K–duality for stratified pseudomanifolds

CLAIRE DEBORD

JEAN-MARIE LESCURE

This paper continues our project started in [11] where Poincaré duality in K–theory was studied for singular manifolds with isolated conical singularities. Here, we extend the study and the results to general stratified pseudomanifolds. We review the axiomatic definition of a smooth stratification S of a topological space X and we define a groupoid $T^S X$, called the S–tangent space. This groupoid is made of different pieces encoding the tangent spaces of strata, and these pieces are glued into the smooth noncommutative groupoid $T^S X$ using the familiar procedure introduced by Connes for the tangent groupoid of a manifold. The main result is that $C^*(T^S X)$ is Poincaré dual to $C(X)$, in other words, the S–tangent space plays the role in K–theory of a tangent space for X.

Introduction

This paper takes place in a longstanding project aiming to study index theory and related questions on stratified pseudomanifolds using tools and concepts from noncommutative geometry.

The key observation at the beginning of this project is that in its K–theoretic form, the Atiyah–Singer index theorem [2] involves ingredients that should survive to the singularities allowed in a stratified pseudomanifold. This is possible, from our opinion, as soon as one accepts reasonable generalizations and new presentation of certain classical objects on smooth manifolds, making sense on stratified pseudomanifolds.

The first instance of these classical objects that need to be adapted to singularities is the notion of *tangent space*. Since index maps in [2] are defined on the K–theory of the tangent spaces of smooth manifolds, one must have a similar space adapted to stratified pseudomanifolds. Moreover, such a space should satisfy natural requirements. It should coincide with the usual notion on the regular part of the pseudomanifold and incorporate in some way copies of usual tangent spaces of strata, while keeping enough smoothness to allow interesting computations. Moreover, it should be Poincaré dual in K–theory (shortly, K–dual) to the pseudomanifold itself. This K–theoretic

property involves bivariant K–theory and was proved between smooth manifolds and their tangent spaces by Kasparov [19] and Connes and Skandalis [7].

In [11], we introduced a candidate to be the tangent space of a pseudomanifold with isolated conical singularities. It appeared to be a smooth groupoid, leading to a noncommutative C^*–algebra, and we proved that it fulfills the expected K–duality.

In [21], the second author interpreted the duality proved in [11] as a principal symbol map, thus recovering the classical picture of Poincaré duality in K–theory for smooth manifolds. This interpretation used a notion of noncommutative elliptic symbols, which appeared to be the cycles of the K–theory of the noncommutative tangent space.

In [13], with Nistor, we used the noncommutative tangent space together with other deformation groupoids to construct analytical and topological index maps, and their equality was proved. As expected, these index maps are straight generalizations of those of Atiyah and Singer [2] for manifolds.

The present paper is devoted to the construction of the noncommutative tangent space for a general stratified pseudomanifold and the proof of the K–duality. It is thus a sequel of [11], but can be read independently. At first glance, one should have expected that the techniques of [11] iterate easily to give the general result. In fact, although the definition of the groupoid giving the noncommutative tangent space itself is natural and intuitive in the general case, its smoothness is quite intricate and brings issues that did not exist in the conical case. We have given here a detailed treatment of this point, since we believe that this material will be useful in further studies about the geometry of stratified spaces. Another difference with [11] is that we have given up the explicit construction of a dual Dirac element. Instead, we use an easily defined Dirac element and then prove the Poincaré duality by an induction, based on an operation called unfolding which consists in removing the minimal strata in a pseudomanifold and then "doubling" it to get a new pseudomanifold, less singular. The difficulty in this approach is moved to the proof of the commutativity of certain diagrams in K–theory, necessary to apply the five lemma and to continue the induction.

The interpretation of this K–duality in terms of noncommutative symbols and pseudodifferential operators, as well as the construction of index maps together with the statement of an index theorem, is postponed to forthcoming papers.

This approach of index theory on singular spaces in the framework of noncommutative geometry takes place in a long history of past and present research works. But the specific issues about Poincaré duality, bivariant K–theory, topological index maps and statement of Atiyah–Singer–like theorems are quite recent and attract an increasing interest; see Monthubert and Nistor [27], Melrose and Rochon [25], Emerson and Meyer [14; 15], Savin [31] and Nazaĭkinskiĭ, Savin and Sternin [29].

1 Basic definitions

1.1 Around Lie groupoids

We refer to Renault [30], Cannas da Silva and Weinstein [32], Mackenzie [22] and Debord and Lescure [12] for the classical definitions and constructions related to groupoids, their Lie algebroids and C^*–algebras of groupoids. In this section, we fix the notation and recall the less classical definitions and results needed in the sequel. Some material presented here is already in our works [11; 13].

1.1.1 Pull back groupoids Let $G \rightrightarrows M$ be a locally compact Hausdorff groupoid with source s and range r. If $f: N \to M$ is a surjective map, the *pull back* groupoid $^*f^*(G) \rightrightarrows N$ of G by f is by definition the set

$$^*f^*(G) := \{(x, \gamma, y) \in N \times G \times N \mid r(\gamma) = f(x), \ s(\gamma) = f(y)\}$$

with the structural morphisms given by

(1) the unit map $x \mapsto (x, f(x), x)$,

(2) the source map $(x, \gamma, y) \mapsto y$ and range map $(x, \gamma, y) \mapsto x$,

(3) the product $(x, \gamma, y)(y, \eta, z) = (x, \gamma\eta, z)$ and inverse $(x, \gamma, y)^{-1} = (y, \gamma^{-1}, x)$.

The results of Muhly, Renault and Williams [28] apply to show that the groupoids G and $^*f^*(G)$ are Morita equivalent when f is surjective and open.

Let us assume for the rest of this subsection that G is a smooth groupoid and that f is a surjective submersion, then $^*f^*(G)$ is also a Lie groupoid. Let $(\mathcal{A}(G), q, [\ ,\ ])$ be the Lie algebroid of G. Recall that $q: \mathcal{A}(G) \to TM$ is the anchor map. Let $(\mathcal{A}(^*f^*(G)), p, [\ ,\ ])$ be the Lie algebroid of $^*f^*(G)$ and $Tf: TN \to TM$ be the differential of f. Then there exists an isomorphism

$$\mathcal{A}(^*f^*(G)) \simeq \{(V, U) \in TN \times \mathcal{A}(G) \mid Tf(V) = q(U) \in TM\}$$

under which the anchor map $p: \mathcal{A}(^*f^*(G)) \to TN$ identifies with the projection $TN \times \mathcal{A}(G) \to TN$. (In particular, if $(V, U) \in \mathcal{A}(^*f^*(G))$ with $V \in T_x N$ and $U \in \mathcal{A}_y(G)$, then $y = f(x)$.)

1.1.2 Subalgebras and exact sequences of groupoid C^*–algebras To any smooth groupoid G are associated two C^*–algebras corresponding to two different completions of the involutive convolution algebra $C_c^\infty(G)$, namely the reduced and maximal C^*–algebras; see Connes [5; 6] and Renault [30]. We will denote respectively these C^*–algebras by $C_r^*(G)$ and $C^*(G)$. Recall that the identity on $C_c^\infty(G)$ induces a surjective morphism from $C^*(G)$ onto $C_r^*(G)$ which is an isomorphism if the groupoid G is amenable. Moreover in this case the C^* algebra of G is nuclear by Anantharaman-Delaroche and Renault [1].

Let

$$G \underset{r}{\overset{s}{\rightrightarrows}} G^{(0)}$$

be a smooth groupoid with source s and range r. If U is any subset of $G^{(0)}$, we use the following usual notation:

$$G_U := s^{-1}(U) , \ G^U := r^{-1}(U) \text{ and } G_U^U = G|_U := G_U \cap G^U .$$

To an open subset O of $G^{(0)}$ corresponds an inclusion i_O of $C_c^\infty(G|_O)$ into $C_c^\infty(G)$ which induces an injective morphism, again denoted by i_O, from $C^*(G|_O)$ into $C^*(G)$.

When O is saturated, $C^*(G|_O)$ is an ideal of $C^*(G)$. In this case, $F := G^{(0)} \setminus O$ is a saturated closed subset of $G^{(0)}$ and the restriction of functions induces a surjective morphism r_F from $C^*(G)$ to $C^*(G|_F)$. Moreover, according to Hilsum and Skandalis [17], the following sequence of C^*–algebras is exact:

$$0 \longrightarrow C^*(G|_O) \overset{i_O}{\longrightarrow} C^*(G) \overset{r_F}{\longrightarrow} C^*(G|_F) \longrightarrow 0 .$$

1.1.3 KK–elements associated to deformation groupoids A smooth groupoid G is called a *deformation groupoid* if:

$$G = G_1 \times \{0\} \cup G_2 \times]0,1] \rightrightarrows G^{(0)} = M \times [0,1] ,$$

where G_1 and G_2 are smooth groupoids with unit space M. That is, G is obtained by gluing $G_2 \times]0,1] \rightrightarrows M \times]0,1]$, which is the cartesian product of the groupoid $G_2 \rightrightarrows M$ with the space $]0,1]$, with the groupoid $G_1 \times \{0\} \rightrightarrows M \times \{0\}$.

In this situation one can consider the saturated open subset $M \times]0,1]$ of $G^{(0)}$. Using the isomorphisms $C^*(G|_{M \times]0,1]}) \simeq C^*(G_2) \otimes C_0(]0,1])$ and $C^*(G|_{M \times \{0\}}) \simeq C^*(G_1)$, we obtain the following exact sequence of C^*–algebras:

$$0 \longrightarrow C^*(G_2) \otimes C_0(]0,1]) \overset{i_{M \times]0,1]}}{\longrightarrow} C^*(G) \overset{\mathrm{ev}_0}{\longrightarrow} C^*(G_1) \longrightarrow 0$$

where $i_{M \times]0,1]}$ is the inclusion map and ev_0 is the *evaluation map* at 0, that is ev_0 is the map coming from the restriction of functions to $G|_{M \times \{0\}}$.

We assume now that $C^*(G_1)$ is nuclear. Since the C^*–algebra $C^*(G_2) \otimes C_0(]0, 1])$ is contractible, the long exact sequence in KK–theory shows that the group homomorphism $(\mathrm{ev}_0)_* = \cdot \otimes [\mathrm{ev}_0]$: $KK(A, C^*(G)) \to KK(A, C^*(G_1))$ is an isomorphism for each C^*–algebra A; see Kasparov [19] and Cuntz and Skandalis [9].

In particular with $A = C^*(G_1)$ and $A = C^*(G)$ we get that $[\mathrm{ev}_0]$ is invertible in KK–theory: there is an element $[\mathrm{ev}_0]^{-1}$ in $KK(C^*(G_1), C^*(G))$ such that $[\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_0] = 1_{C^*(G_1)}$ and $[\mathrm{ev}_0] \otimes [\mathrm{ev}_0]^{-1} = 1_{C^*(G)}$.

Let $\mathrm{ev}_1 \colon C^*(G) \to C^*(G_2)$ be the evaluation map at 1 and $[\mathrm{ev}_1]$ the corresponding element of $KK(C^*(G), C^*(G_2))$.

The *KK–element associated to the deformation groupoid* G is defined by:

$$\delta = [\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1] \in KK(C^*(G_1), C^*(G_2)) .$$

One can find examples of such elements related to index theory in our papers [11; 12], the papers of Connes [6] and Hilsum and Skandalis [17] and our paper with Nisitor [13].

1.2 Generalities about *K*–duality

We give in this paragraph some general facts about Poincaré duality in bivariant K-theory. Most of them are well known and proofs are only added when no self contained proof could be found in the literature. All C^*–algebras are assumed to be separable and σ–unital.

Let us first recall what means the Poincaré duality in K–theory; see Kasparov [20], Connes and Skandalis [7] and Connes [6]:

Definition 1 Let A, B be two C^*–algebras. One says that A and B are Poincaré dual, or shortly K–dual, when there exists $\alpha \in K^0(A \otimes B) = KK(A \otimes B, \mathbb{C})$ and $\beta \in KK(\mathbb{C}, A \otimes B) \simeq K_0(A \otimes B)$ such that

$$\beta \underset{B}{\otimes} \alpha = 1 \in KK(A, A) \quad \text{and} \quad \beta \underset{A}{\otimes} \alpha = 1 \in KK(B, B)$$

Such elements are then called Dirac and dual-Dirac elements.

It follows that for A, B two K–dual C^*–algebras and for any C^*–algebras C, D, the following isomorphisms hold:

$$\beta \underset{B}{\otimes} \cdot \colon KK(B \otimes C, D) \longrightarrow KK(C, A \otimes D) ,$$
$$\beta \underset{A}{\otimes} \cdot \colon KK(A \otimes C, D) \longrightarrow KK(C, B \otimes D) ,$$

with inverses given respectively by $\cdot \underset{A}{\otimes} \alpha$ and $\cdot \underset{B}{\otimes} \alpha$.

Example 1   A basic example is $A = C(V)$ and $B = C_0(T^*V)$ where V is a closed smooth manifold [20; 7] (see also our paper [11] for a description of the Dirac element in terms of groupoids). This duality allows to recover that the usual quantification and principal symbol maps are mutually inverse isomorphisms in K–theory:

$$\Delta_V = (\cdot \underset{C_0(T^*V)}{\otimes} \alpha): K_0(C_0(T^*V)) \xrightarrow{\simeq} K^0(C(V))$$

$$\Sigma_V = (\beta \underset{C(V)}{\otimes} \cdot): K^0(C(V)) \xrightarrow{\simeq} K_0(C_0(T^*V))$$

We observe that:

Lemma 1   *Let A, B be two C^*–algebras. Assume that there exists $\alpha \in KK(A \otimes B, \mathbb{C})$ and $\beta, \beta' \in KK(\mathbb{C}, A \otimes B)$ satisfying*

$$\beta \underset{B}{\otimes} \alpha = 1 \in KK(A, A) \quad and \quad \beta' \underset{A}{\otimes} \alpha = 1 \in KK(B, B) .$$

Then $\beta = \beta'$ so A, B are K–dual.

