A geometric model for Hochschild
homology of Soergel bimodules

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An important step in the calculation of the triply graded link homology of Khovanov and Rozansky is the determination of the Hochschild homology of Soergel bimodules for $SL(n)$. We present a geometric model for this Hochschild homology for any simple group G, as B–equivariant intersection cohomology of $B \times B$–orbit closures in G. We show that, in type A, these orbit closures are equivariantly formal for the conjugation B–action. We use this fact to show that, in the case where the corresponding orbit closure is smooth, this Hochschild homology is an exterior algebra over a polynomial ring on generators whose degree is explicitly determined by the geometry of the orbit closure, and to describe its Hilbert series, proving a conjecture of Jacob Rasmussen.

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1 Introduction

In this paper, we consider the Hochschild homology of Soergel bimodules, and construct a geometric interpretation of it. This will allow us to explicitly compute the Hochschild homology of a special class of Soergel bimodules.

Soergel bimodules are bimodules over a polynomial ring, which appear naturally both in the study of perverse sheaves on flag varieties and of the semiring of projective functors on the BGG category $\mathcal{O}$. Recently interest in them has been rekindled by the appearance of connections with link homology as shown by Khovanov [11].

Khovanov's work showed that one aspect of Soergel bimodules which had not been carefully studied up to that date was, in fact, of great importance: their Hochschild homology. While the operation of taking Hochschild homology is hard to motivate from a representation theoretic perspective, we argue that it is, in fact, naturally geometric.

Throughout, we let

- G be a connected reductive algebraic group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$,

- B be a Borel subgroup of G,
- $T \subset B$ be a Cartan subgroup of G, with Lie algebra $\mathfrak{t}$,
- $n = \dim T$ be the rank of G,
- $W = N_G(T)/T$ be the Weyl group of G.

For any $w \in W$, we let $G_w = \overline{BwB}$.

As the closure of a group orbit, G_w is an irreducible closed subvariety of G which is smooth if and only if the corresponding Schubert variety $\mathcal{B}_w = G_w/B$ is (thus G_w is typically singular). The subgroup $B \times B$ acts on G_w by left and right multiplication. Restricting this action to the diagonal gives the action of B by conjugation. Of course, we also have left and right actions of B but we will never consider these separately. Whenever we refer to a B–action on G_w, we will *always* mean the conjugation action.

Given a complex algebraic variety X, we write $H^*(X)$ and $IH^*(X)$ for its cohomology and intersection cohomology with complex coefficients. If X is in addition a G–variety we let $H_G^*(X)$ and $IH_G^*(X)$ denote the G–equivariant cohomology and G–equivariant intersection cohomology of X (again with coefficients in $\mathbb{C}$) which is a module over $H_G^*(\mathrm{pt})$. We normalize intersection cohomology by requiring that $H^*(X) \cong IH^*(X)$ and $H_G^*(X) \cong IH_G^*(X)$ if X is smooth.

We observe that T is a deformation retract of B, so we have $H_B^*(X) \cong H_T^*(X)$ and $IH_B^*(X) \cong IH_T^*(X)$ for all B–spaces X. We will freely switch between B– and T–equivariant cohomology and intersection cohomology throughout this paper.

Now consider the graded ring $S = H_T^*(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}^*]$ (where the elements of $\mathfrak{t}^*$ have degree 2), which is endowed with a W–action. Given a simple reflection $s \in W$, denote by R_s the S–bimodule $S \otimes_{S^s} S$ where S^s is the subring of invariants under the reflection s. If M is a graded S–bimodule denote by $M[a]$ the shifted module $(M[a])^i = M^{i+a}$. (Later we will think of Soergel bimodules as differential graded modules over $S \otimes S$, which is why we use $[a]$ to denote grading shift.)

We now come to the definition of Soergel bimodules:

Definition 1 We call an S–bimodule R a *Soergel bimodule* if each indecomposable direct summand of R is a direct summand of a tensor product of the form $R_{\mathbf{i}}[a] = R_s \otimes_S R_t \otimes_S \cdots \otimes_S R_u[a]$ in the category of graded S–bimodules, where $\mathbf{i} = (s, t, \ldots, u)$ is a sequence of (not necessarily distinct) simple reflections, and a is an arbitrary grading shift.

The motivation for studying Soergel bimodules comes from representation theory, geometry and connections between the two. In fact, one has the following theorem:

Theorem 1.1 (Soergel) *The indecomposable Soergel bimodules are parametrized (up to a shift in the grading) by the Weyl group [15, Satz 6.14].*

The indecomposable Soergel bimodule corresponding to $w \in W$ may be obtained as $R_w \cong IH^*_{B \times B}(G_w)$ [14, Lemma 5].

Implicit in the theorem is the fact that $IH^*_{B \times B}(G_w)$ is a module over $H^*_{B \times B}(\text{pt}) = S \otimes S$ and hence may be regarded as an S–bimodule. More generally, Soergel bimodules can be identified with the $B \times B$–equivariant hypercohomology $\mathbb{H}^*_{B \times B}(G, \mathcal{F})$, where $\mathcal{F}$ is a direct sum of shifts of $B \times B$–equivariant simple perverse sheaves on G.

Our central result is a geometric description of Hochschild homology $HH_*(-)$ of these bimodules.

Theorem 1.2 *Let $\mathcal{F}$ be a semisimple $B \times B$–equivariant perverse sheaf on G and let R be its $B \times B$–equivariant cohomology. Then*

$$HH_*(R) \cong \mathbb{H}^*_B(G, \mathcal{F}),$$

where B acts by conjugation (ie, by the diagonal inclusion $B \hookrightarrow B \times B$). In particular, we have

$$HH_*(R_w) \cong IH^*_B(G_w).$$

Unfortunately, $IH^*_B(G_w)$ has a single grading, whereas $HH_*(R_w)$ has two: one by decomposition into the components HH_i ("the Hochschild grading"), and one coming from the grading on R_w ("the polynomial grading"). This isomorphism takes the single grading on $IH^*_B(G_w)$ to the difference of the two gradings on $HH_*(R_w)$.

We can give a geometric interpretation of these gradings, but in a somewhat round-about manner, using the geometry of the intersection cohomology complex $\mathbf{IC}(BwB)$ (readers unfamiliar with intersection cohomology should refer to Section 3 and the references given there). We will first prove the following:

Theorem 1.3 *Assume $G = \text{SL}(n)$ or $\text{GL}(n)$. Then the complex $\mathbf{IC}_B(BwB)$ is equivariantly formal. Thus, if we let $j_B \colon B = G_e \to G_w$ denote the inclusion, the map*

$$j_B^* \colon IH^*_B(G_w) \to \mathbb{H}^*_B(B, j_B^*\mathbf{IC}_B(BwB))$$

is injective and becomes an isomorphism after tensoring with the fraction field of S.

