Essential curves in handlebodies and topological contractions

VIATCHEVSLAV GRINES
FRANÇOIS LAUDENBACH

If X is a compact set, a *topological contraction* is a self-embedding f such that the intersection of the successive images $f^k(X)$, $k > 0$, consists of one point. In dimension 3, we prove that there are smooth topological contractions of the handlebodies of genus ≥ 2 whose image is essential.

57M25; 37D15

1 Introduction

For a compact set X and a topological embedding $f\colon X \to X$, we shall say that f is a *topological contraction* if $\bigcap_{k\geq 0} f^k(X)$ consists of one point. We shall show that such a contraction can be very complicated when X is a 3–dimensional handlebody. Namely, we have the following result for which some more classical definitions will be recalled thereafter.

Theorem A *There exists a North–South diffeomorphism f of the 3–sphere S^3 and a Heegaard decomposition $S^3 = P_- \cup P_+$ of genus $g \geq 2$ with the following properties:*

(1) *$f|P_+$ is a topological contraction;*

(2) *$f(P_+)$ is essential in P_+.*

We shall limit ourselves to $g = 2$, since the generalization will be clear. We recall that a 3–dimensional *handlebody* of genus 2 is diffeomorphic to the regular neighborhood P in $\mathbb{R}^3$ of the planar figure eight Γ. A *compression disk* of P is a smooth embedded disk in P whose boundary lies in ∂P in which it is not homotopic to a point. Among the compression disks are the *meridian* disks $\pi^{-1}(x)$, where x is a regular point[1] in Γ and $\pi\colon P \to \Gamma$ is the regular neighborhood projection (that is, a submersion over the smooth part of Γ). A subset X of P is said to be *essential* in P if it intersects every compression disk[2] .

[1] Any point other than the center of the figure eight.
[2] This definition goes back to Rolfsen's book [2, p 110].

Published: 12 May 2008 DOI: 10.2140/gt.2008.12.981

A diffeomorphism f of S^3 is a *North–South diffeomorphism* if it has two fixed points only, one source $\alpha \in P_-$ and one sink $\omega \in P_+$, every other orbit going from α to ω.

A *Heegaard splitting* of S^3 is made of an embedded surface dividing S^3 into two handlebodies. According to a famous theorem of F Waldhausen such a decomposition is unique up to isotopy [3]. It is not hard to prove that the phenomenon mentioned in Theorem A does not happen with a Heegaard splitting of genus 1: if T is a solid torus and f is a topological contraction of T, then there is a compression disk of T avoiding $f(T)$.

The example which we are going to construct for proving Theorem A is based on the next theorem, for which some more notation is introduced. Let $\Gamma_0 \subset \Gamma$ be a simple closed curve. There exists a solid torus $T \subset \mathbb{R}^3$ which contains P and which is a tubular neighborhood of Γ_0. Let $i_0\colon P \to T$ be this inclusion. We say that a simple curve is unknotted in T if it bounds an embedded disk in T.

Theorem B *There exists an essential simple curve C in P such that $i_0(C)$ is unknotted in T.*

The second author is grateful to Sylvain Gervais and Nathan Habegger for interesting conversations on link invariants, in particular on Milnor's invariants [1]. He is also grateful to the organizers[3] of the conference in Toulouse in memory of Heiner Zieschang (*Braids, groups and manifolds*, Sept 2007), who offered him the opportunity to give a short talk on the subject of this paper. We also thank the anonymous referee for very valuable comments.

2 Essential curves

Our candidate for C in Theorem B is pictured in Figure 1.

It is clear that $i_0(C)$ is unknotted in T (or, equivalently, in the complement of the vertical axis which is drawn in Figure 1). A way of proving that C is essential in P is to prove the following lemma (actually equivalent as the referee noticed[4]).

Lemma 1 *Let $p\colon \widetilde{P} \to P$ be the universal cover of P and let $\widetilde{C}$ be the preimage $p^{-1}(C)$. Then $\widetilde{C}$ is essential in $\widetilde{P}$.*

---

[3] Michel Boileau, Thomas Fiedler, John Guaschi and Claude Hayat

[4] According to the Loop theorem and Dehn's lemma, C essential in P implies $\widetilde{C}$ essential in $\widetilde{P}$.

