Cannon–Thurston maps for pared manifolds of bounded geometry

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Let $N^h \in H(M, P)$ be a hyperbolic structure of bounded geometry on a pared manifold such that each component of $\partial_0 M = \partial M - P$ is incompressible. We show that the limit set of N^h is locally connected by constructing a natural Cannon–Thurston map. This provides a unified treatment, an alternate proof and a generalization of results due to Cannon and Thurston, Minsky, Bowditch, Klarreich and the author.

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1 Introduction

This is one in a series of papers (in between [36] and [39]) leading to the existence of Cannon–Thurston maps for, and local connectivity of limit sets of, finitely generated Kleinian groups. The project is completed in [38] and [40]. The main aim of this paper is to develop a *reduction technique*. Given the existence of Cannon–Thurston maps for closed surface groups of bounded geometry (cf [36]), this paper develops techniques to generalize this first to punctured surfaces and then to bounded geometry pared manifolds with incompressible boundary. While this is not a reduction theorem per se, the *techniques* of this paper shall be used mutatis mutandis in [39] and [40] to obtain generalizations of Cannon–Thurston theorems for surface groups to theorems for pared manifolds with incompressible boundary. The main focus of [38; 39] and [40] will be to describe geometries of closed surface groups for which one can prove the existence of Cannon–Thurston maps. The main theorem of this paper is as follows.

Theorem (Theorem 5.12) *Suppose that $N^h \in H(M, P)$ is a hyperbolic structure of* bounded geometry *on a pared manifold (M, P) with incompressible boundary $\partial_0 M = (\partial M - P)$ (ie each component of $\partial_0 M$ is incompressible). Let M_{gf} denote a geometrically finite hyperbolic structure adapted to (M, P). Then the map $i\colon \widetilde{M_{\mathrm{gf}}} \to \widetilde{N^h}$ extends continuously to the boundary $\hat{i}\colon \widehat{M_{\mathrm{gf}}} \to \widehat{N^h}$. If Λ denotes the limit set of $\widetilde{M}$, then Λ is locally connected.*

Examples to which Theorem 5.12 applies include the following.

Published: 1 January 2009 DOI: 10.2140/gt.2009.13.189

(1) The cover corresponding to the fiber subgroup of a closed hyperbolic 3–manifold fibering over the circle (Cannon and Thurston [15]),

(2) Hyperbolic 3 manifolds of bounded geometry, which correspond to simply or doubly degenerate Kleinian groups isomorphic to closed surface groups (Minsky [28]).

(3) Hyperbolic 3 manifolds of bounded geometry without parabolics and freely indecomposable fundamental group (Mitra [36, Section 4.3] and Klarreich [23]).

(4) Hyperbolic 3 manifolds of bounded geometry, arising from simply or doubly degenerate Kleinian groups corresponding to punctured surface groups (Bowditch [11; 12]).

The main issue addressed in this paper has been raised in various forms by Minsky [32], McMullen [26] and the author [33; 3].

Let M be a closed hyperbolic 3–manifold fibering over the circle with fiber F. Let $\widetilde{F}$ and $\widetilde{M}$ denote the universal covers of F and M respectively. Then $\widetilde{F}$ and $\widetilde{M}$ are quasi-isometric to $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. Now let $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}^1_\infty$ and $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ denote the standard compactifications. In [15] Cannon and Thurston show that the usual inclusion of $\widetilde{F}$ into $\widetilde{M}$ extends to a continuous map from $\mathbb{D}^2$ to $\mathbb{D}^3$.

This was generalized by Minsky [28] and Klarreich [23] and independently (and using different techniques) by the author [36, Section 4] to prove that if M is a geometrically tame hyperbolic 3–manifold with freely indecomposable fundamental group and injectivity radius bounded below, and if M_{cc} denotes the (topological) compact core of M then the inclusion of $\widetilde{M}_{cc}$ into $\widetilde{M}$ extends to a continuous map from $\widehat{M}_{cc}$ to $\widehat{M}$ where $\widetilde{M}_{cc}$ and $\widetilde{M}$ denote the universal covers of M_{cc} and M respectively and $\widehat{M}_{cc}$ and $\widehat{M}$ denote the (Gromov) compactifications of $\widetilde{M}_{cc}$ and $\widetilde{M}$ respectively. However, all these results left unanswered the case of manifolds with parabolics.

In [26] McMullen proved that the corresponding result holds for punctured torus groups, using the model manifold built by Minsky for these groups in [29]. In [39], we shall unify the framework of this paper with a certain notion of *i–bounded geometry* to give a simultaneous generalization of the results of this paper and those of McMullen [26].

The present paper was born of an attempt to find a new proof (along the lines of [36]) of a result of Bowditch. In [11; 12], Bowditch proved the existence of Cannon–Thurston maps for punctured surface groups of bounded geometry using some of the ideas from the first paper in the present series by the author [36]. The main result of this paper

simultaneously generalizes Bowditch's results [11; 12] and [36] and hence includes the results of [15; 28; 23; 12; 11; 36]. When (M, P) is the pair $S \times I, \delta S \times I$, for S a surface group, we get Bowditch's result [11; 12]. However, even in this case, our proof is different and circumvents the use of locally infinite (Farey) graphs as in the proofs of [11; 12].

We are grateful to Brian Bowditch for pointing out that local connectivity (the second statement in Theorem 5.12 above) also follows by combining Bowditch's result [12] with a result of Anderson–Maskit [2]. See [12, Section 10] for details. The first part of Theorem 5.12 answers a question attributed to Thurston by Bowditch in the same paper [12, Section 10] (see also [1]).

In a sense, Theorem 5.12 above is a direct generalization of the following which we proved in [36].

Theorem (Theorem 4.7 of [36]) *Let M be a compact 3–manifold with incompressible boundary ∂M without torus components. Let M_{hyp} be a hyperbolic structure without parabolics on M such that each end of the manifold has bounded geometry. Also suppose that M_{gf} is a geometrically finite structure on M. Then*

- *the inclusion of $\widetilde{M}_{\text{gf}}$ into $\widetilde{M}_{\text{hyp}}$ extends to a continuous map from $\widehat{M}_{\text{gf}}$ to $\widehat{M}_{\text{hyp}}$*
- *the limit set of $\widetilde{M}_{\text{hyp}}$ is locally connected.*

This problem is a part of a more general problem. A natural question seems to be the following.

Question [36; 33; 3] Suppose H is a hyperbolic group acting freely and properly discontinuously by isometries on a hyperbolic metric space X. Does the continuous proper map $i\colon \Gamma_H \to X$ extend to a continuous map $\hat{i}\colon \widehat{\Gamma_H} \to \widehat{X}$?

We can ask the same question for relatively hyperbolic groups (in the sense of Gromov [20], Farb [17] and Bowditch [8]). A convenient framework for formulating this question is that of convergence groups acting on compacta (Bowditch [10]). Hyperbolic and relatively hyperbolic groups have been characterized in this setting by Bowditch [9] and Yaman [49].

Question Suppose a relatively hyperbolic group H acts on a compact set K such that the action is a convergence action and such that each 'cusp group' of H leaves a point of K fixed. Does there exist an equivariant continuous map from ∂H to K? (Here ∂H denotes the boundary of the relatively group H, which is well-defined as per [8].)

A closely related question was raised by McMullen [26] in the context of Kleinian groups. However, his question deals with the Floyd completion rather than the Gromov completion as we have done here. There is some difference between the two notions for manifolds with cusps. Nevertheless, modulo Floyd's result [18], the two questions are equivalent in the context of Kleinian groups. This question has also gained attention after the recent resolution of the Ending Lamination Conjecture by Minsky [31] and Brock–Canary–Minsky [14]. In fact the question in the context of Kleinian groups has also been raised by Minsky in [32]. Both McMullen and Minsky ask if the limit sets of Kleinian groups are locally connected given that they are connected. This will be completely answered in the affirmative in [40].

Perhaps the most general context in which the question of local connectivity makes sense is that of convergence group actions on compacta [10].

Question Suppose a finitely generated group H acts on a compact connected metrizable perfect set K such that the action is a convergence action. Is K locally connected? (ie does 'admitting a convergence action' promote a connected compactum to a Peano continuum?)

1.1 Punctured surface groups of bounded geometry: A new proof

As a motivational example, we sketch the proof for Kleinian groups G that correspond to punctured surface groups, such that the 3–manifold $N^h = \mathbb{H}^3/G$ has bounded geometry away from the cusps, ie closed geodesics have length uniformly bounded away from zero. These are precisely the examples handled by Bowditch in [12] and [11] . We give a sketch in this particular case because these are the simplest new non-trivial examples and also to underscore the difference in our approach.

We first excise the cusps (if any) of N^h leaving us a manifold that has one or two ends. Let N denote N^h minus cusps. Then N is quasi-isometric to the universal curve over a Lipschitz path in Teichmuller space from which cusps have been removed. (In fact, in [36] we had proven that the path in question is a Teichmuller geodesic. The proof there was for closed surfaces, but can be extended painlessly to that of surfaces with punctures.) This path is semi-infinite or bi-infinite according as N is one-ended or two-ended. Fix a reference finite volume hyperbolic surface S^h. Let S denote S^h minus cusps. Then $\widetilde{S}$ is quasi-isometric to the Cayley graph of $\pi_1(S)$ which is (Gromov) hyperbolic. We fix a base surface in N and identify it with S. Now look at $\widetilde{S} \subset \widetilde{N}$. Let $\lambda = [a, b]$ be a geodesic segment in $\widetilde{S}$. We 'flow' λ out the end(s) of $\widetilde{N}$ to generate a *hyperbolic ladder-like set* B_λ (thinking of $\widetilde{S}$ as a horizontal sheet, see Figure 4) that looks topologically like $[a, b] \times [0, \infty)$ or $[a, b] \times (-\infty, \infty)$ according

as N has one or two ends. This is exactly a reproduction of our construction in [36] or [35].

A few details are in order. Regarding N as a universal curve (minus cusps) over a Lipschitz path in Teichmuller space, we can assume that each fibre over the path is topologically S with hyperbolic structure varying over the path. The union of all points that correspond to a (or b) is a quasigeodesic ray (or bi-infinite quasigeodesic) in $\widetilde{N}$. Join the pairs of points that lie in the lift of a single copy of $\widetilde{S} \subset \widetilde{N}$, giving a geodesic in each copy of $\widetilde{S}$. This is what is meant by 'flowing' λ out the end(s) with the two quasigeodesic rays mentioned above as 'guides' (or boundary lines), generating B_λ. The main technical theorems [35, Theorem 3.7] or [36, Theorem 3.8] ensure that there is a retraction from $\widetilde{N}$ to B_λ which does not increase distances much. From this it follows that B_λ is quasi-isometrically embedded in $\widetilde{N}$. [A brief proof of this last assertion, viz qi-embeddedness follows from retract, was given in [35] but we had omitted it in [36]. We are grateful to Brian Bowditch for pointing out this gap in [12]. See Gromov [20, Section 7.3J] for related result and [30] for a detailed proof of the same.] Note that for this we do not need $\widetilde{N}$ to be hyperbolic (in fact $\widetilde{N}$ is hyperbolic only when S^h has no cusps and $\widetilde{N^h} = \widetilde{N}$. This is what we had used to prove the existence of Cannon–Thurston maps in [35] and [36]).

Now if λ lies outside a large ball about a fixed reference point p in $\widetilde{S}$, then so does B_λ in $\widetilde{N}$. Since B_λ is qi-embedded in $\widetilde{N}$, there exists an ambient $\widetilde{N}$–quasigeodesic μ lying in a bounded neighborhood of B_λ joining the end-points of λ. If S^h had no cusps, we immediately conclude that for any geodesic segment λ in $\widetilde{S^h}$ lying outside large balls around p, there is a quasigeodesic in $\widetilde{N^h}$ joining its endpoints and lying outside a large ball around p in $\widetilde{N^h}$. This gives us a continuous extension of the inclusion of $\widetilde{S^h}$ into $\widetilde{N^h}$ to the boundary. At this stage we have recaptured the results of Cannon and Thurston [15] and Minsky [28].

However, when S^h has cusps, $\widetilde{S^h}$ and $\widetilde{S}$ are different. So a little more work is necessary. Suppose as before that λ_0 is a geodesic in $\widetilde{S^h}$ lying outside a large ball around p. For ease of exposition we assume that the end-points of λ_0 lie outside cusps. Let $\lambda \subset \widetilde{S}$ be the geodesic in (the path-metric on) $\widetilde{S}$ joining the same pair of points. Then off horodisks, λ_0 and λ track each other. Construct B_λ as before, and let μ be an ambient $\widetilde{N}$–quasigeodesic lying in a bounded neighborhood of B_λ joining the end-points of λ. Then off horoballs in $\widetilde{N^h}$, μ lies outside a large ball around p. Let μ_0 be the hyperbolic geodesic joining the end points of μ. Off horoballs, μ and μ_0 track each other. Hence, off horoballs, μ_0 lies outside large balls about p. The points at which μ_0 intersects a particular horoball therefore lie outside large balls about p. But then the hyperbolic segment joining them must do the same. This shows that μ_0

must itself lie outside large balls around p. As before we conclude that there exists a continuous extension of the inclusion of $\widetilde{S^h}$ into $\widetilde{N^h}$ to the boundary. At this stage we have recaptured the result of Bowditch for punctured surfaces [11; 12].

We note here that the main purpose of this paper is to continue with our parallel approach to handling problems like the existence of a Cannon–Thurston map, and local connectivity of limit sets, initiated in [35] and [36]. These papers and the current one circumvent much of the sophisticated machinery specific to 2– and 3 manifolds developed by Thurston et al. Thus, they provide a direct approach to the existence of a Cannon–Thurston map, (as also local connectivity), without having to deal with laminations and associated geometries. Of course, this means that we lose out on the explicit description that Cannon and Thurston [15] or Minsky [28] or Bowditch [12] provide in terms of laminations. McMullen [26] also uses the notion of foliations in his proof to derive explicit information about locations of closed geodesics. However, the large-scale or coarse nature of our approach makes it more general, and suitable for application to cases where the surface machinery of Thurston is not available, eg general hyperbolic or relatively hyperbolic spaces and groups in the sense of Gromov, and more specifically, free groups where the notion of laminations developed by Bestvina, Feighn and Handel is substantially different [5; 6]. However, once the Cannon–Thurston map is in place, some notions of laminations can be resurrected, as in [34]. Thus, in principle, a long term aim of the present approach would be to derive further parallels between 3 manifolds and surfaces on the one hand, and discrete groups and their subgroups on the other. The latter context being vastly more general, we have tried to avoid the techniques specific to 2 and 3 dimensions.

The main technical difficulty in the first part of this paper (up to Section 4) stems from the fact that a closed geodesic on a component of the boundary of the pared manifold $\partial_0 M = (\partial M - P)$ may be homotopic to a curve on a component of P, ie it is an accidental parabolic in any hyperbolic structure on (M, P). This results in a somewhat trying case-by-case analysis in Section 4 of this paper. The resulting notation becomes a bit elaborate at times and we pause at a few moments during the course of the paper to summarize notation and keep it clear. There are essentially 3 cases we need to handle.

- Z**–cusps** these are relatively easy to handle as the above sketch shows.
- $(Z + Z)$ **cusps** where no curve on any component of $\partial_0 M$ is homotopic to a curve on the boundary torus corresponding to the $(Z + Z)$ cusp.
- Z **and** $(Z + Z)$ **cusps**, where some curve(s) on component(s) of $\partial_0 M$ is(are) homotopic to a particular curve on the boundary torus corresponding to the $(Z + Z)$ cusp or some multiple of the core curve of the Z–cusp.

We shall have occasion to introduce a certain extra hypothesis of *p–incompressibility*, but we stick to the general case, ie we *do not* introduce the extra hypothesis of *p–incompressibility* (or absence of accidental parabolics on components of $\partial_0 M$) till Section 5.2, because the construction of B_λ goes through even in the absence of this simplifying assumption. Further, once the proof is completed for the *p–incompressible* case, we use it (towards the end of Section 5) to prove the general case.

1.2 Outline of the paper

A brief outline of the paper follows. Section 2 deals with preliminaries on (Gromov) hyperbolic spaces. We also recall a result from McMullen [26] which says roughly that hyperbolic geodesics and ambient geodesics (geodesics in spaces obtained by removing some horoballs) track each other off horoballs.

In Section 3, we recall Thurston's definition of pared manifolds [47; 48] and show that the universal cover of a hyperbolic 3–manifold M, whose compact core is a pared manifold, is quasi-isometric to a tree T of hyperbolic metric spaces with possibly exceptional vertex, once the cusps of M have been removed. Further, if α denotes the root vertex, we demand that α be the possibly exceptional vertex. Let X_α denote the corresponding vertex space.We shall choose a geometrically finite structure M_{gf} on M and identify X_α with ($\widetilde{M_{\mathrm{gf}}}$ minus Z–cusps). This converts X_α into a hyperbolic metric space, but we relax the requirement that the embeddings of the edge spaces into X_α be qi-embeddings (*this is what makes α exceptional*). All other vertex and edge spaces are hyperbolic and all other inclusions of edge spaces into vertex spaces are qi-embeddings.

In Section 4, we modify the construction of [36] to construct a quasi-isometrically embedded *hyperbolic ladder-like* set B_λ (see Figure 4) corresponding to a geodesic λ. We show that B_λ is qi-embedded in ($\widetilde{M} - Z$–cusps). Z–cusps and $(Z+Z)$–cusps are treated differently.The construction of B_λ does not require *p–incompressibility* of the boundary components, but only their incompressibility.

It is in Section 5, that we first restrict the scope to pared manifolds with *p–incompressible* boundary components. We show that if λ lies outside a large ball about a fixed reference point p in ($X_\alpha \cup$ cusps) modulo horoballs then so does B_λ in $\widetilde{M}$.

Finally, we use the tracking properties of ambient quasigeodesics vis a vis hyperbolic geodesics and assemble the proof of the main theorem in the case of *p–incompressible boundary*. We deduce from this that the limit sets of the corresponding Kleinian groups are locally connected. Once this case is proven, we use it to prove the result for pared manifolds of incompressible boundary, thus relaxing the assumption on

p–incompressibility. The concluding Section 6 deals with examples to which our theorem applies, notably Brock's example [13]. We also indicate possible directions of generalization. It would be worth bearing in mind that most of the arguments of this paper are relevant to the considerably more general framework of relatively hyperbolic groups a la Gromov [20], Farb [17] and Bowditch [8].

Apologia In *From Beowulf to Virginia Woolf: An astounding and wholly unauthorized history of English literature* [42] Robert Manson Myers claims that the plays usually attributed to Shakespeare are not in fact written by him but by another person of the same name. Being fully aware of our literary deficiencies we frankly admit that the papers [35; 36] and the present one are written by the same person under different names.

Acknowledgements I am grateful to Brian Bowditch for several helpful comments on a previous version of this paper. I would also like to thank the referee for carefully reading the manuscript and suggesting several corrections and improvements. Research partly supported by a UGC Major Research Project grant.

Dedication To S V on his birthday.

