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Minimality of the well-rounded retract

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We prove that the well-rounded retract of $\mathrm{SO}_n \backslash \mathrm{SL}_n \mathbb{R}$ is a minimal $\mathrm{SL}_n \mathbb{Z}$–invariant spine.

20F65; 12G10, 53C35, 11F75

1 Introduction

In this note we are interested in a certain $\mathrm{SL}_n \mathbb{Z}$–invariant deformation retract of the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbb{R}$. To every element $A \in \mathrm{SL}_n \mathbb{R}$ one can associate the lattice $A\mathbb{Z}^n$ in $\mathbb{R}^n$. The element A is *well-rounded* if the set of shortest nonzero vectors of the lattice $A\mathbb{Z}^n$ generate $\mathbb{R}^n$ as a real vector space. This property is invariant under the left action of SO_n and hence there is no ambiguity in saying that an element in S_n is well-rounded. The subset $\mathcal{X}$ of S_n consisting of well-rounded elements is homeomorphic to an $(n(n-1)/2)$–dimensional CW-complex and the right action of $\mathrm{SL}_n \mathbb{Z}$ on S_n induces a cocompact action on $\mathcal{X}$. Observe that if $n = 2$ then $\mathcal{X}$ is the dual to the Farey tesselation of $S_2 = \mathbb{H}^2$ and hence homeomorphic to the Bass–Serre tree of $\mathrm{SL}_2 \mathbb{Z}$. For larger n, the set $\mathcal{X}$ does not have such a simple description, but Lannes and Soulé proved that $\mathcal{X}$ is a deformation retract of S_n and hence contractible (see Soulé [8] for the case of $n = 3$, and Ash [3] for all n, treated in a more general setting). This is why the subset $\mathcal{X}$ is known as the *well-rounded retract* of S_n. Our goal is to show that $\mathcal{X}$ is a minimal $\mathrm{SL}_n \mathbb{Z}$–invariant spine of S_n.

Definition 1.1 Let Γ be a group acting discretely on a contractible space S. We say that a closed subset X of S is a *minimal* Γ*–invariant spine* if it is Γ–invariant, contractible and does not properly contain any closed set with these properties.

We prove:

Theorem 1.2 *The well-rounded retract $\mathcal{X}$ is a minimal $\mathrm{SL}_n \mathbb{Z}$–invariant spine of the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbb{R}$.*

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It has long been known that the well-rounded retract does not contain any smaller dimensional $\mathrm{SL}_n\,\mathbb{Z}$–invariant spines. This follows namely from the fact due to Borel–Serre [5] that the group $\mathrm{SL}_n\,\mathbb{Z}$ has virtual cohomological dimension

$$\mathrm{vcdim}(\mathrm{SL}_n\,\mathbb{Z}) = \frac{n(n-1)}{2} = \dim \mathcal{X}.$$

In order to appreciate the difference between this statement and the claim of Theorem 1.2 it should be observed that the well-rounded retract contains interesting $\mathrm{SL}_n\,\mathbb{Z}$–invariant subsets of dimension $n(n-1)/2$. For instance, recall that an element $A \in \mathrm{SL}_n\,\mathbb{R}$ is well-rounded if the set of shortest nonzero vectors of the lattice $A\mathbb{Z}^n$ generate $\mathbb{R}^n$ as a vector space; equivalently, they generate, as a group, a finite index lattice of $A\mathbb{Z}^n$. We will say that $A \in \mathrm{SL}_n\,\mathbb{R}$ is *extremely well-rounded* if the shortest nonzero vectors of $A\mathbb{Z}^n$ generate the whole lattice $A\mathbb{Z}^n$. The subset $\mathcal{X}'$ of S_n consisting of extremely well-rounded elements is $\mathrm{SL}_n\,\mathbb{Z}$–invariant and has dimension $n(n-1)/2$. While $\mathcal{X}' = \mathcal{X}$ for $n = 2, 3$ and 4, the set $\mathcal{X}'$ is a proper subset of the well-rounded retract for $n \geq 5$. In [7] we proved that $\mathcal{X}'$ is not contractible for $n \geq 5$. This result follows now directly from Theorem 1.2:

Corollary 1.3 [7] *The subset $\mathcal{X}' \subset S_n$ of extremely well-rounded elements is not contractible.* □

In order to prove Theorem 1.2 it suffices to show that whenever $\mathcal{Y}$ is a closed proper $\mathrm{SL}_n\,\mathbb{Z}$–invariant subset of $\mathcal{X}$, there is a torsion-free, finite index subgroup $\Gamma \subset \mathrm{SL}_n\,\mathbb{Z}$ such that the inclusion $\mathcal{Y}/\Gamma \hookrightarrow \mathcal{X}/\Gamma$ is not a homotopy equivalence. We proceed as follows: First we show that there is $A \in \mathcal{X} \setminus \mathcal{Y}$ with the property that there is a torsion-free, finite index subgroup Γ of $\mathrm{SL}_n\,\mathbb{Z}$ and a nontrivial homology class $[\alpha] \in H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ represented by a cycle α which intersects the well-rounded retract exactly at A. Here $\bar{M}_\Gamma$ is the Borel–Serre compactification of the locally symmetric space $M_\Gamma = S_n/\Gamma$ and the homology is with coefficients in the ring $\mathbb{Z}/2\mathbb{Z}$. The class $[\alpha]$ is dual to some class $[\beta] \in H_{n(n-1)/2}(M_\Gamma)$. The fact that the cycle α does not intersect $\mathcal{Y}$ implies that $[\beta]$ is not in the image of $H_*(\mathcal{Y}/\Gamma)$ in $H_*(\mathcal{X}/\Gamma)$. This shows that the inclusion $\mathcal{Y}/\Gamma$ in $\mathcal{X}/\Gamma$ is not a homotopy equivalence.

