & \frac{v+\beta}{2} & 0 \\ 0 & 1 & 0 & \frac{v-\beta}{2} \end{bmatrix} \qquad \rho(b) = \begin{bmatrix} \frac{p}{8\beta} & \frac{q}{32\beta} & \frac{r}{16\beta^2} & \frac{s}{32\beta} \\ \frac{-q^*}{12\beta} & \frac{-p^*}{8\beta} & \frac{-s^*}{12\beta} & \frac{r^*}{16\beta^2} \\ 0 & 0 & \frac{t}{8\beta} & \frac{u}{32\beta} \\ 0 & 0 & \frac{-u^*}{12\beta} & \frac{-t^*}{8\beta} \end{bmatrix}$$

where

$$p = -3v - \alpha + 2\beta + \gamma(-v + \alpha + 2\beta)$$
$$q = 4 + 5v^2 - 3v\alpha - 3v\beta + 5\alpha\beta + \gamma(4 - v^2 - v\alpha - v\beta - \alpha\beta)$$
$$r = 18v + 9v^3 - 2\alpha - v^2\alpha - 4\beta + v^2\beta - v\alpha\beta$$
$$\qquad + \gamma(2v + v^3 - 2\alpha - v^2\alpha + 4\beta + v^2\beta - v\alpha\beta)$$
$$s = 5v + 4v^3 - 3\alpha - 6\beta + 4v\alpha\beta + \gamma(-3v + \alpha - 2\beta)$$
$$t = -3v + \alpha + 2\beta + \gamma(v + \alpha - 2\beta)$$
$$u = -4 - 5v^2 - 3v\alpha - 3v\beta - 5\alpha\beta + \gamma(4 - v^2 + v\alpha + v\beta - \alpha\beta)$$

Here $x \mapsto x^*$ is application of the $\mathbb{Q}(v)$–automorphism of K that fixes α, γ and negates β.

This representation has the advantage that its entries are of small degree ($=$ eight) over $\mathbb{Q}(v)$. For many practical purposes, this is very convenient. The complete representation occurs at $v = 2$ and the complex hyperbolic representations occur for $|v| < 2$. One disadvantage of this representation is that the Hermitian form is somewhat complicated to express as a function of v, although this can be done.

Another useful description of a path of representations can be obtained from the fact that *vol3* admits an automorphism of order four given by $a \to B.A$ and $b \to a.b.a$. This automorphism can be induced by conjugacy by a matrix u. Since the automorphism u^2 conjugates a to a^{-1}, one sees the orbifold group of *vol3*/$\langle u \rangle$ is generated by two elements of finite order, namely u and $c = u^2 a$. In this way one obtains the path of representations

$$\Psi(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{v^2 - 4}/\sqrt{v^2 + 8} & \sqrt{2}\sqrt{-v^2 - 2}/\sqrt{v^2 + 8} \\ 0 & 0 & \sqrt{2}\sqrt{-v^2 - 2}/\sqrt{v^2 + 8} & -\sqrt{v^2 - 4}/\sqrt{v^2 + 8} \end{bmatrix}$$

and $\Psi(c)$ is given by

$$\begin{bmatrix} (v + \sqrt{v^2 + 8})/4 & 0 & -\frac{\sqrt{4 - v^2 - v\sqrt{8 + v^2}}}{2\sqrt{2}} & 0 \\ 0 & (v - \sqrt{v^2 + 8})/4 & 0 & -\frac{\sqrt{4 - v^2 + v\sqrt{8 + v^2}}}{2\sqrt{2}} \\ -\frac{\sqrt{4 - v^2 - v\sqrt{8 + v^2}}}{2\sqrt{2}} & 0 & (-v - \sqrt{v^2 + 8})/4 & 0 \\ 0 & -\frac{\sqrt{4 - v^2 + v\sqrt{8 + v^2}}}{2\sqrt{2}} & 0 & (-v + \sqrt{v^2 + 8})/4 \end{bmatrix}$$

Then in the notation above, if one sets $a = u^2.c$ and $b = (a.u.a)^{-1}.u$, this is a one parameter family of representations of the group $\pi_1(vol3)$; the discrete faithful representation continues to occur at $v = 2$.

