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Novikov-symplectic cohomology and exact Lagrangian embeddings

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Let N be a closed manifold satisfying a mild homotopy assumption. Then for any exact Lagrangian $L \subset T^*N$ the map $\pi_2(L) \to \pi_2(N)$ has finite index. The homotopy assumption is either that N is simply connected, or more generally that $\pi_m(N)$ is finitely generated for each $m \geq 2$. The manifolds need not be orientable, and we make no assumption on the Maslov class of L.

We construct the Novikov homology theory for symplectic cohomology, denoted $SH^*(M; \underline{\Lambda}_\alpha)$, and we show that Viterbo functoriality holds. We prove that the symplectic cohomology $SH^*(T^*N; \underline{\Lambda}_\alpha)$ is isomorphic to the Novikov homology of the free loopspace. Given the homotopy assumption on N, we show that this Novikov homology vanishes when $\alpha \in H^1(\mathcal{L}_0 N)$ is the transgression of a nonzero class in $H^2(\widetilde{N})$. Combining these results yields the above obstructions to the existence of L.

57R17; 57R58

1 Introduction

Consider a disc cotangent bundle $(DT^*N, d\theta)$ of a closed manifold N^n together with its canonical symplectic form. We want to find obstructions to the existence of embeddings $j\colon L^n \hookrightarrow DT^*N$ for which $j^*\theta$ is exact. These are called *exact Lagrangian embeddings*. For now assume that all manifolds are orientable and that we use $\mathbb{Z}$–coefficients in (co)homology.

Denote by $p\colon L \to N$ the composite of j with the projection to the base. Recall that the ordinary transfer map $p_!\colon H_*(N) \to H_*(L)$ is obtained by Poincaré duality and the pullback p^*, by composing

$$p_!\colon H_*(N) \to H^{n-*}(N) \to H^{n-*}(L) \to H_*(L).$$

For the space $\mathcal{L}_0 N$ of smooth contractible loops in N, such transfer maps need not exist, as Poincaré duality no longer holds. However, using techniques from symplectic topology, Viterbo [11; 12] showed that there is a transfer homomorphism

$$\mathcal{L}p_!\colon H_*(\mathcal{L}_0 N) \to H_*(\mathcal{L}_0 L)$$

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which commutes with the ordinary transfer map for p,

$$\begin{tikzcd} H_*(\mathcal{L}_0 L) & H_*(\mathcal{L}_0 N) \arrow[l, "\mathcal{L}p_!"'] \\ H_*(L) \arrow[u, hook, "c_*"] & H_*(N) \arrow[l, "p_!"'] \arrow[u, hook, "c_*"'] \end{tikzcd}$$

where $c\colon N \to \mathcal{L}_0 N$ denotes the inclusion of constant loops.

For any $\alpha \in H^1(\mathcal{L}_0 N)$, we can define the associated Novikov homology theory, which is in fact homology with twisted coefficients in the bundle of Novikov rings $\Lambda = \mathbb{Z}(\!(t)\!)$ associated to a singular cocycle representing α. We denote the bundle by $\underline{\Lambda}_\alpha$ and the Novikov homology by $H_*(\mathcal{L}_0 N; \underline{\Lambda}_\alpha)$.

Main Theorem *For all exact $L \subset T^*N$ and all $\alpha \in H^1(\mathcal{L}_0 N)$, there exists a commutative diagram:*

$$\begin{tikzcd} H_*(\mathcal{L}_0 L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & H_*(\mathcal{L}_0 N; \underline{\Lambda}_\alpha) \arrow[l, "\mathcal{L}p_!"'] \\ H_*(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) \arrow[u, "c_*"] & H_*(N; c^*\underline{\Lambda}_\alpha) \arrow[l, "p_!"'] \arrow[u, "c_*"'] \end{tikzcd}$$

If $c^*\alpha = 0$ then the bottom map becomes $p_! \otimes 1\colon H_*(L) \otimes \Lambda \leftarrow H_*(N) \otimes \Lambda$.

Suppose now that N is simply connected. Then a nonzero class $\beta \in H^2(N)$ defines a nonzero transgression $\tau(\beta) \in H^1(\mathcal{L}_0 N)$. The associated bundles $\underline{\Lambda}_{\tau(\beta)}$ on $\mathcal{L}_0 N$ and $\underline{\Lambda}_{\tau(p^*\beta)}$ on $\mathcal{L}_0 L$ restrict to trivial bundles on N and L.

Suppose $\tau(p^*\beta) = 0 \in H^1(\mathcal{L}_0 L)$. Then the above twisted diagram becomes

$$\begin{tikzcd} H_*(\mathcal{L}_0 L) \otimes \Lambda \arrow[d, shift left, "q_*"] & H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) \arrow[l, "\mathcal{L}p_!"'] \\ H_*(L) \otimes \Lambda \arrow[u, hook, shift left, "c_*"] & H_*(N) \otimes \Lambda \arrow[l, "p_!"'] \arrow[u, "c_*"'] \end{tikzcd}$$

where $q\colon \mathcal{L}_0 N \to N$ is the evaluation at 0 map. If N is simply connected and $\beta \neq 0$, then we will show that $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) = 0$, so the fundamental class $[N] \in H_n(N)$ maps to $c_*[N] = 0$. But $\mathcal{L}p_!(c_*[N]) = c_* p_![N] = c_*[L] \neq 0$ since c_* is injective on $H_*(L)$. Therefore $\tau(p^*\beta) = 0$ cannot be true. This shows that $\tau \circ p^*\colon H^2(N) \to$

$H^1(\mathcal{L}_0 L)$ is injective. Now, from the commutative diagram

$$\begin{tikzcd} H^2(N) \arrow[r, "\tau", "\sim"'] \arrow[d, "p^*"] & \mathrm{Hom}(\pi_2(N),\mathbb{Z}) \cong H^1(\mathcal{L}_0 N) \arrow[d, "(\mathcal{L}p)^*"] \\ H^2(L) \arrow[r, "\tau"] & \mathrm{Hom}(\pi_2(L),\mathbb{Z}) \subset H^1(\mathcal{L}_0 L) \end{tikzcd}$$

we deduce that $p^*\colon H^2(N) \to H^2(L)$ and $\mathrm{Hom}(\pi_2(N),\mathbb{Z}) \to \mathrm{Hom}(\pi_2(L),\mathbb{Z})$ must be injective. Thus we deduce:

Main Corollary (See Corollary 11.) *If $L \subset T^*N$ is exact and N is simply connected, then the image of $p_*\colon \pi_2(L) \to \pi_2(N)$ has finite index and $p^*\colon H^2(N) \to H^2(L)$ is injective.*

We emphasize that there is no assumption on the Maslov class of L in the statement – this is in contrast to the results of Nadler [6] and Fukaya, Seidel and Smith [3]: the vanishing of the Maslov class is crucial for their argument. Also observe that if $H^2(N) \neq 0$ then the corollary overlaps with Viterbo's result [11] that there is no exact Lagrangian $K(\pi,1)$ embedded in a simply connected cotangent bundle.

We will prove that the corollary holds even when N and L are not assumed to be orientable. A concrete application of the Corollary is that there are no exact tori and no exact Klein bottles in T^*S^2. We will also generalize the Corollary to obtain a result in the non–simply connected setup:

Corollary (See Corollary 13.) *Let N be a closed manifold with finitely generated $\pi_m(N)$ for each $m \geq 2$. If $L \subset T^*N$ is exact then the image of $p_*\colon \pi_2(L) \to \pi_2(N)$ has finite index.*

This is innovative since in [3], [6] and [11] it is crucial that N is simply connected.

The outline of the proof of the corollary required showing that the Novikov homology $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)})$ vanishes for nonzero $\beta \in H^2(\widetilde{N})$. The idea is as follows. A class $\tau(\beta) \in H^1(\mathcal{L}\widetilde{N}) = H^1(\mathcal{L}_0 N)$ gives rise to a cyclic covering $\overline{\mathcal{L}_0 N}$ of $\mathcal{L}_0 N$. Let t be a generator for the group of deck transformations. The Novikov ring $\Lambda = \mathbb{Z}(\!(t)\!) = \mathbb{Z}[\![t]\!][t^{-1}]$ is the completion in t of the group ring $\mathbb{Z}[t,t^{-1}]$ of the cover. The Novikov homology is isomorphic to $H_*(C_*(\overline{\mathcal{L}_0 N}) \otimes_{\mathbb{Z}[t,t^{-1}]} \Lambda)$.

Using the homotopy assumptions on N it is possible to prove that $H_*(\overline{\mathcal{L}_0 N})$ is finitely generated in each degree. It then easily follows from the flatness of Λ over $\mathbb{Z}[t,t^{-1}]$ and from Nakayama's lemma that

$$H_*(C_*(\overline{\mathcal{L}_0 N}) \otimes_{\mathbb{Z}[t,t^{-1}]} \Lambda) \cong H_*(\overline{\mathcal{L}_0 N}) \otimes_{\mathbb{Z}[t,t^{-1}]} \Lambda = 0.$$

The outline of the paper is as follows. In Section 2 we recall the construction of symplectic cohomology and we explain how the construction works when we use twisted coefficients in the Novikov bundle of some $\alpha \in H^1(\mathcal{L}N)$, which we call Novikov-symplectic cohomology. In Section 3 we recall Abbondandolo and Schwarz's construction [1] of the isomorphism between the symplectic cohomology of T^*N and the singular homology of the free loopspace $\mathcal{L}N$, and we adapt the isomorphism to Novikov-symplectic cohomology. In Section 4 we review the construction of Viterbo's commutative diagram, and we show how this carries over to the case of twisted coefficients. In Section 5 we prove the Main Theorem and in Section 6 we prove the Main Corollary. In Section 7 we generalize the corollary to the case of non–simply connected cotangent bundles, and in Section 8 we extend the results to the case when N and L are not assumed to be orientable.

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2 Symplectic cohomology

We review the construction of symplectic cohomology, and refer to Viterbo [12] for details and to Seidel [9] for a survey and for more references. We assume the reader is familiar with Floer homology for closed manifolds; for instance, see Salamon [7].

2.1 Liouville domain setup

Let (M^{2n}, θ) be a Liouville domain, that is $(M, \omega = d\theta)$ is a compact symplectic manifold with boundary and the Liouville vector field Z, defined by $i_Z\omega = \theta$, points strictly outwards along ∂M. The second condition is equivalent to requiring that $\alpha = \theta|_{\partial M}$ is a contact form on ∂M, that is $d\alpha = \omega|_{\partial M}$ and $\alpha \wedge (d\alpha)^{n-1} > 0$ with respect to the boundary orientation on ∂M.

The Liouville flow of Z is defined for all negative time r, and it parametrizes a collar $(-\infty, 0] \times \partial M$ of ∂M inside M. So we may glue an infinite symplectic cone $([0, \infty) \times \partial M, d(e^r\alpha))$ onto M along ∂M, so that Z extends to $Z = \partial_r$ on the cone. This defines the completion $\widehat{M}$ of M,

$$\widehat{M} = M \cup_{\partial M} [0, \infty) \times \partial M.$$

We call $(-\infty, \infty) \times \partial M$ the collar of $\widehat{M}$. We extend θ to the entire collar by $\theta = e^r\alpha$, and ω by $\omega = d\theta$. Later on, it will be convenient to change coordinates from r to $\mathrm{x} = e^r$. The collar will then be parametrized as the tubular neighbourhood $(0, \infty) \times \partial M$ of ∂M in $\widehat{M}$, where ∂M corresponds to $\{\mathrm{x} = 1\}$.

Let J be an ω–compatible almost complex structure on $\widehat{M}$ which is of contact type on the collar, that is $J^*\theta = e^r dr$ or equivalently $J\partial_r = \mathcal{R}$ where $\mathcal{R}$ is the Reeb vector field (we only need this to hold for $e^r \gg 0$ so that a certain maximum principle applies there). Denote by $g = \omega(\cdot, J\cdot)$ the J–invariant metric.

2.2 Reeb and Hamiltonian dynamics

The Reeb vector field $\mathcal{R} \in C^\infty(T\partial M)$ on ∂M is defined by $i_{\mathcal{R}} d\alpha = 0$ and $\alpha(\mathcal{R}) = 1$. The periods of the Reeb vector field form a countable closed subset of $[0, \infty)$.

For $H \in C^\infty(\widehat{M}, \mathbb{R})$ we define the Hamiltonian vector field X_H by

$$\omega(X_H, \cdot) = -dH.$$

If inside M the Hamiltonian H is a C^2–small generic perturbation of a constant, then the 1–periodic orbits of X_H inside M are constants corresponding precisely to the critical points of H.

Suppose $H = h(e^r)$ depends only on e^r on the collar. Then $X_H = h'(e^r)\mathcal{R}$. It follows that every nonconstant 1–periodic orbit $x(t)$ of X_H which intersects the collar must lie in $\{e^r\} \times \partial M$ for some e^r and must correspond to a Reeb orbit $z(t) = x(t/T)\colon [0, T] \to \partial M$ with period $T = h'(e^r)$. Since the Reeb periods are countable, if we choose h to have a generic constant slope $h'(e^r)$ for $e^r \gg 0$ then there will be no 1–periodic orbits of X_H outside of a compact set of $\widehat{M}$.

2.3 Action functional

We define the action functional for $x \in C^\infty(S^1, M)$ by

$$A_H(x) = -\int x^*\theta + \int_0^1 H(x(t))\, dt.$$

If $H = h(e^r)$ on the collar and x is a 1–periodic orbit of X_H in $\{e^r\} \times \partial M$, then

$$A_H(x) = -e^r h'(e^r) + h(e^r).$$

Let $\mathcal{L}\widehat{M} = C^\infty(S^1, \widehat{M})$ be the space of free loops in $\widehat{M}$. The differential of A_H at $x \in \mathcal{L}\widehat{M}$ in the direction $\xi \in T_x\mathcal{L}\widehat{M} = C^\infty(S^1, x^*T\widehat{M})$ is

$$dA_H \cdot \xi = -\int_0^1 \omega(\xi, \dot{x} - X_H)\, dt.$$

Thus the critical points $x \in \mathrm{Crit}(A_H)$ of A_H are precisely the 1–periodic Hamiltonian orbits $\dot{x}(t) = X_H(x(t))$. Moreover, we deduce that with respect to the L^2–metric $\int_0^1 g(\cdot, \cdot)\, dt$ the gradient of A_H is $\nabla A_H = J(\dot{x} - X_H)$.

