

Lacunary hyperbolic groups

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We call a finitely generated group *lacunary hyperbolic* if one of its asymptotic cones is an \mathbb{R} -tree. We characterize lacunary hyperbolic groups as direct limits of Gromov hyperbolic groups satisfying certain restrictions on the hyperbolicity constants and injectivity radii. Using central extensions of lacunary hyperbolic groups, we solve a problem of Gromov by constructing a group whose asymptotic cone \mathcal{C} has countable but nontrivial fundamental group (in fact \mathcal{C} is homeomorphic to the direct product of a tree and a circle, so $\pi_1(\mathcal{C}) = \mathbb{Z}$). We show that the class of lacunary hyperbolic groups contains non-virtually cyclic elementary amenable groups, groups with all proper subgroups cyclic (Tarski monsters) and torsion groups. We show that Tarski monsters and torsion groups can have so-called graded small cancellation presentations, in which case we prove that all their asymptotic cones are hyperbolic and locally isometric to trees. This allows us to solve two problems of Druţu and Sapir and a problem of Kleiner about groups with cut points in their asymptotic cones. We also construct a finitely generated group whose divergence function is not linear but is arbitrarily close to being linear. This answers a question of Behrstock.

[20F65](#); [20F69](#)

1 Introduction

1.1 Asymptotic cones and lacunary hyperbolic groups

Asymptotic cones were introduced by Gromov in [19]; a definition via ultrafilters was given by van den Dries and Wilkie [9]. An asymptotic cone of a metric space is, roughly speaking, what one sees when one looks at the space from infinitely far away. More precisely, any asymptotic cone of a metric space (X, dist) corresponds to an ultrafilter ω , a sequence of observation points $e = (e_n)_{n \in \mathbb{N}}$ from X and a sequence of scaling constants $d = (d_n)_{n \in \mathbb{N}}$ diverging to ∞ . The cone $\text{Con}^\omega(X; e, d)$ corresponding to

ω , e and d is the ultralimit of the sequence of spaces with basepoints $(X, \text{dist}/d_n, e_n)$ (see [Section 2.3](#) for a precise definition).

In particular, if X is the Cayley graph of a group G with a word metric then the asymptotic cones of X are called asymptotic cones of G . For every finitely generated group G , its asymptotic cones are complete geodesic homogeneous metric spaces. Since asymptotic cones of a group do not depend on the choice of observation points, we shall omit it from the notation.

The power of asymptotic cones stems from the fact that they capture both geometric and logical properties of the group, since a large subgroup G_e^ω of the ultrapower G^ω of the group G acts transitively by isometries on the asymptotic cone $\text{Con}^\omega(G; d)$. Since a large part of the first order theory of G is inherited by G_e^ω , the isometry group G_e^ω of the asymptotic cone “looks” like G . One of the simple but fundamental applications of asymptotic cones is the following statement by Gromov [\[21\]](#): if all asymptotic cones of a group are simply connected then the group has polynomial isoperimetric and linear isodiametric functions.

On the other hand, the asymptotic cone captures the coarse properties of the word metric in G . In particular, the asymptotic cones of two quasi-isometric groups are bi-Lipschitz equivalent. This makes asymptotic cones very useful tools in proving quasi-isometric rigidity of some classes of groups; see Kleiner and Leeb [\[30\]](#), Kapovich and Leeb [\[28\]](#), Kapovich, Kleiner and Leeb [\[27\]](#), Druţu [\[10; 11\]](#) and Druţu and Sapir [\[13\]](#).

Using asymptotic cones, one can characterize several important classes of groups. For example, groups of polynomial growth are precisely groups with all asymptotic cones locally compact; see Gromov [\[19\]](#), Druţu [\[11\]](#) and Point [\[48\]](#).

Another well-known result of Gromov is the following: a finitely generated group is hyperbolic if and only if all its asymptotic cones are \mathbb{R} -trees [\[20\]](#).