Proof   A simple calculation shows that for all $x \in KK(C, A \otimes D)$ we have:

$$\beta \underset{B}{\otimes} (x \underset{A}{\otimes} \alpha) = x \underset{A \otimes B}{\otimes} (\beta \underset{B}{\otimes} \alpha) .$$

Applying this to $C = \mathbb{C}$, $D = A$ and $x = \beta'$ we get:

$$\beta' = \beta \underset{B}{\otimes} (\beta' \underset{A}{\otimes} \alpha) = \beta \underset{B}{\otimes} 1 = \beta . \qquad \square$$

Corollary 1   (1)   *Given two K–dual C^*–algebras and a Dirac element α, the dual-Dirac element β satisfying the Definition 1 is unique.*

(2)   *If there exists $\alpha \in KK(A \otimes B, \mathbb{C})$ such that*

$$\cdot \underset{B}{\otimes} \alpha: KK(\mathbb{C}, A \otimes B) \longrightarrow KK(A, A) \quad and \quad \cdot \underset{A}{\otimes} \alpha: KK(\mathbb{C}, A \otimes B) \longrightarrow KK(B, B)$$

are onto, then A, B are K–dual and α is a Dirac element.

The two lemmas below have been communicated to us by the referee.

Lemma 2   *Let J_1 and J_2 be two closed two sided ideals in a nuclear C^*–algebra A such that $J_1 \cap J_2 = \{0\}$ and set $B = A/(J_1 + J_2)$. Denote by $\partial_k \in KK_1(B, J_k)$, $k = 1, 2$, the KK–elements associated respectively with the exact sequences $0 \longrightarrow J_1 \longrightarrow A/J_2 \longrightarrow B \longrightarrow 0$ and $0 \longrightarrow J_2 \longrightarrow A/J_1 \longrightarrow B \longrightarrow 0$. Let also $i_k: J_k \longrightarrow A$ denote the inclusions. Then the following equality holds:*

$$(i_1)_*(\partial_1) + (i_2)_*(\partial_2) = 0 .$$

Proof Let $\partial \in KK_1(B, J_1 + J_2)$ denote the KK–element associated with the exact sequence $0 \longrightarrow J_1 + J_2 \longrightarrow A \longrightarrow B \longrightarrow 0$. Denote by $j_k \colon J_k \longrightarrow J_1 + J_2$ and $i \colon J_1 + J_2 \longrightarrow A$ the inclusions and by $p_k \colon J_1 + J_2 \longrightarrow J_k$ the projections, $k = 1, 2$. Since the diagrams $(k = 1, 2)$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 + J_2 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle p_k} & & \downarrow & & \parallel & & \\
0 & \longrightarrow & J_k & \longrightarrow & A/J_{3-k} & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

commute, it follows that $(p_k)_*(\partial) = \partial_k$.

Moreover, $(p_1)_* \times (p_2)_* \colon KK_1(B, J_1 + J_2) \longrightarrow KK_1(B, J_1) \times KK_1(B, J_2)$ is an isomorphism whose inverse is $(j_1)_* + (j_2)_*$. It follows that $\partial = (j_1)_*(\partial_1) + (j_2)_*(\partial_2)$. Moreover the six-term exact sequence associated to $0 \longrightarrow J_1 + J_2 \longrightarrow A \longrightarrow B \longrightarrow 0$ leads to $i_*(\partial) = 0$. The result follows now from the equalities $i_k = i \circ j_k$, $k = 1, 2$. $\qquad \square$

Lemma 3 *Let X be a compact space and A be a nuclear $C(X)$–algebra. Let U_1 and U_2 be disjoint open subsets of X. Set $X_1 = X \setminus U_2$ and $J_k = C_0(U_k)A$, $k = 1, 2$. Let $\Psi \colon C(X) \otimes A \longrightarrow A$ be the homomorphism defined by $\Psi(f \otimes a) = fa$ and let $\varphi \colon C(X_1) \otimes J_1 \longrightarrow A$, $\psi \colon C_0(U_2) \otimes A/J_1 \longrightarrow A$ be the homomorphisms induced by Ψ.*

Denote by $\overline{\partial}_1 \in KK_1(A/J_1, J_1)$ and $\widetilde{\partial}_2 \in KK_1(C(X_1), C_0(U_2))$ the KK–elements associated respectively with the exact sequences $0 \longrightarrow J_1 \longrightarrow A \longrightarrow A/J_1 \longrightarrow 0$ and $0 \longrightarrow C_0(U_2) \longrightarrow C(X) \longrightarrow C(X_1) \longrightarrow 0$. Then the following equality holds:

$$(\varphi)_*(\overline{\partial}_1 \otimes 1_{C(X_1)}) + (\psi)_*(1_{A/J_1} \otimes \widetilde{\partial}_2) = 0 .$$

Proof We use the notation of Lemma 2. We have commuting diagrams

(1-1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 \otimes C(X_1) & \longrightarrow & A \otimes C(X_1) & \longrightarrow & A/J_1 \otimes C(X_1) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \varphi_1} & & \downarrow & & \downarrow{\scriptstyle \chi} & & \\
0 & \longrightarrow & J_1 & \longrightarrow & A/J_2 & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

(1-2)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A/J_1 \otimes C_0(U_2) & \longrightarrow & A/J_1 \otimes C(X) & \longrightarrow & A/J_1 \otimes C(X_1) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \psi_2} & & \downarrow & & \downarrow{\scriptstyle \chi} & & \\
0 & \longrightarrow & J_2 & \longrightarrow & A/J_1 & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

where vertical arrows are induced by Ψ. It follows that $(\varphi_1)_*(\overline{\partial}_1 \otimes 1_{C(X_1)}) = \chi^*(\partial_1)$ and $(\psi_2)_*(1_{A/J_1} \otimes \widetilde{\partial}_2) = \chi^*(\partial_2)$. We then use the equalities $\varphi = i_1 \circ \varphi_1$ and $\psi = i_2 \circ \psi_2$ and apply Lemma 2 to conclude. $\qquad \square$

It yields the following, with the notation of Lemma 3:

Lemma 4 *Let δ be in $K^0(A)$ and set $D = \Psi^*(\delta)$, $D_1 = \varphi^*(\delta)$, $D_2 = \psi^*(\delta)$. Then for any C^*–algebras C and D, the two following long diagrams commute:*

(1-3) $\cdots KK_i(C, D{\otimes}C_0(U_2)) \rightarrow KK_i(C, D{\otimes}C(X)) \rightarrow KK_i(C, D{\otimes}C(X_1)) \rightarrow KK_{i+1}(C, D{\otimes}C_0(U_2)) \cdots$

$$\Big\downarrow {\underset{C_0(U_2)}{\otimes}}^{c_i D_2} \qquad \Big\downarrow {\underset{C(X)}{\otimes}}^{c_i D} \qquad \Big\downarrow {\underset{C(X_1)}{\otimes}}^{c_i D_1} \qquad \Big\downarrow {\underset{C_0(U_2)}{\otimes}}^{c_{i+1} D_2}$$

$\cdots KK_i(C{\otimes}A/J_1, D) \longrightarrow KK_i(C{\otimes}A, D) \longrightarrow KK_i(C{\otimes}J_1, D) \longrightarrow KK_{i+1}(C{\otimes}A/J_1, D) \cdots$

(1-4) $\cdots KK_i(C, D{\otimes}J_1) \longrightarrow KK_i(C, D{\otimes}A) \longrightarrow KK_i(C, D{\otimes}A/J_1) \longrightarrow KK_{i+1}(C, D{\otimes}J_1) \cdots$

$$\Big\downarrow {\underset{J_1}{\otimes}}^{c_i D_1} \qquad \Big\downarrow {\underset{A}{\otimes}}^{c_i D} \qquad \Big\downarrow {\underset{A/J_1}{\otimes}}^{c_i D_2} \qquad \Big\downarrow {\underset{J_1}{\otimes}}^{c_{i+1} D_1}$$

$\cdots KK_i(C{\otimes}C(X_1), D) \rightarrow KK_i(C{\otimes}C(X), D) \rightarrow KK_i(C{\otimes}C_0(U_2), D) \rightarrow KK_{i+1}(C{\otimes}C(X_1), D) \cdots$

where the c_i belong to $\{-1, 1\}$ and are chosen such that $c_i = (-1)^{i+1}c_{i+1}$.

In particular, if two of three elements D_1, D_2, D are Dirac elements, so is the third one.

Proof Observe first that Lemma 3 reads: $\overline{\partial}_1 \underset{J_1}{\otimes} [\varphi] = -\widetilde{\partial}_2 \underset{C_0(U_2)}{\otimes} [\psi]$, which gives:

$$\overline{\partial}_1 \underset{J_1}{\otimes} D_1 = \overline{\partial}_1 \underset{J_1}{\otimes} ([\varphi] \otimes \delta) = (\overline{\partial}_1 \underset{J_1}{\otimes} [\varphi]) \otimes \delta = (-\widetilde{\partial}_2 \underset{C_0(U_2)}{\otimes} [\psi]) \otimes \delta = -\widetilde{\partial}_2 \underset{C_0(U_2)}{\otimes} D_2$$

Now from the skew-commutativity of the product $\underset{\mathbb{C}}{\otimes}$, for any $x \in KK_i(C, D{\otimes}C(X_1))$:

$$\begin{aligned}
\overline{\partial}_1 \underset{J_1}{\otimes} (x \underset{C(X_1)}{\otimes} D_1) &= (\overline{\partial}_1 \underset{\mathbb{C}}{\otimes} x) \underset{J_1 \otimes C(X_1)}{\otimes} D_1 \\
&= (-1)^i (x \underset{\mathbb{C}}{\otimes} \overline{\partial}_1) \underset{J_1 \otimes C(X_1)}{\otimes} D_1 \\
&= (-1)^i x \underset{C(X_1)}{\otimes} (\overline{\partial}_1 \underset{J_1}{\otimes} D_1) \\
&= (-1)^i x \underset{C(X_1)}{\otimes} (-\widetilde{\partial}_2 \underset{C_0(U_2)}{\otimes} D_2) \\
&= (-1)^{i+1} (x \underset{C(X_1)}{\otimes} \widetilde{\partial}_2) \underset{C_0(U_2)}{\otimes} D_2
\end{aligned}$$

This yields, thanks to the choice of the sign c_i, the commutativity for the squares involving boundary homomorphisms in Diagram (1-3). The other squares in Diagram (1-3) commute by definition of D_1, D_2, D and by functoriality of KK–theory. The commutativity of Diagram (1-4) is proved by the same arguments. The last assertion is then a consequence of Corollary 1 and the five lemma. $\square$

2 Stratified pseudomanifolds

We are interested in studying stratified pseudomanifolds; refer to Whitney [34], Mather [23] and Goresky and MacPherson [16]. We will use the notation and equivalent descriptions given by Verona in [33] or used by Brasselet, Hector and Saralegi in [3]. The reader should also look at Hughes and Weinberger [18] for a helpful survey of the subject.

2.1 Definitions

Let X be a locally compact separable metrizable space.

Definition 2 A C^∞–stratification of X is a pair (S, N) such that:

(1) $\mathsf{S} = \{s_i\}$ is a locally finite partition of X into locally closed subsets of X, called the strata, which are smooth manifolds and which satisfies

$$s_0 \cap \bar{s_1} \neq \varnothing \text{ if and only if } s_0 \subset \bar{s_1} \ .$$

In that case we will write $s_0 \leq s_1$ and $s_0 < s_1$ if moreover $s_0 \neq s_1$.

(2) $N = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s \in \mathsf{S}}$ is the set of control data or tube system:
$\mathcal{N}_s$ is an open neighborhood of s in X, $\pi_s \colon \mathcal{N}_s \to s$ is a continuous retraction and $\rho_s \colon \mathcal{N}_s \to [0, +\infty[$ is a continuous map such that $s = \rho_s^{-1}(0)$. The map ρ_s is either surjective or constant equal to 0.
Moreover if $\mathcal{N}_{s_0} \cap s_1 \neq \varnothing$ then the map

$$(\pi_{s_0}, \rho_{s_0}) \colon \mathcal{N}_{s_0} \cap s_1 \to s_0 \times]0, +\infty[$$

is a smooth proper submersion.

(3) For any strata s, t such that $s < t$, the inclusion $\pi_t(\mathcal{N}_s \cap \mathcal{N}_t) \subset \mathcal{N}_s$ is true and the equalities:

$$\pi_s \circ \pi_t = \pi_s \quad \text{and} \quad \rho_s \circ \pi_t = \rho_s$$

hold on $\mathcal{N}_s \cap \mathcal{N}_t$.

(4) For any two strata s_0 and s_1 the following equivalences hold:

$$s_0 \cap \bar{s_1} \neq \varnothing \text{ if and only if } \mathcal{N}_{s_0} \cap s_1 \neq \varnothing \ ,$$
$$\mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \neq \varnothing \text{ if and only if } s_0 \subset \bar{s_1}, \ s_0 = s_1 \text{ or } s_1 \subset \bar{s_0} \ .$$

A stratification gives rise to a filtration: let X_j be the union of strata of dimension $\leq j$, then:

$$\varnothing \subset X_0 \subset \cdots \subset X_n = X \ .$$

We call n the *dimension* of X and $X^\circ := X \setminus X_{n-1}$ the *regular* part of X. The strata included in X° are called *regular* while strata included in $X \setminus X^\circ$ are called *singular*. The set of singular (resp. regular) strata is denoted S_{sing} (resp. S_{reg}).

For any subset A of X, A° will denote $A \cap X^\circ$.

A crucial notion for our purpose will be the notion of *depth*. Observe that the binary relation $s_0 \le s_1$ is a partial ordering on S.