Applying our deformation retract from B to T, we see that

$$\mathbb{H}^*_B(B, j_B^*\mathbf{IC}_B(BwB)) \cong \mathbb{H}^*_T(T, j_T^*\mathbf{IC}_B(BwB))$$

where T acts trivially on itself (that is, by the conjugation action). Thus by the Künneth theorem, we have a further isomorphism

$$\mathbb{H}_T^*(T, j_T^* \mathbf{IC}_T(BwB)) \cong S \otimes_{\mathbb{C}} \mathbb{H}^*(T, j_T^* \mathbf{IC}(BwB)),$$

which we can use to equip this S–module with a bigrading, with an "equivariant" grading from the first factor, and a "topological" grading from the second. We can use these to define "topological" and "equivariant" gradings on $HH_*(R_w)$, using transport of structure by the isomorphism of Theorem 1.2.

Thus, $HH_*(R_w)$ carries four natural gradings, which we call "Hochschild," "polynomial," "topological," and "equivariant."

Theorem 1.4 *These gradings are related by the equations*

$$\deg_t(\gamma) = \deg_h(\gamma) \qquad\qquad \deg_e(\gamma) = \deg_p(\gamma) - 2\deg_h(\gamma)$$

where $\deg_*(x)$ denotes the degree of x in the grading whose name begins with the letter $*$.

The case where G_w is smooth is of special interest to us. Any reader who is unhappy with the presence of intersection cohomology and perverse sheaves may only consider the case where G_w is smooth, in which case intersection cohomology is canonically isomorphic to Čech cohomology.

Theorem 1.5 *If G_w is smooth (in any type), then as a bigraded S–module, the Hochschild homology of R_w is an exterior algebra over S. That is, we have an isomorphism*

$$HH_*(R_w) = \wedge^\bullet(\gamma_1, \dots, \gamma_n) \otimes_{\mathbb{C}} S$$

where $\{\gamma_i\}_{i=1,\dots,m}$ are generators with

$$\deg_h(\gamma_i) = 1 \qquad\qquad \deg_p(\gamma_i) = 2k_i$$

for positive integers k_i determined by the geometry of G_w.

These integers can be calculated using the action of w on the root system or, in the case of $\mathrm{SL}(n)$, by presenting G_w/B as an iterated Grassmannian bundle.

While the indecomposable modules R_w are perhaps most natural from the perspective of geometry or representation theory, Definition 1 (and the study of knot homology, which we discuss briefly in Section 2) encourages us to concentrate on the tensor product modules $R_{\mathbf{i}}$. We call these particular Soergel bimodules *Bott–Samelson* for reasons which will be clarified in Section 5.

Bott–Samelson bimodules are naturally identified with the equivariant cohomology of the "groupy" Bott–Samelson space

$$G_{\mathbf{i}} \cong P_s \times_B P_t \times_B \cdots \times_B P_u,$$

where P_s is the parabolic associated to the reflection s. The analogues of theorems connecting the $B \times B$–orbit closures in G with Soergel bimodules are true here.

Theorem 1.6 *If $G = \mathrm{SL}(n)$ or $\mathrm{GL}(n)$, then for all $\mathbf{i}$, we have*

$$R_{\mathbf{i}} \cong H^*_{B \times B}(G_{\mathbf{i}}) \qquad\qquad HH_*(R_{\mathbf{i}}) \cong H^*_B(G_{\mathbf{i}}).$$

The T–conjugation on $G_{\mathbf{i}}$ is equivariantly formal, and the injection

$$i_T^*\colon H^*_T(G_{\mathbf{i}}) \to H^*_T(G_{\mathbf{i}}^T)$$

induces a bigrading on $H^*_T(G_{\mathbf{i}})$ matching that on $HH_*(R_{\mathbf{i}})$ as in Theorem 1.4.

The structure of the paper is as follows: In Section 2, we discuss the importance of the Hochschild homology of Soergel bimodules in knot theory. In Section 4, we prove Theorem 1.2, using the formalism of dg-modules, the relevant points of which we will summarize in Section 3. In Section 5, we will cover in more detail how to construct Soergel bimodules as equivariant intersection cohomology of various varieties. Finally, in Section 6, we prove Theorems 1.3–1.6.

2   Knot homology and Soergel bimodules

While Soergel bimodules merit study on the basis of their connections to representation theory and geometry, we also have applications in knot theory in mind, as we will now briefly describe. The interested reader can find more details in the papers of Khovanov [11], Rasmussen [12] and Webster [16].

The braid group B_G of G is a finitely presented group, with generators σ_s for each simple reflection $s \in W$, which is defined by the presentation:

$$
\begin{aligned}
\sigma_s \sigma_t &= \sigma_t \sigma_s && \text{(when } (st)^2 = e) \\
\sigma_s \sigma_t \sigma_s &= \sigma_t \sigma_s \sigma_t && \text{(when } (st)^3 = e) \\
(\sigma_s \sigma_t)^2 &= (\sigma_t \sigma_s)^2 && \text{(when } (st)^4 = e) \\
(\sigma_s \sigma_t)^3 &= (\sigma_t \sigma_s)^3 && \text{(when } (st)^6 = e)
\end{aligned}
$$

Note that if $G = \mathrm{SL}(n)$, then $B_n = B_G$ is the standard braid group familiar from knot theory.

There are several natural weak actions of the braid group on category $\mathcal{O}$ by families of functors (see, for example, Andersen and Stroppel [2] and Khomenko and Mazorchuk [9]), which have an avatar on the bimodule side of the picture in the form of a complex of bimodules attached to each braid group element. The description of these bimodule complexes can be found in various sources, for example Khovanov [11], or for general Coxeter groups in Rouquier [13].

Define the complexes of S–bimodules:

$$
\begin{aligned}
F(\sigma_s) &= \cdots \longrightarrow S[-1] \longrightarrow R_s[1] \longrightarrow \quad 0 \quad \longrightarrow \cdots \\
F(\sigma_s^{-1}) &= \cdots \longrightarrow \quad 0 \quad \longrightarrow R_s[1] \longrightarrow S[1] \longrightarrow \cdots
\end{aligned}
$$

where the maps between nonzero spaces are the unique (up to scalar) nonzero maps of degree 0. These maps are defined by (respectively) the pushforward and pullback in $B \times B$–equivariant cohomology for the inclusion $B \hookrightarrow \overline{BsB}$.