In fact, results of Gromov from [\[20\]](#) imply that a finitely presented group is hyperbolic if just one of the asymptotic cones is an \mathbb{R} -tree. This was discovered by Kapovich and Kleiner who give a detailed proof in the [Appendix](#) to this paper (see [Theorem 8.1](#)). On the other hand, there are nonhyperbolic finitely generated (but not finitely presented) groups with one asymptotic cone an \mathbb{R} -tree and another one not an \mathbb{R} -tree by Thomas and Velickovic [\[51\]](#). We call a group *lacunary hyperbolic* if one of its asymptotic cones is an \mathbb{R} -tree. The term is originated in [\[22\]](#) where sparse sequences of relations satisfying certain small cancellation conditions as in [\[51\]](#) are called lacunary. Thus a finitely presented lacunary hyperbolic group is hyperbolic.

The following theorem characterizes lacunary hyperbolic groups as certain direct limits of hyperbolic groups. The proof is not too difficult modulo [Theorem 8.1](#), but the result has never been formulated before.

Let $\alpha: G \rightarrow G'$ be a homomorphism, $G = \langle S \rangle$. The *injectivity radius* of α is the maximal radius of a ball in the Cayley graph $\Gamma(G, S)$ where α is injective.

Theorem 1.1 *Let G be a finitely generated group. Then the following conditions are equivalent.*

- (1) G is lacunary hyperbolic.
- (2) There exists a scaling sequence $d = (d_n)$ such that $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree for any nonprincipal ultrafilter ω .
- (3) G is the direct limit of a sequence of hyperbolic groups $G_i = \langle S_i \rangle$ (S_i is finite) and epimorphisms

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots,$$

where $\alpha_i(S_i) = S_{i+1}$, and the hyperbolicity constant of G_i (relative to S_i) is “little o ” of the injectivity radius of α_i .

Note that not every direct limit of hyperbolic groups is lacunary hyperbolic. For example, the free noncyclic Burnside group of any sufficiently large odd exponent and the wreath product $(\mathbb{Z}/n\mathbb{Z}) \text{ wr } \mathbb{Z}$ are direct limits of hyperbolic groups (see Ivanov [25] and Osin [45], respectively) but are not lacunary hyperbolic by Druţu and Sapir [13].

Groups constructed by Thomas and Velickovic [51] and more general small cancellation groups given by relations whose lengths form lacunary sequences of numbers from Gromov [22, Section 1.7] are lacunary hyperbolic (see also Section 3.2 below).

In this paper, we prove that the class of lacunary hyperbolic groups is very large: non-virtually cyclic groups in that class can be elementary amenable, can have infinite centers, can have all proper subgroups cyclic (Tarski monsters) and can be torsion groups.

We also show (Theorem 4.17) that the class of lacunary hyperbolic groups contains all groups given by *graded small cancellation* presentations, a notion originated by the first author [41; 42]. Moreover, all asymptotic cones of groups given by graded small cancellation presentations are hyperbolic and locally isometric to trees (Theorem 4.17). Thus methods from [41; 42] can be used to construct lacunary hyperbolic groups with unusual properties (see Section 4 and Section 6).

Theorem 1.1 implies that the torsion-free group G with all proper subgroups cyclic from the paper [39] of the first author is lacunary hyperbolic. Indeed, G is a direct limit of hyperbolic groups G_i by [39, Lemma 9.13]. The same lemma shows that the injectivity radius of the homomorphism $G_{i-1} \rightarrow G_i$ can be chosen arbitrary large relative to the hyperbolicity constant of G_{i-1} . Similarly, the finitely generated infinite

torsion group with all proper subgroups of prime orders constructed by the first author in [40] is lacunary hyperbolic as well.

Although the class of lacunary hyperbolic groups is very large, these groups share some common algebraic properties (see Sections 3.3 and 3.4). In particular in [Theorem 3.18](#), [Corollary 3.21](#) and [Proposition 3.15](#), we show that

- an undistorted subgroup of a lacunary hyperbolic group is lacunary hyperbolic itself;
- a lacunary hyperbolic group cannot contain a copy of \mathbb{Z}^2 , an infinite finitely generated subgroup of bounded torsion and exponential growth, a copy of the lamplighter group, etc.;
- every lacunary hyperbolic group is embedded into a relatively hyperbolic 2-generated lacunary hyperbolic group as a peripheral subgroup;
- any group that is hyperbolic relative to a lacunary hyperbolic subgroup is lacunary hyperbolic itself.