In [7], we used this strategy to prove Corollary 1.3. In that particular case we faced much simpler technical problems since it was possible to explicitly find a rational maximal flat intersecting $\mathcal{X}$ exactly once, at a point outside of $\mathcal{X}'$. Even in the case $n = 2$, it is easy to see that for a generic point $A \in \mathcal{X}$, every maximal flat through A intersects $\mathcal{X}$ many times. To bypass this problem we give an elementary, though somewhat involved, construction of the cycle α.

The paper is organized as follows: In Section 2 we review some facts about the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbb{R}$ and its quotients. In Section 3 we discuss some properties of the well-rounded retract, proving that a generic well-rounded element in S_n has exactly $2n$ shortest vectors. In Section 4 we show that certain homology classes are nontrivial; all the results in this section are surely well known. In Section 5 we derive Theorem 1.2 from a result, Proposition 5.1, proved in Section 6. Proposition 5.1, the key point of this paper, yields nontrivial cycles in $C_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ which intersect the well-rounded retract at a single point.

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Notation We denote by $\{e_1, \dots, e_n\}$ and $|\cdot|$ the standard basis and Euclidean norm of $\mathbb{R}^n$. Sometimes we will write elements in $\mathbb{R}^n$ as columns and sometimes as rows; we hope that this does not cause any confusion. If U is a linear subspace of $\mathbb{R}^n$, denote by $U^\perp$ its orthogonal complement with respect to the standard Euclidean product. We will use the same symbol to denote both an equivalence class and a representative of the equivalence class. For example, we use the same notation for an element in $\mathrm{SL}_n \mathbb{R}$ and for the corresponding element in the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbb{R}$, or in even smaller quotients such as $S_n / \mathrm{SL}_n \mathbb{Z}$. We will however consistently denote the homology class corresponding to a cycle α by $[\alpha]$. All the homology groups considered below have coefficients in the field $\mathbb{Z}/2\mathbb{Z}$ of two elements, although everything remains true with respect to any other commutative ring with unit.

2 The symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbb{R}$

Up to scaling, the manifold $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbb{R}$ admits a unique symmetric metric invariant under the right action of $\mathrm{SL}_n \mathbb{R}$; we shall always assume S_n to be endowed with such a metric. The restriction of the right action of $\mathrm{SL}_n \mathbb{R}$ on S_n to $\mathrm{SL}_n \mathbb{Z}$ is discrete. Moreover, any torsion-free subgroup Γ of $\mathrm{SL}_n \mathbb{Z}$ acts freely and hence the quotient $M_\Gamma = S_n / \Gamma$ is a smooth locally symmetric manifold. It is well known that

$\mathrm{SL}_n\,\mathbb{Z}$ contains torsion-free finite index subgroups. If $\Gamma \subset \mathrm{SL}_n\,\mathbb{Z}$ is any such subgroup, then the manifold M_Γ is not compact, but is homeomorphic to the interior of a compact manifold $\bar{M}_\Gamma$, the so-called Borel–Serre compactification of M_Γ [5].

For every $v \in \mathbb{R}^n$, the *length function*

$$l_v\colon S_n \to \mathbb{R}, \quad l_v(A) = |Av|$$

is well-defined, analytic and convex. In particular we have

$$l_v(A'') \leq \max\{l_v(A), l_v(A')\} \tag{2–1}$$

for all $A, A' \in S_n$ and every A'' in the unique geodesic segment $[A, A']$ joining A and A' in S_n. It should be observed that for every $B \in \mathrm{SL}_n\,\mathbb{R}$ we have $l_v(AB) = l_{Bv}(A)$. Since $\mathrm{SL}_n\,\mathbb{Z}$ acts on the set $\mathbb{Z}^n \setminus \{0\}$, this implies that the function

$$\mathrm{syst}_1\colon S_n \to (0,\infty), \quad \mathrm{syst}_1(A) = \min_{v\in\mathbb{Z}^n, v\neq 0} l_v(A) \tag{2–2}$$

is $\mathrm{SL}_n\,\mathbb{Z}$–invariant. The quantity $\mathrm{syst}_1(A)$ is said to be the *systole*, or *first minimum*, of $A \in S_n$. The elements of the set

$$\mathcal{S}_1(A) = \{v \in \mathbb{Z}^n \mid l_v(A) = \mathrm{syst}_1(A)\} \tag{2–3}$$

are said to be the *systoles* or *shortest vectors* of A.