Theorem 2.1   *The shuffling complex*

$$
F(\sigma) \overset{\text{def}}{=} F(\sigma_{i_1}^{\epsilon_1}) \otimes_S \cdots \otimes_S F(\sigma_{i_m}^{\epsilon_m})
$$

of a braid $\sigma = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_m}^{\epsilon_m}$ (where $\epsilon_i = \pm 1$) depends up to homotopy equivalence only on σ, and not on its factorization. In particular,

$$
F(\sigma \sigma') \cong F(\sigma) \otimes_S F(\sigma'),
$$

so F defines a categorification of B_G.

The maps in this complex also have a geometric interpretation: each degree is a direct sum of Bott–Samelson modules for subsequences of **i**, and the "matrix coefficients" of the differential between these are induced by pullback or pushforward maps on $B \times B$–equivariant cohomology for inclusions $G_{\mathbf{i'}} \to G_{\mathbf{i''}}$ of Bott–Samelson spaces where **i'** and **i''** are subsequences of **i** which differ by a single index.

Even better, this complex can be used to define a knot invariant, as was shown by Khovanov [11]. Given an S–bimodule R let $HH_*(R)$ be the Hochschild homology of R, which can be defined (using the standard equivalence between S–bimodules and $S \otimes S^{\mathrm{op}}$–modules) by

$$HH_i(R) = \mathrm{Tor}^i_{S \otimes S^{\mathrm{op}}}(S, R).$$

This can be calculated by the Hochschild complex of S (which is often used as a definition), or by the Koszul complex, both of which are free resolutions of S as an S–bimodule (that is, as an $S \otimes S^{\mathrm{op}}$–module).

In the case where $G = \mathrm{SL}(n)$, Hochschild homology is a categorification of the trace on the braid group defined by Jones [8]. Remarkably, combining these creates a categorification of knot polynomials. The categorification obtained in this way had previously been defined by Khovanov and Rozansky.

Theorem 2.2 (Khovanov [11]) *As a graded vector space, the homology $\mathcal{KR}(\overline{\sigma})$ of the complex $HH_i(F(\sigma))$ depends (after appropriate grading shift) only on the knot $\overline{\sigma}$ formed by closing σ and is in fact the triply graded homology defined by Khovanov and Rozansky in [10].*

Combining Theorem 1.2 and our remarks above, we can understand the differentials of the complex $HH_i(F(\sigma))$ in terms of pullback and pushforward on B–equivariant cohomology.

3 The equivariant derived category and dg-modules

Since our readers may not be well-acquainted with the formalism of equivariant derived categories and its connection with dg-modules, as developed by Bernstein and Lunts, we will provide a brief overview of the necessary background. This material is discussed in more detail and greater generality in their monograph [4].

We will begin with some motivation. Suppose a Lie group G operates on a space X. We have maps:

$$m\colon G \times X \to X \qquad\qquad m(g, x) = g \cdot x$$
$$\pi\colon G \times X \to X \qquad\qquad \pi(g, x) = x$$

A function f on X is G–invariant if and only if $m^* f = \pi^* f$. It is therefore natural to define a G–equivariant sheaf on X to be a sheaf $\mathcal{F}$ on X together with an isomorphism $\theta \colon m^* \mathcal{F} \to \pi^* \mathcal{F}$. (There is also a cocycle condition that we don't mention here).

One can show that if G operates topologically freely on X with quotient X/G then the categories of G–equivariant sheaves on X and sheaves on X/G are equivalent.

Faced with a G–space, one would like to define an "equivariant derived category". This should associate to a pair (G, X) a triangulated category $D_G^b(X)$ together with a "forgetting G–equivariance" functor $D_G^b(X) \to D^b(X)$. For any reasonable definition of $D_G^b(X)$, whenever G acts topologically freely, there should be a natural equivalence $D_G^b(X) \cong D^b(X/G)$, as well as notions of pullback and pushforward for equivariant maps.

The trick is to notice that, at least up to homotopy, we may assume that the action is free: we "liberate" X (ie make it free) by replacing it with $X \times EG$ where EG is the total space of the universal G–bundle (ie any contractible space on which G acts topologically freely). The first projection $p \colon X \times EG \to X$ is a homotopy equivalence (because EG is contractible) and the diagonal operation of G on $X \times EG$ is free. Thus, we can consider the quotient map $q \colon X \times EG \to X \times_G EG$ as the "liberation" of $X \to X/G$.

We will now try to make this more precise. For the rest of this section, assume that G is a connected complex linear algebraic group and X a complex G–variety (both equipped with the classical topology). There exists an G–space EG such that:

(1) G acts topologically freely on EG,

(2) EG is contractible,

(3) we may write EG as a direct limit of smooth complex G–varieties $E_n G$, where each inclusion $E_n G \hookrightarrow E_{n+1} G$ is a G–equivariant closed immersion.

For example, we may embed G in some $GL(n)$ and then take a direct limit of Stiefel manifolds. We let $X \times_G EG$ denote the quotient of $X \times EG$ by the diagonal action. In the case where X is a single point, we have $X \times_G EG \cong EG/G$, which we will denote by BG. Given a space Y, we denote by $D^b(Y)$ the bounded derived category of sheaves of $\mathbb{C}$–vector spaces on Y.

Definition 2 The *(bounded) equivariant derived category* $D_G^b(X)$ is the full subcategory of $D^b(X \times_G EG)$ consisting of complexes $\mathcal{F} \in D^b(X \times_G EG)$ such that $q^* \mathcal{F} \cong p^* \mathcal{G}$ for some complex $\mathcal{G} \in D^b(X)$.

Remark 1 This is not exactly Bernstein and Lunts' definition. Consider the following diagram of spaces:

$$X \xleftarrow{p} X \times EG \xrightarrow{q} X \times_G EG.$$

They construct a category $D^b_G(X)^{\mathrm{BL}}$ as follows: objects are tuples $(\mathcal{G}, \mathcal{F}, \alpha)$ where $\mathcal{G} \in D^b(X)$, $\mathcal{F} \in D^b(X \times_G EG)$, and $\alpha\colon q^*\mathcal{F} \to p^*\mathcal{G}$ is an isomorphism; and morphisms between two objects $(\mathcal{G}, \mathcal{F}, \alpha)$ and $(\mathcal{G}', \mathcal{F}', \alpha')$ consist of pairs (g, f) where $g\colon \mathcal{G} \to \mathcal{G}'$ and $f\colon \mathcal{F} \to \mathcal{F}'$ are morphisms such that the diagram

$$\begin{CD} q^*\mathcal{F} @>{q^*f}>> q^*\mathcal{F}' \\ @V{\alpha}VV @VV{\alpha'}V \\ p^*\mathcal{G} @>{p^*g}>> p^*\mathcal{G}' \end{CD}$$

commutes. One may check that the functor $D^b_G(X)^{\mathrm{BL}} \to D^b_G(X)$ which maps an object $(\mathcal{G}, \mathcal{F}, \alpha)$ to $\mathcal{F}$ and a morphism (g, f) to f is an equivalence of categories.