2.4 Floer's equation

For $u\colon \mathbb{R}\times S^1 \to M$, the negative L^2–gradient flow equation $\partial_s u = -\nabla A_H(u)$ in the coordinates $(s,t)\in\mathbb{R}\times S^1$ is Floer's equation

$$\partial_s u + J(\partial_t u - X_H) = 0.$$

The action $A_H(u(s,\cdot))$ decreases in s along Floer solutions, since

$$\partial_s(A_H(u(s,\cdot))) = dA_H\cdot\partial_s u = -\int_0^1 \omega(\partial_s u, \partial_t u - X_H)\,dt = -\int_0^1 |\partial_s u|_g^2\,dt.$$

Let $\mathcal{M}'(x_-,x_+)$ denote the moduli space of solutions u to Floer's equation, which at the ends converge uniformly in t to the 1–periodic orbits $x_\pm$:

$$\lim_{s\to\pm\infty} u(s,t) = x_\pm(t).$$

These solutions u occur in $\mathbb{R}$–families because we may reparametrize the $\mathbb{R}$ coordinate by adding a constant. We denote by $\mathcal{M}(x_-,x_+) = \mathcal{M}'(x_-,x_+)/\mathbb{R}$ the space of unparametrized solutions.

2.5 Energy

For a Floer solution u the energy is defined as

$$E(u) = \int |\partial_s u|^2\,ds\,dt = \int \omega(\partial_s u, \partial_t u - X_H)\,ds\,dt = -\int \partial_s(A_H(u))\,ds.$$

Thus for $u\in\mathcal{M}'(x_-,x_+)$ there is an a priori energy estimate,

$$E(u) = A_H(x_-) - A_H(x_+).$$

2.6 Compactness and the maximum principle

The only danger in this setup, compared to Floer theory for closed manifolds, is that there may be Floer trajectories $u\in\mathcal{M}(x_-,x_+)$ which leave any given compact set in $\widehat{M}$. However, for any Floer trajectory u, a maximum principle applies to the function $e^r\circ u$ on the collar, namely: on any compact subset $\Omega\subset\mathbb{R}\times S^1$ the maximum of $e^r\circ u$ is attained on the boundary $\partial\Omega$. Therefore, if the $x_\pm$ lie inside $M\cup([0,R]\times\partial M)$ then also all the Floer trajectories in $\mathcal{M}'(x_-,x_+)$ lie in there.

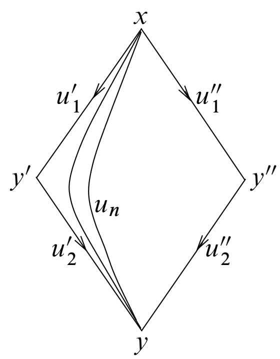

Figure 1: The x, y', y'', y are 1–periodic orbits of X_H, the lines are Floer solutions in $\widehat{M}$. The $u_n \in \mathcal{M}_1(x,y)$ are converging to the broken trajectory $(u_1', u_2') \in \mathcal{M}_0(x,y') \times \mathcal{M}_0(y',y)$.

2.7 Transversality and compactness

Thanks to the maximum principle and the a priori energy estimates, the same analysis as for Floer theory for closed manifolds can be applied to show that for a generic time-dependent perturbation (H_t, J_t) of (H,J) the corresponding moduli spaces $\mathcal{M}(x_-,x_+)$ are smooth manifolds and have compactifications $\overline{\mathcal{M}}(x_-,x_+)$ whose boundaries are defined in terms of broken Floer trajectories (Figure 1). We write $\mathcal{M}_k(x_-,x_+) = \mathcal{M}'_{k+1}(x_-,x_+)/\mathbb{R}$ for the k–dimensional part of $\mathcal{M}(x_-,x_+)$.

The perturbation of (H,J) ensures that the differential $D\phi^1_{X_H}$ of the time 1 return map does not have eigenvalue 1, where $\phi^t_{X_H}$ is the flow of X_H. This nondegeneracy condition ensures that the 1–periodic orbits of X_H are isolated and it is used to prove the transversality results. In the proofs of compactness, the exactness of ω is used to exclude the possibility of bubbling-off of J–holomorphic spheres.

To keep the notation under control, we will continue to write (H,J) even though we are using the perturbed (H_t,J_t) throughout.

2.8 Floer chain complex

The Floer chain complex for a Hamiltonian $H \in C^\infty(\widehat{M}, \mathbb{R})$ is the abelian group freely generated by 1–periodic orbits of X_H,

$$CF^*(H) = \bigoplus \big\{ \mathbb{Z}x : x \in \mathcal{L}\widehat{M},\ \dot{x}(t) = X_H(x(t)) \big\},$$

and the differential ∂ on a generator $y \in \mathrm{Crit}(A_H)$ is defined as

$$\partial y = \sum_{u \in \mathcal{M}_0(x,y)} \epsilon(u)\, x,$$

where $\mathcal{M}_0(x,y)$ is the 0–dimensional part of $\mathcal{M}(x,y)$ and the sign $\epsilon(u)\in\{\pm 1\}$ is determined by the choices of compatible orientations.

We may also filter the Floer complexes by action values $A,B\in\mathbb{R}\cup\{\pm\infty\}$:

$$CF^*(H;A,B)=\bigoplus\big\{\mathbb{Z}x : x\in\mathcal{L}\widehat{M},\ \dot{x}(t)=X_H(x(t)),\ A<A_H(x)<B\big\}.$$

This is a quotient complex of $CF^*(H)$ if $B\neq\infty$. Observe that increasing A gives a subcomplex, $CF^*(H;A',B)\subset CF^*(H;A,B)$ for $A<A'<B$. Moreover there are natural action-restriction maps $CF^*(H;A,B)\to CF^*(H;A,B')$ for $A<B'<B$, because the action decreases along Floer trajectories.

Standard methods show that $\partial^2=0$, and we denote by $HF^*(H)$ and $HF^*(H;A,B)$ the cohomologies of these complexes.

2.9 Continuation maps

One might hope that the continuation method of Floer homology can be used to define a homomorphism between the Floer complexes $CF^*(H_-)$ and $CF^*(H_+)$ obtained for two Hamiltonians $H_\pm$. This involves solving the parametrized version of Floer's equation

$$\partial_s u + J_s(\partial_t u - X_{H_s})=0,$$

where J_s are ω–compatible almost complex structures of contact type and H_s is a homotopy from H_- to H_+ (ie an s–dependent Hamiltonian with $(H_s,J_s)=(H_-,J_-)$ for $s\ll 0$ and $(H_s,J_s)=(H_+,J_+)$ for $s\gg 0$). If x and y are respectively 1–periodic orbits of X_{H_-} and X_{H_+}, then we can define a moduli space $\mathcal{M}(x,y)$ of such solutions u which converge to x and y at the ends. This time there is no freedom to reparametrize u in the s–variable.

The action $A_{H_s}(u(s,\cdot))$ along such a solution u will vary as follows:

$$\partial_s(A_{H_s}(u(s,\cdot)))=-\int_0^1|\partial_s u|^2\,dt+\int_0^1(\partial_s H_s)(u)\,dt$$

So the action decreases if H_s is monotone decreasing, $\partial_s H_s\leq 0$. The energy is

$$E(u)=\int|\partial_s u|^2_{g_s}\,ds\wedge dt=A_{H_-}(x_-)-A_{H_+}(x_+)+\int(\partial_s H_s)(u)\,ds\wedge dt,$$

so an a priori bound will hold if $\partial_s H_s\leq 0$ outside of a compact set in $\widehat{M}$.

If $H_s=h_s(e^r)$ on the collar and $\partial_s h_s'\leq 0$, then a maximum principle for $e^r\circ u$ as before will hold on the collar (we refer to Seidel [9] for a very clear proof) and therefore it automatically guarantees a bound on $(\partial_s H_s)(u)$ and thus an a priori energy bound.

Thus, if outside of a compact in $\widehat{M}$ we have $H_s = h_s(e^r)$ and $\partial_s h_s' \leq 0$, then (after a generic C^2–small time-dependent perturbation of (H_s, J_s)) the moduli space $\mathcal{M}(x, y)$ will be a smooth manifold with a compactification $\overline{\mathcal{M}}(x, y)$ by broken trajectories and a continuation map ϕ: $CF^*(H_+) \to CF^*(H_-)$ can be defined: on a generator $y \in \mathrm{Crit}(A_{H_+})$,

$$\phi(y) = \sum_{v \in \mathcal{M}_0(x,y)} \epsilon(v)\, x,$$

where $\mathcal{M}_0(x, y)$ is the 0–dimensional part of $\mathcal{M}(x, y)$ and $\epsilon(v) \in \{\pm 1\}$ depends on orientations. Standard methods show that ϕ is a chain map and that these maps compose well: given homotopies from H_- to K and from K to H_+, each satisfying the condition $\partial_s h_s' \leq 0$ outside of a compact in $\widehat{M}$, then the composite $CF^*(H_+) \to CF^*(K) \to CF^*(H_-)$ is chain homotopic to ϕ. So on cohomology, ϕ: $HF^*(H_+) \to HF^*(H_-)$ equals the composite $HF^*(H_+) \to HF^*(K) \to HF^*(H_-)$.

For example, a "compactly supported homotopy" is one where H_s is independent of s outside of a compact ($\partial_s H_s = 0$ for $s \gg 0$). Continuation maps for H_s and H_{-s} can then be defined and they will be inverse to each other up to chain homotopy.

2.10 Symplectic cohomology using only one Hamiltonian

We change coordinates from r to $\mathrm{x} = e^r$, so the collar is now $(0, \infty) \times \partial M \subset \widehat{M}$ and $\partial M = \{\mathrm{x} = 1\}$.

Take a Hamiltonian H_∞ with $H_\infty = h(\mathrm{x})$ for $\mathrm{x} \gg 0$, such that $h'(\mathrm{x}) \to \infty$ as $\mathrm{x} \to \infty$. The symplectic cohomology is defined as the cohomology of the corresponding Floer complex (after a C^2–small time-dependent perturbation of (H_∞, J)),

$$SH^*(M; H_\infty) = HF^*(H_\infty).$$

The technical difficulty lies in showing that it is independent of the choices (H_∞, J).

2.11 Symplectic cohomology with action bounds

Similarly one defines the groups $SH^*(M; H_\infty; A, B) = HF^*(M; H_\infty; A, B)$, but these now depend on the choice of H_∞. However, for $B = \infty$, taking the direct limit as $A \to -\infty$ yields

$$\varinjlim SH^*(M; H_\infty; A, \infty) = SH^*(M; H_\infty),$$

since $CF^*(H_\infty; A, \infty)$ are subcomplexes exhausting $CF^*(H_\infty; -\infty, \infty)$ as $A \to -\infty$.

If we use action bounds, one can sometimes vary the Hamiltonian without continuation maps. Let $H_1 = h_1(\mathrm{x})$ for $\mathrm{x} \geq \mathrm{x}_0$, and suppose $A_{h_1}(\mathrm{x}) = -\mathrm{x}h_1'(\mathrm{x}) + h_1(\mathrm{x}) < A$ for $\mathrm{x} \geq \mathrm{x}_0$. Let $H_2 = H_1$ on $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$ and $H_2 = h_2(\mathrm{x})$ with $A_{h_2}(\mathrm{x}) < A$ for $\mathrm{x} \geq \mathrm{x}_0$ (eg if $h_2'' \geq 0$). Then

$$CF^*(H_1; A, B) = CF^*(H_2; A, B)$$

are equal as complexes: the orbits in $\{\mathrm{x} \geq \mathrm{x}_0\}$ get discarded by the action bounds; the orbits agree in $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$ since $H_1 = H_2$ there; and the differential on these common orbits is the same because the maximum principle forces the Floer trajectories to lie in $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$, where $H_1 = H_2$, so the Floer equations agree.

For example, let $H_1 = h_1(\mathrm{x}) = \frac{1}{2}\mathrm{x}^2$ on $\mathrm{x} > 0$, so $A_{h_1}(\mathrm{x}) = -\frac{1}{2}\mathrm{x}^2$. Take $H_2 = H_1$ on $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$ and extend H_2 linearly on $\{\mathrm{x} \geq \mathrm{x}_0\}$. Then $CF^*(H_1; -\frac{1}{2}\mathrm{x}_0^2; \infty) = CF^*(H_2; -\frac{1}{2}\mathrm{x}_0^2; \infty)$. By this trick, $SH^*(M; H_\infty; A, \infty)$ can be computed by a Hamiltonian which is linear at infinity, and so $SH^*(M; H_\infty)$ can be computed as a direct limit using Hamiltonians which are linear at infinity and whose slopes at infinity become steeper and steeper. We now make this precise.

2.12 Hamiltonians linear at infinity

Consider Hamiltonians H which equal

$$h^m_{c,C}(\mathrm{x}) = m(\mathrm{x} - c) + C$$

for $\mathrm{x} \gg 0$. We assume that the slope $m > 0$ does not occur as the value of the period of any Reeb orbit. If H_s is a homotopy from H_- to H_+ among such Hamiltonians, ie $H_s = h^{m_s}_{c_s,C_s}(\mathrm{x})$ for $x \gg 0$, then the maximum principle (and hence a priori energy bounds for continuation maps) will hold if

$$\partial_s \partial_{\mathrm{x}} h^{m_s}_{c_s,C_s} = \partial_s m_s \leq 0.$$

Suppose that $\partial_s m_s \leq 0$, satisfying $m_s = m_-$ for $s \ll 0$ and $m_s = m_+$ for $s \gg 0$, and suppose that the action values $A_{H_s}(x)$ of 1–periodic orbits x of X_{H_s} never cross the action bounds A, B. Then a continuation map can be defined:

$$\phi\colon CF^*(H_+; A, B) \to CF^*(H_-; A, B).$$

These maps compose well: $\phi' \circ \phi''$ is chain homotopic to ϕ (where to define ϕ', ϕ'' we use m_s' varying from m_- to some m, m_s'' varying from m to m_+, and the analogous assumptions as above hold). For example if we vary only c, C, and not m, then $\partial_s m_s = 0$ outside of a compact and the continuation map ϕ for H_s can be inverted (up to chain homotopy) by using the continuation map for H_{-s}. Thus, up to isomorphism, $HF^*(H)$ is independent of the choice of the constants c, C in $h^m_{c,C}$.

