Geometry & Topology 12 (2008) 299–350 **299**

Floer homology and surface decompositions

ANDRÁS JUHÁSZ

Sutured Floer homology, denoted by SFH, is an invariant of balanced sutured manifolds previously defined by the author. In this paper we give a formula that shows how this invariant changes under surface decompositions. In particular, if $(M,\gamma) \rightsquigarrow (M',\gamma')$ is a sutured manifold decomposition then $SFH(M',\gamma')$ is a direct summand of $SFH(M,\gamma)$. To prove the decomposition formula we give an algorithm that computes $SFH(M,\gamma)$ from a balanced diagram defining (M,γ) that generalizes the algorithm of Sarkar and Wang.

As a corollary we obtain that if (M,γ) is taut then $SFH(M,\gamma) \neq 0$. Other applications include simple proofs of a result of Ozsváth and Szabó that link Floer homology detects the Thurston norm, and a theorem of Ni that knot Floer homology detects fibred knots. Our proofs do not make use of any contact geometry.

Moreover, using these methods we show that if K is a genus g knot in a rational homology 3–sphere Y whose Alexander polynomial has leading coefficient $a_g \neq 0$ and if $\mathrm{rk}\widehat{HFK}(Y,K,g) < 4$ then $Y \setminus N(K)$ admits a depth ≤ 2 taut foliation transversal to $\partial N(K)$.

57M27, 57R58

1 Introduction

In [7] we defined a Floer homology invariant for balanced sutured manifolds. In this paper we study how this invariant changes under surface decompositions. We need some definitions before we can state our main result. Recall that Spin^c structures on sutured manifolds were defined in [7]; all the necessary definitions can also be found in Section 3 of the present paper.

Definition 1.1 Let (M,γ) be a balanced sutured manifold and let $(S,\partial S) \subset (M,\partial M)$ be a properly embedded oriented surface. An element $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ is called *outer* with respect to S if there is a unit vector field v on M whose homology class is $\mathfrak{s}$ and $v_p \neq -(\nu_S)_p$ for every $p \in S$. Here ν_S is the unit normal vector field of S with respect to some Riemannian metric on M. Let O_S denote the set of outer Spin^c structures.

Published: 12 March 2008 DOI: 10.2140/gt.2008.12.299

Definition 1.2 Suppose that R is a compact, oriented, open surface. Let C be an oriented simple closed curve in R. If $[C] = 0$ in $H_1(R;\mathbb{Z})$ then $R \setminus C$ can be written as $R_1 \cup R_2$, where R_1 is the component of $R \setminus C$ that is disjoint from ∂R and satisfies $\partial R_1 = C$. We call R_1 the *interior* and R_2 the *exterior* of C.

We say that the curve C is *boundary-coherent* if either $[C] \neq 0$ in $H_1(R;\mathbb{Z})$, or if $[C] = 0$ in $H_1(R;\mathbb{Z})$ and C is oriented as the boundary of its interior.

Theorem 1.3 *Let (M,γ) be a balanced sutured manifold and let $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ be a sutured manifold decomposition. Suppose that S is open and for every component V of $R(\gamma)$ the set of closed components of $S \cap V$ consists of parallel oriented boundary-coherent simple closed curves. Then*

$$SFH(M',\gamma') = \bigoplus_{\mathfrak{s} \in O_S} SFH(M,\gamma,\mathfrak{s}).$$

In particular, $SFH(M',\gamma')$ is a direct summand of $SFH(M,\gamma)$.

In order to prove Theorem 1.3 we generalize the algorithm of Sarkar and Wang [16] to give an algorithm that computes $SFH(M,\gamma)$ from any given balanced diagram of (M,γ).

From Theorem 1.3 we will deduce the following two theorems. These provide us with positive answers to Question 9.19 and Conjecture 10.2 of [7].

Theorem 1.4 *Suppose that the balanced sutured manifold (M,γ) is taut. Then*

$$\mathbb{Z} \le SFH(M,\gamma).$$

If Y is a closed connected oriented 3–manifold and $R \subset Y$ is a compact oriented surface with no closed components then we can obtain a balanced sutured manifold $Y(R) = (M,\gamma)$, where $M = Y \setminus \mathrm{Int}(R \times I)$ and $\gamma = \partial R \times I$; see [7, Example 2.6]. Furthermore, if $K \subset Y$ is a knot, $\alpha \in H_2(Y,K;\mathbb{Z})$, and $i \in \mathbb{Z}$ then let

$$\widehat{HFK}(Y,K,\alpha,i) = \bigoplus_{\substack{\mathfrak{s} \in \mathrm{Spin}^c(Y,K): \\ \langle c_1(\mathfrak{s}),\alpha\rangle = 2i}} \widehat{HFK}(Y,K,\mathfrak{s}).$$

Theorem 1.5 *Let K be a null-homologous knot in a closed connected oriented 3–manifold Y and let $S \subset Y$ be a Seifert surface of K. Then*

$$SFH(Y(S)) \approx \widehat{HFK}(Y,K,[S],g(S)).$$

Remark 1.6 Theorem 1.5 implies that the invariant $\widehat{HFS}$ of balanced sutured manifolds defined by Ni [9] is equal to SFH.

Putting these two theorems together we get a new proof of the fact proved by Ozsváth and Szabó [13] that knot Floer homology detects the genus of a knot. In particular, if Y is a rational homology 3–sphere then $\widehat{HFK}(K, g(K))$ is nonzero and $\widehat{HFK}(K,i)=0$ for $i > g(K)$.

Further applications include a simple proof of a theorem that link Floer homology detects the Thurston norm, which was proved for links in S^3 in Ozsváth and Szabó [12]. We generalize this result to links in arbitrary 3–manifolds. Here we do not use any symplectic or contact geometry. We also show that the Murasugi sum formula proved in [10] is an easy consequence of Theorem 1.3. The main application of our apparatus is a simplified proof that shows knot Floer homology detects fibred knots. This theorem was conjectured by Ozsváth and Szabó and first proved by Ni [9]. Here we avoid the contact topology of Ghiggini [6] and this allows us to simplify some of the arguments in [9].

To show the strength of our approach we prove the following extension of the main result of [9]. First we review a few definitions about foliations; see Gabai [4, Definition 3.8].

Definition 1.7 Let $\mathcal{F}$ be a codimension one transversely oriented foliation. A leaf of $\mathcal{F}$ is of *depth* 0 if it is compact. Having defined the depth $< p$ leaves we say that a leaf L is depth p if it is proper (ie, the subspace topology on L equals the leaf topology), L is not of depth $< p$, and $\overline{L} \setminus L$ is contained in the union of depth $< p$ leaves. If $\mathcal{F}$ contains nonproper leaves then the depth of a leaf may not be defined.

If every leaf of $\mathcal{F}$ is of depth at most n and $\mathcal{F}$ has a depth n leaf then we say that $\mathcal{F}$ is *depth n*.

A foliation $\mathcal{F}$ is *taut* if there is a single circle C transverse to $\mathcal{F}$ which intersects every leaf.

Theorem 1.8 *Let K be a null-homologous genus g knot in a rational homology 3–sphere Y. Suppose that the coefficient a_g of the Alexander polynomial $\Delta_K(t)$ of K is nonzero and*

$$\operatorname{rk} \widehat{HFK}(Y, K, g) < 4.$$

Then $Y \setminus N(K)$ has a depth ≤ 2 taut foliation transverse to $\partial N(K)$.

This constitutes progress in the direction of solving [5, Problem 8.4]. The problem has several variations; the following is one of them.

Problem 1.9 Let K be a knot in S^3. Compute the minimal n such that there is a depth n foliation on $S^3 \setminus N(K)$.

Acknowledgements I am grateful for the guidance of Zoltán Szabó during the course of this work. I would also like to thank David Gabai, Paolo Ghiggini, and Yi Ni for the helpful discussions.

This research was partially supported by OTKA grant no T49449.

2 Preliminary definitions

First we briefly review the basic definitions concerning balanced sutured manifolds and the Floer homology invariant which we defined for them in [7].

Definition 2.1 A *sutured manifold* (M,γ) is a compact oriented 3–manifold M with boundary together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a *suture*, ie, a homologically nontrivial oriented simple closed curve. We denote the union of the sutures by $s(\gamma)$.

Finally every component of $R(\gamma) = \partial M \setminus \mathrm{Int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M \setminus \mathrm{Int}(\gamma)$ whose normal vectors point out of (into) M. The orientation on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, ie, if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture.

Definition 2.2 A sutured manifold (M,γ) is called *balanced* if M has no closed components, $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$, and the map $\pi_0(A(\gamma)) \to \pi_0(\partial M)$ is surjective.

Notation 2.3 Throughout this paper we are going to use the following notation. If K is a submanifold of the manifold M then $N(K)$ denotes a regular neighborhood of K in M.

For the following see Examples 2.3, 2.4, and 2.5 in [7].

Definition 2.4 Let Y be a closed connected oriented 3–manifold. Then the balanced sutured manifold $Y(1)$ is obtained by removing an open ball from Y and taking an annular suture on its boundary.

Suppose that L is a link in Y. The balanced sutured manifold $Y(L) = (M,\gamma)$, where $M = Y \setminus N(L)$ and for each component L_0 of L the sutures $\partial N(L_0) \cap s(\gamma)$ consist of two oppositely oriented meridians of L_0.

Finally, if S is a Seifert surface in Y then the balanced sutured manifold $Y(S) = (N,\nu)$, where $N = Y \setminus \mathrm{Int}(S \times I)$ and $\nu = \partial S \times I$.

The following definition can be found for example in Scharlemann [17].

Definition 2.5 Let S be a compact oriented surface (possibly with boundary) whose components are $S_1, \dots, S_n$. Then define the *norm* of S to be

$$x(S) = \sum_{i:\chi(S_i)<0} |\chi(S_i)|.$$

Let M be a compact oriented 3–manifold and let N be a subsurface of ∂M. For $s \in H_2(M, N; \mathbb{Z})$ we define its *norm* $x(s)$ to be the minimum of $x(S)$ taken over all properly embedded surfaces $(S, \partial S)$ in (M, N) such that $[S, \partial S] = s$.

If $(S, \partial S) \subset (M, N)$ is a properly embedded oriented surface then we say that S is *norm minimizing* in $H_2(M, N)$ if S is incompressible and $x(S) = x([S, \partial S])$ for $[S, \partial S] \in H_2(M, N; \mathbb{Z})$.

Definition 2.6 A sutured manifold (M, γ) is *taut* if M is irreducible and $R(\gamma)$ is norm minimizing in $H_2(M, \gamma)$.

Next we recall the definition of a sutured manifold decomposition; see Gabai [2, Definition 3.1].

Definition 2.7 Let (M, γ) be a sutured manifold. A *decomposing surface* is a properly embedded oriented surface S in M such that for every component λ of $S \cap \gamma$ one of (1)–(3) holds:

(1) λ is a properly embedded nonseparating arc in γ such that $|\lambda \cap s(\gamma)| = 1$.
(2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A \cap s(\gamma)$.
(3) λ is a homotopically nontrivial curve in a torus component T of γ, and if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

Then S defines a *sutured manifold decomposition*

$$(M, \gamma) \rightsquigarrow^S (M', \gamma'),$$

where

$$\begin{aligned} M' &= M \setminus \mathrm{Int}(N(S)), \\ \gamma' &= (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma)), \\ R_+(\gamma') &= ((R_+(\gamma) \cap M') \cup S'_+) \setminus \mathrm{Int}(\gamma'), \\ R_-(\gamma') &= ((R_-(\gamma) \cap M') \cup S'_-) \setminus \mathrm{Int}(\gamma'), \end{aligned}$$

where S'_+ (S'_-) is the component of $\partial N(S) \cap M'$ whose normal vector points out of (into) M'.

Definition 2.8 A decomposing surface S in (M, γ) is called a *product disk* if S is a disk such that $|D \cap s(\gamma)| = 2$. A surface decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ is called a *product decomposition* if S is a product disk.

Definition 2.9 A decomposing surface S lying in the sutured manifold (M, γ) is called a *product annulus* if S is an annulus, one component of ∂S is contained in $R_+(\gamma)$, and the other component is contained in $R_-(\gamma)$.

Definition 2.10 A *sutured Heegaard diagram* is a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where Σ is a compact oriented surface with boundary and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two sets of pairwise disjoint simple closed curves in $\mathrm{Int}(\Sigma)$.

Every sutured Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ uniquely *defines* a sutured manifold (M, γ) using the following construction. Suppose $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_m\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_n\}$. Let M be the 3–manifold obtained from $\Sigma \times I$ by attaching 3–dimensional 2–handles along the curves $\alpha_i \times \{0\}$ and $\beta_j \times \{1\}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. The sutures are defined by taking $\gamma = \partial\Sigma \times I$ and $s(\gamma) = \partial\Sigma \times \{1/2\}$.

Definition 2.11 A sutured Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is called *balanced* if $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and the maps $\pi_0(\partial\Sigma) \to \pi_0(\Sigma \setminus \bigcup \boldsymbol{\alpha})$ and $\pi_0(\partial\Sigma) \to \pi_0(\Sigma \setminus \bigcup \boldsymbol{\beta})$ are surjective.

The following is [7, Proposition 2.14].

Proposition 2.12 *For every balanced sutured manifold (M, γ) there exists a balanced diagram defining it.*

Definition 2.13 For a balanced diagram let $\mathcal{D}_1, \dots, \mathcal{D}_m$ denote the closures of the components of $\Sigma \setminus (\bigcup \boldsymbol{\alpha} \cup \bigcup \boldsymbol{\beta})$ disjoint from $\partial\Sigma$. Then let $D(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be the free abelian group generated by $\{\mathcal{D}_1, \dots, \mathcal{D}_m\}$. This is of course isomorphic to $\mathbb{Z}^m$. We call an element of $D(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ a *domain*. An element $\mathcal{D}$ of $\mathbb{Z}^m_{\geq 0}$ is called a *positive* domain, we write $\mathcal{D} \geq 0$. A domain $\mathcal{P} \in D(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is called a *periodic domain* if the boundary of the 2–chain $\mathcal{P}$ is a linear combination of full α– and β–curves.

Definition 2.14 A balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is called *admissible* if every periodic domain $\mathcal{P} \neq 0$ has both positive and negative coefficients.

The following proposition is [7, Corollary 3.12].

Proposition 2.15 *If (M,γ) is a balanced sutured manifold such that*

$$H_2(M;\mathbb{Z})=0$$

and if $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is an arbitrary balanced diagram defining (M,γ) then there are no nonzero periodic domains in $D(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$. Thus any balanced diagram defining (M,γ) is automatically admissible.

For a surface Σ let $\mathrm{Sym}^d(\Sigma)$ denote the d–fold symmetric product $\Sigma^{\times d}/S_d$. This is a smooth $2d$–manifold. A complex structure j on Σ naturally endows $\mathrm{Sym}^d(\Sigma)$ with a complex structure. Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ be a balanced diagram, where $\boldsymbol{\alpha}=\{\alpha_1,\dots,\alpha_d\}$ and $\boldsymbol{\beta}=\{\beta_1,\dots,\beta_d\}$. Then the tori $\mathbb{T}_\alpha=(\alpha_1\times\dots\times\alpha_d)/S_d$ and $\mathbb{T}_\beta=(\beta_1\times\dots\times\beta_d)/S_d$ are d–dimensional totally real submanifolds of $\mathrm{Sym}^d(\Sigma)$.

Definition 2.16 Let $\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$. A domain $\mathcal{D}\in D(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is said to *connect* $\mathbf{x}$ *to* $\mathbf{y}$ if for every $1\le i\le d$ the equalities $\partial(\alpha_i\cap\partial\mathcal{D})=(\mathbf{x}\cap\alpha_i)-(\mathbf{y}\cap\alpha_i)$ and $\partial(\beta_i\cap\partial\mathcal{D})=(\mathbf{x}\cap\beta_i)-(\mathbf{y}\cap\beta_i)$ hold. We are going to denote by $D(\mathbf{x},\mathbf{y})$ the set of domains connecting $\mathbf{x}$ to $\mathbf{y}$.

Notation 2.17 Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$ and let $e_1=\{z\in\partial\mathbb{D}:\mathrm{Re}(z)\ge 0\}$ and $e_2=\{z\in\partial\mathbb{D}:\mathrm{Re}(z)\le 0\}$.

Definition 2.18 Let $\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ be intersection points. A *Whitney disc connecting* $\mathbf{x}$ *to* $\mathbf{y}$ is a continuous map $u\colon\mathbb{D}\to\mathrm{Sym}^d(\Sigma)$ such that $u(-i)=\mathbf{x}$, $u(i)=\mathbf{y}$ and $u(e_1)\subset\mathbb{T}_\alpha$, $u(e_2)\subset\mathbb{T}_\beta$. Let $\pi_2(\mathbf{x},\mathbf{y})$ denote the set of homotopy classes of Whitney discs connecting $\mathbf{x}$ to $\mathbf{y}$.

Definition 2.19 If $z\in\Sigma\setminus(\bigcup\boldsymbol{\alpha}\cup\bigcup\boldsymbol{\beta})$ and if u is a Whitney disc then choose a Whitney disc u' homotopic to u such that u' intersects the hypersurface $\{z\}\times\mathrm{Sym}^{d-1}(\Sigma)$ transversally. Define $n_z(u)$ to be the algebraic intersection number $u'\cap(\{z\}\times\mathrm{Sym}^{d-1}(\Sigma))$.

Definition 2.20 Let $\mathcal{D}_1,\dots,\mathcal{D}_m$ be as in Definition 2.13. For every $1\le i\le m$ choose a point $z_i\in\mathcal{D}_i$. Then the *domain of a Whitney disc* u is defined as

$$\mathcal{D}(u)=\sum_{i=1}^m n_{z_i}(u)\mathcal{D}_i\in D(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}).$$

If $\phi\in\pi_2(\mathbf{x},\mathbf{y})$ and if u is a representative of the homotopy class ϕ then let $\mathcal{D}(\phi)=\mathcal{D}(u)$.

Definition 2.21 We define the Maslov index of a domain $\mathcal{D} \in D(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ as follows. If there is a homotopy class ϕ of Whitney discs such that $\mathcal{D}(\phi) = \mathcal{D}$ then let $\mu(\mathcal{D}) = \mu(\phi)$. Otherwise we define $\mu(\mathcal{D})$ to be $-\infty$. It follows from Definition 2.20 that μ is additive on $D(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Furthermore, let $\mathcal{M}(\mathcal{D})$ denote the moduli space of holomorphic Whitney discs u such that $\mathcal{D}(u) = \mathcal{D}$ and let $\widehat{\mathcal{M}}(\mathcal{D}) = \mathcal{M}(\mathcal{D})/\mathbb{R}$.

Let (M, γ) be a balanced sutured manifold and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ an admissible balanced diagram defining it. Fix a coherent system of orientations as in [15, Definition 3.11]. Then for a generic almost complex structure the following holds: for each domain $\mathcal{D}$ such that $\mu(\mathcal{D}) = 1$ the moduli space $\widehat{\mathcal{M}}(\mathcal{D})$ is a compact oriented manifold of dimension 0, ie, a finite set of points with signs. We denote by $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ the free abelian group generated by the points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. We define an endomorphism $\partial\colon CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ such that on each generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ it is given by the formula

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \; \sum_{\{\, \mathcal{D} \in D(\mathbf{x}, \mathbf{y}) \colon \mu(\mathcal{D}) = 1 \,\}} \# \widehat{\mathcal{M}}(\mathcal{D}) \cdot \mathbf{y}.$$

Then $(CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), \partial)$ is a chain complex whose homology depends only on the underlying sutured manifold (M, γ). We denote this homology group by $SFH(M, \gamma)$.

For the following see [7, Proposition 9.1] and [7, Proposition 9.2].

Proposition 2.22 *If Y is a closed connected oriented 3–manifold then*

$$SFH(Y(1)) \approx \widehat{HF}(Y).$$

Furthermore, if L is a link in Y and $\vec{L}$ is an arbitrary orientation of L then

$$SFH(Y(L)) \otimes \mathbb{Z}_2 \approx \widehat{HFL}(\vec{L}).$$

3 Spin^c structures and relative Chern classes

First we review the definition of a Spin^c structure on a balanced sutured manifold (M, γ) that was introduced in [7]. Note that in a balanced sutured manifold none of the sutures are tori. Fix a Riemannian metric on M.

Notation 3.1 Let v_0 be a nowhere vanishing vector field along ∂M that points into M along $R_-(\gamma)$, points out of M along $R_+(\gamma)$, and on γ it is the gradient of the height function $s(\gamma) \times I \to I$. The space of such vector fields is contractible.