2 Preliminaries

We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [20]. For details, see [16; 19]. Let (X,d) be a hyperbolic metric space. The *Gromov boundary* of X, denoted by ∂X, is the collection of equivalence classes of geodesic rays $r\colon [0,\infty)\to\Gamma$ with $r(0)=x_0$ for some fixed $x_0\in X$, where rays r_1 and r_2 are equivalent if $\sup\{d(r_1(t),r_2(t))\}<\infty$. Let $\widehat{X}=X\cup\partial X$ denote the natural compactification of X topologized the usual way (cf [19, page 124]).

The *Gromov inner product* of elements a and b relative to c is defined by

$$(a,b)_c = 1/2[d(a,c)+d(b,c)-d(a,b)].$$

Definition 2.1 A subset Z of X is said to be *k–quasiconvex* if any geodesic joining points of Z lies in a k–neighborhood of Z. A subset Z is *quasiconvex* if it is k–quasiconvex for some k.

For simply connected real hyperbolic manifolds this is equivalent to saying that the convex hull of the set Z lies in a bounded neighborhood of Z. We shall have occasion to use this alternate characterization.

Definition 2.2 A map f from one metric space (Y, d_Y) into another metric space (Z, d_Z) is said to be a (K, ϵ)–*quasi-isometric embedding* if

$$\frac{1}{K}(d_Y(y_1, y_2)) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \epsilon.$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$ then f is said to be a *quasi-isometry*. A (K, ϵ)–quasi-isometric embedding that is a quasi-isometry will be called a (K, ϵ)–quasi-isometry.

A (K, ϵ)–*quasigeodesic* is a (K, ϵ)–quasi-isometric embedding of a closed interval in $\mathbb{R}$. A $(K, 0)$–quasigeodesic will also be called a K–quasigeodesic.

Let (X, d_X) be a hyperbolic metric space and Y be a subspace that is hyperbolic with the inherited path metric d_Y. By adjoining the Gromov boundaries ∂X and ∂Y to X and Y, one obtains their compactifications $\widehat{X}$ and $\widehat{Y}$ respectively.

Let $i\colon Y \to X$ denote inclusion.

Definition 2.3 Let X and Y be hyperbolic metric spaces and $i\colon Y \to X$ be an embedding. A *Cannon–Thurston map* $\hat{i}$ from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of i.

The following lemma [35, Lemma 2.1] says that a Cannon–Thurston map exists if for all $M > 0$ and $y \in Y$, there exists $N > 0$ such that if λ lies outside an N ball around y in Y then any geodesic in X joining the end-points of λ lies outside the M ball around $i(y)$ in X. For convenience of use later on, we state this somewhat differently.

Lemma 2.4 *A Cannon–Thurston map from $\widehat{Y}$ to $\widehat{X}$ exists if the following condition is satisfied. Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N–ball around $y_0 \in Y$ any geodesic segment in X joining the end-points of $i(\lambda)$ lies outside the $M(N)$–ball around $i(y_0) \in X$.*

The above result can be interpreted as saying that a Cannon–Thurston map exists if the space of geodesic segments in Y embeds properly in the space of geodesic segments in X.

We shall also be requiring certain properties of hyperbolic spaces minus horoballs. These were studied by Farb [17] under the garb of 'electric geometry'. We combine Farb's results with a version that is a (slight variant of a) theorem due to McMullen [26, Theorem 8.1].

Definition 2.5 A path $\gamma\colon I \to Y$ to a path metric space Y is an ambient K–quasigeodesic if we have

$$L(\beta) \leq KL(A) + K$$

for any subsegment $\beta = \gamma|[a,b]$ and any path $A\colon [a,b] \to Y$ with the same endpoints.

The following definitions are adapted from [17].

Definition 2.6 Let M be a convex hyperbolic manifold. Let Y be the universal cover of M minus cusps and $X = \widetilde{M}$. γ is said to be a K–quasigeodesic in X *without backtracking* if

- γ is a K–quasigeodesic in X
- γ does not return to any horoball $\mathbf{H}$ after leaving it.

γ is said to be an ambient K–quasigeodesic in Y *without backtracking* if

- γ is an ambient K–quasigeodesic in Y
- γ is obtained from a K–quasigeodesic without backtracking in X by replacing each maximal subsegment with end-points on a horosphere by a quasigeodesic lying on the surface of the horosphere.

Note that in the above definition, we allow the behavior to be quite arbitrary on horospheres (since Euclidean quasigeodesics may be quite wild); however, we do not allow wild behavior off horoballs.

$B_R(Z)$ will denote the R–neighborhood of the set Z. Let $\mathcal{H}$ be a locally finite collection of horoballs in a convex subset X of $\mathbb{H}^n$ (where the intersection of a horoball, which meets ∂X in a point, with X is called a horoball in X). The following theorem is due to McMullen [26].

Theorem 2.7 [26] *Let $\gamma\colon I \to X \setminus \bigcup \mathcal{H} = \mathcal{Y}$ be an ambient K–quasigeodesic for a convex subset X of $\mathbb{H}^n$ and let $\mathcal{H}$ denote a collection of horoballs. Let η be the hyperbolic geodesic with the same endpoints as γ. Let $\mathcal{H}(\equiv)$ be the union of all the horoballs in $\mathcal{H}$ meeting η. Then $\eta \cup \mathcal{H}(\eta)$ is (uniformly) quasiconvex and $\gamma(I) \subset B_R(\eta \cup \mathcal{H}(\equiv))$, where R depends only on K.*

The above theorem is similar in flavor to certain theorems about relative hyperbolicity a la Gromov [20], Farb [17] and Bowditch [8]. We give below a related theorem that is derived from Farb's 'Bounded Horosphere Penetration' property.

Let $\gamma_1 = \overline{pq}$ be a hyperbolic K–quasigeodesic without backtracking starting from a horoball $\mathbf{H_1}$ and ending within (or on) a *different* horoball $\mathbf{H_2}$. Let $\gamma = [a, b]$ be the hyperbolic geodesic minimizing distance between $\mathbf{H_1}$ and $\mathbf{H_2}$. Following [17] we put the zero metric on the horoballs that γ penetrates. The resultant pseudo-metric is called the electric metric. Let $\widehat{\gamma}$ and $\widehat{\gamma_1}$ denote the paths represented by γ and γ_1 respectively in this pseudometric. It is shown in [17] that γ, $\widehat{\gamma}$ and $\widehat{\gamma_1}$ have similar intersection patterns with horoballs, ie there exists C_0 such that the following hold.

(1) If only one of γ and $\widehat{\gamma_1}$ penetrates a horoball $\mathbf{H}$, then it can do so for a distance $\leq C_0$.

(2) If both $\widehat{\gamma_1}$ and γ enter (or leave) a horoball $\mathbf{H}$ then their entry (or exit) points are at a distance of at most C_0 from each other. [Here by 'entry' (resp 'exit') point of a quasigeodesic we mean a point at which the path switches from being in the complement of or 'outside' (resp in the interior of or 'inside') a closed horoball to being inside (resp. outside) it].

The point to observe here is that quasigeodesics without backtracking in our definition gives rise to quasigeodesics without backtracking in Farb's sense. Since this is true for arbitrary γ_1 we give below a slight strengthening of this fact. Further, by our construction of ambient quasigeodesics without backtracking, we might just as well consider ambient quasigeodesics without backtracking in place of quasigeodesics.

Theorem 2.8 [17] *Given $C > 0$, there exists C_0 such that if*

(1) *either two quasigeodesics without backtracking γ_1, γ_2 in X, or*

(2) *two ambient quasigeodesics without backtracking γ_1, γ_2 in Y or*

(3) *γ_1 – an ambient quasigeodesic without backtracking in Y and γ_2 – a quasigeodesic without backtracking in X,*

start and end

(1) *either on (or within) the same horoball, or*

(2) *a distance C from each other*

then they have similar intersection patterns with horoballs (except possibly the first and last ones), ie there exists C_0 such that

(1) *if only γ_1 penetrates or travels along the boundary of a horoball $\mathbf{H}$, then it can do so for a distance $\leq C_0$,*

(2) *if both γ_1 and γ_2 enter (or leave) a horoball $\mathbf{H}$ then their entry (or exit) points are at a distance of at most C_0 from each other.*

3 Trees of hyperbolic metric spaces and pared manifolds

3.1 Definitions

We start with a notion closely related to one introduced in Bestvina–Feighn [4].

Definition 3.1 A *tree T of hyperbolic metric spaces satisfying the q(uasi) i(so-metrically) embedded condition* is a metric space (X, d) admitting a map $P\colon X \to T$ onto a simplicial tree T, such that there exist δ,ϵ and $K > 0$ satisfying the following.

(1) For all vertices $v \in T$, $X_v = P^{-1}(v) \subset X$ with the induced path metric d_v is a δ–hyperbolic metric space. Further, the inclusions $i_v\colon X_v \to X$ are uniformly proper, ie for all $M > 0$, there exists $N > 0$ such that for all $v \in T$ and $x, y \in X_v$, $d(i_v(x), i_v(y)) \leq M$ implies $d_v(x, y) \leq N$.

(2) Let e be an edge of T with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under P of the mid-point of e. Then X_e with the induced path metric is δ–hyperbolic.

(3) There exist maps $f_e\colon X_e \times [0, 1] \to X$, such that $f_e|_{X_e \times (0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric.

(4) $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ are (K, ϵ)–quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively. $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ will occasionally be referred to as f_{v_1} and f_{v_2} respectively.

d_v and d_e will denote path metrics on X_v and X_e respectively. i_v, i_e will denote inclusion of X_v, X_e respectively into X.

We need a version of the above definition adapted to 3 manifolds with cusps. For convenience of exposition, T shall be assumed to be rooted, ie equipped with a base vertex α.

Definition 3.2 A tree T of hyperbolic metric spaces with *possibly exceptional vertex* satisfying the q(uasi) i(sometrically) embedded condition is a metric space (X, d) admitting a map $P\colon X \to T$ onto a rooted simplicial tree T with root α, such that there exist δ, ϵ and $K > 0$ satisfying the following.

(1) For all vertices $v \in T$, $X_v = P^{-1}(v) \subset X$ with the induced path metric d_v is a δ–hyperbolic metric space. Further, the inclusions $i_v\colon X_v \to X$ are uniformly proper, ie for all $M > 0$, $v \in T$ and $x, y \in X_v$, there exists $N > 0$ such that $d(i_v(x), i_v(y)) \leq M$ implies $d_v(x, y) \leq N$.

(2) Let e be an edge of T with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under P of the mid-point of e. Then X_e with the induced path metric is δ–hyperbolic.

(3) There exist maps $f_e\colon X_e\times[0,1]\to X$, such that $f_e|_{X_e\times(0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric.

(4) $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ are (K,ϵ)–quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively for all $v_1, v_2 \neq \alpha$. When one of v_1, v_2 is α, this restriction is relaxed for the corresponding inclusion of edge spaces. $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ will occasionally be referred to as f_{v_1} and f_{v_2} respectively.

We shall work in the framework of pared manifolds in the sense of Thurston [47; 48].

Definition 3.3 A *pared manifold* is a pair (M,P), where $P\subset\delta M$ is a (possibly empty) 2–dimensional submanifold with boundary such that

(1) the fundamental group of each component of P injects into the fundamental group of M,

(2) the fundamental group of each component of P contains an abelian subgroup with finite index,

(3) any cylinder $C\colon (S^1\times I,\delta S^1\times I)\to(M,P)$ with $C_*\colon \pi_1(S^1\times I)\to\pi_1(M)$ injective is homotopic rel. boundary to P,

(4) P contains every component of δM which has an abelian subgroup of finite index.

The terminology is meant to suggest that certain parts of the skin of M have been pared off to form parabolic cusps in hyperbolic structures for M. $H(M,P)$ will denote the set of hyperbolic structures on (M,P). (Note that this means that the elements of P and the elements of P alone are taken to cusps.)

Definition A pared manifold (M,P) is said to have *incompressible boundary* if each component of $\partial_0 M=\partial M\setminus P$ is incompressible in M.

Further, (M,P) is said to have *p–incompressible boundary* if

(1) it has incompressible boundary,

(2) if some curve σ on a component of $\partial_0 M$ is freely homotopic in M to a curve α on a component of P, then σ is homotopic to α in ∂M.

P_0, P_1 will denote the components of P whose fundamental groups are virtually $Z, (Z+Z)$ respectively. The adjective 'virtually' shall sometimes be omitted and we shall refer to the components of P_0 (resp. P_1) as Z–cusps (resp. $(Z+Z)$–cusps).

3.2 3 manifold as a tree of spaces

The *convex core* of a hyperbolic 3–manifold N^h is the smallest convex submanifold $C(N^h) \subset N^h$ for which inclusion is a homotopy equivalence.

If an ϵ neighborhood of $C(N^h)$ has finite volume, N^h is said to be *geometrically finite*.

There exists a compact 3–dimensional submanifold $M_{cc} \subset N^h$, the *compact core* or *Scott core* [44] whose inclusion is a homotopy equivalence. M_{cc} can be thought of as $C(N^h)$ minus cusps for geometrically finite N^h. N^h minus cusps will be denoted by N and $C(N^h)$ minus cusps will be denoted by $C(N)$. The ends of N are in one-to-one correspondence with the components of $(N - M_{cc})$ or, equivalently, the components of $\partial_0 M$.

We say that an end of N is *geometrically finite* if it has a neighborhood missing $C(N)$.

Note The notion of ends here is slightly non-standard, as we do not want to regard a cusp as an end.

An end E of N is *simply degenerate* if it has a neighborhood homeomorphic to $S_0 \times \mathbb{R}$, where S_0 is the corresponding component of $\partial_0 M$, and if there is a sequence of pleated surfaces (with cusps removed) homotopic in this neighborhood to the inclusion of S_0, and exiting every compact set. Let S^h denote a hyperbolic surface of finite volume, from which S_0 is obtained by excising cusps. We shall refer to $E^h = S^h \times \mathbb{R}$ (respecting the parameterization of E) as an *end of* N^h. Note that we may think of E^h as obtained from E by adjoining "half" a Z–cusp.

N is called *geometrically tame* if all of its ends are either geometrically finite or simply degenerate. Note that N^h and the interior of N are homeomorphic to the interior of M. For a more detailed discussion of pleated surfaces and geometrically tame ends, see Canary–Epstein–Green [46] or Minsky [27].

A hyperbolic structure on (M, P) is a complete hyperbolic metric on the interior of M which takes precisely the elements of P to cusps. A manifold N^h will be said to be *adapted to a pared manifold* (M, P) if N^h corresponds to such a hyperbolic structure on (M, P).

Note Since the flaring ends of N contribute nothing to our discussion, we shall (abusing notation somewhat) regarding $\mathbf{N = C(N)}$ and refer to N as a hyperbolic manifold, though it should really be called a convex submanifold (minus cusps) homotopy equivalent to the big manifold (minus cusps).

A manifold M has *bounded geometry* if on a complement of the cusps, the injectivity radius of the manifold is bounded below by some number ϵ greater than 0. Equivalently, all closed geodesics have length greater than ϵ.

We want to first show that the universal cover of N^h minus Z–cusps is quasi-isometric to a tree of hyperbolic metric spaces with possibly exceptional vertex corresponding to the core.

Let E^h be a simply degenerate end of N^h. Then E^h is homeomorphic to $S^h \times [0, \infty)$ for some surface S^h of negative Euler characteristic. Cutting off a neighborhood of the cusps of S^h we get a surface with boundary denoted as S. Let E denote E^h minus a neighborhood of the Z–cusps. We assume that each Z–cusp has the standard form coming from a quotient of a horoball in $\mathbb{H}^3$ by Z. Also, we shall take our pleated surfaces to be such that the pair (S, cusps) is mapped to the pair (E, cusps) for each pleated S^h. We shall now show that each $\widetilde{E}$ is quasi-isometric to a ray of hyperbolic metric spaces satisfying the qi-embedded condition. In [36] we had shown this for manifolds without cusps. Each edge and vertex space will be a copy of $\widetilde{S}$ and the edge to vertex space inclusions shall be quasi-isometries. Note that each $\widetilde{S}$ can be thought of as a copy of $\mathbb{H}^2$ minus an equivariant family of horodisks.

Lemma 3.4 [46] *There exists $D_1 > 0$ such that for all $x \in E$, there exists a pleated surface $g\colon (S^h, \sigma) \to E^h$ with $g(S) \cap B_{D_1}(x) \neq \varnothing$. Also g maps (S, cusps) to (E, cusps).*

The following lemma follows easily from the fact that $\text{inj}_N(x) > \epsilon_0$.

Lemma 3.5 [7; 46] *There exists $D_2 > 0$ such that if $g\colon (S^h, \sigma) \to N^h$ is a pleated surface, then the diameter of the image of S is bounded, ie $\text{dia}(g(S)) < D_2$.*

The following lemma due to Thurston [46, Theorems 9.2 and 9.6.1] and Minsky [27] follows from compactness of pleated surfaces.

Lemma 3.6 [27] *Fix S^h and $\epsilon > 0$. Given $a > 0$ there exists $b > 0$ such that if $g\colon (S^h, \sigma) \to E^h$ and $h\colon (S^h, \rho) \to E^h$ are homotopic pleated surfaces which are isomorphisms on π_1 and E^h is of bounded geometry, then*

$$d_E(g(S), h(S)) \leq a \Rightarrow d_{\text{Teich}}(\sigma, \rho) \leq b,$$

where d_{Teich} denotes Teichmuller distance.

In Lemma 3.6 d_E denotes the path-metric on E inherited from N^h. More precisely, the complete hyperbolic metric on N^h gives rise to a path-metric on N^h. $E \subset N \subset N^h$ inherits a path-metric d_E given by

$$d_E(x, y) = \inf\{l(\sigma) : \sigma \text{ is a path in } E \text{ joining } x, y\}.$$

In [27] a special case of Lemma 3.6 is proven for closed surfaces. However, the main ingredient, a theorem due to Thurston is stated and proven in [46, Theorems 9.2 and 9.6.1] – *'algebraic limit is geometric limit'* for finite area surfaces. The arguments given by Minsky to prove the above lemma from Thurston's Theorems (Lemma 4.5, Corollary 4.6 and Lemma 4.7 of [27]) go through with very little change for surfaces of finite area. In [12], Bowditch gives an alternate approach to this using (quite general) Gromov–Hausdorff limit arguments.

Note that in the above lemma, pleated surfaces are not assumed to be embedded. This is because immersed pleated surfaces with a uniform lower bound on injectivity radius are uniformly quasi-isometric to the corresponding Riemann surfaces.

Construction of equispaced pleated surfaces exiting the end We next construct a sequence of *equispaced* pleated surfaces $S^h(i) \subset E^h$ exiting the end. Assume that $S^h(0), \ldots, S^h(n)$ have been constructed such that:

(1) $S(i)$, cusps is mapped to E, cusps,

(2) if $E(i)$ is the component of $E \backslash S(i)$ for which $E(i)$ is non-compact, then $S(i+1) \subset E(i)$,

(3) Hausdorff distance between $S(i)$ and $S(i+1)$ is bounded above by $3(D_1 + D_2)$,

(4) $d_E(S(i), S(i+1)) \geq D_1 + D_2$,

(5) from Lemma 3.6 and condition (2) above there exists D_3 depending on D_1, D_2 and S such that $d_{\text{Teich}}(S(i), S(i+1)) \leq D_3$.