2.13 Symplectic cohomology as a direct limit

Suppose $H_\infty = h(\mathrm{x})$ for $\mathrm{x} \gg 0$ and $h'(\mathrm{x}) \to \infty$ as $\mathrm{x} \to \infty$. Suppose also that $\mathrm{x}h''(\mathrm{x}) > \delta > 0$ for $x \gg 0$. This implies that $\partial_{\mathrm{x}} A_h = -\mathrm{x}h''(\mathrm{x}) < -\delta$ so A_h decreases to $-\infty$ as $\mathrm{x} \to \infty$.

Given $A \in \mathbb{R}$, suppose $A_h(\mathrm{x}) = -\mathrm{x}h'(\mathrm{x}) + h(\mathrm{x}) < A$ for $\mathrm{x} \geq \mathrm{x}_0$. Define $H = H_\infty$ on $M \cup \{\mathrm{x} \leq \mathrm{x}_0\}$ and extend H linearly in x for $\mathrm{x} \geq \mathrm{x}_0$. Then $CF^*(H; A, B) = CF^*(H_\infty; A, B)$, and $CF^*(H; A, B)$ is a subcomplex of $CF^*(H_\infty; -\infty, B)$.

Decreasing A to $A' < A$ defines some Hamiltonian H' which is steeper at infinity, and it induces a continuation map $CF^*(H; A, B) \to CF^*(H'; A', B)$. The direct limit over these continuation maps yields a chain isomorphism

$$\varinjlim CF^*(H; A, B) \to CF^*(H_\infty; -\infty, B),$$

which by the exactness of direct limits induces an isomorphism on cohomology

$$\varinjlim HF^*(H; A, B) \to SH^*(M; H_\infty; -\infty, B).$$

So an alternative definition is

$$SH^*(M) = \varinjlim HF^*(H),$$

where the direct limit is over the continuation maps for all the Hamiltonians which are linear at infinity, ordered by increasing slopes $m > 0$. In the above argument, we chose particular H which approximated H_∞ on larger and larger compacts. However, the direct limit can be taken over any family of H with slopes at infinity $m \to \infty$ because, up to an isomorphism induced by a continuation map, $HF^*(H)$ is independent of the choice of H for fixed m, so any two cofinal families ($m \to \infty$) will give the same limit up isomorphism.

2.14 Novikov bundles of coefficients

We recommend Whitehead [14] as a reference on local systems. Let $\mathcal{L}N = C^\infty(S^1, N)$ denote the free loopspace of a manifold N, and let $\mathcal{L}_0 N$ be the component of contractible loops. The Novikov ring

$$\Lambda = \mathbb{Z}(\!(t)\!) = \mathbb{Z}[\![t]\!][t^{-1}]$$

is the ring of formal Laurent series. Let α be a singular cocycle representing $a \in H^1(\mathcal{L}N)$. The Novikov bundle $\underline{\Lambda}_\alpha$ is the local system of coefficients on $\mathcal{L}N$ defined by a copy Λ_γ of Λ over each loop $\gamma \in \mathcal{L}N$ and by the multiplication isomorphism $t^{\alpha[u]}\colon \Lambda_\gamma \to \Lambda_{\gamma'}$ for each path u in $\mathcal{L}N$ connecting γ to γ', where $\alpha[\cdot]\colon C_1(\mathcal{L}N) \to \mathbb{Z}$

is evaluation on singular one-chains. A different choice of representative α for a gives an isomorphic local system, so by abuse of notation we write $\underline{\Lambda}_a$ instead of $\underline{\Lambda}_\alpha$ and $a[u]$ instead of $\alpha[u]$.

We will be using the Novikov bundle $\underline{\Lambda}_{\tau(\beta)}$ on $\mathcal{L}_0 N$ corresponding to the transgression $\tau(\beta) \in H^1(\mathcal{L}_0 N)$ of some $\beta \in H^2(N)$ (see Section 6.1). This bundle pulls back to a trivial bundle under the inclusion of constant loops $c\colon N \to \mathcal{L}_0 N$, since the transgression $\tau(\beta)$ vanishes on $\pi_1(N) \subset \pi_1(\mathcal{L}_0 N)$. Therefore we just get ordinary cohomology with coefficients in the ring Λ,

$$H^*(N; c^*\underline{\Lambda}_{\tau(\beta)}) \cong H^*(N;\Lambda).$$

Moreover, for any map $j\colon L \to T^*N$ the projection $p\colon L \to T^*N \to N$ induces a map $\mathcal{L}p\colon \mathcal{L}_0 L \to \mathcal{L}_0 N$, and the pullback of the Novikov bundle is

$$(\mathcal{L}p)^*\underline{\Lambda}_{\tau(\beta)} \cong \underline{\Lambda}_{(\mathcal{L}p)^*(\tau(\beta))} \cong \underline{\Lambda}_{\tau(p^*\beta)}.$$

If $\tau(p^*\beta) = 0 \in H^1(\mathcal{L}_0 L)$, then this is a trivial bundle and

$$H_*(\mathcal{L}_0 L; (\mathcal{L}p)^*\underline{\Lambda}_{\tau(\beta)}) \cong H_*(\mathcal{L}_0 L) \otimes \Lambda.$$

2.15 Novikov–Floer cohomology

Let (M^{2n},θ) be a Liouville domain Section 2.1. Let α be a singular cocycle representing a class in $H^1(\mathcal{L}M) \cong H^1(\mathcal{L}\widehat{M})$. We define the Novikov–Floer chain complex for $H \in C^\infty(\widehat{M},\mathbb{R})$ with twisted coefficients in $\underline{\Lambda}_\alpha$ to be the Λ–module freely generated by the 1–periodic orbits of X_H:

$$CF^*(H;\underline{\Lambda}_\alpha) = \bigoplus \bigl\{\Lambda x : x \in \mathcal{L}\widehat{M},\ \dot{x}(t) = X_H(x(t))\bigr\},$$

and the differential δ on a generator $y \in \mathrm{Crit}(A_H)$ is defined as

$$\delta y = \sum_{u \in \mathcal{M}_0(x,y)} \epsilon(u)\, t^{\alpha[u]}\, x,$$

where $\mathcal{M}_0(x,y)$ and $\epsilon(u) \in \{\pm 1\}$ are the same as in Section 2.8. The new factor $t^{\alpha[u]}$ which appears in the differential is precisely the multiplication isomorphism $\Lambda_x \to \Lambda_y$ of the local system $\underline{\Lambda}_\alpha$ which identifies the Λ–fibres over x and y.

As in the untwisted case, we assume that a generic C^2–small time-dependent perturbation of (H,J) has been made so that the transversality and compactness results of Section 2.7 for the moduli spaces $\mathcal{M}(x,y)$ are achieved.

Proposition 1 $(CF^*(H;\underline{\Lambda}_\alpha);\delta)$ *is a chain complex, ie* $\delta\circ\delta = 0$.

Proof We mimic the proof that $\partial^2 = 0$ in Floer homology (see Salamon [7]). Observe Figure 1. A sequence $u_n \in \mathcal{M}'_2(x,y)$ converges to a broken trajectory $(u'_1, u'_2) \in \mathcal{M}'_1(x,y') \times \mathcal{M}'_1(y',y)$, in the sense that there are $s_n \to -\infty$ and $S_n \to \infty$ with

$$u_n(s_n + \cdot, \cdot) \to u'_1 \text{ and } u_n(S_n + \cdot, \cdot) \to u'_2 \text{ both in } C^\infty_{\mathrm{loc}};$$

Conversely given such (u'_1, u'_2) there is a curve $u\colon [0,1) \to \mathcal{M}'_2(x,y)$, unique up to reparametrization and up to the choice of $u(0) \in \mathcal{M}'_2(x,y)$, which approaches (u'_1, u'_2) as $r \to 1$, and the curve is orientation preserving if and only if $\epsilon(u'_1)\epsilon(u'_2) = 1$.

So the boundary of $\overline{\mathcal{M}}_1(x,y)$ is parametrized by $\mathcal{M}_0(x,y') \times \mathcal{M}_0(y',y)$. The value of $d\alpha = 0$ on the connected component of $\mathcal{M}_1(x,y)$ shown in Figure 1 is equal to the sum of the values of α over the broken trajectories,

$$\alpha[u'_1] + \alpha[u'_2] = \alpha[u''_1] + \alpha[u''_2],$$

and since $\epsilon(u'_1)\epsilon(u'_2) = -\epsilon(u''_1)\epsilon(u''_2)$, we conclude that

$$\epsilon(u'_1)\, t^{\alpha[u'_1]}\, \epsilon(u'_2)\, t^{\alpha[u'_2]} = -\epsilon(u''_1)\, t^{\alpha[u''_1]}\, \epsilon(u''_2)\, t^{\alpha[u''_2]}.$$

Thus the broken trajectories contribute opposite Λ–multiples of x to $\delta(\delta y)$ for each connected component of $\mathcal{M}_1(x,y)$. Hence, summing over x, y',

$$\delta(\delta y) = \sum_{(u'_1,u'_2) \in \mathcal{M}_0(x,y') \times \mathcal{M}_0(y',y)} \epsilon(u'_1)\, t^{\alpha[u'_1]}\, \epsilon(u'_2)\, t^{\alpha[u'_2]}\, x = 0. \qquad \square$$

Denote by $HF^*(H; \underline{\Lambda}_\alpha)$ the Λ–modules corresponding to the cohomology groups of the complex $(CF^*(H; \underline{\Lambda}_\alpha); \delta)$. We call these the Novikov–Floer cohomology groups. By filtering the chain complex by action as in Section 2.8, we can define

$$HF^*(H; \underline{\Lambda}_\alpha; A, B) = H^*(CF^*(H; \underline{\Lambda}_\alpha; A, B); \delta).$$

2.16 Twisted continuation maps

We now show that the continuation method described in Section 2.9 can be used in the twisted case under the same assumptions that we made in the untwisted case. Recall that this involves solving

$$\partial_s v + J_s(\partial_t v - X_{H_s}) = 0,$$

and that under suitable assumptions on (H_s, J_s) the moduli spaces $\mathcal{M}(x,y)$ of solutions v joining 1–periodic orbits x, y of X_{H_-} and X_{H_+} are smooth manifolds with compactifications $\overline{\mathcal{M}}(x,y)$ whose boundaries are given by broken trajectories.

So far, using a twisted differential does not change the setup. However, to make the continuation map ϕ: $CF^*(H_+;\underline{\Lambda}_\alpha) \to CF^*(H_-;\underline{\Lambda}_\alpha)$ into a chain map we need to define it on a generator $y \in \mathrm{Crit}(A_{H_+})$ by

$$\phi(y) = \sum_{v \in \mathcal{M}_0(x,y)} \epsilon(v)\, t^{\alpha[v]}\, x,$$

where $\mathcal{M}_0(x,y)$ and $\epsilon(v) \in \{\pm 1\}$ are as in Section 2.9.

Proposition 2 ϕ: $CF^*(H_+;\underline{\Lambda}_\alpha) \to CF^*(H_-;\underline{\Lambda}_\alpha)$ *is a chain map.*

Proof We mimic the proof that ϕ is a chain map in the untwisted case [7]. Denote by $\mathcal{M}^{H_\pm}(\cdot,\cdot)$ the moduli spaces of Floer trajectories for $H_\pm$. Observe Figure 2.

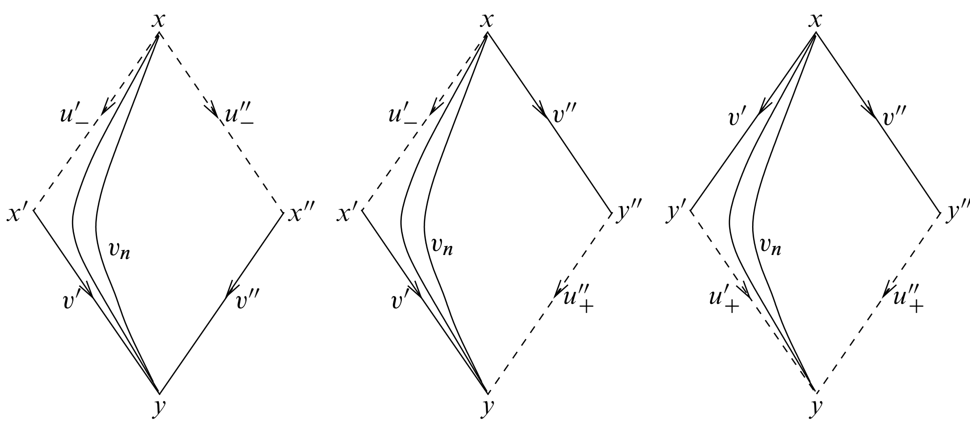

Figure 2: The dashed lines $u_\pm$ are Floer solutions converging to 1–periodic orbits of $X_{H_\pm}$, the solid lines are continuation map solutions and the $v_n \in \mathcal{M}_1(x,y)$ are converging to broken trajectories.

A compactness result in Floer homology shows that a sequence of solutions $v_n \in \mathcal{M}_1(x,y)$ will converge to a broken trajectory

$$(u'_-, v') \in \mathcal{M}_0^{H_-}(x,x') \times \mathcal{M}_0(x',y) \ \text{ or } \ (v', u'_+) \in \mathcal{M}_0(x,y') \times \mathcal{M}_0^{H_+}(y',y).$$

Conversely, given such (u'_-, v') or (v', u'_+) there is a smooth curve v: $[0,1) \to \mathcal{M}_1(x,y)$, unique up to reparametrization and up to the choice of $v(0)$, which approaches the given broken trajectory as $r \to 1$, and the curve is orientation preserving if and only if respectively $\epsilon(u'_-)\epsilon(v') = -1$ and $\epsilon(v')\epsilon(u'_+) = 1$.