Definition 3.2 Let v and w be nowhere vanishing vector fields on M that agree with v_0 on ∂M. We say that v and w are *homologous* if there is an open ball $B \subset \mathrm{Int}(M)$ such that $v|(M \setminus B)$ is homotopic to $w|(M \setminus B)$ through nowhere vanishing vector fields rel ∂M. We define $\mathrm{Spin}^c(M,\gamma)$ to be the set of homology classes of nowhere vanishing vector fields v on M such that $v|\partial M = v_0$.

Definition 3.3 Let (M,γ) be a balanced sutured manifold and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ a balanced diagram defining it. To each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we assign a Spin^c structure $\mathfrak{s}(\mathbf{x}) \in \mathrm{Spin}^c(M,\gamma)$ as follows. Choose a Morse function f on M compatible with the given balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Then $\mathbf{x}$ corresponds to a multi-trajectory $\gamma_{\mathbf{x}}$ of $\mathrm{grad}(f)$ connecting the index one and two critical points of f. In a regular neighborhood $N(\gamma_{\mathbf{x}})$ we can modify $\mathrm{grad}(f)$ to obtain a nowhere vanishing vector field v on M such that $v|\partial M = v_0$. We define $\mathfrak{s}(\mathbf{x})$ to be the homology class of this vector field v.

Proposition 3.4 *The vector bundle $v_0^\perp$ over ∂M is trivial if and only if for every component F of ∂M the equality $\chi(F \cap R_+(\gamma)) = \chi(F \cap R_-(\gamma))$ holds.*

Proof Since $v_0^\perp|R_+(\gamma) = TR_+(\gamma)$ and $v_0^\perp|R_-(\gamma) = -TR_-(\gamma)$ we get that

$$\left\langle e(v_0^\perp|F), [F] \right\rangle = \chi(F \cap R_+(\gamma)) - \chi(F \cap R_-(\gamma)).$$

Moreover, the rank two bundle $v_0^\perp|F$ is trivial if and only if its Euler class vanishes. □

Definition 3.5 We call a sutured manifold (M,γ) *strongly balanced* if for every component F of ∂M the equality $\chi(F \cap R_+(\gamma)) = \chi(F \cap R_-(\gamma))$ holds.

Remark 3.6 Note that if (M,γ) is balanced then we can associate to it a strongly balanced sutured manifold (M',γ') such that (M,γ) can be obtained from (M',γ') by a sequence of product decompositions. We can construct such an (M',γ') as follows. If F_1 and F_2 are distinct components of ∂M then choose two points $p_1 \in s(\gamma) \cap F_1$ and $p_2 \in s(\gamma) \cap F_2$. For $i = 1, 2$ let D_i be a small neighborhood of p_i homeomorphic to a closed disc. We get a new sutured manifold by gluing together D_1 and D_2. Then (M,γ) can be retrieved by decomposing along $D_1 \sim D_2$. By repeating this process we get a sutured manifold (M',γ') with a single boundary component. Since (M,γ) was balanced (M',γ') is strongly balanced. By adding such product one-handles we can even achieve that γ is connected.

Definition 3.7 Suppose that (M,γ) is a strongly balanced sutured manifold. Let t be a trivialization of $v_0^\perp$ and let $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$. Then we define

$$c_1(\mathfrak{s}, t) \in H^2(M, \partial M; \mathbb{Z})$$

to be the relative Euler class of the vector bundle $v^\perp$ with respect to the trivialization t. In other words, $c_1(\mathfrak{s}, t)$ is the obstruction to extending t from ∂M to a trivialization of $v^\perp$ over M.

Definition 3.8 Let S be a decomposing surface in a balanced sutured manifold (M,γ) such that the positive unit normal field ν_S of S is nowhere parallel to v_0 along ∂S. This holds for generic S. We endow ∂S with the boundary orientation. Let us denote the components of ∂S by $T_1, \dots, T_k$.

Let w_0 denote the projection of v_0 into TS, this is a nowhere zero vector field. Moreover, let f be the positive unit tangent vector field of ∂S. For $1 \le i \le k$ we define the *index* $I(T_i)$ to be the number of times w_0 rotates with respect to f as we go around T_i. Then define

$$I(S) = \sum_{i=1}^{k} I(T_k).$$

Let $p(\nu_S)$ be the projection of ν_S into $v^\perp$. Observe that $p(\nu_S)|\partial S$ is nowhere zero. For $1 \le i \le k$ we define $r(T_i, t)$ to be the rotation of $p(\nu_S)|\partial T_i$ with respect to the trivialization t as we go around T_i. Moreover, let

$$r(S,t) = \sum_{i=1}^{k} r(T_i, t).$$

We introduce the notation

$$c(S,t) = \chi(S) + I(S) - r(S,t).$$

Lemma 3.9 *Let (M,γ) be a balanced sutured manifold and let S be a decomposing surface as in Definition 3.8.*

(1) *If T is a component of ∂S such that $T \not\subset \gamma$ then*

$$I(T) = -\frac{|T \cap s(\gamma)|}{2}.$$

(2) *Suppose that $T_1, \dots, T_a$ are components of ∂S such that $\mathcal{T} = T_1 \cup \dots \cup T_a \subset \gamma$ is parallel to $s(\gamma)$ and ν_S points out of M along $\mathcal{T}$. Then $I(T_j) = 0$ for $1 \le j \le a$; moreover,*

$$\sum_{j=1}^{a} r(T_j, t) = \chi(R_+(\gamma)).$$

Proof First we prove part (1). We can suppose that w_0 is tangent to T exactly at the points of $\partial T \cap s(\gamma)$. Then at a point $p \in T \cap s(\gamma)$ we have $w_0/|w_0| = f$ if and only if T goes from $R_-(\gamma)$ to $R_+(\gamma)$ and in that case w_0 rotates from the inside of S to the outside; see Figure 1. Thus w_0 rotates $-|T \cap s(\gamma)|/2$ times with respect to f as we go around T.

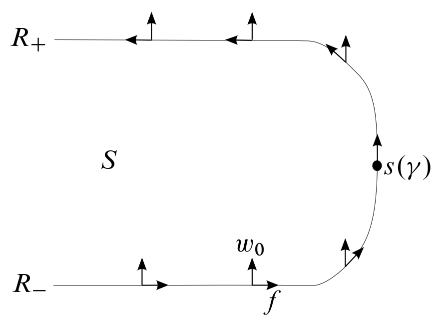

Figure 1: If $T \not\subset \gamma$ then the index $I(T)$ is $-|T \cap s(\gamma)|/2$.

Now we prove part (2). Let $1 \le j \le a$. Since ν_S points out of M along T_j we get that w_0 points into S along T_j. So w_0 and f are nowhere equal along T_j, and thus $I(T_j) = 0$.

Since $\mathcal{T}$ is parallel to $s(\gamma)$ it bounds a surface $\mathcal{R}_+ \subset \partial M$ which is diffeomorphic to $R_+(\gamma)$ and contains $R_+(\gamma)$. Since ν_S points out of M along $\mathcal{T}$ there is an isomorphism $i\colon v_0^\perp|\mathcal{R}_+ \to T\mathcal{R}_+$ such that $i(p(\nu_S))$ is an outward normal field of $\mathcal{R}_+$ along $\partial\mathcal{R}_+$. Moreover, $i(t|\mathcal{R}_+)$ gives a trivialization of $T\mathcal{R}_+$. Using the Poincaré–Hopf theorem we get that $p(\nu_S)$ rotates $\chi(\mathcal{R}_+) = \chi(R_+(\gamma))$ times with respect to t as we go around $\mathcal{T}$. $\square$

Recall that we defined the notion of an outer Spin^c structure in Definition 1.1.

Lemma 3.10 *Suppose that (M, γ) is a strongly balanced sutured manifold. Let t be a trivialization of $v_0^\perp$, let $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$, and let S be a decomposing surface in (M, γ) as in Definition 3.8. Denote the components of S by $S_1, \dots, S_k$. Then $\mathfrak{s}$ is outer with respect to S if and only if*

$$\langle c_1(\mathfrak{s}, t), [S_i] \rangle = c(S_i, t) \text{ for every } 1 \le i \le k. \tag{3–1}$$

In particular, if $\mathfrak{s} \in O_S$ then

$$\langle c_1(\mathfrak{s}, t), [S] \rangle = \sum_{i=1}^{k} c(S_i, t) = c(S, t).$$

Proof Endow M with an arbitrary Riemannian metric. First we show that if $\mathfrak{s} \in O_S$ then Equation (3–1) holds. Since $\mathfrak{s} \in O_S$ implies that $\mathfrak{s} \in O_{S_i}$ for every $1 \le i \le k$ we can assume that S is connected. Using the naturality of Chern classes it is sufficient to prove that if v is a unit vector field over S that agrees with v_0 over ∂S and is nowhere equal to $-\nu_S$ then $\langle\, c_1(v^\perp, t), [S] \,\rangle = c(S,t)$.

If we project ν_S into $v^\perp$ we get a section $p(\nu_S)$ of $v^\perp$ that vanishes exactly where $\nu_S = v$. We can perturb v slightly to make all tangencies between $v^\perp$ and S nondegenerate. Let e and h denote the number of elliptic, respectively hyperbolic tangencies between $v^\perp$ and S. At each such tangency the orientation of $v^\perp$ and TS agree. Thus $\langle\, c_1(v^\perp, t_1), [S] \,\rangle = e - h$, where $t_1 = p(\nu_S)|\partial S$. Since

$$\left\langle c_1(v^\perp, t_1) - c_1(v^\perp, t), [S] \right\rangle = r(S,t)$$

we get that

$$\left\langle c_1(v^\perp, t), [S] \right\rangle = e - h - r(S,t).$$

On the other hand, if we project v into TS we get a vector field w on S that is zero exactly at the points where $\nu_S = v$ as well. Note that w has index 1 exactly where $v^\perp$ and S have an elliptic tangency and has index -1 at hyperbolic tangencies. Moreover, $w|\partial S = w_0$. If we extend f to a vector field f_1 over S the sum of the indices of f_1 will by $\chi(S)$ by the Poincaré–Hopf theorem. Putting these observations together we get that

$$I(S) = (e-h) - \chi(S).$$

So we conclude that

$$\left\langle c_1(v^\perp, t), [S] \right\rangle = \chi(S) + I(S) - r(S,t) = c(S,t).$$

Now we prove that if for $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ Equation (3–1) holds then $\mathfrak{s} \in O_S$. Let STM denote the unit sphere bundle of TM. Then $v_0|\partial S$ is a section over ∂S of $(STM|S) \setminus (-\nu_S)$, which is a bundle over S with contractible fibers. Thus $v_0|\partial S$ extends to a section $v_1\colon S \to STM|S$ that is nowhere equal to $-\nu_S$. In the first part of the proof we showed that for such a vector field v_1 the equation $\langle\, c_1(v_1^\perp, t), [S] \,\rangle = c(S,t)$ holds.

Let v' be a unit vector field over M whose homology class is $\mathfrak{s}$ and let $v = v'|S$. Since $\mathfrak{s}$ satisfies Equation (3–1) we get that

$$\left\langle c_1(v^\perp, t) - c_1(v_1^\perp, t), [S_i] \right\rangle = 0 \text{ for every } 1 \le i \le k.$$

The obstruction class $o(v, v_1) \in H^2(S, \partial S; \mathbb{Z})$ vanishes if and only if the sections v and v_1 of $STM|S$ are homotopic relative to ∂S. A cochain o representing $o(v, v_1)$ can be obtained as follows. First take a triangulation of S and a trivialization of

$STM|S$. Then v and v_1 can be considered to be maps from S to S^2. One can homotope v rel ∂S to agree with v_1 on the one-skeleton of S. The value of o on a 2–simplex Δ is the difference of $v|\Delta$ and $v_1|\Delta$, which is an element of $\pi_2(S^2) \approx \mathbb{Z}$. Since $2o(v, v_1) = c_1(v^\perp, t) - c_1(v_1^\perp, t)$ and $H^2(S, \partial S; \mathbb{Z}) \approx \mathbb{Z}^k$ is torsion free we get that $o(v, v_1) = 0$, ie, v is homotopic to v_1 rel ∂S. By extending this homotopy of v' fixing $v'|\partial M$ we get a vector field v_1' on M that agrees with v_1 on S. Thus $\mathfrak{s}$ can be represented by the vector field v_1' that is nowhere equal to $-\nu_S$, and so $\mathfrak{s} \in O_S$. □

In light of Lemma 3.10 we can reformulate Theorem 1.3 for strongly balanced sutured manifolds as follows.

Theorem 3.11 *Let (M, γ) be a strongly balanced sutured manifold; furthermore, let $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ be a sutured manifold decomposition. Suppose that S is open and for every component V of $R(\gamma)$ the set of closed components of $S \cap V$ consists of parallel oriented boundary-coherent simple closed curves. Denote the components of S by $S_1, \dots, S_k$ and choose a trivialization t of $v_0^\perp$. Then*

$$SFH(M', \gamma') = \bigoplus_{\substack{\mathfrak{s} \in Spin^c(M,\gamma): \\ \langle c_1(\mathfrak{s},t), [S_i] \rangle = c(S_i,t)\ \forall 1 \le i \le k}} SFH(M, \gamma, \mathfrak{s}).$$

In particular,

$$SFH(M', \gamma') \le \bigoplus_{\substack{\mathfrak{s} \in Spin^c(M,\gamma): \\ \langle c_1(\mathfrak{s},t), [S] \rangle = c(S,t)}} SFH(M, \gamma, \mathfrak{s}).$$

4 Finding a balanced diagram adapted to a decomposing surface

Definition 4.1 We say that the decomposing surfaces S_0 and S_1 are *equivalent* if they can be connected by an isotopy through decomposing surfaces.

Remark 4.2 During an isotopy through decomposing surfaces the number of arcs of $S \cap \gamma$ can never change. Moreover, if S_0 and S_1 are equivalent then decomposing along them give the same sutured manifold.

Definition 4.3 A balanced diagram *adapted* to the decomposing surface S in (M, γ) is a quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ that satisfies the following conditions. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a balanced diagram of (M, γ); furthermore, $P \subset \Sigma$ is a quasi-polygon (ie, a closed subsurface of Σ whose boundary is a union of polygons) such that $P \cap \partial\Sigma$ is exactly the set of vertices of P. We are also given a decomposition $\partial P = A \cup B$, where both A and B are unions of pairwise disjoint edges of P. This decomposition has to satisfy the property that $\alpha \cap B = \varnothing$ and $\beta \cap A = \varnothing$ for every $\alpha \in \boldsymbol{\alpha}$ and $\beta \in \boldsymbol{\beta}$. Finally, S is given up to equivalence by smoothing the corners of the surface $(P \times \{1/2\}) \cup (A \times [1/2, 1]) \cup (B \times [0, 1/2]) \subset (M, \gamma)$ (see Definition 2.10). The orientation of S is given by the orientation of $P \subset \Sigma$. We call a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ satisfying the above conditions a *surface diagram*.

Proposition 4.4 *Suppose that S is a decomposing surface in the balanced sutured manifold (M, γ). If the boundary of each component of S intersects both $R_+(\gamma)$ and $R_-(\gamma)$ (in particular S is open) and ∂S has no closed component lying entirely in γ then there exists a Heegaard diagram of (M, γ) adapted to S.*

Proof We are going to construct a self-indexing Morse function f on M with no minima and maxima as in the proof of [7, Proposition 2.13] with some additional properties. In particular, we require that $f|R_-(\gamma) \equiv -1$ and $f|R_+(\gamma) \equiv 4$. Furthermore, $f|\gamma$ is given by the formula $p_2 \circ \varphi$, where $\varphi\colon \gamma \to s(\gamma) \times [-1, 4]$ is a diffeomorphism such that $\varphi(s(\gamma)) = s(\gamma) \times \{3/2\}$ and $p_2\colon s(\gamma) \times [-1, 4] \to [-1, 4]$ is the projection onto the second factor. We choose φ such that each arc of $S \cap \gamma$ maps to a single point under $p_1 \circ \varphi\colon \gamma \to s(\gamma)$.

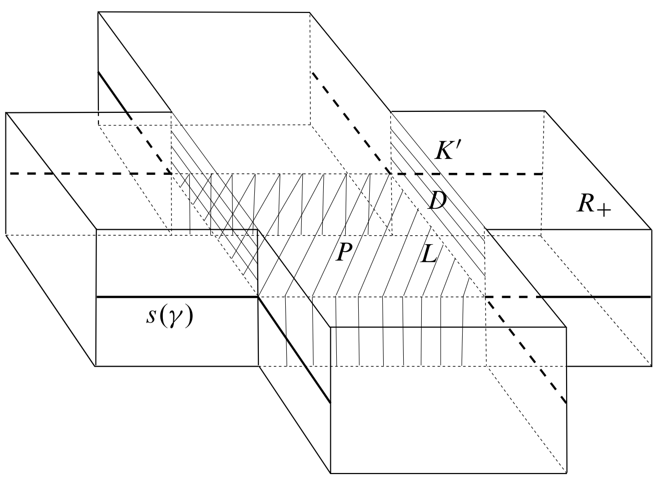

Figure 2: This diagram shows a decomposing surface which is a disk that intersects $s(\gamma)$ in four points.

We are going to define a quasi-polygon $P \subset S$ such that $S \cap s(\gamma)$ is the set of vertices of P; see Figure 2. Let $K_1, \dots, K_{m+n}$ be the closures of the components of $\partial S \setminus s(\gamma)$ enumerated such that K_i is an arc for $1 \le i \le m$ and K_i is a circle for $m+1 \le i \le m+n$.

For every $1 \le i \le m$ choose an arc L_i whose interior lies in $\operatorname{int}(S)$ parallel to K_i and such that $\partial L_i = \partial K_i$. Moreover, let D_i be the closed bigon bounded by K_i and L_i and define $K_i' = K_i \cap R(\gamma)$. Also choose a diffeomorphism $d_i \colon D_i \to I \times I$ that takes K_i' to $I \times \{0\}$ and L_i to $I \times \{1\}$ and such that for each $t \in [0,1]$ we have $f \circ d_i^{-1}(0,t) = f \circ d_i^{-1}(1,t)$. Note that f is already defined on ∂M. We define f on D_i by the formula

$$f(d_i^{-1}(u,t)) = f(d_i^{-1}(0,t)).$$

If $m+1 \le i \le m+n$ then let L_i be a circle parallel to K_i lying in the interior of S. Let D_i be the annulus bounded by K_i and L_i. Choose a diffeomorphism

$$d_i \colon D_i \to S^1 \times J_i,$$

where $J_i = [3/2, 4]$ if $K_i \subset R_+(\gamma)$ and $J_i = [-1, 3/2]$ otherwise. In both cases we require that $d_i(L_i) = 3/2$. Then let $f|D_i = \pi_2 \circ d_i$, where $\pi_2 \colon S^1 \times J_i \to J_i$ is the projection onto the second factor.

We take

$$\partial P = \bigcup_{i=1}^{m+n} L_i,$$

and L_i will be an edge of ∂P for every $1 \le i \le m+n$. The decomposition $\partial P = A \cup B$ is given by taking A to be the union of those edges L_i of ∂P for which $K_i \cap R_+(\gamma) \neq \varnothing$.

Let P be the closure of the components of $S \setminus \partial P$ that are disjoint from ∂S. For $p \in P$ let $f(p) = 3/2$. Note that the function $f|S$ is not smooth along ∂P, so we modify S by introducing a right angle edge along ∂P (such that we get back S after smoothing the corners). There are essentially two ways of creasing S along an edge L_i of P. Let $\nu_P = \nu_S|P$ be the positive unit normal field of P in M. If $L_i \subset A$ then we choose the crease such that $\nu_P|L_i$ points into D_i and if $L_i \subset B$ then we require that $\nu_P|L_i$ points out of D_i.

Now extend f from $\partial M \cup S$ to a Morse function f_0 on M. Then

$$P = S \cap f_0^{-1}(3/2).$$

We choose the extension f_0 as follows. For $1 \le i \le m+n$ let $N(D_i)$ be a regular neighborhood of D_i and let $T_i \colon N(D_i) \to D_i \times [-1,1]$ be a diffeomorphism. Then for $(x,t) \in D_i \times [-1,1]$ let

$$f_0(T_i^{-1}(x,t)) = f(x).$$

Due to the choice of the creases we can define f_0 such that $\operatorname{grad}(f)|P \neq -\nu_S$. Thus we have achieved that for each $a \in A$ the gradient flow line of f_0 coming out of a ends on $R_+(\gamma)$ and for each $b \in B$ the negative gradient flow line of f_0 going through b ends on $R_-(\gamma)$.

