Teichmüller geodesics that do not have a limit in $\mathcal{PMF}$

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We construct a Teichmüller geodesic which does not have a limit on the Thurston boundary of the Teichmüller space. We also show that for this construction the limit set is contained in a one-dimensional simplex in $\mathcal{PMF}$.

30F60, 32G15; 32F45, 57M50

1 Introduction

In this paper we consider a problem in Teichmüller geometry at infinity. Recall that Teichmüller space $\mathcal{T}_g$ of a closed oriented surface of genus g equipped with Teichmüller metric is a complete geodesic metric space. There is a natural way to compactify $\mathcal{T}_g$, by fixing a base point and considering all geodesic rays through that point. However Kerckoff proved that the Teichmüller, or visual, compactification was base point dependent by showing that the action of the mapping class group on $\mathcal{T}_g$ does not extend to the boundary. Another natural compactification is due to Thurston, by projective measured foliations $\mathcal{PMF}$, to which the action of the mapping class group does extend.

Teichmüller geodesic rays are associated with quadratic differentials which in turn are closely related to measured foliations: geodesics are described by scaling horizontal and vertical measured foliations of a quadratic differential. Therefore it is natural to compare the two compactifications, in particular, to study the behavior of a Teichmüller ray with respect to the Thurston compactification. The question is quite nontrivial. A geodesic is defined by deforming the flat metric. On the other hand, the Thurston compactification is defined using the hyperbolic metric, and there is no obvious way to compare the two metrics. Masur [8] proved that in almost every direction through every point, geodesic rays have a limit on the boundary. In particular Masur considered geodesic rays given by quadratic differentials with uniquely ergodic vertical measured foliation and showed that a ray defined by a differential q with vertical measured foliation F converges to the class of F. Quadratic differentials with closed vertical trajectories were considered in the same paper, and it was shown that corresponding geodesic rays also converge. However the limit in this case might not be the class of F

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itself. It turned out that the ray approaches the barycenter of the simplex of measures on F, ie in the limit all the leaves have the same weight.

In this paper we give a first example of a geodesic ray which does not have a limit on the Thurston's boundary. Our main result is the following:

Theorem 1 *There exists a Teichmüller geodesic ray which does not converge in Thurston's compactification.*

The proof is by construction of a divergent ray, which is found by gluing two copies of a square torus cut along slits whose slopes satisfy certain conditions, to obtain a surface of genus 2, and then scaling the vertical and horizontal foliations of the corresponding quadratic differential. We also describe the limit set of the divergent ray. More precisely, we show that it is contained in a one dimensional simplex on the boundary.

2 Preliminaries

We refer the reader to Imayoshi and Tanaguchi [4] and *Travaux* [3] for more information on Teichmüller theory. Let M be a closed surface of genus $g \geq 2$. Recall that the *Teichmüller space* $\mathcal{T}_g$ is the space of equivalence classes of conformal structures X on M. The equivalence relation is defined by considering two structures X_1 and X_2 equivalent if there is a biholomorphic map from X_1 to X_2 which is isotopic to the identity on M.

Let X_1 and X_2 be two points in $\mathcal{T}_g$. The *Teichmüller distance* between X_1 and X_2 is defined to be
$$d(X_1, X_2) = \frac{1}{2} \log K$$
where K is the smallest number such that there is a homeomorphism homotopic to the identity on M which is a K–quasiconformal map between X_1 and X_2. There is a unique quasiconformal map from X_1 to X_2 realizing this distance, called *Teichmüller mapping*.

A *holomorphic quadratic differential* q (see Strebel [11] for details) on a Riemann surface X is an assignment to each chart (U_α, z_α) of X a holomorphic function q_{z_α} with the property
$$q_{z_\beta}(z_\beta)\left(\frac{dz_\alpha}{dz_\beta}\right)^2 = q_{z_\alpha}(z_\alpha)$$
in $U_{z_\alpha} \cap U_{z_\beta}$. The norm or area of q is defined by $\|q\| = \int_X |q(z)||dz|^2$. The vector space Q_0 of all holomorphic quadratic differentials on X is a $6g-6$ dimensional vector space.

In a neighborhood of every regular point P of q one can introduce a local parameter w, in terms of which q is identically equal to 1. This parameter, called the *natural parameter* of q near P, is determined by the integral

$$w = \int_X \sqrt{q(z)}dz$$

uniquely up to a transformation $w \to \pm w + \text{const}$.

The vertical trajectories of q are the arcs along which $q(z)dz^2 < 0$, and the horizontal trajectories are the arcs where $q(z)dz^2 > 0$. Hence every quadratic differential q determines a pair of transverse measured foliations: the *vertical* foliation, where the leaves are the vertical trajectories, and the *horizontal* foliation, with leaves being the horizontal trajectories of q. One can also obtain the vertical and horizontal foliations by pulling back the vertical and horizontal foliations of $\mathbb{C}$ via a natural parameter. Every quadratic differential q determines a flat metric with the length element $|q(z)|^{1/2}|dz|$. Again, the flat metric can be obtained from the natural parameter of q by pulling back the Euclidean metric from $\mathbb{C}$.

Geodesic rays can be described as follows. Each direction at a point X in $\mathcal{T}_g$ is associated to a quadratic differential q on X. For $t \in \mathbb{R}$, let q_t be 1–parameter family of quadratic differentials obtained from q so that if $z = x + iy$ are natural coordinates for q away from zeroes then $z_t = e^{-t/2}x + ie^{t/2}y$ are natural coordinates for q_t. Let X_t be the conformal structure corresponding to q_t. Then $\{X_t\}$ is a geodesic.

Let $\mathcal{S}$ be the set of homotopy classes of essential simple closed curves on M with the discrete topology. Here by essential we mean homotopically nontrivial and non-peripheral. The space of functionals $R_+^{\mathcal{S}}$ is given the product topology. Let $PR_+^{\mathcal{S}}$ be the corresponding projective space. Recall that the Teichmüller space $\mathcal{T}_g$ can be identified with the space of equivalence classes of hyperbolic metrics ρ on M of constant curvature -1, where $\rho_1 \sim \rho_2$ if there exists an isometry from (M, ρ_1) to (M, ρ_2) isotopic to the identity. Thurston showed that the hyperbolic lengths of curves can be approximated, as one goes to infinity in $\mathcal{T}_g$, by their intersection numbers with a measured foliation. Therefore one can define a compactification in terms of ratios of hyperbolic lengths. The map $\mathcal{T}_g \mapsto PR_+^{\mathcal{S}}$ defined by $\rho \to (\gamma \to \ell_\rho(\gamma))$, where $\ell_\rho(\gamma)$ is the length of the unique geodesic in the hyperbolic metric in the class of γ, is injective. It is called the Thurston embedding of Teichmüller space. The boundary of $\mathcal{T}_g$ in $PR_+^{\mathcal{S}}$ is the sphere $\mathcal{PMF}$ of projective measured foliations on M. The union of $\mathcal{T}_g$ and $\mathcal{PMF}$ is denoted by $\widehat{\mathcal{T}_g}$ and is called Thurston compactification. It is homeomorphic to a closed $6g-6$ dimensional ball, where $\mathcal{PMF}$ is the $6g-7$ dimensional sphere.