Remark 2 The space $X \times_G EG$ is usually infinite dimensional. However, one may choose the spaces $E_n G$ such that the quotient $X \times_G E_n G$ is an algebraic variety. In this case $D^b_G(X)$ has an alternative definition as a certain limit of subcategories in $D^b(X \times_G E_n G)$ (see Bernstein and Lunts [4]).

As previously mentioned, it is natural to expect that there will exist a "forgetting G–equivariance functor" For$\colon D^b_G(X) \to D^b(X)$. With p and q as in the Remark 1, one defines $\mathrm{For}(\mathcal{F}) = p_*q^*\mathcal{F}$.

If $H \subset G$ is a subgroup, then H acts topologically freely on EG, so we may take EG for EH. Thus, we have a natural map

$$\varphi^G_H\colon X \times_H EG \to X \times_G EG$$

commuting with the projection to X. The pullback and pushforward by this map induce functors

$$(\varphi^G_H)^* = \mathrm{res}^G_H\colon D^b_G(X) \to D^b_H(X), \qquad (\varphi^G_H)_* = \mathrm{ind}^H_G\colon D^b_H(X) \to D^b_G(X).$$

Similarly, for any G–space we have a map $X \times_G EG \to BG$. Because X has finite dimensional cohomology, pushforward yields a functor

$$\pi_*\colon D^b_G(X) \to D^b_G(\mathrm{pt})$$

which commutes with the induction and restriction functors.

In this work, we are interested in equivariant cohomology for connected complex linear algebraic groups. This emerges as the cohomology of objects living in $D_G(\mathrm{pt})$. The first key observation of Bernstein and Lunts is the following:

Proposition 3.1 *If G is a connected complex algebraic group, then $D_G^b(\mathrm{pt})$ is the full triangulated subcategory of $D^b(BG)$ generated by the constant sheaf.*

It turns out that this observation allows Bernstein and Lunts to give an algebraic description of $D_G(\mathrm{pt})$. For this we need the language of differential graded algebras and modules.

Definition 3 A *differential graded algebra* (or *dg-algebra*) is a unital, graded associative algebra $\mathcal{A} = \oplus_{i \in \mathbb{Z}} A_i$ together with an additive endomorphism $d \colon \mathcal{A} \to \mathcal{A}$ of degree 1 such that:

(1) d is a differential: ie $d^2 = 0$,

(2) d satisfies the Leibniz rule: $d(ab) = (da)b + (-1)^{\deg a} a(db)$,

(3) $d(1_\mathcal{A}) = 0$, where $1_\mathcal{A}$ denotes the identity of $\mathcal{A}$.

A *left differential graded module* (or *left dg-module*) over a differential graded algebra $\mathcal{A}$ is a graded left $\mathcal{A}$–module M together with a differential $d_M \colon M \to M$ of degree 1 satisfying:

(1) $d_M^2 = 0$,

(2) $d_M(am) = (da)m + (-1)^{\deg a} a(d_M m)$ for $a \in \mathcal{A}$ and $m \in M$.

A morphism of dg-modules is a graded $\mathcal{A}$–module homomorphism $f \colon M \to M'$ commuting with the differentials.

Remark 3 If $\mathcal{A} = A_0$ is concentrated in degree zero, then a differential graded module is just a chain complex of $\mathcal{A}$–modules.

Given any dg-module M, we may consider its cohomology $H^*(M)$, which is a graded module over the graded algebra $H^*(\mathcal{A})$. As with the category of modules over an algebra, the category of dg-modules over a dg-algebra has a homotopy category and a derived category, as defined by Bernstein and Lunts [4]. A map $f \colon M \to M'$ of dg-modules is a *quasi-isomorphism* if the induced map $H^*(M) \to H^*(M')$ on cohomology is an isomorphism. After localizing quasi-isomorphisms, one obtains the *derived category* of dg-modules for the dg-algebra $\mathcal{A}$, which we denote by $\mathrm{dgMod}\mathcal{A}$.

We denote by $\mathrm{dgMod}^f \mathcal{A}$ the full subcategory consisting of objects isomorphic to dg-modules which are finitely generated over $\mathcal{A}$.

Given a morphism $\mathcal{A} \to \mathcal{A}'$ of dg-algebras, we would like to define functors of restriction and extension of scalars between the categories $\mathrm{dgMod}\mathcal{A}$ and $\mathrm{dgMod}\mathcal{A}'$. Restriction of scalars is unproblematic (acyclic complexes are mapped to acyclic complexes) but more care is needed in defining extension of scalars. Just as in the normal derived category, one needs a special class of objects in order to define functors. In $\mathrm{dgMod}\mathcal{A}$ these are the $\mathcal{K}$–projective objects [4], which we will not discuss in complete generality. In the sequel, we will only be interested in a special class of dg-algebras in which it is possible to construct $\mathcal{K}$–projective objects rather explicitly.

Proposition 3.2 [4, Proposition 11.1.1] *Let $\mathcal{A} = \mathbb{C}[x_1, \dots x_n]$ be viewed as a dg-algebra by setting $d_\mathcal{A} = 0$ and requiring that each x_i have even degree. Then all dg-modules which are free as $\mathcal{A}$–modules are $\mathcal{K}$–projective.*

As it will be important for later arguments, we will describe how to construct a $\mathcal{K}$–projective resolution of a dg-module M when $\mathcal{A}$ is as in the proposition, and M has trivial differential. We may choose a free resolution in the category of graded $\mathcal{A}$–modules:

$$P_{-n} \to \cdots \to P_{-2} \to P_{-1} \to M$$

We then consider the direct sum $P = \oplus_i P_{-i}[i]$ as a dg-module which carries a natural differential: elements in $P_{-i}[i]$ are mapped under d_P into $P_{-i+1}[i-1]$ using the corresponding map in the above resolution. The natural morphism $P \to M$ (again induced from the resolution above) is a quasi-isomorphism.