By making f_0 self-indexing we obtain a Morse function f. Suppose that the Heegaard diagram corresponding to f is $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. We have two partitions $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 \cup \boldsymbol{\alpha}_1$ and $\boldsymbol{\beta} = \boldsymbol{\beta}_0 \cup \boldsymbol{\beta}_1$, where curves in $\boldsymbol{\alpha}_1$ correspond to index one critical points p of f_0 for which $f_0(p) > 3/2$ and $\boldsymbol{\beta}_1$ comes from those index two critical points q of f_0 for which $f_0(q) < 3/2$. Then $f^{-1}(3/2)$ differs from $f_0^{-1}(3/2)$ as follows. Add an S^2 component to $f_0^{-1}(3/2)$ for each index zero critical point of f_0 lying above $3/2$ and for each index three critical point of f_0 lying below $3/2$. Then add two-dimensional one-handles to the previous surface whose belt circles are the curves in $\boldsymbol{\alpha}_1 \cup \boldsymbol{\beta}_1$.

Let $P' = S \cap f^{-1}(3/2)$. Then $\partial P'$ is the union of ∂P and some of the feet of the additional tubes. Next we are going to modify P' such that it becomes disjoint from these additional tubes and it defines a surface equivalent to S.

Let S_0 be a component of S and let $P_0' = P' \cap S_0$. Since ∂S_0 intersects both $R_+(\gamma)$ and $R_-(\gamma)$ we see that $A \cap P_0' \neq \varnothing$ and $B \cap P_0' \neq \varnothing$. Because S_0 is connected P_0' is also connected. Note that for $\alpha \in \boldsymbol{\alpha}_1$ we have $\alpha \cap P' = \varnothing$. Thus we can achieve using isotopies that every arc of $\alpha \cap P'$ for each $\alpha \in \boldsymbol{\alpha}_0$ intersects A. Indeed, for every component P_0' of P' choose an arc $\varphi_0 \subset P_0'$ whose endpoint lies on A and intersects every α–arc lying in P_0'. Then simultaneously apply a finger move along φ_0 to all the α–arcs that intersect φ_0. Similarly, we can achieve that each arc of $\beta \cap P'$ for every $\beta \in \boldsymbol{\beta}_0$ intersects B. This can be done keeping both the α– and the β–curves pairwise disjoint.

Let $F \subset \partial P'$ be the foot of a tube whose belt circle is a curve $\alpha_1 \in \boldsymbol{\alpha}_1$. Pick a point $p \in F$. Since every arc of $\beta \cap P'$ for $\beta \in \boldsymbol{\beta}_0$ intersects B each component of $P' \setminus (\bigcup \boldsymbol{\beta}_0)$ intersects B. Thus we can connect p to B with an arc φ lying in $P' \setminus (\bigcup \boldsymbol{\beta})$. Now handleslide every $\alpha \in \boldsymbol{\alpha}_0$ that intersects φ over α_1 along φ. Then we can handleslide B over α_1 along φ. To this handleslide corresponds an isotopy of S through decomposing surfaces such that $S \cap f^{-1}(3/2)$ changes the required way (given by taking the negative gradient flow lines of f flowing out of B). Thus we have removed F from P'. The case when the belt circle of the tube lies in $\boldsymbol{\beta}_1$ is completely analogous. By repeating this process we can remove all the additional one-handles from P'. Call this new quasi-polygon P.

Finally, cancel every index zero critical point with an index one critical point and every index three critical point with an index two critical point and delete the corresponding α– and β–curves. The balanced diagram obtained this way, together with the quasi-polygon P, defines S. $\square$

Lemma 4.5 *Let $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ be a surface decomposition such that for every component V of $R(\gamma)$ the set of closed components of $S \cap V$ consists of parallel oriented boundary-coherent simple closed curves. Then S is isotopic to a decomposing surface S' such that each component of $\partial S'$ intersects both $R_+(\gamma)$ and $R_-(\gamma)$ and decomposing (M,γ) along S' also gives (M',γ'). Furthermore, $O_S = O_{S'}$.*

Proof We call a tangency between two curves positive if their positive unit tangent vectors coincide at the tangency point. Our main observation is the following. Isotope a small arc of ∂S on ∂M using a finger move through γ such that during the isotopy we have a positive tangency between ∂S and $s(\gamma)$ (thus introducing two new intersection points between ∂S and $s(\gamma)$). Let the resulting isotopy of ∂S be $\{\, s_t \colon 0 \le t \le 1 \,\}$. Attach the collar $\partial M \times I$ to M to get a new manifold $\widetilde{M}$ and attach $\bigcup_{t \in I} (s_t \times \{t\})$ to S to obtain a surface $\widetilde{S} \subset \widetilde{M}$. Then decomposing

$$(\widetilde{M}, \gamma \times \{1\}) \approx (M,\gamma)$$

along $\widetilde{S}$ we also get (M',γ'); see Figure 3. Furthermore, $\widetilde{S}$ is isotopic to S.

Let γ_0 be a component of γ such that $\gamma_0 \cap \partial S$ consists of closed curves $\sigma_1,\dots,\sigma_k$. First isotope S in a neighborhood of $\partial S \cap \gamma_0$ through decomposing surfaces such that after the isotopy $\sigma_1,\dots,\sigma_k$ are all parallel to $s(\gamma)$ and ν_S points out of M along $\partial S \cap \gamma_0$. This new decomposing surface is equivalent to the original. Then isotope $\sigma_1,\dots,\sigma_k$ into $R_-(\gamma)$. Decomposing along S still gives (M',γ'). Let δ be an oriented arc that intersects $\sigma_1,\dots,\sigma_k$, and $s(\gamma)$ exactly once and its endpoint lies in $R_+(\gamma)$. Applying a finger move to $\sigma_1,\dots,\sigma_k$ simultaneously along δ we get a positive tangency between each σ_i and $s(\gamma)$ since they are oriented coherently.

Let V be a component of $R(\gamma)$ and let $C_1,\dots,C_k$ be the parallel oriented closed components of $S \cap V$. Choose a small arc T that intersects every C_i in a single point. Let $\partial T = \{x,y\}$. First suppose that $[C_1] \ne 0$ in $H_1(V;\mathbb{Z})$. Then we can connect both x and y to $s(\gamma)$ by an arc whose interior lies in $\partial M \setminus (\partial S \cup s(\gamma))$. This is possible since C_1 does not separate ∂V and now $\partial S \cap \gamma$ has no closed components. This way we obtain an arc $\delta \subset \partial M$ such that for every $1 \le i \le k$ we have $|\delta \cap C_i| = 1$ and $\partial\delta = \delta \cap s(\gamma)$; moreover,

$$\delta \cap \partial S = \delta \cap (C_1 \cup \dots \cup C_k).$$

Recall that $s(\gamma)$ is oriented coherently with ∂V (this is especially important if $s(\gamma)$ is disconnected and δ connects two distinct components of $s(\gamma)$) and the curves $C_1,\dots,C_k$ are also oriented coherently. Thus with exactly one of the orientations of δ if we apply a finger move to all the C_i simultaneously we get a positive tangency between each C_i and ∂V, and thus also $s(\gamma)$.

Now suppose that $[C_1] = 0$ in $H_1(V;\mathbb{Z})$ and C_1 is oriented as the boundary of its interior. Then exactly one of x and y can be connected to $s(\gamma)$ by an arc δ_0 whose interior lies in $\partial M \setminus (\partial S \cup s(\gamma))$. The arc $T \cup \delta_0$ defines an oriented arc δ whose endpoint lies on $s(\gamma)$. If we apply a finger move to each C_i along δ we get positive tangencies with $s(\gamma)$ because every C_i is oriented as the boundary of its interior and $s(\gamma)$ is oriented coherently with respect to ∂V.

Continuing this process we get a surface S' isotopic to S such that each component of $\partial S'$ intersects $s(\gamma)$ and decomposing (M,γ) along S' we still get (M',γ').

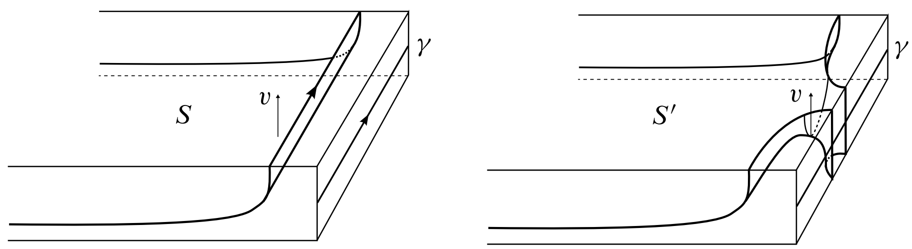

Figure 3: Making a decomposing surface good

To show that $O_S = O_{S'}$ first observe that if S_0 and S_1 are equivalent then $O_{S_0} = O_{S_1}$. Now suppose that for some component γ_0 of γ the components of $\partial S \cap \gamma_0$ are curves $\sigma_1,\dots,\sigma_k$ parallel to $s(\gamma)$ such that ν_S points out of M along them. Moreover, suppose that S' only differs from S by isotoping $\sigma_1,\dots,\sigma_k$ into $R_-(\gamma)$. If $\mathfrak{s}$ is a Spin^c structure and v is a vector field representing it, then in a standard neighborhood of γ_0 we have $v \neq \pm\nu_S$ and $v \neq \pm\nu_{S'}$. So $\mathfrak{s} \in O_S$ if and only if $\mathfrak{s} \in O_{S'}$.

Thus we only have to show that $O_S = O_{S'}$ when S and S' are related by a small finger move of ∂S that crosses $s(\gamma)$ through a positive tangency. Let $\mathfrak{s}$ be a Spin^c structure on (M,γ) and v a vector field representing it. Then in a standard neighborhood U of the tangency point we can perform the isotopy such that in U we have $v \neq \pm\nu_S$; furthermore, $v^\perp$ and S' only have a single hyperbolic tangency, where $v = \nu_{S'}$ (see Figure 3). Thus $\mathfrak{s} \in O_S$ if and only if $\mathfrak{s} \in O_{S'}$. Note that if the tangency of ∂S and $s(\gamma)$ is negative during the isotopy then at the hyperbolic tangency $v = -\nu_{S'}$.

If (M,γ) is strongly balanced then $O_S = O_{S'}$ also follows from Lemma 3.10. Indeed, $\langle c_1(\mathfrak{s},t),[S]\rangle$ is invariant under isotopies of S. As before, we can suppose that the closed components of $\partial S \cap \gamma$ are parallel to $s(\gamma)$ and ν_S points out of M along them. In the above proof I and r are unchanged when we isotope σ_i from γ_0 to $R_-(\gamma)$ since we can achieve that ν_S and v are never parallel along ∂S, so I and r change continuously. When we do a finger move I decreases by 1 according to part (1) of Lemma 3.9 and r also decreases by 1, as can be seen from Figure 3. Thus $c(S,t) = c(S',t)$. □

Definition 4.6 We call a decomposing surface $S \subset (M,\gamma)$ *good* if it is open and each component of ∂S intersects both $R_+(\gamma)$ and $R_-(\gamma)$. We call a surface diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ *good* if A and B have no closed components.

Remark 4.7 Because of Lemma 4.5 it is sufficient to prove Theorem 1.3 for good decomposing surfaces. According to Proposition 4.4 for each good decomposing surface we can find a good surface diagram adapted to it.

Proposition 4.8 *Suppose that S is a good decomposing surface in the balanced sutured manifold (M,γ). Then there exists an* admissible *surface diagram of (M,γ) adapted to S.*

Proof According to Remark 4.7 we can find a good surface diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ adapted to S.

Here we improve on the idea of the proof of [7, Proposition 3.15]. Choose pairwise disjoint arcs $\gamma_1, \dots, \gamma_k \subset \Sigma \setminus B$ whose endpoints lie on $\partial\Sigma$ and together generate $H_1(\Sigma \setminus B, \partial(\Sigma \setminus B); \mathbb{Z})$. This is possible because each component of $\partial(\Sigma \setminus B)$ intersects $\partial\Sigma$. Choose curves $\gamma_1', \dots, \gamma_k'$ such that γ_i and γ_i' are parallel and oriented oppositely.

Then wind the $\boldsymbol{\alpha}$ curves along $\gamma_1, \gamma_1', \dots, \gamma_k, \gamma_k'$ as in the proof of [7, Proposition 3.15]. A similar argument as there gives that after the winding $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ will be admissible. Note that every $\alpha \in \boldsymbol{\alpha}$ lies in $\Sigma \setminus B$. Thus if a linear combination $\mathcal{A}$ of α–curves intersects every γ_i algebraically zero times then $\mathcal{A}$ is null-homologous in $\Sigma \setminus B$, and thus also in Σ. Since the winding is done away from B the new diagram is still adapted to S. □

5 Balanced diagrams and surface decompositions

Definition 5.1 Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ be a surface diagram (see Definition 4.3). Then we can uniquely associate to it a tuple $D(P) = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', P_A, P_B, p)$, where $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is a balanced diagram, $p\colon \Sigma' \to \Sigma$ is a smooth map, and $P_A, P_B \subset \Sigma'$ are two closed subsurfaces (see Figure 4).

To define Σ' take two disjoint copies of P that we call P_A and P_B together with diffeomorphisms $p_A\colon P_A \to P$ and $p_B\colon P_B \to P$. Cut Σ along ∂P and remove P. Then glue A to P_A using p_A^{-1} and B to P_B using p_B^{-1} to obtain Σ'. The map

$p\colon \Sigma' \to \Sigma$ agrees with p_A on P_A and p_B on P_B, and it maps $\Sigma' \setminus (P_A \cup P_B)$ to $\Sigma \setminus P$ using the obvious diffeomorphism. Finally, let $\boldsymbol{\alpha}' = \{\, p^{-1}(\alpha) \setminus P_B : \alpha \in \boldsymbol{\alpha} \,\}$ and $\boldsymbol{\beta}' = \{\, p^{-1}(\beta) \setminus P_A : \beta \in \boldsymbol{\beta} \,\}$.

$D(P)$ is uniquely characterized by the following properties. The map p is a local diffeomorphism in $\mathrm{int}(\Sigma')$; furthermore, $p^{-1}(P)$ is the disjoint union of P_A and P_B. Moreover, $p|P_A\colon P_A \to P$, and $p|P_B\colon P_B \to P$, and also

$$p|(\Sigma' \setminus (P_A \cup P_B))\colon \Sigma' \setminus (P_A \cup P_B) \to \Sigma \setminus P$$

are diffeomorphisms. Furthermore, $p(\mathrm{int}(\Sigma') \cap \partial P_A) = \mathrm{int}(A)$ and $p(\mathrm{int}(\Sigma') \cap \partial P_B) = \mathrm{int}(B)$. Finally, $p|(\cup\boldsymbol{\alpha}')\colon \bigcup \boldsymbol{\alpha}' \to \bigcup \boldsymbol{\alpha}$ and $p|(\bigcup \boldsymbol{\beta}')\colon \bigcup \boldsymbol{\beta}' \to \bigcup \boldsymbol{\beta}$ are diffeomorphisms. Thus $(\bigcup \boldsymbol{\alpha}') \cap P_B = \varnothing$ and $(\bigcup \boldsymbol{\beta}') \cap P_A = \varnothing$.

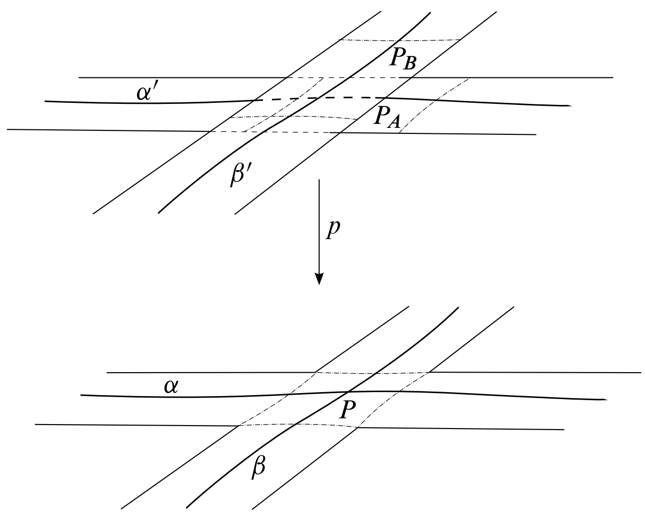

Figure 4: Balanced diagrams before and after a surface decomposition

There is a unique holomorphic structure on Σ' that makes the map p holomorphic. Since p is a local diffeomorphism in $\mathrm{int}(\Sigma')$ it is even conformal.

So p is 1:1 over $\Sigma \setminus P$, it is 2:1 over P, and α curves are lifted to P_A and β curves to P_B.

Proposition 5.2 *Let (M,γ) be a balanced sutured manifold and*

$$(M,\gamma) \rightsquigarrow^S (M',\gamma')$$

a surface decomposition. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ is a surface diagram adapted to S and if $D(P) = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', P_A, P_B, p)$ then $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is a balanced diagram defining (M',γ').

Proof Let (M_1, γ_1) be the sutured manifold defined by the diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$. We are going to construct an orientation preserving homeomorphism $h\colon (M_1, \gamma_1) \to (M', \gamma')$ that takes $R_+(\gamma_1)$ to $R_+(\gamma')$. Figure 5 is a schematic illustration of the proof.

Let N_A and N_B be regular neighborhoods of P_A and P_B in Σ' so small that $\alpha' \cap N_B = \varnothing$ and $\beta' \cap N_A = \varnothing$ for every $\alpha' \in \boldsymbol{\alpha}'$ and $\beta' \in \boldsymbol{\beta}'$. Furthermore, let $N = N_A \cup N_B$. Define $\lambda\colon \Sigma' \to I$ to be a smooth function such that $\lambda(x) = 1$ for $x \in \Sigma' \setminus N$ and $\lambda(x) = 1/2$ for $x \in P_A \cup P_B$. Moreover, let $\mu\colon \Sigma' \to I$ be a smooth function such that $\mu(x) = 1 - \lambda(x)$ for $x \in N_B$ and $\mu(x) = 0$ for $x \in \Sigma' \setminus N_B$.

The homeomorphism h is constructed as follows. For $(x, t) \in \Sigma' \times I$ let

$$h(x, t) = (p(x), \mu(x) + \lambda(x)t)\,.$$

Since for every $x \in \Sigma'$ and $t \in I$ the inequality $0 \leq \mu(x) + \lambda(x)t \leq 1$ holds the map h takes $\Sigma' \times I$ into $\Sigma \times I \subset (M, \gamma)$. Choose an $\alpha' \in \boldsymbol{\alpha}'$ and let $\alpha = p(\alpha') \in \boldsymbol{\alpha}$. Let $D_{\alpha'}$ be the 2–handle attached to $\Sigma' \times I$ along $\alpha' \times \{0\}$ and D_α the 2–handle attached to $\Sigma \times I$ along $\alpha \times \{0\}$. Since $\alpha' \cap N_B = \varnothing$ and because $\mu(x) + \lambda(x) \cdot 0 = 0$ for $x \in \Sigma' \setminus N_B$ we see that $h(\alpha' \times \{0\}) = \alpha \times \{0\}$. Thus h naturally extends to a map from $(\Sigma' \times I) \cup D_{\alpha'}$ to $(\Sigma \times I) \cup D_\alpha$. Similarly, for $\beta' \in \boldsymbol{\beta}'$ we have $\beta' \cap N_A = \varnothing$. Furthermore, $\mu(x) + \lambda(x) \cdot 1 = 1$ for $x \in \Sigma' \setminus N_A$. Thus h also extends to the 2–handles attached along the β–curves. So now we have a local homeomorphism from (M_1, γ_1) into (M, γ).

Recall that $S \subset (M, \gamma)$ is equivalent to the surface obtained by smoothing

$$(P \times \{1/2\}) \cup (A \times [1/2, 1]) \cup (B \times [0, 1/2]) \subset \Sigma \times I.$$

Since $h(N_B \times \{0\}) \cup h(N_A \times \{1\})$ is a smoothing of the above surface we can assume that it is in fact equal to S. Indeed, for $x \in P_A$ we have that $\mu(x) + \lambda(x) \cdot 1 = 1/2$ and for $x \in P_B$ the equality $\mu(x) + \lambda(x) \cdot 0 = 1 - \lambda(x) = 1/2$ holds. Moreover, $p(\partial N_A \setminus \partial \Sigma') = A'$ is a curve parallel to A, thus for $x \in \partial N_A \setminus \partial \Sigma'$ we have $h(x, 1) \in A' \times \{1\}$. Similarly, $h(x, 0) \in B' \times \{0\}$ for $x \in \partial N_B \setminus \partial \Sigma'$, where B' is a curve parallel and close to B.