We note that since the computation for the infinitesimal deformations of $\pi_1(vol3)$ gives that there is at most one essential dimension of deformations for representations into $SL(2, \mathbb{C})$, the computation above yields the following:

Corollary 2.11 *The complete representation ρ_0 is a smooth point of the variety* $\text{Hom}(\pi_1(vol3), SL(4, \mathbb{C}))$.

An unexpected consequence of the representation above which seems to be worth remarking upon is the following. One can regard the family of representations with parameter v as a single tautological representation $\rho: \pi_1(vol3) \longrightarrow SL(4, K)$ where K is a field of degree eight over $\mathbb{Q}(v)$ and therefore has transcendence degree one over $\mathbb{Q}$. We can therefore apply the construction described by Brown [4] to obtain an action of the group $\pi_1(vol3)$ on a 3–dimensional Euclidean building which we shall denote by Δ. We have the surprising:

Proposition 2.12 *The action of $\pi_1(vol3)$ on Δ is free.*

Sketch proof The action of the group $\pi_1(vol3)$ is by maps which preserve the simplicial structure, so if there is a fixed point for some element γ lying in the interior of some 3–simplex, then this 3–simplex must be mapped to itself by γ and hence some power of γ preserves a vertex. A similar argument applied to the simplices of lower dimension shows that if γ fixes a point, some power of γ fixes a vertex. Since $\pi_1(vol3)$ is torsion free, this element we have constructed is nontrivial.

The vertex stabilisers for the action correspond to matrices that can be conjugated into $SL(4, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers for the valuation which in this setting corresponds to functions which are bounded as $v \to \infty$. The proposition will follow, since we will show that no element of the group has a characteristic polynomial which remains bounded as $v \to \infty$.

Firstly, we note that it may be checked (we include an elegant proof in [8], or one can compute with the matrices above) that the coefficients of any characteristic polynomial lie in the field $\mathbb{Q}(v, \alpha)$ where $\alpha = \sqrt{v^2 - 4}$; moreover, since the image of the representation is Hermitian, the coefficients of Q^3 and Q in any characteristic polynomial differ by the involution $*: \mathbb{Q}(v, \alpha) \to \mathbb{Q}(v, \alpha)$ given by $\alpha \to -\alpha$ and the coefficient of Q^2 lies in the fixed field of this involution, namely $\mathbb{Q}(v)$.

Secondly, we note the two descriptions of the representations above have different denominators so that in fact no coefficient in any characteristic polynomial has a pole. (Alternatively one can argue considering only the first representation, that the only possible denominators in a characteristic polynomial of a group element are $\sqrt{v^2 + 2}$ and compute directly that limiting representations exist at both these points.) It follows that the coefficients of any characteristic polynomial are all of the shape $p + q\alpha$ where p, q are *polynomials* in v with rational coefficients.

Suppose some group element has bounded characteristic polynomial as $v \to \infty$. The coefficient of Q^2 is therefore a bounded polynomial and so constant. The coefficients of Q^3 and Q are $p + q\alpha$ and $p - q\alpha$ and these are both bounded; taking the sum gives the polynomial p is constant, from which is follows that $q\alpha$ is bounded as $v \to \infty$ so that $q = 0$. We deduce that the only possibility for bounded characteristic polynomial is that this polynomial must have constant coefficients.

Finally, we show that this is impossible: Using the matrices above, one checks that the representation as $v = 0$ has image a finite group, in particular all the characteristic polynomials are cyclotomic. However, at $v = 2$ we have the discrete faithful representation of a closed manifold into $SO_0(3, 1)$ and there, no such polynomials are possible. This completes the proof.   □

Remarks (i) This theorem gives an alternative proof, independent of arithmetic considerations that flexing in this example does not come from any of the classical constructions, since these constructions invariably leave some traces invariant and this proof shows that $\pi_1(vol3)$ contains no such traces.