Using standard techniques from homological algebra, one may construct $\mathcal{K}$–projective resolutions for dg-modules over $\mathcal{A}$ with nontrivial differential, but this will not be necessary for our results.

We may now define the extension of scalars functor. Suppose we have a morphism $\mathcal{A} \to \mathcal{A}'$ of dg-algebras, and that $\mathcal{A}$ and $\mathcal{A}'$ are as in the proposition. For any $N \in \mathrm{dgMod}\mathcal{A}$, we define extension of scalars by

$$\mathcal{A}' \overset{L}{\otimes}_\mathcal{A} N \overset{\mathrm{def}}{=} \mathcal{A}' \otimes_\mathcal{A} P$$

where P is a $\mathcal{K}$–projective resolution of P and the differential on the later module is given by $d(a \otimes p) = a \otimes d_P(p)$ for $a \in \mathcal{A}'$ and $p \in P$. (Alternatively we may also take a $\mathcal{K}$–projective resolution of $\mathcal{A}'$ as an $\mathcal{A}$ dg-module).

We can now return to a discussion of the equivariant derived category. Abelian and triangulated categories can often be described by "module categories" over endomorphism

rings of generators. We have already seen that $D_G^b(\text{pt})$ is the triangulated subcategory of $D^b(BG)$ generated by the constant sheaf. Hence this category should be described in terms of the endomorphisms of $\underline{\mathbb{C}}_{BG}$ in the derived category (which is naturally a dg-algebra).

First, we must describe this algebra: there is a quasi-isomorphism

$$\mathcal{A}_G \stackrel{\text{def}}{=} H^*(BG) \to \text{End}^\bullet_{D^b(BG)}(\underline{\mathbb{C}}_{BG})$$

of dg-algebras. This yields a functor

$$\Gamma_G = \text{Hom}^\bullet_{D^b(BG)}(\underline{\mathbb{C}}_{BG}, -)\colon D_G^b(\text{pt}) \to \text{dgMod}^f \mathcal{A}_G.$$

Bernstein and Lunts then show:

Theorem 3.3 (Main Theorem of Bernstein–Lunts [4]) *Assume as before that G is a connected complex algebraic group. The above functor gives an equivalence of triangulated categories commuting with the cohomology functor:*

$$\Gamma_G\colon D_G^b(\text{pt}) \to \text{dgMod}^f \mathcal{A}_G$$

Moreover if $\varphi\colon G \to H$ is an inclusion and $\mathcal{A}_H \to \mathcal{A}_G$ is the induced homomorphism, then the restriction and induction functors have an algebraic description in terms of dg-modules:

$$
\begin{array}{ccc}
D_G^b(\text{pt}) & \xrightarrow{\text{ind}_G^H} & D_H^b(\text{pt}) \\
\downarrow{\Gamma_G} & & \downarrow{\Gamma_H} \\
\text{dgMod}^f \mathcal{A}_G & \xrightarrow[\text{res. of sc.}]{} & \text{dgMod}^f \mathcal{A}_H
\end{array}
\qquad
\begin{array}{ccc}
D_H^b(\text{pt}) & \xrightarrow{\text{res}_H^G} & D_G^b(\text{pt}) \\
\downarrow{\Gamma_H} & & \downarrow{\Gamma_G} \\
\text{dgMod}^f \mathcal{A}_H & \xrightarrow{\mathcal{A}_G \otimes_{\mathcal{A}_H} -} & \text{dgMod}^f \mathcal{A}_G
\end{array}
$$

Remark 4 Note that if G is a connected complex linear algebraic group, then $\mathcal{A}_G$ is always a polynomial ring on even generators. In this case, by Proposition 3.2 and the above discussion, we can describe the $\mathcal{K}$–projective objects and hence compute the effect of the restriction and induction functors in terms of dg-modules.

We will now describe equivariant intersection cohomology complexes, which will be important in the sequel. Given a complex variety X and a smooth locally closed subvariety $U \subset X$, there is a complex $\mathbf{IC}(U) \in D^b(X)$ called the *intersection cohomology complex*, with remarkable properties (see for example Beilinson, Bernstein and Deligne [3] and Goresky and MacPherson [7]). We normalize $\mathbf{IC}(U)$ so that, if $j\colon U \hookrightarrow X$ denotes the inclusion, then

$$j^*\mathbf{IC}(U) \cong \underline{\mathbb{C}}_U.$$

where $\mathbb{C}_U$ denotes the constant sheaf on U in degree 0. If $U \subset X$ is in addition dense, we may define the *intersection cohomology* of X, denoted $IH^*(X)$, to be the hypercohomology of $\mathbf{IC}(U)$. In other words

$$IH^*(X) \overset{\text{def}}{=} \mathbb{H}^*(\mathbf{IC}(U)).$$

It does not depend on the choice of the dense smooth subvariety $U \subset X$.

It is possible to construct equivariant analogues of the intersection cohomology complexes, as described in Chapter 5 of [4]: If X is furthermore a G–variety for a complex algebraic group G and U is a smooth G–stable subvariety, there exists an *equivariant intersection cohomology complex* which we denote by $\mathbf{IC}_G(U)$. One then defines the *equivariant intersection cohomology* by

$$IH_G^*(X) \overset{\text{def}}{=} \mathbb{H}^*(\mathbf{IC}_G(U)).$$

where $U \subset X$ is dense. The complex $\mathbf{IC}_G(U)$ is determined up to isomorphism in $D_G^b(X)$ by requiring that $\mathrm{For}(\mathbf{IC}_G(U)) \cong \mathbf{IC}(U)$. It follows that equivariant intersection cohomology complexes behave well with respect to restriction:

Lemma 3.4 *If $H \hookrightarrow G$ is an inclusion of complex linear algebraic groups, X is a G–variety and U is a smooth G–stable subvariety then:*

$$\mathrm{res}_G^H \mathbf{IC}_G(U) \cong \mathbf{IC}_H(U)$$

Remark 5 In dealing with equivariant intersection cohomology complexes, it is more convenient to work with the equivalent definition of the equivariant derived category mentioned in Remark 2.

4   Hochschild homology and dg-algebras

Recall that the Hochschild homology of an S–bimodule R can be defined as

$$HH_*(R) = S \overset{L}{\otimes}_{S \otimes S^{\mathrm{op}}} R,$$

where S has been made into an $S \otimes S^{\mathrm{op}}$ algebra by left and right multiplication. Since $S \cong \mathcal{A}_B$ and $S \otimes S \cong \mathcal{A}_{B \times B}$, this map is that induced by the diagonal group homomorphism $B \hookrightarrow B \times B$. Thus, we expect that the geometric analogue of taking Hochschild homology is restricting from a $B \times B$–action to the diagonal B.