Definition 3 The depth $d(s)$ of a stratum s is the biggest k such that one can find k different strata $s_0, \cdots, s_{k-1}$ such that

$$s_0 < s_1 < \cdots < s_{k-1} < s_k := s .$$

The depth of the stratification (S, N) of X is:

$$d(X) := \sup\{d(s), \ s \in \mathsf{S}\} .$$

A stratum whose depth is 0 will be called minimal.

We have followed the terminology of [3], but remark that the opposite convention for the depth also exists [33].

Finally we can define stratified pseudomanifolds:

Definition 4 A stratified pseudomanifold is a triple (X, S, N) where X is a locally compact separable metrizable space, (S, N) is a C^∞–stratification on X and the regular part X° is a dense open subset of X.

If (X, S_X, N_X) and (Y, S_Y, N_Y) are two stratified pseudomanifolds an homeomorphism $f \colon X \to Y$ is an isomorphism of stratified pseudomanifold if:

(1) $\mathsf{S}_Y = \{f(s), \ s \in \mathsf{S}_X\}$ and the restriction of f to each stratum is a diffeomorphism onto its image.

(2) $\pi_{f(s)} \circ f = f \circ \pi_s$ and $\rho_s = \rho_{f(s)} \circ f$ for any stratum s of X.

Let us make some basic remark on the previous definitions.

Remark 1   (1) At a first sight, the definition of a stratification given here seems more restrictive than the usual one. In fact according to [33] these definitions are equivalent.

(2) Usually, for example in [16], the extra assumption $X_{n-1} = X_{n-2}$ is required in the definition of stratified pseudomanifold. Our constructions remain without this extra assumption.

(3)   A stratum s is regular if and only if $\mathcal{N}_s = s$ and then $\rho_s = 0$.

(4)   Pseudomanifolds of depth 0 are smooth manifolds, and the strata are then union of connected components.

The following simple consequence of the axioms will be useful enough in the sequel to be pointed out:

Proposition 1   *Let (X, S, N) be a stratified pseudomanifold. Any subset $\{s_i\}_I$ of distinct elements of S is totally ordered by $<$ as soon as the intersection $\bigcap_{i \in I} \mathcal{N}_{s_i}$ is nonempty. In particular if the strata s_0 and s_1 are such that $\mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \neq \varnothing$ then $d(s_0) \neq d(s_1)$ or $s_0 = s_1$.*

By a slight abuse of language we will sometime talk about a stratified pseudomanifold X while we only have a partition S on the space X. This means that one can find at least one control data N such that (X, S, N) is a stratified pseudomanifold in the sense of our Definition 4.

2.2   Examples

(1)   Smooth manifolds are, without other mention, pseudomanifolds of depth 0 and with a single stratum.

(2)   Stratified pseudomanifolds of depth one are *wedges* and are obtained as follows.

Take M to be a manifold with a compact boundary L and let π be a surjective submersion of L onto a manifold s. Consider the *mapping cone* of (L, π) :

$$c_\pi L := L \times [0, 1] / \sim_\pi$$

where $(z, t) \sim_\pi (z', t')$ if and only if $(z, t) = (z', t')$ or $t = t' = 0$ and $\pi(z) = \pi(z')$. The image of $L \times \{0\}$ identifies with s and by a slight abuse of notation we will denote it s. Now glue $c_\pi L$ and M along their boundary in order to get X. The space X with the partition $\{s, X \setminus s\}$ is a stratified pseudomanifold.

Two extreme examples are obtained by considering π either equal to identity, with $s = L$ or equal to the projection on one point c. In the first case X is a manifold with boundary L isomorphic to M and the stratification corresponds to the partition of X by $\{L, X \setminus L\}$. In the second case X is a *conical manifold* and the stratification corresponds to the partition of X by $\{c, X \setminus c\}$, where c is the singular point.

(3)   Manifolds with corners with their partition into faces are stratified pseudomanifolds; see Melrose [24] and Monthubert [26].

(4) If (X, S, N) is a pseudomanifold and M is a smooth manifold then $X \times M$ is naturally endowed with a structure of pseudomanifold of same depth as X whose strata are $\{s \times M, \ s \in \mathsf{S}\}$.

(5) If (X, S, N) is a pseudomanifold of depth k then $CrX := X \times S^1 / X \times \{p\}$ is naturally endowed with a structure of pseudomanifold of depth $k + 1$, whose strata are $\{s \times \,]0, 1[, \ s \in \mathsf{S}\} \cup \{[p]\}$. Here we have identified $S^1 \setminus \{p\}$ with $]0, 1[$ and we have denoted by $[p]$ the image of $X \times \{p\}$ in CrX.

For example, if X is the square we get the following picture:
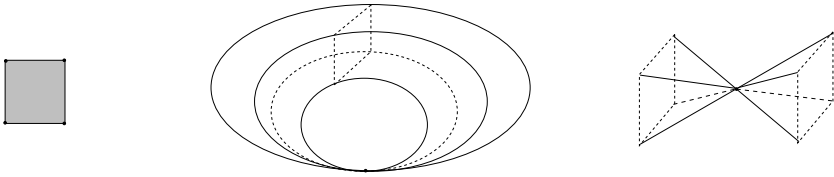
2.3 The unfolding process

Let (X, S, N) be a stratified pseudomanifold. If s is a singular stratum, we let $L_s := \rho_s^{-1}(1)$. Then L_s inherits from X a structure of stratified pseudomanifold.

One can then define the *open mapping cone* of (L_s, π_s):

$$c_{\pi_s} L_s := L_s \times [0, +\infty[/ \sim_{\pi_s}$$

where $\sim_{\pi_s}$ is as before.

According to [33] (also [3]) the open mapping cone is naturally endowed with a structure of stratified pseudomanifold whose strata are $\{(t \cap L_s) \times \,]0, +\infty[, \ t \in \mathsf{S}\} \cup \{s\}$. Here we identify s with the image of $L_s \times \{0\}$ in $c_{\pi_s} L_s$. Moreover, up to isomorphism, the control data on X can be chosen such that one can find a continuous retraction $f_s \colon \mathcal{N}_s \setminus s \to L_s$ for which the map

$$
\begin{array}{rl}
\Psi_s \colon \ \mathcal{N}_s & \to \ c_{\pi_s} L_s \\
z & \mapsto \begin{cases} [f_s(z), \rho_s(z)] & \text{if } z \notin s \\ z & \text{elsewhere} \end{cases}
\end{array}
$$

(2-1)

is an isomorphism of stratified pseudomanifolds. Here $[y, t]$ denotes the class in $c_{\pi_s} L_s$ of $(y, t) \in L_s \times [0, +\infty[$.

This result of local triviality around strata will be crucial for our purpose. In particular it enables one to make the unfolding process [3] which consists in replacing each minimal stratum s by L_s. Precisely suppose that $d(X) = k > 0$ and let S_0 be the

set of strata of depth 0. Define $O_0 := \bigcup_{s \in S_0} \{z \in \mathcal{N}_s \mid \rho_s(z) < 1\}$, $X_b = X \setminus O_0$ and $L := \bigcup_{s \in S_0} \{z \in \mathcal{N}_s \mid \rho_s(z) = 1\} \subset X_b$. Notice that it follows from Remark 1 that the L_s's where $s \in S_0$ are disjoint and thus $L = \sqcup_{s \in S_0} L_s$. We let

$$2X = X_b^- \cup L \times [-1, 1] \cup X_b^+$$

where $X_b^{\pm} = X_b$ and X_b^- (respectively X_b^+) is glued along L with $L \times \{-1\} \subset L \times [-1, 1]$ (respectively $L \times \{1\} \subset L \times [-1, 1]$).

Let s be a stratum of X which is not minimal and which intersects O_0. We define the following subset of $2X$:

$$\widetilde{s} := (s \cap X_b^-) \cup (s \cap L) \times [-1, 1] \cup (s \cap X_b^+)$$

We then define

$$\mathsf{S}_{2X} := \{\widetilde{s}; \ s \in \mathsf{S} \text{ and } s \cap O_0 \neq \varnothing\} \cup \{s^-, s^+; s^{\pm} = s \in \mathsf{S} \text{ and } s \cap O_0 = \varnothing\}.$$

The space $2X$ inherits from X a structure of stratified pseudomanifold of depth $k - 1$ whose set of strata is S_{2X}.

Notice that there is a natural map p from $2X$ onto X. The restriction p to any copy of X_b is identity and for $(z, t) \in L_s \times [-1, 1]$, $p(z, t) = \Psi_s^{-1}([z, |t|])$. The strata of $2X$ are the connected components of the preimages by p of the strata of X.

The interested reader can find all the details related to the unfolding process in [3] and [33] where it is called decomposition. In particular starting with a compact pseudomanifold X of depth k, one can iterate this process k times and obtain a compact smooth manifold $2^k X$ together with a continuous surjective map $\pi \colon 2^k X \to X$ whose restriction to $\pi^{-1}(X^\circ)$ is a trivial 2^k–fold covering.

Example 2 Look at the square C with stratification given by its vertices, edges and its interior. It can be endowed with a structure of stratified pseudomanifold of depth 2. Applying once the unfolding process gives a sphere with 4 holes: $S := S^2 \setminus \{D_1, D_2, D_3, D_4\}$ where the D_i's are disjoint and homeomorphic to open disks. The set of strata of S^2 is then int $S, S_1, S_2, S_3, S_4\}$ where S_i is the boundary of D_i and int S the interior of S. Applying the unfolding process once more gives the torus with three holes.
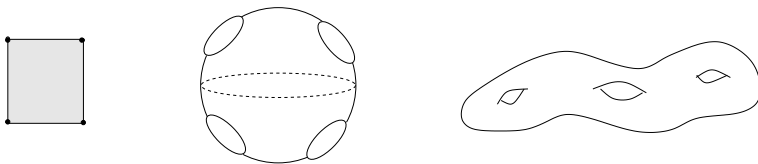
3 The tangent groupoid and S–tangent space of a compact stratified pseudo-manifold

3.1 The set construction

We begin by the description at the set level of the S–*tangent groupoid* and the S–*tangent space* of a compact stratified pseudo-manifold.

We keep the notation of the previous section: X is a compact stratified pseudo-manifold, S the set of strata, X° the regular part and $N = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s \in S}$ the set of control data.

For each $s \in S$ we let

$$O_s := \{z \in \mathcal{N}_s \mid \rho_s(z) < 1\} \quad \text{and} \quad F_s := O_s \setminus \bigcup_{s_0 < s} O_{s_0} .$$

Note that $F_s = O_s$ if and only if s is a minimal stratum and $O_s = s$ when s is regular.

Lemma 5 *The set $\{F_s\}_{s \in \mathcal{S}}$ form a partition of X.*

Proof If z belongs to X, let $R_z := \{s \in S \mid z \in \mathcal{N}_s \text{ and } \rho_s(z) < 1\}$. It follows from Proposition 1 that R_z is a finite set totally ordered by $<$. Since the set R_z contains the stratum passing through z, it is nonempty. Let s_0^z be the minimal element of R_z. Then z belongs to $F_{s_0^z}$. Moreover, for all stratum $s \in S$, if $s \neq s_0^z$ and $z \in O_s$, then $s \in R_z$, whence $s_0^z < s$, so that $s \notin F_s$. $\qquad\qquad\square$

Recall that $O_s^\circ = O_s \cap X^\circ$. We denote again by $\pi_s \colon O_s^\circ \to s$ the projection. When s is a stratum, π_s is a proper submersion and one can consider the pullback groupoid $^*\pi_s^*(Ts) \rightrightarrows O_s^\circ$ of the usual tangent space $Ts \rightrightarrows s$ by π_s. It is naturally endowed with a structure of smooth groupoid. When s is a regular stratum, $s = O_s = O_s^\circ$ and π_s is the identity map, thus $^*\pi_s^*(Ts) \simeq TO_s^\circ$ in a canonical way.

At the set level, the S-*tangent space* of X is the groupoid:

$$T^S X = \bigcup_{s \in S} {}^*\pi_s^*(TS)|_{F_s^\circ} \rightrightarrows X^\circ$$

where $F_s^\circ = F_s \cap X^\circ$. Following the cases of smooth manifolds [6] and isolated conical singularities [11], the S-*tangent groupoid* of X is defined to be a deformation of the pair groupoid of the regular part of X onto its S–tangent space:

$$\mathcal{G}_X^t := T^S X \times \{0\} \cup X^\circ \times X^\circ \times\,]0, 1] \rightrightarrows X^\circ \times [0, 1] .$$

Examples 1   (1)   When X has depth 0, we recover the usual tangent space and tangent groupoid.

(2) Suppose that X is a trivial wedge (see Section 2.2):

$$X = c_\pi L \cup M$$

where M is a manifold with boundary L and L is the product of two manifolds $L = s \times Q$ with $\pi\colon L \to s$ being the first projection. We have denoted by $c_\pi L = L \times [0, 1]/ \sim_\pi$ the mapping cone of (L, π). In other word $c_\pi L = s \times cQ$ where $cQ := Q \times [0, 1]/Q \times \{0\}$ is the cone over Q. We denote again by s the image of $L \times \{0\}$ in X. Then X admits two strata: s and $X^\circ = X \setminus s$, $F_s = O_s = L \times]0, 1[$ and $F_{X^\circ} = X^\circ \setminus O_s = M$. The tangent space is

$$T^S X = Ts \times (Q \times]0, 1[) \times (Q \times]0, 1[) \sqcup TM \rightrightarrows X^\circ$$

where $Ts \times (Q \times]0, 1[) \times (Q \times]0, 1[)$ is the product of the tangent space $Ts \rightrightarrows s$ with the pair groupoid over $Q \times]0, 1[$ and TM denotes the restriction of the usual tangent bundle TX° to the submanifold with boundary M.