Thus the boundary of $\overline{\mathcal{M}}_1(x,y)$ is parametrized by $-\mathcal{M}_0^{H_-}(x,x') \times \mathcal{M}_0(x',y)$ and by $\mathcal{M}_0(x,y') \times \mathcal{M}_0^{H_+}(y',y)$. The value of $d\alpha = 0$ on a connected component of

$\mathcal{M}_1(x,y)$ as in Figure 2 is equal to the sum of the values of α over the broken trajectories. For instance, in the second figure

$$\alpha[u'_-]+\alpha[v'] = \alpha[v'']+\alpha[u''_+],$$

and since $\epsilon(u'_-)\epsilon(v') = \epsilon(v'')\epsilon(u''_+)$,

$$\epsilon(u'_-)\, t^{\alpha[u'_-]}\, \epsilon(v')\, t^{\alpha[v']} = \epsilon(v'')\, t^{\alpha[v'']}\, \epsilon(u''_+)\, t^{\alpha[u''_+]}.$$

Thus the broken trajectories contribute equal Λ–multiples of x to $\delta(\phi(y))$ and $\phi(\delta y)$ for that component of $\mathcal{M}_1(x,y)$. A similar computation shows that in the first or third figures, the two broken trajectories contribute opposite Λ–multiples of x and so in total give no contribution to $\delta(\phi(y))$ or $\phi(\delta y)$. We deduce that

$$\begin{aligned}\delta(\phi(y)) &= \sum_{(u'_-,v')\in\mathcal{M}_0^{H_-}(x,x')\times\mathcal{M}_0(x',y)} \epsilon(u'_-)\, t^{\alpha[u'_-]}\, \epsilon(v')\, t^{\alpha[v']}\, x \\ &= \sum_{(v',u'_+)\in\mathcal{M}_0(x,y')\times\mathcal{M}_0^{H_+}(y',y)} \epsilon(v')\, t^{\alpha[v']}\, \epsilon(u'_+)\, t^{\alpha[u'_+]}\, x \ = \phi(\delta y),\end{aligned}$$

where we sum respectively over x, x' and x, y'. Hence ϕ is a chain map. □

A similar argument, by mimicking the proof of the untwisted case, shows that the twisted continuation maps compose well: given homotopies from H_- to K and from K to H_+ satisfying the conditions required in the untwisted case, the composite $CF^*(H_+;\underline{\Lambda}_\alpha)\to CF^*(K;\underline{\Lambda}_\alpha)\to CF^*(H_-;\underline{\Lambda}_\alpha)$ is chain homotopic to ϕ.

2.17 Novikov-symplectic cohomology

If we use the groups $HF^*(H;\underline{\Lambda}_\alpha)$ from Section 2.15 in place of $HF^*(H)$ in our discussion (Sections 2.10–2.13) of the symplectic cohomology groups of a Liouville domain, and we use the twisted continuation maps constructed in Section 2.16, then we obtain the Λ–modules

$$SH^*(M;H_\infty;\underline{\Lambda}_\alpha) \text{ and } SH^*(M;H_\infty;\underline{\Lambda}_\alpha;A,B),$$

which we call Novikov-symplectic cohomology groups.

So for H_∞ such that $H_\infty = h(\mathrm{x})$ for $\mathrm{x}\gg 0$ and $h'(\mathrm{x})\to\infty$ as $\mathrm{x}\to\infty$, we define

$$SH^*(M;\underline{\Lambda}_\alpha) = HF^*(H_\infty;\underline{\Lambda}_\alpha).$$

Alternatively, we may use the Hamiltonians H which equal $h^m_{c,C}(\mathrm{x}) = m(\mathrm{x}-c)+C$ for $\mathrm{x}\gg 0$, and we take the direct limit over the twisted continuation maps between the corresponding twisted Floer cohomologies as the slopes $m > 0$ increase:

$$SH^*(M; \underline{\Lambda}_\alpha) = \varinjlim HF^*(H; \underline{\Lambda}_\alpha).$$

3 Abbondandolo–Schwarz isomorphism

For a closed (oriented) manifold N^n, the symplectic cohomology of the cotangent disc bundle $M^{2n} = DT^*N$ is isomorphic to the homology of the free loopspace:

$$SH^*(DT^*N) \cong H_{n-*}(\mathcal{L}N).$$

This was first proved by Viterbo [13], and there are now two alternative approaches by Abbondandolo and Schwarz [1] and Salamon and Weber [8]. We will use the Abbondandolo–Schwarz isomorphism and show that it carries over to twisted coefficients, but similar arguments could be carried out using either of the other approaches. We will recall the construction [1] of the chain isomorphism

$$(CM_*(\mathcal{E}), \partial^{\mathcal{E}}) \to (CF^{n-*}(H), \partial^H)$$

between the Morse complex of the Hilbert manifold $\mathcal{L}^1 N = W^{1,2}(S^1, N)$ with respect to a certain Lagrangian action functional $\mathcal{E}$ and the Floer complex of T^*N with respect to an appropriate Hamiltonian $H \in C^\infty(S^1 \times T^*N, \mathbb{R})$.

Let $\pi\colon T^*N \to N$ denote the projection. We use the standard symplectic structure $\omega = d\theta$ and Liouville field Z on T^*N, which in local coordinates (q, p) are

$$\theta = p\, dq \qquad \omega = dp \wedge dq \qquad Z = p\, \partial_p.$$

A metric on N induces metrics and Levi-Civita connections on TN and T^*N, and it defines a splitting $T_{(q,p)}T^*N \cong T_qN \oplus T_q^*N \cong T_qN \oplus T_qN$ into horizontal and vertical vectors and a connection $\nabla = \nabla_q \oplus \nabla_p$, and similarly for $T_{(q,v)}TN$. For this splitting our preferred ω–compatible almost complex structure is $J\partial_q = -\partial_p$.

Remark Our action A_H is opposite to the action $\mathcal{A}$ used in [1], so our Floer trajectory $u(s,t)$ corresponds to $u(-s,t)$ in [1]. Our grading is $\mu(x) = n - \mu_{CZ}(x)$ (see Salamon [7], where the sign of H is opposite to ours), the one used in [1] is $\mu_{CZ}(x)$ and that in [9] is $-\mu_{CZ}(x)$. In our convention the index $\mu(x)$ agrees with the Morse index $\mathrm{ind}_H(x)$ for $x \in \mathrm{Crit}(H)$ when H is a C^2–small Morse Hamiltonian.

3.1 The Lagrangian Morse functional

The Morse function one considers on $\mathcal{L}^1 N = W^{1,2}(S^1, N)$ is the Lagrangian action functional

$$\mathcal{E}(q) = \int_0^1 L(t, q(t), \dot{q}(t))\, dt,$$

where the Lagrangian $L \in C^\infty(S^1 \times TN, \mathbb{R})$ is generic and satisfies certain growth conditions and a strong convexity assumption that ensure that: $\mathcal{E}$ is bounded below; the critical points of $\mathcal{E}$ are nondegenerate with finite Morse index; and $\mathcal{E}$ satisfies the Palais–Smale condition (any sequence of $q_n \in \mathcal{L}^1 N$ with bounded actions $\mathcal{E}(q_n)$ and with energies $\|\nabla \mathcal{E}(q_n)\|_{W^{1,2}} \to 0$ has a convergent subsequence). By an appropriate generic perturbation it is possible to obtain a metric G which is uniformly equivalent to the $W^{1,2}$ metric on $\mathcal{L}^1 N$ and for which $(\mathcal{E}, G)$ is a Morse–Smale pair. Denote by $\mathcal{M}^{\mathcal{E}}(q_-, q_+) = \mathcal{M}'_{\mathcal{E}}(q_-, q_+)/\mathbb{R}$ the unparametrized trajectories, where

$$\mathcal{M}'_{\mathcal{E}}(q_-, q_+) = \left\{ v \colon \mathbb{R} \to \mathcal{L}^1 N : \partial_s v(s) = -\nabla \mathcal{E}(v(s)),\ \lim_{s \to \pm\infty} v(s) = q_\pm \right\}.$$

Under these assumptions, infinite dimensional Morse theory can be applied to the space $(\mathcal{L}^1 N, \mathcal{E}, G)$ and the Morse homology is isomorphic to the singular homology of $\mathcal{L}^1 N$ (which is isomorphic to the singular homology of $\mathcal{L} N$, since $\mathcal{L}^1 N$ and $\mathcal{L} N$ are homotopy equivalent). This isomorphism respects the filtration by action: the homology of the Morse complex generated by the $x \in \mathrm{Crit}(\mathcal{E})$ with $\mathcal{E}(x) < a$ is isomorphic to $H_*(\{q \in \mathcal{L}^1 N : \mathcal{E}(q) < a\})$. The isomorphism also respects the splitting of the Morse complex and the singular complex into subcomplexes corresponding to the components of $\mathcal{L}^1 N$ (which are indexed by the conjugacy classes of $\pi_1(N)$).

3.2 Legendre transform

L defines a Hamiltonian $H \in C^\infty(S^1 \times T^* N, \mathbb{R})$ by

$$H(t, q, p) = \max_{v \in T_q N} (p \cdot v - L(t, q, v)).$$

The strong convexity assumption on L ensures that there is a unique maximum precisely where $p = d_v L(t, q, v)$ is the differential of L restricted to the vertical subspace $T^{\mathrm{vert}}_{(q,v)} TN \cong T_q N$, and it ensures that the Legendre transform

$$\mathfrak{L} \colon S^1 \times TN \to S^1 \times T^* N,\ (t, q, v) \mapsto (t, q, d_v L(t, q, v))$$

is a fiber-preserving diffeomorphism.

Pull back (ω, H, X_H) via $\mathfrak{L}$ to obtain $(\mathfrak{L}^*\omega, H \circ \mathfrak{L}, Y_L)$, so $\mathfrak{L}^*\omega(Y_L, \cdot) = -d(H \circ \mathfrak{L})$. The critical points of $\mathcal{E}$ are precisely the 1–periodic orbits $(q, \dot{q})$ of Y_L in TN, and these bijectively correspond to 1–periodic orbits x of X_H in T^*N via

$$(t,x) = \mathfrak{L}(t,q,\dot{q}).$$

Under this correspondence the Morse index of q is $m(q) = n - \mu(x)$ (in the conventions of [1], $m(q) = \mu_{CZ}(x)$). Moreover, for any $W^{1,2}$–path $x\colon [0,1] \to T^*N$,

$$\mathcal{E}(\pi x) \geq -A_H(x),$$

which becomes an equality if and only if $(t,x) = \mathfrak{L}(t, \pi x, \partial_t(\pi x))$ for all t.

3.3 The moduli spaces $\mathcal{M}^+(\mathbf{q},\mathbf{x})$

For 1–periodic orbits q of Y_L and x of X_H, define $\mathcal{M}^+(q,x)$ to be the collection of all maps $u \in C^\infty((-\infty,0)\times S^1, T^*N)$ which are of class $W^{1,3}$ on $(-1,0)\times S^1$ and which solve Floer's equation

$$\partial_s u + J(t,u)(\partial_t u - X_H(t,u)) = 0,$$

with the following boundary conditions:

(i) As $s \to -\infty$, $u(s,\cdot) \to x$ uniformly in t.

(ii) As $s \to 0$, u will converge to some loop $u(0,\cdot)$ of class $W^{2/3,3}$, and we require that the projection $\overline{q}(t) = \pi \circ u(0,t)$ in N flows backward to $q \in \mathrm{Crit}(\mathcal{E})$ along the negative gradient flow $\phi^s_{-\nabla\mathcal{E}}$ of $\mathcal{E}$: $\phi^s_{-\nabla\mathcal{E}}(\overline{q}) \to q$ as $s \to -\infty$.

Loosely speaking, $\mathcal{M}^+(q,x)$ consists of pairs of trajectories (w, u_+) where w is a $-\nabla\mathcal{E}$ trajectory in N flowing out of q, and u_+ is a Floer solution in T^*N flowing out of x, such that w and πu_+ intersect in a loop $\overline{q}(t) = \pi u_+(0,t)$ in N.

3.4 Transversality and compactness

The assumption on H and L is that there are constants $c_i > 0$ such that for all $(t,q,p) \in S^1 \times T^*N$, $(t,q,v) \in S^1 \times TN$,

$$dH(p\partial_p) - H \geq c_0|p|^2 - c_1, \;\; |\nabla_p H| \leq c_2(1+|p|), \;\; |\nabla_q H| \leq c_2(1+|p|^2);$$
$$\nabla_{vv} L \geq c_3\,\mathrm{Id}, \;\; |\nabla_{vv} L| \leq c_4, \;\; |\nabla_{qv} L| \leq c_4(1+|v|), \;\; |\nabla_{qq} L| \leq c_4(1+|v|^2).$$

We also assume that a small generic perturbation of L (and hence H) are made so that the nondegeneracy condition (see Section 2.7) holds for 1–periodic orbits of Y_L and X_H. We call such H, L regular. For regular H, there are only finitely many 1–periodic orbits x of X_H with action $A_H(x) \geq a$, for $a \in \mathbb{R}$. After a small generic perturbation of J, the compactness and transversality results of Section 2.7 hold for the spaces $\mathcal{M}^H(x,y) = \mathcal{M}'(x,y)/\mathbb{R}$ of unparametrized Floer solutions in T^*N

converging to $x, y \in \mathrm{Crit}(A_H)$ at the ends, and similar results hold for $\mathcal{M}^+(q,x)$ by using the $W^{1,3}$ condition in the definition to generalize the proofs used for $\mathcal{M}'(x,y)$.