3 Construction and idea of the proof

Start with a square torus, ie the unit square with lower left vertex at $(0,0)$, with opposite sides identified. Cut the torus along a line segment (call it a slit) of length $0 < s < 1$ and a slope $\theta_1 > 0$. Take a second copy of the torus, cut it along a slit of same length s and a slope $\theta_2 > 0$. Rotate the squares counterclockwise so that the slits are vertical. Now identify the left side of the slit on one copy with the right side on the other copy. This defines a flat structure with a parallel line field which corresponds to a quadratic differential q_{θ_1,θ_2} with 2 zeroes of order 2 on a surface X of genus 2. For more details on this construction, flat structures and quadratic differentials see Masur and Tabachnikov [9]. X is partitioned into 2 sheets S_1 and S_2 separated from each other by the union of the two slits. If both θ_1 and θ_2 are irrational, then the vertical foliation of q_{θ_1,θ_2} has one closed leaf, which is the union of the two slits, and all other leaves are dense in S_1 or S_2. Let $\{X_t\}$ be the Teichmüller geodesic ray from X determined by the differential q_{θ_1,θ_2}.

Recall that each $x \in \mathbb{R}$ admits a continued fraction expansion of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

with $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$, $i \geq 1$ and that $x \in \mathbb{R} - \mathbb{Q}$ iff infinitely many a_i's are nonzero. We will also use the notation $x = [a_0; a_1, a_2, \ldots]$. We will call a_i's the *elements* of x.

We will prove:

Theorem 2 *Suppose $\theta_1 \in \mathbb{R}-\mathbb{Q}$ has bounded elements $a_{1,n} \geq 3$, $n \geq 1$, and $\theta_2 \in \mathbb{R}-\mathbb{Q}$ has unbounded elements $a_{2,n} \geq 3$, $n \geq 1$. Let $\{X_t\}$ be the Teichmüller geodesic ray constructed as above. Then $\{X_t\}$ does not converge in $\widehat{\mathcal{T}}_2$.*

Theorem 1 follows immediately from Theorem 2.

Idea of the proof By the definition of Thurston compactification, a sequence $\{X_n\}$ in Teichmüller space converges to projective measured foliation $\mathcal{F}$ in $\mathcal{PMF}$ if there exists a sequence $\{r_n\}$ such that $r_n \to 0$, such that for any nontrivial simple closed curve α we have $r_n \cdot \ell_n(\alpha) \to i(\alpha, F)$ as $n \to \infty$. Here $\ell_n(\alpha)$ is the length of the shortest curve in the hyperbolic metric of X_n which is homotopic to a curve α, and $i(\alpha, F)$ denotes the measure of α with respect to a representative F in $\mathcal{F}$. Hence, to prove that the geodesic does not have a limit in $\mathcal{PMF}$, it suffices to find a pair of simple closed curves α_1 and α_2 on M and, after ruling out the possibility that

$i(\alpha_j, F) = 0$ for both $j = 1$ and $j = 2$, show that the limit $\lim_{t\to\infty} \ell_t(\alpha_1)/\ell_t(\alpha_2)$ does not exist. In our proof α_j is the curve represented by the vector $(1, 0)$ before the rotation on S_i, $j = 1, 2$.

To prove the theorem, we need to be able to estimate hyperbolic lengths of simple closed curves on X_t for large values of t. The geodesic ray $\{X_t\}$ is constructed using the flat metric. However, in general there is no easy way to get a good estimate of the hyperbolic length of a curve. Our method is to identify the shortest and the second shortest curves in the flat metric on each torus (we think of the surface X_t as a surface glued out of two tori S_1 and S_2), and estimate their hyperbolic length. This information, together with the notion of intersection number, makes it possible to find good lower and upper bounds on the lengths of the curves α_1 and α_2.

The idea behind the choice of θ_1 and θ_2 is as follows. We will show (Lemma 1) that at any time the shortest curve on S_i is a curve whose slope is a convergent of θ_i. Moreover, we will prove that the hyperbolic length of the shortest curve depends on the elements of θ_i. More precisely, the smallest hyperbolic length of the curve whose slope is n–th convergent of θ_i is roughly $1/a_{i,n+1}$, where $a_{i,n+1}$ is the $(n+1)$–st element of θ_i (Lemma 3). Hence assuming that elements of θ_2 are unbounded and those of θ_1 are bounded we easily find a sequence of times when the hyperbolic length of the shortest curve on S_2 goes to 0, while on the other side the length of the shortest curve stays bounded below. By the Collar Lemma, each time one intersects an extremely short curve, one has to cross a collar of width approximately $\log(1/\ell(\text{shortcurve}))$. Hence one would think that the curve α_2 must become exceedingly long compared to the curve α_1. However, we will see that the curve α_1 intersects a curve of bounded length on S_1 significantly more than the curve α_2 crosses the long collar in S_2. As a result, the ratio of the hyperbolic lengths along the sequence becomes arbitrarily large. On the other hand, after a shortest curve on S_2 reaches its minimal length, it has to grow. As its length becomes compatible with the length of the next short curve, the curve α_2 grows faster and catches up with α_1. At that time the ratio of the hyperbolic lengths is bounded above. We then conclude that the limit $\lim_{t\to\infty} \ell_t(\alpha_1)/\ell_t(\alpha_2)$ does not exist.

The main result is proven in Section 6. In Section 4 we consider the shortest curves in the flat metric. Estimates of the hyperbolic lengths of these curves are made in Section 5.

Section 7 is dedicated to the limit set of the geodesic ray. More precisely, in Section 7 we show that it a connected subset of one-dimensional simplex L_{θ_1,θ_2}. Each point in L_{θ_1,θ_2} is a projective class of measured foliations topologically equivalent to the vertical foliation of q_{θ_1,θ_2}, with possibly different weights on S_1 and S_2.

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4 Continued fractions and the flat metric on a torus

In this section we focus on one torus only, ignoring the slit and the other torus. Let $\theta = [a_0; a_1, a_2, a_3, \ldots]$ be any positive irrational number, with $a_i \geq 3$. The k–th convergent of θ is the reduced fraction

$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}.$$

A few standard facts (see for example Khinchin [5]) about continued fractions are:

(1a) $\quad p_{n+1} = a_{n+1} p_n + p_{n-1}$ and $q_{n+1} = a_{n+1} q_n + q_{n-1}$

(1b) $\quad \dfrac{1}{q_n + q_{n+1}} \leq |p_n - \theta q_n| \leq \dfrac{1}{q_{n+1}}$

(1c) $\quad \dfrac{p_{2n}}{q_{2n}} \nearrow \theta$ and $\dfrac{p_{2n+1}}{q_{2n+1}} \searrow \theta$

(1d) $\quad |p_{n+1} q_n - q_{n+1} p_n| = 1$

Consider a standard lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. Let g_t^θ be a map given by a matrix

$$\frac{1}{\sqrt{1+\theta^2}} \begin{pmatrix} \theta e^{t/2} & -e^{t/2} \\ e^{-t/2} & \theta e^{-t/2} \end{pmatrix}$$

which is a rotation by an angle of $\frac{\pi}{2} - \tan^{-1}\theta$, followed by a horizontal stretch by a factor of $e^{t/2}$ and vertical contraction by $e^{t/2}$. For every t we get a new lattice in $\mathbb{R}^2$. We will refer to the image of any vector $(q, p) \in \mathbb{Z}^2$ under the map g_t^θ as (q, p)–vector or (q, p)–curve at time t.