Ash proved in [2] that the systole function is a topological Morse function (see also Bavard [4] and Akrout [1]). Moreover, the induced function on $S_n/\mathrm{SL}_n\,\mathbb{Z}$ is proper by the following theorem:

Mahler's compactness theorem *A closed subset $K \subset S_n/\mathrm{SL}_n\,\mathbb{Z}$ is compact if and only if there is $\epsilon > 0$ with $\mathrm{syst}_1(A) \geq \epsilon$ for all $A \in K$.*

We deduce from (2–1) and Mahler's compactness theorem the following important observation:

Lemma 2.1 *Let Γ be a torsion-free subgroup of $\mathrm{SL}_n\,\mathbb{Z}$, N a manifold, and $f, g\colon N \to S_n$ two continuous maps such that for all $\epsilon > 0$ there is a compact set $K_\epsilon \subset N$ with the following property:*

(∗) *For all $x \notin K_\epsilon$ there is $v \in \mathbb{Z}^n \setminus \{0\}$ with $l_v(f(x)), l_v(g(x)) < \epsilon$.*

Then the compositions of f and g with the projection $\pi\colon S_n \to M_\Gamma$ are properly homotopic.

Proof Let $H: N \times [0,1] \to S_n$ be the geodesic homotopy from f to g, ie $t \to H_t(x)$ traverses with constant velocity the geodesic segment $[f(x), g(x)]$. We claim that $h = \pi \circ H$ is proper. Let C be a compact subset of $M_\Gamma = S_n/\Gamma$. By Mahler's compactness theorem there is some ϵ positive with $\mathrm{syst}_1(A) \geq \epsilon$ for all $A \in C$. For such an ϵ, let $K_\epsilon \subset N$ be the compact subset provided by $(*)$. Then for $x \notin K_\epsilon$ there is some $v_x \in \mathbb{Z}$, $v_x \neq 0$, with $l_{v_x}(f(x)), l_{v_x}(g(x)) < \epsilon$. By (2–1) we have then $l_{v_x}(H_t(x)) < \epsilon$ for all $t \in [0,1]$. This implies that $h^{-1}(C) \subset K_\epsilon \times [0,1]$, proving that it is proper. □

We will use Lemma 2.1 several times in the following situation.

Corollary 2.2 *Assume that Γ is a finite index subgroup of $\mathrm{SL}_n \mathbb{Z}$, and that $N \subset \mathrm{SL}_n \mathbb{R}$ projects properly to $M_\Gamma = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbb{R}/\Gamma$. Then for every $B \in \mathrm{SL}_n \mathbb{R}$ the projections of N and of $BN = \{Bx, x \in N\}$ to M_Γ are properly homotopic.* □

3 The well-rounded retract

In this section we discuss briefly some of the properties of the well-rounded retract. Recall the definition of the systole (2–2) and of the set of systoles (2–3) of a point $A \in S_n$. Let also

$$\Lambda_1(A) = \mathrm{Span}_{\mathbb{R}}(\mathcal{S}_1(A)) \tag{3–1}$$

be the linear subspace of $\mathbb{R}^n$ generated by the set of systoles of A.

Definition 3.1 An element $A \in S_n$ is *well-rounded* if $\Lambda_1(A) = \mathbb{R}^n$. The subset $\mathcal{X}$ of S_n consisting of all well-rounded elements is called the *well-rounded retract*.

As mentioned in the introduction, Soulé [8] and Ash [3] proved that $\mathcal{X}$ is an $\mathrm{SL}_n \mathbb{Z}$–invariant deformation retract. The idea behind this result is simple and beautiful, and so we explain it briefly here:

Theorem 3.2 (Soulé, Ash) *The well-rounded retract $\mathcal{X}$ is a deformation retract of S_n.*

For $k = 1, \dots, n$ let $\mathcal{X}_k$ be the set of those $A \in S_n$ for which we have $\dim \Lambda_1(A) \geq k$. We have the following chain of nested $\mathrm{SL}_n \mathbb{Z}$–invariant subspaces:

$$\mathcal{X} = \mathcal{X}_n \subset \mathcal{X}_{n-1} \subset \cdots \subset \mathcal{X}_1 = S_n$$

In order to prove Theorem 3.2 it suffices to show that for $k = 1, \dots, n-1$ the space $\mathcal{X}_{k+1}$ is an $\mathrm{SL}_n \mathbb{Z}$–equivariant spine of $\mathcal{X}_k$; we construct a retraction. Given $A \in \mathcal{X}_k$ and $\lambda \in \mathbb{R}$, consider the one-parameter family of linear maps:

$$T_A^\lambda \in \mathrm{SL}_n \mathbb{R}, \quad T_A^\lambda(v) = \begin{cases} e^{(n-k)\lambda} v & \text{for } v \in A\Lambda_1(A) \\ e^{-k\lambda} v & \text{for } v \in (A\Lambda_1(A))^\perp \end{cases}$$

In other words, for positive λ the map T_A^λ expands the subspace generated by the image of the shortest vectors of A, while contracting the orthogonal complement. Observe that for $U \in \mathrm{SO}_n$ we have $T_{UA}^\lambda UA = UT_A^\lambda A$; hence the point $T_A^\lambda A \in S_n$ depends only on A and not on the choice of representative.