Next choose $x \in E(n)$, such that $d_E(x, S_n) = 2(D_1 + D_2)$. Then by Lemma 3.4, there exists a pleated surface $g\colon (S^h, \tau) \to E^h$ such that $d_E(x, g(S)) \leq D_1$. Let $S^h(n+1) = g(S^h)$. Then by the triangle inequality and Lemma 3.5, if $p \in S(n)$ and $q \in S(n+1)$,

$$D_1 + D_2 \leq d_E(p, q) \leq 3(D_1 + D_2).$$

This allows us to continue inductively. $S(i)$ corresponds to a point x_i of $\text{Teich}(S)$. Joining the x_i in order, one gets a Lipschitz path σ in $\text{Teich}(S)$.

Definition 3.7 A sequence of pleated surfaces satisfying conditions (1–5) above will be called an *equispaced sequence of pleated surfaces*. The corresponding sequence of $S(i) \subset E$ will be called an *equispaced sequence of truncated pleated surfaces*.

Since all $S^h(i)$ have bounded geometry away from cusps, they all lie in the thick part of Teichmuller space. After quotienting by the mapping class group, their images lie in a compact subset of the moduli space. Hence, by acting on $S^h(i)$ by some uniformly quasi-conformal map ψ_i, we may assume that $\psi_i(S^h(i))$ gives rise to a fixed point S^h in moduli space after quotienting by the mapping class group. Then $\psi_{i-1} \circ \psi_i^{-1}\colon (S^h(i)) \to (S^h(i-1))$ is C–quasiconformal for some fixed C. We may assume that $\psi_i \circ \psi_{i-1}^{-1}\colon (S(i-1)) \to (S(i))$ is a bijective map by excising cusps appropriately. Also, assume that all these excised surfaces correspond to a fixed S obtained from S^h by excising cusps.

Definition 3.8 The *universal curve* over $X \subset \mathrm{Teich}(S^h)$ is a fiber bundle over X whose fiber over $x \in X$ is the Riemann surface corresponding to x. (Topologically this is $X \times S^h$.)

Assuming that the Z–cusps are invariant under Teichmuller maps, we may assume that there is an induced *universal curve of truncated hyperbolic surfaces* obtained by excising cusps.

Each $S(i)$ being compact (with or without boundary), $\widetilde{S(i)}$ is a hyperbolic metric space. We want to regard the universal cover $\widetilde{E}$ of E as being quasi-isometric to a ray T of hyperbolic metric spaces. To this end, we construct a quasi-isometric model of $\widetilde{E}$. Let the following hold.

(1) $T = [0, \infty)$.

(2) Vertex set $\mathcal{V} = \{n : n \in \mathbb{N} \cup \{0\}\}$.

(3) Edge set $\mathcal{E} = \{[n-1, n] : n \in \mathbb{N}\}$.

(4) $X_n = \Gamma = X_{[n-1,n]}$, where Γ is a Cayley graph of $\pi_1(S)$ with some fixed generating set.

(5) There exists K, ϵ and a map η_n such that $\eta_n\colon \Gamma \to \widetilde{S(n)}$ is a (K, ϵ) quasi-isometry for all i. Let η_n^{-1} denote its quasi-isometric inverse.

(6) The qi-embeddings of edge sets into vertex sets are given by:
- $\phi_n\colon X_{[n-1,n]} \to X_n$ is the identity map on Γ,
- $\phi_{n-1}\colon X_{[n-1,n]} \to X_{n-1}$ is the change of marking induced by sending $\psi_{n-1}^{-1}(S)$ to $\psi_n^{-1}(S)$.

By Lemma 3.6 and the fact that σ is a Lipschitz path in Teichmuller space, this tree of hyperbolic metric spaces satisfies the quasi-isometrically embedded condition. In fact, we get more. Each $S(i)$ corresponds, via $S^h(i)$ to a point x_i of Teich(S). Joining the x_i in order, one gets the Lipschitz path σ obtained above in Teich(S). Mapping the fiber over x_i to *an embedded incompressible surface* lying in a (uniformly) bounded neighborhood of the corresponding pleated surface and extending over product regions (using a metric product structure), we get a homeomorphism between the model and E. Further, since the Teichmuller distance between $S^h(i)$ and $S^h(i+1)$ is uniformly bounded above, the metric product is uniformly bi-Lipschitz to the region trapped between them in E. Pasting these homeomorphisms together and lifting to the universal cover, we get the following lemma.

Lemma 3.9 *If E^h is a simply degenerate end of a hyperbolic 3 manifold N^h with bounded geometry, then there is a sequence of equispaced pleated surfaces exiting E^h and hence a sequence of truncated pleated surfaces exiting $\widetilde{E}$. Further, $\widetilde{E}$ is quasi-isometric to a ray of hyperbolic metric spaces satisfying the qi-embedded condition.*

This lemma allows us to pass between E and its quasi-isometric model, the ray of hyperbolic metric spaces satisfying the qi-embedded condition.

Z–cusps in N correspond to Z–cusps in the boundary components of $\partial_0 M$. But this is not true for $(Z+Z)$–cusps. Recall that P_0 (resp. P_1) denotes the components of P whose fundamental group is virtually Z (resp. $(Z+Z)$). Let $N_0 = (N \cup (Z+Z)\text{ cusps})$. We shall now describe the universal cover $\widetilde{N_0}$ as a tree of hyperbolic metric spaces with possibly exceptional vertex. It is at this stage that we need to assume that N^h is adapted to a pared manifold (M, P) with incompressible boundary. Recall that the incompressibility of the boundary $\partial_0 M$ of a pared manifold does not require that ∂M be incompressible, but only that the components of $\partial_0 M = \partial M \setminus P$ be incompressible.

We give (M, P_1) a geometrically finite structure (with no extra parabolics as per definition of a hyperbolic structure adapted to (M, P_1)) and denote the *convex core* of this geometrically finite manifold as M_0. Note that M_0 has no Z–cusps corresponding to P_0 but continues to have $(Z+Z)$ cusps corresponding to P_1. Further, since all Z–cusps have been excised in N_0, all the $(Z+Z)$–cusps in N_0 are retained in M_0.

Let $E(1), E(2), \ldots, E(k)$ denote the simply degenerate ends of $(N_0 - \text{cusps})$. M_0 is homeomorphic by a homeomorphism that is a quasi-isometry to (the closure of) $N_0 \setminus \bigcup_i E(i)$. We identify M_0 with its image under this map and denote $M_0 \cap E(i) = F(i)$, where $F(i)$ is both *an embedded surface* in $E(i)$ cutting off the end, and

an incompressible boundary component of the pared manifold (M_0, P_1). Note that $F(i)$ need not be a truncated pleated surface; but we have constructed $\widetilde{E}$ as a ray of hyperbolic metric spaces, and hence we only need $\widetilde{F(i)}$ to be quasi-isometric to Γ, a Cayley graph of the fundamental group $\pi_1(S)$.

Remark That there exists such a geometrically finite hyperbolic manifold homeomorphic to N is part of Thurston's monster theorem. See McMullen [24; 25] for a different proof of the fact. Also, the limit set of a geometrically finite manifold is locally connected (Anderson–Maskit [2]). This shall be of use later.

Summary of notation We summarize here the notation introduced so far.

- (M, P) – pared manifold.
- $P_0 \subset P$ – components of P whose fundamental group is virtually Z.
- $P_1 \subset P$ – components of P whose fundamental group is virtually $(Z+Z)$.
- $N^h \in H(M, P)$. Since flaring ends of N^h do not contribute anything to the discussion, we identify N^h with its convex core $C(N^h)$.
- $N = (N^h - \text{cusps})$.
- $N_0 = N$ with $(Z+Z)$–cusps (corresponding to P_1) adjoined.
- M_0 – geometrically finite structure on (M, P_1).
- $E(i)$ for $i = 1 \dots k$ denote the ends of N. (Alternately, $E(i) = (E^h(i) - \text{cusps})$).
- M_0 is identified with its quasi-isometric image and so is thought of as a subset of N_0.
- $N_0 = M_0 \cup \bigcup_i E(i)$.
- $F(i) = M_0 \cap E(i)$.

Lemma 3.10 *$\widetilde{N_0}$ is quasi-isometric to a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition with possibly exceptional vertex α corresponding to $\widetilde{M_0} \subset \widetilde{N_0}$.*

Proof Note that $\widetilde{M_0} \subset \widetilde{N_0}$ is the universal cover of the convex core of a geometrically finite manifold and hence is a hyperbolic metric space. Let $\widetilde{F(i)} \subset \widetilde{N_0}$ represent a lift of $F(i)$ to $\widetilde{N_0}$. Then, $\widetilde{F(i)}$, being quasi-isometric to the fundamental group of a compact surface (with or without boundary) is a word-hyperbolic metric space. If $\widetilde{E(i)}$ is a lift of $E(i)$ containing $\widetilde{F(i)}$ then from Lemma 3.9, $\widetilde{E(i)}$ is a ray of hyperbolic metric spaces satisfying the qi-embedded condition. Since there are only finitely many ends E_i, we can thus regard $X = \widetilde{N_0}$ as a tree (T) of hyperbolic metric spaces such that the following hold.

- α is the root of T. $X_\alpha = \widetilde{M_0}$ is a hyperbolic metric space.
- T consists of a (finite or infinite) collection of rays emanating from α.
- Each copy of $\widetilde{E(i)}$ in $\widetilde{N_0}$ is quasi-isometric to a ray (T_i) of hyperbolic metric spaces satisfying the qi-embedded condition (Lemma 3.9).
- $N_0 = M_0 \cup \bigcup_i E(i)$.
- $F(i) = M_0 \cap E(i)$.
- As of now no restrictions are imposed on the inclusion of each $\widetilde{F_i}$ into $\widetilde{M_0}$.

These are precisely the defining conditions of a tree of hyperbolic metric spaces with possibly exceptional vertex satisfying the qi-embedded condition. □

Note T has a root vertex α which is possibly exceptional. The rest of T consists of a number of rays emanating from α.

3.3 A topological property of pared manifolds

Before we enter into the construction of quasiconvex sets, we shall describe a basic topological property of pared manifolds.

Lemma 3.11 *Let (M, P) be a pared manifold with incompressible boundary. Then*

- *no annulus component of P is freely homotopic to a curve on a torus component or on another annulus component of P,*
- *if two curves (which are not non-trivial powers of any other curves) on $\partial_0 M = \partial M \setminus P$ are freely homotopic to curves on the same torus component of P, then they are in fact freely homotopic to the same curve on a torus component of P and hence to each other.*

Proof Statement 1 Let Δ be a boundary torus. If possible, let A be an annulus in the boundary of M such that its core curve is freely homotopic to a curve σ on Δ. The complement of a small neighborhood of σ in Δ is an annulus A_1. Connecting the boundary curves of A_1 to those of A by the free homotopy we get an immersed annulus with boundary on the boundary of M, but not homotopic to ∂M. This proves that the core curve of A cannot be homotopic to the core curve of an annular component of P as this would contradict condition 3 of the definition of a pared manifold. If two core curves of annuli components of P are homotopic, we get a new annulus interpolating between these curves, again contradicting Condition 3 of the definition of a pared manifold. This proves the first part of the lemma.

Statement 2 Let M_Δ denote the cover of M corresponding to $\pi_1(\Delta)$. Let σ_1 and σ_2 be the curves homotopic to curves α_1 and α_2 on Δ. Let B denote the annulus between σ_1 and α_1, then the intersection number between α_1 and α_2 in Δ is the same as the intersection number between α_2 and B in M_Δ (see for instance [46] or [7]). Since α_2, σ_2 are homotopic in M, this homotopy lifts to M_Δ. Hence, the intersection number between α_2 and B in M_Δ is the same as the intersection number between σ_2 and B in M_Δ (by homotopy invariance of intersection numbers). If σ_1 and σ_2 are on different components then this latter number is zero.

Else, they belong to the same component K of $\partial_0(M)$. $\pi_1(K)$ is either free non-abelian or else K is a closed surface of genus greater than 1. Since the fundamental group of K injects into that of M, and since elements represented by σ_1, σ_2 commute, therefore σ_1 and σ_2 must be powers of the same curve. But the hypothesis says that the curves σ_1 and σ_2 are not powers of any other curves. Hence the two denote the same curve.

In either case, σ_1 and σ_2 are freely homotopic to the same curve on Δ and hence to each other. □

4 Construction of qi-embedded sets

From Lemma 3.10 we have that $\widetilde{N_0}$ is quasi-isometric to a tree T of hyperbolic metric spaces satisfying the qi-embedded condition with possibly exceptional vertex α corresponding to $\widetilde{M_0} \subset \widetilde{N_0}$. In fact T has a root vertex α and consists of a number of distinct rays emanating from α. Let X denote the tree of spaces, X_v denote vertex space corresponding to vertex v, X_e denote edge space corresponding to edge e.

For convenience of exposition, we shall sometimes need to modify X, X_v, X_e by quasi-isometric perturbations and regard them as graphs. Given a geodesically complete metric space (Z, d) of bounded geometry, choose a maximal disjoint collection $\{B_1(z')\}$ of disjoint 1–balls. Then by maximality, for all $z \in Z$ there exist z' in the collection such that $d(z, z') < 2$. Construct a graph Z_1 with vertex set $\{z'\}$ and edge set consisting of distinct vertices z_a, z_b such that $d(z_a, z_b) \leq 4$. Then Z_1 equipped with the path-metric is quasi-isometric to (Z, d) (see for instance [21]). Henceforth we shall move back and forth between descriptions of spaces as Riemannian manifolds and as graphs, assuming that there is a quasi-isometry between them. The quasi-isometry will usually be suppressed.

Let v be a vertex of T. Let $v_- \neq \alpha$ be the penultimate vertex on the geodesic edge path from α to v. Let e_- denote the directed edge from v_- to v. Define a quasi-isometry

ϕ_v: $f_{e_-}(X_{e_-}\times\{0\}) \to f_{e_-}(X_{e_-}\times\{1\})$ as follows. If $p \in f_{e_-}(X_{e_-}\times\{0\}) \subset X_{v_-}$, choose $x \in X_{e_-}$ such that $p = f_{e_-}(x\times\{0\})$ and define

$$\phi_v(p) = f_{e_-}(x\times\{1\}).$$

Note that in the above definition, x is arbitrarily chosen from a set of bounded diameter.

Let μ be a geodesic in X_{v_-}, joining $a, b \in f_{e_-}(X_{e_-}\times\{0\})$. $\Phi_v(\mu)$ will denote a geodesic in X_v joining $\phi_v(a)$ and $\phi_v(b)$. For our purposes since all the edge and vertex spaces on a ray, (apart from X_α) are identical, we might as well identify $f_{e_-}(X_{e_-}\times\{0\}) \subset X_{v_-}$ with X_{v_-} itself. Similarly we may identify $f_{e_-}(X_{e_-}\times\{1\}) \subset X_v$ with X_v itself. Thus, ϕ_v is regarded as a quasi-isometry from X_{v_-} to X_v for $v, v_- \neq \alpha$. We define a corresponding map Φ_v from geodesics in X_{v_-} to geodesics in X_v by taking a geodesic joining $a, b \in X_{v_-}$ to one joining $\phi_v(a), \phi_v(b) \in X_v$. From Lemma 3.9, there exist $k, \epsilon > 0$ such that for all v, ϕ_v is a (k, ϵ)–quasi-isometry.

In Section 4.1, we shall construct a *hyperbolic ladder-like* set B_λ containing λ. In Section 4.2, we shall construct a retract $\Pi_\lambda\colon \widetilde{N_0} \to B_\lambda$. In Section 4.3, we shall prove that Π_λ *is* a retract, ie it fixes B_λ and does not stretch distances much. This will show that B_λ is quasi-isometrically embedded in $\widetilde{N_0}$.

4.1 Construction of the hyperbolic ladder-like set B_λ

The quasi-isometrically embedded set B_λ that we intend to construct will contain the images of a geodesic under such quasi-isometries. Suppose $F(i)$ cuts off the end $E(i)$ and $\mu \subset \widetilde{F(i)} \subset \widetilde{E(i)}$. Then denote the union of the images of μ under the quasi-isometries taking it to the different vertex spaces as $B(\mu)$. That is to say, if $v_0, v_1, \ldots$ denote the vertices of the ray exiting the end, then the union of μ, $\Phi_{v_1}(\mu)$, $\Phi_{v_2} \circ \Phi_{v_1}(\mu)$, $\Phi_{v_3} \circ \Phi_{v_2} \circ \Phi_{v_1}(\mu)$, etc is denoted by $B(\mu)$.

Recall that M_0 denotes a convex geometrically finite hyperbolic manifold with hyperbolic structure adapted to the pared manifold (M, P_1), where $P_1 \subset P$ denotes the set of $(Z+Z)$–cusps. Also $\widetilde{M_0}$ the universal cover of M_0 is identified with $\widetilde{N_0} - \bigcup_i E(i)$. M_{cc} will denote M_0 minus a neighborhood of the $(Z+Z)$–cusps. Note that M_{cc} can be thought of as the Scott core of the manifold. M_{gf} will denote a geometrically finite hyperbolic structure adapted to the pared manifold (M, P). The difference between M_{gf} and M_0 is that while M_{gf} is adapted to the pared manifold (M, P), M_0 is adapted to the pair (M, P_1). Thus M_{gf} has incompressible boundary as a pared manifold, but the same may not be true of M_0. One may also think of M_{gf} as M_0 with Z–cusps (corresponding to P_0) adjoined.

Fix a set of neighborhoods of the cusps of M_0, which are sufficiently separated from each other. Recall that M_{cc} denotes M_0 minus these cusps. Then $\widetilde{M_{cc}} = \widetilde{M_0} \setminus \bigcup \mathcal{H}$ where $\mathcal{H}$ denotes an equivariant system of horoballs corresponding to lifts of $(Z+Z)$–cusps. Note that $\widetilde{M_{cc}}$ is quasi-isometric to the Cayley graph of $\pi_1(M)$ as the quotient is compact [19; 20].

Let $\lambda^h = [a, b]$ be a hyperbolic geodesic in $\widetilde{M_{gf}}$. Let β^h be a hyperbolic geodesic in $\widetilde{N^h}$ joining a, b. Here $\widetilde{M_{gf}}$ is identified with its image in $\widetilde{N^h}$. We shall show that if λ^h lies outside a large ball around a fixed reference point $p \in \widetilde{M_{gf}}$ then so does $\beta^h \in \widetilde{N^h}$. Recall that $F(i)$ for $i = 1, \dots, k$ are components of $\partial_0 M = \partial M - P$. We can identify each $F(i)$ with a boundary component of M_0 so that the inclusion of $F(i)$ into M_{cc} induces an injection at the level of the fundamental group. We identify each $F(i)$ with the first *truncated pleated surface* exiting the end $E(i)$.

Summary of Notation:

- M_0 – convex geometrically finite hyperbolic structure on (M, P_1).
- M_{gf} – convex geometrically finite hyperbolic structure adapted to the pared manifold (M, P).
- M_{cc} – M_0 minus $(Z+Z)$–cusps.
- $N^h \in H(M, P)$ has bounded geometry and is geometrically tame.
- N – (N^h – cusps).
- N_0 – N with $(Z+Z)$ cusps adjoined.
- M_{gf} is identified with its homeomorphic image in N^h taking cusps to cusps.
- M_0 is identified with its homeomorphic image in N^0.
- M_{cc} is identified with its homeomorphic image in N.