Remark 2 For any stratum s, the restriction of $\mathcal{G}_X^t$ to F_s° is equal to

$$^*\pi_s^*(TS)|_{F_s^\circ} \times \{0\} \cup F_s^\circ \times F_s^\circ \times]0, 1] \rightrightarrows F_s^\circ \times [0, 1]$$

which is also the restriction to F_s° of $^*(\pi_s \times \mathrm{Id})^*(\mathcal{G}_s^t)$, the pullback by $\pi_s \times \mathrm{Id}\colon O_s^\circ \times [0, 1] \to S \times [0, 1]$ of the (usual) tangent groupoid of s:

$$\mathcal{G}_s^t = Ts \times \{0\} \cup s \times s \times]0, 1] \rightrightarrows s \times [0, 1] \,.$$

In the following, we will denote by $\mathcal{A}_{\pi_s \times \mathrm{Id}}^t$ the Lie algebroid of $^*(\pi_s \times \mathrm{Id})^*(\mathcal{G}_s^t)$.

3.2 The recursive construction

Thanks to the unfolding process described in Section 2.3, one can also construct the S–tangent spaces of stratified pseudomanifolds by an induction on the depth.

If X is of depth 0, it is a smooth manifold and the S–tangent space is the usual tangent space TX viewed as a groupoid on X.

Let k be an integer and assume that the S–tangent space of any pseudomanifold of depth smaller than k is defined. Let X be a stratified pseudomanifold of depth $k + 1$ and let $2X$ be the stratified pseudomanifold of depth k obtained from X by applying Section 2.3. With the notation of Section 2.3 we define

$$T^S X = T^S 2X|_{2X^\circ \cap X_b^+} \underset{s \in S_0}{\cup} {}^*\pi_s^*(Ts)|_{O_s^\circ} \rightrightarrows X^\circ$$

where $T^S 2X$ is the S–tangent space of the stratified pseudomanifold $2X$. Here we have identified $2X^\circ \cap X_b^+$ with the subset $X^\circ \setminus O_0 = X_b \cap X^\circ$ of X°. It is a simple

exercise to see that this construction leads to the same definition of S–tangent space as the previous one.

3.3   The smooth structure

In this subsection we prove that the S–tangent space of a stratified pseudomanifold, as well as its S–tangent groupoid, can be endowed with a smooth structure which reflects the local structure of the pseudomanifold itself.

Let (X, S, N) be a stratified pseudomanifold. The smooth structure of $T^\mathsf{S} X$ will depend on the stratification and a smooth, decreasing, positive function $\tau \colon \mathbb{R} \to \mathbb{R}$ such that $\tau([0, +\infty[) = [0, 1]$, $\tau^{-1}(0) = [1, +\infty[$ and τ' does not vanish on $]0, 1[$. The function τ will be called a *gluing function*. We will also use functions associated with τ and defined on $\mathcal{N}_s$ for any singular stratum s by: for each singular stratum:

$$\tau_s = \tau \circ \rho_s$$

Observe that $\tau_s = 0$ outside O_s°.

Before coming into the details of the smooth structure of $T^\mathsf{S} X$, let us describe its consequences for the convergence of sequences:

A sequence $(x_n, V_n, y_n) \in {}^*\pi_{s_n}^*(T s_n)|_{F_{s_n}}$ where n belongs to $\mathbb{N}$, goes to $(x, V, y) \in {}^*\pi_s^*(Ts)|_{F_s}$ if and only if:

$$(3\text{-}1) \qquad\qquad x_n \to x, \;\; y_n \to y, \;\; V_n + \frac{\pi_s(x_n) - \pi_s(y_n)}{\tau_{s_n}(x_n)} \to V$$

The first two convergences have an obvious meaning, and they imply that for n big enough, $s_n \leq s$. The third one needs some explanations. Let us note $z = \pi_s(x) = \pi_s(y)$ and $z_n = \pi_{s_n}(x_n) = \pi_{s_n}(y_n)$. Since $\pi_s(x_n)$ and $\pi_s(y_n)$ become close to z, we can interpret $w_n = \pi_s(x_n) - \pi_s(y_n)$ as a vector in $T_{\pi_s(y_n)}s$ (use any local chart of s around z). Moreover, using $\pi_{s_n} \circ \pi_s = \pi_{s_n}$, we see that this vector w_n is vertical for π_{s_n}, that is, belongs to the kernel K_n of the differential of π_{s_n} (suitably restricted to s_n). Now, the meaning of last convergence in (3-1) is $T\pi_{s_n}(V - w_n/\tau_{s_n}(x_n)) - V_n \to 0$ which has to be interpreted for each subsequences of $(x_n, V_n, y_n)_n$ with $s_n = s_{n_0}$ for all $n \geq n_0$ big enough.

The smooth structure of $T^\mathsf{S} X$ will be obtained by an induction on the depth of the stratification, and a concrete atlas will be given. For the sake of completeness, we also explicit a Lie algebroid whose integration gives the tangent groupoid $\mathcal{G}_X^t$. We begin by describing the local structure of X° around its strata, then we will prove inductively the existence of a smooth structure on the S–tangent space. Next, an atlas of the resulting

smooth structure is given by brut computations. A similar construction is easy to guess for the tangent groupoid $\mathcal{G}_X^t$. In the last part the previous smooth structure is recovered in a more abstract approach using an integrable Lie algebroid.

These parts are quite technical and can be left out as soon as you believe that the tangent space and the tangent groupoid can be endowed with a smooth structure compatible with the topology described above.

Before going into the details, we should point out that the constructions described above depend on the set of control data together with the choice of τ, the gluing function. As far as we know, there is no way to get rid of these extra data. Nevertheless, a consequence of the last chapter is that up to K–theory the S–tangent space $T^S X$ only depends on X.

3.3.1 The local structure of X° We now describe local charts of X° adapted to the stratification, called *distinguished charts*.

Let $z \in X^\circ$ and consider the set

$$S_z := \{s \in S \mid z \in \mathcal{N}_s \text{ and } \rho_s(z) \leq 1\} .$$

It is a nonempty finite set, totally ordered according to [Proposition 1](#), thus we can write

$$S_z = \{s_0, \cdots, s_\kappa\}, \quad s_0 < s_1 < \cdots < s_\kappa$$

where $s_\kappa \subset X^\circ$ must be regular. Let n_i be the dimension of s_i, $i \in \{0, 1, \ldots, \kappa\}$ and $n = n_\kappa = \dim X^\circ$.

Let U_z be an open neighborhood of z in X° such that the following hold:

$$(3\text{-}2) \qquad U_z \subset \bigcap_{s \in S_z} \mathcal{N}_s \quad \text{and} \quad \forall s \in S_{\text{sing}}, \ U_z \cap O_s \neq \varnothing \Leftrightarrow s \in S_z$$

In particular, the following hold on U_z:

$$(3\text{-}3) \qquad \text{for } 0 \leq i \leq j \leq \kappa : \pi_{s_i} \circ \pi_{s_j} = \pi_{s_i} \text{ and } \rho_{s_i} \circ \pi_{s_j} = \rho_{s_i} .$$

Without loss of generality, we can also assume that U_z is the domain of a local chart of X°.

If $\kappa = 0$, any local chart of X° with domain U_z will be called distinguished. When $\kappa \geq 1$, we can take successively canonical forms of the submersions $\pi_{s_0}, \pi_{s_1} \ldots, \pi_{s_\kappa}$ available on a possibly smaller U_z, that is, one can shrink U_z enough and find diffeomorphisms:

$$(3\text{-}4) \qquad \phi_i \colon \pi_{s_i}(U_z) \to \mathbb{R}^{n_i} \text{ for all } i \in \{0, 1, \ldots, \kappa\}$$

such that the diagram:

(3-5)
$$
\begin{array}{ccc}
\pi_{s_i}(U_z) & \xrightarrow{\phi_i} & \mathbb{R}^{n_i} \\
{\scriptstyle \pi_{s_j}}\downarrow & & \downarrow{\scriptstyle \sigma_{n_j}} \\
\pi_{s_j}(U_z) & \xrightarrow{\phi_j} & \mathbb{R}^{n_j}
\end{array}
$$

commutes for all $i, j \in \{0, 1, \ldots, \kappa\}$ such that $i \geq j$. Above, for any integers $p \geq d$, the map $\sigma_d \colon \mathbb{R}^p \to \mathbb{R}^d$ denotes the canonical projection onto the last d coordinates.

Remember that s_κ is regular so π_{s_κ} is the identity map and $\phi := \phi_\kappa$ is a local chart around z of X°. Now we set:

Definition 5 A *distinguished chart* of X° around $z \in X^\circ$ is a local chart (U_z, ϕ) around z such that U_z satisfies (3-2) together with diffeomorphisms (3-4) satisfying (3-5) and $\phi = \phi_\kappa$.

From now on, a Riemannian metric is chosen on X° (any adapted metric in the sense of Brasselet, Hector and Saralegi [4] is suitable for our purpose). Recall that for any stratum s, the map $\pi_s \colon \mathcal{N}_s^\circ \to s$ is a smooth submersion. Thus, if $K_s \subset T\mathcal{N}_s^\circ$ denotes the kernel of the differential map $T\pi_s$ and $q_s \colon T\mathcal{N}_s^\circ \to T\mathcal{N}_s^\circ$ the orthogonal projection on K_s, the map

$$(q_s, T\pi_s) \colon T\mathcal{N}_s^\circ \to K_s \oplus \pi_s^*(Ts)$$

is an isomorphism and the vector bundle $\pi_s^*(Ts)$ can be identified with the orthogonal complement of K_s into $T\mathcal{N}_s = TX^\circ|_{\mathcal{N}_s}$.

Now, let (U_z, ϕ) be a distinguished chart around some $z \in X^\circ$. Set $S_z = \{s_0, s_1, \ldots, s_\kappa\}$ with $s_0 < s_1 < \ldots < s_\kappa$, and set $K_i = K_{s_i}|_{U_z}$, $U_i = \pi_{s_i}(U_z)$ for all $i = 0, 1, \ldots, \kappa$.

By (3-3) we have:

(3-6) $$U_z \times \{0\} = K_\kappa \subset K_{\kappa-1} \subset \cdots \subset K_1 \subset K_0 \subset TU_z .$$

Rewriting the diagram (3-5) for the differential maps and $i = \kappa$, we get for all $j \leq \kappa$:

(3-7)
$$
\begin{array}{ccc}
TU_z & \xrightarrow{T\phi} & \mathbb{R}^n \times \mathbb{R}^n \\
{\scriptstyle T\pi_{s_j}}\downarrow & & \downarrow{\scriptstyle \sigma_{n_j} \times \sigma_{n_j}} \\
TU_j & \xrightarrow{T\phi_j} & \mathbb{R}^{n_j} \times \mathbb{R}^{n_j}
\end{array}
$$

and we see that $T\phi$ sends the filtration (3-6) to the following filtration:

(3-8) $$\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^{n-n_{\kappa-1}} \subset \cdots \subset \mathbb{R}^n \times \mathbb{R}^{n-n_0} \subset \mathbb{R}^n \times \mathbb{R}^n,$$

where $\mathbb{R}^{n-n_i}$ is included in $\mathbb{R}^n$ by the map $v \mapsto (v, 0) \in \mathbb{R}^{n-n_i} \times \mathbb{R}^{n_i} \simeq \mathbb{R}^n$. This property can be reformulated in terms of natural graduations associated with (3-6) and (3-8) (and will be used in this latter form). Indeed, let T^i be the orthogonal complement of K_i into K_{i-1} for all $i = 0, \ldots, \kappa$ (with the convention $K_{-1} = TU_z$). Moreover, on the Euclidean side, let us embed $\mathbb{R}^{n_i-n_{i-1}}$ into $\mathbb{R}^n$ by the map:

$$v \in \mathbb{R}^{n_i-n_{i-1}} \longmapsto (0, v, 0) \in \mathbb{R}^{n-n_i} \times \mathbb{R}^{n_i-n_{i-1}} \times \mathbb{R}^{n_{i-1}} \simeq \mathbb{R}^n$$

for all $i = 0, 1, \ldots, \kappa$ (by convention $n_{-1} = 0$). With these notation and conventions, the filtrations (3-6) and (3-8) give rise to the following decompositions:

(3-9) $$TU_z = T^\kappa \oplus T^{\kappa-1} \oplus \cdots \oplus T^0$$
(3-10) $$\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^{n-n_\kappa} \oplus \mathbb{R}^{n_\kappa-n_{\kappa-1}} \oplus \cdots \oplus \mathbb{R}^{n_1-n_0} \oplus \mathbb{R}^{n_0})$$

Now, that $T\phi$ respects the filtrations (3-6) and (3-8) means that for all $x \in U_z$ the linear map $T\phi_x$ is upper triangular with respect to the decompositions (3-9) and (3-10).

The *diagonal blocks* of $T\phi$ are the maps:

$$\delta^j \phi : T^j \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n_j-n_{j-1}}; \quad j = 0, 1, \ldots, \kappa,$$

obtained by composing $T\phi$ on the left and on the right respectively by the projections:

$$TU_z = T^\kappa \oplus T^{\kappa-1} \oplus \cdots \oplus T^0 \longrightarrow T^j$$
$$\mathbb{R}^n \times (\mathbb{R}^{n-n_\kappa} \oplus \mathbb{R}^{n_\kappa-n_{\kappa-1}} \oplus \cdots \oplus \mathbb{R}^{n_1-n_0} \oplus \mathbb{R}^{n_0}) \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n_j-n_{j-1}}.$$

The *diagonal part* of $T\phi$ will be defined by $\Delta\phi = (\delta^\kappa \phi, \delta^{\kappa-1}\phi, \ldots, \delta^0\phi)$. Of course, the inverse of $T\phi$ is also upper triangular with diagonal blocks given by $(\delta^j \phi)^{-1}$, $j = 0, 1, \ldots, \kappa$.