Now $T_A^0 A = A$, and if $A \in \mathcal{X}_k \setminus \mathcal{X}_{k+1}$, there is some λ positive with $T_A^\lambda A \in \mathcal{X}_{k+1}$. For $A \in \mathcal{X}_k$, let $\tau(A) \geq 0$ be maximal such that

$$T_A^\lambda A \in \mathcal{X}_k \setminus \mathcal{X}_{k+1} \quad \text{for all } \lambda \in [0, \tau(A)).$$

By definition $\tau(A) = 0$ for $A \in \mathcal{X}_{k+1}$. The function $A \mapsto \tau(A)$ is continuous on $\mathcal{X}_k$, which implies that

$$[0,1] \times \mathcal{X}_k \to \mathcal{X}_k, \quad (t, A) \mapsto T_A^{t\tau(A)} A$$

is continuous as well. By definition, this homotopy is $\mathrm{SL}_n \mathbb{Z}$–equivariant, starts with the identity, and ends with a projection of $\mathcal{X}_k$ to $\mathcal{X}_{k+1}$. This proves that $\mathcal{X}_{k+1}$ is an $\mathrm{SL}_n \mathbb{Z}$–equivariant spine of $\mathcal{X}_k$ for $k = 1, \dots, n-1$, concluding the sketch of the proof of Theorem 3.2. □

It is not difficult to prove that $\mathcal{X}_k$ is a codimension $k-1$ semi-algebraic set, ie, that it is given by a locally finite collection of inequalities and (quadratic) algebraic equations. Hence $\mathcal{X}$ is homeomorphic to a CW-complex of dimension $\dim(\mathcal{X}) = \dim S_n - (n-1) = n(n-1)/2$. It is also easy to see that $\mathcal{X}/\Gamma$ is compact. We prove now that a generic point in $\mathcal{X}$ has exactly $2n$ shortest vectors:

Proposition 3.3 *The set of those $A \in \mathcal{X}$ for which there are $v_1, \dots, v_n \in \mathbb{Z}^n$ linearly independent with $\mathcal{S}_1(A) = \{\pm v_1, \dots, \pm v_n\}$ is dense in $\mathcal{X}$.*

In order to prove Proposition 3.3 we will use the following not very surprising but also not completely obvious geometric lemma.

Lemma 3.4 *Assume that $\mathcal{S}$ is a finite subset of the sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ with the property that $\mathbb{R}^n = \mathrm{Span}_{\mathbb{R}} \mathcal{S}$ and assume that if $v \in \mathcal{S}$ then $-v \in \mathcal{S}$ as well. Then there is basis $\mathcal{B}$ of $\mathbb{R}^n$ contained in $\mathcal{S}$ and a linear map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ close to the identity such that for $v \in \mathcal{S}$ we have $|Fv| = |v|$ if $\pm v \in \mathcal{B}$ and $|Fv| > |v|$ otherwise.*

Assuming Lemma 3.4, we prove Proposition 3.3. Given $A \in \mathcal{X}$ choose a representative in $\mathrm{SL}_n \mathbb{R}$, again denoted by A. By definition, the image $A\mathcal{S}_1(A)$ of the set of systoles of A generates $\mathbb{R}^n$ and is contained in the round sphere $\mathbb{S}^{n-1}_{\mathrm{syst}_1(A)}$ of radius $\mathrm{syst}_1(A)$. Let $\mathcal{B} \subset A\mathcal{S}_1(A)$ and $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be the basis and the linear map provided by Lemma 3.4. We set $A^{-1}\mathcal{B} = \{v_1, \dots, v_n\}$ and $A' = (1/\sqrt[n]{\det(F)})FA$. Since we may assume that F is very close to the identity, we have that A' is very close to A, and hence $\mathcal{S}_1(A') \subset \mathcal{S}_1(A)$. It follows now from Lemma 3.4 that $\mathcal{S}_1(A') = \{\pm v_1, \dots, \pm v_n\}$. This concludes the proof of Proposition 3.3. □

We prove now Lemma 3.4:

Proof of Lemma 3.4 We use induction on the number of elements in $\mathcal{S}$. There is nothing to show if $\mathcal{S}$ has $2n$ elements, so assume that we have proved the lemma for all sets with at most $2k \geq 2n$ elements, and that $\mathcal{S}$ has $2(k+1)$ elements. Observe that there is a codimension one linear subspace $U \subset \mathbb{R}^n$ generated by $U \cap \mathcal{S}$ such that there are at least four elements in $\mathcal{S}$ which don't belong to U (recalling that if $v \in \mathcal{S}$, then $-v \in \mathcal{S}$ as well). We first describe a map $F_1\colon \mathbb{R}^n \to \mathbb{R}^n$ which will allow us to apply our inductive hypothesis.

We choose $v \in \mathcal{S}$, $v \notin U$ with minimal angle $\angle(U, v) = \theta \in (0, \pi/2)$. Let V be the codimension one linear subspace containing v and the intersection $(\mathbb{R}v)^\perp \cap U$ of the orthogonal complement of $\mathbb{R}v$ and U. The planes U and V have angle θ and divide $\mathbb{R}^n$ into two open sectors, C_1 and C_2 with angle θ, and two also open sectors, C_3 and C_4 with angle $\pi - \theta$. By the minimality of θ, any vector in S which is not in $U \cup \{\pm v\}$ has angle at least θ with U and so is not contained in V. Moreover, for the same reason, we have $\mathcal{S} \cap (C_1 \cup C_2) = \varnothing$, but $\mathcal{S} \cap (C_3 \cup C_4) \neq \varnothing$.