However, we must be careful about the difference between dg-modules and modules. Hochschild homology is an operation on S–bimodules, not dg-bimodules. Thus, to make a precise statement requires us to restrict to formal complexes.

Definition 4 Let $M \in \mathrm{dgMod}\mathcal{A}_G$ be a dg-module. If $M \cong H^*(M)$ in $\mathrm{dgMod}\mathcal{A}_G$ we say that M is *formal*. Similarly, $\mathcal{F} \in D_G(\mathrm{pt})$ is *formal* if $\Gamma_G(\mathcal{F})$ is.

The following proposition connects the Hochschild cohomology of formal equivariant sheaves with another equivariant cohomology. This is our main technical tool.

Proposition 4.1 *Suppose $\mathcal{F} \in D_{B \times B}(\mathrm{pt})$ is formal. Then one has an isomorphism:*

$$\bigoplus_i HH_i(\mathbb{H}^*_{B \times B}(\mathcal{F}))[i] \cong \mathbb{H}^*_B(\mathrm{res}^B_{B \times B}\mathcal{F})$$

as graded S–modules.

Furthermore this isomorphism is functorial. That is, if $\mathcal{F}$ and $\mathcal{G}$ are formal sheaves in $D_{B \times B}(\mathrm{pt})$, and $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism, the maps $\mathbb{H}^*_B(\varphi)$ and $HH_*(\mathbb{H}^*_{B \times B}(\varphi))$ commute with this isomorphism.

Proof of Proposition 4.1 In order to work out the Hochschild homology of $\mathbb{H}^*_{B \times B}(\mathcal{F})$ we may take a free resolution of $\mathbb{H}^*_{B \times B}(\mathcal{F})$ by $S \otimes S$–modules:

$$0 \to P_{-2n} \to \cdots \to P_{-1} \to \mathbb{H}^*_{B \times B}(\mathcal{F})$$

We then apply $S \otimes_{S \otimes S} -$ and take cohomology. However, because $\mathbb{H}^*_{B \times B}(\mathcal{F}) \cong \Gamma_{B \times B}(\mathcal{F})$ in $\mathrm{dgMod}\mathcal{A}_{B \times B}$, we may also regard $\bigoplus P_i[-i]$ as a $\mathcal{K}$–projective resolution of $\Gamma(\mathcal{F})$. By Theorem 3.3 we have:

$$\mathbb{H}^*_B(\mathrm{res}^B_{B \times B}\mathcal{F}) \cong H^*(S \overset{L}{\otimes}_{S \otimes S} \mathbb{H}^*_{B \times B}(\mathcal{F})) \cong \bigoplus_i HH_i(\mathbb{H}^*_{B \times B}(\mathcal{F}))[i] \qquad \square$$

Proof of Theorem 1.2 Let $\mathcal{F}$ denote the image of the intersection cohomology sheaf on $\overline{BwB}$ in $D_{B \times B}(\mathrm{pt})$. By Proposition 5.2, $\mathcal{F}$ is formal. Hence we can apply Proposition 4.1. However, we also know that $\mathbb{H}^*_{B \times B}(\mathcal{F})$ is the indecomposable Soergel bimodule R_w. Thus:

$$\bigoplus_i HH_i(R_w)[i] \cong \mathbb{H}^*_B(\mathrm{res}^B_{B \times B}\mathcal{F}).$$

But $\mathrm{res}^B_{B \times B}$ commutes with the map to a point and by Lemma 3.4,

$$\mathrm{res}^B_{B \times B}(\mathbf{IC}_{B \times B}(BwB)) \cong \mathbf{IC}_B(BwB).$$

Therefore

$$\bigoplus_i HH_i(R_w)[i] \cong \mathbb{H}^*_B(\mathbf{IC}_B(BwB)) \cong IH^*_B(G_w)$$

which yields the main theorem. $\qquad \square$

5 The geometry of Bott–Samelson bimodules

In this section, we discuss Bott–Samelson bimodules, calculate their $B \times B$–equivariant cohomology and obtain the formality results needed in the previous section.

Since we have already described the geometric realization of indecomposable bimodules as intersection cohomology of subvarieties of G, we know abstractly that the Bott–Samelson bimodule $R_{\mathbf{i}}$ must be the hypercohomology of a perverse sheaf obtained by taking a direct sum of **IC**–sheaves of these subvarieties with appropriate multiplicities.

However, this is deeply dissatisfying from a geometric viewpoint, and totally at odds with our viewpoint that Bott–Samelson bimodules are very natural objects. Thus we would like a more natural geometric realization of them.

For each simple reflection s, let P_s be the minimal parabolic containing s. For a sequence $\mathbf{i} = (s, t, \dots, u)$ of simple reflections, let

$$G_{\mathbf{i}} = P_s \times_B P_t \dots \cdots \times_B P_u.$$

We call this the *Bott–Samelson variety* corresponding to $\mathbf{i}$. Note that this variety still carries a $B \times B$–action, and thus a diagonal B–action.

Furthermore, we have a projective $B \times B$–equivariant map $m_{\mathbf{i}}: G_{\mathbf{i}} \to G$ given by multiplication, which intertwines the diagonal B–action on $G_{\mathbf{i}}$ with the conjugation B–action on G.

The quotient of $G_{\mathbf{i}}$ by the right Borel action is the familiar projective Bott–Samelson variety which is used to construct resolutions of singularities for Schubert varieties. It is worth noting that just like in the flag variety case, if $\mathbf{i}$ is a reduced expression (ie if $\ell(st \dots u)$ is the length of $\mathbf{i}$), then the multiplication map is a resolution of singularities.

Let us explain how to calculate the $B \times B$–equivariant cohomology of the Bott–Samelson varieties. Actually, we will calculate the corresponding dg-module over $S \otimes S$. We start with a lemma:

Lemma 5.1 *Suppose $\mathcal{F} \in D^b_{B \times B}(\mathrm{pt})$ is formal and let s be a simple reflection. Then*

$$\Gamma_{B \times B}(\mathrm{res}^{B \times B}_{P_s \times B} \mathrm{ind}^{P_s \times B}_{B \times B} \mathcal{F}) \cong S \otimes_{S^s} \Gamma_{B \times B}(\mathcal{F})$$

as objects in $\mathrm{dgMod}^f_{A_{B \times B}}$.