Construction of $B_0(\lambda)$ Given λ^h we shall first construct an ambient quasigeodesic λ in $\widetilde{M_0}$. Since λ^h is a hyperbolic geodesic in $\widetilde{M_{gf}}$ there are unique entry and exit points for each horoball that λ^h meets and hence unique Euclidean geodesics joining them on the corresponding horosphere. Replacing the segments of λ^h lying inside Z–horoballs by the corresponding Euclidean geodesics, we obtain an ambient quasigeodesic λ in $\widetilde{M_0}$ by Theorem 2.7. See Figure 1.

Again, since λ coincides with λ^h outside horoballs corresponding to Z–cusps, there exist unique entry and exit points of λ into horoballs corresponding to $(Z+Z)$–cusps, and hence again, unique Euclidean geodesics joining them on the corresponding horosphere. Replacing the segments of λ lying inside $(Z+Z)$–horoballs by the

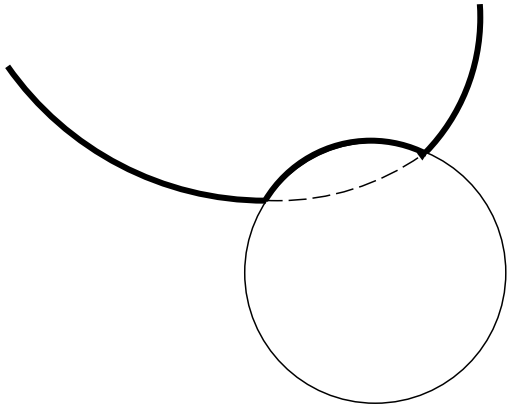

Figure 1

corresponding Euclidean geodesics, we obtain an ambient quasigeodesic λ_{cc} in $\widetilde{M_{cc}}$ by Theorem 2.7. Each of the Euclidean geodesics mentioned above along with the hyperbolic geodesic joining its end-points (and lying entirely within the horoball) bounds a totally geodesic disk. (See Figure 2).

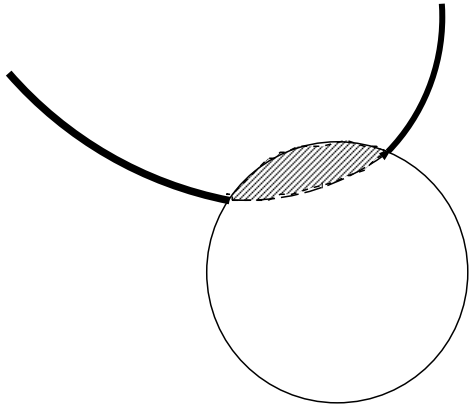

Figure 2

The union of λ and all these totally geodesic disks is denoted by $B_0(\lambda)$. There exists $C_1 > 0$ such that each $B_0(\lambda)$ is C_1–quasiconvex in $\widetilde{M_0}$. (See for instance McMullen [26, Section 8].)

$B_0^{\mathrm{aug}}(\lambda)$ will denote $\lambda \cup \mathcal{H}(\lambda)$, where $\mathcal{H}(\lambda) \subset \mathcal{H}$ denotes the collection of horoballs in $\mathcal{H}$ that λ meets. Again, from Theorem 2.7, $B_0^{\mathrm{aug}}(\lambda)$ can be assumed to be C_1–quasiconvex in $\widetilde{M_0}$.

Note We will be using $\widetilde{N_0}$ rather than $\widetilde{N^h}$ for the construction of B_λ. Hence $\widetilde{M_0}$ rather than $\widetilde{M_{\mathrm{gf}}}$ is the relevant space. Recall (Lemma 3.10) that $\widetilde{N_0}$ is a tree of hyperbolic metric spaces with possibly exceptional vertex corresponding to $\widetilde{M_0}$ satisfying the qi-embedded property.

Construction of $B_1(\lambda)$ Technically, (though not conceptually) this step is the most intricate one, and is the one new construction that is required to handle parabolics. The values of the constants chosen here become clear only through hindsight. The reason behind the choices made here will become clear only while constructing the projection in the next subsection. As mentioned in Section 1.1, we have to be quite careful while handling the different kinds of cusps that arise:

(1) **Z–cusps**,

(2) **$(Z+Z)$ cusps**, where no curve on any component of $\partial_0 M$ is homotopic to a curve on the boundary torus corresponding to the $(Z+Z)$ cusp,

(3) **Z and $(Z+Z)$ cusps**, where some curve(s) on component(s) of $\partial_0 M$ is (are) homotopic to a particular curve on the boundary torus corresponding to the $(Z+Z)$ cusp or to some multiple of the core curve of a Z–cusp.

Next we need to consider the parts of λ that can have substantial overlap with the ends $\widetilde{E}$, that is to say those pieces of λ that follow some $\widetilde{F(i)}$ for a considerable length. There are geodesic segments on each $\widetilde{F(i)}$ parallel to these pieces. The union of all these parallel geodesic segments along with $B_0(\lambda)$ will be denoted by $B_1(\lambda)$. Details follow.

Lemma 4.1 *Let X be a convex subset of $\mathbb{H}^n$. Let Y, Z be closed convex subsets of X. Let $Y_\delta = N_\delta(Y)$, $Z_\delta = N_\delta(Z)$ be closed δ–neighborhoods of Y, Z respectively where δ denotes the hyperbolicity constant of the ambient space (in this case a convex subset of $\mathcal{H}^n$). Let Π denote nearest point projection of X onto Y. Then $d_X(\Pi(Z), Y_\delta \cap Z_\delta)$ is bounded in terms of δ.*

Proof Case (a) $(Y_\delta)\cap(Z_\delta) \neq \varnothing$. Let $z \in Z$. We want to show that there exists $D = D(\delta) > 0$ independent of z such that $d_X(\Pi(z), (Y_\delta)\cap(Z_\delta)) < D$. Let Π_0 denote nearest point retraction of X onto the convex set $(Y_\delta)\cap(Z_\delta)$ (which, being an intersection of convex sets, is convex).

As usual, $[a, b]$ denotes a geodesic joining a, b.

$$[z, \Pi(z)] \subset N_\delta([z, \Pi_0(z)] \cup [\Pi_0(z), \Pi(z)]).$$

But $[z, \Pi_0(z)] \subset Z_\delta$ and $[\Pi_0(z), \Pi(z)] \subset Y_\delta$. Hence,

$$[z, \Pi(z)] \subset N_\delta(Y \cup Z).$$

- $z \in Z \subset Z_\delta$.
- $\Pi_0(z) \in Y_\delta \cap Z_\delta$.
- $\Pi(z) \in Y_\delta$.

The geodesic $[z, \Pi(z)]$ has to cross into Y_δ at some point $p \in Y_\delta \cap Z_\delta$. (Note that here we are using the fact that $Y_\delta \cap Z_\delta$ is closed.) But then $d(\Pi(z), p) \leq \delta$ (else we would be able to find another point q at a distance of less than δ from p in Y contradicting the definition of $\Pi(z)$.) This proves Case (a).

Case (b) $Y_\delta \cap Z_\delta = \varnothing$. Let $z_1, z_2 \in Z$ and let $y_i = \Pi(z_i)$ for $i = 1, 2$. Since local quasi-geodesics are global quasi-geodesics [20; 19] in hyperbolic metric spaces, there exists $D > 0$ such that if $d(y_1, y_2) \geq D$, then $[z_1, y_1]\cup[y_1, y_2]\cup[y_2, z_2]$ is a 2δ–quasigeodesic. (To see this, first note that [36, Lemma 3.1] ensures that $[z_1, y_1]\cup[y_1, y_2]\cup[y_2, z_2]$ is a $C = C(2\delta)$–quasigeodesic.That $C = 2\delta$ comes from applying δ–thinness to triangles (z_1, y_1, y_2) and (y_1, y_2, z_2) in succession.) In this case, $([z_1, y_1]\cup[y_1, y_2]\cup[y_2, z_2]) \subset N_{2\delta}[z_1, z_2]$. But $[z_1, z_2] \subset Z$ as Z is convex. In particular $y_1, y_2 \in N_{2\delta}(Z)$. This contradicts the assumption that $Y_\delta \cap Z_\delta = \varnothing$. Hence, $d(y_1, y_2) \leq D$, proving the lemma. □

The above lemma can be 'quasi-fied' as follows. (The scheme of proof being a 'quasification' of the above, we omit it.)

Lemma 4.2 *Given $\delta, C > 0$, there exists $k > 0$ such that the following holds. Let X be a δ–hyperbolic metric space and Y, Z be C–quasiconvex subsets of X. Let Y_k, Z_k denote k–neighborhoods of Y, Z respectively. Let Π denote a nearest point projection of X onto Y. Then $d_X(\Pi(Z), (Y_k \cap Z_k))$ is uniformly bounded. Hence there exists $k > 0$ such that $(\Pi(Z) \subset (Y_k \cap Z_k))$ if the latter set is non-empty and $\Pi(Z)$ has diameter bounded by $k > 0$ if $Y_k \cap Z_k = \varnothing$.*

Let F be one of the $F(i)$. Either $\widetilde{F}$ is quasiconvex in $\widetilde{M_0}$ (when there are no accidental parabolics) or there exist disjoint curves σ_i for $i = 1, \dots, l$ (by Lemma 3.11) homotopic to curves on components of P. Therefore σ_i correspond to parabolics. Each σ_i lies at a bounded distance from some Euclidean geodesic η_i on a torus or annular component of P in M_{cc}. Each η_i forms the boundary of a totally geodesic Z cusp κ_i (possibly a totally geodesic subset of a $Z + Z$–cusp). Further, the two curves η_i and σ_i bound between themselves an annulus A_i. Let $G = F \cup \bigcup_i (A_i \cup \kappa_i)$. Then $\widetilde{G}$ is quasiconvex in $\widetilde{M_0}$. Choose a constant $C_2 > 0$ such that each such $\widetilde{G}$ is C_2–quasiconvex in $\widetilde{M_0}$. Recall that $B_0(\lambda)$ and $B_0^{\text{aug}}(\lambda)$ are C_1–quasiconvex.

We are now in a position to start constructing $B_1(\lambda)$ from $B_0(\lambda)$.

Choosing $Y = B_0(\lambda)$ and $Z = \widetilde{G(i)}$ (one lift of $G(i)$ is considered in turn, and choosing k to be the maximum of the corresponding finite collection of k), we obtain a $k > 0$ from Lemma 4.2 above, so that $(\Pi(Z) \subset (Y_k \cap Z_k))$. Let $\widetilde{F(i)}$, $i = 1, \dots, s$ denote the different lifts of the F_i that intersect a $(3k + 4\delta)$ neighborhood of $B_0(\lambda)$. Since λ is a finite segment, the number s is finite. Choose $p_i, q_i \in \widetilde{F(i)} \cap N_{3k+4\delta}(B_0(\lambda))$ such that $d(p_i, q_i)$ is maximal.

Fix $D > 0$. Choose the copies of $\widetilde{F(i)}$ for which $d(p_i, q_i) \geq D$.

Note The number D picks up significance in the *p–incompressible* case. We shall come back to this in Section 5.2, when we simplify our problem under this extra assumption.

Redefining s if necessary, we let $\widetilde{F(1)}, \dots, \widetilde{F(s)}$ denote this collection. Let μ_i denote the geodesic in $\widetilde{F(i)}$ joining p_i and q_i. Let $\mathcal{E}$ denote the corresponding collection of $\widetilde{E(i)}$. Let $\mathcal{E}'$ denote the collection of the remaining $\widetilde{E(i)}$. We would like to define

$$B_1(\lambda) = B_0(\lambda) \cup \bigcup \mu_i.$$

The choice of $\mathcal{E}$, $\mathcal{E}'$ is dependent on k. We want each $\widetilde{E}$ to be such that

- either $\widetilde{F}$ is quasiconvex in $\widetilde{M_0}$ (case where no curve on F is parabolic in $\widetilde{M_0}$),
- or, $\widetilde{G}$ (which is always quasiconvex in $\widetilde{M_0}$) intersects at most one $\mathbf{H} \in \mathcal{H}(\curlyvee)$.

Lemma 4.3 *There exists $k' > 0$ such that if $\widetilde{G}$ intersects more than one horoball in $\mathcal{H}(\lambda)$, then $N_{k'}(B_0(\lambda)) \cap \widetilde{F}$ has non-zero diameter.*

Proof Suppose $\widetilde{G}$ intersects more than one horoball $\mathbf{H} \in \mathcal{H}(\backsim)$, say $\mathbf{H}_1$, $\mathbf{H}_2$. Let $p \in \mathbf{H}_1 \cap \widetilde{G}$ and $q \in \mathbf{H}_2 \cap \widetilde{G}$. Let $[p,q]_0$ denote the geodesic in $\widetilde{G}$ joining p,q, and let $[p,q]$ denote the geodesic in $\widetilde{M_0}$ joining p,q. $[p,q]_0$ lies in a C_1–neighborhood of $[p,q]$ as $\widetilde{G}$ is C_1–quasiconvex. Since horoballs are sufficiently separated in both $\widetilde{G}$ and $\widetilde{M_0}$, therefore some part of $[p,q]_0$ lies on $\widetilde{F}$. Hence $N_{C_1}([p,q]) \cap \widetilde{F} \neq \varnothing$ and has non-zero diameter.

Next, there is a subsegment of λ starting in $\mathbf{H}_1$ and ending in $\mathbf{H}_2$. Let λ' be the part of this subsegment lying outside the interior of the horoballs $\mathbf{H}_1$ and $\mathbf{H}_2$. Then, off horoballs, λ' and $[p,q]$ track each other (by Theorem 2.8), ie there exists $C_2 > 0$ such that $[p,q] \subset N_{C_2}(\lambda \cup \mathcal{H}(\lambda))$ and the subsegment of $[p,q]$ lying outside the horoballs $\mathbf{H}_1$ and $\mathbf{H}_2$ is contained in a C_2–neighborhood of $B_0(\lambda)$. Choosing $k' = C_1 + C_2$, we are through. □

The next corollary follows.

Corollary 4.4 *Given $k > 0$, there exists $k' > 0$ such that if the $2k$–neighborhood of $\widetilde{G}$ intersects more than one horoball in $\mathcal{H}(\lambda)$, then the diameter of $N_{k'}(B_0(\lambda)) \cap \widetilde{F}$ is non-zero.*

Here the term $2k$ occurs so as to recall the k of Lemma 4.2. Choose $K = 3k + 4\delta + k'$, where k is as in Lemma 4.2 and k' is as in Corollary 4.4 above.

We now return to our construction of $B_1(\lambda)$, or more specifically the μ_i that we had mentioned after Lemma 4.2.

Let $\widetilde{F(i)}$, $i = 1, \ldots, s$ (redefining s if necessary) denote the different lifts of the $F(i)$ that intersect a K neighborhood of $B_0(\lambda)$. Since λ is a finite segment, the number s is finite. Choose $p_i, q_i \in \widetilde{F(i)} \cap N_{3k+4\delta}(B_0(\lambda))$ such that $d(p_i, q_i)$ is maximal.

Recall that $p_i, q_i \in \widetilde{F(i)} \cap N_{3k+4\delta}(B_0(\lambda))$ for some copies of $\widetilde{F(i)}$. Choose the copies of $\widetilde{F(i)}$ for which $d(p_i, q_i) > 0$. Redefining s once more, if necessary, we let $\widetilde{F(1)}, \ldots, \widetilde{F(s)}$ denote this collection. Let μ_i denote the geodesic in $\widetilde{F(i)}$ joining p_i and q_i. Let $\mathcal{E}$ denote the corresponding collection of $\widetilde{E(i)}$. Let $\mathcal{E}'$ denote the collection of the remaining $\widetilde{E(i)}$. We are finally in a position to define

$$B_1(\lambda) = B_0(\lambda) \cup \bigcup \mu_i.$$

See Figure 3 below. λ lies on $\widetilde{M_0}$. 'Parallel' segments $\mu_1, \ldots, \mu_k$ are constructed lying on $\widetilde{F(1)}, \ldots, \widetilde{F(k)}$.

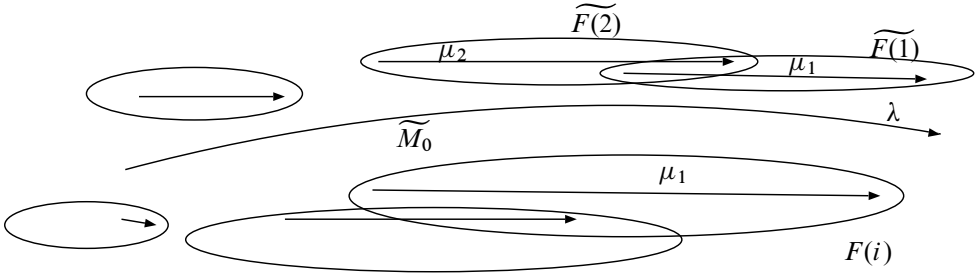

Figure 3

Construction of $B(\mu)$ Recall from Lemma 3.9 that each $\widetilde{E}$ is quasi-isometric to a ray $[0,\infty)$ of hyperbolic metric spaces with integer points corresponding to vertices and $[n-1,n]$ (for $n\in\mathbb{N}$) corresponding to edges. Fix a particular $\widetilde{F}$ cutting off the end $\widetilde{E}$ and a geodesic segment $\mu\subset\widetilde{F}$. Let X_i denote the vertex spaces for $i=0,1,\dots$. Let ϕ_i denote the quasi-isometry from X_{i-1} to X_i and Φ_i denote the corresponding map from geodesic segments in X_{i-1} to those in X_i. Define

$$\mu^i=\Phi_i\circ\cdots\circ\Phi_1(\mu) \text{ and } B(\mu)=\bigcup\mu^i.$$

See Figure 4 below.

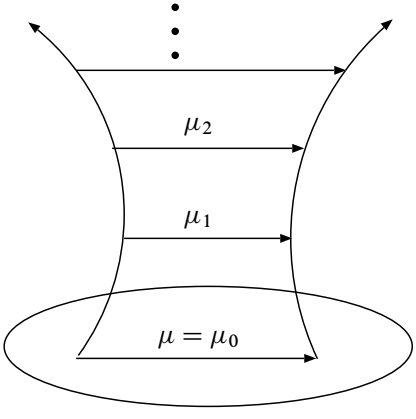

Figure 4: Hyperbolic ladder-like Set

Definition of B_λ Finally define

$$B_\lambda=B_1(\lambda)\cup\bigcup_{i=1\dots s}B(\mu_i).$$

4.2 Definition of Π_λ

Recall $\mathcal{E} = \{\widetilde{E(1)}, \ldots, \widetilde{E(s)}\}$ and $\mathcal{E}' = \{\widetilde{E(s+1)}, \widetilde{E(s+2)}, \ldots\}$. We next want to show that B_λ is quasi-isometrically embedded. To prove this, we shall construct a retraction Π_λ. Let X_{ij} denote the vertex spaces in $\widetilde{E(i)} \in \mathcal{E}$. Below, we shall have need to replace $\widetilde{E(i)}$ by the union of vertex spaces $\bigcup_j X_{ij}$ (fixing i and letting j vary from 0 to ∞) with the understanding that the latter has the metric induced from $\widetilde{E(i)}$. The difference between $\widetilde{E(i)}$ and $\bigcup_j X_{ij}$ is just that the edge spaces of $\widetilde{E(i)}$ are not explicitly present in the latter.