When all of the above assumptions are satisfied, we call (L, G, H, J) regular. In this case, $\mathcal{M}^{\mathcal{E}}(p,q)$, $\mathcal{M}^H(x,y)$ and $\mathcal{M}^+(q,x)$ are smooth manifolds with compactifications by broken trajectories, and their dimensions are:

$$\begin{aligned}
\dim \mathcal{M}^{\mathcal{E}}(p,q) &= m(p) - m(q) - 1\\
\dim \mathcal{M}^{H}(x,y) &= \mu(x) - \mu(y) - 1\\
\dim \mathcal{M}^{+}(q,x) &= m(q) + \mu(x) - n
\end{aligned}$$

and we denote by $\mathcal{M}^{\mathcal{E}}_k(p,q)$, $\mathcal{M}^H_k(x,y)$ and $\mathcal{M}^+_k(q,x)$ the k–dimensional ones.

Theorem (Abbondandolo–Schwarz [1]) *If (L, G, H, J) is regular then there is a chain-complex isomorphism φ: $(CM_*(\mathcal{E}), \partial^{\mathcal{E}}) \to (CF^{n-*}(H), \partial^H)$, which on a generator $q \in \mathrm{Crit}(\mathcal{E})$ is defined as*

$$\varphi(q) = \sum_{u_+ \in \mathcal{M}^+_0(q,x)} \epsilon(u_+)\, x,$$

where $\epsilon(u_+) \in \{\pm 1\}$ are orientation signs. The isomorphism is compatible with the splitting into subcomplexes corresponding to different conjugacy classes of $\pi_1(N)$, and it is compatible with the action filtrations: for any $a \in \mathbb{R}$ it induces an isomorphism on the subcomplexes generated by the q, x with $\mathcal{E}(q) < a$ and $-A_H(x) < a$.

3.5 Proof that φ is an isomorphism

Since actions decrease along orbits and $\mathcal{E}(\pi x) \geq -A_H(x)$ with equality if and only if $(t,x) = \mathfrak{L}(t, \pi \circ x, \partial_t \pi x)$, we deduce that

$$\mathcal{E}(q) \geq \mathcal{E}(\overline{q}) \geq -A_H(u_+(0,\cdot)) \geq -A_H(x),$$

so $\mathcal{E}(q) \geq -A_H(x)$ with equality if and only if $q \equiv \overline{q}$, $u_+ \equiv x$, $q = \pi x$ and $(t,x) = \mathfrak{L}(t,q,\dot{q})$. Therefore if $\mathcal{E}(q) < -A_H(x)$ then $\mathcal{M}^+(q,x) = \varnothing$, and if $\mathcal{E}(q) = -A_H(x)$ then $\mathcal{M}^+(q,x)$ is either empty or, when $(t,x) = \mathfrak{L}(t,q,\dot{q})$, it consists of $u_+ \equiv x$. Now order the generators of $CM_*(\mathcal{E})$ according to increasing action and those of $CF^*(H)$ according to decreasing action, and so that the order is compatible with the correspondence $(t,x) = \mathfrak{L}(t,q,\dot{q})$. Then φ is a (possibly infinite) upper triangular matrix with ± 1 along the diagonal, so φ is an isomorphism.

3.6 Proof that φ is a chain map

The differentials for the complexes $(CM_*(\mathcal{E}), \partial^{\mathcal{E}})$ and $(CF^*(H), \partial^H)$ are defined on generators $q \in \mathrm{Crit}(\mathcal{E})$, $y \in \mathrm{Crit}(A_H)$ by

$$\partial^{\mathcal{E}}(q) = \sum_{v \in \mathcal{M}_0^{\mathcal{E}}(q,p)} \epsilon(v)\, p \quad \text{and} \quad \partial^H(y) = \sum_{u \in \mathcal{M}_0^H(x,y)} \epsilon(u)\, x$$

where $\epsilon(v), \epsilon(u) \in \{\pm 1\}$ depend on orientations. Observe Figure 3. A compactness

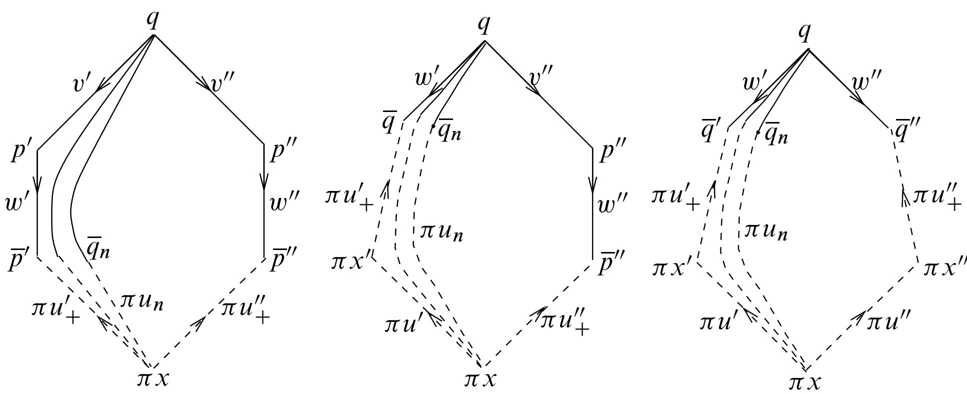

Figure 3: Solid lines are $-\nabla\mathcal{E}$ trajectories in N and dotted lines are the projections under $\pi\colon T^*N \to N$ of Floer solutions.

argument shows that the broken trajectories that compactify $\mathcal{M}_1^+(q,x)$ are of two types: either (i) the $-\nabla\mathcal{E}$ trajectory breaks, or (ii) the Floer trajectory breaks. More precisely, if $u_n \in \mathcal{M}_1^+(q,x)$ and $\overline{q}_n(t) = \pi(u_n(0,t))$, then one of the following holds:

(i) There are $[v] \in \mathcal{M}_0^{\mathcal{E}}(q,p)$; $u'_+ \in \mathcal{M}_0^+(p,x)$; and reals $t_n \to -\infty$ with

$$\phi^{t_n}_{-\nabla\mathcal{E}}(\overline{q}_n) \to v(0) \text{ in } W^{1,2}, \text{ and } u_n \to u'_+ \text{ in } C^\infty_{\mathrm{loc}}.$$

(ii) There are $[u'] \in \mathcal{M}_0^H(x,x')$; $u'_+ \in \mathcal{M}_0^+(q,x')$; and reals $s_n \to -\infty$ with

$$u_n(s_n + \cdot, \cdot) \to u' \text{ and } u_n \to u'_+ \text{ both in } C^\infty_{\mathrm{loc}}.$$

Conversely, given (v, u'_+) or (u', u'_+) as above, there is a smooth curve $u\colon [0,1) \to \mathcal{M}_1^+(q,x)$, unique up to reparametrization and up to the choice of $u(0)$, which approaches the given broken trajectory as $r \to 1$, and the curve is orientation preserving if and only if respectively $\epsilon(v)\epsilon(u'_+) = 1$ and $\epsilon(u')\epsilon(u'_+) = -1$.

Thus the boundary of $\mathcal{M}_1^+(q,x)$ is parametrized by $\mathcal{M}_0^{\mathcal{E}}(q,p) \times \mathcal{M}_0^+(p,x)$ and by $-\mathcal{M}_0^H(x,x') \times \mathcal{M}_0^+(q,x')$. Figure 3 shows the possible components of $\mathcal{M}_1^+(q,x)$:

in the first and third figures, the broken trajectories contribute zero respectively to $\varphi(\partial^{\mathcal{E}}(q))$ and $\partial^H(\varphi(q))$; in the second figure we see that $\epsilon(u')\epsilon(u'_+) = \epsilon(v'')\epsilon(u''_+)$, so the broken trajectories contribute $\pm x$ to both $\partial^H(\varphi(q))$ and $\varphi(\partial^{\mathcal{E}}(q))$. Therefore $\partial^H(\varphi(q)) = \varphi(\partial^{\mathcal{E}}(q))$, so φ is a chain map.

3.7 The twisted version of the Abbondandolo–Schwarz isomorphism

Let α be a singular cocycle representing a class in $H^1(\mathcal{L}^1 N) \cong H^1(\mathcal{L}N)$. We will use the bundles $\underline{\Lambda}_\alpha$ on $\mathcal{L}^1 N$ and $\underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}$ on $\mathcal{L}^1 T^*N$ (see Section 2.14), where $\mathcal{L}\pi\colon \mathcal{L}^1 T^*N \to \mathcal{L}^1 N$ is induced by $\pi\colon T^*N \to N$. The twisted complexes $(CM_*(\mathcal{E};\underline{\Lambda}_\alpha), \delta^{\mathcal{E}})$ and $(CF^*(H;\underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}), \delta^H)$ are freely generated over Λ respectively by the $q \in \mathrm{Crit}(\mathcal{E})$ and the $y \in \mathrm{Crit}(A_H)$. The twisted differentials are defined by

$$\delta^{\mathcal{E}}(q) = \sum_{v \in \mathcal{M}_0^{\mathcal{E}}(q,p)} \epsilon(v)\, t^{-\alpha[v]}\, p \quad \text{and} \quad \delta^H(y) = \sum_{u \in \mathcal{M}_0^H(x,y)} \epsilon(u)\, t^{\alpha[\mathcal{L}\pi(u)]}\, x$$

since $\alpha[\mathcal{L}\pi(u)] = (\mathcal{L}\pi)^*\alpha[u]$. The sign difference in the powers of t arises because $\delta^{\mathcal{E}}$ is a differential and δ^H is a codifferential. For simplicity, we write $\pi u = \mathcal{L}\pi(u)$.

Theorem 3 *If (L,G,H,J) is regular then for all $\alpha \in H^1(\mathcal{L}N)$ there is a chain-complex isomorphism $\varphi\colon (CM_*(\mathcal{E};\underline{\Lambda}_\alpha), \delta^{\mathcal{E}}) \to (CF^{n-*}(H;\underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}), \delta^H)$, which on a generator q is defined as*

$$\varphi(q) = \sum_{u_+ \in \mathcal{M}_0^+(q,x)} \epsilon(u_+)\, t^{-\alpha[w]+\alpha[\pi u_+]}\, x,$$

where $w\colon (-\infty,0] \to \mathcal{L}^1 N$ is the negative gradient trajectory $w(s) = \phi^s_{-\nabla\mathcal{E}}(\overline{q})$ connecting q to $\overline{q}(\cdot) = \pi u_+(0,\cdot)$. The isomorphism is compatible with the splitting into subcomplexes corresponding to different conjugacy classes of $\pi_1(N)$, and it is compatible with the action filtrations: for any $a \in \mathbb{R}$ it induces an isomorphism on the subcomplexes generated by the q, x with $\mathcal{E}(q) < a$ and $-A_H(x) < a$.

After identifying Morse cohomology with singular cohomology, the map φ induces an isomorphism

$$SH^*(DT^*N;\underline{\Lambda}_\alpha) \cong H_{n-*}(\mathcal{L}N;\underline{\Lambda}_\alpha).$$

Proof Figure 3 shows the possible connected components of $\mathcal{M}_1^+(q,x)$. Evaluating $d\alpha = 0$ on a component equals the sum of the values of α on the broken trajectories. For instance, in the second figure

$$-\alpha[w'] + \alpha[\pi u'_+] + \alpha[\pi u'] = -\alpha[v''] - \alpha[w''] + \alpha[\pi u''_+],$$

and therefore, since $\epsilon(u')\epsilon(u'_+) = \epsilon(v'')\epsilon(u''_+)$,

$$\epsilon(u')\epsilon(u'_+)\, t^{-\alpha[w'] + \alpha[\pi u'_+]}\, t^{\alpha[\pi u']} = \epsilon(v'')\epsilon(u''_+)\, t^{-\alpha[v'']}\, t^{-\alpha[w''] + \alpha[\pi u''_+]}.$$

Thus the broken trajectories contribute equally to $\delta^H(\varphi(q))$ and $\varphi(\delta^{\mathcal{E}}(q))$. A similar computation shows that in the first and third figures the broken trajectories contribute zero respectively to $\varphi(\delta^{\mathcal{E}}(q))$ and $\delta^H(\varphi(q))$. Hence

$$\begin{aligned}
\delta^H(\varphi(q)) &= \sum_{(u',u'_+)\in\mathcal{M}_0^H(x,x')\times\mathcal{M}_0^+(q,x')} \epsilon(u')\, t^{\alpha[\pi u']} \cdot \epsilon(u'_+)\, t^{-\alpha[w'] + \alpha[\pi u'_+]}\, x \\
&= \sum_{(v'',u''_+)\in\mathcal{M}_0^{\mathcal{E}}(q,p)\times\mathcal{M}_0^+(p,x)} \epsilon(v'')\, t^{-\alpha[v'']} \cdot \epsilon(u''_+)\, t^{-\alpha[w''] + \alpha[\pi u''_+]}\, x \\
&= \varphi(\delta^{\mathcal{E}}(q)),
\end{aligned}$$

where we sum respectively over x, x' and over x, p, and where w', w'' are the $-\nabla\mathcal{E}$ trajectories ending in $\pi u'_+(0,\cdot)$, $\pi u''_+(0,\cdot)$. Hence φ is a chain map.