Let $E_A \subset \Sigma \times I$ be the set of points (y, s) such that $y = p(x)$ for some $x \in N_A \setminus P_A$ and $s \geq \mu(x) + \lambda(x)$. Define $E_B \subset \Sigma \times I$ to be the set of those points (y, s) such that $y = p(x)$ for some $x \in N_B \setminus P_B$ and $s \leq \mu(x)$. Now we are going to show that the map

$$h|(\Sigma' \times I \setminus (N_A \times \{1\} \cup N_B \times \{0\})) \to (\Sigma \times I) \setminus (S \cup E_A \cup E_B)$$

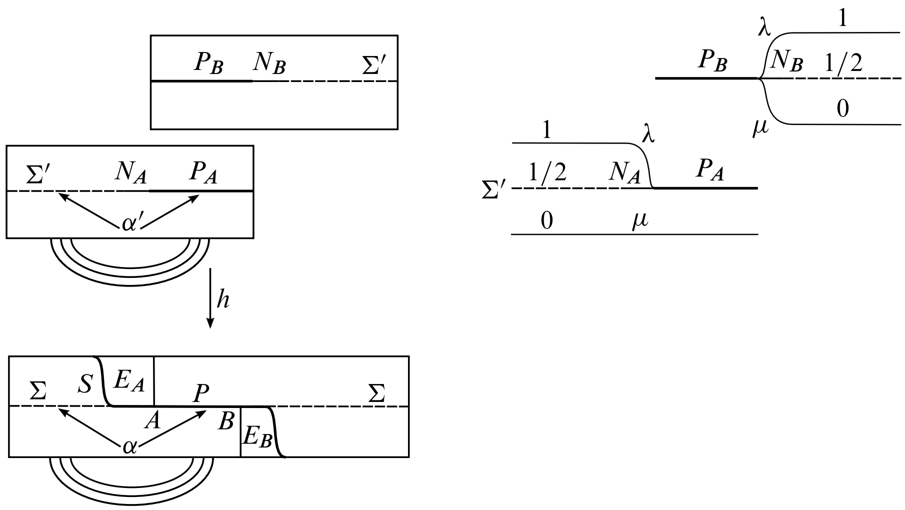

Figure 5: The left hand side shows the homeomorphism h. On the right we can see the functions λ and μ.

is a homeomorphism by constructing its continuous inverse. Let

$$(y,s) \in (\Sigma \times I) \setminus (S \cup E_A \cup E_B).$$

If $y \in \Sigma \setminus p(N)$ then $h^{-1}(y,s) = (p^{-1}(y), s)$. If $y \in P$ and $s < 1/2$ then $h^{-1}(y,s) = (p^{-1}(y) \cap P_A, 2s)$ and for $s > 1/2$ we have $h^{-1}(y,s) = (p^{-1}(y) \cap P_B, 2s-1)$. In the case when $y \in p(N_A \setminus P_A)$ and $s < \mu(x) + \lambda(x)$ we let $h^{-1}(y,s) = (x,t)$, where $x = p^{-1}(y)$ and $t = (s - \mu(x))/\lambda(x) < 1$. Note that here $\mu(x) = 0$, and thus $t \geq 0$. Finally, for $y \in p(N_B \setminus P_B)$ and $s > \mu(x)$ define $h(y,s) = (x,t)$, where $x = p^{-1}(y)$ and $t = (s - \mu(x))/\lambda(x) > 0$. Here $t \leq 1$ because $s \leq 1$ and $\mu(x) = 1 - \lambda(x)$.

Recall that we defined the surfaces S'_+ and S'_- in Definition 2.7. Since S is oriented coherently with $P \times \{1/2\}$ thickening $S'_+ \cap R_-(\gamma)$ in $\partial M'$ can be achieved by cutting off its neighborhood E_B and taking $B \times [0,1/2] \subset \partial E_B$ to belong to γ'. Similarly, E_A is a neighborhood of $S'_- \cap R_+(\gamma)$ in M', and cutting it off from M' we can add $A \times [1/2, 1]$ to γ'. Thus we can identify M' with the metric completion of $M \setminus (S \cup E_A \cup E_B)$ and γ' with $(\gamma \cap M') \cup (A \times [1/2,1]) \cup (B \times [0,1/2])$.

What remains is to show that $h(\gamma_1) = \gamma'$. If $x \in (\partial \Sigma') \setminus (P_A \cup P_B)$ then for any $t \in I$ we have

$$h(x,t) = (p(x), \mu(x) + \lambda(x)t) \in \gamma \cap M' \subset \gamma'$$

because $p(x) \in \partial \Sigma$. On the other hand, for $x \in \partial \Sigma' \cap P_A$ and $t \in I$ we have $h(x,t) \in B \times [0,1/2]$, which is part of γ' by the above construction. The case $x \in \partial \Sigma' \cap P_B$ is similar. □

Remark 5.3 It is not hard to show that the homeomorphism h constructed in the above proof is a actually a diffeomorphism.

Definition 5.4 Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ be a surface diagram. We call an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *outer* if $\mathbf{x} \cap P = \varnothing$. We denote by O_P the set of outer intersection points. Then $I_P = (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \setminus O_P$ is called the set of *inner* intersection points.

Lemma 5.5 *Let $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ be a surface decomposition and suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ is a surface diagram adapted to S. Let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Then $\mathbf{x} \in O_P$ if and only if $\mathfrak{s}(\mathbf{x}) \in O_S$. Furthermore, if $D(P) = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', P_A, P_B, p)$ then p gives a bijection between $\mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ and O_P.*

Proof Let f be a Morse function on M compatible with the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. If $\mathbf{x} \in O_P$ then the multi-trajectory $\gamma_{\mathbf{x}}$ (see Definition 3.3) is disjoint from S. Consequently, the regular neighborhood $N(\gamma_{\mathbf{x}})$ can be chosen to be disjoint from S. Thus $\mathfrak{s}(\mathbf{x})$ can be represented by a unit vector field v that agrees with $\operatorname{grad}(f)/\,\|\operatorname{grad}(f)\|$ in a neighborhood of S. Since the orientation of S is compatible with the orientation of $P \subset \Sigma$, even after smoothing the corners of $(P \times \{1/2\}) \cup (A \times [1/2, 1]) \cup (B \times [0, 1/2])$ we have that v is nowhere equal to $-\nu_S$. So we see that $\mathfrak{s}(\mathbf{x}) \in O_S$.

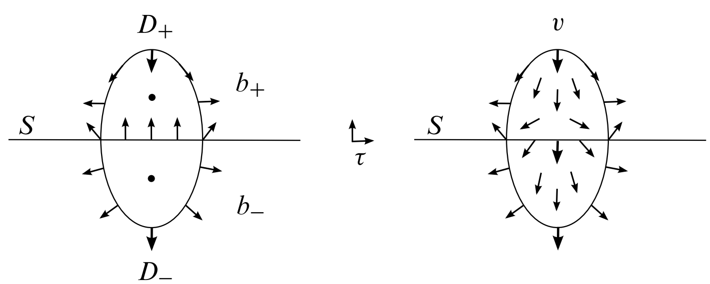

Figure 6: This is a schematic two-dimensional picture illustrating the proof of Lemma 5.5.

Now suppose that $\mathbf{x} \in I_P$. Let $\gamma_{\mathbf{x}}$ be the multi-trajectory associated to $\mathbf{x}$. Since S is open its tangent bundle TS is trivial. Thus there is a trivialization $\tau = (\tau_1, \tau_2, \tau_3)$ of $TM|(S \cup N(\gamma_{\mathbf{x}}))$ such that $\tau_3|S = \nu_S$ and $(\tau_1|S, \tau_2|S)$ is a trivialization of TS. The Spin^c structure $\mathfrak{s}(\mathbf{x})$ can be represented by a unit vector field v such that $v|(M \setminus N(\gamma_{\mathbf{x}}))$ agrees with

$$g = \frac{\operatorname{grad}(f)|(M \setminus N(\gamma_{\mathbf{x}}))}{\|\operatorname{grad}(f)|(M \setminus N(\gamma_{\mathbf{x}}))\|}.$$

If v was outer then for any ball $B^3 \subset M \setminus S$ the vector field $v|(M \setminus B^3)$ would be homotopic through unit vector fields rel ∂M to a field v' such that $v'|S$ is nowhere equal to $-\nu_S$. So to prove that $\mathfrak{s}(\mathbf{x}) \notin O_S$ it is sufficient to show that $v|S$ is not homotopic through unit vector fields rel ∂S to a vector field v' on S that is nowhere equal to $-\nu_S$. In the trivialization τ we can think of $v|(S \cup N(\gamma_{\mathbf{x}}))$ as a map from $S \cup N(\gamma_{\mathbf{x}})$ to S^2 and $-\nu_S$ corresponds to the South Pole $s \in S^2$. If we put S in generic position $v_0 = v|\partial M$ is nowhere equal to $-\nu_S$. Thus v maps ∂S into $S^2 \setminus \{s\}$.

Let $x \in \mathbf{x}$ and let γ_x be the component of $\gamma_{\mathbf{x}}$ containing x. Then $\gamma_x \cap S = \varnothing$ if $x \notin P$ and $\gamma_x \cap S = \{x\}$ if $x \in P$. So suppose that $x \in P$. We denote $N(\gamma_x)$ by B and let B_+ and B_- be the closures of the two components of $B \setminus S$; an index one critical point of f lies in B_- and an index two critical point in B_+. Moreover, let $D_\pm = \partial B_\pm \setminus S$. The vector field $\mathrm{grad}(f)|B$ is a map from B to $\mathbb{R}^3$ in the trivialization τ. Let

$$b_\pm = \frac{\mathrm{grad}(f)|\partial B_\pm}{\|\mathrm{grad}(f)|\partial B_\pm\|}$$

(see Figure 6). Since $B_\pm$ contains an index ± 1 singularity of $\mathrm{grad}(f)$ we see that $\# b_\pm^{-1}(s) = \pm 1$. Here $\#$ denotes the algebraic number of points in a given set. Since $\mathrm{grad}(f)|(S \cap B)$ is equal to ν_S we even get that $\#(b_\pm^{-1}(s) \cap D_\pm) = \pm 1$. Let $v_\pm = v|\partial B_\pm$. Then $\# v_\pm^{-1}(s) = 0$ because v is nowhere zero. The co-orientation of S is given by $\mathrm{grad}(f)$, so $S \cap B \subset S$ is oriented coherently with ∂B_-. Moreover, $v|D_- = b_-|D_-$, so we see that $\#(v_-^{-1}(s) \cap S) = 1$. We have seen that $g|(S \setminus P) = v|(S \setminus P)$ is nowhere equal to $-\nu_S$. So we conclude that $\#(v|S)^{-1}(s) = |\mathbf{x} \cap P|$. Thus if $\mathbf{x} \in I_P$ then $v|S$ is not homotopic to a map $S \to S^2 \setminus \{s\}$ through a homotopy fixing ∂S. This means that $\mathfrak{s}(\mathbf{x}) \notin O_S$.

The last part of the statement follows from the fact that p is a diffeomorphism between $\Sigma' \setminus (P_A \cup P_B)$ and $\Sigma \setminus P$, furthermore $(\bigcup \boldsymbol{\alpha}') \cap P_B = \varnothing$ and $(\bigcup \boldsymbol{\beta}') \cap P_A = \varnothing$. $\square$

Remark 5.6 We can slightly simplify the proof of Lemma 5.5 when $O_P \neq \varnothing$. Suppose that $\mathbf{x} \in I_P$ and let $\mathbf{y} \in O_P$ be an arbitrary intersection point. Using [7, Lemma 4.7] we get that $\mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y}) = PD[\gamma_{\mathbf{x}} - \gamma_{\mathbf{y}}]$. Since the co-orientation of $P \subset S$ is given by $\mathrm{grad}(f)$ we get that

$$\langle\, \mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y}), [S]\,\rangle = |\gamma_{\mathbf{x}} \cap S| - |\gamma_{\mathbf{y}} \cap S| = |\mathbf{x} \cap P| - |\mathbf{y} \cap P| \neq 0.$$

If $\mathfrak{s}(\mathbf{x})$ was outer then both $\mathfrak{s}(\mathbf{x})$ and $\mathfrak{s}(\mathbf{y})$ could be represented by unit vector fields that are homotopic over S rel ∂S since $(STM|S) \setminus (-\nu_S)$ is a bundle with contractible fibers. And that would imply that $\langle\, \mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y}), [S]\,\rangle = 0$. Thus $\mathfrak{s}(\mathbf{x})$ is not outer.

Notation 5.7 We will also denote by O_P and I_P the subgroups of $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ generated by the outer and inner intersection points, respectively.

Corollary 5.8 *For a surface diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ such that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is admissible the chain complex $(CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), \partial)$ is the direct sum of the subcomplexes $(O_{\boldsymbol{P}}, \partial|O_{\boldsymbol{P}})$ and $(I_{\boldsymbol{P}}, \partial|I_{\boldsymbol{P}})$.*

6 An algorithm providing a nice surface diagram

In this section we generalize the results of Sarkar and Wang [16] to sutured Floer homology and surface diagrams. Our argument is an elaboration of the Sarkar–Wang algorithm. The basic approach is the same, but there are some important differences. The definition of distance had to be modified to work in this generality. Additional technical difficulties arise because when we would like to make a surface diagram nice we have to assure that the property $A \cap B = \varnothing$ is preserved. Moreover, $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ might not span $H_1(\Sigma; \mathbb{Z})$, which makes some of the arguments more involved.

Definition 6.1 We say that the surface diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ is *nice* if every component of $\Sigma \setminus (\bigcup \boldsymbol{\alpha} \cup \bigcup \boldsymbol{\beta} \cup A \cup B)$ whose closure is disjoint from $\partial\Sigma$ is a bigon or a square. In particular, a balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is called *nice* if the surface diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \varnothing)$ is nice.

Definition 6.2 Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ be a surface diagram. Then a *permissible move* is an isotopy or a handle slide of the α–curves in $\Sigma \setminus B$ or the β–curves in $\Sigma \setminus A$.

Lemma 6.3 *Let $\mathcal{S}$ be a surface diagram adapted to the decomposing surface $S \subset (M, \gamma)$. If the surface diagram $\mathcal{S}'$ is obtained from $\mathcal{S}$ using permissible moves then $\mathcal{S}'$ is also adapted to S.*

Proof This is a simple consequence of the definitions. □

Theorem 6.4 *Every good surface diagram $\mathcal{S} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ can be made nice using permissible moves. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ was admissible our algorithm gives an admissible diagram.*

Proof Let $\mathbb{A} = (\bigcup \boldsymbol{\alpha}) \cup B$ and $\mathbb{B} = (\bigcup \boldsymbol{\beta}) \cup A$. The set of those components of $\Sigma \setminus (\mathbb{A} \cup \mathbb{B})$ whose closure is disjoint from $\partial\Sigma$ is denoted by $C(\mathcal{S})$.

First we achieve that every element of $C(\mathcal{S})$ is homeomorphic to D^2. Let $R(\mathcal{S})$ denote the set of those elements of $C(\mathcal{S})$ which are *not* homeomorphic to D^2 and let $a(\mathcal{S}) = \sum_{R \in R(\mathcal{S})} (1 - \chi(R))$. Choose a component $R \in R(\mathcal{S})$. Then $H_1(R, \partial R) \neq 0$, thus there exists a curve $(\delta, \partial\delta) \subset (R, \partial R)$ such that $[\delta] \neq 0$ in $H_1(R, \partial R)$. Moreover, we can choose δ such that either $\delta(0) \in \bigcup \boldsymbol{\alpha}$ and $\delta(1) \in \mathbb{B}$, or $\delta(0) \in \bigcup \boldsymbol{\beta}$ and $\delta(1) \in \mathbb{A}$, as follows. Since our surface diagram is good there are no closed components of A and B, and note that $A \cap B = \varnothing$. Furthermore, $\partial R \cap \mathbb{A} \neq \varnothing$ and $\partial R \cap \mathbb{B} \neq \varnothing$ since otherwise R would give a linear relation between either the α–curves or the β–curves. So if ∂R is disconnected we can even find two distinct components C and C' of ∂R such that $C \cap \mathbb{A} \neq \varnothing$ and $C' \cap \mathbb{B} \neq \varnothing$. Thus we can choose δ such that $\partial\delta \cap \mathbb{A} \neq \varnothing$ and $\partial\delta \cap \mathbb{B} \neq \varnothing$. If $\partial\delta \cap A \neq \varnothing$ and $\partial\delta \cap B \neq \varnothing$ then move the endpoint of δ lying on A to the neighboring α–arc. Possibly changing the orientation of δ we obtain a curve with the required properties.

Now perform a finger move of the α– or β–arc through $\delta(0)$, pushing it all the way along δ. Since $R' = R \setminus \delta$ is connected we obtain a surface diagram $\mathcal{S}'$ where R is replaced by a component homeomorphic to R', plus an extra bigon. The homeomorphism type of every other component remains unchanged. Observe that $\chi(R') = \chi(R) + 1$, so we have $a(\mathcal{S}') = a(\mathcal{S}) - 1$. If we repeat this process we end up in a finite number of steps with a diagram, also denoted by $\mathcal{S}$, where $a(\mathcal{S}) = 0$. Note that for every connected surface F with nonempty boundary we have $\chi(F) \leq 1$, and $\chi(F) = 1$ if and only if $F \approx D^2$. Thus $a(\mathcal{S}) = 0$ implies that $R(\mathcal{S}) = \varnothing$.

Next we achieve that every component $D \in C(\mathcal{S})$ is a bigon or a square. All the operations that follow preserve the property that $R(\mathcal{S}) = \varnothing$.

Definition 6.5 If D is a component of $\Sigma \setminus (\mathbb{A} \cup \mathbb{B})$ then its *distance* $d(D)$ from $\partial\Sigma$ is defined to be the minimum of $|\varphi \cap (\bigcup \boldsymbol{\alpha} \cup \bigcup \boldsymbol{\beta})|$ taken over those curves $\varphi \subset \Sigma$ for which $\varphi(0) \in \partial\Sigma$ and $\varphi(1) \in \mathrm{int}(D)$; furthermore, $\varphi(t) \in \Sigma \setminus (A \cup B)$ for $0 < t \leq 1$. If φ passes through an intersection point between an α– and a β–curve we count that with multiplicity two in $|\varphi \cap (\bigcup \boldsymbol{\alpha} \cup \bigcup \boldsymbol{\beta})|$.

If $D \in C(\mathcal{S})$ is a $2n$–gon, then its *badness* is defined to be $\max\{n-2, 0\}$. The *distance of a surface diagram* $\mathcal{S}$ is

$$d(\mathcal{S}) = \max\{\, d(D) : D \in C(\mathcal{S}), b(D) > 0\,\}.$$

For $d > 0$ the *distance d complexity* of the surface diagram $\mathcal{S}$ is defined to be the tuple

$$\left(\sum_{i=1}^{m} b(D_i), -b(D_1), \dots, -b(D_m)\right),$$

where $D_1, \dots, D_m$ are all the elements of $C(\mathcal{S})$ with $d(D) = d$ and $b(D) > 0$, enumerated such that $b(D_1) \geq \dots \geq b(D_m)$. We order the set of distance d complexities lexicographically. Finally, let $b_d(\mathcal{S}) = \sum_{i=1}^{m} b(D_i)$.

Lemma 6.6 *Let $\mathcal{S}$ be a surface diagram of distance $d(\mathcal{S}) = d > 0$ and $R(\mathcal{S}) = \varnothing$. Then we can modify $\mathcal{S}$ using permissible moves to get a surface diagram $\mathcal{S}'$ with $R(\mathcal{S}') = \varnothing$, distance $d(\mathcal{S}') \le d(\mathcal{S})$, and $c_d(\mathcal{S}') < c_d(\mathcal{S})$.*

Proof Let $D_1, \dots, D_m$ be an enumeration of the distance d bad elements of $C(\mathcal{S})$ as in Definition 6.5. Then D_m is a $2n$–gon for some $n \ge 3$. Let D_* be a component of $\Sigma \setminus (\mathbb{A} \cup \mathbb{B})$ with $d(D_*) = d - 1$ and having at least one common α– or β–edge with D_m. Without loss of generality we can suppose that they have a common β–edge b_*. Let $a_1, \dots, a_n$ be an enumeration of the edges of D_m lying in $\mathbb{A}$ starting from b_* and going around ∂D_m counterclockwise.

Let $1 \le i \le n$. We denote by $R_i^1, \dots, R_i^{k_i}$ the following distinct components of $\Sigma \setminus (\mathbb{A} \cup \mathbb{B})$. For every $1 \le j \le k_i - 1$ the component R_i^j is a square of distance $d(R_i^j) \ge d$, but $R_i^{k_i}$ does not have this property. Furthermore, $a_i \cap R_i^1 \ne \varnothing$ and $R_i^j \cap R_i^{j+1} \subset \mathbb{A}$ for $1 \le j \le k_i - 1$. Then $R_i^{k_i}$ is either a bigon or a component of distance $d(R_i^{k_i}) \le d$. Note that it is possible that $R_i^{k_i} = D_m$, in which case $R_i^j = R_l^{k_i - j}$ for some $a_l \subset R_i^{k_i - 1} \cap R_i^{k_i}$ and every $1 \le j \le k_i - 1$.