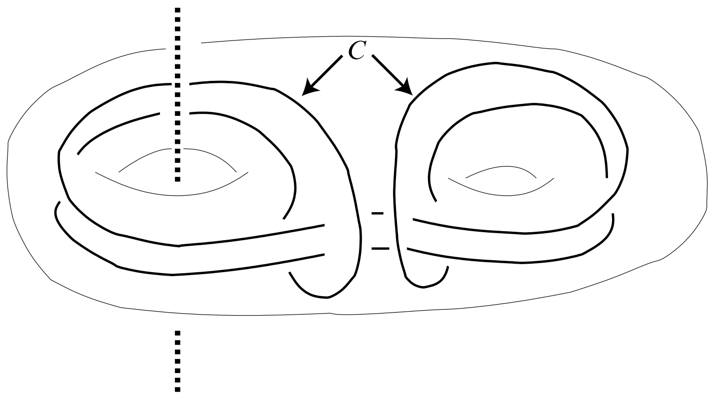

Figure 1

Proof We have the following description of $\widetilde{P}$: it is a 3–ball with a Cantor set E removed from its bounding 2–sphere[5]. This Cantor set is the set of ends of $\widetilde{P}$. A simple curve in $\partial\widetilde{P}$ is not homotopic to zero if it divides E into two nonempty parts. We get a fundamental domain F for the action of $\pi_1(P)$ on $\widetilde{P}$ by cutting P along two non-parallel meridian disks D_0 and D_1. Figure 2 shows what the pair $(F, \widetilde{C} \cap F)$ looks like: F is a 3–ball whose boundary consists of the union $\partial_1 F$ of four disks d_0, d_0', d_1, d_1' and a punctured sphere $\partial_0 F$, where $p(d_0) = p(d_0') = D_0$ and $p(d_1) = p(d_1') = D_1$; $\widetilde{C} \cap F$ is made of four strands with end points in $\partial_1 F$ and pairwise linked as it is shown.

One can show easily that (i) $\partial_1 F \setminus \widetilde{C}$ is incompressible and boundary incompressible in $F \setminus \widetilde{C}$, and (ii) $\partial_0 F$ is incompressible in $F \setminus \widetilde{C}$. Now suppose on the contrary that $\widetilde{C}$ is not essential and consider a compression disk Δ of $\widetilde{P}$ avoiding $\widetilde{C}$. We take Δ to be transversal to $\widetilde{D} := p^{-1}(D_0 \cup D_1)$. A standard innermost circle/arc argument, using (i), shows that we may assume that Δ is contained in $F \setminus \widetilde{C}$, contradicting (ii). □

Remarks 1 (1) Globally $\widetilde{C}$ looks like an *infinite Borromean* chain: any finite number of components is unlinked. We would like to know whether there exists a topological algebraic tool proving that C is essential in P.

(2) Our referee proposed another example where $[C] = aba^{-1}b^{-1}$ with respect to the obvious basis a, b of $\pi_1(P)$. In this case C is essential in P by an algebraic argument: the quotient $\pi_1(P)/\langle[C]\rangle$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. If C were inessential in P this quotient would be isomorphic to $\mathbb{Z} * \mathbb{Z}_n$ for some $n \geq 0$.

[5]Take the universal cover of Γ properly embedded in the hyperbolic plane and take a 3–dimensional thickening of it.