For $\eta > \theta$ with $\eta - \theta$ small we can consider the linear map $F_1\colon \mathbb{R}^n \to \mathbb{R}^n$ which is the identity on U, an isometry when restricted to V, and which opens C_1 and C_2 to angle η. The map F_1 preserves the length of vectors in $U \cup V$, reduces the length of vectors in $C_1 \cup C_2$ and increases the length of vectors in $C_3 \cup C_4$. In particular, F_1 maps $(\mathcal{S} \cap U) \cup \{\pm v\}$ to the subset $(\mathcal{S} \cap U) \cup \{\pm F_1(v)\}$ of $\mathbb{S}^n$ which still generates $\mathbb{R}^n$, and increases the length of the (at least two) remaining vectors in $\mathcal{S}$.

The induction hypothesis now applies to the set $(\mathcal{S} \cap U) \cup \{\pm F_1(v)\}$ of cardinality at most $2k$: there is a basis $\mathcal{B}_1$ of $\mathbb{R}^n$ contained in $(\mathcal{S} \cap U) \cup \{\pm F_1(v)\}$, and a map $F_2\colon \mathbb{R}^n \to \mathbb{R}^n$ which preserves the lengths of the elements of $\mathcal{B}_1$ (and their negatives) and increases the lengths of all other vectors in $(\mathcal{S} \cap U) \cup \{\pm F_1(v)\}$. We require that F_2 be close enough to the identity that the vectors in $F_1(\mathcal{S})$ of length greater than one remain so after applying F_2. Now the basis $\mathcal{B} = F_1^{-1}(\mathcal{B}_1)$ and the map $F = F_2 \circ F_1\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfy the requirements of the lemma for the set $\mathcal{S}$. □

4 A bit of homology

In this section we give elementary proofs of some homological results which are probably well known to experts and nonexperts alike.

As mentioned above, $\mathrm{SL}_n \mathbb{Z}$ contains torsion-free subgroups of finite index, and any such subgroup acts freely and discretely on S_n; as always, we denote the quotient manifold by $M_\Gamma = S_n/\Gamma$ and its Borel–Serre compactification by $\bar{M}_\Gamma$. If $U \subset \bar{M}_\Gamma$ is a regular neighborhood of $\partial \bar{M}_\Gamma$, we have $H_*(\bar{M}_\Gamma, U) \simeq H_*(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$. In particular, we can consider every properly immersed submanifold of M_Γ as a cycle in $C_*(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$. Recall that we always consider homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Before stating the main result of this section, we recall that by Lefschetz duality there is a nondegenerate pairing

$$\iota\colon H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma) \times H_{n(n-1)/2}(M_\Gamma) \to \mathbb{Z}/2\mathbb{Z}$$

which can be computed as follows. Given homology classes $[\alpha] \in H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ and $[\beta] \in H_{n(n-1)/2}(M_\Gamma)$, represent them by cycles α and β in general position. Then $\iota([\alpha],[\beta])$ is just the parity of the cardinality of the set $\alpha \cap \beta$. Observe that in order to prove that a cycle $\beta \in C_{n(n-1)/2}(M_\Gamma)$ represents a nontrivial homology class, it suffices to find a cycle $\alpha \in C_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ which intersects β transversally at a single point; if this is the case we will say that the two classes $[\alpha]$ and $[\beta]$ are dual to each other. This is the argument used in [7] to prove:

Proposition 4.1 *Let Γ be a finite index torsion-free subgroup of $\mathrm{SL}_n \mathbb{Z}$, Δ the connected component of the identity in the diagonal subgroup of $\mathrm{SL}_n \mathbb{R}$ and Nil the subgroup of $\mathrm{SL}_n \mathbb{R}$ consisting of upper triangular matrices with units in the diagonal. Then the projection of Δ and Nil to M_Γ represent dual, and hence nontrivial, homology classes in $H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ and $H_{n(n-1)/2}(M_\Gamma)$, respectively.* □

Proposition 4.1 is surely well known, as is the following slightly more general version.

Corollary 4.2 *Given $B \in \mathrm{GL}_n \mathbb{Q}$ assume that $\Gamma \subset \mathrm{SL}_n \mathbb{Z}$ is a finite index torsion-free subgroup with $B^{-1}\Gamma B \subset \mathrm{SL}_n \mathbb{Z}$, and that Δ and Nil are as in Proposition 4.1. Then the projections of $B\Delta B^{-1}$ and $B\,\mathrm{Nil}\,B^{-1}$ to M_Γ represent dual, and hence nontrivial, homology classes in $H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ and $H_{n(n-1)/2}(M_\Gamma)$, respectively.*

Proof The map $\phi\colon S_n \to S_n$ given by $\phi(X) = XB^{-1}$ induces a diffeomorphism $\Phi\colon M_{B^{-1}\Gamma B} \to M_\Gamma$. By Proposition 4.1 the projections of Δ and Nil represent dual homology classes in $M_{B^{-1}\Gamma B}$. Pushing forward with Φ, we obtain dual cycles ΔB^{-1} and $\mathrm{Nil}\,B^{-1}$. By Corollary 2.2, these cycles are properly homotopic, and hence homologous, to the cycles $B\Delta B^{-1}$ and $B\,\mathrm{Nil}\,B^{-1}$. The claim follows. □