Proof Thanks to Theorem 3.3 we know that $\Gamma_{P_s \times B}(\mathrm{ind}^{P_s \times B}_{B \times B} \mathcal{F})$ is equal to $\Gamma_{B \times B}(\mathcal{F})$ regarded as dg-module over $S^s \otimes S$. Hence

$$\Gamma_{B \times B}(\mathrm{res}^{B \times B}_{P_s \times B} \mathrm{ind}^{P_s \times B}_{B \times B} \mathcal{F}) = (S \otimes S) \overset{L}{\otimes}_{S^s \otimes S} \Gamma_{B \times B}(\mathcal{F}).$$

However, $S \otimes S$ is free as a module over $S^s \otimes S$ and is hence $\mathcal{K}$–projective. Thus the derived tensor product coincides with the naive tensor product and the result follows. □

We can now prove the crucial "formality" claim mentioned above:

Proposition 5.2 *The direct images of the sheaves $\mathbb{C}_{G_i}$ and $\mathbf{IC}_{B \times B}(BwB)$ in $D^b_{B \times B}(\mathrm{pt})$ are formal.*

Proof First notice that we can write the sheaf $(m_{\mathbf{i}})_* \mathbb{C}_{G_i}$ as an iterated induction and restriction:

$$(m_{\mathbf{i}})_* \mathbb{C}_{G_i} \cong \mathrm{res}^{B \times B}_{P_s \times B} \mathrm{ind}^{P_s \times B}_{B \times B} \cdots \mathrm{res}^{B \times B}_{P_u \times B} \mathrm{ind}^{P_u \times B}_{B \times B} (\mathbb{C}_B).$$

Hence by the above lemma, letting p be the projection to a point:

$$\Gamma_{B \times B}(p_* \mathbb{C}_{G_i}) = S \otimes_{S^s} S \otimes_{S^t} \otimes \cdots \otimes_{S^u} S \qquad \text{in } \mathrm{dgMod}^f_{S \otimes S}.$$

Thus the proposition is true for $\mathbb{C}_{G_i}$. Now, by the decomposition theorem of [3] (or more precisely, its equivariant version in [4]), we may obtain $\mathbf{IC}_{B \times B}(BwB)$ as a direct summand of $(m_{\mathbf{i}})_* G_{\mathbf{i}}$, where $\mathbf{i} = (s, \ldots, u)$ is a reduced expression for w. Thus $p_* \mathbf{IC}_{B \times B}(BwB)$ is a direct summand of $p_* \mathbb{C}_{G_i}$ and is also formal. □

6 Equivariant formality

Now, we will carry out some actual computations of B–equivariant cohomology, and thus of Hochschild homology.

Of course, the best setting in which to compute the equivariant cohomology of a variety is when that variety (or more precisely, the sheaf one intends to compute the hypercohomology of) is equivariantly formal.

Definition/Theorem 6.1 We call $\mathcal{F} \in D^b_T(X)$ on a T–variety X *equivariantly formal* if one of the following equivalent conditions holds:

 (1) The S module $\mathbb{H}^*_T(X, \mathcal{F})$ is free.

 (2) The differentials in the spectral sequence

$$\mathbb{H}^*(X, \mathcal{F}) \otimes S \Rightarrow \mathbb{H}^*_T(X, \mathcal{F})$$

   are trivial, that is, if $\mathbb{H}^*(X, \mathcal{F}) \otimes S \cong \mathbb{H}^*_T(X, \mathcal{F})$ as S–modules.

 (3) We have the equality $\dim_{\mathbb{C}} H^*(X) = \dim_{\mathbb{C}} H^*(X^T)$.

The equivariant formality of the Bott–Samelsons of SL(n) has been proven by Rasmussen in different language.

Proposition 6.2 *If $G = \mathrm{SL}(n)$ or $\mathrm{GL}(n)$, the T–space $G_{\mathbf{i}}$ is equivariantly formal for all* $\mathbf{i}$.

Proof By Theorem 1.2, $HH_*(R_{\mathbf{i}})$ is free as an S–module if and only if $H_T^*(G_{\mathbf{i}})$ is. By [12, Propositon 4.6], the module $HH_*(R_{\mathbf{i}})$ is free in type A, so by Definition/Theorem 6.1 above $G_{\mathbf{i}}$ is equivariantly formal. $\qquad\square$

This in turn implies that $m_*\mathbb{C}_{G_{\mathbf{i}}}$ is equivariantly formal, where $m\colon G_{\mathbf{i}} \to G$ is the multiplication map. Since all summands of equivariant formal sheaves are themselves equivariantly formal, and each $\mathbf{IC}_{B \times B}(BwB)$ appears as a summand of such a sheaf (if, for example, $\mathbf{i}$ is a reduced word for w), this completes the proof of Theorem 1.3 and the first part of Theorem 1.6.

While the most obvious consequence of equivariant formality, calculating the equivariant cohomology from the ordinary cohomology or vice versa, is a useful one, there are less obvious ones as well.

Proposition 6.3 (Goresky, Kottwitz and MacPherson [6, Theorem 6.3]) *If $\mathcal{F}$ is equivariantly formal, and $i\colon X^T \to X$ is the natural inclusion, then the pullback map*

$$i_T^*\colon H_T^*(X,\mathcal{F}) \to H_T^*(X^T, i^*\mathcal{F})$$

is injective.

As we mentioned earlier, we are interested in the Hochschild homology of Soergel bimodules as a bigraded object (so that we get a triply-graded knot homology theory), but the grading on equivariant hypercohomology is only one of these. From now on, we consider $H_T^*(G_w)$ as a bigraded S–module, with the bigrading defined by the isomorphism with Hochschild homology given by Proposition 4.1.

Proof of Theorem 1.4 and Theorem 1.6 Since the pullback map $H_T^*(G_{\mathbf{i}}) \to H_T^*(G_{\mathbf{i}}^T)$ is induced by a map of Soergel bimodules, it is homogeneous in both gradings. Similar reasoning applies to the map induced by the inclusion of a summand $\mathbf{IC}_{B \times B}(BwB) \hookrightarrow m_*\mathbb{C}_{G_{\mathbf{i}}}$. Thus we need only establish the theorem for $G_{\mathbf{i}}^T$. As this is a union of complex tori with the trivial action, we need only establish the theorem for T.