(ii) The same general considerations show that the fundamental group of any flexible manifold acts on a Euclidean building, however this action is not usually free. For example the manifold $s518(4, 1)$ is two generator and flexes with both generators having *constant* characteristic polynomial.

Finally, we can ask about the degenerations of the *vol3* example. The parametrized path of representations of *vol3* constructed above is real hyperbolic at $v = 2$ and becomes complex hyperbolic as one decreases the parameter v through real values. An easy check at e.g. $v = 1$ shows the element b has order 3 there so the representation is not faithful. It follows from Theorem 2.6 that there is a degeneration in the interval $(1, 2)$.

Exactly where the first degeneration occurs remains open and seems to be a very difficult problem, although there is some evidence that it is $v = \sqrt{2}$; one finds that at this value the element $a.b.a.a.b^{-1}$ is parabolic and this is the first value of v for which any of the first hundred or so shortest geodesics in *vol3* become parabolic. The

result of Proposition 2.4 shows that we can continue to deform until the orbit and the Cayley graph stop being quasi-isometric. Ruling out some sort of geometrically infinite degeneration, that is, the Cayley graph is not quasi-isometric to the orbit without a parabolic having developed, seems fiendishly hard.

Nonetheless, the representation at the value of $v = \sqrt{2}$ can be conjugated into a particularly beautiful form which exhibits it as a discrete group:

$$\rho_{\sqrt{2}}(a) = \begin{bmatrix} (1+i)/\sqrt{2} & 0 & 0 & -\sqrt{2} \\ -(1+i)/\sqrt{2} & -i\sqrt{2} & (1-i)\sqrt{2} & (3-3i)/\sqrt{2} \\ (3-i)/\sqrt{2} & \sqrt{2} & (3+i)/\sqrt{2} & (2+i)\sqrt{2} \\ -(1-i)/\sqrt{2} & -(1-i)/\sqrt{2} & -\sqrt{2} & -\sqrt{2} \end{bmatrix}$$

and

$$\rho_{\sqrt{2}}(b) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

where these matrices are isometries for the Hermitian form

$$q = \begin{bmatrix} 0 & -1+i & -1+i & -1+3i \\ -1-i & 0 & -1+i & -1+i \\ -1-i & -1-i & 0 & -1+i \\ -1-3i & -1-i & -1-i & 0 \end{bmatrix}$$

In fact the group generated by the matrices $\{\rho_{\sqrt{2}}(a), \rho_{\sqrt{2}}(b)\}$ is discrete since one can verify directly that the unique subgroup of index two, which turns out to be generated by a^2 and b, actually represents into the finite covolume discrete complex hyperbolic group $PU(3, 1; q, \mathbb{Z}[i])$.

Remark One can show that the group $PU(3, 1; q, \mathbb{Z}[i])$ is commensurable with the more traditional group $PU(3, 1; J, \mathbb{Z}[i])$ where J is the standard Hermitian form given by the diagonal matrix $(-1, 1, 1, 1)$. We briefly sketch the argument for this: By making a change of basis using matrices drawn from $GL(4, \mathbb{Q}[i])$ (for example using the Gram Schmidt process) one can easily compute that the form q is equivalent over $\mathbb{Q}[i])$ with the diagonal form $(-14, 7, 1, 2)$. One may now verify that this form is equivalent over the rationals to $(-1, 1, 1, 1)$; this uses the technology of Hasse invariants: A simple family of equivalences of quadratic forms is $(-14, 7, 1, 2) \sim (-2, 1, 1, 2) \sim (-1, 1, 1, 1)$.) To sum up, we may find a matrix M in $GL(4, \mathbb{Q}[i])$ so that $M^*q.M = J$ and so $PU(3, 1; q, \mathbb{Q}[i]) = M.PU(3, 1; J, \mathbb{Q}[i]).M^{-1}$. Although M might not have a unit determinant, by passing to a sufficiently deep congruence subgroup in $PU(3, 1; J, \mathbb{Z}[i])$ we see that there is a subgroup of finite index for which integral