That φ is an isomorphism follows just as in the untwisted case, because for $\mathcal{E}(q) \leq -A_H(x)$ the only nonempty $\mathcal{M}_0^+(q,x)$ occurs when $(t,x) = \mathfrak{L}(t,q,\dot{q})$, and in this case $\mathcal{M}_0^+(q,x) = \{u_+\}$ where $u_+ \equiv x$ and $w \equiv q$ are independent of $s \in \mathbb{R}$ and so the coefficient of x in $\varphi(q)$ is

$$\epsilon(u_+)\, t^{-\alpha[w] + \alpha[\pi u_+]} = \epsilon(u_+) = \pm 1.$$

The last statement in the claim is a consequence of the identification of the Morse cohomology of $(\mathcal{L}^1 N, \mathcal{E}, G)$ with the singular cohomology of $\mathcal{L}^1 N$ just as in [1], after introducing the system $\underline{\Lambda}_\alpha$ of local coefficients. □

4 Viterbo functoriality

Let (M^{2n}, θ) be a Liouville domain Section 2.1, and suppose

$$i\colon (W^{2n}, \theta') \hookrightarrow (M^{2n}, \theta)$$

is a Liouville embedded subdomain, that is we require that $i^*\theta - e^\rho\theta'$ is exact for some $\rho \in \mathbb{R}$. For example the embedding $DT^*L \hookrightarrow DT^*N$, obtained by extending an exact Lagrangian embedding $L \hookrightarrow DT^*N$ to a neighbourhood of L, is of this type. We fix $\delta > 0$ with

$$0 < \delta < \min\{\text{periods of the nonconstant Reeb orbits on } \partial M \text{ and } \partial W\}.$$

We will now recall the construction of Viterbo's commutative diagram [12]:

$$\begin{tikzcd} SH^*(W) & SH^*(M) \arrow[l, "SH^*(i)"'] \\ H^*(W) \arrow[u, "c_*"] & H^*(M) \arrow[l, "i^*"'] \arrow[u, "c_*"] \end{tikzcd}$$

4.1 Hamiltonians with small slopes

We now consider Hamiltonians H^0 as in Section 2.12, which are C^2–close to a constant on $\widehat{M} \setminus (0,\infty) \times \partial M$; $H^0 = h(\mathrm{x})$ with slopes $h'(\mathrm{x}) \le \delta$ for $\mathrm{x} \ge 0$; and which have constant slope $h'(\mathrm{x}) = m > 0$ for $\mathrm{x} \ge \mathrm{x}_0$.

A standard result in Floer homology is that (after a generic C^2–small time-independent perturbation of (H^0, J)) the 1–periodic orbits of X_{H^0} and the Floer trajectories connecting them inside $\widehat{M} \setminus \{\mathrm{x} \ge \mathrm{x}_0\}$ are both independent of $t \in S^1$, and so these orbits correspond to critical points of H^0 and these Floer trajectories correspond to negative gradient trajectories of H^0. By the maximum principle, the Floer trajectories connecting these orbits do not enter the region $\{\mathrm{x} \ge \mathrm{x}_0\}$, and by the choice of δ there are no 1–periodic orbits in $\{\mathrm{x} \ge \mathrm{x}_0\}$ since there $0 < h'(\mathrm{x}) \le \delta$.

The Floer complex $CF^*(H^0)$ is therefore canonically identified with the Morse complex $CM^*(H^0)$, which is generated by $\mathrm{Crit}(H^0)$ and whose differential counts the $-\nabla H^0$ trajectories. The Morse cohomology $HM^*(H^0)$ is isomorphic to the singular cohomology of $\widehat{M}$ (which is homotopy equivalent to M), so

$$HF^*(H^0) \cong HM^*(H^0) \cong H^*(M).$$

Moreover, by Morse cohomology, a different choice $H^{0'}$ of H^0 yields an isomorphism $HM^*(H^{0'}) \cong H^*(M)$ which commutes with $HM^*(H^0) \cong H^*(M)$ via the continuation isomorphism $HM^*(H^0) \to HM^*(H^{0'})$.

4.2 Construction of $\mathbf{c_*}$

Recall from Section 2.13 that

$$SH^*(M) = \varinjlim HF^*(H),$$

where the direct limit is over the continuation maps for Hamiltonians H which equal $h^m_{c,C}(\mathrm{x}) = m(\mathrm{x} - c) + C$ for $\mathrm{x} \gg 0$, ordered by increasing slopes $m > 0$.

Since H^0 is such a Hamiltonian, there is a natural map $HF^*(H^0) \to \lim HF^*(H)$ arising as a direct limit of continuation maps. By Section 4.1, this defines a map

$$c_*\colon H^*(M) \to SH^*(M).$$

A different choice $H^{0'}$ yields a map $HF^*(H^{0'}) \to SH^*(M)$ which commutes with the map $HF^*(H^0) \to SH^*(M)$ via the continuation isomorphism $HF^*(H^0) \to HF^*(H^{0'})$. Together with Section 4.1, this shows that c_* is independent of the choice of H^0.

4.3 Diagonal-step shaped Hamiltonians

We now consider the Liouville subdomain $i\colon W \hookrightarrow M$. The ∂_r–Liouville flow for θ' defines a tubular neighbourhood $(0, 1+\epsilon) \times \partial W$ of ∂W inside $\widehat{M}$, where ∂W corresponds to $\mathrm{x} = e^r = 1$. This coordinate x may not extend to $\widehat{M} \setminus W$, and it should not be confused with the x we previously used to parametrize $(0, \infty) \times \partial M \subset \widehat{M}$.

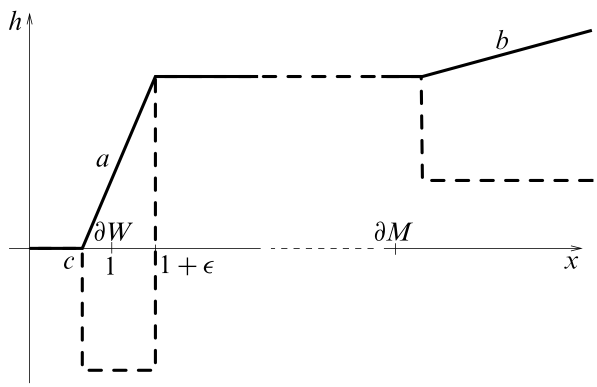

Figure 4: The solid line is a diagonal-step shaped Hamiltonian $h = h_c^{a,b}$ with slopes $a \gg b$. The dashed line is the action function $A_h(\mathrm{x}) = -\mathrm{x}h'(\mathrm{x}) + h(\mathrm{x})$.

We consider diagonal-step shaped Hamiltonians H as in Figure 4, which are zero on $W \setminus \{\mathrm{x} \geq c\}$ and which equal $h_c^{a,b}(\mathrm{x})$ on $\{\mathrm{x} \geq c\}$, where $h_c^{a,b}$ is piecewise linear with slope b at infinity; with slope $a \gg b$ on $(c, 1+\epsilon)$; and which is constant elsewhere. We assume that $0 \leq c \leq 1$ and that a, b are chosen generically so that they are not periods of Reeb orbits (see Section 2.2).

As usual, before we take Floer complexes we replace H by a generic C^2–small time-dependent perturbation of it, and the orbits and action values that we will mention take this into account. Let $M' \subset \widehat{M}$ be the compact subset where h does not have slope b. Observe Figure 4: the 1–periodic orbits of X_H that can arise are:

(1) critical points of H inside $W \setminus \{\mathrm{x} \geq c\}$ of action very close to 0;

(2) nonconstant orbits near $\mathrm{x} = c$ of action in $(-ac, -\delta c)$;

(3) nonconstant orbits near $\mathrm{x} = 1+\epsilon$ of action in $(-ac, a(1+\epsilon-c))$;

(4) critical points of H in $M' \setminus (W \cup \{\mathrm{x} \leq 1+\epsilon\})$ of action close to $a(1+\epsilon-c)$;

(5) nonconstant orbits near $\partial M'$ of action $\gg 0$ provided $a \gg b$.

Since the complement of the Reeb periods is open, there are no Reeb periods in $(a-\nu_a, a+\nu_a)$ for some small $\nu_a > 0$. Thus the actions in case (3) will be at least

$$-(a-\nu_a)(1+\epsilon) + a(1+\epsilon-c) = \nu_a(1+\epsilon) - ac,$$

and for sufficiently small c, depending on a, we can ensure that this is at least ν_a. Hence (after a suitable perturbation of H) we can ensure that if $a \gg b$ and $c \ll a^{-1}$ then the actions of (1) and (2) are negative and those of (3), (4) and (5) are positive.

4.4 Construction of $SH^*(\mathrm{i})$

Suppose H is a (perturbed) diagonal-step shaped Hamiltonian, with $a \gg b$ and $c \ll a^{-1}$ so that the orbits in W have negative actions and those outside W have positive actions. We write $CF^*(M,H)$ to emphasize that the Floer complex is computed for M. Consider the action-restriction map Section 2.8

$$CF^*(M,H;-\infty,0) \leftarrow CF^*(M,H).$$

Given two diagonal-step shaped Hamiltonians H, H' with $H \leq H'$ everywhere, pick a homotopy H_s from H' to H which is monotone ($\partial_s H_s \leq 0$). The induced continuation map $\phi\colon CF^*(M,H) \to CF^*(M,H')$ restricts to a map on the quotient complexes $\phi\colon CF^*(M,H;-\infty,0) \to CF^*(M,H';-\infty,0)$ because the action decreases along Floer trajectories when H_s is monotone (see Section 2.9).

Consider the Hamiltonian H_W on the completion $\widehat{W} = W \cup_{\partial W} [0,\infty) \times \partial W$ which equals H inside W and which is linear with slope a outside W. Then the quotient complex $CF^*(M,H;-\infty,0)$ can be identified with $CF^*(W,H_W)$ by showing that there are no Floer trajectories connecting 1–periodic orbits of X_H in $\widehat{M}$ which exit $W \cup \{\mathrm{x} \leq 1+\epsilon\}$. Therefore we obtain the commutative diagram

$$\begin{tikzcd}
CF^*(W,H'_W) & CF^*(M,H') \arrow[l] \\
CF^*(W,H_W) \arrow[u] & CF^*(M,H) \arrow[l] \arrow[u]
\end{tikzcd}$$

where the vertical maps are continuation maps and where the horizontal maps arose from action-restriction maps. Taking cohomology, and then taking the direct limit as $a \gg b \to \infty$ (so $c \ll a^{-1} \to 0$) defines the map $SH^*(i)$:

$$SH^*(i)\colon SH^*(W) \leftarrow SH^*(M).$$

4.5 Viterbo functoriality

Consider a (perturbed) diagonal-step shaped Hamiltonian $H = H^0$ with $c = 1$ and slopes $0 < b \ll a < \delta$ so that the orbits inside W have negative actions and those outside W have positive actions. Then H^0 and the corresponding H^0_W are of the type described in Section 4.1 for M and W respectively. The action-restriction map $CF^*(W, H^0_W) \leftarrow CF^*(M, H^0)$ is then identified with the map on Morse complexes $CM^*(W, H^0|_W) \leftarrow CM^*(M, H^0)$ which restricts to the generators $x \in \mathrm{Crit}(H^0)$ with $H^0(x) < 0$. In cohomology this map corresponds to the pullback on singular cohomology $i^*\colon H^*(W) \leftarrow H^*(M)$.

This identifies $CM^*(W, H^0|_W) \leftarrow CM^*(M, H^0)$ with the bottom map of the diagram in Section 4.4 when we take $H = H^0$, and so taking the direct limit over the H' we obtain Viterbo's commutative diagram in cohomology:

$$\begin{tikzcd} SH^*(W) & SH^*(M) \arrow[l, "SH^*(i)"'] \\ H^*(W) \arrow[u, "c_*"] & H^*(M) \arrow[l, "i^*"'] \arrow[u, "c_*"] \end{tikzcd}$$

4.6 Twisted Viterbo functoriality

We now introduce the twisted coefficients $\underline{\Lambda}_\alpha$ for some $\alpha \in H^1(\mathcal{L}\widehat{M}) \cong H^1(\mathcal{L}M)$, as explained in Section 2.15 and Section 2.17. Recall that we have constructed twisted continuation maps Section 2.16 which compose well, so the discussion of Section 4.2 and Section 4.4 will hold in the twisted case provided that we understand how the local systems restrict.

Suppose H^0 is a Hamiltonian with small slope as in Section 4.1. In the twisted case the canonical identification of $CF^*(H^0)$ with the Morse complex $CM^*(H^0)$ becomes

$$CF^*(H^0; \underline{\Lambda}_\alpha) = CM^*(H^0; c^*\underline{\Lambda}_\alpha),$$

where $c^*\underline{\Lambda}_\alpha$ is the restriction of $\underline{\Lambda}_\alpha$ to the local system on $\widehat{M} \subset \mathcal{L}_0\widehat{M}$ which consists of a copy Λ_m of Λ over each $m \in \widehat{M}$ and of the multiplication isomorphism $t^{\alpha[c \circ v]} = t^{c^*\alpha[v]}\colon \Lambda_m \to \Lambda_{m'}$ for every path $v(s)$ in $\widehat{M}$ joining m to m', and where the twisted Morse differential is defined on $q_+ \in \mathrm{Crit}(H^0)$ analogously to the Floer case:

$$\delta q_+ = \sum \left\{ \epsilon(v)\, t^{c^*\alpha[v]}\, q_- : q_- \in \mathrm{Crit}(H^0),\, \partial_s v = -\nabla H^0(v),\, \lim_{s\to\pm\infty} v(s) = q_\pm \right\}.$$

By mimicking the proof that $HM^*(H^0) \cong H^*(M)$, for twisted coefficients we have $HM^*(H^0; c^*\underline{\Lambda}_\alpha) \cong H^*(M; c^*\underline{\Lambda}_\alpha)$ (singular cohomology with coefficients in the local system $c^*\underline{\Lambda}_\alpha$, as defined in [14]).

As in Section 4.2, we get twisted continuation maps $CF^*(H^0; \underline{\Lambda}_\alpha) \to CF^*(H; \underline{\Lambda}_\alpha)$ for Hamiltonians H linear at infinity. In cohomology these maps yield a morphism $HF^*(H^0; \underline{\Lambda}_\alpha) \to \lim HF^*(H; \underline{\Lambda}_\alpha)$, where the direct limit is taken over twisted continuation maps as the slopes at infinity of the H increase. This defines

$$c_*\colon H^*(M; c^*\underline{\Lambda}_\alpha) \to SH^*(M; \underline{\Lambda}_\alpha).$$

We get action-restriction maps $CF^*(M, H; \underline{\Lambda}_\alpha; -\infty, 0) \leftarrow CF^*(M, H; \underline{\Lambda}_\alpha)$ in Section 4.4, and two choices of diagonal-step shaped Hamiltonians H, H' with $H \le H'$ induce a continuation map $\phi\colon CF^*(M, H; \underline{\Lambda}_\alpha) \to CF^*(M, H'; \underline{\Lambda}_\alpha)$ which restricts to the quotient complexes $\phi\colon CF^*(M, H; \underline{\Lambda}_\alpha; -\infty, 0) \to CF^*(M, H'; \underline{\Lambda}_\alpha; -\infty, 0)$.