This case follows directly from applying HH_0 to the Koszul resolution of $H_{T \times T}(T) \cong S$ as a bimodule over itself. $\qquad\square$

Let us turn to the case where G_w is smooth. Since $H^*_T(G_w) \cong S \otimes H^*(G_w)$, we should address the structure of $H^*(G_w)$. Surprisingly, no description of this cohomology seems to be in the literature, but, in fact, there is a very beautiful one.

As is well known (and we reprove in the course of Lemma 6.6 below), there exists a unique decreasing sequence of positive integers $k_1, \cdots k_n$ such that the Hilbert series of $H^*(G_w/B)$ is of the form

$$\sum_{i=1}^{\ell(w)} q^{i/2} \dim H^i(G_w/B) = \prod_{j=1}^{n} \frac{1 - q^{k_j}}{1 - q}.$$

When $w_0 = w$, we have $G_w = G$, and the numbers k_i are the exponents of the group G.

Theorem 6.4 *If G_w is smooth, then as an algebra*

$$H^*(G_w) \cong \wedge^\bullet(\gamma_1, \ldots \gamma_n),$$

where $\deg(\gamma_i) = 2k_i - 1$, and as an S–algebra

$$H^*_T(G_w) \cong HH_i(R_w) \cong S \otimes H^*(G_w).$$

In the standard double grading on $HH_i(R_w)$, we have $\deg(1 \otimes \gamma_i) = (1, 2k_i)$.

This immediately implies Theorem 1.5. Furthermore, this explicitly describes the Hilbert series of $HH_i(R_w)$, proving a conjecture of Rasmussen.

Corollary 6.5 *The Hilbert series of $HH_i(R_w)$ is given by:*

$$\sum_{i,j} a^i q^j \dim_{\mathbb{C}} HH_i(R_w)_{2j} = \prod_{\ell=1}^{n} \frac{1 + a q^{k_\ell}}{1 - q}$$

Since $\dim_{\mathbb{C}} H^*(G_w) = 2^{\mathrm{rk}(G)}$, Definition/Theorem 6.1(3) establishes the equivariant formality of G_w independently of the earlier results of this paper (and for all types).

As usual in Lie theory, we define the height $h(\alpha) = \langle \rho, \alpha \rangle$ of a root α to be its evaluation against the sum of the fundamental weights, that is, the height is the linear extension of the function which assigns 1 to each simple root.

Let $\mathcal{R}_+$ be the set of positive roots of G, and $\mathcal{R}_-$ be the set of negative ones.

Lemma 6.6 *The cohomology ring $H^*(G_w/B)$ is a quotient of the polynomial ring S by a regular sequence $(f_1, \ldots, f_n)$.*

Proof By results of Akyıldız and Carrell [1], the ring $H^*(G_w/B)$ is a quotient of $\mathbb{C}[BwB/B]$ by a regular sequence $(g_1, \ldots, g_k)$, where $k = \ell(w)$.

In this grading, $\mathbb{C}[BwB/B]$ is a polynomial algebra generated by elements of degree $2h(\alpha)$ for each root $\alpha \in \mathcal{R}_+ \cap w^{-1}(\mathcal{R}_-)$, and the degrees of g_i are given by $2h(\alpha) + 2$ as α ranges over the same roots. Corresponding to the simple roots are m generators $y_1, \ldots y_m$ of degree 2, which are the first Chern classes of line bundles on G/B corresponding the fundamental weights, and for the other roots we have $\ell(w) - m$ other generators $y_{m+1}, \ldots, y_{\ell(w)}$ of higher degree. Here, we assume these are in increasing order by degree.

It is a well known fact that the cohomology ring $H^*(G/B)$ is generated by the Chern classes x_i. Since the natural pullback map $H^*(G/B) \to H^*(G_w/B)$ is onto, the ring $H^*(G_w/B)$ is also generated by the x_i, which span $H^2(G_w/B)$. That is, if $p = \deg y_k > 2$, then for some $\beta_i \in \mathbb{C}$, we have the equality in $\mathbb{C}[BwB/B]$

$$y_k - \sum_{\deg(g_i)=p} \beta_i g_i \in S + \sum_{\deg(g_j)<p} S g_j.$$

Since $y_k \notin \sum_{\deg(g_j)<p} S g_j$, we can eliminate y_k and any single relation g_i such that $\beta_i \neq 0$. Obviously, $(g_1, \ldots, g_n) \setminus \{g_i\}$ is again a regular sequence in $\mathbb{C}[y_1, \ldots, y_{k-1}]$. Applying this argument inductively, we obtain a subsequence $(g_{i_1}, \ldots, g_{i_n})$ which is regular in S, which is the desired sequence. $\square$

Define $k_i = \deg(f_i)$.

Proposition 6.7 *The number of times the integer m appears in the list $k_1, \ldots, k_n$ is precisely the number of roots in $\mathcal{R}^+ \cap w^{-1}(\mathcal{R}^-)$ of height $m-1$ minus the number of such roots of height m.*

Proof Since the number of relations of degree j from the original presentation of Akyıldız and Carrell which have been eliminated is the number of generators of degree j, the remaining number of relations is precisely the difference between these, which is also the number of roots of height $j/2 - 1$ minus the number of height $j/2$ in $\mathcal{R}^+ \cap w^{-1}(\mathcal{R}^-)$. $\square$

Proof of Theorem 6.4 Applying the Hirsch lemma (as stated in the paper [5]) to the fibration $B \to G_w \to G_w/B$, we see that the cohomology ring $H^*(G_w)$ is the cohomology of the dg-algebra $H^*(G_w/B) \otimes_S \mathcal{K}_T$, where $\mathcal{K}_T$ is the Koszul complex of S (the natural free resolution of $\mathbb{C}$ as an S-module). The complex $\mathcal{K}_T$ gains a dg-algebra structure from the Yoneda product.

Since $(f_1, \ldots, f_n)$ is regular, $H^*(G_w/B)$ is quasi-isomorphic to the Koszul complex $\mathcal{K}_{\mathbf{f}}$. Thus, we have quasi-isomorphisms:

$$H^*(G_w/B) \otimes_S \mathcal{K}_T \simeq \mathcal{K}_{\mathbf{f}} \otimes_S \mathcal{K}_T \simeq \mathcal{K}_{\mathbf{f}} \otimes_S \mathbb{C}$$

The right-hand side is just an exterior algebra over $\mathbb{C}$ with generators $\gamma_1, \ldots, \gamma_n$ the degrees of which are given by $\deg(\gamma_i) = 2k_i - 1$.

Since we used a Koszul complex, the generators γ land in Hochschild degree 1 under the restriction map to T, so the degree of γ_i is $(2k_i, 1)$. □

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