matrices of $PU(3, 1; J, \mathbb{Z}[i])$ are conjugated to integral matrices in $PU(3, 1; q, \mathbb{Z}[i])$; this subgroup has finite covolume therefore finite index. (See Long and Reid [15] for more details).

Acknowledgements

Appendix A   Sketch of the computations and Examples

In this section we sketch, very roughly, the method used to compute the deformed representations and in particular there are closed hyperbolic 3–manifolds to which the results of Section 2 apply. We set forth a much more detailed account in [8].

The methodology here is in the spirit of Snap, initially one does floating-point computations that are heuristic (albeit very accurate), and these are used to construct candidate exact solutions to the equations, (typically by means of the celebrated LLL algorithm of Lenstra–Lenstra–Lovász). The resulting candidate representation can then be verified to be correct by putting the computer in integer mode and substituting directly, verifying that group relations hold exactly. In this sense, the computations may be regarded as mathematically proven.

We recall that the census compiled by C Hodgson and J Weeks [12] lists 10986 low-volume closed orientable 3–manifolds having no geodesic of length less than 0.3, we restricted to two generator manifolds (of which there are 7254) and computed which of these admit (essential) infinitesimal deformations. One finds this cuts down the number substantially.

(These computations should really be regarded as heuristic as they were performed in floating-point mode, though as 1000 decimal places of accuracy were used it would be surprising if round-off error caused a discrepancy in the linear algebra). A list of the first twenty such manifolds, together with an exact computation of the dimension of the space of essential infinitesimal deformations is provided in Section A.1.

We shall describe (more or less) all the techniques necessary for doing any of the computations tabulated in Section A.1, referring occasionally to *vol3* by way of illustration.

Having done the infinitesimal computation, it is prudent to first test the possibility that the canonical discrete, faithful representation (which we shall denote here by ϕ_0) might

lie on a curve V of essential deformations. In order to do this, one randomly perturbs the generating matrices $\phi_0(a)$, $\phi_0(b)$ slightly to matrices a_1 and b_1 (which will no longer satisfy the group relations exactly), and then perform a Newton iteration on these matrices to try to converge to a representation ϕ_1 distinct from ϕ_0.

One finds that for *vol3*, with several different initial perturbations, the outcome is always a representation, accurate to 1000 decimal places, and that the generating matrices a_1, b_1 have characteristic polynomials of form $1 - vx - vx^3 + x^4$ and $1 - x - (v^2 - 1)x^2 - x^3 + x^4$ (respectively) with v differing slightly from 2 (which is the value for the canonical representation). It follows that this new (apparent) representation is not conjugate to the canonical representation and one can plausibly pass to the next step of constructing and hence confirming the existence of the curve V of essential representations passing through ϕ_0.

Remark There are small volume manifolds (for example $m149(-4, 1)$) which have infinitesimal deformations, but for which this step fails; one finds that Newton's method always gives a representation of the same character. In this case, one uses the obstruction theory developed in Section 2.3.

We can confirm the existence of the curve V of representations passing through ϕ_0 if we can find a continuous family of matrix pairs $\{(a(v), b(v))\}$ satisfying the relations and containing the $SO_0(3, 1)$–pair (a_0, b_0) corresponding to ϕ_0. Since if it exists V is an algebraic set, our task is to find a single representation Φ into $SL(4, \mathbb{Q}(v))$ that specializes to ϕ_0 at the appropriate parameter value. The broad plan is then to compute highly accurate approximations to $(a(v), b(v))$ for a large sequence of rational values of the parameter v, guess exact values for the matrix entries using LLL, and then use polynomial interpolation to obtain the exact entries of $\Phi(a)$, $\Phi(b)$. Finally, as described above, one can verify the relators exactly, using the formal algebra capabilities of either Maple or Mathematica.