Let ϕ_{ij} denote the quasi-isometry from $X_{i,j-1}$ to X_{ij} and let Φ_{ij} denote the induced map from geodesics in $X_{i,j-1}$ to geodesics in X_{ij}. Let $\mu_{ij} \subset X_{ij}$ denote the image of μ_i under the composition of the maps Φ_{ik} as k runs from 1 to j.

- Let Π_{ij} denote a nearest point retraction onto $\mu_{ij} \subset X_{ij}$. On $\widetilde{E(i)} = \bigcup_j X_{ij}$, Π_i is defined by
$$\Pi_i(x) = \Pi_{ij}(x) \text{ for } x \in X_{ij}.$$
- On $\widetilde{M_0} = X_\alpha$ (recall that α corresponds to the possibly exceptional vertex of the tree of spaces) let Π_0 denote a nearest point retraction onto $B_0(\lambda)$.

On the remaining set $\mathcal{E}'$, Π_λ needs to be defined with some caution. Suppose $\widetilde{E} \in \mathcal{E}'$. Observe first that no point on $\widetilde{F} \subset \widetilde{E}$ lies at a distance less than K from $B_0(\lambda)$, where K is as chosen after Corollary 4.4. Also let k be as in Lemma 4.2.

Case (a) $N_{2k}(\widetilde{G})$ does not intersect $\mathcal{H}(\lambda)$. In this case, choose $y \in \widetilde{F}$ arbitrarily and define
$$\Pi_E(x) = \Pi_0(y) \text{ for all } x \in \widetilde{E}.$$
For $\widetilde{E} = \widetilde{E(i)}$, denote Π_E by Π_i.

Case (b) $N_{2k}(\widetilde{G})$ intersects precisely one $\mathbf{H} \in \mathcal{H}(\lambda)$. In this case, there exists a unique lift $\widetilde{\sigma}$ of a curve σ on F, (parabolic in M_0) lying on $\widetilde{F}$ at a bounded distance from $\mathbf{H}$.

As in the construction of $B(\mu)$ for $\mu \subset \widetilde{F} \subset \widetilde{E}$, construct $B(\widetilde{\sigma}) = \bigcup_j \widetilde{\sigma}_j \subset \bigcup_j \widetilde{S(j)}$ where $S(j)$ denotes the jth member of a sequence of truncated pleated surfaces exiting E and $\widetilde{\sigma}_j = \Phi_j(\widetilde{\sigma}_{j-1})$ (Φ_j is the map on geodesics induced by the quasi-isometry ϕ_j from $\widetilde{S(j-1)}$ to $\widetilde{S(j)}$). Note that this construction works just as well for infinite geodesics. On $\widetilde{S(j)}$, define $\Pi_{\sigma j}$ to be a nearest point projection onto $\widetilde{\sigma}_j$. Define
$$\Pi^0_\sigma(x) = \Pi_{\sigma j}(x) \text{ for } x \in \widetilde{S(j)}.$$

Next, let Π_σ denote nearest point projection of $\widetilde{E}$ onto $\widetilde{\sigma}$. Define

$$\Pi_E(x) = \Pi_0 \circ \Pi_\sigma(x) \circ \Pi^0_\sigma(x) \text{ for all } x \in \widetilde{E}.$$

For $\widetilde{E} = \widetilde{E(i)}$, denote Π_E by Π_i.

The reason for factoring the projection $\Pi_\sigma(x)$ through $\Pi^0_\sigma(x)$ is that on $\widetilde{F(i)}$ we want Π_i to coincide with Π_σ. (Else nearest point projections will have to be taken in the $\widetilde{E(i)}$ metric and $\widetilde{F(i)}$ is not quasiconvex in this metric, so that nearest point projections might well differ substantially in the two metrics.)

Non-Definition We would like to define

$$\begin{aligned}\Pi_\lambda(x) &= \Pi_i(x), x \in \widetilde{E(i)} \in \mathcal{E} \\ &= \Pi_0(x), x \in X_\alpha \\ &= \Pi_i(x), x \in \widetilde{E(i)} \in \mathcal{E}'.\end{aligned}$$

Caveat Our non-definition above is not yet a real definition as Π_λ has been putatively defined twice on each $\widetilde{F(i)}$, once regarding $\widetilde{F(i)}$ as a subset of $\widetilde{M_0}$ and again, regarding it as a subset of $\widetilde{E(i)}$. We will show that there is at most a bounded amount of discrepancy between the two definitions and so any choice will work. So we make the following definition.

Definition

$$\begin{aligned}\Pi_\lambda(x) &= \Pi_i(x), x \in \widetilde{E(i)} - X_{i0}, \widetilde{E(i)} \in \mathcal{E} \\ &= \Pi_0(x), x \in X_\alpha \\ &= \Pi_i(x), x \in \widetilde{E(i)} \in \mathcal{E}'.\end{aligned}$$

4.3 Π_λ is a retract

We will show in this subsection that there exists $C > 0$ such that for all $x, y \in X = \widetilde{N_0}, d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d(x, y)$ There are three parts in the definition of Π_λ as described above. We discuss the three cases separately.

Since we are dealing with graphs, it suffices to prove the result for $d(x, y) = 1$. We need to check the following.

(1) $x \in \widetilde{F(i)} = \widetilde{E(i)} \cap \widetilde{M_0}$ for some $\widetilde{E(i)} \in \mathcal{E}$. We want to show that

$$d(\Pi_i(x), \Pi_0(x)) \leq C.$$

(2) $x, y \in \widetilde{E(i)}$ for some $\widetilde{E(i)} \in \mathcal{E}$ and $d(x, y) = 1$. We want to show that

$$d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C.$$

(3) $x, y \in \widetilde{M_0} = X_\alpha$ and $d(x, y) = 1$. We want to show that

$$d(\Pi_0(x), \Pi_0(y)) \leq C.$$

(4) $x \in \widetilde{F(i)} = \widetilde{E(i)} \cap \widetilde{M_0}$ for some $\widetilde{E(i)} \in \mathcal{E}'$. We want to show that

$$d(\Pi_i(x), \Pi_0(x)) \leq C.$$

(5) $x, y \in \widetilde{F(i)} = \widetilde{E(i)} \cap \widetilde{M_0}$ for some $\widetilde{E(i)} \in \mathcal{E}'$ and $d(x, y) = 1$. We want to show that

$$d(\Pi_i(x), \Pi_i(y)) \leq C.$$

In the above check-list, the first and fourth steps ensure that there is approximate agreement on $\widetilde{F_i}$ for the two possible definitions of Π_λ occurring in the non-definition given earlier. This ensures smooth passage from the non-definition to the definition. The rest of the steps are required to prove the three cases in the definition of Π_λ.

Step 1: Bounded discrepancy on $\widetilde{F(i)}$ when $\widetilde{E(i)} \in \mathcal{E}$ The next lemma follows easily from the fact that local quasigeodesics in a hyperbolic metric space are quasigeodesics [19] (See also [36, Lemma 3.1]). If x, y are points in a hyperbolic metric space, $[x, y]$ will denote a geodesic joining them.

Lemma 4.5 *Given $\delta, C_1 > 0$, there exist D, C_0 such that if a, b, c, d are vertices of a δ–hyperbolic metric space (Z, d), and $W \subset Z$ is a C_1–quasiconvex set, with $d(a, W) = d(a, b)$, $d(d, W) = d(c, d)$ and $d(b, c) \geq D$ then $[a, b] \cup W \cup [c, d]$ is C_0–quasiconvex. Also, if $[b, c]_W$ denotes the shortest path in W joining b, c, then $[a, b] \cup [b, c]_W \cup [c, d]$ is a C_0–quasigeodesic. Further, if $[b, c]_{\text{amb}}$ denotes an ambient quasigeodesic in $(Z - W)$ (adding on b, c as the initial and terminal points), then $[a, b] \cup [b, c]_{\text{amb}} \cup [c, d]$ is an ambient C_0–quasigeodesic in $(Z - W)$.*

The next couple of lemmas are taken from [36], where they are stated in the general framework of trees of hyperbolic metric spaces satisfying the qi-embedded condition. X_v, X_e, f_v denote respectively vertex space, edge space and qi-embedding of edge space in vertex space.

Lemma 4.6 [36, Lemma 3.6] *Let X be a tree T of hyperbolic metric spaces and v be a vertex of T. Let $C > 0$. Let $\mu_1 = [a, b] \subset X_v$ be a geodesic and let e be an edge of T incident on v. Let $p, q \in N_C(\mu_1) \cap f_v(X_e)$ be such that $d_v(p, q)$ is maximal. Let μ_2 be a geodesic in X_v joining p, q. If $r \in N_C(\mu_1) \cap f_v(X_e)$, then $d_v(r, \mu_2) \leq D_1$ for some constant D_1 depending only on C, δ.*

Proof Let π denote a nearest point projection onto μ_1. Since μ_2 and $[\pi(p), \pi(q)] \subset \mu_1$ are geodesics whose end-points lie at distance at most C apart, there exists C' such that $[\pi(p), \pi(q)] \subset N_{C'}(\mu_2)$. If $\pi(r) \in [\pi(p), \pi(q)]$, then

$$d(r, \mu_2) \leq C + C'.$$

If $\pi(r) \notin [\pi(p), \pi(q)]$, then without loss of generality, assume $\pi(r) \in [a, \pi(p)] \subset [a, \pi(q)]$. Then

$$\begin{aligned} d(p,q) &\geq d(r,q) \\ &\geq d(\pi(r), \pi(q)) - 2C \\ &= d(\pi(r), \pi(p)) + d(\pi(p), \pi(q)) - 2C \\ &\geq d(\pi(r), \pi(p)) + d(p,q) - 4C \\ \Rightarrow d(\pi(r), \pi(p)) &\leq 4C \\ \Rightarrow d(r,p) &\leq 6C \\ \Rightarrow d(r, \mu_2) &\leq 6C. \end{aligned}$$

Choosing $D_1 = \max\{C + C', 6C\}$, we are through. □

Lemma 4.7 [36, Lemma 3.7] *Let μ_1, μ_2 be as in Lemma 4.6 above. Let π_i denote nearest point projections onto μ_i $(i = 1, 2)$. If $p \in f_v(X_e)$, then $d(\pi_1(p), \pi_2(p)) \leq C_6$ for some constant C_6 depending on δ alone.*

Proof If $d(\pi_1(p), \pi_1 \cdot \pi_2(p)) \leq D$, then $d(\pi_1(p), \pi_2(p)) \leq C + D$.

Else, suppose $d(\pi_1(p), \pi_1 \cdot \pi_2(p)) > D$. Then $[p, \pi_1(p)] \cup [\pi_1(p), \pi_1 \cdot \pi_2(p)] \cup [\pi_1 \cdot \pi_2(p), \pi_2(p)]$ is a (uniform) quasigeodesic by Lemma 4.5 . But $p, \pi_2(p) \in f_v(X_e)$ which is quasiconvex. Hence there exists (uniform) C_1 and $r \in f_v(X_e)$ such that $d(r, \pi_1(p)) \leq C_1$.

Then, by Lemma 4.6 above, there exists $s \in \mu_2$ such that $d(s, \pi_1(p)) \leq C_1 + D_1$.

Again (see for instance [36, proof of Lemma 3.2]), $(p, s)_{\pi_2(p)} \leq 2\delta$. Hence,

$$(p, \pi_1(p))_{\pi_2(p)} \leq 2\delta + C_1 + D_1.$$

Similarly, $(p, \pi_1 \cdot \pi_2(p))_{\pi_1(p)} \leq 2\delta$. Hence,

$$(p, \pi_2(p))_{\pi_1(p)} \leq 2\delta + C.$$

Therefore, using the definition of Gromov inner product,

$$d(\pi_1(p), \pi_2(p)) = (p, \pi_1(p))_{\pi_2(p)} + (p, \pi_2(p))_{\pi_1(p)} \leq 4\delta + 2C + D_1.$$

Choosing $C_6 = 4\delta + 2C + D_1$ we are through. □

Though we have stated and proved Lemma 4.7 and Lemma 4.6 for geodesic segments, the proof goes through for quasiconvex sets. What Lemma 4.7 effectively says is the following.

We start with a geodesic μ_1 in a hyperbolic metric space $X = X_v$. $W = f_v(X_e) \subset X_v$ is a C_1–quasiconvex set. We consider the set of points

$$P = \{p \in W : d(p.\mu_1) \leq C_2\}.$$

Choose $x, y \in P$ such that $d(x, y)$ is maximal. Let μ_2 be the geodesic joining x, y. Then nearest-point retractions onto μ_1, μ_2 almost agree for points in W.

This becomes easier to grasp if all sets in sight are convex subsets of $\mathbb{H}^n$. We start with $X = \mathbb{H}^n$. $W \subset X$ is convex. μ_1 is replaced by another convex set $V \subset X$. We consider the intersection of neighborhoods $N_\epsilon(V) \cap N_\epsilon(W) = V_1 \neq \varnothing$. Then nearest point retractions onto V_1 and V almost agree for $p \in W$. This is the context of Lemma 4.7 in general. We quasify this below.

Corollary 4.8 *Given $\delta, C, C_1 > 0$, there exists $D > 0$ such that if*

(1) *$A \subset Y$, $B \subset Z$, $Y \subset Z$ are inclusions of C_1–quasiconvex sets into δ hyperbolic metric spaces,*

(2) *A and $(N_C(B)) \cap Y)$ are within a Hausdorff distance of D of each other*

(3) *Π_A denote a nearest point projection of Y onto A, and Π_B denote a nearest point projection of Z onto B,*

then for all $y \in Y$, $d(\Pi_A(y), \Pi_B(y)) \leq D$.

There exist (possibly) curves σ_{ij} on $F(i)$ homotopic to some curves on components Δ_j of P in (M, P). Then σ_{ij} is one boundary component of a (possibly immersed) annulus A_{ij} whose other boundary component lies on Δ_j. Let $J(i) = F(i) \cup \bigcup A_{ij} \subset M_0$. Lift $J(i)$ to the universal cover $\widetilde{M_0}$ to get copies of $\widetilde{J(i)}$.

We will show that on $\widetilde{F(i)}$, the two putative definitions of Π_λ almost agree. Suppose $\mu_i = [a_i, b_i]_F \subset \widetilde{F(i)} \subset \widetilde{G(i)}$ is a geodesic in the path-metric on $\widetilde{F(i)}$. As in the construction of $B_0(\lambda)$, we can construct a 'hyperbolic' geodesic μ_i^h joining a_i, b_i in $\widetilde{G(i)}$. Such a geodesic has unique entry and exit points for every horodisk and hence unique 'Euclidean' geodesics joining them on the bounding horocycles. Replacing the hyperbolic segments by Euclidean segments, we obtain an ambient quasigeodesic μ_i^a

(by Theorem 2.7) in $\widetilde{J(i)}$. For $x \in \widetilde{F(i)}$, let Π_{i1} denote a nearest point retraction onto μ_i in the path metric on $\widetilde{F(i)}$. Also, let Π_{i2} denote a nearest point retraction onto $\mu_i^a \subset \widetilde{J(i)}$ in the path metric on $\widetilde{J(i)}$. Then from Lemma 4.6 and Lemma 4.7 above, we get the following lemma.

Lemma 4.9 *There exists $C > 0$ such that for all i and $x \in \widetilde{F(i)} \subset \widetilde{J(i)}$,*

$$d(\Pi_{i1}(x), \Pi_{i2}(x)) \leq C.$$

Again, as in the construction of $B_0(\lambda)$, each of the Euclidean geodesic segments in μ_i^a along with the hyperbolic geodesic segments of μ_i^h joining its end-points (and lying entirely within the corresponding horodisk) bounds a totally geodesic disk. The union of μ_i^a and all these totally geodesic disks is denoted by $B_0(\mu_i)$. There exists $C_1 > 0$ such that each $B_0(\mu_i)$ is C_1–quasiconvex in $\widetilde{G_i}$ and hence in $\widetilde{M_0}$. (See for instance McMullen [26, Section 8].)

Let Π_{i3} denote a nearest point projection onto $B_0(\mu_i)$, in $\widetilde{G(i)}$.

Lemma 4.10 *There exists $C > 0$ such that for all i and $x \in \widetilde{F(i)} \subset \widetilde{J(i)}$,*

$$d(\Pi_{i2}(x), \Pi_{i3}(x)) \leq C.$$

Proof For the purposes of this lemma, $[a, b]$ will denote a geodesic in $\widetilde{G(i)}$. First, $[x, \Pi_{i2}(x)] \cup \mu_i^a$ is a tripod in $\widetilde{J(i)}$ and hence it is uniformly quasiconvex from Lemma 4.5. Again, by the same lemma, $[x, \Pi_{i3}(x)] \cup B_0(\mu_i)$ is quasiconvex. Further, $\Pi_{i3}(x) \in \mu_i^a$ (since μ_i^a forms the boundary of $B_0(\mu_i)$ in $\widetilde{G(i)}$ and separates it from the $\widetilde{J(i)}$). Therefore, $[x, \Pi_{i2}(x)] \cup [\Pi_{i2}(x), a_i]$ and (from Theorem 2.7) $[x, \Pi_{i3}(x)] \cup [\Pi_{i3}(x), a_i]$ are both ambient quasigeodesics in the hyperbolic metric space $\widetilde{G(i)}$. In the same way, $[x, \Pi_{i2}(x)] \cup [\Pi_{i2}(x), b_i]$ and $[x, \Pi_{i3}(x)] \cup [\Pi_{i3}(x), b_i]$ are both ambient quasigeodesics in the hyperbolic metric space $\widetilde{G(i)}$.

Hence, $\Pi_{i3}(x)$ must lie in a bounded neighborhood of $[x, \Pi_{i2}(x)] \cup [\Pi_{i2}(x), a_i]$, as also $[x, \Pi_{i2}(x)] \cup [\Pi_{i2}(x), b_i]$, as also μ_i^a. Hence, it must lie close to the intersection of these three sets, which is $\Pi_{i2}(x)$. This proves the lemma. □

Recall that Π_0 denotes nearest point projection onto $B_0(\lambda)$. Now, $B_0(\mu_i)$ lies in a bounded neighborhood of $B_0(\lambda)$, since μ_i lies in a bounded neighborhood of λ. Now from Corollary 4.8, choosing $Y = \widetilde{G(i)}$, $Z = \widetilde{M_0}$, $A = B_0(\mu_i)$, $B = B_0(\lambda)$, we obtain the following corollary.

Corollary 4.11 *There exists $C > 0$ such that for all $x \in \widetilde{F(i)} \subset \widetilde{J(i)} \subset \widetilde{G(i)} \subset \widetilde{M_0}$, $d(\Pi_0(x), \Pi_{i3}(x)) \leq C$.*

Combining Lemma 4.9, Lemma 4.10 and Corollary 4.11, we get the following lemma.