Thus if we leave D_m through a_i and move through opposite edges we visit the sequence of squares $R_i^1, \dots, R_i^{k_i - 1}$ until we reach a component $R_i^{k_i}$ which is not a square of distance $\ge d$.

Let $I = \{\, 1 \le i \le n : R_i^{k_i} \ne D_m \,\}$. We claim that $I \ne \varnothing$. Indeed, otherwise take the domain $\mathcal{D}$ that is the sum of those components of $\Sigma \setminus (\mathbb{A} \cup \mathbb{B})$ that appear as some R_i^j for $1 \le i \le n$ and $1 \le j \le k_i$, each taken with coefficient one. Then $\partial \mathcal{D}$ is a sum of closed components of $\mathbb{B}$. Since B has no closed components $\partial \mathcal{D}$ is a sum of full β–curves, contradicting the fact that the elements of $\boldsymbol{\beta}$ are linearly independent in $H_1(\Sigma; \mathbb{Z})$.

First suppose that $\exists i \in I \cap \{2, \dots, n-1\}$. Then choose a properly embedded arc $\delta \subset D_m \cup (R_i^1 \cup \dots \cup R_i^{k_i})$ such that $\delta(0) \in b_*$ and $\delta(1) \in \mathrm{int}(R_i^{k_i})$; furthermore, $|\delta \cap \partial R_i^j| = 2$ for $1 \le j < k_i$. Observe that $\delta(t) \cap \mathbb{B} = \varnothing$ for $0 < t \le 1$. Do a finger move of the b_* arc along δ and call the resulting surface diagram $\mathcal{S}'$. The finger cuts D_m into two pieces called D_m^1 and D_m^2, and D_* becomes a new component D'_*.

We claim that $\mathcal{S}'$ satisfies the required properties. Indeed, $d(\mathcal{S}') \le d(\mathcal{S})$ because δ does not enter any region of distance $< d$ except possibly $R_i^{k_i}$ for which $R_i^{k_i} \setminus \delta$ is still connected. Thus $d(D'_*) < d$ and the only new bad regions that we possibly make, D_m^1 and D_m^2, have a common edge with D'_*. All the other new components are bigons or squares. To show that $c_d(\mathcal{S}') < c_d(\mathcal{S})$ we distinguish three cases. Observe that we have

$$b(D_m^1) + b(D_m^2) = b(D_m) - 1. \tag{6–1}$$

Indeed, if D_m^1 is a $2n_1$–gon and D_m^2 is a $2n_2$–gon then $n_1 > 1$ and $n_2 > 1$ since $1 < i < n$. Thus $b(D_m^1) = n_1 - 2$ and $b(D_m^2) = n_2 - 2$. Since the finger cuts a_i into two distinct arcs we have that $n_1 + n_2 = n + 1$, ie, $(n_1 - 2) + (n_2 - 2) = (n-2) - 1$. Furthermore, the finger cuts R_i^j for $1 \le j < k_i$ into three squares.

Case 1 $R_i^{k_i}$ is a bigon of distance $\ge d$. Then $R_i^{k_i} \ne D_*$ because their distances are different. Thus the finger cuts $R_i^{k_i}$ into a bigon and a square, both have badness 0. So Equation (6–1) implies that $b_d(\mathcal{S}') = b_d(\mathcal{S}) - 1$, showing that $c_d(\mathcal{S}') < c_d(\mathcal{S})$.

Case 2 $d(R_i^{k_i}) < d$. Then the finger cuts $R_i^{k_i}$ into a bigon and a component of distance $< d$. Thus again we have that $b_d(\mathcal{S}') = b_d(\mathcal{S}) - 1$.

Case 3 $R_i^{k_i} = D_l$ for some $1 \le l < m$. Then the finger cuts D_l into a bigon and a component D_l' such that $d(D_l') = d$ and $b(D_l') = b(D_l) + 1$. Thus $b_d(\mathcal{S}') = b_d(\mathcal{S})$. But we still have $c_d(\mathcal{S}') < c_d(\mathcal{S})$ because $D_1, \dots, D_{l-1}$ remained unchanged, $-b(D_l') < -b(D_l)$, and every other distance d region in $\mathcal{S}'$ has badness $< b(D_l')$.

Now suppose that $I \cap \{2, \dots, n-1\} = \varnothing$. Since $I \ne \varnothing$ we have $1 \in I$ or $n \in I$. We can suppose without loss of generality that $1 \in I$. Then we have two cases.

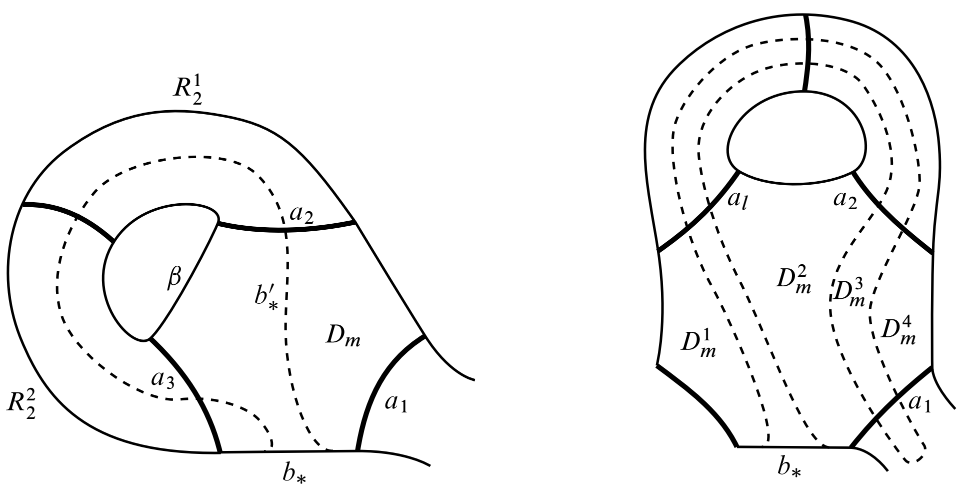

Figure 7: The handle slide of Case A is shown on the left. Subcase B2 is illustrated on the right.

Case A $n = 3$; for an illustration see the left hand side of Figure 7. Then $R_2^{k_2} = D_m$, and thus $R_2^{k_2-1} \cap D_m \supset a_3$, so $I = \{1\}$. Let b be the $\mathbb{B}$–arc of ∂D_m lying between a_2 and a_3. Then the component C of $\partial(R_2^1 \cup \cdots \cup R_2^{k_2})$ containing b is a closed curve such that $C \subset \mathbb{B}$. Since B has no closed components $C = \beta \in \boldsymbol{\beta}$ is disjoint from b_*. Then handle slide b_* over β to get a new surface diagram $\mathcal{S}'$. In $\mathcal{S}'$ the component D_* becomes D_*' with $b(D_*') = b(D_*) + 2$. Let b_*' denote b_* after the handle slide. Since $d(R_2^j) \geq d$ for $1 \leq j \leq k_2$ we see that $d(\mathcal{S}') \leq d(\mathcal{S})$; furthermore, $d(D'_*) < d$. The arc b'_* cuts D_m into a bigon and a square; moreover, it cuts each R_2^j for $1 \leq j < k_2 - 1$ into two squares. Thus we got rid of the distance d bad component D_m, so $b_d(\mathcal{S}') < b_d(\mathcal{S})$.

Case B $n > 3$. Then for some $2 < l \leq n$ we have $a_l \subset R_2^{k_2-1} \cap D_m$.

Subcase B1 $l < n$; for an illustration see the right hand side of Figure 7. Let

$$\delta \subset (R_1^1 \cup \cdots \cup R_1^{k_1}) \cup (R_2^1 \cup \cdots \cup R_2^{k_2})$$

be a properly embedded arc that starts on b_*, enters $R_2^{k_2-1}$ through a_l, crosses each R_2^j for $1 \leq j < k_2 - 1$ exactly once, reenters D_m through a_2, leaves D_m through a_1 and ends in $R_1^{k_1}$. Note that $R_1^{k_1} \neq D_m$ since $1 \in I$. Do a finger move of b_* along δ, we obtain a surface diagram $\mathcal{S}'$. The finger cuts D_m into four components $D_m^1, \dots, D_m^4$ and D_* becomes a component D'_*. Observe that D_m^3 and D_m^4 are squares, $d(D'_*) < d$, and both D_m^1 and D_m^2 have a common edge with D'_*. Moreover, the only component δ enters that can be of distance $< d$ is $R_1^{k_1}$. Thus $d(\mathcal{S}') \leq d(\mathcal{S})$. Furthermore, $b(D_m^1) + b(D_m^2) = b(D_m) - 1$. So we can conclude that $c_d(\mathcal{S}') < c_d(\mathcal{S})$ in a manner analogous to cases 1–3 above, according to the type of $R_1^{k_1}$.

Subcase B2 $l = n$. Then $a_p \subset R_{n-1}^{k_{n-1}-1} \cap D_m$ for some $2 < p < n-1$. We define a properly embedded arc

$$\delta \subset (R_1^1 \cup \cdots \cup R_1^{k_1}) \cup (R_2^1 \cup \cdots \cup R_2^{k_2}) \cup (R_p^1 \cup \cdots \cup R_p^{k_p})$$

as follows (see Figure 8). The curve δ starts on b_*, enters R_p^1 through a_p, reenters D_m through a_{n-1}, goes into $R_n^1 = R_2^{k_2-1}$ through a_n, reenters D_m through a_2, leaves across a_1, and ends in $R_1^{k_1}$. Furthermore, $\delta \cap R_i^j$ consists of a single arc for $i \in \{1, 2, p\}$ and $1 \leq j < k_i$. Note that all these squares R_i^j are pairwise distinct, so δ can be chosen to be embedded. Do a finger move of b_* along δ to obtain a surface diagram $\mathcal{S}'$. The component D_* becomes D'_* and the finger cuts D_m into six pieces $D_m^1, \dots, D_m^6$. Observe that D_m^1, D_m^2, D_m^5, and D_m^6 are all squares; moreover, both D_m^3 and D_m^4 have a common edge with D'_*. Since $d(D'_*) < d$ we have $d(D_m^3) \leq d$ and $d(D_m^4) \leq d$. Furthermore, $b(D_m^3) + b(D_m^4) = b(D_m) - 1$. Thus we get, similarly to Subcase B1, that $\mathcal{S}'$ has the required properties. $\square$

Applying Lemma 6.6 to $\mathcal{S}$ a finite number of times we get a surface diagram $\mathcal{S}' = (\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', P)$ with $d(\mathcal{S}') = 0$, which means that $\mathcal{S}'$ is nice. All that remains to show is that $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is admissible if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ was admissible.

The proof of the fact that isotopies of the α– and β–curves do not spoil admissibility is a local computation that is analogous to the one found in [16, Section 4.3]. Handleslides

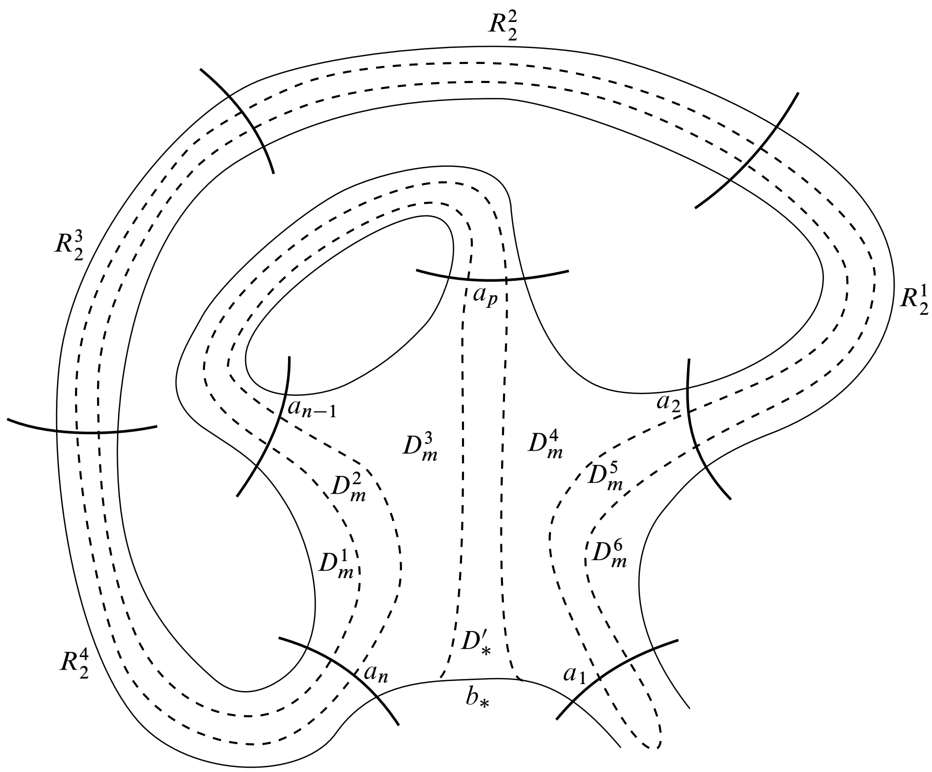

Figure 8: The finger move of Subcase B2

only happen in Case A of Lemma 6.6. The local computation of [16, Section 4.3] happens in $\mathcal{D} = R_2^1 \cup \cdots \cup R_2^{k_2}$, which satisfies $\partial\mathcal{D} \cap \mathbb{B} \subset \bigcup \boldsymbol{\beta}$ because both b_* and b belong to a β–curve. The computation does not depend on whether an arc of $\partial\mathcal{D} \cap \mathbb{A}$ belongs to $\bigcup \boldsymbol{\alpha}$ or B, so the same proof works here too.

This concludes the proof of Theorem 6.4. □

7 Holomorphic disks in nice surface diagrams

In this section we give a complete description of Maslov index one holomorphic disks in nice balanced diagrams. Using that result we prove Theorem 1.3. First we state a generalization of [8, Corollary 4.3].

Definition 7.1 Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a balanced diagram and let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. For $\mathcal{D} \in D(\mathbf{x}, \mathbf{y})$ we define $\Delta(\mathcal{D})$ as follows. Let ϕ be a homotopy class of Whitney disks such that $D(\phi) = \mathcal{D}$. Then $\Delta(\mathcal{D})$ is the algebraic intersection number of ϕ and the diagonal in $\mathrm{Sym}^d(\Sigma)$.

Suppose that $\mathcal{D} = \sum_{i=1}^{m} a_i \mathcal{D}_i$; see Definition 2.13. If $p \in (\bigcup \boldsymbol{\alpha}) \cap (\bigcup \boldsymbol{\beta})$ and $\mathcal{D}_{i_1}, \dots, \mathcal{D}_{i_4}$ are the four components that meet at p then we define

$$n_p(\mathcal{D}) = \frac{1}{4}(a_{i_1} + \cdots + a_{i_4}).$$

Furthermore, if $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$ then let $n_{\mathbf{x}}(\mathcal{D}) = \sum_{i=1}^{d} n_{x_i}(\mathcal{D})$ and $n_{\mathbf{y}}(\mathcal{D}) = \sum_{i=1}^{d} n_{y_i}(\mathcal{D})$.

To define the Euler measure $e(\mathcal{D})$ of $\mathcal{D}$ choose a metric of constant curvature $1, 0$, or -1 on Σ such that $\partial\mathcal{D}$ is geodesic and such that the corners of $\mathcal{D}$ are right angles. Then $e(\mathcal{D})$ is $1/2\pi$ times the area of $\mathcal{D}$.

Remark 7.2 The Euler measure is additive under disjoint unions and gluing of components along boundaries. Moreover, the Euler measurer of a $2n$–gon is $1 - n/2$.

Proposition 7.3 *If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a balanced diagram, $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and $\mathcal{D} \in D(\mathbf{x}, \mathbf{y})$ is a positive domain then*

$$\mu(\mathcal{D}) = e(\mathcal{D}) + n_{\mathbf{x}}(\mathcal{D}) + n_{\mathbf{y}}(\mathcal{D});$$

furthermore,

$$\Delta(\mathcal{D}) = n_{\mathbf{x}}(\mathcal{D}) + n_{\mathbf{y}}(\mathcal{D}) - e(\mathcal{D}).$$

Proof Observe that the proof of [8, Corollary 4.3] does not use the fact that the number of elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ equals the genus of Σ. $\square$

Theorem 7.4 *Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a nice balanced diagram, $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and $\mathcal{D} \in D(\mathbf{x}, \mathbf{y})$ is a positive domain with $\mu(\mathcal{D}) = 1$. Then for a generic almost complex structure, $\widehat{\mathcal{M}}(\mathcal{D})$ consists of a single element which is represented by an embedding of a disk with two or four marked points into Σ.*

Proof In light of Proposition 7.3 the proof is completely analogous to the proofs of Theorems 3.2 and 3.3 of [16]. $\square$

Proposition 7.5 *If the surface diagram $\mathcal{S} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ is nice and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is admissible then the balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is also nice.*

Proof As before, let $C(\mathcal{S})$ denote the set of those components of $\Sigma \setminus (\mathbb{A} \cup \mathbb{B})$ whose closure is disjoint from $\partial\Sigma$. Since $\mathcal{S}$ is nice each component $R \in C(\mathcal{S})$ is a bigon or a square, and thus its Euler measure $e(R) \geq 0$. Let $\mathcal{S}' = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \varnothing)$. Then every component $R' \in C(\mathcal{S}')$ is a sum of elements of $C(\mathcal{S})$, each taken with multiplicity one. Thus $e(R') \geq 0$, which implies that R' is a bigon, a square, an annulus, or a disk. It cannot be an annulus or a disk because that would give a nontrivial positive periodic domain in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. $\square$

Proposition 7.6 *Let $\mathcal{S} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ be a good, nice, and admissible surface diagram and let $D(P) = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', P_A, P_B, p)$. Then the balanced diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is admissible and*

$$CF(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}') \approx (O_P, \partial|O_P).$$

Proof Suppose that Q' is a periodic domain in $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ with either no positive or no negative multiplicities. Then $Q = p(Q')$ is a periodic domain in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ since $p(\partial Q') = \partial Q$ will be a linear combination of full α– and β–curves. Furthermore, Q has either no positive or no negative multiplicities, thus by the admissibility of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ we get that $Q = 0$. So Q' is also zero since all of its coefficients have the same sign.

According to Lemma 5.5 the map p induces a bijection between $\mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ and O_P, which we denote by p_*. We claim that p_* is an isomorphism of chain complexes.

Let $\mathbf{x}', \mathbf{y}' \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ and let $\mathbf{x} = p_*(\mathbf{x}')$ and $\mathbf{y} = p_*(\mathbf{y}')$. Then $\mathbf{x}, \mathbf{y} \in O_P$. Take a positive domain $\mathcal{D}' \in D(\mathbf{x}', \mathbf{y}')$ such that $\mu(\mathcal{D}') = 1$ and let $\mathcal{D} = p(\mathcal{D}')$. Observe that $n_{\mathbf{x}}(\mathcal{D}) = n_{\mathbf{x}}(\mathcal{D}')$, $n_{\mathbf{y}}(\mathcal{D}) = n_{\mathbf{y}}(\mathcal{D}')$, and $e(\mathcal{D}) = e(\mathcal{D}')$. Then $\mathcal{D}$ is a positive domain with $\mu(\mathcal{D}) = 1$ due to Proposition 7.3. Thus p induces a map p_0 from

$$L' = \{\, \mathcal{D}' \in D(\mathbf{x}', \mathbf{y}') : \mathcal{D}' \geq 0 \text{ and } \mu(\mathcal{D}') = 1 \,\}$$

to

$$L = \{\, \mathcal{D} \in D(\mathbf{x}, \mathbf{y}) : \mathcal{D} \geq 0 \text{ and } \mu(\mathcal{D}) = 1 \,\}.$$

We claim that p_0 is a bijection by constructing its inverse r_0.

Let $\mathcal{A} = (\bigcup \boldsymbol{\alpha}) \cup A$ and $\mathcal{B} = (\bigcup \boldsymbol{\beta}) \cup B$. Suppose that $\mathcal{D} \in L$. Then $\mathcal{D}$ is an embedded square or bigon according to Theorem 7.4. Let C be a component of $\mathcal{D} \cap P$. We claim that either $\partial C \subset \mathcal{A}$ or $\partial C \subset \mathcal{B}$. Indeed, C is a sum of elements of $C(\mathcal{S})$ (recall that $C(\mathcal{S})$ was defined in the proof of Theorem 6.4), which are all bigons and squares. Thus the Euler measure $e(C) \geq 0$. The component C cannot be an annulus or a disk since A and B have no closed components and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is admissible. Thus C is either a bigon or a square. Since $\mathbf{x}, \mathbf{y} \in O_P$ and because $\mathcal{D}$ is an embedded bigon or square no corner of C can be an intersection of an α– and a β–edge of ∂C. Thus if C is a bigon it can either have an α– and an A–edge, or a β– and a B–edge. On the other hand, if C is a square it can have two opposite α– and two opposite A–edges, or two opposite β– and two opposite B–edges. Note that in all these cases if $\partial C \subset \mathcal{A}$ then $\partial C \cap A \neq \varnothing$ and if $\partial C \subset \mathcal{B}$ then $\partial C \cap B \neq \varnothing$.