[Theorem 1.1](#) implies that lacunary hyperbolic groups satisfy the Strong Novikov Conjecture (that is the Baum–Connes assembly map with trivial coefficients is injective) since all direct limits of hyperbolic groups satisfy that conjecture (hyperbolic groups satisfy it by Skandalis, Tu and Yu [50], and direct limits respect the conjecture by Rosenberg [49, Proposition 2.4]).

It is also easy to see that the class of lacunary hyperbolic groups is closed under quasi-isometry: indeed, asymptotic cones of quasi-isometric groups are bi-Lipschitz equivalent if they correspond to the same ultrafilter and the same sequence of scaling constants. Hence if an asymptotic cone of one of these groups is an \mathbb{R} -tree, then the other group also has an asymptotic cone that is a \mathbb{R} -tree.

1.2 Central extensions of lacunary hyperbolic groups and fundamental groups of asymptotic cones

One of the interesting properties of hyperbolic groups was established by Neumann and Reeves [37]: every finitely generated central extension H of a hyperbolic group G is quasi-isometric to the direct product of G and the center $Z(H)$. Using some general properties of asymptotic cones of group extensions ([Theorem 5.2](#)) we establish an asymptotic analog of this result for central extensions of lacunary hyperbolic groups.

Theorem 1.2 ([Theorem 5.6](#)) *Let N be a central subgroup of a finitely generated group G endowed with the induced metric. Suppose that for some nonprincipal*

ultrafilter ω and some scaling sequence $d = (d_n)$, $\text{Con}^\omega(G/N, d)$ is an \mathbb{R} -tree. Then $\text{Con}^\omega(G, d)$ is bi-Lipschitz equivalent to $\text{Con}^\omega(N, d) \times \text{Con}^\omega(G/N, d)$ endowed with the product metric.

This theorem opens many opportunities to construct asymptotic cones of groups with unusual properties. Recall that one of the main problems about asymptotic cones of groups is the following question by Gromov.

Problem 1.3 (Gromov [21]) Is it true that the fundamental group of an asymptotic cone of a finitely generated group is either trivial or of cardinality continuum?

Here is what was known about [Problem 1.3](#) before.

- As we have mentioned above, the triviality of the fundamental group of all asymptotic cones of a group G implies that the group is finitely presented, its Dehn function is polynomial and its isodiametric function is linear. Thus fundamental groups of asymptotic cones carry important algorithmic information about the group.
- By Papasoglu [47] if the Dehn function of a finitely presented group is at most quadratic, then all asymptotic cones are simply connected. By Ol’shanskii and Sapir [43], one cannot replace in the previous statement “quadratic” by, say, $n^2 \log n$.
- By Burillo [8], in many cases asymptotic cones of groups contain a π_1 -embedded Hawaiian earring, and their fundamental groups are of order continuum (that is true, for example, for solvable Baumslag–Solitar groups and the Sol group).
- Non-simply connected asymptotic cones are non-locally compact by Druţu [11] and Point [48] but homogeneous, and the isometry groups act on them with uncountable point stabilizers. Hence every nontrivial loop in the asymptotic cone typically has uncountably many copies sharing a common point. This makes a positive answer to [Problem 1.3](#) plausible.
- In all cases when the nontrivial fundamental groups of asymptotic cones of groups could be computed, these groups had cardinality continuum. Druţu and Sapir [13] proved that for every countable group C there exists a finitely generated group G and an asymptotic cone of G whose fundamental group is the free product of continuously many copies of C .

Nevertheless, by carefully choosing a central extension of a lacunary hyperbolic group, we answer [Problem 1.3](#) negatively.

Theorem 1.4 (Theorem 5.12) *There exists a finitely generated group G and a scaling sequence $d = (d_n)$ such that for any ultrafilter ω , $\text{Con}^\omega(G, d)$ is bi-Lipschitz equivalent to the product of an \mathbb{R} -tree and \mathbb{S}^1 . In particular, $\pi_1(\text{Con}^\omega(G, d)) = \mathbb{Z}$.*

1.3 Cut points in asymptotic cones

The examples of lacunary hyperbolic groups constructed in this paper solve several problems of Druţu and Sapir [13] and of Kleiner (see below).

Recall that one of the main applications of asymptotic cones of groups is the following: if a finitely generated group H has infinitely many homomorphisms into a locally compact (say, finitely generated) group G that are pairwise nonconjugate in G , then H acts on an asymptotic cone of G without a global fixed point. If the asymptotic cone is an \mathbb{R} -tree, this implies (using the theory of groups acting on \mathbb{R} -trees due to Rips, Sela, Bestvina and Feighn, Dunwoody and others) that H splits into a graph of groups.