Lemma 4.12 *Recall that for $x \in \widetilde{F(i)}$, Π_i denotes nearest point retraction onto μ_i and Π_0 denotes nearest point retraction onto $B_0(\lambda)$. Then $d(\Pi_i(x), \Pi_\alpha(x) \leq C_4$ for some $C_4 \geq 0$.*

Step 2: Retract on $\widetilde{E(i)}$ for $\widetilde{E(i)} \in \mathcal{E}$ A number of lemmas will be necessary. These are lifted directly from [36]. We state them here without proof.

The following lemma says nearest point projections in a δ–hyperbolic metric space do not increase distances much.

Lemma 4.13 [36, Lemma 3.2] *Let (Y, d) be a δ–hyperbolic metric space and let $\mu \subset Y$ be a C–quasiconvex subset, eg a geodesic segment. Let $\pi\colon Y \to \mu$ map $y \in Y$ to a point on μ nearest to y. Then $d(\pi(x), \pi(y)) \leq C_3 d(x, y)$ for all $x, y \in Y$ where C_3 depends only on δ, C.*

The following lemma says that nearest point projections and quasi- isometries in hyperbolic metric spaces 'almost commute'.

Lemma 4.14 [36, Lemma 3.5] *Suppose (Y, d) is δ–hyperbolic. Let μ_1 be some geodesic segment in Y joining a, b and let p be any vertex of Y. Also let q be a vertex on μ_1 such that $d(p, q) \leq d(p, x)$ for any $x \in \mu_1$. Let ϕ be a (K, ϵ)–quasi-isometry from Y to itself. Let μ_2 be a geodesic segment in Y joining $\phi(a)$ to $\phi(b)$. Let r be a point on μ_2 such that $d(\phi(p), r) \leq d(\phi(p), x)$ for $x \in \mu_2$. Then $d(r, \phi(q)) \leq C_4$ for some constant C_4 depending only on K, ϵ and δ.*

Theorem 4.15 *There exists $C > 0$, such that if $x, y \in \widetilde{E(i)}$ for some $\widetilde{E(i)} \in \mathcal{E}$, and $d(x, y) = 1$, then $d(\Pi_i(x), \Pi_i(y)) \leq C$.*

Proof **Case (a)** $x, y \in X_{ij}$ for some $i = 1, \ldots, s$, $j = 0, 1, \ldots$. This follows directly from Lemma 4.13.

Case (b) $x \in X_{i,j-1}$ and $y \in X_{ij}$ for some $i = 1, \ldots, s$, $j = 0, 1, \ldots$. Recall that $\mu_{ik} = B(\mu_i) \cap X_{ik}$ for all i, k. Then $\Pi_i(z) = \Pi_{ik}(x)$ for $z \in X_{ik}$. Also, from Lemma 3.9, there exist $K, \epsilon > 0$ such that for all i, k, ϕ_{ik} is a (K, ϵ)–quasi-isometry. Hence, $\phi_{ij}(\mu_{i,j-1})$ is a (K, ϵ)–quasigeodesic lying in a bounded K'–neighborhood of $\Phi_{ij}(\mu_{i,j-1}) = \mu_{ij}$ where K' depends only on K, ϵ, δ.

Now $\phi_{ij}\circ\Pi_{i,j-1}(x)$ lies on this quasi-geodesic. By Lemma 4.14, there exists $C_0>0$ such that $d(\phi_{ij}\circ\Pi_{i,j-1}(x),\Pi_{ij}\circ\phi_{i,j-1}(x))\le C_0$. Also $d(x,y)=1=d(x,\phi_{ij}(x))$. Hence, $d(\phi_{ij}(x),y)\le 2$ and $\phi_{ij}(x),y\in X_{ij}$. Therefore, by Lemma 4.13, there exists C_1 such that $d(\Pi_{ij}\circ\phi_{ij}(x),\Pi_{ij}(y))\le 2C_1$. Hence

$$\begin{aligned} d(\Pi_i(x),\Pi_i(y)) &= d(\Pi_{i,j-1}(x),\Pi_{ij}(y)) \\ &\le d(\Pi_{i,j-1}(x),\phi_{ij}\circ\Pi_{i,j-1}(x)) + \\ &\quad d(\phi_{ij}\circ\Pi_{i,j-1}(x),\Pi_{ij}\circ\phi_{i,j-1}(x)) + d(\Pi_{ij}\circ\phi_{ij}(x),\Pi_{ij}(y)) \\ &\le 1+C_0+2C_1. \end{aligned}$$

Choosing $C=(1+C_0+2C_1)$, we are through. □

Step 3 Retract on $\widetilde{M_0}=X_\alpha$. This case follows directly from Lemma 4.13. We state a special case required for our specific purposes.

Lemma 4.16 *There exists $C>0$ such that if $x,y\in\widetilde{M_0}=X_\alpha$ and $d(x,y)=1$, then $d(\Pi_0(x),\Pi_0(y))\le C$.*

Step 4 Bounded discrepancy on $\widetilde{F(i)}$ when $\widetilde{E(i)}\in\mathcal{E}'$. We want to show that for $x,y\in\widetilde{F(i)}\subset\widetilde{E(i)}\in\mathcal{E}'$, $d(\Pi_0(x),\Pi_i(x))$ is uniformly bounded.

Recall from the definition of $\widetilde{G(i)}$, two cases may arise.

Case (a) $N_{2k}(\widetilde{G(i)})$ does not intersect $\mathcal{H}(\lambda)$. In this case, recall that $y\in\widetilde{F(i)}$ is chosen arbitrarily, and we had defined

$$\Pi_i(x)=\Pi_0(y)\text{ for all }x\in\widetilde{E}.$$

Next, by the choice of $K=3k+4\delta+k'$ in the definition of $\mathcal{E}'$, and Corollary 4.4, we have that the diameter of the set $\Pi_0(\widetilde{G(i)})$ and hence $\Pi_0(\widetilde{F(i)})$ is bounded by k (from Lemma 4.2).

Hence, $d(\Pi_0(x),\Pi_i(x))=d(\Pi_0(x),\Pi_0(y))\le k$.

Case (b) $N_{2k}(\widetilde{G(i)})$ intersects precisely one $\mathbf{H}\in\mathcal{H}(\check{})$. In this case, there exists a unique lift $\widetilde{\sigma}$ of a curve σ on $F(i)$, (parabolic in M_0) lying on $\widetilde{F(i)}$ at a bounded distance from $\mathbf{H}$. Recall that Π_σ denotes nearest point projection of $\widetilde{E(i)}$ onto $\widetilde{\sigma}$ and $\Pi^0_\sigma(x)$ denotes nearest point projection onto $B(\widetilde{\sigma})$. On $\widetilde{F(i)}$, Π^0_σ coincides with $\Pi_\sigma\circ\Pi^0_\sigma$. Also recall that $\Pi_i(x)=\Pi_0\circ\Pi_\sigma\circ\Pi^0_\sigma(x)$ for all $x\in\widetilde{E(i)}$.

On $\widetilde{F(i)}$, Π^0_σ coincides with $\Pi_\sigma\circ\Pi^0_\sigma$. Thus, we want to show that $d(\Pi_0(x),\Pi_0\circ\Pi^0_\sigma(x))$ is uniformly bounded on $\widetilde{F(i)}$.

Here, $N_k(\widetilde{G(i)}) \cap N_k(B_0(\lambda))$ lies in some horoball $\mathbf{H} \in \mathcal{H}(\breve{})$ while $N_k(\widetilde{F(i)}) \cap N_k(B_0(\lambda)) = \varnothing$ (as per definition of $\mathcal{E}'$). By Lemma 4.2 $\Pi_0(\widetilde{F(i)}) \subset N_k(\widetilde{(A \cup \kappa)} \subset N_k(\mathbf{H})$, where $\kappa \subset G(i)$ is the cusp whose boundary η bounds the annulus A along with σ in M_0. (Recall from the discussion following Lemma 4.2 that each η forms the boundary of a totally geodesic 2 dimensional subset κ of a Z cusp or a $Z + Z$–cusp of M_0.) Also, $\widetilde{(A \cup \kappa)} = \widetilde{A_\kappa}$ (say) is C_3–quasiconvex for some C_3.

Let Π_κ denote nearest point projection of $\widetilde{G(i)}$ onto $\widetilde{A_\kappa}$ in the metric on $\widetilde{M_0}$. Also, let Π_0^{aug} denote nearest point projection onto $B_0^{\text{aug}}(\lambda)$. Then by Corollary 4.8 there exists $C_4 > 0$ such that

$$d(\Pi_\kappa(x), \Pi_0^{\text{aug}}(x)) \leq C_4 \text{ for all } x \in \widetilde{G(i)}.$$

Next, $B_0(\lambda) \subset B_0^{\text{aug}}(\lambda) \subset \widetilde{M_0}$ and both $B_0(\lambda)$ and $B_0^{\text{aug}}(\lambda)$ are C_1–quasiconvex. Both Π_0 and $\Pi_0 \circ \Pi_0^{\text{aug}}$ are large-scale Lipschitz retracts onto $B_0(\lambda)$, the only difference being that the latter factors through a large-scale Lipschitz retract Π_0^{aug} onto $B_0^{\text{aug}}(\lambda)$. Hence the two maps must send x to close by points, ie there exists $C_5 > 0$ such that

$$d(\Pi_0 \circ \Pi_0^{\text{aug}}(x), \Pi_0(x)) \leq C_5 \text{ for all } x \in \widetilde{M_0}.$$

Using the above two inequalities along with Lemma 4.13 we get a $C_6 > 0$ such that

$$d(\Pi_0 \circ \Pi_\kappa(x), \Pi_0(x)) \leq C_6 \text{ for all } x \in \widetilde{M_0}.$$

Finally, we observe that for $x \in \widetilde{F(i)}$, since Π_σ^0 denotes nearest point projection onto $\widetilde{\sigma}$, then $d(\Pi_\kappa(x), \Pi_\sigma^0(x))$ is bounded. This is because $\widetilde{\sigma}$ separates $\widetilde{G(i)}$ into $\widetilde{F(i)}$ and $\widetilde{A_\kappa}$. Hence, the geodesic joining x to $\Pi_\kappa(x)$ in $\widetilde{G(i)}$ must cut $\widetilde{\sigma}$ at some point which therefore must coincide with $\Pi_\sigma^0(x)$, provided we take nearest point projections in $\widetilde{G(i)}$. Since $\widetilde{G(i)}$ is quasiconvex in $\widetilde{M_0}$, therefore by Corollary 4.8, we might as well take projections in the $\widetilde{M_0}$ metric and we have a $C_7 > 0$ such that

$$d(\Pi_\kappa(x), \Pi_\sigma^0(x)) \leq C_7 \text{ for all } x \in \widetilde{F(i)}.$$

Combining the above two inequalities and using Lemma 4.13 for Π_0, we obtain finally the next lemma.

Lemma 4.17 *There exists $C > 0$ such that*

$$d(\Pi_0 \circ \Pi_\sigma^0(x), \Pi_0(x)) \leq C \text{ for all } x \in \widetilde{F(i)}.$$

Since on such a $\widetilde{F(i)}$, $\Pi_0 \circ \Pi_\sigma^0$ is denoted by Π_i, we have $d(\Pi_i(x), \Pi_0(x)) \leq C$ for all $x \in \widetilde{F(i)}$.

The three results Lemma 4.12, Case (a) of Step 4 above and Lemma 4.17 show that Π_0 and Π_i almost agree, ie have at most a bounded amount of discrepancy for all $\widetilde{F(i)}$. We summarize these three in the following useful proposition.

Proposition 4.18 *There exists $C > 0$ such that for all $\widetilde{F(i)}$ and all $x \in \widetilde{F(i)}$,*

$$d(\Pi_i(x), \Pi_0(x)) \leq C.$$

Step 5 Retract on $\widetilde{E(i)}$ for $\widetilde{E(i)} \in \mathcal{E}'$. $\Pi_i(x) = \Pi_0 \circ \Pi_\sigma \circ \Pi^0_\sigma(x)$ for all $x \in \widetilde{E(i)}$. By the proof of Theorem 4.15, there exists $C_1 > 0$ such that for all $x, y \in \widetilde{E(i)} \in \mathcal{E}'$ with $d(x, y) = 1$, we have

$$d(\Pi^0_\sigma(x), \Pi^0_\sigma(y)) \leq C_1.$$

(Though Theorem 4.15 is stated for $\widetilde{E(i)} \in \mathcal{E}$, all that we need for the above assertion is the intrinsic metric on $\widetilde{E(i)}$ and here the proof is the same.)

Also, by Lemma 4.13, there exists $C_2 > 0$ such that for $x, y \in \widetilde{M_0}$,

$$d(\Pi_0(x), \Pi_0(y)) \leq C_2 d(x, y).$$

Hence, it would be enough to prove that there exists $C_3 > 0$ such that

$$d(\Pi_\sigma(x), \Pi_\sigma(y)) \leq C_3 d(x, y)$$

for all $x, y \in \widetilde{E(i)}$.

For this, it would suffice to show that $\widetilde{\sigma}$ is quasiconvex in $\widetilde{E(i)}$. Recall that there exists a simply degenerate end E^h, one of whose lifts to the universal cover is $\widetilde{E^h(i)}$. $\widetilde{E^h(i)}$ with cusps removed is $\widetilde{E(i)}$. We shall show that $\widetilde{\sigma}$ is quasiconvex in $\widetilde{E^h(i)}$. We give E^h a convex hyperbolic structure by taking the cover of N^h corresponding to $\pi_1(E^h)$ and looking at its convex core. This makes the universal cover of E^h (equipped with such a convex hyperbolic structure) quasi-isometric to $\widetilde{E^h(i)}$. For the purposes of this step, therefore, we assume that $\widetilde{E^h(i)}$ is the universal cover of a simply degenerate hyperbolic manifold (so as to avoid the extra expository complication of mapping via quasi-isometries). Now, $\sigma \subset E^h$ is a closed curve, and is therefore homotopic by a bounded homotopy to a closed geodesic σ'. Then any lift $\widetilde{\sigma'}$ is a hyperbolic geodesic in the convex hyperbolic manifold $\widetilde{E^h(i)}$. Since $\widetilde{\sigma}$ lies at a bounded distance from $\widetilde{\sigma'}$, we conclude that $\widetilde{\sigma}$ is quasiconvex in $\widetilde{E^h(i)}$ and hence there exists a nearest point projection of $\widetilde{E^h(i)}$ onto $\widetilde{\sigma}$ which stretches distances by a bounded factor by Lemma

4.13. Restricting this map to $\widetilde{E(i)}$, the retraction property persists. Since the path metric on $\widetilde{E(i)}$ dominates the metric induced from $\widetilde{E^h(i)}$, and since the metric on $\widetilde{\sigma}$ remains undisturbed on removing cusps, we find (reverting to our description of spaces as graphs) that there exists $C_3 > 0$ such that

$$d(\Pi_\sigma(x), \Pi_\sigma(y)) \leq C_3 d(x, y)$$

for all $x, y \in \widetilde{E(i)}$.

Combining the three above equations, we arrive at the following proposition.

Proposition 4.19 *There exists $C > 0$, such that if $x, y \in \widetilde{E(i)}$ for some $\widetilde{E(i)} \in \mathcal{E}'$, and $d(x, y) = 1$, then $d(\Pi_i(x), \Pi_i(y)) \leq C$.*

We next prove that Π_λ is a retract. We combine the five steps above to conclude that Π_λ is a retract. Recall the definition of Π_λ.

Definition

$$\begin{aligned}\Pi_\lambda(x) &= \Pi_i(x), x \in \widetilde{E(i)} - X_{i0}, \widetilde{E(i)} \in \mathcal{E} \\ &= \Pi_0(x), x \in X_\alpha \\ &= \Pi_i(x), x \in \widetilde{E(i)} - X_{i0}, \widetilde{E(i)} \in \mathcal{E}'.\end{aligned}$$

Also recall that $\widetilde{N_0}$ is quasi-isometric to a graph X which can be regarded as a tree of hyperbolic metric spaces with possibly exceptional vertex α satisfying the qi-embedded condition. (See Lemma 3.9, Lemma 3.10 and the discussion preceding Lemma 3.9 where this is elaborated.)

We are now in a position to prove our main technical theorem.

Theorem 4.20 *There exists $C_0 \geq 0$ such that $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d(x, y)$ for x, y vertices of X.*

Proof It suffices to prove the theorem when $d(x, y) = 1$.

(1) For $x, y \in \widetilde{E(i)} - \widetilde{M_0}$, $\widetilde{E(i)} \in \mathcal{E}$, this follows from Theorem 4.15 in Step 2 above.

(2) For $x, y \in X_\alpha = \widetilde{M_0}$, this follows from Lemma 4.16 in Step 3 above.

(3) For $x, y \in \widetilde{E(i)} - \widetilde{M_0}$, $\widetilde{E(i)} \in \mathcal{E}'$, this follows from Proposition 4.19 in Step 5 above.

(4) $x \in X_\alpha, y \in X_{i1} \subset \widetilde{E(i)}$ and $d(x, y) = 1$ for $\widetilde{E(i)} \in \mathcal{E} \cup \mathcal{E}'$.

Here,

$$\begin{aligned} d(\Pi_\lambda(x), \Pi_\lambda(y)) &= d(\Pi_0(x), \Pi_i(y)) \\ &\leq d(\Pi_0(x), \Pi_i(x)) + d(\Pi_i(x), \Pi_i(y)). \end{aligned}$$

Choose constants C_1, C_2 and C_3 from Proposition 4.18, Theorem 4.15 and Proposition 4.19. Let $C = C_1 + \max(C_2, C_3)$. Then we get

$$d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C.$$

This proves the result. □

5 Proof of main theorem

Recall that we started off with a geodesic λ^h on $\widetilde{M_{\mathrm{gf}}}$ outside an n–ball around a fixed reference point p, joining a, b. Replacing the maximal segments of λ^h lying inside Z–horoballs by 'Euclidean' geodesics lying on the corresponding horosphere, we obtained an ambient quasigeodesic λ in $\widetilde{M_0}$ joining a, b. λ agrees with λ^h off Z–horoballs. Let β^h be the geodesic in $\widetilde{N^h}$ joining a, b. Performing the same operation on β^h for Z–horoballs in $\widetilde{N^h}$, we obtain an ambient geodesic β^0_{amb} in $\widetilde{N_0}$ joining the end-points of λ. We have proved the existence of a retraction Π_λ in the preceding section (Theorem 4.20). Project β^0_{amb} onto B_λ, using Π_λ to get another ambient quasigeodesic β_{amb}. Thus, $\beta_{\mathrm{amb}} = \Pi_\lambda(\beta^0_{\mathrm{amb}})$.

Our starting point for this section is therefore the ambient quasigeodesic $\beta_{\mathrm{amb}} \subset B_\lambda \subset \widetilde{N_0}$.

5.1 Quasigeodesic rays

The purpose of this subsection is to construct for any $\widetilde{E(i)} \in \mathcal{E}$ and $x \in B(\mu_i) = B_\lambda \cap \widetilde{E(i)}$ a quasi-geodesic ray $r_x \subset B_\lambda$ passing through x. r_x can be regarded both as a function $r_x\colon \{0\} \cup \mathbb{N} \to B(\mu_i)$ or as a subset of $B(\mu_i)$ (the image of the function r_x). If x lies away from cusps, so will r_x.