We have similar properties for all the underlying maps ϕ_i, $i = 0, 1, \ldots, \kappa-1$ coming with the distinguished chart. To fix notation and for future references, let U_i denote $\pi_{s_i}(U_z)$, and T_i^j denote $T\pi_{s_i}(T^j)$ for all $j \le i < \kappa$. Applying now $T\pi_{s_i}$ to (3-9) yields:

$$TU_i = T_i^i \oplus T_i^{i-1} \oplus \cdots \oplus T_i^0,$$

It follows that the differential maps:

$$T\phi_i : T_i^i \oplus T_i^{i-1} \oplus \cdots \oplus T_i^0 \longrightarrow \mathbb{R}^{n_i} \times (\mathbb{R}^{n_i-n_{i-1}} \oplus \cdots \oplus \mathbb{R}^{n_1-n_0} \oplus \mathbb{R}^{n_0})$$

for all $i = 0, 1, \ldots, \kappa-1$ are upper triangular with diagonal blocks $\delta^j \phi_i$ defined as above. Note that for all $j \le i \le k \le \kappa$, $(T\pi_{s_i})(T_k^j) = T_i^j$ and that applying the correct

restrictions and projections in (3-7) gives the following commutative diagram:

$$
\begin{array}{ccc}
T^j & \xrightarrow{\ \delta^j \phi_k\ } & \mathbb{R}^{n_k} \times \mathbb{R}^{n_j - n_j - 1} \\
{\scriptstyle T\pi_{s_i}}\Big\downarrow & & \Big\downarrow{\scriptstyle \sigma_{n_i} \times \mathrm{Id}} \\
T_i^j & \xrightarrow{\ \delta^j \phi_i\ } & \mathbb{R}^{n_i} \times \mathbb{R}^{n_j - n_j - 1}
\end{array}
$$

3.3.2 The smooth structure by induction   We show that $T^S X$ can be provided with a smooth structure by a simple recursive argument.

Let us first introduce the *s–exponential maps*. Let s be a stratum. The corresponding *s–exponential map* will be an exponential along the fibers of π_s. Precisely, recall that the map $\pi_s \colon \mathcal{N}_s^\circ \to s$ is a smooth submersion, $K_s \subset T\mathcal{N}_s^\circ$ denotes the kernel of the differential map $T\pi_s$ and $q_s \colon T\mathcal{N}_s^\circ \to T\mathcal{N}_s^\circ$ the orthogonal projection on K_s. The subbundle K_s of TX° inherits from TX° a Riemannian metric whose associated Riemannian connection is $\nabla^s = q_s \circ \nabla$, where ∇ is the Riemannian connection of the metric on X°. The associated exponential map

$$
\mathrm{Exp}^s \colon V_s \subset K_s \to \mathcal{N}_s^\circ
$$

is smooth and defined on an open neighborhood V_s of the zero section of K_s. Moreover it satisfies:

- $\pi_s \circ \mathrm{Exp}^s = \pi_s$.
- For any fiber L^s of π_s, the restriction of Exp^s to L^s is the usual exponential map for the submanifold L^s of X° with the induced Riemannian structure.

If X is a stratified pseudomanifold of depth 0 it is smooth and its S–tangent space is the usual tangent space TX equipped with its usual smooth structure.

Suppose that the S–tangent space of any stratified pseudomanifold of depth strictly smaller than k is equipped with a smooth structure for some integer $k > 0$. Let X be a stratified pseudomanifold of depth k and take $2X$ be the stratified pseudomanifold of depth $k - 1$ obtained from X by the unfolding process Section 2.3. According to Section 3, with the notation of Section 2.3 we have

$$
T^S X = T^S 2X|_{2X^\circ \cap X_b^+} \underset{s \in S_0}{\cup} {}^*\pi_s^*(Ts)|_{O_s^\circ} \rightrightarrows X^\circ .
$$

Let L° be the boundary of $2X^\circ \cap X_b^+$ in X°. We equip the restriction of $T^S X$ to $2X^\circ \cap X_b^+ \setminus L^\circ$ with the smooth structure coming from $T^S 2X$ and its restriction to any O_{s_0}, $s_0 \in S_0$, with the usual smooth structure. It remains to describe the gluing

over L°. One can find an open subset W of $T^S 2X$ which contains the restriction of $T^S 2X$ to L° such that the following map is defined:

$$\Theta: \quad W \quad \longrightarrow \quad T^S X$$
$$(x, u, y) \mapsto \begin{cases} (x, T\pi_{s_0}(u), \mathrm{Exp}^{s_0}(y, -\tau_{s_0}(x)q_{s_0}(u))) & \text{if } x \in O^\circ_{s_0}, \ s_0 \in S_0 \\ (x, u, y) & \text{elsewhere} \end{cases}$$

Here, if s denotes the unique stratum such that $x, y \in F_s$, the vector bundle $\pi_s^*(Ts)$ is identified with the orthogonal complement of K_s into $T\mathcal{N}^\circ_s$, in other words $q_{s_0}(y, u) = q_{s_0}(W - q_s(W))$ where $W \in T_y X^\circ$ satisfies $T\pi_s(W) = u$.

Then, we equip $T^S X$ with the unique smooth structure compatible with the one previously defined on $T^S X|_{X^\circ \setminus L^\circ}$ and such that the map Θ is a smooth diffeomorphism onto its image. The nontrivial point is to check that the restriction of the map Θ over $O^\circ_{s_0}$ is a diffeomorphism onto its image for any $s_0 \in S_0$. This will follows from the following lemma.

Lemma 6 *If $s_0 < s$, for any $x_0 \in s_0$ and $x \in s$ with $\pi_{s_0}(x) = x_0$. The following assertions hold:*

(1) $E := q_{s_0}(\pi_s^*(Ts))$ *is a subbundle of K_{s_0} of dimension $\dim(s) - \dim(s_0)$.*

(2) *Let $E^x := q_{s_0}(\pi_s^*(Ts))|_{\pi_s^{-1}(x)}$ be the restriction of E to the submanifold $\pi_s^{-1}(x)$. There exists a neighborhood W of the zero section of E^x such that the restriction of Exp^{s_0} to W is a diffeomorphism onto a neighborhood of $\pi_s^{-1}(x)$ in $\pi_{s_0}^{-1}(x_0)$.*

Proof (1) The first assertion follows from the inclusion: $\pi_{s_0}^*(Ts_0) = K_{s_0}^\perp \subset K_s^\perp = \pi_s^*(Ts)$ which ensures that the dimension of the fibers of $q_{s_0}(\pi_s^*(Ts))$ is constant equal to $\dim(s) - \dim(s_0)$.

The same argument shows that $K_{s_0} = K_s \oplus E$.

(2) If Ψ denotes the restriction of Exp^{s_0} to E^x then $T\Psi(z, 0)(U, V) = U + V$ where $(z, U) \in K_s$ and $V \in E_z$. Since $K_s \cap E$ is the trivial bundle we get that $T\Psi$ is injective and since E^x and $\pi_{s_0}^{-1}(x_0)$ have same dimension, it is bijective. We conclude with the local inversion theorem. $\square$

3.3.3 An atlas for $T^S X$

The atlas will contain two kinds of local charts. The kind of charts will depend on the fact that their domains meet or not a gluing between the different pieces composing the tangent space $T^S X$, that is the boundary of some F_s.

The first kind of charts, called *regular charts* are charts whose domain is contained in $T^S X|_{\mathrm{int}\, F_s}$ for a given stratum s of the stratification. We observe that $T^S X|_{\mathrm{int}\, F_s}$ is

a smooth groupoid as an open subgroupoid of $^*\pi_s^*(Ts) \rightrightarrows \mathcal{N}_s^\circ$. Thus, regular charts have domains contained in

$$\bigsqcup_{s\in\mathcal{S}} \text{int } F_s$$

and coincide with the usual local charts of the (disjoint) union of the smooth groupoids $\bigsqcup_{s\in\mathcal{S}} {}^*\pi_s^*(Ts)$.

The second kind of charts, called *deformation charts* (adapted to a stratum s), are charts whose domain meets $T^S X|_{\partial F_s}$ for a given stratum s, that is, charts around points in

$$\bigcup_{s\in S} T^S X|_{\partial F_s}.$$

Their description is more involved. Let $(p, u, q) \in T^S X$. Thus there is a stratum s such that p and q belong to F_s with $\pi_s(p) = \pi_s(q)$ and $u \in T_{\pi_s}(p)s$. Assume that $p \in \partial F_s$. This means that $\rho_s(p) < 1$, that $\rho_t(p) \geq 1$ for all strata $t < s$ and that the set of strata t such that $t < s$ and $\rho_t(p) = 1$ is not empty. Using again the axioms of the stratification, we see that this set is totally ordered and we denote $s_0, s_1, \ldots, s_{l-1}$ its elements listed by increasing order. We also set $s_l = s$. Observe that:

$$\{s_0, s_1, \ldots, s_l\} = \mathsf{S}_p \cap \{t \in \mathsf{S} \mid t \leq s\}$$

and that, thanks to the compatibility conditions (3-3), this set only depends on $\pi_s(p)$ and thus is equal with the corresponding set associated with q.

Let us take distinguished charts $\phi\colon U_p \to \mathbb{R}^n$ around p and $\phi'\colon U_q \to \mathbb{R}^n$ around q. Since $\pi_s(p) = \pi_s(q)$, we can also assume without loss of generality that:

(3-11) $\pi_{s_i}(U_p) = \pi_{s_i}(U_q)$ and $\phi_i = \phi_i'$ for $i = 0, \ldots, l$.

We will use the same notation as in Section 3.3.1: $n_i = \dim s_i$, $U_i = \pi_{s_i}(U_p)$, $K_i = \ker(T\pi_{s_i})|_{U_p}$, $T^i = K_i^{\perp K_{i-1}}$ for all $i = 0, 1, \ldots, l$ (here again $K_{-1} = TU_p$). The main difference with the settings of Section 3.3.1 is that we forget the strata bigger than s in S_p and S_q to concentrate on the lower (and common) strata in S_p and S_q. It amounts to forget the tail of the filtration (3-6) up to the term K_l:

$$K_l \subset K_{l-1} \subset \cdots \subset K_0 \subset TU_p$$

and this leads to a less fine graduation:

$$TU_p = K_l \oplus T^l \oplus T^{l-1} \oplus \cdots \oplus T^0$$

Let us also introduce the positive smooth functions:

$$t_i = \sum_{j=0}^{i} \tau \circ \rho_{s_j} \ , \ i = 0, 1, \ldots, l \ ; \qquad \theta_i = \prod_{j=i-1}^{l} t_j \ , \ i = 1, \ldots, l$$

Note that t_j (resp. θ_j) is strictly positive on F_{s_i} if $j \geq i$ (resp. $j > i$) and vanishes identically if $j < i$ (resp. $j \leq i$).

Finally we will write:

$$\forall x \in U_p, \quad \phi(x) = (x^{l+1}, x^l, \ldots, x^1, x^0) \in \mathbb{R}^{n-n_l} \times \mathbb{R}^{n_l-n_{l-1}} \times \cdots \times \mathbb{R}^{n_1-n_0} \times \mathbb{R}^{n_0} ,$$

and for all $j = 0, 1, \ldots, l$,

$$\pi_{s_j}(x) = x_j \ ,$$

thus $\phi_j(x_j) = (x^j, x^{j-1}, \ldots, x^0) \in \mathbb{R}^{n_j}$; and we adopt similar notation for ϕ' and $y \in U_q$.

We can now define a deformation chart around the point (p, u, q). The domain will be:

$$\tilde{U} = T^S X|_{U_q}^{U_p}$$

and the chart itself:

$$\tilde{\phi} \colon \tilde{U} \to \mathbb{R}^{2n}$$

is defined as follows. Up to a shrinking of U_p and U_q, the following is true: for all $(x, v, y) \in \tilde{U}$, there exists a unique $i \in \{0, 1, \ldots, l\}$ such that $x \in F_{s_i}$. Then $(x, v) \in \pi_{s_i}^*(T U_i)$, and we set:

$$\tilde{\phi}(x, v, y) = \left(\phi(x), \frac{x^{l+1} - y^{l+1}}{\theta_{l+1}(x)}, \ldots, \frac{x^{i+1} - y^{i+1}}{\theta_{i+1}(x)}, \Delta\phi_i(x_i, v) \right)$$

The map $\tilde{\phi}$ is clearly injective with inverse defined as follows. For $(\mathbf{x}, \mathbf{w}) \in \tilde{\phi}(\tilde{U})$ and i such that $\phi^{-1}(\mathbf{x}) \in F_{s_i}$:

$$\tilde{\phi}^{-1}(\mathbf{x}, \mathbf{w}) = \left(\phi^{-1}(\mathbf{x}), (\Delta\phi_i)^{-1}(\mathbf{x}_i, \mathbf{w}), \phi'^{-1}(\mathbf{x} - \Theta^{[i+1]}(\phi^{-1}(\mathbf{x})) \cdot \mathbf{w}) \right)$$

where $\mathbf{x}_i = \sigma_{n_i}(\mathbf{x})$ and, using the decomposition

$$\mathbf{w} = (\mathbf{w}^{l+1}, \mathbf{w}^l \cdots, \mathbf{w}^0) \in \mathbb{R}^{n-n_l} \times \mathbb{R}^{n_l-n_{l-1}} \times \cdots \times \mathbb{R}^{n_1-n_0} \times \mathbb{R}^{n_0},$$

we have set

$$\Theta^{[i+1]}(x) \cdot \mathbf{w} = \theta_{l+1}(x)\mathbf{w}^{l+1} + \cdots + \theta_{i+1}(x)\mathbf{w}^{i+1} \in \mathbb{R}^{n-n_i} \times \{0\} \subset \mathbb{R}^n.$$