Notation To simplify our presentation we use $\approx, \Theta$ and O defined as follows: for two sequences $x_n > 0, y_n > 0$, $x_n \approx y_n$ means $x_n/y_n \to 1$ as $n \to \infty$; $x_n = O(y_n)$ iff $\sup x_n/y_n < \infty$; $x_n = \Theta(y_n)$ iff $x_n = O(y_n)$ and $y_n = O(x_n)$.

Lemma 1 *Suppose a (q, p)–vector is the shortest vector in Euclidean length at some time t. Suppose $t \geq \log\big((1 + a_0\theta)/(\theta - a_0)\big)$. Then p/q is a convergent for θ, ie*

$p = p_n$ and $q = q_n$ for some n. Also at time $T_n = \log\big((p_n\theta + q_n)/|q_n\theta - p_n|\big)$, the (q_n, p_n)–vector is the shortest. For $t \in [T_n, T_{n+1}]$ the shortest vector is either (q_n, p_n) or (q_{n+1}, p_{n+1}).

Proof By taking reciprocals if needed we can assume that $p/q < \theta$. Suppose first that $p_0/q_0 < p/q$. If p/q is not a convergent of θ, then there is a unique n such that

$$\frac{p_n}{q_n} < \frac{p}{q} < \frac{p_{n+2}}{q_{n+2}}. \tag{2}$$

We claim that the (q_{n+1}, p_{n+1})–vector is always shorter than the vector (q, p). At time t the image of (q, p) is

$$g_t^\theta(q, p) = \frac{q + \theta p}{e^{t/2}\sqrt{1+\theta^2}}(0,1) + \frac{e^{t/2}(q\theta - p)}{\sqrt{1+\theta^2}}(1,0).$$

The Euclidean length of $g_t^\theta(q, p)$ satisfies

$$l_t((q,p))^2 = \frac{1}{1+\theta^2}\left(\frac{(q+\theta p)^2}{e^t} + e^t(p - \theta q)^2\right).$$

Since $p_{n+1}q_n - q_{n+1}p_n = 1$ and (2) implies

$$\frac{p_n}{q_n} < \frac{p}{q} < \frac{p_{n+1}}{q_{n+1}}, \tag{3}$$

we can show that $p \geq p_n + p_{n+1}$ and $q \geq q_n + q_{n+1}$. Indeed, (3) implies

$$pq_n - p_n q \geq 1$$

and

$$p_{n+1}q - pq_{n+1} \geq 1.$$

Multiplying the first inequality by p_{n+1} and the second by p_n, and adding them we obtain

$$p_{n+1}q_n p - p_n q_{n+1} p \geq p_n + p_{n+1}.$$

Therefore $p \geq p_n + p_{n+1}$. Similar argument shows that $q \geq q_n + q_{n+1}$.

Going back to the proof of the lemma, we see that

$$q + \theta p > q_{n+1} + \theta p_{n+1}.$$

Also $$|p - \theta q| = q\left|\frac{p}{q} - \theta\right| > q\left|\frac{p}{q} - \frac{p_{n+2}}{q_{n+2}}\right| > \frac{1}{q_{n+2}} \geq |p_{n+1} - \theta q_{n+1}|.$$

The claim now follows from these inequalities and we conclude that p/q is a convergent of θ. Now suppose $p_0/q_0 > p/q$. Then we compute that $l_t((q,p))^2 > l_t((q_0,p_0))^2$ if $t \geq \log\big((1+a_0\theta)(\theta - a_0)\big)$. Hence taking t large enough guarantees that the claim holds.

The function $l_t((q_n, p_n))^2$ reaches its minimum at $T_n = \log\big((p_n\theta + q_n)/|q_n\theta - p_n|\big)$ and

$$l^2_{T_n}((q_n, p_n)) = 2\frac{(q_n + p_n\theta)|q_n\theta - p_n|}{1+\theta^2}.$$

We have for all $k < n$

$$\begin{aligned} l_{T_n}((q_k, p_k))^2 > \frac{(q_k\theta - p_k)^2 e^{T_n}}{1+\theta^2} &= \frac{(q_k\theta - p_k)^2(p_n\theta + q_n)}{(1+\theta^2)|q_n\theta - p_n|} \\ &> 4\frac{(p_n\theta + q_n)}{q_{n+1}(1+\theta^2)} \geq 4\frac{(p_n\theta + q_n)|q_n\theta - p_n|}{1+\theta^2}. \end{aligned}$$

In the estimates above we used (1a), (1b) and the assumption that the elements of θ satisfy $a_k > 2$. Similarly, for all $k > n$

$$\begin{aligned} l_{T_n}((q_k, p_k))^2 > \frac{(q_k + p_k\theta)^2}{(1+\theta^2)e^{T_n}} &= \frac{(q_k + p_k\theta)^2|q_n\theta - p_n|}{(1+\theta^2)(p_n\theta + q_n)} \\ &> 4\frac{(q_n + p_n\theta)^2|q_n\theta - p_n|}{(1+\theta^2)(p_n\theta + q_n)} \\ &= 4\frac{(q_n + p_n\theta)|q_n\theta - p_n|}{1+\theta^2}. \end{aligned}$$

We conclude that $l_{T_n}((q_n, p_n)) < \min\{l_{T_n}((q_k, p_k)) : k \neq n\}$. To verify the last claim

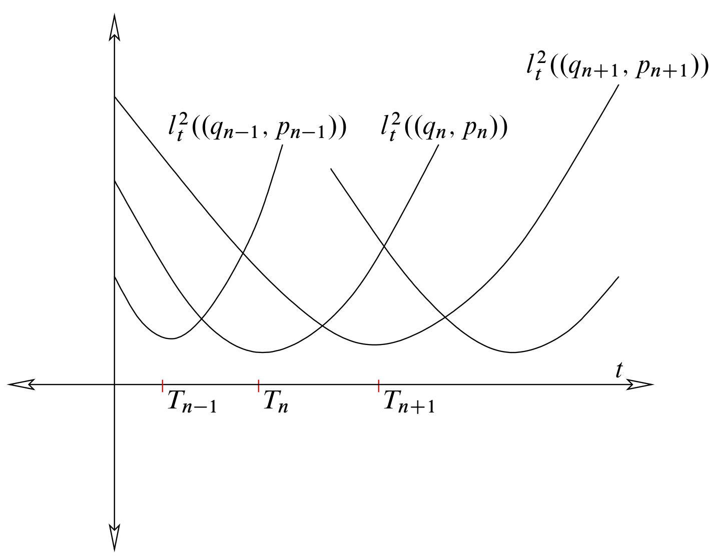

Figure 1

of the lemma, we notice that the sequence $\{T_n\}$ is strictly increasing and $T_n \to \infty$. Also we see that, for each n, the function $l_t^2((q_n, p_n))$ is strictly convex and that the graphs of $l_t^2((q_n, p_n))$ and $l_t^2((q_k, p_k))$ intersect once (see Figure 1). Since $n < k$ implies $l_0^2((q_n, p_n)) < l_0^2((q_k, p_k))$, we see that at time $t \in [T_n, T_{n+1}]$ only (q_n, p_n) or (q_{n+1}, p_{n+1}) can be the shortest. □

To find the second shortest vector is a more complicated task. There are more curves which can become second shortest than those whose slopes are convergents of θ.