Fix $\widetilde{E} \in \mathcal{E}$. Recall that

- $\widetilde{E} \cap \widetilde{M_0} = \widetilde{F}$,
- $\widetilde{F} \cap B_\lambda = \mu = \mu_0$,
- $\widetilde{E} = \bigcup_i \widetilde{S(i)}$,
- $B(\mu) = \cup_i \mu_i$,

- $\mu_i = \Phi_i(\mu_{i-1}) \subset \widetilde{S(i)}$.

Also note that Φ_i is the map on geodesics induced by the quasi-isometry $\phi_i\colon \widetilde{S(i-1)} \to \widetilde{S(i)}$. The quasi-isometry ϕ_i is induced by a cusp-preserving homeomorphism of truncated pleated surfaces. So we can assume that ϕ_i restricted to each horocycle boundary component of $\widetilde{S(i-1)}$ is an orientation-preserving isometry. Since each such horocycle may naturally be identified with $\mathbb{R}$, we might as well assume that the map on horocycles induced by ϕ_i is the identity map. Now, extend ϕ_i to a map $\phi_i^h\colon \widetilde{S^h(i-1)} \to \widetilde{S^h(i)}$ by demanding that ϕ_i^h restricted to each horodisk is an isometry. Note that ϕ_i^h is thus an equivariant quasi-isometry from $\mathbb{H}^2$ to itself, taking horodisks isometrically to horodisks. Next, let μ_i^h be the hyperbolic geodesic joining the end-points of μ_i. As usual, let μ_i^a denote the ambient quasigeodesic obtained by replacing hyperbolic segments in horodisks by 'Euclidean' segments on the boundary horocycles. Then μ_i^a lies in a uniform (independent of i) C_0–neighborhood of μ_i as they are both ambient quasigeodesics in $\widetilde{S(i)}$. Let

$$B^a(\mu) = \cup_i \mu_i^a.$$

Then $B^a(\mu)$ lies in a C_0–neighborhood of $B(\mu)$. Let μ_i^c denote the union of the segments of μ_i^a which lie along horocycles and let $\mu_i^b = \mu_i^a - \mu_i^c$. Let

$$B^c(\mu) = \cup_i \mu_i^c$$
$$B^b(\mu) = \cup_i \mu_i^b.$$

We want to show that for all $x \in B^b(\mu)$ there exists a C–quasigeodesic $r_x\colon \{0\} \cup \mathbb{N} \to B^b(\mu)$ such that $x \in r_x(\{0\} \cup \mathbb{N})$ and $r_x(i) \in \mu_i^b$. Suppose $x \in \mu_k^b \subset B^b(\mu)$. We define r_x by starting with $r_x(k) = x$ and construct $r_x(k-i)$ and $r_x(k+i)$ inductively (of course $(k-i)$ stops at 0 but $(k+i)$ goes on to infinity). For the sake of concreteness, we prove the existence of such a $r_x(k+1)$. The same argument applies to $(k-1)$ and inductively for the rest.

Lemma 5.1 *There exists $C > 0$ such that if $r_x(k) = x \in \mu_k^b$ then there exists $x' \in \mu_{k+1}^b$ such that $d(x, x') \leq C$. We denote $r_x(k+1) = x'$.*

Proof Let $[a, b]$ be the maximal connected component of μ_k^b on which x lies. Then there exist two horospheres $\mathbf{H_1}$ and $\mathbf{H_2}$ such that $a \in \mathbf{H_1}$ (or is the initial point of μ_k) and $b \in \mathbf{H_2}$ (or is the terminal point of μ_k). Note that $[a, b]$ does not intersect any of the horodisks of $\widetilde{S_k^h}$. Since ϕ_{k+1}^h preserves horodisks, $\phi_{k+1}(a)$ lies on a horocycle (or is the initial point of μ_{k+1}) as does $\phi_{k+1}(b)$ (or is the terminal point

of μ_{k+1}). Further, the image of $[a,b]$ is a hyperbolic quasigeodesic (which we now denote as $\phi_{k+1}([a,b])$) lying outside horoballs. Let $\Phi^h_{k+1}([a,b])$ denote the hyperbolic geodesic joining $\phi_{k+1}(a)$ and $\phi_{k+1}(b)$. Let $\Phi_{k+1}([a,b])$ denote the ambient geodesic in $\widetilde{S(k+1)}$ joining $\phi_{k+1}(a)$ and $\phi_{k+1}(b)$. Therefore $\Phi^h_{k+1}([a,b])$ lies in a bounded neighborhood of $\phi_{k+1}([a,b])$ (which in turn lies at a bounded distance from $\Phi_{k+1}([a,b])$) and hence by Theorem 2.8 there exists an upper bound on how much $\Phi^h_{k+1}([a,b])$ can penetrate horoballs, ie there exists $C_1 > 0$ such that for all $z \in \Phi^h_{k+1}([a,b])$, there exists $z' \in \Phi^h_{k+1}([a,b])$ lying outside horoballs with $d(z,z') \leq C_1$. Further, since ϕ_{k+1} is a quasi-isometry there exists $C_2 > 0$ such that $d(\phi_{k+1}(x), \Phi^h_{k+1}([a,b])) \leq C_2$. Hence there exists $x' \in \Phi^h_{k+1}([a,b])$ such that

- $d(\phi_{k+1}(x), x') \leq C_1 + C_2$,
- x' lies outside horoballs.

Again, $\Phi_{k+1}([a,b])$ lies at a uniformly bounded distance $\leq C_3$ from μ_{k+1} and so, if $c,d \in \mu_{k+1}$ such that $d(a,c) \leq C_3$ and $d(b,d) \leq C_3$ then the segment $[c,d]$ can penetrate only a bounded distance into any horoball. Hence there exists $C_4 > 0$ and $x'' \in [c,d]$ such that

- $d(x', x'') \leq C_4$,
- x'' lies outside horoballs.

Hence, $d(\phi_{k+1}(x), x'') \leq C_1 + C_2 + C_4$. Since $d(x, \phi_{k+1}(x)) = 1$, we have, by choosing $r_{k+1}(x) = x''$,

$$d(r_k(x), r_{k+1}(x)) \leq 1 + C_1 + C_2 + C_4.$$

Choosing $C = 1 + C_1 + C_2 + C_4$, we are through. □

Using Lemma 5.1 repeatedly (inductively replacing x with $r_x(k+i)$ we obtain the values of $r_x(i)$ for $i \geq k$. By an exactly similar symmetric argument, we get $r_x(k-1)$ and proceed down to $r_x(0)$. Now for any i, $z \in \widetilde{S(i)}$ and $y \in \widetilde{S(i+1)}$, $d(z,y) \geq 1$. Hence, for any $z \in \widetilde{S(i)}$ and $y \in \widetilde{S(j)}$, $d(z,y) \geq |i-j|$. This gives the following corollary.

Corollary 5.2 *There exist $K, \epsilon > 0$ such that for all $x \in \mu^b_k \subset B^b(\mu)$ there exists a (K,ϵ) quasigeodesic ray r_x such that $r_x(k) = x$ and $r_x(i) \in \mu^b_i$ for all $i = 0, 1, 2, \ldots$.*

5.2 p–incompressible boundary

Till this point we have not used the hypothesis of *p–incompressibility*. We need a modification of Corollary 5.2 above to go down from a point in $\mu^b = \mu_0^b$ to a point in λ^b.

We can change B_λ to B^a_λ by replacing each $B(\mu) \subset \widetilde{E(i)}$ by $B^a(\mu)$ for $\widetilde{E(i)} \in \mathcal{E}$. Recall that we had started off with λ being an ambient quasigeodesic in $\widetilde{M_0}$ constructed from λ^h by replacing hyperbolic geodesic segments within Z–horoballs by 'Euclidean' geodesic segments along horospheres. So, there is no need to freshly construct a λ^a (we can think that $\lambda = \lambda^a$). But we do need to construct λ_{cc} by performing the same modifications for $(Z+Z)$–horoballs. Just as in the construction of $\mu_i^a, \mu_i^c, \mu_i^b$, we can define λ^c to be the collection of subsegments of λ_{cc} lying along horoballs (Z or $(Z+Z)$), and $\lambda^b = \lambda_{cc} - \lambda^c$. Adjoining λ_{cc}, λ^c and λ^b to $B^a(\mu)$, $B^c(\mu)$ and $B^b(\mu)$ we get respectively, $B^a(\mu,\lambda)$, $B^c(\mu,\lambda)$ and $B^b(\mu,\lambda)$. Then we can extend Corollary 5.2 above so that λ is included.

For the convenience of the reader, we summarize the notation for the different types of geodesics and quasigeodesics that we have introduced and will use henceforth.

- λ^h = hyperbolic geodesic in $\widetilde{M_{gf}}$ joining a, b.
- $\lambda = \lambda^a$ = ambient quasigeodesic in $\widetilde{M_0}$ constructed from $\lambda^h \subset \widetilde{M_{gf}}$.
- λ_{cc} = ambient quasigeodesic in $\widetilde{M_{cc}}$ constructed from $\lambda \subset \widetilde{M_0}$.
- λ^c = part of λ_{cc} lying along Z–cusps or $Z+Z$–cusps.
- $\lambda^b = \lambda_{cc} - \lambda^c$.
- μ_i = ambient geodesic in $\widetilde{S(i)}$.
- μ_i^a = ambient quasigeodesic in $\widetilde{S(i)}$ constructed from μ_i.
- μ_i^c = part of μ_i^a lying along Z–cusps.
- $\mu_i^b = \mu_i^a - \mu_i^c$.

Recall that in the construction of $B_1(\lambda)$ from $B_0(\lambda)$ we construct μ from λ by taking a K (as in the discussion following Corollary 4.4) and choosing $p, q \in \widetilde{F} \cap N_K(B_0(\lambda))$ with $d(p,q)$ maximal.

Now suppose (M, P) is p–incompressible. (This assumption has effect from now till the end of Section 5.4, unless explicitly stated otherwise.) Then $\widetilde{F}$ is quasiconvex in the hyperbolic metric space $\widetilde{M_0}$. Hence, μ is a quasigeodesic in $\widetilde{M_0}$ and therefore lies in a K_1–neighborhood of $B_0(\lambda)$ (where K_1 depends on K). Further, since μ does not penetrate horoballs, the hyperbolic geodesic μ' in $\widetilde{M_0}$ joining the end-points of μ

can penetrate horoballs only for some bounded length D_1. Also, there is a subsegment $\lambda_\mu = [p', q'] \subset \lambda$ such that $d(p, p') \leq K_1$ and $d(q, q') \leq K_1$. By Theorem 2.8, μ' and λ_μ must have similar intersections with horoballs, ie there exists C_0 such that the following hold.

- If only one of μ' and λ_μ penetrates or travels along the boundary of a horoball $\mathbf{H}$, then it can do so for a distance $\leq C_0$.
- If both μ' and λ_μ enter (or leave) a horoball $\mathbf{H}$ then their entry (or exit) points are at a distance of at most C_0 from each other.

Then as in Corollary 5.2 if $x \in \mu^b$ there exists $C_1 > 0$, $x'' \in \mu'$ and $x' \in \lambda^b$ such that

$$d(x, x'') \leq C_1$$
$$d(x', x'') \leq C_1.$$

Hence, $d(x', x) \leq 2C_1$. We have thus shown, (using Corollary 5.2) the following corollary.

Corollary 5.3 *Suppose (M, P) is a pared manifold with* p–incompressible boundary. *There exist $K, \epsilon > 0$ such that for all $x \in \mu_k^b \subset B^b(\mu, \lambda)$ there exists a (K, ϵ) quasigeodesic ray $r_x\colon \{-1, 0\} \cup \mathbb{N} \to B^b(\mu, \lambda)$ such that $r_x(k) = x$, $r_x(i) \in \mu_i^b$ for all $i = 0, 1, 2, \ldots$ and $r_x(-1) \in \lambda^b$.*

Note In the discussion preceding Corollary 5.3 above, all that we really required was that μ be a quasigeodesic in $\widetilde{M_0}$. Quasiconvexity of $\widetilde{F}$ (following from p–incompressibility) ensured this. We therefore state the more general version below, as we shall require it to prove our main Theorem 5.12 where p–incompressibility is relaxed to incompressibility.

Corollary 5.4 *Suppose (M, P) is a pared manifold with* incompressible boundary. *Given D, δ there exist K, ϵ such that the following holds: Suppose μ is a (D, δ) hyperbolic quasigeodesic in $\widetilde{M}_0$ lying on $\widetilde{F}$ for some F. Then for all $x \in \mu_k^b \subset B^b(\mu, \lambda)$ there exists a (K, ϵ) quasigeodesic ray $r_x\colon \{-1, 0\} \cup \mathbb{N} \to B^b(\mu, \lambda)$ such that $r_x(k) = x$, $r_x(i) \in \mu_i^b$ for all $i = 0, 1, 2, \ldots$ and $r_x(-1) \in \lambda^b$.*

The hypothesis of Corollary 5.4 is satisfied if μ moves along boundaries of horoballs for uniformly bounded distances. Equivalently, this is satisfied if the hyperbolic geodesic μ^h joining the end-points of μ penetrates horoballs by a uniformly bounded amount. Actually, we do not need even this much. μ^h should penetrate *exceptional horoballs* (see definition below) by a bounded amount.

Definition A cusp of M_{gf} is said to be *exceptional* if there exist closed curves carried by the cusp (ie lying on its boundary horocycle or horosphere) which are homotopic to non-peripheral curves on some other boundary component of M_{gf}. *Exceptional horoballs* are lifts of exceptional cusps. A geodesic is said to *penetrate a horoball H by at most D* if any subsegment of it lying inside H has length less than or equal to D.

The sufficient condition for the hypothesis of Corollary 5.4 to hold is set forth in the following lemma.

Lemma 5.5 *Let F be some boundary component of M_0 and $\mu \subset \widetilde{F}$ be a geodesic segment in the path metric on $\widetilde{F}$. Let μ^h be the hyperbolic geodesic in $\widetilde{M_{\mathrm{gf}}}$ joining the end-points of μ. Given $D > 0$ there exist K, ϵ such that if μ^h penetrates exceptional horoballs by at most D then μ is a (K, ϵ) quasigeodesic in $\widetilde{M_0}$.*

5.3 Ambient and hyperbolic geodesics

Recall

- λ^h = hyperbolic geodesic in $\widetilde{M_{\mathrm{gf}}}$ joining a, b,
- $\lambda = \lambda^a$ = ambient quasigeodesic in $\widetilde{M_0}$ constructed from $\lambda^h \subset \widetilde{M_{\mathrm{gf}}}$,
- β^h = geodesic in $\widetilde{N^h}$ joining a, b,
- β^0_{amb} = ambient quasigeodesic in $\widetilde{N_0}$ obtained from β^h by replacement of hyperbolic by 'Euclidean' geodesic segments for Z–horoballs in $\widetilde{N^h}$,
- $\beta_{\mathrm{amb}} = \Pi_\lambda(\beta^0_{\mathrm{amb}})$.

By construction, the hyperbolic geodesic β^h and the ambient quasigeodesic β^0_{amb} agree exactly off horoballs. β_{amb} is constructed from β^0_{amb} by projecting it onto B_λ and so by Theorem 4.20, it is an ambient quasigeodesic. But it might 'backtrack'. Hence, we shall modify it such that it satisfies the no backtracking condition. First, observe by Theorem 2.8 that β^h, β^0_{amb} track each other off some K–neighborhood of horoballs.

The advantage of working with β_{amb} is that it lies on B_λ. However, it might backtrack. Now, recall that $B^a(\lambda)$ was constructed from B_λ by replacing each $B(\mu)$ by $B^a(\mu)$. We can therefore choose an ambient quasigeodesic lying on $B^a(\lambda)$ that tracks β_{amb} throughout its length. The advantage of switching to $B^a(\lambda)$ is that it is constructed from ambient geodesics without backtracking lying on the universal covers of equispaced pleated surfaces. We shall (abusing notation) call this new ambient quasigeodesic β_{amb}. Thus β_{amb} now lies on $B^a(\lambda)$ rather than B_λ.

Lemma 5.6 *There exists $C > 0$ such that for all $x \in \mu_i^b \subset B^a(\mu, \lambda) \subset B^a(\lambda)$ if λ^h lies outside $B_n(p)$ for a fixed reference point $p \in \widetilde{M^h}$, then x lies outside an $\frac{n-C}{C+1}$ ball about p in $\widetilde{N_0}$.*

Proof By Corollary 5.3, $r_x(-1) \in \lambda^b$. Since λ^b is a part of λ^h, therefore $r_x(-1)$ lies outside $B_n(p)$. By Corollary 5.3, there exists $C > 0$ such that for all $i, j \in \{-1, 0, 1, 2, \dots\}$,

$$|i - j| \le d(r_x(i), r_x(j)) \le C|i - j|.$$

Also, $d(x, p) \ge i$ since $x \in \mu_i^b$. (Here distances are all measured in $\widetilde{N_0}$.) Hence,

$$\begin{aligned} d(x, p) &\ge \max\{i, n - C(i + 1)\} \\ &\ge \frac{n - C}{C + 1} \end{aligned}$$

This proves the result. □

If $x \in B^a(\lambda)$, then $x \in \lambda^b$ or $x \in \lambda^c$ or $x \in B^b(\mu)$ or $x \in B^c(\mu)$ for some μ. Hence $x \in B^a(\lambda)$ implies that either x lies on some horosphere bounding some $\mathbf{H} \in \mathcal{H}$ or, from Lemma 5.6 above, $d(x, p) \ge \frac{n-C}{C+1}$. Since β_{amb} lies on $B^a(\lambda)$, we conclude that β_{amb} is an ambient quasigeodesic in $\widetilde{N_0}$ such that every point x on β_{amb} either lies on some horosphere bounding some $\mathbf{H} \in \mathcal{H}$ or, from Lemma 5.6 above, $d(x, p) \ge \frac{n-C}{C+1}$.

McMullen [26] shows (cf Theorem 2.7) that in $\widetilde{N^h}$, any such ambient quasigeodesic β_{amb} lies in a bounded neighborhood of $\beta^h \cup \mathcal{H}(\beta^h)$. We do not as yet know that β_{amb} does not backtrack, but we can convert it into one without much effort. (Note that Theorem 2.7 does not require "no backtracking".) Let Π denote nearest point projection of $\widetilde{N^h}$ onto $\lambda^h \cup \mathcal{H}$. Since Π is a large-scale Lipschitz retract (Theorem 4.20), $\Pi(\beta_{\text{amb}}) = \beta_1$ is again an ambient quasigeodesic in $\widetilde{N_0}$. Further, β_1 tracks β_{amb} throughout its length as Π moves points on β_{amb} through a uniformly bounded distance (Theorem 2.7). Now β_1 might backtrack, but it can do so in a trivial way, ie if β_1 re-enters a horoball after leaving it, it must do so at exactly the point where it leaves it. Removing these 'trivial backtracks', we obtain an *ambient quasigeodesic without backtracking* β which tracks β_{amb} throughout its length.