In [14], Druţu and Sapir showed that similar conclusions can be drawn if all asymptotic cones of G are not trees but only have global cut points (ie points whose removal makes the cones disconnected). Such groups are called *constricted*. In that case the asymptotic cone is tree-graded in the sense of [13], and an action on a tree-graded space under some mild assumptions leads to an action on an \mathbb{R} -tree. In [14], this program has been carried out for relatively hyperbolic groups (all asymptotic cones of relatively hyperbolic groups have cut points by a result of Druţu and Sapir [13]). It is quite plausible that the program will work also for mapping class groups (where existence of cut points in asymptotic cones has been proved by Behrstock [5]), fundamental groups of graph manifolds (their asymptotic cones have cut points by a result of Kapovich, Kleiner and Leeb [27]), groups acting k -acylindrically on trees by Druţu, Mozes and Sapir [12] and other groups.

On the other hand many groups do not have cut points in any of their asymptotic cones. Such groups were called *wide* in [13]. Among them are non-virtually cyclic groups satisfying nontrivial laws [13], lattices in classical semisimple Lie groups of rank > 1 [12], groups having infinite cyclic central subgroups [13], direct products of infinite groups and so on.

Metric spaces whose asymptotic cones do not have cut points (ie, wide spaces) are characterized internally in terms of divergence in [12]. A metric space is *wide* if and only if there are constants $C, \varepsilon > 0$ such that for every three points a, b, c there exist a path of length at most $C \text{dist}(a, b)$ connecting a with b and avoiding a ball of radius $\varepsilon \text{dist}(c, \{a, b\})$ about c .

One can formulate this condition more precisely in terms of the divergence function of a metric space.

Definition 1.5 Let (X, dist) be a 1-ended geodesic metric space, $0 < \delta < 1, \lambda > 0$. Let $a, b, c \in X, \min(\text{dist}(c, a), \text{dist}(c, b)) = r$. Define $\text{Div}_\lambda(a, b, c; \delta)$ as the infimum of lengths of paths connecting a, b and avoiding the ball $\text{Ball}(c, \delta r - \lambda)$ (a ball of nonpositive radius is defined to be empty). Now define the *divergence function* $\text{Div}_\lambda(n; \delta): \mathbb{R} \rightarrow \mathbb{R}$ of the space X as the supremum of all numbers $\text{Div}_\lambda(a, b, c; \delta)$ where $\text{dist}(a, b) \leq n$.

Clearly, the smaller δ and the bigger λ , the smaller the functions $\text{Div}_\lambda(a, b, c; \delta)$ and $\text{Div}_\lambda(n; \delta)$. For 1-ended Cayley graphs, and any $\delta, \delta' < \frac{1}{2}, \lambda, \lambda' > 2$, the functions $\text{Div}_\lambda(n; \delta)$ and $\text{Div}_{\lambda'}(n, \delta')$ are equivalent [12]. (Recall that two nondecreasing functions $f, g: \mathbb{N} \rightarrow \mathbb{N}$ are called equivalent if for some constant $C > 1$, we have

$$f(n/C - C) - Cn - C \leq g(n) < f(Cn + C) + Cn + C$$

for every n .)

Hence we can talk about the divergence function $\text{Div}(n)$ of a 1-ended Cayley graph (setting $\delta = \frac{1}{3}, \lambda = 2$).

It is proved in [12] that a 1-ended Cayley graph X is wide if and only if $\text{Div}(n)$ is bounded by a linear function; and asymptotic cones $\text{Con}^\omega(X, (d_n))$ do not have cut points for all ω if and only if for every $C > 1$, the divergence function $\text{Div}(n)$ is uniformly (in n) bounded by a linear function on the intervals $[d_k/C, Cd_k]$.

The divergence function $\text{Div}(n)$ is an interesting quasi-isometry invariant of a group. It is essentially proved in Alonso et al [2] that for every hyperbolic group $\text{Div}(n)$ is at least exponential. On the other hand, for the mapping class groups (that also have cut points in all asymptotic cones; see Behrstock [5]), the divergence function is quadratic. The following question was asked by J Behrstock.