Lemma 2 *Suppose that a (q, p)–curve is second shortest at time t, and (q_n, p_n) is the shortest. Then either p/q is a convergent of θ (namely, (q_{n-1}, p_{n-1}) or (q_{n+1}, p_{n+1})) or the following holds:*

$$q_{n-1} + q_n \le q < q_{n+1}$$

and

$$p_{n-1} + p_n \le p < p_{n+1}$$

Proof Suppose p/q is not a convergent of θ. Assume that $p/q < \theta$. Then there is a (unique) k such that $p_{k-1}/q_{k-1} < p/q < p_{k+1}/q_{k+1}$. Using the argument from Lemma 1 we can show that the (q_k, p_k)–curve is shorter then (q, p). Therefore $k = n$. Since the shortest and the second shortest curves intersect once, $p_n q - p q_n = 1$. Also $p_n q_{n-1} - q_n p_{n-1} = 1$. Putting these two things together we get

$$p_n(q - q_{n-1}) - q_n(p - p_{n-1}) = 0.$$

This implies that $p_n/q_n = (p - p_{n-1})/(q - q_{n-1})$. Hence $p = a p_n + p_{n-1}$ and $q = a q_n + q_{n-1}$, where $a \in \mathbb{Q}$. It is easy to see that $a < a_{n+1}$: if $a \ge a_{n+1}$ then

$$\frac{p}{q} - \frac{p_{n+1}}{q_{n+1}} = \frac{a p_n + p_{n-1}}{a q_n + q_{n-1}} - \frac{a_{n+1} p_n + p_{n-1})}{a_{n+1} q_n + q_{n-1}} = \frac{(a_{n+1} - a)(p_{n-1} q_n - q_{n-1} p_n)}{(a q_n + q_{n-1})(a_{n+1} q_n + q_{n-1})} \ge 0$$

which is impossible. Therefore $p < p_{n+1}$ and $q < q_{n+1}$. Also $p_{n-1}/q_{n-1} < p/q < p_n/q_n$ and $p_n q_{n-1} - q_n p_{n-1} = 1$ imply $p \ge p_{n-1} + p_n$ and $q \ge q_{n-1} + q_n$. A similar argument works if $p/q > \theta$.

If p/q is a convergent of θ, then Lemma 1 implies that (q, p) is either (q_{n-1}, p_{n-1}) or (q_{n+1}, p_{n+1}). □

5 The comparison of the flat and the hyperbolic metrics on the branched cover

In the previous section we considered shortest vectors on a torus. It follows from Lemma 1 that the vectors on S_i with smallest length in the flat metric are those whose slopes are n–th convergents of θ_i, which we denote by $p_{i,n}/q_{i,n}$. We claim that the same result holds for the flat surface glued from the two tori S_1 and S_2, each with a slit $(0, s)$. Indeed, since the slope of the $(p_{i,n}, q_{i,n})$–curve is a convergent, the curve can be made disjoint from the slit.

Hence at any time we know the short curves in S_1 and S_2. We want to estimate the hyperbolic length of these curves. To do so, we first prove a claim about extremal lengths of curves in S_i which get short. We will denote by $\mathrm{Ext}_t(\alpha)$ the extremal length at time t of a family of curves homotopic to a curve α. We refer the reader to Ahlfors [1] for the basic facts concerning the extremal length.

Lemma 3 *Let α be a $(q_{i,n}, p_{i,n})$ -curve on S_i, $i = 1, 2$. Then the extremal length of α at time t satisfies*

$$\frac{\mathrm{Ext}_t(\alpha)}{l_t^2(\alpha)} \xrightarrow[n,t\to\infty]{} 1$$

Proof By definition of extremal length, $\mathrm{Ext}(\alpha) \geq \inf_{\beta\sim\alpha} \left(\int_\beta \rho|dz|\right)^2/A(\rho)$ for any metric of the form $\rho|dz|$ where $\rho \geq 0$ is Borel measurable. Assume α is a $(p_{1,n}, q_{1,n})$–

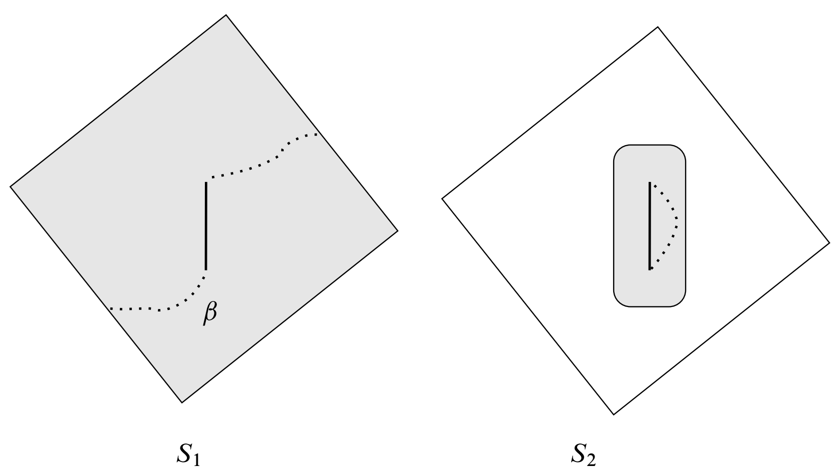

Figure 2: $\rho_t|dz|$ coincides with the flat metric in the shaded area

curve on S_1. Let $\rho_t|dz|$ be the metric which coincides with the flat metric of S_1 at time t, and with the flat metric of S_2 at time t on the set of points in S_2 which are at most $se^{-t/2}$ (the length of the slit) away from the slit (see Figure 2). On the rest of the S_2 we define $\rho_t = 0$. We need to find the shortest curve with respect to this metric in the homotopy class of α. We claim that a geodesic with respect to $\rho_t|dz|$ is contained in S_1, ie it does not cross the slit. The point is that any curve β homotopic to α which crosses the slit will contain an arc in S_2 with endpoints in the slit that is homotopic relative to its endpoints into the slit. The arc is longer than the line segment connecting the endpoints. Hence there is a curve in the class of α which is shorter than β. Since α is a geodesic in the flat metric of S_1 at time t, we conclude that it is a shortest curve with respect to $\rho_t|dz|$ in its homotopy class. The length of α in this metric is

$$\sqrt{\frac{1}{1+\theta_1^2}\left(\frac{(q_{1,n}+\theta_1 p_{1,n})^2}{e^t}+e^t(p_{1,n}-\theta_1 q_{1,n})^2\right)}.$$

We then have

$$\mathrm{Ext}_t(\alpha) \geq \frac{1}{1+\theta_1^2}\frac{\frac{(q_{1,n}+\theta_1 p_{1,n})^2}{e^t}+e^t(p_{1,n}-\theta_1 q_{1,n})^2}{1+(\pi+2)s^2e^{-t}}. \tag{4}$$

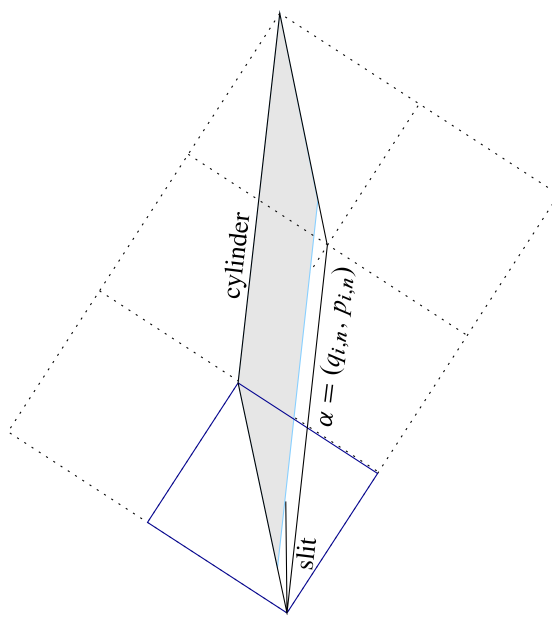

Figure 3

On the other hand, the extremal length of a simple closed curve α can be defined as $\inf_C(1/\operatorname{Mod}(C))$ where the infimum is taken over all cylinders C with α a core curve, and where $\operatorname{Mod}(C)$ denotes the modulus of a cylinder C.