Note On the one hand β is an ambient quasigeodesic without backtracking. Hence, it reflects the intersection pattern of β^h with horoballs. On the other hand, it *tracks* β_{amb} whose properties we already know from Corollary 5.3 above.

Since, of β and β^h, one is an ambient quasigeodesic without backtracking, and the other a hyperbolic geodesic joining the same pair of points, we conclude from Theorem 2.8 that they have similar intersection patterns with horoballs, ie there exists C_0 such that

- if only one of β and β^h penetrates or travels along the boundary of a horoball $\mathbf{H}$, then it can do so for a distance $\leq C_0$,
- if both β and β^h enter (or leave) a horoball $\mathbf{H}$ then their entry (or exit) points are at a distance of at most C_0 from each other.

Again, since β tracks β_{amb}, we conclude that there exists $C > 0$ such that β lies in a C–neighborhood of β_{amb} and hence from Lemma 5.6,

- every point x on β either lies on some horosphere bounding some $\mathbf{H} \in \mathcal{H}$ or, $d(x, p) \geq \frac{n-C}{C+1} - C$.

The above three conditions on β and β^h allow us to deduce the following (identical to the third) condition for β^h in the p–incompressible case.

Proposition 5.7 *Suppose (M, P) has p–incompressible boundary. Then every point x on β^h either lies inside some horoball $\mathbf{H} \in \mathcal{H}$ or, $d(x, p) \geq \frac{n-C}{C+1} - C = m(n)$.*

We have denoted $\frac{n-C}{C+1} - C$ by $m(n)$, so that $m(n) \to \infty$ as $n \to \infty$. The above Proposition asserts that the geodesic β^h lies outside large balls about p modulo horoballs. By Lemma 2.4 this is almost enough to guarantee the existence of a Cannon–Thurston map.

Again as in Corollary 5.4 it is not necessary to restrict ourselves to the p–incompressible case. We deduce, using Corollary 5.4 and Lemma 5.5 the following proposition.

Proposition 5.8 *Suppose (M, P) has incompressible boundary $\partial_0 P$. Given D, n there exist $m(n, D)$ such that the following holds: Let λ^h, β^h, λ and μ be as before. If each μ penetrates exceptional horoballs by at most D, then every point x on β^h either lies inside some horoball $\mathbf{H} \in \mathcal{H}$ or, $d(x, p) \geq m(n, D)$, for some function $m(n, D)$ where $m(n, D) \to \infty$ as $n \to \infty$ for each fixed D.*

We can divide β^h into two subsets β^c and β^b as earlier. $\beta^c = \beta^h \cap \mathcal{H}$ is the intersection of β^h with horoballs, and $\beta^b = \beta^h - \beta^c$. The main theorem of this paper under the assumption of *p–incompressibility* follows. Recall that M_{gf} is identified with its homeomorphic image in N^h taking cusps to cusps.

Theorem 5.9 *Suppose that $N^h \in H(M,P)$ is a hyperbolic structure of bounded geometry on a pared manifold (M,P) with p–incompressible boundary. Let M_{gf} denotes a geometrically finite hyperbolic structure adapted to (M,P), then the map $i\colon \widetilde{M_{\mathrm{gf}}} \to \widetilde{N^h}$ extends continuously to the boundary $\hat{i}\colon \widehat{M_{\mathrm{gf}}} \to \widehat{N^h}$. If Λ denotes the limit set of $\widetilde{M}$, then Λ is locally connected.*

Proof Let λ^h be a geodesic segment in $\widetilde{M_{\mathrm{gf}}}$ lying outside $N_n(p)$ for some fixed reference point p. Fix neighborhoods of the cusps and lift them to the universal cover. Let $\mathcal{H}$ denote the set of horoballs. Assume without loss of generality that p lies outside horoballs. Let β^h be the hyperbolic geodesic in $\widetilde{N^h}$ joining the endpoints of λ^h. Further, let $\beta^h = \beta^b \cup \beta^c$ as above. Then by Proposition 5.7, β^b lies outside an $m(n)$ ball about p, with $m(n) \to \infty$ as $n \to \infty$.

Next, let $\mathbf{H}_1$ be some horoball that β^h meets. Then the entry and exit points u and v of β^h into and out of $\mathbf{H}_1$ lie outside an $m(n)$ ball about p. Let z be the point on the boundary sphere that $\mathbf{H}_1$ is based at. Then for any sequence $x_i \in \mathbf{H}_1$ with $d(p,x_i) \to \infty$, $x_i \to z$. If $\{x_i\}$ and $\{y_i\}$ denote two such sequences, then the visual diameter of the set $\{x_i, y_i\}$ must go to zero. Hence, if $[x_i, y_i]$ denotes the geodesic joining x_i, y_i then $d(p,[x_i,y_i]) \to \infty$. Since, u, v lie outside an $m(n)$ ball, there exists some function ψ, such that the geodesic $[u,v]$ lies outside a $\psi(m(n))$ ball around p, where $\psi(k) \to \infty$ as $k \to \infty$.

We still need to argue that the function ψ is independent of the horoball $\mathbf{H}_1$. Assume that the lifts of horoballs miss the base-point p. If the conclusion fails, there exist a sequence of horoballs $\mathbf{H}_i$, points x_i, y_i on the boundary of $\mathbf{H}_i$, such that $[x_i, y_i] \subset \mathbf{H}_i$ passes through some fixed N–ball $B_N(p)$ about p. Passing to a subsequence and extracting a limit we may assume that the horoballs $\mathbf{H}_i$ converge to some horoball $\mathbf{H}$ cutting $B_N(p)$ (possibly touching p). Then the above discussion for $\mathbf{H}_1$ furnishes the required contradiction. Hence, the choice of this function ψ does not depend on $\mathbf{H}_1$. We conclude that there exists such a function for all of β^c. We have thus established:

- β^b lies outside an $m(n)$ ball about p,
- β^c lies outside a $\psi(m(n))$ ball about p,
- $m(n)$ and $\psi(m(n))$ tend to infinity as $n \to \infty$.

Define $f(n) = \min(m(n), \psi(m(n)))$. Then β^h lies outside an $f(n)$ ball about p and $f(n) \to \infty$ as $n \to \infty$.

By Lemma 2.4 $i\colon \widetilde{M_{\mathrm{gf}}} \to \widetilde{N^h}$ extends continuously to the boundary $\hat{i}\colon \widehat{M_{\mathrm{gf}}} \to \widehat{N^h}$. This proves the first statement of the theorem.

Now, for any geometrically finite Kleinian group, its limit set is locally connected.(See, for instance, Anderson–Maskit [2].) Hence, the limit set of $\widetilde{M_{gf}}$ is locally connected. Further, the continuous image of a compact locally connected set is locally connected Hocking–Young [22]. Hence, if Λ denotes the limit set of $\widetilde{N^h}$, then Λ is locally connected. This proves the theorem. □

Again as in Corollary 5.4 and Proposition 5.8 it is not necessary to restrict ourselves to the p–incompressible case. In Theorem 5.9 above we removed the clause on being within horoballs from Proposition 5.7. In the proof of Theorem 5.9 above the one new thing we had shown was that geodesics entering and leaving horoballs at a large distance from the reference point p, itself lies at a large distance from p. Using this observation along with Proposition 5.8 we easily deduce the following corollary.

Corollary 5.10 *Suppose (M, P) has incompressible boundary $\partial_0 P$. Given D, n there exist $m(n, D)$ such that the following holds: Let λ^h, β^h be as before. If λ^h penetrates exceptional horoballs by at most D and if λ^h lies outside $B_n(p)$ in $\widetilde{M_{gf}}$ then β^h lies outside a ball of radius $m(n, D)$, for some function $m(n, D)$ where $m(n, D) \to \infty$ as $n \to \infty$ for each fixed D.*

The above proposition will be useful in the next subsection when we go from *p–incompressibility* to *incompressibility*.

5.4 From p–incompressibility to incompressibility

In this subsection we shall use Corollary 5.10 to relax the hypothesis of p–incompressibility. Recall that we started with a hyperbolic geodesic λ^h in $\widetilde{M_{gf}}$ and then modified it along horoballs to obtain λ. In this subsection we first consider subsegments $\lambda^h_1, \ldots, \lambda^h_k$ where each λ^h_i starts and ends on horospheres bounding horoballs, and the complementary segments (of λ^h) lie inside horoballs. This decomposition is made in such a way that the starting and ending points of λ^h_i lie on *exceptional horoballs* (see previous subsection for definitions.) Having completed this decomposition we consider hyperbolic geodesics β^h_i in $\widetilde{N^h}$ joining the end-points of λ^h_i. If we ensure that the geodesics λ^h_i penetrate exceptional horoballs by a uniformly bounded amount, then by Corollary 5.10 above, we can ensure that the segments β^h_i lie outside a large ball about p.

If further, we can ensure that

- the β^h_i penetrate the horoballs they start and end on by a bounded amount,

- and that the terminal point of β_i^h and the initial point of β_{i+1}^h are separated by more than a critical amount,

then (by appealing to the fact that local quasigeodesics are global quasigeodesics [20; 19]) we conclude that the union of the β_i^h with the hyperbolic geodesics interpolating between them (and lying entirely within horoballs) is a (uniform) hyperbolic quasigeodesic.

Since each β_i^h lies outside a large ball about p, it follows that their union along with interpolating geodesics does so too. This union being a quasigeodesic, the geodesic β^h joining the end-points of λ must also lie outside a large ball about p. By Lemma 2.4 the existence of a Cannon–Thurston map follows.

We now furnish the details of the above argument.

First note from Corollary 5.10 that given D there exists $m(n, D)$ such that if λ_1^h be a subsegment of λ^h penetrating exceptional horoballs by at most D, and if λ^h (and hence λ_1^h) lies outside $B_n(p)$ then β_1^h (the hyperbolic geodesic in $\widetilde{N^h}$ joining the endpoints of λ_1^h) lies outside an $m(n, D)$–ball about p.

The next proposition follows from the fact that local quasigeodesics are global quasigeodesics in hyperbolic space. (See Gromov [20, Proposition 7.2C] and Coornaert–Delzant–Papadopoulos [16].)

Proposition 5.11 *There exist D, K, ϵ such that the following holds.*

Suppose β^x is a path in $\widetilde{N^h}$ such that β^x can be decomposed into finitely many geodesic segments $\beta_1, \ldots, \beta_k$. Further suppose that the starting and ending point of each β_i lie on exceptional horospheres (except possibly the starting point of β_1 and the ending-point of β_k) meeting the horospheres at right angles. Also suppose that the 'even segments' β_{2i} lie entirely within exceptional horoballs and have length greater than D. Then β^x is a (K, ϵ) quasigeodesic.

Proof Idea Since geodesic segments lying outside horoballs meet horospheres at right angles, successive pieces meet at an angle that is bounded below. Hence, if $x_i, z_i \in \beta_i, \beta i + 1$ respectively and $\beta_i \cap \beta i + 1 = y_i$, then $[x_i, y_i] \cup [y_i, z_i]$ is a (uniform) (K_1, ϵ_1)–quasigeodesic of length $\geq D + 1$ assuming that any two horoballs are separated by a distance of at least one. That is every segment of length $D + 1$ is a (K_1, ϵ_1)–quasigeodesic, ie see [16, Chapter 3]) it is a local quasigeodesic. If D is sufficiently long, stability of quasigeodesics (Theorems 1.2, 1.3, 1.4 of [16, Chapter 3]) ensures that β^x is a (K, ϵ) quasigeodesic for some K, ϵ independent of β.

In fact we do not need that the horoballs be exceptional, but that they be uniformly separated, ie the distance between any two horospheres is uniformly bounded below. This is clearly true for the exceptional horoballs since they are lifts of certain given cusps in M_{gf} which in turn are finitely many in number and have been chosen uniformly separated.

Next, we need to look closely at how we use Corollary 5.3 to prove Theorem 5.9. Note that β_{amb} is an ambient quasigeodesic independent of the hypothesis of *p–incompressibility*. It is in concluding that β_{amb} lies outside a large $m(n)$–ball about the reference point p that we needed to construct quasigeodesic rays going down to λ.

Now let $\lambda^h = [a, x]$ be a hyperbolic geodesic in $\widetilde{M_{\text{gf}}}$ having the end-point x on an exceptional horosphere bounding a horoball H. By our construction of B_λ we note that B_λ is quasi-isometrically embedded in $\widetilde{N}$ and further that β_{amb} (the ambient quasigeodesic corresponding to λ^h) meets H either at x (if H is a $(Z+Z)$–horoball) or at some point on r_x (for a quasigeodesic ray r_x constructed through x and lying in the universal cover $\widetilde{E}$ of an end E of the manifold) if H be a Z–horoball.

Now suppose $\lambda^h_1 = [a, x]$, $\lambda^h_2 = [x, y]$ and $\lambda^h_3 = [y, b]$ be three hyperbolic geodesic segments such that $[x, y] \subset \mathbf{H}$ and λ^h_3 meets H at y. Again, as before the corresponding ambient quasigeodesic β^1_{amb} meets $\mathbf{H}$ at x or at some point along an r_x and β^3_{amb} meets $\mathbf{H}$ at y or at some point along an r_y.

In any case, the distance between the entry point of β^h_1 (the hyperbolic geodesic joining the end-points of β^1_{amb}) into H and the exit point of β^h_3 (the hyperbolic geodesic joining the end-points of β^3_{amb}) from H is greater than $d(x, y) - 2C_0$ for some C_0 depending on the quasiconvexity constant of β^i_{amb} (i=1,3) by Theorem 2.8. Thus C_0 depends only on the quasi-Lipschitz constant of Π_λ.

We are now in a position to break λ^h into pieces. Let λ^h_{2i} denote the maximal subsegments of λ^h lying inside exceptional horoballs and having length greater than $(2C_0 + D_0)$. Let λ^h_{2i-1} denote the complementary segments. Now, let $\overline{\beta^h_{2i-1}}$ be the hyperbolic geodesic in $\widetilde{N^h}$ joining the endpoints of λ^h_{2i-1}. Then the entry point of $\overline{\beta^h_{2i-1}}$ into the *exceptional horoball* $\mathbf{H}$ it terminates on lies at a distance greater than $(2C_0 + D_0 - 2C_0) = D_0$ from the exit point of $\overline{\beta^h_{2i+1}}$ from the same horoball $\mathbf{H}$. Shorten the hyperbolic geodesic $\overline{\beta^h_{2i-1}}$ if necessary by cutting it off at the entry point into $\mathbf{H}$. Let β^h_{2i-1} denote the resultant geodesic. By Corollary 5.10 β^h_{2i-1} being a subsegment of $\overline{\beta^h_{2i-1}}$ lies at a distance of at least $m(n, (2C_0 + D_0)) = m_1(n)$ from the reference point p.

Next denote by β^h_{2i} the hyperbolic geodesic lying entirely within $\mathbf{H}$ joining the entry point of β^h_{2i-1} into $\mathbf{H}$ to the exit point of β^h_{2i+1} from $\mathbf{H}$. The initial and terminal points of β^h_{2i} lying on $\mathbf{H}$ are at a distance of at least $m_1(n)$ from p. Therefore each β^h_{2i} and hence the union of all the β^h_i lie outside an $m_2(n)$ ball about p where $m_2(n) \to \infty$ as $n \to \infty$. Further, the union of the segments β^h_i is a hyperbolic quasigeodesic by Proposition 5.11 and hence lies at a bounded distance D' from the hyperbolic geodesic β^h joining the endpoints of λ (ie the end-points of λ^h). Let $m_3(n) = m_2(n) - D'$. Then what we have shown amounts to the following.

- If λ^h lies outside $B_n(p)$, then β^h lies outside a ball of radius $m_3(n)$ about p.
- $m_3(n) \to \infty$ as $n \to \infty$.

Coupled with Lemma 2.4 this proves the main theorem of this paper.

Theorem 5.12 *Suppose that $N^h \in H(M, P)$ is a hyperbolic structure of bounded geometry on a pared manifold (M, P) with incompressible boundary $\partial_0 M$. Let M_{gf} denotes a geometrically finite hyperbolic structure adapted to (M, P). Then the map $i\colon \widetilde{M_{\mathrm{gf}}} \to \widetilde{N^h}$ extends continuously to the boundary $\hat{i}\colon \widehat{M_{\mathrm{gf}}} \to \widehat{N^h}$. If Λ denotes the limit set of $\widetilde{M}$, then Λ is locally connected.*

6 Examples and consequences

As mentioned in the Introduction, the simplest non-trivial examples to which Theorem 5.12 applies are hyperbolic three manifolds of finite volume fibering over the circle. These include the original examples of Cannon and Thurston [15] as well as those of Bowditch [11; 12].

The next set of examples are those three manifolds homeomorphic to the product of a surface and $\mathbb{R}$. These were dealt with in Minsky's work [28] and the punctured surface case was dealt with by Bowditch [11; 12].

The case of three manifolds of freely indecomposable fundamental group were dealt with independently by Klarreich [23] and the author [36]. In fact, Klarreich's theorem is really a reduction theorem which effectively says that if one can prove Cannon–Thurston for closed surface groups (of some given geometry) then one can also prove it for 3 manifolds whose ends have the same geometry. In combination with Minsky's adaptation of the original proof of Cannon–Thurston, this proves the theorem for bounded geometry 3–manifolds with incompressible (closed surface) boundary. In [36] we had approached this problem directly and had given different proofs of these results. In this paper we have continued the approach in [36] to prove the analogue of the above results in the presence of parabolics.

Incompressibility and compressibility One problem that has not been addressed by this paper or its predecessor [36] is the case of compressible $\partial_0 M$. Miyachi [37] (see also Souto [45]) has solved this problem in the bounded geometry case, when there are no cusps. In [40] we shall combine the reduction techniques of this paper with a coarse version of Miyachi's argument to settle affirmatively the question of existence of Cannon–Thurston maps for finitely generated Kleinian groups.

Coarse framework Many of the arguments of this paper rightfully belong to the domain of 'coarse' or 'asymptotic' or 'large-scale' geometry in spirit and it is more than likely that they may be generalized to this setting. In [35] and [36], we had proven the following theorems.

Theorem 6.1 [35] *Let G be a hyperbolic group and let H be a hyperbolic subgroup that is normal in G. Let $i\colon \Gamma_H \to \Gamma_G$ be the continuous proper embedding of Γ_H in Γ_G described above. Then i extends to a continuous map $\hat{i}$ from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$.*

Theorem 6.2 [36] *Let (X, d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Let v be a vertex of T. If X is hyperbolic $i_v\colon X_v \to X$ extends continuously to $\hat{i_v}\colon \widehat{X_v} \to \widehat{X}$.*

In this paper we have described a fairly general way of handling cusps. The generalization to 'coarse geometry' involves dealing with relatively hyperbolic groups a la Gromov [20], Farb [17] and Bowditch [8].

In [41], the author and Abhijit Pal generalize Theorem 6.2 to the context of trees of (strongly) relatively hyperbolic metric spaces. In [43], Pal adapts the techniques of this paper and generalizes Theorem 6.1 to (strongly) relatively hyperbolic normal subgroups of (strongly) relatively hyperbolic groups.

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Received: 17 February 2006
Revised: 18 September 2008