5 Proof of Theorem 1.2

In the next section we will show:

Proposition 5.1 *Assume that $A \in \mathcal{X}$ is such that there are $v_1, \dots, v_n \in \mathbb{Z}^n$ linearly independent with $\mathcal{S}_1(A) = \{\pm v_1, \dots, \pm v_n\}$. Let $B \in \mathrm{GL}_n \mathbb{Q}$ be the matrix with columns $v_1, \dots, v_n$, and let Γ be a finite index torsion-free subgroup of $\mathrm{SL}_n \mathbb{Z} \cap B \, \mathrm{SL}_n \mathbb{Z} B^{-1}$. Then the nontrivial homology class $[B\Delta B^{-1}]$ is represented by a cycle $\alpha \in C_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ whose support intersects the well-rounded retract $\mathcal{X}$ only in A.*

Assuming Proposition 5.1, we prove the main theorem:

Theorem 1.2 *The well-rounded retract $\mathcal{X}$ is a minimal $\mathrm{SL}_n \mathbb{Z}$–invariant spine of the symmetric space $S_n = \mathrm{SO}_n \backslash \mathrm{SL}_n \mathbb{R}$.*

Proof Assume that $\mathcal{Y} \subset \mathcal{X}$ is a proper, closed, $\mathrm{SL}_n \mathbb{Z}$–invariant subset of $\mathcal{X}$. As mentioned in the introduction, in order to show that $\mathcal{Y}$ is not contractible, it suffices to prove that for some $\Gamma \subset \mathrm{SL}_n \mathbb{Z}$ the induced map $\mathcal{Y}/\Gamma \to \mathcal{X}/\Gamma$ is not a homotopy equivalence.

By Proposition 3.3 there is $A \in \mathcal{X} \setminus \mathcal{Y}$ and a linearly independent subset $\{v_1, \dots, v_n\} \subset \mathbb{Z}$ with $\mathcal{S}_1(A) = \{\pm v_1, \dots, \pm v_n\}$. Let $B \in \mathrm{GL}_n \mathbb{Q}$ be the matrix with columns $v_1, \dots, v_n$. The subgroups $\mathrm{SL}_n \mathbb{Z}$ and $B \, \mathrm{SL}_n \mathbb{Z} B^{-1}$ are commensurable and hence there is a torsion-free finite index subgroup $\Gamma \subset \mathrm{SL}_n \mathbb{Z} \cap B \, \mathrm{SL}_n \mathbb{Z} B^{-1}$. By Proposition 5.1, the homology class $[B\Delta B^{-1}] \in H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ is represented by a cycle α with $\alpha \cap \mathcal{X} = \{A\}$. On the other hand, the class $[B\Delta B^{-1}]$ is dual to some class $[\beta] \in H_{n(n-1)/2}(M_\Gamma)$ by Corollary 4.2. Since α represents $[B\Delta B^{-1}]$ and intersects $\mathcal{X}$ only at A, we deduce that every cycle contained in $\mathcal{X}/\Gamma$ and representing $[\beta]$ has to contain A in its support. In particular, the map

$$H_{n(n-1)/2}(\mathcal{Y}/\Gamma) \to H_{n(n-1)/2}(\mathcal{X}/\Gamma)$$

is not surjective. This implies that the map $\mathcal{Y}/\Gamma \to \mathcal{X}/\Gamma$ is not a homotopy equivalence. □

6 Flags of systoles

In this section we prove Proposition 5.1. The first step is to construct a certain continuous map

$$\Phi\colon S_n \times [0, \infty) \to S_n \tag{6–1}$$

which essentially pushes points in $S_n \setminus \mathcal{X}$ away from $\mathcal{X}$.

To begin with, recall the definition of the systole $\mathrm{syst}_1(A)$ of $A \in S_n$. We can extend this definition as follows: for $i = 1, \dots, n$, the *i–th systole* of A is given by

$$\mathrm{syst}_i(A) = \inf\{r \mid \dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}\{v \in \mathbb{Z} \text{ with } |Av| < r\}) \geq i\}. \tag{6–2}$$

In other words, $\mathrm{syst}_i(A)$ is the infimum of those r for which the set of vectors v in $\mathbb{Z}^n$ whose image Av has length less than r generates a subspace of $\mathbb{R}^n$ with dimension at least i. Equivalently,

$$\mathrm{syst}_i(A) = \sup\{r \mid \dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}\{v \in \mathbb{Z} \text{ with } |Av| < r\}) < i\}. \tag{6–3}$$

The i–th systole coincides with Minkowski's i–th successive minimum of the lattice $A\mathbb{Z}^n$ with respect to the ball B_1 of radius 1 in $\mathbb{R}^n$. See Martinet [6] for more about successive minima.