So we need to find a cylinder for a good upper bound. We consider the largest cylinder swept out by curves parallel to α which avoid the slit (see Figure 3). Curves parallel to α and crossing the slit make up a parallelogram spanned by the vectors representing the slit and α. At time t the slit is represented by a vector $u=(0,s/e^{t/2})$, and α is represented by

$$v=\frac{q_{1,n}+\theta p_{1,n}}{e^{t/2}\sqrt{1+\theta_1^2}}(0,1)+\frac{e^{t/2}|q_{1,n}\theta-p_{1,n}|}{\sqrt{1+\theta_1^2}}(1,0).$$

Then the area of our cylinder is

$$1-|u\times v|=1-\frac{s}{e^{t/2}}\frac{e^{t/2}|q_{1,n}\theta_1-p_{1,n}|}{\sqrt{1+\theta_1^2}}=1-\frac{s|q_{1,n}\theta_1-p_{1,n}|}{\sqrt{1+\theta_1^2}}.$$

The length of the cylinder is $\|v\|$, and the height is

$$\frac{\text{area}}{\text{length}}=\frac{1-\frac{s|q_{1,n}\theta-p_{1,n}|}{\sqrt{1+\theta_1^2}}}{\|v\|}.$$

Thus the modulus is

$$\frac{1-\frac{s|q_{1,n}\theta-p_{1,n}|}{\sqrt{1+\theta_1^2}}}{\|v\|^2}.$$

It follows that

$$\text{Ext}_t(\alpha)\le\frac{1}{1+\theta_1^2}\left(\frac{(q_{1,n}+\theta_1 p_{1,n})^2}{e^t}+e^t(p_{1,n}-\theta_1 q_{1,n})^2\right)\frac{1}{1-\frac{s|q_{1,n}\theta-p_{1,n}|}{\sqrt{1+\theta_1^2}}}. \tag{5}$$

The claim of the lemma now follows from (5), (4) and (1b). □

Corollary 1 *Suppose θ_i is such that $a_{i,n}\to\infty$ for $i=1$ or $i=2$. Then at time*

$$T_n=\log\frac{p_{i,n}\theta_i+q_{i,n}}{|q_{i,n}\theta_i-p_{i,n}|}$$

the hyperbolic length of a curve $\alpha=(q_{i,n},p_{i,n})$ satisfies $\ell_{T_n}(\alpha)=\Theta(1/a_{i,n+1})$.

Proof By Lemma 3 we have $\text{Ext}_{T_n}(\alpha)\approx l^2_{T_n}(\alpha)$. Furthermore, the assumption $a_{i,n}\to\infty$ and (1b) imply $l^2_{T_n}(\alpha)\approx(2q_{i,n})/q_{i,n+1}\approx 2/a_{i,n+1}$. On the other hand, by Proposition 1 and Corollary 3 in Maskit [7]

$$2e^{-\ell(\gamma)/2}\le\frac{\ell(\gamma)}{\text{Ext}(\gamma)}\le\pi \tag{6}$$

where $\mathrm{Ext}(\gamma)$ is the extremal length and $\ell(\gamma)$ the hyperbolic length of any simple closed curve γ. In particular, if the extremal length of α becomes very small, then so does the hyperbolic length, and their ratio is bounded above and below. Therefore, the hyperbolic length of α at time T_n satisfies $\ell_{T_n}(\alpha) = \Theta(1/a_{i,n+1})$. □

6 Proof of the main theorem

Proof We begin by choosing a subsequence $\{a_{2,n_k}\}$ such that $a_{2,n_k} \to \infty$. Let for $k \in \mathbb{Z}, k \geq 1$,

$$t_{2k} = \log \frac{p_{2,n_k-1}\theta_2 + q_{2,n_k-1}}{|q_{2,n_k-1}\theta_2 - p_{2,n_k-1}|}$$
$$t_{2k+1} = \log \Big((1+\theta_2^2)(q_{2,n_k})^2\Big).$$

We are going to show first that the limit $\lim_{i\to\infty} \ell_{t_i}(\alpha_1)/\ell_{t_i}(\alpha_2)$ does not exist. By $\alpha_{1,1}(i)$ and $\alpha_{1,2}(i)$ (respectively $\alpha_{2,1}(i)$ and $\alpha_{2,2}(i)$) we denote curves on S_1 (respectively S_2) which are the first and second shortest in the flat metric at time t_i.

$\mathbf{i = 2k}$ We will show $\sup_k \ell_{t_{2k}}(\alpha_1)/\ell_{t_{2k}}(\alpha_2) = \infty$. We have for sufficiently large k,

$$\ell_{t_{2k}}(\alpha_2) \leq \ell_{t_{2k}}(\alpha_{2,1}(2k))\, i(\alpha_{2,2}(2k), \alpha_2) + \ell_{t_{2k}}(\alpha_{2,2}(2k))\, i(\alpha_{2,1}(2k), \alpha_2). \tag{7}$$

Lemma 1 implies that $\alpha_{2,1}(2k)$ is a $(q_{2,n_k-1}, p_{2,n_k-1})$–curve. Thus

$$i(\alpha_{2,1}(2k), \alpha_2) = p_{2,n_k-1}. \tag{8}$$

Corollary 1 implies that

$$\ell_{t_{2k}}(\alpha_{2,1}(2k)) = \Theta(1/a_{2,n_k}). \tag{9}$$

If the second shortest curve is a (q, p)–curve, then by Lemma 2 we have $p \leq p_{2,n_k}$. We can assume $\alpha_{2,2}(2k)$ is the (q_{2,n_k}, p_{2,n_k})–curve. Therefore

$$i(\alpha_{2,2}(2k), \alpha_2) = p = p_{2,n_k}. \tag{10}$$

We need to estimate $\ell_{t_{2k}}(\alpha_{2,2}(2k))$, and for our purposes it is enough to show that

$$\ell_{t_{2k}}(\alpha_{2,2}(2k)) = \Theta(\log(a_{2,n_k})). \tag{11}$$

An easy calculation shows that the flat length of $\alpha_{2,2}(2k)$ satisfies

$$l^2_{t_{2k}}(\alpha_{2,2}(2k)) \approx a_{2,n_k}.$$

By the Lemma 3, for t and n large enough, the extremal length

$$\mathrm{Ext}_{t_{2k}}(\alpha_{2,2}(2k)) \approx l^2_{t_{2k}}(\alpha_{2,2}(2k)) \approx a_{2,n_k}.$$

We will be using the well-known Collar Lemma (Theorem 4.1.1 in Buser [2]) several times in this paper. The lemma implies that any simple closed geodesic β on a closed Riemann surface X of genus $g \geq 2$ has a collar of width

$$w(\beta) = \operatorname{arcsinh} \frac{1}{\sinh(1/2\ell_X(\beta))} \tag{12}$$

where $\ell_X(\beta)$ is the hyperbolic length of β. The collar is isometric to the cylinder

$$[-w(\beta), w(\beta)] \times \mathbb{S}^1$$

with the Riemannian metric $ds^2 = d\rho^2 + \ell_X^2(\beta) \cosh^2 \rho dt^2$. Hence the hyperbolic length of any curve which intersects β nontrivially is at least $2w(\beta)$.