Now we define a map $h = h_{\mathcal{D}} \colon \mathcal{D} \to \Sigma'$ as follows. Let $x \in \mathcal{D}$. If $x \in \mathcal{D} \setminus P$ then let $h(x) = p^{-1}(x)$. If x lies in a component C of $\mathcal{D} \cap P$ such that $\partial C \subset \mathcal{A}$ then let $h(x) = p^{-1}(x) \cap P_A$; finally, let $h(x) = p^{-1}(x) \cap P_B$ if $\partial C \subset \mathcal{B}$. The map h is continuous because if $x \in A$ (or $x \in B$) and the sequence $(x_n) \subset \mathcal{D} \setminus P$ converges to x then the sequence $(p^{-1}(x_n))$ converges to $p^{-1}(x) \cap P_A$ (or $p^{-1}(x) \cap P_B$). See Figure 4. The map p is conformal, thus h is holomorphic. Furthermore, $p \circ h = \mathrm{id}_{\mathcal{D}}$ and thus h is an embedding. So h is a conformal equivalence between $\mathcal{D}$ and $h(\mathcal{D})$, which implies that $h(\mathcal{D}) \in L'$. We define $r_0(\mathcal{D})$ to be $h(\mathcal{D})$. Then it is clear that $p_0 \circ r_0 = \mathrm{id}_L$.

Now we prove that $r_0 \circ p_0 = \mathrm{id}_{L'}$. Let $\mathcal{D}' \in L'$ and let $\mathcal{D} = p_0(\mathcal{D}')$; furthermore, $h = h_{\mathcal{D}}$. Since $\mathcal{D}' \geq 0$ and $\mathcal{D}$ has only 0 and 1 multiplicities we see that $\mathcal{D}'$ also has only 0 and 1 multiplicities. Since p is conformal the map $p|\mathcal{D}' \colon \mathcal{D}' \to \mathcal{D}$ is a conformal equivalence. Let

$$h' = (p|\mathcal{D}')^{-1} \colon \mathcal{D} \to \mathcal{D}'.$$

It suffices to show that $h = h'$ because this would imply that

$$r_0(\mathcal{D}) = h(\mathcal{D}) = h'(\mathcal{D}) = \mathcal{D}'.$$

Since $p\colon (\Sigma' \setminus P) \to (\Sigma \setminus P)$ is a conformal equivalence we get that $h|(\mathcal{D} \setminus P) = h'|(\mathcal{D} \setminus P)$. Let C be a component of $\mathcal{D} \cap P$. Without loss of generality we can suppose that $\partial C \subset \mathcal{A}$, and thus $\partial C \cap A \neq \varnothing$. Let $x \in \partial C \cap A$. Then $h'(C)$ is connected, so either $h'(C) \subset P_A$ or $h'(C) \subset P_B$. But $h'(C) \subset P_B$ cannot happen. Indeed, then we had

$$h'(x) \in p^{-1}(A) \cap P_B \subset \partial \Sigma'.$$

Moreover, the multiplicity of $\mathcal{D}'$ at $h'(x)$ is one, but $\mathcal{D}'$ has multiplicity zero along $\partial\Sigma'$, a contradiction. So $h'(C) \subset P_A$, which means that $h|C = h'|C$.

Thus p_0 is indeed a bijection between L' and L. We have seen that if $\mathcal{D}' \in L'$ and $\mathcal{D} = p_0(\mathcal{D}')$ then both $\mathcal{D}$ and $\mathcal{D}'$ are either embedded bigons or embedded squares; moreover, $h_{\mathcal{D}}$ is a conformal equivalence between them. In both cases $\widehat{\mathcal{M}}(\mathcal{D})$ and $\widehat{\mathcal{M}}(\mathcal{D}')$ have a single element.

This implies that p_* is an isomorphism between the chain complexes $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ and $(O_P, \partial|O_P)$. $\square$

Proof of Theorem 1.3 According to Lemma 4.5 it is sufficient to prove the theorem for good decomposing surfaces. Because of Proposition 4.4 for each good decomposing surface we can find a good surface diagram $\mathcal{S} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ adapted to it. This surface diagram can be made admissible using isotopies according to Proposition 4.8. According to Theorem 6.4 we can achieve that $\mathcal{S}$ is nice using permissible moves, and it still defines (M, γ) because of Lemma 6.3. Now Proposition 5.2 says that if $D(P) = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', P_A, P_B, p)$ then $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is a balanced diagram defining

(M', γ'). From Proposition 7.6 we see that $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is admissible; furthermore,

$$SFH(M', \gamma') = SFH(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}') \approx H(O_P, \partial | O_P).$$

Finally, Lemma 5.5 implies that $(O_P, \partial | O_P)$ is the subcomplex of $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ generated by those $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ for which $\mathfrak{s}(\mathbf{x}) \in O_S$. So

$$H(O_P, \partial | O_P) \approx \bigoplus_{\mathfrak{s} \in O_S} SFH(M, \gamma, \mathfrak{s}),$$

which concludes the proof. $\square$

8 Applications

First we are going to remind the reader of Definition 4.11 and Theorem 4.2 of Gabai [2]. See also Theorem 4.19 of Scharlemann [17].

Definition 8.1 A *sutured manifold hierarchy* is a sequence of decompositions

$$(M_0, \gamma_0) \rightsquigarrow^{S_1} (M_1, \gamma_1) \rightsquigarrow^{S_2} \cdots \rightsquigarrow^{S_n} (M_n, \gamma_n),$$

where (M_n, γ_n) is a product sutured manifold, ie, $(M_n, \gamma_n) = (R \times I, \partial R \times I)$ and $R_+(\gamma_n) = R \times \{1\}$ for some surface R. We define the *depth* of (M_0, γ_0) to be the minimum m for which there is a hierarchy where exactly m of $S_1, \dots, S_n$ are not disjoint unions of horizontal surfaces (see Definition 9.3).

Theorem 8.2 *Let (M, γ) be a connected taut sutured manifold (see Definition 2.6), where M is not a rational homology sphere containing no essential tori. Then (M, γ) has a sutured manifold hierarchy such that each S_i is connected, $S_i \cap \partial M_{i-1} \neq \varnothing$ if $\partial M_{i-1} \neq \varnothing$, and for every component V of $R(\gamma_i)$ the intersection $S_{i+1} \cap V$ is a union of parallel oriented nonseparating simple closed curves or arcs.*

Proof of Theorem 1.4 According to Theorem 8.2 every taut balanced sutured manifold $(M, \gamma) = (M_0, \gamma_0)$ admits a sutured manifold hierarchy

$$(M_0, \gamma_0) \rightsquigarrow^{S_1} (M_1, \gamma_1) \rightsquigarrow^{S_2} \cdots \rightsquigarrow^{S_n} (M_n, \gamma_n).$$

Note that by definition M is open. So every surface S_i in the hierarchy satisfies the requirements of Theorem 1.3. Thus for every $1 \leq i \leq n$ we get that

$$SFH(M_i, \gamma_i) \leq SFH(M_{i-1}, \gamma_{i-1}).$$

Finally, since (M_n, γ_n) is a product it has a balanced diagram with $\boldsymbol{\alpha} = \varnothing$ and $\boldsymbol{\beta} = \varnothing$, and thus $SFH(M_n, \gamma_n) \approx \mathbb{Z}$ (also see [7, Proposition 9.4]). So we conclude that $\mathbb{Z} \approx SFH(M_n, \gamma_n) \leq SFH(M_0, \gamma_0)$. $\square$

Proof of Theorem 1.5 Let $Y(K)$ be the balanced sutured manifold (M,γ), where M is the knot complement $Y \setminus N(K)$ and $s(\gamma)$ consists of a meridian of K and a parallel copy of it oriented in the opposite direction; see Definition 2.4. Let ξ be a tangent vector field along $\partial N(K)$ pointing in the meridional direction. Then ξ lies in $v_0^{\perp}$, and thus gives a canonical trivialization t_0 of $v_0^{\perp}$. Observe that there is a surface decomposition

$$Y(K) \rightsquigarrow^S Y(S).$$

Since $Y(S)$ is strongly balanced we can apply Theorem 3.11 to get that

$$SFH(Y(S)) = \bigoplus_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(Y(K)):\\ \langle c_1(\mathfrak{s},t_0),[S]\rangle = c(S,t_0)}} SFH(Y(K),\mathfrak{s}).$$

Recall that

$$c(S,t_0) = \chi(S) + I(S) - r(S,t_0).$$

Since $\partial S \subset \partial N(K)$ is a longitude of K we see that the rotation of $p(\nu_S)$ with respect to ξ is zero. Furthermore, $\chi(S) = 1-2g(S)$ and $I(S) = -1$ by part (1) of Lemma 3.9, thus $c(S,t_0) = -2g(S)$. So we get that

$$SFH(Y(S)) = \bigoplus_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(Y(K)):\\ \langle c_1(\mathfrak{s},t_0),[S]\rangle = -2g(S)}} SFH(Y(K),\mathfrak{s}),$$

which in turn is isomorphic to $\widehat{HFK}(Y,K,[S],-g(S)) \approx \widehat{HFK}(Y,K,[S],g(S))$; see [11]. Note that we get $\widehat{HFK}(Y,K,[S],g(S))$ if we decompose along $-S$ instead of S. $\square$

Using our machinery we give a simpler proof of the fact that knot Floer homology detects the genus of a knot, which was first proved in [13].

Corollary 8.3 *Let K be a null-homologous knot in a rational homology 3–sphere Y whose Seifert genus is $g(K)$. Then*

$$\widehat{HFK}(K,g(K)) \neq 0;$$

moreover, $$\widehat{HFK}(K,i) = 0 \text{ for } i > g(K).$$

Proof First suppose that $Y \setminus N(K)$ is irreducible. Let S be a Seifert surface of K. Then $Y(S)$ is taut if and only if $g(S) = g(K)$. Thus, according to Theorem 1.4, if $g(S) = g(K)$ then $\mathbb{Z} \le SFH(Y(S))$ and because of [7, Proposition 9.18] we have that $SFH(Y(S)) = 0$ if $g(S) > g(K)$. Since for every $i \ge g(K)$ we can find a Seifert surface S such that $g(S) = i$, together with Theorem 1.5 we are done for the case when $Y \setminus N(K)$ is irreducible.

Now suppose that $Y(K)$ can be written as a connected sum $(M,\gamma)\#Y_1$, where (M,γ) is irreducible and Y_1 is a rational homology 3–sphere. Since we can find a minimal genus Seifert surface S lying entirely in (M,γ) (otherwise we can do cut-and-paste along the connected sum sphere) we can apply the connected sum formula [7, Proposition 9.15] to get that $SFH(Y(S)) \approx SFH(M,\gamma) \otimes \widehat{HF}(Y_1)$ over $\mathbb{Q}$. Since $\operatorname{rk} \widehat{HF}(Y_1) \neq 0$ [14, Proposition 5.1], we can finish the proof as in the previous case. □

Next we are going to give a new proof of [12, Theorem 1.1]. Let L be a link in S^3, then

$$x\colon H_2(S^3, L; \mathbb{R}) \to \mathbb{R}$$

denotes the Thurston semi-norm. Link Floer homology provides a function

$$y\colon H^1(S^3 \setminus L; \mathbb{R}) \to \mathbb{R}$$

defined by

$$y(h) = \max_{\substack{\{\,\mathfrak{s}\in H_1(L;\mathbb{Z}):\\ \widehat{HFL}(L,\mathfrak{s})\neq 0\,\}}} |\langle\, \mathfrak{s}, h\,\rangle|.$$

Theorem 8.4 *For a link $L \subset S^3$ with no trivial components and every $h \in H^1(S^3 \setminus L)$ we have that*

$$2y(h) = x(PD[h]) + \sum_{i=1}^{l} |\langle\, h, \mu_i\,\rangle|,$$

where μ_i is the meridian of the i^{th} component L_i of L.

Proof Let ξ be a unit vector field along $\partial N(L)$ that points in the direction of the meridian μ_i along $\partial N(L_i)$. Consider the balanced sutured manifold $(M,\gamma) = S^3(L)$, then ξ is a section of $v_0^{\perp}$, and consequently it defines a canonical trivialization t_0 of $v_0^{\perp}$. Let R be a Thurston norm minimizing representative of $PD[h]$ having no S^2 components. Note that R has no D^2 components because no component of L is trivial.

We claim that $r(R,t_0) = 0$. Indeed, $K_i = R \cap \partial N(L_i)$ is a torus link. We can arrange that K_i and ξ make a constant angle and that R is perpendicular to $\partial N(L_i)$ along K_i. Then $\nu_i = \nu_{\boldsymbol{R}}|K_i$ is the positive unit normal field of K_i in $\partial N(L_i)$ and $\langle\, \nu_i, \xi\,\rangle_q$ is some constant c_i for every $q \in K_i$; see Figure 9. First suppose that $c_i = 0$. Then K_i is a meridian of L_i and we can suppose that $K_i \subset R(\gamma)$. Thus $p(\nu_{\boldsymbol{R}})|K_i$ is always perpendicular to ξ. Now suppose that $c_i \neq 0$. We define the function

$$a_i(q) = \langle\, p(\nu_{\boldsymbol{R}}) / \, \| p(\nu_{\boldsymbol{R}}) \| \,, \xi\,\rangle_q$$

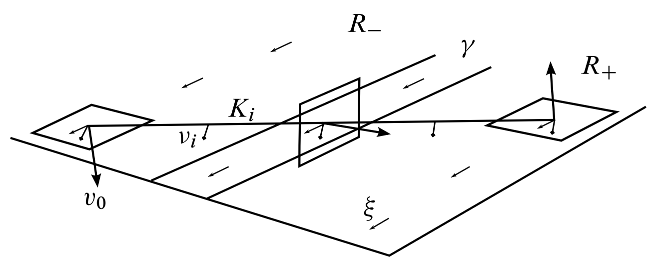

Figure 9: A portion of the torus $\partial N(L_i)$, together with the trivialization ξ of $v_0^\perp$ and v_i

for $q \in K_i$. Then $a_i(q) = \operatorname{sgn}(c_i)$ for $q \in K_i \cap s(\gamma)$ and $a_i(q) = c_i$ for every $q \in K_i \cap R(\gamma)$ such that v_0 is perpendicular to $R(\gamma)$. Moreover, the range of a_i is $[c_i, \operatorname{sgn}(c_i)]$; see Figure 9. So in both cases the rotation of $p(v_R)|K_i$ in the trivialization t_0 is zero as we go around K_i.

Furthermore, we can achieve that

$$|\partial R \cap s(\gamma)| = 2 \sum_{i=1}^{l} |\langle h, \mu_i \rangle|.$$

Since R is norm minimizing and has no S^2 and D^2 components $\chi(R) = -x(PD[h])$. So using part (1) of Lemma 3.9 we get that

$$c(R, t_0) = -x(PD[h]) - \sum_{i=1}^{l} |\langle h, \mu_i \rangle|.$$

Note that $c(R, t_0) \leq 0$.

Now observe that $S^3(R)$ can be obtained from $S^3(L)$ by decomposing along R. Since R is norm minimizing $S^3(R)$ is a connected sum of taut balanced sutured manifolds, thus combining Theorem 1.4 with the connected sum formula [7, Proposition 9.15] we get that $\operatorname{rk} SFH(S^3(R)) \neq 0$. So if we apply Theorem 3.11 to the decomposition

$$S^3(L) \rightsquigarrow^R S^3(R)$$

we see that there is an $\mathfrak{s}_0 \in \operatorname{Spin}^c(S^3(L))$ such that $\mathfrak{s}_0 \in O_R$, hence $\langle c_1(\mathfrak{s}_0, t_0), h \rangle = c(R, t_0)$ and $\widehat{HFL}(L, \mathfrak{s}_0) \approx SFH(S^3(L), \mathfrak{s}_0) \otimes \mathbb{Z}_2 \neq 0$; see [7, Proposition 9.2]. Thus

$$2y(h) = \max_{\substack{\{\, \mathfrak{s} \in H_1(L;\mathbb{Z}): \\ \widehat{HFL}(L,\mathfrak{s}) \neq 0 \,\}}} |\langle c_1(\mathfrak{s}, t_0), h \rangle| \geq x(PD[h]) + \sum_{i=1}^{l} |\langle h, \mu_i \rangle|.$$

To prove that we have an equality let $\mathfrak{s}$ be a Spin^c structure on $S^3(L)$ for which

$$|\langle\, c_1(\mathfrak{s}, t_0), h\,\rangle| > x(PD[h]) + \sum_{i=1}^{l} |\langle\, h, \mu_i\,\rangle| = -c(R, t_0).$$

Denote the components of R by $R_1, \dots, R_k$ and let $h_j = [R_j]$. Since

$$\sum_{j=1}^{k} |\langle\, c_1(\mathfrak{s}, t_0), h_j\,\rangle| \geq |\langle\, c_1(\mathfrak{s}, t_0), h\,\rangle| > -c(R, t_0) = -\sum_{j=1}^{k} c(R_j, t_0)$$

there is a j for which $|\langle\, c_1(\mathfrak{s}, t_0), h_j\,\rangle| > -c(R_j, t_0)$. Thus

$$|\langle\, c_1(\mathfrak{s}, t_0), h_j\,\rangle| + c(R_j, t_0) = 2d > 0.$$

The above sum is even since $\langle\, c_1(\mathfrak{s}_0, t_0), h_j\,\rangle = c(R_j, t_0)$ and

$$\langle c_1(\mathfrak{s}, t_0) - c_1(\mathfrak{s}_0, t_0), h_j\rangle = \langle 2(\mathfrak{s} - \mathfrak{s}_0), h_j\rangle.$$

Let R_j^d be a Seifert surface of L obtained from R_j by d stabilizations and oriented such that $\langle c_1(\mathfrak{s}, t_0), [R_j^d]\rangle < 0$. Observe that $[R_j^d] = \pm h_j$, thus

$$\langle c_1(\mathfrak{s}, t_0), [R_j^d]\rangle = c(R_j, t_0) - 2d = c(R_j^d, t_0),$$

which implies that $\mathfrak{s} \in O_{R_j^d}$. Now $R(S^3(R_j^d))$ is not Thurston norm minimizing, thus according to [7, Proposition 9.19] we have that $SFH(S^3(R_j^d)) = 0$. So if we apply Theorem 3.11 again we see that

$$\widehat{HFL}(L, \mathfrak{s}) \approx SFH(S^3(L), \mathfrak{s}) \otimes \mathbb{Z}_2 \leq SFH(S^3(R_j^d)) \otimes \mathbb{Z}_2 = 0$$

for such an $\mathfrak{s}$. $\square$

Remark 8.5 Suppose that Y is an oriented 3–manifold and $L \subset Y$ is a link such that $Y \setminus N(L)$ is irreducible. Let $x\colon H_2(Y, L; \mathbb{R}) \to \mathbb{R}$ be the Thurston semi-norm and for $h \in H_2(Y, L; \mathbb{R})$ let

$$z(h) = \max_{\substack{\{\, \mathfrak{s} \in \mathrm{Spin}^c(Y, L): \\ \widehat{HFL}(Y, L, \mathfrak{s}) \neq 0\,\}}} |\langle\, c_1(\mathfrak{s}), h\,\rangle|.$$

Then an analogous proof as above gives that

$$z(h) = x(h) + \sum_{i=1}^{l} |\langle\, h, \mu_i\,\rangle|,$$

where μ_i is the meridian of the i^{th} component of L.

The following proposition generalizes the horizontal decomposition formula [9, Theorem 3.4].

Proposition 8.6 *Let (M,γ) be a balanced sutured manifold. Suppose that*

$$(M,\gamma) \rightsquigarrow^{S} (M',\gamma')$$

is a decomposition such that S is connected satisfies the requirements of Theorem 1.3, (M',γ') is taut, and $[S]=0$ in $H_2(M,\partial M)$. The surface S separates (M',γ') into two parts denoted by (M_1,γ_1) and (M_2,γ_2). Then

$$SFH(M,\gamma) \approx SFH(M',\gamma') \approx SFH(M_1,\gamma_1) \otimes SFH(M_2,\gamma_2)$$

over any field $\mathbb{F}$.