For $i = 1, \dots, n$, the i–th systole function

$$\mathrm{syst}_i \colon S_n \to (0, \infty)$$

is well-defined and $\mathrm{SL}_n \mathbb{Z}$–equivariant. We claim that it is continuous. In fact, if (A_k) is a sequence in S_n converging to some $A \in S_n$ then for all r the finite sets $\{v \in \mathbb{Z}^n, |A_k v| < r\}$ converge in the Gromov-Hausdorff topology to the (again finite) set $\{v \in \mathbb{Z}^n, |Av| < r\}$. Since $\mathbb{Z}^n$ is discrete, we have that for all sufficiently large k

$$\{v \in \mathbb{Z}^n, |A_k v| < r\} = \{v \in \mathbb{Z}^n, |Av| < r\}.$$

Together with (6–2), this implies that syst_i is lower semi-continuous. Likewise (6–3) and the same argument yield upper semi-continuity.

Lemma 6.1 *The function* $\mathrm{syst}_i \colon S_n \to (0, \infty)$ *is continuous and* $\mathrm{SL}_n \mathbb{Z}$*–equivariant for* $i = 1, \dots, n$. □

Recall now the definition of $\Lambda_1(A)$ given in (3–1). We extend this definition, setting for $i = 1, \dots, n$

$$\Lambda_i(A) = \mathrm{Span}_{\mathbb{R}}(\{v \in \mathbb{Z}^n, |Av| \leq \mathrm{syst}_i(A)\}).$$

In order to avoid treating special cases we set $\Lambda_0(A) = 0$ for all $A \in S_n$. By definition

$$0 \subsetneq \Lambda_1(A) \subset \cdots \subset \Lambda_n = \mathbb{R}^n \tag{6–4}$$

and $\dim_{\mathbb{R}}(\Lambda_i(A)) \geq i$. Observe that for $i < n$ this last inequality is strict if A is well-rounded. In particular, we cannot expect that the subspaces $\Lambda_i(A)$ depend continuously on A. However we have the following weak continuity, which can be proved with essentially the same argument as Lemma 6.1:

Lemma 6.2 *Assume that (A_k) is a sequence in S_n converging to some $A \in S_n$. Then there is k_0 such that for all $k \geq k_0$ and $i \in \{1, \dots, n\}$ there is a unique $\kappa(k,i) \in \{1, \dots, n\}$ with*

- $\Lambda_{\kappa(k,i)}(A_k) = \Lambda_i(A)$, *and*
- *if $\kappa(k,i) \neq n$ then $\Lambda_{\kappa(k,i)+1}(A_k) \neq \Lambda_i(A)$.*

If moreover i' is minimal with $\mathrm{syst}_{i'}(A) = \mathrm{syst}_i(A)$ then

$$\lim_{k \to \infty} \mathrm{syst}_{j_k}(A_k) = \mathrm{syst}_i(A)$$

for all choices of j_k with $\kappa(k, i'-1) < j_k \leq \kappa(k,i)$. □

We use the flag (6–4) to construct the continuous map (6–1). To begin with we consider for $i = 1, \dots, n$ the subspace

$$\Theta_i(A) = (A\Lambda_{i-1}(A))^{\perp} \cap (A\Lambda_i(A)).$$

In more plain language, $\Theta_i(A)$ is the orthogonal complement of the image of $\Lambda_{i-1}(A)$ under A within the image of $\Lambda_i(A)$. We have thus the orthogonal decomposition

$$\mathbb{R}^n = \Theta_1(A) \oplus \cdots \oplus \Theta_n(A)$$

together with the associated orthogonal projections

$$\pi_{\Theta_i(A)} \colon \mathbb{R}^n \to \Theta_i(A).$$

We define now for $x \in \mathbb{R}^n$

$$\Phi_t(A)x = \frac{1}{\sqrt[n]{\prod_{i=1}^n \mathrm{syst}_i(A)^{t \dim_{\mathbb{R}} \Theta_i(A)}}} \sum_{i=1}^n \mathrm{syst}_i(A)^t \pi_{\Theta_i(A)}(Ax). \tag{6–5}$$

The multiplicative factor in (6–5) ensures that $\Phi_t(A) \in \mathrm{SL}_n \mathbb{R}$ for all $A \in \mathrm{SL}_n \mathbb{R}$. Moreover, for all $U \in \mathrm{SO}_n$ we have $\Phi_t(UA) = U\Phi_t(A)$. In particular, we have a well-defined map

$$\Phi_t \colon S_n \times [1, \infty) \to S_n \tag{6–6}$$

It is easy to check that the map (6–6) is $\mathrm{SL}_n \mathbb{Z}$–equivariant, and its continuity follows from Lemma 6.2. Moreover, since $\mathrm{syst}_1(A) \le \mathrm{syst}_i(A)$ for all i, we have for all $x \in \mathbb{R}^n$

$$|\Phi_t(A)x| \ge \left(\frac{\mathrm{syst}_1(A)}{\sqrt[n]{\prod_{i=1}^n \mathrm{syst}_i(A)^{\dim_{\mathbb{R}} \Theta_i(A)}}} \right)^t |Ax| \tag{6–7}$$

with equality if and only if $x \in \Lambda_1(A)$. In particular we see that $\Lambda_1(\Phi_t(A)) = \Lambda_1(A)$ for all $t \ge 0$. Moreover, if $\Lambda_1(A) \ne \mathbb{R}^n$ then the exponentiated quantity in (6–7) is less than 1 and hence

$$\lim_{t\to\infty} \mathrm{syst}_1(\Phi_t(A)) = 0$$

On the other hand, if $\Lambda_1(A) = \mathbb{R}^n$ then $\Phi_t(A) = A$ for all t.