Let α be the shortest curve in the hyperbolic metric at time t_{2k} such that

$$i(\alpha, \alpha_{2,2}(2k)) = 1.$$

Its length by (12) is at least

$$2 \cdot \operatorname{arcsinh}\Big(\frac{1}{\frac{1}{2}\sinh(\ell_{t_{2k}}(\alpha_{2,1}(2k)))}\Big) \approx \log a_{2,n_k}.$$

In fact, it is easy to see that outside the collar around $\alpha_{2,1}(2k)$ the curve α is bounded. Therefore $\ell(\alpha) \approx \log a_{2,n_k}$. By the estimate (6),

$$\operatorname{Ext}(\alpha) \leq \frac{1}{2}\ell(\alpha)\exp^{\ell(\alpha)/2}.$$

Minsky has shown in [10] that

$$i^2(\alpha, \alpha_{2,2}(2k)) \leq \operatorname{Ext}(\alpha)\operatorname{Ext}(\alpha_{2,2}(2k)).$$

Hence $$i(\alpha, \alpha_{2,2}(2k)) \leq \Theta(a_{2,n_k} \log^{1/2}(a_{2,n_k})).$$

Using (9), and the fact that $i(\alpha_{2,1}(2k), \alpha_{2,2}(2k)) = 1$ we see that

$$\ell_{t_{2k}}(\alpha_{2,2}(2k)) = \Theta(\log a_{2,n_k}) \tag{13}$$

and using (7), (8), (9), (10) and (13) we conclude that

$$\ell_{t_{2k}}(\alpha_2) = O(\log(a_{2,n_k}) p_{2,n_k-1}). \tag{14}$$

Now we want to estimate $\ell_{t_{2k}}(\alpha_1)$. Recall that θ_1 is such that the elements satisfy $\sup_i a_{1,i} < \infty$. Then Lemma 1 and Lemma 3 imply that $\operatorname{Ext}_{t_{2k}} \alpha_{1,1}(2k) = \Theta(1)$. Hence $\ell_{t_{2k}}(\alpha_{1,1}(2k)) = \Theta(1)$. By (12) the collar around $\alpha_{1,1}(2k)$ is of bounded width and therefore the hyperbolic length of (α_1) is bounded below by the number of its intersections with $\alpha_{1,1}(2k)$:

$$\ell_{t_{2k}}(\alpha_1) \geq \Theta(i(\alpha_{1,1}(2k), \alpha_1)) \tag{15}$$

If $\alpha_{1,1}(2k)$ is a (q_{1,j_k}, p_{1,j_k})–curve, then $i(\alpha_{1,1}(2k), \alpha_1) = p_{1,j_k} \approx q_{1,j_k}\theta_1$. Since $\alpha_{1,1}(2k)$ is the shortest curve on S_1 at time t_{2k}, it follows from Lemma 1 that the curve $(q_{1,j_k+1}, p_{1,j_k+1})$ has not reached its minimal length yet, ie

$$t_{2k} \leq \log \frac{p_{1,j_k+1}\theta_1 + q_{1,j_k+1}}{|q_{1,j_k+1}\theta_1 - p_{1,j_k+1}|}.$$

Hence for sufficiently large k, using (1b), we get

$$\begin{aligned} q_{2,n_k} q_{2,n_k-1}(1+\theta_2^2) &\leq \frac{p_{2,n_k-1}\theta + q_{2,n_k-1}}{|q_{2,n_k-1}\theta - p_{2,n_k-1}|} \\ &\leq \frac{p_{1,j_k+1}\theta_1 + q_{1,j_k+1}}{|q_{1,j_k+1}\theta_1 - p_{1,j_k+1}|} \\ &\leq (1+\theta_1^2) q_{1,j_k+1}(q_{1,j_k+1} + q_{1,j_k+2}). \end{aligned}$$

We then have $q_{1,j_k} \geq \Theta(q_{2,n_k-1}\sqrt{a_{2,n_k}})$. It follows that

$$\ell_{t_{2k}}(\alpha_1) \geq \Theta(q_{2,n_k-1}\sqrt{a_{2,n_k}}). \tag{16}$$

Putting together (14) and (16) we get

$$\frac{\ell_{t_{2k}}(\alpha_1)}{\ell_{t_{2k}}(\alpha_2)} \geq \frac{\Theta(q_{2,n_k-1}\sqrt{a_{2,n_k}})}{\Theta(\log(a_{2,n_k}) p_{2,n_k-1})} \underset{n\to\infty}{\to} \infty. \tag{17}$$

$\mathbf{i = 2k+1}$ We will demonstrate that $\sup_k \ell_{t_{2k+1}}(\alpha_1)/\ell_{t_{2k+1}}(\alpha_2)$ is bounded above. Recall that $t_{2k+1} = \log\big((1+\theta_2^2)(q_{2,n_k})^2\big)$. Is is easy to see that $t_{2k} < t_{2k+1} < t_{2k+2}$. Hence by Lemma 1, $\alpha_{2,1}(2k)$ is either $(q_{2,n_k-1}, p_{2,n_k-1})$– or (q_{2,n_k}, p_{2,n_k})–curve. By Lemma 3 both $(q_{2,n_k-1}, p_{2,n_k-1})$ and (q_{2,n_k}, p_{2,n_k}) have at time t_{2k+1} extremal (and hyperbolic) length bounded above and below. Then

$$\ell_{t_{2k+1}}(\alpha_2) \geq \Theta(p_{2,n_k}). \tag{18}$$

On the other hand $\alpha_{1,1}(2k+1)$ and $\alpha_{1,2}(2k+1)$ also have bounded extremal length, and therefore

$$\ell_{t_{2k+1}}(\alpha_1) = O(i(\alpha_{1,1}(2k+1), \alpha_1)) + O(i(\alpha_{1,2}(2k+1), \alpha_1)).$$

To estimate $i(\alpha_{1,1}(2k+1),\alpha_1)$ we argue similarly to the case when $i = 2k$. If $\alpha_{1,1}(2k+1)$ is a (q_{1,j_k}, p_{1,j_k})–curve, then Lemma 1 implies

$$t_{2k+1} \geq \log \frac{p_{1,j_k-1}\theta_1 + q_{1,j_k-1}}{|q_{1,j_k-1}\theta_1 - p_{j_k-1}|}.$$