Let $\mathcal{L}i\colon \mathcal{L}W \to \mathcal{L}M$ be the map induced by i. As in Section 4.4, the quotient complex $CF^*(M, H; \underline{\Lambda}_\alpha; -\infty, 0)$ can be identified with $CF^*(W, H_W; \underline{\Lambda}_{(\mathcal{L}i)^*\alpha})$ because there are no Floer trajectories connecting 1–periodic orbits of X_H which exit $W \cup \{x \le 1+\epsilon\}$ in $\widehat{M}$ and so the twisted differentials of the two complexes agree since $(\mathcal{L}i)^*\alpha$ and α agree on the common Floer trajectories inside $W \cup \{x \le 1+\epsilon\}$.

As in Section 4.4, the direct limit over the twisted continuation maps for diagonal-step shaped H of the action-restriction maps

$$CF^*(W, H_W; \underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) \leftarrow CF^*(M, H; \underline{\Lambda}_\alpha)$$

as $a \gg b \to \infty$ will define a twisted map $SH^*(i)$ in cohomology:

$$SH^*(i)\colon SH^*(W; \underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) \leftarrow SH^*(M; \underline{\Lambda}_\alpha).$$

As in Section 4.5, the action-restriction maps fit into a commutative diagram

$$\begin{tikzcd}
CF^*(W, H'_W; \underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) & CF^*(M, H'; \underline{\Lambda}_\alpha) \arrow[l] \\
CM^*(W, H^0_W; c^*\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) \arrow[u] & CM^*(M, H^0; c^*\underline{\Lambda}_\alpha) \arrow[l] \arrow[u]
\end{tikzcd}$$

and taking the direct limit over the H' yields the following result in cohomology.

Theorem 4 *Let (M^{2n},θ) be a Liouville domain. Then for all $\alpha \in H^1(\mathcal{L}M)$ there exists a map $c_*\colon H^*(M;c^*\underline{\Lambda}_\alpha) \to SH^*(M;\underline{\Lambda}_\alpha)$, where $c\colon M \to \mathcal{L}M$ is the inclusion of constant loops. Moreover, for any Liouville embedding $i\colon (W^{2n},\theta') \to (M^{2n},\theta)$ there exists a map $SH^*(i)\colon SH^*(W;\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) \leftarrow SH^*(M;\underline{\Lambda}_\alpha)$ which fits into the commutative diagram:*

$$\begin{tikzcd} SH^*(W;\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) & SH^*(M;\underline{\Lambda}_\alpha) \arrow[l, "SH^*(i)"'] \\ H^*(W;c^*\underline{\Lambda}_{(\mathcal{L}i)^*\alpha}) \arrow[u, "c_*"'] & H^*(M;c^*\underline{\Lambda}_\alpha) \arrow[l, "i^*"'] \arrow[u, "c_*"'] \end{tikzcd}$$

5 Proof of the Main Theorem

Lemma 5 *Let N^n be a closed manifold and let $L \to DT^*N$ be an exact Lagrangian embedding. Then for all $\alpha \in H^1(\mathcal{L}N)$, the composite*

$$H_*(N;c^*\underline{\Lambda}_\alpha) \xrightarrow{\sim} H^{n-*}(N;c^*\underline{\Lambda}_\alpha) \xrightarrow{c_*} SH^{n-*}(DT^*N;\underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}) \xrightarrow{\varphi^{-1}} H_*(\mathcal{L}N;\underline{\Lambda}_\alpha)$$

of Poincaré duality, the map c_* from Section 4.6 and the inverse of φ (Theorem 3), is equal to the ordinary map $c_*\colon H_*(N;c^*\underline{\Lambda}_\alpha) \to H_*(\mathcal{L}N;\underline{\Lambda}_\alpha)$ induced by the inclusion of constants $c\colon N \to \mathcal{L}N$.

In the untwisted case, the lemma was proved by Viterbo [11] using his construction of the isomorphism φ, and it can be proved in the Abbondandolo–Schwarz setup by using small perturbations of $L(q,v)=\frac{1}{2}|v|^2$ and $H(q,p)=\frac{1}{2}|p|^2$ and by considering the restriction of the isomorphism φ to the orbits of action close to zero. The twisted version is proved analogously.

Theorem 6 *Let N^n be a closed manifold and let $L \to DT^*N$ be an exact Lagrangian embedding. Then for all $\alpha \in H^1(\mathcal{L}N)$ there exists a commutative diagram*

$$\begin{tikzcd} H_*(\mathcal{L}L;\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & H_*(\mathcal{L}N;\underline{\Lambda}_\alpha) \arrow[l, "\mathcal{L}p_!"'] \\ H_*(L;c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) \arrow[u, "c_*"'] & H_*(N;c^*\underline{\Lambda}_\alpha) \arrow[l, "p_!"'] \arrow[u, "c_*"'] \end{tikzcd}$$

where $c\colon N \to \mathcal{L}N$ is the inclusion of constant loops, $p\colon L \to T^*N \to N$ is the projection and $p_!$ is the ordinary transfer map. Moreover, the diagram can be restricted to the components $\mathcal{L}_0L$ and $\mathcal{L}_0N$ of contractible loops.

If $c^*\alpha = 0$ then the bottom map becomes $p_! \otimes 1\colon H_*(L)\otimes\Lambda \leftarrow H_*(N)\otimes\Lambda$.

Proof Let θ_N be the canonical 1–form which makes $(DT^*N, d\theta_N)$ symplectic. By Weinstein's theorem a neighbourhood of L is symplectomorphic to a small disc cotangent bundle DT^*L. Therefore the exact Lagrangian embedding $j\colon L^n \hookrightarrow DT^*N$ yields a Liouville embedding $i\colon (DT^*L, \theta_L) \hookrightarrow (DT^*N, \theta_N)$.

By Theorem 3 there are twisted isomorphisms:

$$\begin{aligned} \varphi_N\colon H_*(\mathcal{L}N; \underline{\Lambda}_\alpha) &\to SH^{n-*}(DT^*N; \underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}) \\ \varphi_L\colon H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) &\to SH^{n-*}(DT^*L; \underline{\Lambda}_{(\mathcal{L}i)^*(\mathcal{L}\pi)^*\alpha}) \end{aligned}$$

We define $\mathcal{L}p_! = \varphi_L^{-1} \circ SH^*(i) \circ \varphi_N$ so that the following diagram commutes:

$$\begin{tikzcd} H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & H_*(\mathcal{L}N; \underline{\Lambda}_\alpha) \arrow[l, "\mathcal{L}p_!"'] \arrow[d, "\varphi_N", "\wr"'] \\ SH^{n-*}(DT^*L; \underline{\Lambda}_{(\mathcal{L}i)^*(\mathcal{L}\pi)^*\alpha}) \arrow[u, "\varphi_L^{-1}", "\wr"'] & SH^{n-*}(DT^*N; \underline{\Lambda}_{(\mathcal{L}\pi)^*\alpha}) \arrow[l, "SH^*(i)"'] \end{tikzcd}$$

Recall that the ordinary transfer map $p_!$ is defined using Poincaré duality and the pullback p^* so that the following diagram commutes:

$$\begin{tikzcd} H^{n-*}(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & H^{n-*}(N; c^*\underline{\Lambda}_\alpha) \arrow[l, "p^*"'] \arrow[d, "\wr"] \\ H_*(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) \arrow[u, "\wr"] & H_*(N; c^*\underline{\Lambda}_\alpha) \arrow[l, "p_!"'] \end{tikzcd}$$

Finally, Theorem 4 for the map i yields another commutative diagram whose horizontal maps are the bottom and top rows respectively of the above two diagrams (in the second diagram we use that L, N are homotopy equivalent to DT^*L, DT^*N). By combining these diagrams we obtain a commutative diagram

$$\begin{tikzcd} H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) & H_*(\mathcal{L}N; \underline{\Lambda}_\alpha) \arrow[l, "\mathcal{L}p_!"'] \\ H_*(L; c^*\underline{\Lambda}_{(\mathcal{L}p)^*\alpha}) \arrow[u] & H_*(N; c^*\underline{\Lambda}_\alpha) \arrow[l, "p_!"'] \arrow[u] \end{tikzcd}$$

Lemma 5 shows that the vertical maps are indeed the maps c_* in ordinary homology. Since $c\colon N \to \mathcal{L}N$ maps into the component of contractible loops $\mathcal{L}_0N$, the diagram restricts to $\mathcal{L}_0L$ and $\mathcal{L}_0N$ by restricting $\mathcal{L}p_!$ and projecting to $H_*(\mathcal{L}L; \underline{\Lambda}_{(\mathcal{L}p)^*\alpha})$ (not all loops in T^*L that are contractible in T^*N need be contractible in T^*L). $\square$

6 Proof of the Main Corollary

6.1 Transgressions

For $\beta \in H^2(N)$, let $f\colon N \to \mathbb{CP}^\infty$ be a classifying map for β. Let ev: $\mathcal{L}_0 N \times S^1 \to N$ be the evaluation map. Define

$$\tau = \pi \circ \mathrm{ev}^*\colon H^2(N) \xrightarrow{\mathrm{ev}^*} H^2(\mathcal{L}_0 N \times S^1) \xrightarrow{\pi} H^1(\mathcal{L}_0 N),$$

where π is the projection to the Künneth summand. If N is simply connected, then τ is an isomorphism. Let u be a generator of $H^2(\mathbb{CP}^\infty)$, then $v = \tau(u)$ generates $H^1(\mathcal{L}\mathbb{CP}^\infty) \cong H^1(\Omega\mathbb{CP}^\infty)$ and $\tau(\beta) = (\mathcal{L}f)^* v$. Identify $H^1(\mathcal{L}_0 N) \cong \mathrm{Hom}(\pi_1(\mathcal{L}_0 N), \mathbb{Z})$ and $\pi_1(\mathcal{L}_0 N) \cong \pi_2(N) \rtimes \pi_1(N)$, then the class $\tau(\beta)$ vanishes on $\pi_1(N)$ and corresponds to

$$f_*\colon \pi_2(N) \to \pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}.$$

Similarly, define $\tau_b\colon H^2(N) \to H^1(\Omega_0 N)$ for the space $\Omega_0 N$ of contractible based loops. Then $\Omega f\colon \Omega_0 N \to \Omega\mathbb{CP}^\infty$ is a classifying map for $\tau_b(\beta)$. The inclusion $\Omega_0 N \to \mathcal{L}_0 N$ induces a bijection $\tau(\beta) \mapsto \tau_b(\beta)$ between transgressed forms.

We will assume throughout that the transgression $\alpha = \tau(\beta) \in H^1(\mathcal{L}_0 N)$ is nonzero, or equivalently that $f_*\colon \pi_2(N) \to \mathbb{Z}$ is not the zero map.

6.2 Novikov homology of the free loopspace

Denote by $\overline{\mathcal{L}_0 N}$ the infinite cyclic cover of $\mathcal{L}_0 N$ corresponding to $\alpha\colon \pi_1(\mathcal{L}_0 N) \to \mathbb{Z}$, and let t denote a generator of the group of deck transformations of $\overline{\mathcal{L}_0 N}$. The group ring of the cover is $R = \mathbb{Z}[t, t^{-1}]$, and $\Lambda = \mathbb{Z}(\!(t)\!) = \mathbb{Z}[\![t]\!][t^{-1}]$ is the Novikov ring of α (see Section 2.14).

The Novikov homology of $\mathcal{L}_0 N$ with respect to α is defined as the homology of $\mathcal{L}_0 N$ with local coefficients in the bundle $\underline{\Lambda}_\alpha$, which by [14] can be calculated as

$$H_*(\mathcal{L}_0 N; \underline{\Lambda}_\alpha) \cong H_*(C_*(\overline{\mathcal{L}_0 N}) \otimes_R \Lambda).$$

Say that a space X is of *finite type* if $H_k(X)$ is finitely generated for each k.

Theorem 7 *For a compact manifold N, if $\tau(\beta) \neq 0$ and $\pi_m(N)$ is finitely generated for each $m \geq 2$ then $\overline{\mathcal{L}_0 N}$ is of finite type.*

Proof The theorem follows from five claims.

Claim 1 *If $\overline{\Omega_0 N}$ is of finite type then so is $\overline{\mathcal{L}_0 N}$.*

Proof Consider the fibration $\Omega_0 N \to \mathcal{L}_0 N \to N$, and take cyclic covers corresponding to $\tau_b(\beta)$ and $\tau(\beta)$ to obtain the fibration $\overline{\Omega_0 N} \to \overline{\mathcal{L}_0 N} \to N$. By compactness, N is homotopy equivalent to a finite CW complex and Claim 1 follows by a Leray–Serre spectral sequence argument. $\square$

After replacing N by a homotopy equivalent space, we may assume that we have a fibration $f\colon N \to \mathbb{CP}^\infty$ with fibre $F = f^{-1}(*)$, and taking the spaces of contractible based loops gives a fibration $\Omega f\colon \Omega_0 N \to \Omega\mathbb{CP}^\infty$.