Proof Since (M',γ') is taut we can apply Theorem 1.4 to conclude that

$$SFH(M',\gamma') \neq 0.$$

Together with Theorem 1.3 this implies that $O_S \neq \varnothing$. Fix an element $\mathfrak{s}_0 \in O_S$. Then for every Spin^c structure $\mathfrak{s} \in \text{Spin}^c(M,\gamma)$ the equality

$$\langle\, \mathfrak{s} - \mathfrak{s}_0, [S]\,\rangle = 0$$

holds since $[S]=0$. Thus $\mathfrak{s} \in O_S$; see the proof of Lemma 3.10 and Remark 5.6. So we get that $O_S = \text{Spin}^c(M,\gamma)$, and thus $SFH(M',\gamma') \approx SFH(M,\gamma)$.

Now we sketch an alternative proof. Let $\mathcal{S} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ be a surface diagram adapted to S. Then $D(P) = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', P_A, P_B, p)$ (see Definition 5.1) can be written as the disjoint union of two balanced diagrams $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$ such that $P_A \subset \Sigma_1$ and $P_B \subset \Sigma_2$. Let $\beta_1 \in \boldsymbol{\beta}_1$ and $\alpha_2 \in \boldsymbol{\alpha}_2$ be arbitrary curves. Since $\beta_1 \cap P_A = \varnothing$ and $\alpha_2 \cap P_B = \varnothing$ we get that $p(\beta_1) \cap P = \varnothing$ and $p(\alpha_2) \cap P = \varnothing$. Furthermore, $p(\beta_1) \cap p(\alpha_2) = \varnothing$. Thus for the surface diagram $\mathcal{S}$ the set of inner intersection points $I_P = \varnothing$. So Theorem 1.3 gives that $SFH(M,\gamma) \approx SFH(M',\gamma')$.

Note that $(\Sigma_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)$ is a balanced diagram of (M_i,γ_i) for $i = 1,2$; moreover,

$$CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \approx CF(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1) \otimes CF(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2). \qquad \square$$

As a corollary of this we give a simple proof of [10, Theorem 1.1]. The following definition can be found in [3]:

Definition 8.7 The oriented surface $R \subset S^3$ is a *Murasugi sum* of the compact oriented surfaces R_1 and R_2 in S^3 if the following conditions are satisfied. First, $R = R_1 \cup_E R_2$, where E is a $2n$–gon. Furthermore, there are balls B_1 and B_2 in

S^3 such that $R_1 \subset B_1$ and $R_2 \subset B_2$, the intersection $B_1 \cap B_2 = H$ is a two-sphere, $B_1 \cup B_2 = S^3$, and $R_1 \cap H = R_2 \cap H = E$. We also say that the knot ∂R is a Murasugi sum of the knots ∂R_1 and ∂R_2.

Corollary 8.8 *Suppose that the knot $K \subset S^3$ is the Murasugi sum of the knots K_1 and K_2 along some minimal genus Seifert surfaces. Then*

$$\widehat{HFK}(K, g(K)) \approx \widehat{HFK}(K_1, g(K_1)) \otimes \widehat{HFK}(K_2, g(K_2))$$

over any field $\mathbb{F}$.

Proof Let R_1 and R_2 be minimal genus Seifert surfaces of K_1 and K_2, respectively. The Murasugi sum of R_1 and R_2 is a minimal genus Seifert surface R of K; see Gabai [3]. By the definition of the Murasugi sum there is an embedded 2-sphere $H \subset S^3$ that separates R_1 and R_2 and such that $R_1 \cap H = R_2 \cap H$ is a $2n$–gon E for some $n > 0$. Thus in the balanced sutured manifold $S^3(R)$ the disk $D = H \setminus \mathrm{int}(E)$ is a separating decomposing surface that satisfies the requirements of Theorem 1.3. Decomposition along D gives the disjoint union of $S^3(R_1)$ and $S^3(R_2)$, which is taut. Thus, according to Proposition 8.6,

$$SFH(S^3(R)) \approx SFH(S^3(R_1)) \otimes SFH(S^3(R_2))$$

over $\mathbb{F}$. Using Theorem 1.5 we get the required formula. □

Lemma 8.9 *Suppose that (M, γ) is a balanced sutured manifold such that*

$$H_2(M; \mathbb{Z}) = 0.$$

Let $S \subset (M, \gamma)$ be a product annulus (see Definition 2.9) such that at least one component of ∂S is nonzero in $H_1(R(\gamma); \mathbb{Z})$ or both components of ∂S are boundary-coherent in $R(\gamma)$. If S gives a surface decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ then

$$SFH(M', \gamma') \approx SFH(M, \gamma).$$

Proof With at least one orientation of S both components of ∂S are boundary-coherent in $R(\gamma)$. On the other hand, (M', γ') does not depend on the orientation of S. Thus we can suppose that both components of ∂S are boundary-coherent.

Since S is connected and ∂S intersects both $R_+(\gamma)$ and $R_-(\gamma)$ we can apply Proposition 4.4 to get a surface diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ adapted to S. Here P is an annulus with one boundary component being A and the other one B. Thus we can isotope all the α– and β–curves to be disjoint from P, and so $I_P = \varnothing$ for this new diagram. The balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is admissible due to Proposition 2.15.

Now Lemma 5.5 implies that for every $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we have $\mathbf{x} \in O_P$ if and only if $\mathfrak{s}(\mathbf{x}) \in O_S$. Consequently, $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}) = 0$ for $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma) \setminus O_S$. Thus $SFH(M, \gamma, \mathfrak{s}) = 0$ for $\mathfrak{s} \notin O_S$. The surface S satisfies the conditions of Theorem 1.3, and so $SFH(M', \gamma') \approx SFH(M, \gamma)$. $\square$

The next proposition is an analogue of the decomposition formula for separating product annuli proved in [9, Theorem 4.1] using completely different methods.

Proposition 8.10 *Suppose that (M, γ) is a balanced sutured manifold such that $H_2(M; \mathbb{Z}) = 0$. Let $S \subset (M, \gamma)$ be a product annulus such that at least one component of ∂S does not bound a disk in $R(\gamma)$. Then S gives a surface decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$, where $SFH(M', \gamma') \leq SFH(M, \gamma)$. If we also suppose that S is separating in M then $SFH(M', \gamma') \approx SFH(M, \gamma)$.*

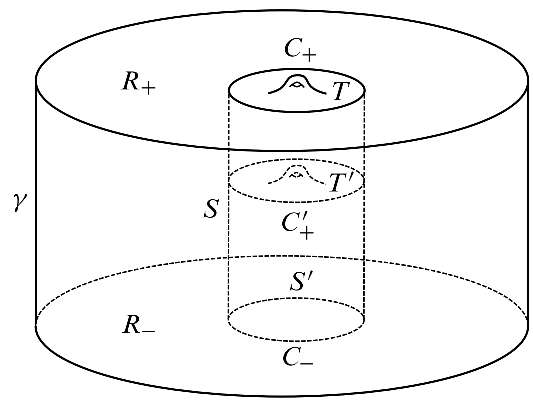

Figure 10: A product annulus

Proof Let $C_\pm = \partial S \cap R_\pm(\gamma)$ and suppose that C_+ does not bound a disk in $R_+(\gamma)$; see Figure 10. According to Lemma 8.9 we only have to consider the case when both $[C_+]$ and $[C_-]$ are zero in $H_1(R(\gamma); \mathbb{Z})$. Since (M', γ') does not depend on the orientation of S we can suppose that S is oriented such that C_- is boundary-coherent in $R_-(\gamma)$. If C_+ is also boundary-coherent in $R_+(\gamma)$ then we are again done due to Lemma 8.9. Thus suppose that C_+ is not boundary-coherent.

The idea of the following argument was communicated to me by Yi Ni. Let T denote the interior of C_+ in $R_+(\gamma)$; then C_+ and ∂T are oriented oppositely; see Definition 1.2. Let C'_+ be a curve lying in $\mathrm{int}(S)$ parallel and close to C_+ and choose a surface T' parallel to T such that $\mathrm{int}(T') \subset \mathrm{int}(M \setminus S)$ and $\partial T = C'_+$. Let S_0 be the component of $S \setminus C'_+$ containing C_-. We define S' to be the surface $S_0 \cup T'$. Note that the orientations of S_0 and T' match along C'_+, so S' has a natural orientation. Let (M_0, γ_0) be the manifold obtained after decomposing (M, γ) along S'. Observe that $\partial S' = C_-$ is boundary-coherent in $R_-(\gamma)$, thus we can apply Theorem 1.3 to S' and get that $SFH(M_0, \gamma_0) \leq SFH(M, \gamma)$. If we also suppose that S is separating then S' is separating and so we have an equality due to Proposition 8.6. Decomposing (M_0, γ_0) along the annulus $S \setminus S_0$ we get a sutured manifold homeomorphic to the disjoint union of (M', γ') and $(T \times I, \partial T \times I)$. Since $T \neq D^2$ we can remove the $(T \times I, \partial T \times I)$ part of (M_0, γ_0) by a series of decompositions along product disks and product annuli having no separating boundary components. Thus $SFH(M', \gamma') \approx SFH(M_0, \gamma_0)$ by [7, Lemma 9.13] and Lemma 8.9. □

9 Fibred knots

Ghiggini [6] (for the genus one case) and Ni [9] recently proved a conjecture of Ozsváth and Szabó that knot Floer homology detects fibred knots. We use the methods developed in this paper to simplify their proof by avoiding the introduction of contact structures. Moreover, we give a relationship between knot Floer homology and the existence of depth two taut foliations on the knot complement.

Definition 9.1 Let (M, γ) be a balanced sutured manifold. Then (M, γ) is called a *homology product* if $H_1(M, R_+(\gamma); \mathbb{Z}) = 0$ and $H_1(M, R_-(\gamma); \mathbb{Z}) = 0$. Similarly, (M, γ) is said to be a *rational homology product* if $H_1(M, R_+(\gamma); \mathbb{Q}) = 0$ and $H_1(M, R_-(\gamma); \mathbb{Q}) = 0$.

Remark 9.2 It follows from the universal coefficient theorem that every homology product is also a rational homology product.

Definition 9.3 Let (M, γ) be a balanced sutured manifold. A decomposing surface $S \subset M$ is called a *horizontal surface* if

(i) S is open,

(ii) $\partial S \subset \gamma$ and $|\partial S| = |s(\gamma)|$,

(iii) $[S] = [R_+(\gamma)]$ in $H_2(M, \gamma)$,

(iv) $\chi(S) = \chi(R_+(\gamma))$.

We say that (M, γ) is *horizontally prime* if every horizontal surface in (M, γ) is parallel to either $R_+(\gamma)$ or $R_-(\gamma)$.

Lemma 9.4 *Suppose that (M,γ) is a balanced sutured manifold and let*

$$(M,\gamma) \rightsquigarrow^S (M',\gamma')$$

be a surface decomposition. Then the following holds.

(1) *If (M,γ) is a rational homology product then $H_2(M,R_\pm(\gamma);\mathbb{Q})=0$, and both $H_2(M;\mathbb{Q})$ and $H_2(M;\mathbb{Z})$ vanish.*

(2) *If S is either a product disk or a product annulus then (M',γ') is a rational homology product if and only if (M,γ) is.*

(3) *If $R_+(\gamma)$ is connected, S is a connected horizontal surface, and (M,γ) is a rational homology product then (M',γ') is also a rational homology product.*

Proof Let $R_\pm = R_\pm(\gamma)$ and $R'_\pm = R_\pm(\gamma')$. Then using Alexander–Poincaré duality we get that

$$H_2(M,R_+;\mathbb{Q}) \approx H^1(M,R_-;\mathbb{Q}) \approx H_1(M,R_-;\mathbb{Q}) = 0.$$

A similar argument shows that $H_2(M,R_-;\mathbb{Q})=0$.

Look at the following segment of the long exact sequence of the pair (M,R_+):

$$H_2(R_+;\mathbb{Q}) \to H_2(M;\mathbb{Q}) \to H_2(M,R_+;\mathbb{Q}) = 0.$$

Since R_+ has no closed components $H_2(R_+;\mathbb{Q})=0$, so $H_2(M;\mathbb{Q})=0$. From Poincaré duality and the universal coefficient theorem

$$H_2(M;\mathbb{Z}) \approx H^1(M,\partial M;\mathbb{Z}) \approx \mathrm{Hom}(H_1(M,\partial M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Tor}(H_0(M,\partial M;\mathbb{Z})),$$

which implies that $H_2(M;\mathbb{Z})$ is torsion free. Thus $H_2(M;\mathbb{Z})=0$. This proves (1).

Now suppose that S is a product disk or a product annulus. Consider the relative Mayer–Vietoris sequence associated to the pairs (M',R'_+) and $(N(S),R_+\cap N(S))$. From the segment

$$\begin{aligned} 0 = H_1(M'\cap N(S), R'_+\cap N(S);\mathbb{Q}) \\ \to H_1(M',R'_+;\mathbb{Q}) \oplus H_1(N(S),R_+\cap N(S);\mathbb{Q}) \\ \to H_1(M,R_+;\mathbb{Q}) \to H_0(M'\cap N(S),R'_+\cap N(S);\mathbb{Q}) = 0 \end{aligned}$$

and since $H_1(N(S),R_+\cap N(S);\mathbb{Q})=0$ we get that $H_1(M',R'_+;\mathbb{Q})=0$ if and only if $H_1(M,R_+;\mathbb{Q})=0$. We can similarly show that $H_1(M',R'_-;\mathbb{Q})=0$ if and only if $H_1(M,R_-;\mathbb{Q})=0$. This proves (2).

Finally, let S be a connected horizontal surface in the balanced sutured manifold (M,γ) with R_+ connected. We denote by (M_1,γ_1) and (M_2,γ_2) the two components of

(M', γ'), indexed such that $R_+ \subset M_1$ and $R_- \subset M_2$. The sutured manifold (M, γ) is a homology product and we have already seen that this implies that $H_2(M;\mathbb{Q}) = 0$. So from the Mayer–Vietoris sequence

$$0 = H_2(S;\mathbb{Q}) \to H_2(M_1;\mathbb{Q}) \oplus H_2(M_2;\mathbb{Q}) \to H_2(M;\mathbb{Q}) = 0$$

we obtain that $H_2(M_i;\mathbb{Q}) = 0$ for $i = 1, 2$. Another segment of the same exact sequence is

$$0 \to H_1(S;\mathbb{Q}) \to H_1(M_1;\mathbb{Q}) \oplus H_1(M_2;\mathbb{Q}) \to H_1(M;\mathbb{Q}) \to \widetilde{H}_0(S;\mathbb{Q}) = 0,$$

thus $\quad \dim H_1(M_1;\mathbb{Q}) + \dim H_1(M_2;\mathbb{Q}) = \dim H_1(S;\mathbb{Q}) + \dim H_1(M;\mathbb{Q}).$

From the long exact sequence of the pair $(M, R_\pm)$ we see that

$$0 = H_2(M, R_\pm;\mathbb{Q}) \to H_1(R_\pm;\mathbb{Q}) \to H_1(M;\mathbb{Q}) \to 0,$$

and so $\dim H_1(M;\mathbb{Q}) = \dim H_1(R_\pm;\mathbb{Q})$. Since S is horizontal $\chi(S) = \chi(R_+)$. Moreover, R_+ and S are both connected, thus $\dim H_1(R_\pm;\mathbb{Q}) = \dim H_1(S;\mathbb{Q})$. Consequently,

$$\dim H_1(M_1;\mathbb{Q}) + \dim H_1(M_2;\mathbb{Q}) = 2\dim H_1(S;\mathbb{Q}). \tag{9–1}$$

From the long exact sequence of the triple (M, M_2, R_-) consider

$$0 = H_1(M, R_-;\mathbb{Q}) \to H_1(M, M_2;\mathbb{Q}) \to H_0(M_2, R_-;\mathbb{Q}).$$

Here $H_0(M_2, R_-;\mathbb{Q}) = 0$ because (M_2, γ_2) is balanced. So, using excision, we get that $H_1(M_1, S;\mathbb{Q}) \approx H_1(M, M_2;\mathbb{Q}) = 0$. Now the exact sequence

$$0 = H_2(M_1;\mathbb{Q}) \to H_2(M_1, S;\mathbb{Q}) \to H_1(S;\mathbb{Q}) \to H_1(M_1;\mathbb{Q}) \to H_1(M_1, S;\mathbb{Q}) = 0$$

implies that $\dim H_1(M_1;\mathbb{Q}) \le \dim H_1(S;\mathbb{Q})$. Using a similar argument we get that $\dim H_1(M_2;\mathbb{Q}) \le \dim H_1(S;\mathbb{Q})$. Together with Equation (9–1) we see that

$$\dim H_1(M_i;\mathbb{Q}) = \dim H_1(S;\mathbb{Q})$$

for $i = 1, 2$. So the map $H_1(S;\mathbb{Q}) \to H_1(M_1;\mathbb{Q})$ is an isomorphism and we can conclude that $H_2(M_1, S;\mathbb{Q}) = 0$. Using Alexander–Poincaré duality we get that

$$H_1(M_1, R_+;\mathbb{Q}) \approx H^1(M_1, R_+;\mathbb{Q}) \approx H_2(M_1, S;\mathbb{Q}) = 0.$$

Together with $H_1(M_1, S;\mathbb{Q}) = 0$ this implies that (M_1, γ_1) is a rational homology product. An analogous argument shows that (M_2, γ_2) is also a rational homology product. This proves (3). □

Observe that the proof of [9, Proposition 3.1] gives the following slightly stronger result.

Lemma 9.5 *Let K be a null-homologous knot in the oriented 3–manifold Y and let S be a Seifert surface of K. If*

$$\mathrm{rk}\,\widehat{HFK}(Y,K,[S],g(S)) = 1$$

then $Y(S)$ is a homology product.

Corollary 9.6 *If (M,γ) is a balanced sutured manifold with γ connected and*

$$\mathrm{rk}\,SFH(M,\gamma) = 1$$

then (M,γ) is a homology product, and thus also a rational homology product.

Proof Since (M,γ) is balanced and γ is connected $R_+(\gamma)$ and $R_-(\gamma)$ are diffeomorphic. Glue $R_+(\gamma)$ and $R_-(\gamma)$ together using an arbitrary diffeomorphism, then do an arbitrary Dehn filling along the torus boundary. This way we get a 3–manifold Y together with a null-homologous knot K (the core of the Dehn filling). Moreover, $R_+(\gamma)$ gives a Seifert surface S of K such that $Y(S) = (M,\gamma)$. Using Theorem 1.5

$$\widehat{HFK}(Y,K,[S],g(S)) \approx SFH(M,\gamma).$$

So Lemma 9.5 implies that $Y(S) = (M,\gamma)$ is a homology product. □

Theorem 9.7 *Suppose that (M,γ) is a taut balanced sutured manifold that is not a product. Then $SFH(M,\gamma) \geq \mathbb{Z}^2$.*

Proof The outline of the proof is the following. First we modify (M,γ) using decompositions along product disks and product annuli, horizontal decompositions, and adding product one-handles. The goal is to make (M,γ) a rational homology product, strongly balanced, and horizontally prime. Moreover, we need a curve in $R_+(\gamma)$ which homologically lies outside the characteristic product region (see Definition 9.8). Then we can find decomposing surfaces S_1 and S_2 which give taut decompositions $(M,\gamma) \rightsquigarrow^{S_i} (M_i,\gamma_i)$ for $i = 1,2$ such that $O_{S_1} \cap O_{S_2} = \varnothing$. To distinguish between Spin^c structures we use Lemma 3.10. According to Theorem 1.4 we have $\mathbb{Z} \leq SFH(M_i,\gamma_i)$. From Theorem 1.3 we get that

$$SFH(M_1,\gamma_1) \oplus SFH(M_2,\gamma_2) \leq SFH(M,\gamma),$$

which concludes the proof.

Throughout the proof we use the fact that if $(N,\nu) \rightsquigarrow^J (N',\nu')$ is a decomposition such that J is either a product disk or product annulus then (N,ν) is taut if and only if (N',ν') is taut. This is [2, Lemma 3.12].

By adding product one-handles to (M,γ) as in Remark 3.6 we can achieve that γ is connected. This new (M,γ) is still taut and is not a product. It was shown in [7, Lemma 9.13] that adding product one-handles does not change $SFH(M,\gamma)$, so it is sufficient to prove the theorem when γ *is connected*. In particular, both $R_+(\gamma)$ and $R_-(\gamma)$ are connected, thus (M,γ) *is strongly balanced*.

By Theorem 1.4 and Corollary 9.6 if the taut balanced sutured manifold (M,γ) is not a rational homology product and if γ is connected then $SFH(M,\gamma) \geq \mathbb{Z}^2$. So in order to prove Theorem 9.7 it is sufficient to consider the case when (M,γ) *is a rational homology product*.