Hence, using (1b), for sufficiently large k we obtain

$$(1+\theta_2^2)q_{2,n_k}^2 \geq \frac{p_{1,j_k-1}\theta_1 + q_{1,j_k-1}}{|q_{1,j_k-1}\theta_1 - p_{j_k-1}|} \geq q_{1,j_k-1}(1+\theta_1^2)q_{1,j_k}.$$

Thus $i(\alpha_{1,1}(2k+1),\alpha_1) = p_{1,j_k} = \Theta(q_{1,j_k}) = O(q_{2,n_k})$. Lemma 2 implies that

$$i(\alpha_{1,2}(2k+1),\alpha_1) \leq p_{1,j_k+1} = \Theta(p_{1,j_k}).$$

We then have

$$\ell_{t_{2k+1}}(\alpha_1) = O(q_{2,n_k}). \tag{19}$$

Putting (18) and (19) together we obtain

$$\sup_n \frac{\ell_{t_{2k+1}}(\alpha_1)}{\ell_{t_{2k+1}}(\alpha_2)} < \infty. \tag{20}$$

To finish the proof we need to rule out the possibility that the limit is a projective measured foliation $[F]$ so that $i(\alpha_1, F) = i(\alpha_2, F) = 0$. This could be the case if the collar around σ grew a lot faster than α_1 and α_2. In this case $[F]=[\sigma]$. By Theorem 3 this is impossible. □

7 The limit set of X_t

In this section we say what the limit set of the geodesic ray is. Note that the only condition we put on θ_1 and θ_2 is that they are both irrational.

Let $\{X_t\}$ be a Teichmüller ray constructed as in Section 1, with $\theta_1, \theta_2 \in \mathbb{R} - \mathbb{Q}$. Let F_{θ_1,θ_2} be the vertical measured foliation of the quadratic differential q_{θ_1,θ_2}. Let Λ be the simplex of all measures on the underlying foliation of F_{θ_1,θ_2}. Also denote by L_{θ_1,θ_2} the limit set of $\{X_t\}$ in $\mathcal{PMF}$.

Theorem 3 *L_{θ_1,θ_2} is a connected subset of one dimensional simplex in $[\Lambda]$. If $[F] \in L_{\theta_1,\theta_2}$ then the weight it puts on σ is 0.*

Suppose the slopes θ_1 and θ_2 are chosen so that one has bounded elements, and the other unbounded. It follows from Theorem 2 and Theorem 3 that L_{θ_1,θ_2} is a one-dimensional simplex. Moreover, the proof of Theorem 2 shows that the ratio $\ell_t(\alpha_1)/\ell_t(\alpha_2)$ gets arbitrary large, and hence L_{θ_1,θ_2} contains the projective class of the measured foliation which puts all the weight on S_1. Hence we have the following:

Corollary 2 *Let $\{X_t\}$ be as above. Suppose also that the elements of θ_1 are bounded and those of θ_2 are unbounded. Then L_{θ_1,θ_2} is a one-dimensional simplex. Further, one of the two endpoints is such that the measure is supported on S_1.*

Proof of Theorem 3 Recall that a sequence $\{X_n\}$ in Teichmüller space converges to a projective measured foliation $[F]$ in $\mathcal{PMF}$ if there is $r_n \to \infty$ such that for any $\alpha \in S$ we have $r_n \ell_n(\alpha) \to i(\alpha, F)$ as $n \to \infty$. The theorem claims that the limit points are projective measured foliations which are the same as $[F_{\theta_1,\theta_2}]$, except for different ratios of weights on S_1 and S_2, which come from different accumulation points of $\{\ell_t(\alpha_1)/\ell_t(\alpha_2)\}$ as $t \to \infty$.

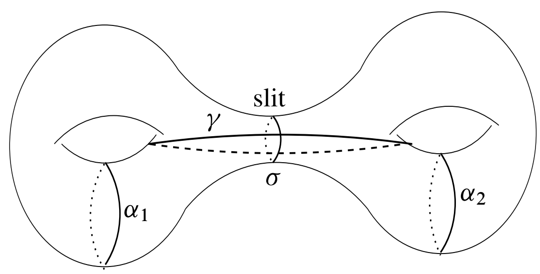

Figure 4

The fact that $L_{\theta_1,\theta_2} \in [\Lambda]$ can be demonstrated as follows. Suppose $X_{t_n} \to [F]$ as $n \to \infty$. We recall first that the space of measured foliations $\mathcal{MF}(M)$ can be identified with the space of measured laminations $\mathcal{ML}(M)$. See G Levitt's paper [6] for details. In particular, if a foliation $G \in \mathcal{MF}$ corresponds to a lamination $\mu_G \in \mathcal{ML}$ then $i(G, \cdot) = i(\mu_G, \cdot)$. Let μ and μ_{θ_1,θ_2} be the measured laminations corresponding to F and F_{θ_1,θ_2}. Then

$$r_n \ell_{t_n}(\mu_{\theta_1,\theta_2}) \to i(\mu, \mu_{\theta_1,\theta_2}) = i(F, F_{\theta_1,\theta_2}).$$

Since $\ell_{t_n}(\mu_{\theta_1,\theta_2}) \to 0$, we conclude that $i(F, F_{\theta_1,\theta_2}) = 0$. Since F_{θ_1,θ_2} intersects every simple closed curve except for σ, and F is not minimal, F has one closed leaf which is in the homotopy class of σ. It is also easy to see that any other leaf of F is dense in S_1 or S_2 and has a slope of θ_1 or θ_2. Therefore $F \in \Lambda$.

To prove that L_{θ_1,θ_2} is contained in the one dimensional simplex in $[\Lambda]$ we consider a curve γ which crosses the dividing curve σ (Figure 4) and estimate its hyperbolic length along the geodesic ray. It suffices to demonstrate that

$$\frac{\ell_t(\gamma)}{\ell_t(\alpha_1)+\ell_t(\alpha_2)} \to 1. \tag{21}$$

At any time t the image of the curve γ can be thought of as a curve made of an arc parallel to the image of α_1 followed by an arc crossing the slit σ and perhaps winding around σ for a while, then followed by an arc parallel to the image of α_2 and then by another arc crossing σ. Hence the length of γ in the hyperbolic metric at time t satisfies

$$\ell_t(\alpha_1)+\ell_t(\alpha_2)-C \leq \ell_t(\gamma) \leq \ell_t(\alpha_1)+\ell_t(\alpha_2)+2c_t+C$$

where c_t is the hyperbolic length of an arc connecting the geodesic (in the flat metric) representatives of α_1 and α_2 at time t and parallel to γ, and C is some positive constant independent of t. Therefore

$$1-\frac{C}{\ell_t(\alpha_1)+\ell_t(\alpha_2)} \leq \frac{\ell_t(\gamma)}{\ell_t(\alpha_1)+\ell_t(\alpha_2)} \leq 1+\frac{2c_t+C}{\ell_t(\alpha_1)+\ell_t(\alpha_2)}. \tag{22}$$

It suffices to show that

$$\frac{c_t}{\ell_t(\alpha_1)+\ell_t(\alpha_2)} \to 0. \tag{23}$$

Consider the cylinder $A \in X_t$ with σ the core curve which boundary components are Euclidean circles C_1 and C_2 of radii R_1 and R_2 (see Figure 5). We want to have an upper bound for its modulus. Since $\mathrm{Mod}(A) = 1/\mathrm{Ext}(\Gamma)$, where Γ is the family of curves homotopic to a core curve of A, we first find a good lower bound for $\mathrm{Ext}(\Gamma)$.