Problem 1.6 Does there exist a group with strictly subquadratic but not linear divergence function?

The answer to this question is given below.

If some asymptotic cones of a group G do not have cut points, then the divergence function of G can be estimated.

Theorem 1.7 (Theorem 6.1) *Let G be a 1-ended finitely generated group. Suppose that for some sequence of scaling constants d_n and every ultrafilter ω , the asymptotic cone $\text{Con}^\omega(G, (d_n))$ does not have cut points. Let $f(n) \geq n$ be a nondecreasing function such that $d_n \leq f(d_{n-1})$ for all sufficiently large n . Then the divergence function $\text{Div}(n)$ of G does not exceed $Cf(n)$ for some constant C (and all n).*

An a priori stronger property than existence of cut points in asymptotic cones is the existence of the so-called Morse quasigeodesics in the Cayley graph of the group [12]. A quasigeodesic q is called *Morse* if every (L, C) -quasigeodesic p with endpoints on the image of q stays M -close to q where M depends only on L, C . By the Morse lemma, every bi-infinite quasigeodesic of a hyperbolic space is Morse. It is proved in [12], that a quasigeodesic q in a metric space X is Morse if and only if in every asymptotic cone \mathcal{C} , and every point m in the ultralimit \bar{q} of q , the two halves of \bar{q} (before m and after m) are in two different connected components of $\mathcal{C} \setminus \{m\}$ (the implication “ \rightarrow ” of this statement was proved in [5]).

Note that similar divergence properties of geodesics have been studied in the case of $\text{CAT}(0)$ spaces with a cocompact group action by Ballmann [4] and Kapovich and Leeb [28]. In particular, linear (and even subquadratic) divergence for a locally compact Hadamard metric space (ie $\text{CAT}(0)$, complete, geodesic, simply connected metric space) implies that every periodic bi-infinite geodesic in it bounds a flat half-plane (recall that a geodesic is called *periodic* if it is stable under an isometry of the space that acts nontrivially on the geodesic). Ballmann proved [4, Theorem 3.5] that if a $\text{CAT}(0)$ space X has at least 3 points on the boundary and contains a bi-infinite periodic geodesic that does not bound a flat half-plane, then any sufficiently large (say, cocompact) group of isometries of X contains a free non-Abelian subgroup.

This leads to the following two problems from [13]. Recall that a finitely generated group is *constricted* if all its asymptotic cones have cut points.

Problem 1.8 [13, Problem 1.17] *Is every nonconstricted group wide? In other words, if one of the asymptotic cones of a group has no cut points, does every asymptotic cone of the group have no cut points?*

Problem 1.9 [13, Problem 1.19] *Does every non-virtually cyclic finitely generated constricted group contain free non-Abelian subgroups? Is there a constricted group with all proper subgroups cyclic?*

Bruce Kleiner asked the following stronger questions:

Problem 1.10 (Kleiner) Can a finitely generated group G without free noncyclic subgroups contain a bi-infinite Morse quasigeodesic that is periodic? Is there a nonwide amenable non-virtually cyclic group?

We show that the answers to these questions are affirmative.

The second part of [Problem 1.10](#) is answered by the following result (because nontrivial trees have cut points).

Theorem 1.11 ([Theorem 3.25](#), [Lemma 3.23](#)) *There exists a finitely generated lacunary hyperbolic non-virtually cyclic elementary amenable group G . The group G satisfies the following additional properties:*

- G is 2-generated.
- G is (locally nilpotent p -group)-by-(infinite cyclic).
- G is residually (finite p -group), in particular it is residually nilpotent.

Note that since G is not hyperbolic (being amenable and non-virtually cyclic), not all of its asymptotic cones are \mathbb{R} -trees. Hence we obtain the first example of an amenable group with two nonhomeomorphic asymptotic cones.

The following result gives a solution of the first half of that problem and of [Problem 1.9](#).

Recall that a geodesic metric space X is called *tree-graded* with respect to a collection of connected proper subsets \mathcal{P} [[13](#)] if any two distinct subsets from \mathcal{P} intersect by at most one point, and every nontrivial simple geodesic triangle of X is contained in one of the sets from \mathcal{P} . In particular, if subsets from \mathcal{P} are circles (with the natural length metric) of diameters bounded both from above and from below, we call X a *circle-tree*. It is easy to see that every circle-tree is a hyperbolic space.