Let $R_0,\dots,R_{k+1}$ be a maximal family of pairwise disjoint and nonparallel horizontal surfaces in (M,γ) such that $R_0 = R_+(\gamma)$ and $R_{k+1} = R_-(\gamma)$. Since γ is connected, R_i is open, and $|\partial R_i| = |s(\gamma)|$ we get that each R_i is connected. Decomposing (M,γ) along $R_1,\dots,R_k$ we get taut balanced sutured manifolds (M_i,γ_i) for $1 \leq i \leq k+1$ such that $R_+(\gamma_i) = R_{i-1}$ and $R_-(\gamma_i) = R_i$. From Proposition 8.6

$$SFH(M,\gamma) = \bigotimes_{i=1}^{k+1} SFH(M_i,\gamma_i)$$

over $\mathbb{Q}$. Furthermore, part (3) of Lemma 9.4 implies that each (M_i,γ_i) is a rational homology product. And (M_i,γ_i) is not a product since R_{i-1} and R_i are not parallel. Thus it is enough to prove Theorem 9.7 for $(M,\gamma) = (M_1,\gamma_1)$. So we can suppose that (M,γ) *is horizontally prime* (see Definition 9.3). Next we recall Definition 6.1 of Ni [9]; also see Cooper and Long [1].

Definition 9.8 Suppose that (M,γ) is an irreducible sutured manifold, $R_-(\gamma)$ and $R_+(\gamma)$ are incompressible and diffeomorphic to each other. A *product region* of (M,γ) is a submanifold $\Phi \times I$ of M such that Φ is a compact (possibly disconnected) surface and $\Phi \times \{0\}$ and $\Phi \times \{1\}$ are incompressible subsurfaces of $R_-(\gamma)$ and $R_+(\gamma)$, respectively.

In [1, Theorem 3.4] it is proven that there is a product region $E \times I$ such that if $\Phi \times I$ is any product region of (M,γ) then there is an ambient isotopy of M which takes $\Phi \times I$ into $E \times I$. We call $E \times I$ a *characteristic product region* of (M,γ).

Let $E \times I$ be a characteristic product region of (M,γ). We can suppose that $\gamma \subset E \times I$. Since (M,γ) is not a product $E \times I \neq M$. Let

$$(M',\gamma') = (M \setminus E \times I, (\partial E \times I) \setminus \gamma).$$

Denote the components of $(\partial E \times I) \setminus \gamma$ by $F_1, \dots, F_m$. Then each F_i is a product annulus in (M,γ). Moreover, no component of ∂F_i bounds a disk in $R(\gamma)$ since $E \times \{0\}$ and $E \times \{1\}$ are incompressible subsurfaces of $R(\gamma)$. After the sequence of decompositions along the product annuli $F_1, \dots, F_m$ we get the disjoint union of (M',γ') and the product sutured manifold $(E \times I, \partial E \times I)$. From part (2) of Lemma 9.4 we get that (M',γ') is also a rational homology product. Moreover, using Proposition 8.10 and the fact that

$$SFH((M',\gamma') \cup (E \times I, \partial E \times I)) \approx SFH(M',\gamma') \otimes \mathbb{Z} \approx SFH(M',\gamma')$$

we obtain that $SFH(M',\gamma') \le SFH(M,\gamma)$. Of course (M',γ') is not a product. Thus it is sufficient to prove that $SFH(M',\gamma') \ge \mathbb{Z}^2$. Note that $E' \times I = N(\gamma')$ is a characteristic product region of (M',γ'). Furthermore, (M',γ') is taut, horizontally prime, and strongly balanced.

If $R_+(\gamma')$ is not planar then let $(M_1,\gamma_1) = (M',\gamma')$ and $E_1 \times I = E' \times I$. If $R_+(\gamma')$ is planar then $\partial R_+(\gamma')$ is disconnected since otherwise we had $\partial M' = S^2$ and (M',γ') would not be irreducible. Connect two different components of γ' with a product one-handle T as in Remark 3.6 to obtain a sutured manifold (M_1,γ_1). Then $E_1 \times I = N(\gamma') \cup T$ is a characteristic product region of (M_1,γ_1). According to part (2) of Lemma 9.4 the sutured manifold (M_1,γ_1) is also a rational homology product. In both cases the map

$$H_1(E_1 \times \{1\}; \mathbb{Q}) \to H_1(R_+(\gamma_1); \mathbb{Q})$$

is not surjective. Indeed, in the second case the curve ω obtained by closing the core of the handle $T \cap R_+(\gamma_1)$ in $R_+(\gamma')$ lies outside $H_1(E_1 \times \{1\}; \mathbb{Q})$. Also, $SFH(M_1,\gamma_1) = SFH(M',\gamma')$ in both cases. Note that (M_1,γ_1) is still taut, horizontally prime, and strongly balanced.

From now on let $(M,\gamma) = (M_1,\gamma_1)$ and $E \times I = E_1 \times I$. Let $\omega_+ \subset R_+(\gamma)$ be a properly embedded oriented curve such that $[\omega_+] \notin H_1(E \times \{1\}; \mathbb{Q})$. Then $n[\omega_+] \notin H_1(E \times I; \mathbb{Z})$ for every $n \in \mathbb{Z}$. Since (M,γ) is a rational homology product the maps

$$i_\pm \colon H_1(R_\pm(\gamma); \mathbb{Q}) \to H_1(M; \mathbb{Q})$$

are isomorphisms; see Lemma 9.4. Thus there exists a properly embedded oriented curve $\omega_- \subset R_-(\gamma)$ such that $[\omega_-] \neq 0$ in $H_1(R_-(\gamma); \mathbb{Q})$ and nonzero integers a, b such that $a \cdot i_+([\omega_+]) = b \cdot i_-([\omega_-])$ in $H_1(M; \mathbb{Z})$. Choose a regular neighborhood $N(\omega_+ \cup \omega_-)$ of $\omega_+ \cup \omega_-$ in $R(\gamma)$. Then

$$N = \gamma \cup N(\omega_+ \cup \omega_-)$$

is a subsurface of ∂M. Let x be the Thurston semi-norm on $H_2(M,N;\mathbb{Z})$; see Definition 2.5. Since $H_2(M;\mathbb{Z})=0$ the map

$$\partial\colon H_2(M,N;\mathbb{Z})\to H_1(N;\mathbb{Z})$$

is injective. Thus there is a unique homology class $s\in H_2(M,N;\mathbb{Z})$ such that $\partial s=a[\omega_+]-b[\omega_-]$. Moreover, let

$$r=[R_+(\gamma)]=[R_-(\gamma)]\in H_2(M,N;\mathbb{Z}),$$

then $\partial r=[s(\gamma)]$. We will need the following definition; see Scharlemann [17].

Definition 9.9 Suppose $(S_1,\partial S_1)$ and $(S_2,\partial S_2)$ are oriented surfaces in general position in $(M,\partial M)$. Then the *double curve sum* of S_1 and S_2 is obtained by doing oriented cut and paste along $S_1\cap S_2$ to get an oriented surface representing the cycle S_1+S_2. The result is an embedded oriented surface coinciding with $S_1\cup S_2$ outside a regular neighborhood of $S_1\cap S_2$.

The following claim is analogous to [9, Lemma 6.5].

Claim 9.10 *For any integers $p,q\geq 0$ we have a strict inequality*

$$x(s+pr)+x(-s+qr)>(p+q)x(r).$$

Proof Let the surfaces S_1 and S_2 be norm minimizing representatives of $s+pr$ and $-s+qr$, respectively. Since M is irreducible and $R(\gamma)$ is incompressible we can assume that S_1 and S_2 have no S^2 or D^2 components. Thus $\chi(S_1)=-x(S_1)$ and $\chi(S_2)=-x(S_2)$. Furthermore, we can suppose that S_1 and S_2 are transversal, $(S_1\cup S_2)\cap\gamma$ consists of $p+q$ parallel copies of $s(\gamma)$, and $S_1\cap R(\gamma)=S_2\cap R(\gamma)$ consists of a parallel copies of ω_+ and b parallel copies of ω_-. Since M is irreducible and S_1 and S_2 are incompressible we can achieve that $(S_1\cup S_2)\setminus(S_1\cap S_2)$ has no disk components. Let P denote the double curve sum of S_1 and S_2; see Definition 9.9. Then $[P]=(p+q)r$ and P has no S^2 or D^2 components. Moreover, for any double curve sum $\chi(P)=\chi(S_1)+\chi(S_2)$. Thus $x(P)=x(S_1)+x(S_2)$. Also note that $P\cap R(\gamma)=\varnothing$ and $P\cap\gamma$ consists of $p+q$ parallel copies of $s(\gamma)$.

Suppose that T is a torus component of P. Then $T=\bigcup_{j=1}^{2m}A_j$, where $A_{2i-1}\subset S_1$ and $A_{2i}\subset S_2$ are annuli for $1\leq i\leq m$. Let $A^1=\bigcup_{i=1}^{m}A_{2i-1}$ and $A^2=\bigcup_{i=1}^{m}A_{2i}$, and define $S_1'=(S_1\setminus A^1)\cup(-A^2)$ and $S_2'=(S_2\setminus A^2)\cup(-A^1)$. With a small isotopy we can achieve that $|S_1'\cap S_2'|<|S_1\cap S_2|$. For $i=1,2$ we have $\partial S_i'=\partial S_i$, and thus $[S_i']=[S_i]$ in $H_2(M,N)$; moreover, $x(S_i')=x(S_i)$. Thus we can suppose that P has no torus components.

Due to the triangle inequality we only have to exclude the case

$$x(s+pr)+x(-s+qr)=(p+q)x(r).$$

Thus suppose that $x(P)=(p+q)x(r)$. We define a function $\varphi\colon M\setminus P\to\mathbb{Z}$ by setting $\varphi(z)$ to be the algebraic intersection number of P with a path connecting z and $R_+(\gamma)$. This is well defined because the image of $[P]=(p+q)r$ in $H_2(M,\partial M)$ is zero, and thus any closed curve in M intersects P algebraically zero times.

Let $J_i=\mathrm{cl}(\varphi^{-1}(i))$ for $0\le i\le p+q$ and let $P_i=J_{i-1}\cap J_i$ for $1\le i\le p+q$. Then $P=\coprod_{i=1}^{p+q}P_i$ and $\bigcup_{k=0}^{i-1}J_i$ is a homology between $R_+(\gamma)$ and P_i in $H_2(M,N)$. Thus $[P_i]=[R_+(\gamma)]=r$ and $x(P_i)\ge x(r)$. Since

$$\sum_{i=1}^{p+q}x(P_i)=x(P)=(p+q)x(r)$$

we must have $x(P_i)=x(r)$ for $1\le i\le p+q$. Each P_i is connected since it has no S^2 and T^2 components, and $H_2(M)=0$ implies that P_i can have no higher genus closed components, otherwise it would not be norm minimizing in r.

So each P_i is a horizontal surface in (M,γ), consequently it is parallel to $R_+(\gamma)$ or $R_-(\gamma)$. Thus for some $0\le k\le p+q$ the surfaces $P_1,\dots,P_k$ are parallel to $R_+(\gamma)$ and $P_{k+1},\dots,P_{p+q}$ are parallel to $R_-(\gamma)$. Let $P_0=R_+(\gamma)$ and $P_{p+q+1}=R_-(\gamma)$.

We can isotope S_1 such that $S_1\cap\mathrm{int}(J_i)$ is a collection of vertical annuli for $0\le i\le p+q$. Thus $S_1\cap\mathrm{int}(J_i)=C_i\times(0,1)$, where C_i is a collection of circles in P_i. Let $\gamma_k=\gamma\cap J_k$. Observe that there is a homeomorphism $h\colon(M,\gamma)\to(J_k,\gamma_k)$ such that $[C_k]=a[h(\omega_+)]$ in $H_1(P_k)$. Since $a[h(\omega_+)]\notin H_1(h(E\times\{1\}))$ there is a component C_k' of C_k such that $[C_k']\notin H_1(h(E\times\{1\}))$. Thus the product annulus $C_k'\times I$ cannot be homotoped into $h(E\times I)$, which contradicts the fact that $h(E\times I)$ is a characteristic product region of (J_k,γ_k). □

From [17, Theorem 2.5] we see that there are decomposing surfaces S_1 and S_2 in (M,γ) such that

1. $[S_1]=s+pr$ and $[S_2]=-s+qr$ in $H_2(M,N)$ for some integers $p,q\ge 0$,
2. if we decompose (M,γ) along S_i for $i=1,2$ we get a *taut* sutured manifold (M_i,γ_i),
3. ν_{S_i} is nowhere parallel to v_0 along ∂S_i for $i=1,2$,
4. $\partial S_1\cap R(\gamma)$ consists of a parallel copies of ω_+ and b parallel copies of $-\omega_-$,
5. $\partial S_2\cap R(\gamma)=-\partial S_1\cap R(\gamma)$,

(6) $\partial S_i \cap \gamma$ consists of parallel copies of $s(\gamma)$ and $\nu_{S_i}|(\partial S_i \cap \gamma)$ points out of M for $i = 1, 2$.

From (2) and Theorem 1.4 we get that

$$\mathbb{Z} \leq SFH(M_i, \gamma_i)$$

for $i = 1, 2$. Since (M, γ) is strongly balanced and S satisfies (3) we can define $c(S_1, t)$ and $c(S_2, t)$ for some trivialization t of $v_0^{\perp}$; see Definition 3.8.

Using part (2) of Lemma 3.9 and (6) we get that $I(S_1) = 0$ and $I(S_2) = 0$. Moreover, $r(S_1, t) = p\chi(R_+(\gamma)) + K$ and $r(S_2, t) = q\chi(R_+(\gamma)) - K$, where K is the contribution of $\partial S_1 \cap R(\gamma)$ to $r(S_1, t)$.

Since (M, γ) is taut $\chi(R_+(\gamma)) = -x(r)$. Thus

$$c(S_1, t) = \chi(S_1) + px(r) - K = -x(s + pr) + px(r) - K$$

and $$c(S_2, t) = \chi(S_2) + qx(r) + K = -x(-s + qr) + qx(r) + K.$$

From Claim 9.10 we get that

$$c(S_1, t) + c(S_2, t) = (p + q)x(r) - (x(s + pr) + x(-s + qr)) < 0.$$

Let $\mathfrak{s}_i \in O_{S_i}$ for $i = 1, 2$. Lemma 3.10 implies that $\langle c_1(\mathfrak{s}_1, t), [S_1]\rangle = c(S_1, t)$ and $\langle c_1(\mathfrak{s}_2, t), [S_2]\rangle = c(S_2, t)$. But $r = 0$ in $H_2(M, \partial M)$, and thus $[S_1] = s = -[S_2]$ in $H_2(M, \partial M)$. So $\langle c_1(\mathfrak{s}_2, t), [S_1]\rangle = -c(S_2, t)$. Together with $c(S_1, t) \neq -c(S_2, t)$ this implies that $\mathfrak{s}_1 \neq \mathfrak{s}_2$, and thus $O_{S_1} \cap O_{S_2} = \varnothing$. Using Theorem 1.3 we get that

$$\mathbb{Z}^2 \leq SFH(M_1, \gamma_1) \oplus SFH(M_2, \gamma_2) \leq SFH(M, \gamma).$$

This concludes the proof of Theorem 9.7. □

Theorem 9.11 *Let K be a null-homologous knot in an oriented 3–manifold Y such that $Y \setminus K$ is irreducible and let S be a Seifert surface of K. If*

$$\mathrm{rk}\, \widehat{HFK}(Y, K, [S], g(S)) = 1$$

then K is fibred with fibre S.

Proof From Theorem 1.5

$$SFH(Y(S)) \approx \widehat{HFK}(Y, K, [S], g(S)).$$

Consequently, $SFH(Y(S)) \neq 0$ and thus $Y(S)$ is taut. So we can apply Theorem 9.7 to $Y(S)$ and conclude that $Y(S)$ is a product, since otherwise we had $\mathbb{Z}^2 \leq SFH(Y(S))$. This implies that the knot K is fibred with fibre S. □

Theorem 9.12 *Let (M,γ) be a taut balanced sutured manifold that is a rational homology product. If* $\operatorname{rk} SFH(M,\gamma) < 4$ *then the depth of (M,γ) is at most two.*

Proof Suppose that the depth of (M,γ) is ≥ 3. Note that decompositions along product disks and product annuli do not decrease the depth of a sutured manifold, and decompositions along disjoint unions of product annuli can decrease the depth by at most one. Thus applying the same procedure to (M,γ) as in the proof of Theorem 9.7 we get two depth ≥ 1 (ie, nonproduct) taut balanced sutured manifolds (M_1,γ_1) and (M_2,γ_2) such that

$$SFH(M,\gamma) \geq SFH(M_1,\gamma_1) \oplus SFH(M_2,\gamma_2).$$

Theorem 9.7 implies that $SFH(M_i,\gamma_i) \geq \mathbb{Z}^2$ for $i=1,2$. Thus $SFH(M,\gamma) \geq \mathbb{Z}^4$. $\square$

Proof of Theorem 1.8 Let S be a genus g Seifert surface of K. Then $(M,\gamma) = Y(S)$ is a taut balanced sutured manifold with $SFH(Y(S)) \approx \widehat{HFK}(Y,K,g)$ due to Theorem 1.5. The linking matrix V of S is a matrix of the map

$$i_+\colon H_1(R_+(\gamma);\mathbb{Q}) \to H_1(M;\mathbb{Q}),$$

thus $\det V = \pm a_g \neq 0$ and i_+ is an isomorphism. From the long exact sequence of the pair $(M, R_+(\gamma))$ we see that $H_1(M, R_+(\gamma);\mathbb{Q}) = 0$. Similarly, $H_1(M, R_-(\gamma);\mathbb{Q})$ is also zero, thus (M,γ) is a rational homology product. Using Theorem 9.12 we conclude that the depth of (M,γ) is ≤ 2. Now using [2] we get a depth ≤ 2 taut foliation on (M,γ) transverse to γ and leaves including $R_\pm(\gamma)$. $\square$

Remark 9.13 If $\operatorname{rk} \widehat{HFK}(Y,K,g) = 3$ then using the fact that $\chi(\widehat{HFK}(Y,K,g)) = a_g$ we see that the condition $a_g \neq 0$ is automatically satisfied.

Question 9.14 Let K be a knot in a rational homology 3–sphere Y and suppose that k is a positive integer. Does

$$\operatorname{rk} \widehat{HFK}(Y,K,g(K)) < 2^k$$

imply that $Y \setminus N(K)$ has a depth $< 2(k-1)$ taut foliation transverse to $\partial N(K)$?

References

[1] **D Cooper**, **D D Long**, *Virtually Haken Dehn-filling*, J. Differential Geom. 52 (1999) 173–187 MR1743462

[2] **D Gabai**, *Foliations and the topology of 3–manifolds*, J. Differential Geom. 18 (1983) 445–503 MR723813

[3] **D Gabai**, *The Murasugi sum is a natural geometric operation*, from: "Low-dimensional topology (San Francisco, CA, 1981)", Contemp. Math. 20, Amer. Math. Soc., Providence, RI (1983) 131–143 MR718138

[4] **D Gabai**, *Foliations and the topology of 3–manifolds III*, J. Differential Geom. 26 (1987) 479–536 MR910018

[5] **D Gabai**, *Problems in foliations and laminations*, from: "Geometric topology (Athens, GA, 1993)", AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc., Providence, RI (1997) 1–33 MR1470750

[6] **P Ghiggini**, *Knot Floer homology detects genus-one fibred knots* `arXiv:math.GT/0603445`

[7] **A Juhász**, *Holomorphic discs and sutured manifolds*, Algebr. Geom. Topol. 6 (2006) 1429–1457 MR2253454

[8] **R Lipshitz**, *A cylindrical reformulation of Heegaard Floer homology*, Geom. Topol. 10 (2006) 955–1097 MR2240908

[9] **Y Ni**, *Knot Floer homology detects fibred knots* `arXiv:math.GT/0607156`

[10] **Y Ni**, *Sutured Heegaard diagrams for knots*, Algebr. Geom. Topol. 6 (2006) 513–537 MR2220687

[11] **P Ozsváth**, **Z Szabó**, *Holomorphic disks, link invariants, and the multi-variable Alexander polynomial* `arXiv:math.GT/0512286`

[12] **P Ozsváth**, **Z Szabó**, *Link Floer homology and the Thurston norm* `arXiv:math.GT/0601618`

[13] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004) 311–334 MR2023281

[14] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) 159 (2004) 1159–1245 MR2113020

[15] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) 159 (2004) 1027–1158 MR2113019

[16] **S Sarkar**, **J Wang**, *An algorithm for computing some Heegaard Floer homologies* `arXiv:math.GT/0607777`

[17] **M Scharlemann**, *Sutured manifolds and generalized Thurston norms*, J. Differential Geom. 29 (1989) 557–614 MR992331

Department of Mathematics, Princeton University
Princeton NJ 08544, USA

`ajuhasz@math.princeton.edu`

Proposed: Peter Ozsváth   Received: 13 November 2006
Seconded: Ron Fintushel, Tom Mrowka   Accepted: 24 November 2007