To ensure that $(\tilde{\phi}, \tilde{U})$ is a local chart, it remains to check that $\tilde{\phi}(\tilde{U})$ is an open subset of $\mathbb{R}^{2n}$. It is easy to see that $\tilde{\phi}(\text{int } F_{s_i})$ is open for every $i \in \{0, \ldots, l\}$ so we

consider $(p, u, q) \in \tilde{U}$ such that $p \in \partial F_{s_i}$ for some integer i. Let $J = \{i_0, \ldots, i_k\} \subset \{0, 1, \ldots, i-1\}$ such that:

$$\forall j \in J, \quad \rho_{s_j}(p) = 1 .$$

Thus we have:

$$\rho_{s_i}(p) < 1; \quad \forall j \in J, \ \rho_{s_j}(p) = 1; \quad \forall j \notin J \text{ and } j < i, \ \rho_{s_j}(p) > 1$$

by construction, q satisfies the same relations. Set $\tilde{\phi}(p, u, q) = (\mathbf{x}_0, \mathbf{v}_0)$. Using the Taylor formula and the fact that θ_{j+1} is negligible with respect to $1 - \rho_{s_j}$ at the region $\rho_{s_j} = 1$, noting also the invariance of ρ_{s_k} with respect to perturbations of points along the fibers of $\pi_{s_{k+1}}, \pi_{s_{k+2}}, \ldots$; we prove that there exist an open ball B_1 of $\mathbb{R}^n$ centered at $\mathbf{x}_0$ and an open ball B_2 of $\mathbb{R}^n$ centered at 0 and containing $\mathbf{v}_0$ such that for all $(\mathbf{x}, \mathbf{v}) \in B_1 \times B_2$, if

$$x = \phi^{-1}(\mathbf{x}) \in F_{s_j} \text{ for } j \in J \text{ or } j = i, \text{ then } y = \phi'^{-1}(\mathbf{x} - \Theta^{[j+1]}(x) \cdot \mathbf{v}) \in F_{s_j} .$$

This proves that $(\mathbf{x}, \mathbf{v}) \in \operatorname{Im} \tilde{\phi}$, thus

$$\tilde{\phi}(p, u, q) \in B_1 \times B_2 \subset \operatorname{Im} \tilde{\phi}$$

and the required assertion is proved. We end with:

Theorem 1 *The collection of regular and deformation charts provides $T^S X$ with a structure of smooth groupoid.*

Proof The compatibility between a regular and a deformation chart contains no issue and is omitted. We need only to check the compatibility between a deformation chart adapted to a stratum s and a deformation chart adapted to a stratum t, when their domains overlap, which implies automatically that $s < t$ or $s > t$ or $s = t$.

Let us work out only the case $s = t$, since the other case is similar. We have here to compare two charts $\tilde{\phi}$ and $\tilde{\psi}$ with common domain $\tilde{U}$ and involving the same chain of strata $s = s_l > s_{l-1} > \cdots > s_0$. The whole notation are as before and ψ, ψ' are the underlying charts of X° allowing the definition of $\tilde{\psi}$. We note, for the sake of concision, u^k (resp. u'^k), $k = l+1, \ldots, 0$, the coordinate functions of $u := \psi \circ \phi^{-1}$ (resp. $\psi' \circ \phi'^{-1}$) with respect to the decomposition (3-10) of $\mathbb{R}^n$. Observe, thanks to the particular assumptions made on ϕ, ϕ', ψ, ψ' (cf (3-5), (3-11)), that $u^k(\mathbf{x})$ only depends on $\mathbf{x}_k := (\mathbf{x}^k, \mathbf{x}^{k-1}, \ldots, \mathbf{x}^0) \in \mathbb{R}^{n_k}$ and that $u^k = u'^k$ for all $k < l+1$. Let $(\mathbf{x}, \mathbf{v}) \in \operatorname{Im} \tilde{\phi}$ and i such that $x = \phi^{-1}(\mathbf{x}) \in F_{s_i}$. Then:

$$\tilde{\psi} \circ \tilde{\phi}^{-1}(\mathbf{x}, \mathbf{v}) = \left(u(\mathbf{x}), \frac{u^{l+1}(\mathbf{x}) - u'^{l+1}(\mathbf{x} - \Theta^{[l+1]} \cdot \mathbf{v})}{\theta^{l+1}}, \frac{u^l(\mathbf{x}) - u^l(\mathbf{x} - \Theta^{[l+1]} \cdot \mathbf{v})}{\theta^l}, \ldots \right.$$
$$\left. \ldots, \frac{u^{i+1}(\mathbf{x}) - u^{i+1}(\mathbf{x} - \Theta^{[l+1]} \cdot \mathbf{v})}{\theta^{i+1}}, (\Delta \psi_i) \circ (\Delta \phi_i)^{-1}(\mathbf{v}) \right)$$

We need to check that the above expression matches smoothly with the corresponding expression for an integer $k \in [i, l]$ when $\theta_k(x)$ (and thus $\theta_{k-1}, \ldots, \theta_{i+1}$) goes to zero. For that, the Taylor formula applied to u^r, $k \geq r \geq i+1$, shows that the map defined below is smooth in $(\mathbf{x}, \mathbf{v}, t)$ where $(\mathbf{x}, \mathbf{v})$ are as before and $t = (t_l, t_{l-1}, \ldots, t_0) \in \mathbb{R}^{l+1}$ is this time an arbitrary $(l+1)$–uple close to 0:

$$
\begin{cases}
\frac{u^r(\mathbf{x}) - u^r(\mathbf{x} - \Theta^{[i+1]} \cdot \mathbf{v})}{\theta^r} & \text{if } \theta_r = \Pi_{r-1}^l t_j \neq 0 \\
d(u^r)_{\mathbf{x}}(\mathbf{v}^r + t_{r-2}\mathbf{v}^{r-1} + \cdots + t_i \mathbf{v}^{i+1}) & \text{if } \exists j \in \{r-1, r, \ldots, l\} \text{ such that } t_j = 0.
\end{cases}
$$

In our case, $t_j = t_j(x)$ and $t_{k-1}, \ldots, t_i$ go to zero, so the second line in the previous expression is just:

$$
d(u^r)_{\mathbf{x}}(\mathbf{v}^r)
$$

and for obvious matricial reasons:

$$
d(u^r)_{\mathbf{x}}(\mathbf{v}^r) = (\Delta \psi_k) \circ (\Delta \phi_k)^{-1}(\mathbf{v}^r)
$$

Summing up these relations for $r = i+1, \ldots, k$, we arrive at the desired identity.

Thus, $T^S X$ is endowed with a structure of smooth manifold. Changing the Riemannian metric on X° modifies the choices of the T_j^i's, but gives rise to compatible charts. Moreover, the smoothness of all algebraic operations associated with this groupoid is easy to check in these local charts. □

3.3.4 The Lie algebroid of the tangent groupoid

We describe here the smooth structure of the tangent space via its infinitesimal structure, namely its Lie algebroid. Precisely, we define

$$
\begin{array}{rcl}
Q_s: & TX^\circ & \longrightarrow \quad TX^\circ \\
& (z, V) & \mapsto \begin{cases} (z, \tau_s(z) q_s(z, V)) & \text{if } z \in \mathcal{N}_s^\circ \\ 0 & \text{elsewhere} \end{cases}
\end{array}
$$

By a slight abuse of notation, we will keep the notation q_s and Q_s for the corresponding maps induced on the set of local tangent vector fields on X°.

Let $\mathcal{A}$ be the smooth vector bundle $\mathcal{A} := TX^\circ \times [0, 1]$ over $X^\circ \times [0, 1]$. We define the following morphism of vector bundle:

$$
\begin{array}{rcl}
\Phi: & \mathcal{A} = TX^\circ \times [0, 1] & \longrightarrow \quad TX^\circ \times T[0, 1] \\
& (z, V, t) & \mapsto \quad (z, tV + \sum_{s \in S_{\text{sing}}} Q_s(z, V); t, 0)
\end{array}
$$

In the sequel we will give an idea of how one can show that there is a unique structure of Lie algebroid on $\mathcal{A}$ such that Φ is its anchor map. The Lie algebroid $\mathcal{A}$ is almost injective and so it is integrable, moreover we will see that at a set level $\mathcal{G}_X^t$ must be a

groupoid which integrates it [8; 10]. In particular $\mathcal{G}_X^t$ can be equipped with a unique smooth structure such that it integrates the Lie algebroid $\mathcal{A}$.

Now we can state the following:

Theorem 2 *There exists a unique structure of Lie algebroid on the smooth vector bundle $\mathcal{A} = TX° \times [0, 1]$ over $X° \times [0, 1]$ with Φ as anchor.*

To prove this theorem we will need several lemmas:

Lemma 7 *Let s_0 and s_1 be two strata such that $d(s_0) \leq d(s_1)$.*

(1) *For any tangent vector field W on $X°$, $Q_{s_1}(W)(\tau_{s_0}) = 0$.*

(2) *For any $(z, V) \in TX°$, the following equality holds:*

$$Q_{s_1} \circ Q_{s_0}(z, V) = Q_{s_0} \circ Q_{s_1}(z, V) = \tau_{s_0}(z) Q_{s_1}(z, V) .$$

Proof First notice that outside $O_{s_0} \cap O_{s_1}$ either Q_{s_1} hence $Q_{s_1}(W)$ or τ_{s_0} and Q_{s_0} vanish thus the equalities in (1) and (2) are simply $0 = 0$.

(1) According to the compatibility conditions (3-3) we have $\rho_{s_0} \circ \pi_{s_1} = \rho_{s_0}$ on $O_{s_0} \cap O_{s_1}$. Thus ρ_{s_0} is constant on the fibers of π_{s_1} and since $\tau_{s_0} = \tau \circ \rho_{s_0}$, τ_{s_0} is also constant on the fibers of π_{s_1}. For any tangent vector field W, and any $z \in O_{s_1}°$ the vector $Q_{s_1}(W)(z)$ is tangent to the fibers of π_{s_1} thus $Q_{s_1}(V)(\tau_{s_0}) = 0$ on $O_{s_0} \cap O_{s_1}$.

(2) The result follows from the first remark and the equality (3-6) of the part above. □

The next lemma ensures that Φ is almost injective, in particular it is injective in restriction to $X° \times]0, 1]$. A simple calculation shows the following:

Lemma 8 *For any $t \in]0, 1]$ the bundle map Φ_t is bijective, moreover*

$$\Phi_t^{-1}(z) = \frac{1}{t} V - \sum_{s \in S_{\text{sing}}} \frac{1}{(t + t_s(z)) \cdot (t + t_s(z) - \tau_s(z))} Q_s(z, V)$$

where for any singular stratum s the map t_s is defined as follows:

$$t_s \colon X° \to \mathbb{R} , \quad t_s(z) = \sum_{s_0 \leq s}^{l} \tau_{s_0}(z) .$$

Thus in order to prove the Theorem 2 it is enough to show that locally the image of the map induced by Φ from the set of smooth local sections of $\mathcal{A}$ to the set of smooth local tangent vector fields on $X° \times [0, 1]$ is stable under the Lie bracket.

Idea of the proof of Theorem 2   First notice that outside the closure of $\bigcup O^\circ_{s_i}$, where $s_i \in \mathsf{S}_{\text{sing}}$, the image under Φ of local tangent vector fields is clearly stable under Lie Bracket.

Thus using decomposition of the form (3-9) described in the last part and standard arguments it remains to show that if s_a and s_b are strata of depth respectively a and b with $s_a \leq s_b$, if U is an open subset of X°, as small as we want contained in $\mathcal{N}_{s_a} \cap \mathcal{N}_{s_b}$, and if $W^\perp, V^\perp, V_a$ and W_b are tangent vector fields on U, satisfying:

$$V^\perp \text{ and } V_a \text{ can be projected by } \pi_{s_b},$$

$$Q_s(W^\perp) = Q_s(V^\perp) = 0 \text{ for any } s \in \mathsf{S},$$

$$Q_s(V_a) = \begin{cases} \tau_s V_a & \text{when } s \leq s_a \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad Q_s(W_a) = \begin{cases} \tau_s W_a & \text{when } s \leq s_b \\ 0 & \text{elsewhere} \end{cases},$$

then $[\Phi(W^\perp + W_b), \Phi(V^\perp)]$ and $[\Phi(W_b), \Phi(V_a)]$ are in the image of Φ. In other word, we have to show that the maps

$$(z,t) \in X^\circ \times]0,1] \mapsto (\Phi_t^{-1}([\Phi(W^\perp + W_b), \Phi(V^\perp)](z)), t)$$

and $\qquad (z,t) \in X^\circ \times]0,1] \mapsto (\Phi_t^{-1}([\Phi(W_b), \Phi(V_a)](z)), t)$

can be extended into smooth local sections of $\mathcal{A}$. The result follows from our preceding lemmas and usual calculations. $\qquad\square$

Now we can state:

Theorem 3   *The groupoid* $\mathcal{G}^t_X$ *can be equipped with a smooth structure such that its Lie algebroid is* $\mathcal{A}$ *with* Φ *as anchor.*

Proof   According to Theorem 2 and Lemma 8, the Lie algebroid $\mathcal{A}$ is almost injective. Thus according to [10] there is a unique s–connected quasi-graphoid $\mathcal{G}(\mathcal{A}) \rightrightarrows X^\circ \times [0,1]$ which integrates $\mathcal{A}$. Suppose for simplicity that for each stratum s, O°_s is connected (which will ensure that $\mathcal{G}^t_X|_{F^\circ_s \times [0,1]}$ is a s–connected quasi-graphoid).

Moreover the map Φ satisfies:

(i) Φ induces an isomorphism from $\mathcal{A}_{]0,1]} := \mathcal{A}|_{X^\circ \times]0,1]}$ to $TX^\circ \times]0,1]$.