Let z_j be the midpoint of the slit on S_j for $j = 1, 2$. Let $A_j \subset S_j$ be the annulus centered at z_j, with inner radius $r_j = \frac{1}{2}se^{-t/2}$ and outer radius R_j. Define a metric $\rho^t(z)$ on A to be

$$\rho^t(z) = \left\{ \begin{array}{c} \frac{1}{2\pi r},\ \text{if } z \in A_j,\ j=1,2 \\ \frac{1}{2\pi r_j},\ \text{if } z \in A \setminus (\cup A_j) \end{array} \right\}.$$

Then it is easy to see that

$$A(\rho^t) = \frac{1}{2\pi}\log\frac{R_1R_2}{r_1r_2}+\frac{1}{2\pi} = \frac{1}{2\pi}(\log\frac{4R_1R_2}{s^2}+t+1)$$

and
$$\inf_{\beta \leadsto \alpha}\int_\beta \rho|dz| = \frac{1}{2\pi r_1}l_t(\sigma) = \frac{1}{2\pi se^{-t/2}}2se^{-t/2} = \frac{1}{\pi}.$$

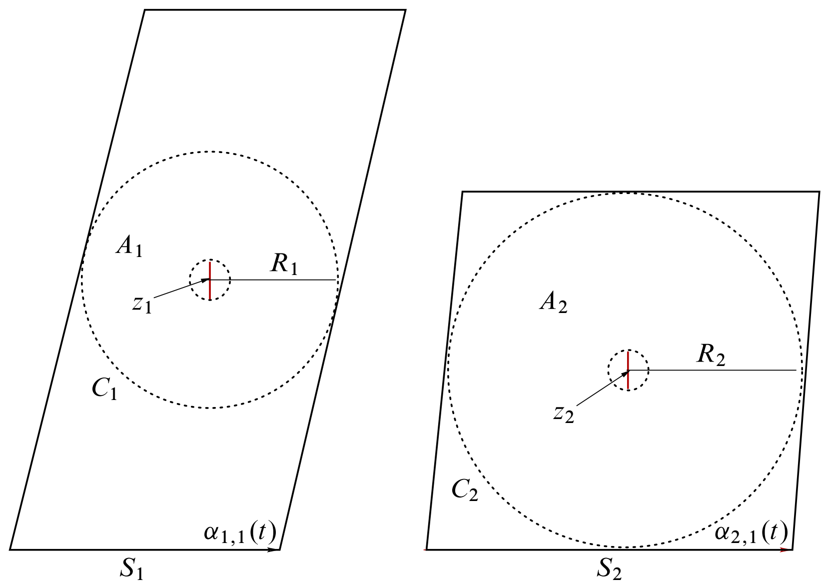

Figure 5

Hence
$$\mathrm{Ext}(\Gamma) \geq \frac{2}{\pi(\log \frac{4R_1R_2}{s^2} + t + 1)}$$

and noticing that $R_j \leq l_t(\alpha_{j,1}(t))$ we now have

$$\mathrm{Mod}(A) \leq \frac{(\log \frac{4R_1R_2}{s^2} + t + 1)}{2} = \Theta(t). \tag{24}$$

Now the cylinder is conformally equivalent to the annulus

$$\mathcal{A} = \{z \in \mathbb{C} | e^{-2\pi\,\mathrm{Mod}(A)} < |z| < 1\}$$

with the hyperbolic metric [8, Lemma 3, p 188]

$$\rho_t(z) = \frac{-\pi|dz|}{|z|(2\,\mathrm{Mod}(A))\sin(\log|z|/(2\,\mathrm{Mod}(A)))}.$$

Further, applying Lemma 4 in [8] we see that c_t is approximately equal to the length of the radius of $\mathcal{A}$. More precisely, given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{\rho_{X_t}(z)}{\rho_t(z)} - 1\right| < \epsilon,\ z \in \mathcal{A}_\delta = \{e^{-2\pi m}/\delta < |z| < \delta\} \tag{25}$$

where ρ_{X_t} is the hyperbolic metric of X_t. A simple calculation shows that a radius of $\mathcal{A}_\delta$ in the metric $\rho_t(z)$ is

$$\left(\log \frac{1+\cos\frac{\log 1/\delta}{2\,\mathrm{Mod}(A)}}{1-\cos\frac{\log 1/\delta}{2\,\mathrm{Mod}(A)}}\right)(1+\epsilon) = \Theta(\log(\mathrm{Mod}(A))).$$

Hence $$c_t = \Theta(\log(\mathrm{Mod}(A))) = O(\log t).$$

On the other hand, if $\alpha_{j,1}(t)$ is a (q_{j,n_j}, p_{j,n_j}) curve for some n_j and $j = 1, 2$ then

$$e^{t/2} = O\Big(\frac{1}{|q_{j,n_j} - \theta_j p_{j,n_j}|}\Big) = O(q_{j,n_j} + q_{j,n_j+1})$$

and hence $$t = O(\log(q_{j,n_j} + q_{j,n_j+1})) = O(\log a_{j,n_j+1} q_{j,n_j}).$$

Therefore we have $$c_t = O(\log\log a_{j,n_j+1} q_{j,n_j})).$$

Since $\ell_t(\alpha_j) \geq \Theta(p_{j,n_j} \log(a_{j,n_j+1}))$, it is clear that (23) holds.

The function $\ell_t(\alpha_1)/\ell_t(\alpha_2)$ is continuous, and therefore the limit set is an interval or a point. This concludes the proof. □

References

[1] **L V Ahlfors**, *Lectures on quasiconformal mappings*, second edition, University Lecture Series 38, American Mathematical Society, Providence, RI (2006) MR2241787With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard

[2] **P Buser**, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics 106, Birkhäuser, Boston (1992) MR1183224

[3] **A Fathi**, **F Laudenbach**, **V Poenaru**, *Travaux de Thurston sur les surfaces*, Astérisque 66, Société Mathématique de France, Paris (1979) MR568308 Séminaire Orsay, with an English summary

[4] **Y Imayoshi**, **M Taniguchi**, *An introduction to Teichmüller spaces*, Springer, Tokyo (1992) MR1215481Translated and revised from the Japanese by the authors

[5] **A Y Khinchin**, *Continued fractions*, russian edition, Dover Publications, Mineola, NY (1997) MR1451873With a preface by B. V. Gnedenko, Reprint of the 1964 translation

[6] **G Levitt**, *Foliations and laminations on hyperbolic surfaces*, Topology 22 (1983) 119–135 MR683752

[7] **B Maskit**, *Comparison of hyperbolic and extremal lengths*, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985) 381–386 MR802500

[8] **H Masur**, *Two boundaries of Teichmüller space*, Duke Math. J. 49 (1982) 183–190 MR650376

[9] **H Masur**, **S Tabachnikov**, *Rational billiards and flat structures*, from: "Handbook of dynamical systems, Vol. 1A", (B Hasselblatt, A Katok, editors), North-Holland, Amsterdam (2002) 1015–1089 MR1928530

[10] **Y N Minsky**, *Teichmüller geodesics and ends of hyperbolic 3-manifolds*, Topology 32 (1993) 625–647 MR1231968

[11] **K Strebel**, *Quadratic differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 5, Springer, Berlin (1984) MR743423

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