This method appears to work in a good deal of generality for determining deformed representations of two generator fundamental groups of closed hyperbolic manifolds.

A.1 Zoo of examples

A.1.1 The first twenty infinitesimally deformable census manifolds

Volume	Chern Simons	Census	Essential Deformations
1.014941606410	0	$m007(3, 1)$	1
2.029883212819	0	$m036(-3, 2)$	1
2.195964118694	0	$m034(-4, 1)$	1
2.595387593687	0	$m160(-3, 2)$	1
2.786804556416	0	$m082(1, 3)$	1
2.816179876318	0	$m078(5, 1)$	1
2.882494387252	0	$m100(2, 3)$	1
3.044824819229	0	$m149(-4, 1)$	Rigid
3.044824819229	0	$m188(2, 3)$	1
3.044824819229	0	$m247(-1, 3)$	1
3.044824819229	1/4	$m159(2, 3)$	Rigid
3.060334984517	0	$m115(5, 2)$	1
3.195780918694	0	$m121(-4, 3)$	1
3.663862376709	0	$m336(-1, 3)$	2
3.663862376709	0	$m303(-1, 3)$	1
3.663862376709	1/24	$s572(1, 2)$	1
3.663862376709	−1/24	$m293(4, 1)$	Rigid
3.663862376709	−1/8	$s645(-1, 2)$	1
3.663862376709	1/8	$m312(-1, 3)$	2
3.663862376709	−1/8	$s778(-3, 1)$	1

A.1.2 Arithmetic commensurability classes Many of the manifolds above are arithmetic and so the question of commensurability is easily settled, for example using *Snap*. Below we give a list of all manifolds in the census commensurable with a flexible manifold (assuming this class contains more than the one element listed above). In particular, this exhibits examples of manifolds which one computes are infinitesimally rigid and therefore not flexible, but have a finite sheeted covering which is flexible: $m010(-4, 3)$ is such a manifold, for example. Commensurability is much harder to determine for nonarithmetic examples and we leave this issue unaddressed.

Flexible manifold $m007(3, 1)$**:** $m007(3,1)$, $m036(-3,2)$, $s778(-3,2)$, $m010(-4,3)$, $m247(-1,3)$, $m358(1,3)$, $m395(-2,3)$, $s440(-1,3)$, $s775(1,2)$, $s779(1,2)$, $s787(1,2)$, $v0825(4,1)$, $v2417(-1,3)$, $v2636(2,3)$, $v3210(3,1)$, $v3216(-4,1)$.

Flexible manifold $m160(-3, 2)$**:** $m160(-3, 2)$, $v3347(3, 1)$.

Flexible manifold $m336(-1, 3)$**:** $m304(5,1)$, $m336(-1,3)$, $s942(-2,1)$, $s960(-1,2)$.

Flexible manifold $s572(1, 2)$**:** $m293(4, 1)$, $s297(-1, 3)$, $s572(1, 2)$, $s645(-1, 2)$, $s682(-3, 1)$, $s775(-1, 2)$, $s778(-3, 1)$, $v3213(-1, 3)$, $v3216(4, 1)$.

A.1.3 Examples with Chern–Simons apparently irrational While it appears that most of the examples currently known have rational Chern–Simons invariant, the examples below apparently do not. (No Chern–Simons invariant currently known is provably irrational) These examples exhibit another interesting feature that they can be generated by two elements whose characteristic polynomials remain constant through the flexing.

Volume	Chern Simons	Census	Essential Deformations
4.400901789291	0.051636653040	$s518(4, 1)$	1
4.598853813756	0.114136653040	$s636(-1, 4)$	1
4.598853813756	0.135863346960	$s618(1, 4)$	1

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