(ii)   For any stratum s, the Lie algebroid $\mathcal{A}$ restricts over $F^\circ_s \times [0,1]$ to a Lie algebroid $\mathcal{A}_s := \mathcal{A}|_{F^\circ_s \times [0,1]}$ which is isomorphic to the restriction of $\mathcal{A}^t_{\pi_s \times \mathrm{Id}}$ over $F^\circ_s \times [0,1]$.

Thus, again by using the uniqueness of s–connected quasi-graphoid integrating a given almost injective Lie algebroid, we obtain:

(i)   The restriction of the groupoid $\mathcal{G}(\mathcal{A})$ over $X^\circ \times \,]0,1]$ is isomorphic to $X^\circ \times X^\circ \times \,]0,1] \rightrightarrows X^\circ \times \,]0,1]$, the pair groupoid on X° parametrized by $]0,1]$.

(ii) To each stratum s the restriction over $F_s^\circ \times [0,1]$ is equal to $\mathcal{G}_X^t|_{F_s^\circ \times [0,1]}$.

Finally $\mathcal{G}(\mathcal{A}) = \mathcal{G}_X^t$ and there is a unique smooth structure on $\mathcal{G}_X^t$ such that $\mathcal{A}$ is its Lie algebroid.

If some O_s° is not connected, we replace in the construction of the tangent space the groupoid ${}^*\pi_s^*(Ts)|_{F_s}$ by its s–connected component. Let CT^SX and $C\mathcal{G}_X^t$ be the corresponding groupoids. The previous arguments apply and the groupoid $C\mathcal{G}_X^t$ admits a unique smooth structure such that $\mathcal{A}$ is its Lie algebroid. One can then show that there is a unique smooth structure on $\mathcal{G}_X^t$ such that $C\mathcal{G}_X^t$ is its s–connected component. Precisely, according to [10] there is a quasi-graphoid $\mathcal{GI}(\mathcal{A}) \rightrightarrows X^\circ \times [0,1]$ which integrates $\mathcal{A}$ and is maximal for the inclusion among quasi-graphoids which integrate $\mathcal{A}$. The groupoid $C\mathcal{G}_X^t$ is then the s–connected component of $\mathcal{GI}(\mathcal{A})$. In particular it is open in $\mathcal{GI}(\mathcal{A})$. Let $X^r := X^\circ \setminus \bigcup_{s \in S} F_s$. The restriction of $\mathcal{G}_X^t$ to $X^r \times [0,1]$ is a quasi-graphoid which integrates the restriction of $\mathcal{A}$ to $X^r \times [0,1]$ and is then clearly an open subgroupoid of $\mathcal{GI}(\mathcal{A})$. Now we have that $\mathcal{G}_X^t$ equals $\{\gamma \cdot \eta \mid \gamma \in C\mathcal{G}_X^t, \ \eta \in \mathcal{G}_X^t|_{X^r \times [0,1]}, \ s(\gamma) = r(\eta)\}$ which is open in $\mathcal{GI}(\mathcal{A})$ and so $\mathcal{G}_X^t$ inherits the required smooth structure. $\square$

Thus T^SX, which is the restriction of $\mathcal{G}_X^t$ to the saturated set $X^\circ \times \{0\}$, inherits from $\mathcal{G}_X^t$ a smooth structure which is equivalent to the one described in previous paragraphs.

3.3.5 Standard projection from the tangent space onto the space
The space of orbits of X°/T^SX is equivalent to X in the sense that there is a canonical isomorphism $C_0(X^\circ/T^SX) \simeq C(X)$.

Definition 6   Let $r, s\colon T^SX \to X^\circ$ be the target and source maps of the S–tangent space of X. A continuous map $p\colon X \to X$ is a *standard projection* for T^SX on X if:

(1) $p \circ r = p \circ s$.

(2) p is homotopic to the identity map of X.

A standard projection p for T^SX on X is *surjective* if $p|_{X^\circ}\colon X^\circ \to X$ is onto.

This definition leads to the following:

Lemma 9   (1)   *There exists a standard surjective projection for T^SX on X.*

(2)   *Two standard projections are homotopic and the homotopy can be done within the set of standard projections.*

Proof (1) If X has depth 0, $X^\circ = X$ and we just take $p = \text{Id}$. Let us consider X with depth $k > 0$. Choose a smooth nondecreasing function $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $f([0,1]) = 0$ and $f|_{[2,+\infty[} = \text{Id}$. Recall that there exists for each singular stratum s an isomorphism (2-1):

$$\Psi_s \colon \mathcal{N}_s \to c_{\pi_s} L_s = L_s \times [0, +\infty[/ \sim_s .$$

we define the map

$$p_s \colon \mathcal{N}_s \longrightarrow \mathcal{N}_s$$

by the formula:

$$\Psi_s \circ p_s \circ \Psi_s^{-1}[x, t] = [x, f(t)] .$$

For each integer $i \in [0, k-1]$, we define a continuous map:

$$p_i \colon X \longrightarrow X$$

by setting $p_i(z) = p_s(z)$ if z belongs to $\mathcal{N}_s$ for some singular stratum of depth i and $p_i(z) = z$ elsewhere. In particular, $p_i|_{O_s} = \pi_s$ for every stratum s of depth i. Finally we set:

$$p = p_0 \circ p_1 \circ \cdots \circ p_{k-1} .$$

This is the map we looked for. Indeed:

Let $\gamma \in T^S X$. There exists a unique stratum s such that $\gamma \in {}^*\pi_s^*(TS)$. If s is regular, then $r(\gamma) = s(\gamma)$ so the result is trivial here. Let us assume that s is singular and let $i < k$ be its depth. By definition, $r(\gamma)$ and $s(\gamma)$ belong to O_s. For each stratum $t \geq s$ of depth $j \geq i$, we have everywhere it makes sense:

$$\pi_t \circ p_t = \pi_t, \quad \pi_s \circ \pi_t = \pi_s, \quad \rho_s \circ \pi_t = \rho_s$$

thus:
$$\rho_s \circ p_t = \rho_s \circ \pi_t \circ p_t = \rho_s \circ \pi_t = \rho_s$$

which proves that $p_j(O_s) = O_s$, and moreover:

$$\pi_s \circ p_t = \pi_s \circ \pi_t \circ p_t = \pi_s \circ \pi_t = \pi_s$$

Recalling that $p_i|_{O_s} = \pi_s|_{O_s}$, this last relation implies:

$$p_i \circ \cdots \circ p_{k-1}|_{O_s} = \pi_s \circ p_{i+1} \circ \cdots \circ p_{k-1}|_{O_s} = \pi_s|_{O_s}$$

Since by definition we also have $\pi_s(r(\gamma)) = \pi_s(s(\gamma))$, we conclude that:

$$p(r(\gamma)) = p_0 \circ \cdots p_{i-1} \circ \pi_s(r(\gamma)) = p_0 \circ \cdots p_{i-1} \circ \pi_s(s(\gamma)) = p(s(\gamma))$$

If in the definition of p, we replace the function f by $t \, \text{Id}_{\mathbb{R}_+} + (1-t) f$, we get a homotopy between p and Id_X.

Finally, p has the required surjectivity property: $p_{k-1}(X^\circ) = X^\circ \cup_{d(s)=k-1} s$ and for all j we have the equality $p_{j-1}(X^\circ \cup_{d(s)\geq j} s) = X^\circ \cup_{d(s)\geq j-1} s$.

(2) Let q be a standard projection and p be the standard projection built in (1). Let also q_t be a homotopy between q and Id_X and p_t the homotopy built in (1) between p and Id_X. Observe that $q_t \circ p$ is a standard projection, providing a path of standard projections between $q \circ p$ and p. Moreover, by construction of p_t, the inclusion $\mathrm{Im}(p_t \circ r, p_t \circ s) \subset \mathrm{Im}(r,s)$ holds for any $1 \geq t > 0$, thus $q \circ p_t$ is a standard projection, providing a path of standard projections between $q \circ p$ and q. Thus, any standard projection q is homotopic to p within the set of standard projections and the result is proved. □

Remark 3 Let p be the surjective standard projection built in the proof of the last proposition. The map $p \circ r \colon T^S X \to X$ provides $T^S X$ with a structure of continuous field of groupoids. Following the arguments of [11, Remark 5], it can be shown that each fiber of this field is amenable, thus $T^S X$ is amenable and $C^*(T^S X) = C_r^*(T^S X)$ is nuclear. The same holds for $\mathcal{G}_X^t$ and all other deformation groupoids used below.

4   Poincaré duality for stratified pseudo-manifolds

Let X be a compact stratified pseudomanifold of depth $k \geq 0$.

The tangent groupoid $\mathcal{G}_X^t$ is a deformation groupoid, thus it provides us with a $K-$homology class, called a *pre-Dirac* element:

$$\delta_X = [e_0]^{-1} \otimes [e_1] \in KK(C^*(T^S X), \mathbb{C}) .$$

Here $e_0 \colon C^*(\mathcal{G}_X^t) \to C^*(T^S X)$ and $e_1 \colon C^*(\mathcal{G}_X^t) \to \mathcal{K}(L^2(X^\circ))$ are the usual evaluation homomorphisms. Now we need:

Lemma 10   (1)   *Let* $p \colon X^\circ \to X$ *be a surjective standard projection for* $T^S X$. *The formula:*

$$\forall a \in C^*(T^S X), \ f \in C(X), \ \gamma \in T^S X, \quad (a \cdot f)(\gamma) = f(p \circ r(\gamma)).a(\gamma)$$

defines a $C(X)$–*algebra structure on* $C^*(T^S X)$.

(2)   *For any standard projection* p *for* $T^S X$, *the formula:*

$$\forall a \in C^*(T^S X), \ f \in C(X), \ \gamma \in T^S X, \quad \Psi_X(a \cdot f)(\gamma) = f(p \circ r(\gamma)).a(\gamma)$$

defines a homomorphism $\Psi_X \colon C^*(T^S X) \otimes C(X) \to C^*(T^S X)$ *whose class* $[\Psi_X] \in KK(C^*(T^S X) \otimes C(X), C^*(T^S X))$ *does not depend on the choice of* p.

The last assertion uses [Lemma 9](#). Note that if $k = 0$, X is smooth and we can choose $p = \mathrm{Id}$, thus:
$$\Psi_X(a \otimes b)(V) = b(x).a(x, V)$$
for all $V \in T_x X$, $x \in X$, $a \in C(X)$ and $b \in C^*(TX)$.

From now on, we choose a surjective standard projection and denote by $\Psi_X \colon C(X) \otimes C^*(T^S X) \to C^*(T^S X)$ the homomorphism defined in the previous lemma. We set:
$$D_X = \Psi_X^*(\delta_X) = [\Psi_X] \otimes \delta_X \in KK(C^*(T^S X) \otimes C(X), \mathbb{C}).$$

This section is devoted to the proof of the main theorem:

Theorem 4 *Let X be a compact stratified pseudomanifold. The K–homology class D_X is a Dirac element, that is, it provides a Poincaré duality between the algebras $C^*(T^S X)$ and $C(X)$.*

We need some notation. If W is an open set of the stratified pseudomanifold X and $\overline{W}$ its closure, we set:
$$T^S W = T^S X|_{W^\circ}, \quad T^S \overline{W} = T^S X|_{\overline{W}^\circ} \quad \text{and} \quad \mathcal{G}_W^t = \mathcal{G}_X^t|_{W^\circ \times [0,1]}.$$

The groupoid $\mathcal{G}_W^t$ is a deformation groupoid which defines the K–homology class $\delta_W \in K^0(C^*(T^S W))$. We define the homomorphisms induced by Ψ_X:
$$\widehat{\Psi}_W \colon C^*(T^S \overline{W}) \otimes C_0(W) \to C^*(T^S W)$$
$$\widehat{\Psi}_{\overline{W}} \colon C^*(T^S W) \otimes C(\overline{W}) \to C^*(T^S W)$$

and we set $\Psi_W = i_W \circ \widehat{\Psi}_W$ and $\Psi_{\overline{W}} = i_W \circ \widehat{\Psi}_{\overline{W}}$ where $i_w \colon C^*(T^S W) \to C^*(T^S X)$ is the natural homomorphism. Finally we let:
$$D_W = (\widehat{\Psi}_W)^*(\delta_W) = (\Psi_W)^*(\delta_X) \in KK(C^*(T^S \overline{W}) \otimes C_0(W), \mathbb{C})$$
$$D_{\overline{W}} = (\widehat{\Psi}_{\overline{W}})^*(\delta_W) = (\Psi_{\overline{W}})^*(\delta_X) \in KK(C^*(T^S W) \otimes C(\overline{W}), \mathbb{C}).$$

In the sequel, we will be interested in the disjoint open sets:

(4-1) $$O_- = \bigcup_{s \in S_0} \{z \in \mathcal{N}_s \mid \rho_s(z) < 2\} \text{ and } O_+ = X \setminus \overline{O_-},$$

as well as in the intersection of their closures:
$$L = \overline{O_+} \cap \overline{O_-} = \bigcup_{s \in S_0} \{z \in X \mid \rho_s(z) = 2\}.$$

We recall from [Section 2.3](#) that S_0 denotes the set of minimal strata.

Proof of Theorem 4 It will be proved by induction on the depth of the stratification and the unfolding process will be used to reduce the depth.

If $\text{depth}(X) = 0$ the content of the theorem is well known, and that D_X is a Dirac element is a consequence of [11]. Let $k \geq 0$, assume that the Theorem 4 holds for all compact stratified pseudomanifolds with depth $\leq k$ and let X be a compact stratified pseudomanifold of depth $k + 1$. The proof of the induction is divided in two parts.

First part of the proof We consider two natural "restrictions" of D_X, namely $D_{O_+} \in K^0(C^*(T^S\overline{O_+}) \otimes C_0(O_+))$ and $D_{\overline{O_-}} \in K^0(C^*(T^S O_-) \otimes C(\overline{O_-}))$. Then, we reduce the proof of the theorem to checking that D_{O_+} is a Dirac element.