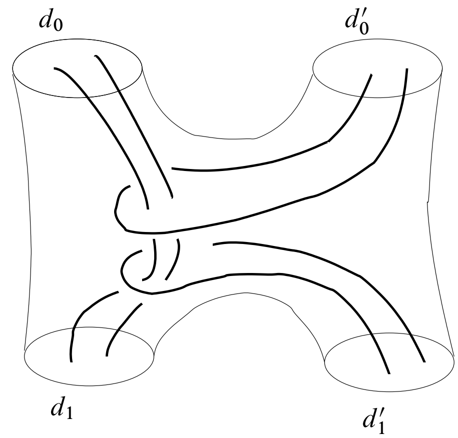

Figure 2

3 Proof of Theorem A

We recall the embedding $i_0\colon P \to \operatorname{int} T$. We start with a curve C in P which meets the conclusion of Theorem B. We equip it with its 0–normal framing (a section of this framing is not linked with C in $\mathbb{R}^3$) and we choose an embedding $j_0\colon T \to P$ whose image is a tubular neighborhood of C. Let B be a small ball in $\operatorname{int} T$. As C is unknotted in T, there is an ambient isotopy, supported in $\operatorname{int} T$, deforming i_0 to $i_1\colon P \to \operatorname{int} T$ such that $i_1 \circ j_0(T)$ is a standard small solid torus in B. One half of the desired Heegaard splitting of genus 2 will be given by $P_+ := i_1(P)$. At the present time f is only defined on T by $f := i_1 \circ j_0\colon T \to \operatorname{int} T$. If we compose i_1 with a sufficiently strong metric contraction of B into itself (with respect to some metric), then f is a metric contraction. Hence $\bigcap_{k>0} f^k(T)$ consists of one point.

Choose a round ball B' containing T in its interior. Since $f|T$ is isotopic to the inclusion $T \hookrightarrow \mathbb{R}^3$, f extends as a diffeomorphism $B' \to B$, and further as a diffeomorphism $S^3 \to S^3$. We are free to choose $f\colon S^3 \setminus B' \to S^3 \setminus B$ as we like. Let B" be the closure of $S^3 \setminus B'$ and $\varphi\colon S^3 \to S^3$ be a diffeomorphism which is the identity on B' and a strong metric contraction on a ball containing $f^{-1}(B")$. If we replace f by $f \circ \varphi^{-1}$ (without changing the notation), then $f^{-1}|B"$ becomes a metric contraction and the intersection $\bigcap\limits_{k>0} f^{-k}(S^3 \setminus B')$ consists of one point. We now have a North–South diffeomorphism f of S^3 which induces a topological contraction of T. Since $f(T) \subset \operatorname{int} P_+ \subset P_+ \subset \operatorname{int} T$, f also induces a topological contraction of P_+.

It remains to prove that $f(P_+)$ is essential in P_+. We know that $i_1(C)$ is essential in P_+. As a consequence, any compression disk Δ of P_+ intersects $f(T)$. We can take Δ to be transversal to $f(\partial T)$ such that no intersection curve is null-homotopic in $f(\partial T)$. Let γ be an intersection curve which is *innermost* in Δ and let δ be the disk that γ bounds in Δ.

Lemma 2 *We have* $\delta \subset f(T)$.

Proof If not, we have $\delta \subset P_+ \setminus f(\operatorname{int} T)$ and the simple curve γ in $f(\partial T)$ is unlinked with the core $i_1(C)$. Therefore, up to isotopy in $f(\partial T)$, it is a section of the 0–framing. In that case, $i_1(C)$ itself bounds an embedded disk in P_+. This is impossible, as $i_1(C)$ is essential in P_+. □

Therefore δ is a compression disk of the solid torus $f(T)$. But $P_+ = i_1(P)$, like P itself, is essential in T. Hence $f(P_+)$ is essential in $f(T)$ and δ must intersect $f(P_+)$. □

References

[1] **J Milnor**, *Isotopy of links. Algebraic geometry and topology*, from: "A symposium in honor of S. Lefschetz", Princeton University Press (1957) 280–306 MR0092150

[2] **D Rolfsen**, *Knots and links*, Math. Lecture Series 7, Publish or Perish, Berkeley, CA (1976) MR0515288

[3] **F Waldhausen**, *Heegaard-Zerlegungen der 3–Sphäre*, Topology 7 (1968) 195–203 MR0227992

N. Novgorod State University
Gagarina 23, N. Novgorod, 603950 Russia

Laboratoire de mathématiques Jean Leray, UMR 6629 du CNRS
Faculté des Sciences et Techniques, Université de Nantes
2, rue de la Houssinière, F-44322 Nantes cedex 3, France

grines@vmk.unn.ru, francois.laudenbach@univ-nantes.fr

Proposed: Cameron Gordon Received: 13 November 2007
Seconded: Martin Bridson, Walter Neumann Accepted: 8 February 2008