Claim 2 *The fibre of Ωf is a union $(\Omega F)_K$ of finitely many components of ΩF, indexed by the finite set $K = \mathrm{Coker}(f_*\colon \pi_2 N \to \pi_2\mathbb{CP}^\infty)$.*

Proof Consider the homotopy LES for the fibration f,

$$\pi_2 N \xrightarrow{f_*} \pi_2\mathbb{CP}^\infty \longrightarrow \pi_1 F \longrightarrow \pi_1 N$$

then $(\Omega f)^{-1}(*) = \Omega F \cap \Omega_0 N$ consists of loops $\gamma \in \Omega F$ whose path component lies in the kernel of $\pi_1 F \to \pi_1 N$, which is isomorphic to the cokernel of f_*. Since $\tau(\beta) \neq 0$, also f_* is nonzero and so K is finite. $\square$

Claim 3 *$\overline{\Omega j}\colon (\Omega F)_K \to \overline{\Omega_0 N}$ is a homotopy equivalence.*

Proof Observe that $\overline{\Omega_0 N}$ is the pullback under Ωf of the cyclic cover of $\Omega\mathbb{CP}^\infty$ corresponding to the transgression $v = \tau_b(u) \in H^1(\Omega\mathbb{CP}^\infty)$ of a generator $u \in H^2(\mathbb{CP}^\infty)$ (see Section 6.1). We obtain the commutative diagram

$$\begin{tikzcd}
(\Omega F)_K \arrow[r, "\overline{\Omega j}"] \arrow[d] & \overline{\Omega_0 N} \arrow[r, "\overline{\Omega f}"] \arrow[d] & \overline{\Omega\mathbb{CP}^\infty} \arrow[r, "\overline{\varphi}", "\simeq"'] \arrow[d] & \mathbb{R} \arrow[d] \\
(\Omega F)_K \arrow[r, "\Omega j"] & \Omega_0 N \arrow[r, "\Omega f"] & \Omega\mathbb{CP}^\infty \arrow[r, "\varphi", "\simeq"'] & S^1
\end{tikzcd}$$

where the homotopy equivalence φ corresponds to the class $\tau_b(u) \in H^1(\Omega\mathbb{CP}^\infty) \cong [\Omega\mathbb{CP}^\infty, S^1]$. The claim follows since $\mathbb{R}$ is contractible. $\square$

Claim 4 *$\overline{\Omega_0 N}$ is of finite type if and only if $\Omega_0 F = \Omega\widetilde{F}$ is of finite type.*

Proof Each component of ΩF is homotopy equivalent to $\Omega_0 F$ via composition with an appropriate fixed loop. The claim follows from Claims 3 and 2 since K is finite. Note that we may identify $\Omega_0 F = \Omega\widetilde{F}$ since the loops of F that lift to closed loops of the universal cover $\widetilde{F}$ are precisely the contractible ones. $\square$

Claim 5 *$\Omega\widetilde{F}$ is of finite type if and only if $\pi_m N$ is finitely generated for each $m \geq 2$.*

Proof Since $\widetilde{F}$ is simply connected, $\Omega\widetilde{F}$ is of finite type if and only if $\widetilde{F}$ is of finite type, by a Leray-Serre spectral sequence argument applied to the path-space fibration $\Omega\widetilde{F} \to P\widetilde{F} \to \widetilde{F}$ (see Spanier [10, 9.6.13]). Moreover $\widetilde{F}$ is of finite type if and only if $\pi_m(\widetilde{F}) = \pi_m(F)$ is finitely generated for all $m \geq 2$ (see [10, 9.6.16]). The claim follows from the homotopy LES for $F \to N \to \mathbb{CP}^\infty$. □

Corollary 8 *For a compact manifold N, if $\tau(\beta) \neq 0$ and $\pi_m(N)$ is finitely generated for each $m \geq 2$, then $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) = 0$.*

Proof We need to show that each $HN_k = H_k(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)})$ vanishes. Since $\mathbb{Z}[t]$ is Noetherian, its (t)–adic completion $\mathbb{Z}[[t]]$ is flat over $\mathbb{Z}[t]$ (see Matsumura [4, Theorem 8.8]). Therefore, localizing at the multiplicative set S generated by t, $\Lambda = S^{-1}\mathbb{Z}[[t]]$ is flat over $R = S^{-1}\mathbb{Z}[t]$. Thus $HN_k \cong H_k(\overline{\mathcal{L}_0 N}) \otimes_R \Lambda$, which is the localization of $H_k = H_k(\overline{\mathcal{L}_0 N}) \otimes_{\mathbb{Z}[t]} \mathbb{Z}[[t]]$. Observe that $t \cdot H_k = H_k$ since t acts invertibly on $H_k(\overline{\mathcal{L}_0 N})$. So if H_k were finitely generated over $\mathbb{Z}[t]$, then $H_k = 0$ by Nakayama's lemma [4, Theorem 2.2] since t lies in the radical of $\mathbb{Z}[[t]]$. By Theorem 7, H_k is in fact finitely generated over $\mathbb{Z}$, so this concludes the proof. □

Remark 9 The idea behind the proof of Corollary 8 is not original. I later realized that it is a classical result that if $H_*(X;\mathbb{Z})$ is finitely generated in each degree then the Novikov homology $H_*(C_*(\overline{X}) \otimes_R \Lambda_\alpha)$ vanishes for $0 \neq \alpha \in H^1(X)$. The basic idea dates back to Milnor [5] and a very general version of this result is proved in Farber [2, Proposition 1.35].

Corollary 10 *If N is a compact simply connected manifold, then $H_*(\mathcal{L}_0 N; \underline{\Lambda}_\alpha) = 0$ for any nonzero $\alpha \in H^1(\mathcal{L}_0 N)$.*

Proof N is simply connected so its homotopy groups are finitely generated in each dimension because its homology groups are finitely generated by compactness (see Spanier [10, 9.6.16]). Since N is simply connected, any α in $H^1(\mathcal{L}_0 N)$ is the transgression of some $\beta \in H^2(N)$. The result now follows from Corollary 8. □

6.3 Proof of the Main Corollary

We recall the Main Corollary:

Corollary 11 *Let N^n be a closed simply connected manifold. Let $L \to DT^*N$ be an exact Lagrangian embedding. Then the image of $p_*\colon \pi_2(L) \to \pi_2(N)$ has finite index and $p^*\colon H^2(N) \to H^2(L)$ is injective.*

Proof A nonzero class $\beta \in H^2(N)$ yields a nonzero transgression $\tau(\beta) \in H^1(\mathcal{L}_0 N)$ (see Section 6.1). Suppose by contradiction that $\tau(p^*\beta) = 0$. Then the local system $(\mathcal{L}p)^* \underline{\Lambda}_{\tau(\beta)}$ is trivial (see Section 2.14). Moreover $c^*\tau(\beta) = 0$ since $\tau(\beta)$ vanishes on $\pi_1(N)$. Therefore the diagram of Theorem 6, restricted to contractible loops, becomes

$$\begin{tikzcd} H_*(\mathcal{L}_0 L) \otimes \Lambda \arrow[d, shift left, "q_*"] & H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) \arrow[l, "\mathcal{L}p_!"'] \\ H_*(L) \otimes \Lambda \arrow[u, hook, shift left, "c_*"] & H_*(N) \otimes \Lambda \arrow[l, "p_!"'] \arrow[u, "c_*"'] \end{tikzcd}$$

where $q\colon \mathcal{L}_0 L \to L$ is the evaluation at 0. By Corollary 10, $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\beta)}) = 0$, so the fundamental class $[N] \in H_n(N)$ maps to $c_*[N] = 0$. But $\mathcal{L}p_!(c_*[N]) = c_* p_![N] = c_*[L] \neq 0$ since c_* is injective on $H_*(L)$.

Therefore $\tau(p^*\beta)$ cannot vanish, and so $\tau_b \circ p^*\colon H^2 N \to H^1(\Omega L)$ is injective. Consider the commutative diagram

$$\begin{tikzcd} H^2(N) \arrow[r, "\tau_b", "\sim"'] \arrow[d, "p^*"] & \mathrm{Hom}(\pi_2(N), \mathbb{Z}) \cong H^1(\Omega N) \arrow[d, "(\Omega p)^*"] \\ H^2(L) \arrow[r, "\tau_b"] & \mathrm{Hom}(\pi_2(L), \mathbb{Z}) \cong H^1(\Omega L) \end{tikzcd}$$

where the top map τ_b is an isomorphism since N is simply connected. We deduce from the injectivity of $\tau_b \circ p^* = (\Omega p)^* \circ \tau_b$ that $p^*\colon H^2(N) \to H^2(L)$ and $\mathrm{Hom}(\pi_2(N), \mathbb{Z}) \to \mathrm{Hom}(\pi_2(L), \mathbb{Z})$ are both injective, so in particular the image of $p_*\colon \pi_2(L) \to \pi_2(N)$ has finite index. □

7 Non–simply connected cotangent bundles

We will prove that for non–simply connected N the map $\pi_2(L) \to \pi_2(N)$ still has finite index provided that the homotopy groups $\pi_m(N)$ are finitely generated for each $m \geq 2$.

This time we consider transgressions induced from the universal cover $\widetilde{N}$ of N,

$$\tau\colon H^2(\widetilde{N}) \to H^1(\mathcal{L}\widetilde{N}) = H^1(\mathcal{L}_0 N) \cong \mathrm{Hom}(\pi_2 N, \mathbb{Z}).$$

The homomorphism $\widetilde{f}_*\colon \pi_2(\widetilde{N}) = \pi_2(N) \to \mathbb{Z}$ corresponding to such a transgression $\tau(\widetilde{\beta})$ is induced by a classifying map $\widetilde{f}\colon \widetilde{N} \to \mathbb{CP}^\infty$ for $\widetilde{\beta} \in H^2(\widetilde{N})$. Since $\Omega\widetilde{N} = \Omega_0 N$ and $\mathcal{L}\widetilde{N} = \mathcal{L}_0 N$, the transgressions $\tau_b(\widetilde{\beta})$ and $\tau(\widetilde{\beta})$ define cyclic covers $\overline{\Omega_0 N}$ and $\overline{\mathcal{L}_0 N}$. We will use these in the construction of the Novikov homology.

Theorem 12 *Let N be a compact manifold with finitely generated $\pi_m(N)$ for each $m \geq 2$. If $\tau(\widetilde{\beta}) \neq 0$ then $\overline{\mathcal{L}_0 N}$ is of finite type and $H_*(\mathcal{L}_0 N; \underline{\Lambda}_{\tau(\widetilde{\beta})}) = 0$.*

Proof Revisit the proof of Theorem 7. It suffices to prove that $\overline{\Omega_0 N}$ has finite type. This time we have the commutative diagram:

$$\begin{tikzcd}
\Omega F \arrow[r, "\overline{\Omega j}"] \arrow[d] & \overline{\Omega_0 N} \arrow[r, "\overline{\Omega \widetilde{f}}"] \arrow[d] & \overline{\Omega \mathbb{CP}^\infty} \simeq \mathbb{R} \arrow[d] \\
\Omega F \arrow[r, "\Omega j"] \arrow[d] & \Omega \widetilde{N} = \Omega_0 N \arrow[r, "\Omega \widetilde{f}"] \arrow[d] & \Omega \mathbb{CP}^\infty \simeq S^1 \arrow[d] \\
F \arrow[r, "j"] & \widetilde{N} \arrow[r, "\widetilde{f}"] & \mathbb{CP}^\infty
\end{tikzcd}$$

Since $\Omega F \simeq \overline{\Omega_0 N}$, it suffices to show that ΩF has finite type. Observe that

$$\Omega F \cong \oplus_K \Omega_0 F$$

where $K = \mathrm{Coker}(\widetilde{f}_*\colon \pi_2 N \to \pi_2 \mathbb{CP}^\infty)$ is a finite set since $\widetilde{f}_* \neq 0$. So we just need to show that $\Omega_0 F = \Omega \widetilde{F}$ is of finite type. The same argument as in Theorem 7 proves that $\Omega \widetilde{F}$ is of finite type if and only if $\pi_m N = \pi_m \widetilde{N}$ is finitely generated for each $m \geq 2$. The same proof as for Corollary 8 yields the vanishing of the Novikov homology. □

Corollary 13 *Let N be a closed manifold with finitely generated $\pi_m(N)$ for each $m \geq 2$. Let $L \to DT^*N$ be an exact Lagrangian embedding. Then the image of $p_*\colon \pi_2(L) \to \pi_2(N)$ has finite index and $\widetilde{p}^*\colon H^2(\widetilde{N}) \to H^2(\widetilde{L})$ is injective.*

Proof The proof is analogous to that of Corollary 11: $(\mathcal{L}p)^*$ in the diagram

$$\begin{tikzcd}
H^2(\widetilde{N}) \arrow[r, "\tau", "\sim"'] \arrow[d, "\widetilde{p}^*"] & \mathrm{Hom}(\pi_2(N), \mathbb{Z}) \cong H^1(\mathcal{L}_0 N) \arrow[d, "(\mathcal{L}p)^*"] \\
H^2(\widetilde{L}) \arrow[r, "\tau", "\sim"'] & \mathrm{Hom}(\pi_2(L), \mathbb{Z}) \cong H^1(\mathcal{L}_0 L)
\end{tikzcd}$$

is injective because if, by contradiction, $\tau(\widetilde{p}^* \widetilde{\beta}) \in H^1(\mathcal{L}_0 L)$ vanished then the functoriality diagram of Theorem 6 would not commute. □

8 Unoriented theory

So far we assumed that all manifolds were oriented. By using $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients instead of $\mathbb{Z}$ coefficients one no longer needs the Floer and Morse moduli spaces to be oriented in order to define the differentials and continuation maps. For the twisted setup, we change the Novikov ring to

$$\Lambda = \mathbb{Z}_2(\!(t)\!) = \mathbb{Z}_2[\![t]\!][t^{-1}],$$

the ring of formal Laurent series with $\mathbb{Z}_2$ coefficients. The bundle $\underline{\Lambda}_\alpha$ is now a bundle of $\mathbb{Z}_2(\!(t)\!)$ rings, however the singular cocycle α is still integral: $[\alpha] \in H^1(\mathcal{L}_0 N; \mathbb{Z})$.

Using these coefficients, all our theorems hold true without the orientability assumption on N and L. The following is an interesting application of Corollary 11 in this setup.

Corollary 14 *There are no unorientable exact Lagrangians in T^*S^2.*

Proof For unorientable L, $H^2(L;\mathbb{Z}) = \mathbb{Z}_2$. Therefore the transgression τ vanishes on $H^2(L;\mathbb{Z})$ since its range $\mathrm{Hom}(\pi_2(L), \mathbb{Z})$ is torsion-free. But for S^2 there is a nonzero transgression. This contradicts the proof of Corollary 11. □

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