Let O_0 be the open set of X obtained by replacing the condition $\rho_s < 2$ by $\rho_s < 1$ in the definition of O_- in (4-1). The C^*–algebra $C^*(T^S O_0)$ is a closed two-sided ideal in $C^*(T^S O_-)$ and the quotient

$$C^*(T^S O_-)/C^*(T^S O_0) \simeq C_0([1,2[) \otimes C^*(T^S L \times \mathbb{R})$$

is contractible in K–theory. It follows that the inclusion $C^*(T^S O_0) \subset C^*(T^S O_-)$ is a KK–equivalence which sends δ_{O_0} to δ_{O_-}. This is obvious once we consider the corresponding tangent groupoids $\mathcal{G}^t_{O_0}$, $\mathcal{G}^t_{O_-}$. As already noted there is a natural Morita equivalence between the groupoid $T^S O_0 = \bigcup_{s \in S_0} {}^*\pi_s^*(Ts)|_{O_s}$ and the tangent space $TS = \bigcup_{s \in S_0} Ts$ of the closed smooth manifold $S = \bigcup_{s \in S_0} s$. Under this Morita equivalence, δ_{O_0} corresponds to δ_S: this follows from the extension of the previous Morita equivalence to the tangent groupoids $\mathcal{G}^t_{O_0}$ and $\mathcal{G}^t_S$. Moreover the control data provide a homotopy equivalence between $\overline{O_-}$ and S and we finally get a KK–equivalence between $C^*(T^S O_-) \otimes C(\overline{O_-})$ and $C^*(T^S S) \otimes C(S)$ under which the class $[\Psi_{\overline{O_-}}]$ coincides with the class $[\Psi_S]$. We have proved:

Lemma 11 *There is a KK–equivalence between $C^*(T^S O_-) \otimes C(\overline{O_-})$ and $C^*(T^S S) \otimes C(S)$ under which the Dirac element D_S corresponds to $D_{\overline{O_-}}$. In particular, $D_{\overline{O_-}}$ is a Dirac element.*

We now apply Lemma 4 to the nuclear $C(X)$–algebra $C^*(T^S X)$, the disjoint open subsets O_- and O_+ and the K–homology class δ_X. Since by Lemma 11 $D_{\overline{O_-}}$ is a Dirac element, we immediately get:

$$D_X \text{ is a Dirac element if and only if } D_{O_+} \text{ is.}$$

Second part of the proof We check that D_{O_+} is a Dirac element. Let us go back to the compact pseudomanifold of depth k coming from the unfolding process: $2X$.

Modifying slightly the definition of Section 2.3, we set:

$$2X = \overline{O_+} \underset{L}{\cup} L \times [-2, +2] \underset{L}{\cup} \overline{O_+}$$

We consider this time the disjoint open subsets $U = L \times]-2, +2[$ and $V = 2X \setminus L \times [-2, +2] = O_+ \sqcup O_+$ of the pseudomanifold $2X$. Let us introduce as before the homomorphisms induced by Ψ_{2X}:

$$\Psi_{\overline{U}} \colon C^*(T^S U) \otimes C(\overline{U}) \longrightarrow C^*(T^S 2X)$$

and

$$\Psi_V \colon C^*(T^S \overline{V}) \otimes C_0(V) \longrightarrow C^*(T^S 2X)$$

where $T^S U = T^S 2X|_{U^\circ}$ and $T^S \overline{V} = T^S 2X|_{\overline{V}^\circ}$. Note that under the natural identification $C^*(T^S \overline{V}) \otimes C_0(V) \simeq M_2(C^*(T^S \overline{O_+}) \otimes C_0(O_+))$, the homomorphism Ψ_V has the following diagonal form: $\Psi_V = \mathrm{diag}(\Psi_{O_+}, \Psi_{O_+})$.

We shall consider three K–homology classes:

$$D_{2X} = \Psi_{2X}^*(\delta_{2X}), \quad D_{\overline{U}} = \Psi_{\overline{U}}^*(\delta_{2X}), \quad D_V = \Psi_V^*(\delta_{2X}).$$

Since $2X$ is a compact stratified pseudomanifold of depth k, we know by induction hypothesis that D_{2X} is a Dirac element.

The space L with the stratification induced by X is also a compact stratified pseudomanifold of depth k. So it has a Dirac element D_L defined as before. Observe that $\Psi_{\overline{U}}$ has range in the ideal $C^*(T^S U)$ of $C^*(T^S 2X)$. We note $\widehat{\Psi}_{\overline{U}}$ the induced homomorphism, $i_U \colon C^*(T^S U) \to C^*(T^S 2X)$ the inclusion and δ_U the KK–element associated with the deformation groupoid $\mathcal{G}_U^t := \mathcal{G}_{2X}^t|_{U^\circ}$. We have $\delta_U = (i_U)^*(\delta_{2X})$, hence $D_{\overline{U}} = (\widehat{\Psi})_{\overline{U}}^*(\delta_U)$. On the other hand, let δ be the KK–element associated with the deformation groupoid $\mathcal{G}_{]-2,2[}^t$. It is clear that δ is a generator of $K^0(C^*(T]-2, 2[)) \simeq \mathbb{Z}$ and its pullback Δ under the homotopy equivalence $C([-2, 2]) \to \mathbb{C}$ is a Dirac element. Now, under the groupoid isomorphism $T^S U \simeq T^S L \times T]-2, 2[$, the element δ_U corresponds to $\delta_L \underset{\mathbb{C}}{\otimes} \delta$ and $D_{\overline{U}}$ to $D_L \underset{\mathbb{C}}{\otimes} \Delta$. It follows that $D_{\overline{U}}$ is a Dirac element.

Since D_{2X} and $D_{\overline{U}}$ are Dirac elements, we get from Lemma 4 applied to the nuclear $C(2X)$–algebra $C^*(T^S 2X)$, to the open sets U, V and to the K–homology class δ_{2X}, that D_V is a Dirac element. Since Ψ_V^* has diagonal form, we have:

$$D_V = D_{O_+} \oplus D_{O_+} \in K^0(C^*(T^S \overline{O_+}) \otimes C_0(O_+))^{\oplus 2} \subset K^0(C^*(T^S \overline{V}) \otimes C_0(V)).$$

It is clear from this formula that D_V is a Dirac element if and only if D_{O_+} is, so we have proved that D_{O_+} is a Dirac element, which ends the proof of the theorem. $\square$

The following remark collect some technical facts which were in the main body of the proof of Theorem 4 before we took into account the Referee's suggestions.

Remark 4 Replace $\rho_s(z) < 2$ by $\rho_s(z) < 1$ in the definition of O_- in (4-1) and modify accordingly the subsequent sets in (4-1). Let $\partial_\pm \in KK_1(C^*(T^S L \times \mathbb{R}), C^*(T^S O_\pm))$ be the KK–elements associated with the exact sequences of C^*–algebras:

$$0 \longrightarrow C^*(T^S O_\pm) \longrightarrow C^*(T^S \overline{O_\pm}) \longrightarrow C^*(T^S L \times \mathbb{R}) \longrightarrow 0 .$$

We can apply Lemma 2 to $A = C^*(T^S X)$, $J_1 = C^*(T^S O_+)$ and $J_2 = C^*(T^S O_-)$. This gives:

$$\partial_+ \otimes \delta_{O_+} = -\partial_- \otimes \delta_{O_-} \in K^1(C^*(T^S L \times \mathbb{R})) .$$

Moreover, one can show that this element is the Dirac element D_L associated with L, modulo sign and Bott periodicity $K^1(C^*(T^S L \times \mathbb{R})) \simeq K^0(C^*(T^S L))$. The idea to prove this is to build a smooth groupoid:

$$\widehat{\mathcal{G}}^t_{O_-} := \mathcal{G}^t_{O_-} \sqcup (\mathcal{G}^t_L \times \mathbb{R}) \rightrightarrows \overline{O_-} \times [0, 1] .$$

such that the following (smooth) isomorphisms hold:

- $\widehat{\mathcal{G}}^t_{O_-}|_{\overline{O_-} \times \{0\}} \simeq T^S \overline{O_-},$

- $\widehat{\mathcal{G}}^t_{O_-}|_{\overline{O_-} \times \{1\}} \simeq (L^\circ \times L^\circ) \times (\mathbb{R} \rtimes_\phi \mathbb{R}|_{]0,1]}),$

where $\mathbb{R} \rtimes_\phi \mathbb{R} \rightrightarrows \mathbb{R}$ is the groupoid of the action of $\mathbb{R}$ onto itself by the complete flow of the vector field $\tau(h)\partial_h$ and τ is the gluing function used in Section 3.3 (this is exactly the tangent space of $[1, +\infty[$ with $\{1\}$ as a conical point). Since $\widehat{\mathcal{G}}^t_{O_-}|_{\overline{O_-} \times \{1\}}$ has vanishing K–theory, the KK–element α associated with the exact sequence:

$$0 \longrightarrow C^*(O_-^\circ \times O_-^\circ) \longrightarrow C^*(\widehat{\mathcal{G}}^t_{O_-}|_{\overline{O_-} \times \{1\}}) \longrightarrow C^*(L^\circ \times L^\circ \times \mathbb{R}) \longrightarrow 0$$

is invertible in KK–theory, thus it corresponds to Bott periodicity modulo a sign and the Morita equivalences between $C^*(O_-^\circ \times O_-^\circ), C^*(L^\circ \times L^\circ)$ and $\mathbb{C}$. Finally, we consider the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(T^S O_-) & \longrightarrow & C^*(T^S \overline{O_-}) & \longrightarrow & C^*(T^S L \times \mathbb{R}) & \longrightarrow & 0 \\
 & & \big\uparrow{\scriptstyle e_0^{O_-}} & & \big\uparrow{\scriptstyle e_0^{\overline{O_-}}} & & \big\uparrow{\scriptstyle e_0^L \otimes 1} & & \\
0 & \longrightarrow & C^*(\mathcal{G}^t_{O_-}) & \longrightarrow & C^*(\widehat{\mathcal{G}}^t_{O_-}) & \longrightarrow & C^*(\mathcal{G}^t_L \times \mathbb{R}) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle e_1^{O_-}} & & \big\downarrow{\scriptstyle e_1^{\overline{O_-}}} & & \big\downarrow{\scriptstyle e_1^L \otimes 1} & & \\
0 & \longrightarrow & C^*(O_-^\circ \times O_-^\circ) & \longrightarrow & C^*(\widehat{\mathcal{G}}^t_{O_-}|_{\overline{O_-} \times \{1\}}) & \longrightarrow & C^*(L^\circ \times L^\circ \times \mathbb{R}) & \longrightarrow & 0
\end{array}
$$

It gives by functoriality: $\partial_- \otimes \delta_{O_-} = \delta_L \underset{\mathbb{C}}{\otimes} \alpha$ which proves the claim.

4.0.6 Stratified pseudomanifold with boundary As a byproduct of the proof of Theorem 4, we have proved that Poincaré duality also holds for compact stratified pseudomanifolds with boundary. Precisely a stratified pseudomanifold with boundary is $(X_b, L, \mathsf{S}_b, N_b)$ where:

(1) X_b is a compact separable metrizable space and L is a compact subspace of X_b.

(2) $\mathsf{S}_b = \{s_i\}$ is a finite partition of X_b into locally closed subset of X_b, which are smooth manifolds possibly with boundary. Moreover for each s_i we have

$$s_i \cap L = \partial s_i .$$

(3) $N_b = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s \in \mathsf{S}_b}$, where $\mathcal{N}_s$ is an open neighborhood of s in X, $\pi_s \colon \mathcal{N}_s \to s$ is a continuous retraction and $\rho_s \colon \mathcal{N}_s \to [0, +\infty[$ is a continuous map such that $s = \rho_s^{-1}(0)$.

(4) The *double*

$$X = X_b \underset{L}{\cup} X_b$$

obtained by gluing two copies of X_b along L together with the partition

$$\mathsf{S} := \{s_i \,|\, \partial s_i = \varnothing\} \cup \{s_i \underset{\partial s_i}{\cup} s_i\} \cup \{s_i \,|\, \partial s_i = \varnothing\}$$

and the set of control data $N = \{\widetilde{\mathcal{N}}_s, \widetilde{\pi}_s, \widetilde{\rho}_s\}_{s \in \mathsf{S}}$ where

$$\mathcal{N}_s = \mathcal{N}_{s_i}, \quad \pi_s = \pi_{s_i}, \quad \rho_s = \rho_{s_i} \text{ if } s = s_i \text{ with } \partial s_i = \varnothing$$

and $\quad \mathcal{N}_s = \mathcal{N}_{s_i} \underset{\mathcal{N}_{s_i} \cap L}{\cup} \mathcal{N}_{s_i}, \quad \pi_s|_{\mathcal{N}_{s_i} \backslash L} = \pi_{s_i}, \quad \rho_s|_{\mathcal{N}_{s_i} \backslash L} = \rho_{s_i} \text{ elsewhere}$

is a stratified pseudomanifold.

We let $O_b := X_b \backslash L$. According to the previous work, one can define the tangent spaces:

$$T^{\mathsf{S}} X_b := T^{\mathsf{S}} X|_{X_b} \text{ and } T^{\mathsf{S}} O_b := T^{\mathsf{S}} X|_{O_b}$$

We deduce the following:

Theorem 5 *The C^*–algebras $C^*(T^{\mathsf{S}} X_b)$ and $C_0(O_b)$ are Poincaré dual as well as the C^*–algebras $C^*(T^{\mathsf{S}} O_b)$ and $C(X_b)$.*

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Laboratoire de Mathématiques, Université Blaise Pascal
Complexe universitaire dse Cézeaux, 24 Av. des Landais, 63177 Aubière cedex, France

debord@math.univ-bpclermont.fr,   lescure@math.univ-bpclermont.fr

http://math.univ-bpclermont.fr/~debord/,
http://math.univ-bpclermont.fr/~lescure/