Summing up, we have:

Proposition 6.3 *There is a continuous map $\Phi\colon S_n \times [0,\infty) \to S_n$, $\Phi(A,t) = \Phi_t(A)$, with the following properties:*

- $\Phi_0(\cdot) = \mathrm{Id}$,
- $\Phi_t(A) \in \mathcal{X}$ *if and only if* $A \in \mathcal{X}$, *and*
- *if* $A \notin \mathcal{X}$ *then* $\lim_{t\to\infty} |\Phi_t(A)v| = 0$ *for all* $v \in \Lambda_1(A)$. □

We are now ready to prove Proposition 5.1:

Proposition 5.1 *Assume that $A \in \mathcal{X}$ is such that there are $v_1, \dots, v_n \in \mathbb{Z}^n$ linearly independent with $\mathcal{S}_1(A) = \{\pm v_1, \dots, \pm v_n\}$, let $B \in \mathrm{GL}_n \mathbb{Q}$ be the matrix with columns $v_1, \dots, v_n$ and Γ a finite index torsion-free subgroup in $\mathrm{SL}_n \mathbb{Z} \cap B\,\mathrm{SL}_n \mathbb{Z}\, B^{-1}$. Then the nontrivial homology class $[B\Delta B^{-1}]$ is represented by a cycle $\alpha \in C_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ whose support intersects the well-rounded retract $\mathcal{X}$ only at A.*

Recall that Δ is the connected component of the identity in the diagonal subgroup of $\mathrm{SL}_n \mathbb{R}$.

Proof In order to construct the cycle α we start with the map

$$g_1\colon \Delta \to M_\Gamma, \quad g_1(X) = BXB^{-1}$$

The cycle $g_1(\Delta)$ represents a nontrivial homology class in $H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ by Corollary 4.2. The point A may not belong to the image of $g_1(\Delta)$, but this can be easily corrected by considering the map

$$g_2\colon \Delta \to M_\Gamma, \quad g_2(X) = ABXB^{-1}$$

Corollary 2.2 implies that $g_1(\Delta)$ and $g_2(\Delta)$ are properly homotopic and hence homologous.

Now we have $g_2(\mathrm{Id}) = A$, but it is not clear at all how many other times $g_2(\Delta)$ may intersect $\mathcal{X}$. We correct this problem by constructing a third map g_3 properly homotopic to g_2. Before going further we identify Δ with $\mathbb{R}^{n-1}$ via the following map

$$(a_1,\dots,a_{n-1}) \mapsto \begin{pmatrix} e^{a_1} & 0 & \dots & 0 & 0 \\ 0 & e^{a_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{a_{n-1}} & 0 \\ 0 & 0 & \dots & 0 & e^{-a_1-a_2-\cdots-a_{n-1}} \end{pmatrix}$$

A simple computation shows:

Lemma 6.4 *There is some $\epsilon > 0$ such that for all $x \in B_\epsilon \subset \mathbb{R}^{n-1} = \Delta$, $g_2(x) \in \mathcal{X}$ if and only if $x = 0$. If moreover $x \in B_\epsilon$, $x \neq 0$ and $v \in \mathcal{S}_1(g_2(x))$ then we have*

$$\lim_{t\to\infty} l_v(g_2(tx)) = 0. \tag{6–8}$$

Here B_ϵ is the ball of radius ϵ centered at 0 in $\mathbb{R}^{n-1} \simeq \Delta$. □

We can now define the map $g_3\colon \mathbb{R}^{n-1} \to M_\Gamma$. With ϵ as in Lemma 6.4 and Φ the map provided by Proposition 6.3, we set

$$g_3(x) = \begin{cases} g_2(x) & |x| \le \epsilon \\ \Phi_{|x|-\epsilon}(g_2(\epsilon\frac{x}{|x|})) & |x| \ge \epsilon. \end{cases}$$

In other words we extend radially, using the map Φ and the restriction of g_2 to B_ϵ. Since $g_2(x) \notin \mathcal{X}$ for x with $|x| = \epsilon$, we deduce from Proposition 6.3 that $g_3(x) \notin \mathcal{X}$ for all x with $|x| \ge \epsilon$. On the other hand, for $|x| \le \epsilon$ we have $g_3(x) = g_2(x)$. Hence

$$g_3(\mathbb{R}^{n-1}) \cap \mathcal{X} = \{A\}.$$

If $v \in \mathbb{Z}^n$ is a systole for $g_2(x)$ with $|x| = \epsilon$, then we have by (6–8)

$$\lim_{t\to\infty} l_v(g_2(tx)) = 0$$

and by Proposition 6.3

$$\lim_{t\to\infty} l_v(g_3(tx)) = \lim_{t\to\infty} l_v(\Phi_{t-1}(g_2(x)) = 0.$$

Lemma 2.1 implies now that the maps g_2 and g_3 are properly homotopic to each other. Hence the cycle $\alpha = g_3(\Delta)$ represents the nontrivial homology class $[B\Delta B^{-1}] \in H_{n-1}(\bar{M}_\Gamma, \partial\bar{M}_\Gamma)$ and $\alpha \cap \mathcal{X} = \{A\}$. □

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