It is proved in [[13](#)] that every (nonsingleton) space that is tree-graded with respect to proper subspaces has cut points. Conversely, every geodesic space with cut points is tree-graded with respect to the collection of maximal connected subsets without (their own) cut points [[13](#)].

Theorem 1.12 ([Theorem 4.26](#), [Remark 4.27](#)) *There exist two lacunary hyperbolic non-virtually cyclic groups Q_1 and Q_2 such that all asymptotic cones of Q_i are circle-trees and*

- (1) Q_1 is a torsion group;

- (2) every proper subgroup of Q_2 is infinite cyclic and every infinite periodic path in the Cayley graph of Q_2 is a Morse quasigeodesic.

Note that circle-trees are locally isometric to trees and are hyperbolic, so all the asymptotic cones of all groups from [Theorem 1.12](#) are locally isometric hyperbolic spaces. But since some of the asymptotic cones of Q_i are trees and some are not, not all of the cones are homeomorphic.

The group Q_1 is a torsion group but the exponents of elements in Q_1 are not bounded. By the cited result from [\[13\]](#) about groups satisfying a law, asymptotic cones of infinite torsion groups of bounded exponent do not have cut points.

The following theorems give two solutions of [Problem 1.8](#). The first theorem uses central extensions of lacunary hyperbolic groups again.

Although any finitely generated group with infinite central cyclic subgroup is wide [\[13\]](#), the next theorem shows that one can construct lacunary hyperbolic groups with infinite (torsion) centers. Such a group has an asymptotic cone with cut points (a tree) and an asymptotic cone without cut points.

In fact the information we get is much more precise.

Theorem 1.13 ([Theorem 5.11](#)) *For every $m \geq 2$, there exists a finitely generated central extension G of a lacunary hyperbolic group such that for any ultrafilter ω and any scaling sequence $d = (d_n)$, exactly one of the following possibilities occurs and both of them can be realized for suitable ω and d .*

- (a) $\text{Con}^\omega(G, d)$ is an m -fold cover of a circle-tree, the fibers of that cover are cut sets, and every finite cut set of $\text{Con}^\omega(G, d)$ contains one of the fibers.
- (b) $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree.

In particular, in both cases $\text{Con}^\omega(G, d)$ is locally isometric to an \mathbb{R} -tree.

Theorem 1.14 ([Theorem 6.3](#)) *There exists a finitely generated torsion lacunary hyperbolic group G such that one of the asymptotic cones of G does not have cut points.*

The construction from the proof of [Theorem 6.3](#) allows us to answer [Problem 1.6](#). Indeed, by carefully choosing exponents of elements of the group G we can control the scaling constants in the asymptotic cones without cut points. Using [Theorem 1.7](#), we prove ([Corollary 6.4](#)) that for every function $f(n)$ with $f(n)/n$ nondecreasing, $\lim f(n)/n = \infty$, there exists a finitely generated torsion group G whose divergence function $\text{Div}(n)$ is

- not linear but bounded by a linear function on an infinite subset of \mathbb{N} ;
- bounded from above by $Cf(n)$ for some constant C and all n .

In addition, one can arrange that the orders of elements $x \in G$ grow with the length $|x|$ as $O(g(|x|))$ for any prescribed in advance nondecreasing unbounded function $g(n)$. (Recall that groups with bounded torsion are wide [13].)

1.4 Plan of the paper

Section 2 (Preliminaries) contains the main properties of tree-graded spaces (this makes this paper as independent of [13] as possible) and the definition and main properties of asymptotic cones. It also contain some useful properties of hyperbolic groups.

Section 3 starts with the definition and basic properties of lacunary hyperbolic groups. In particular, we show (**Lemma 3.1**) that a group is lacunary hyperbolic provided it has a hyperbolic asymptotic cone, or an asymptotic cone that is locally isometric to an \mathbb{R} -tree. Then we prove the characterization of lacunary hyperbolic groups (**Theorem 3.3**).

The easiest examples of lacunary hyperbolic nonhyperbolic groups are groups given by certain infinite small cancellation presentations. **Proposition 3.12** characterizes such presentations.

In **Section 3.3** we present several observations connecting relative hyperbolicity and lacunary hyperbolicity. In particular **Proposition 3.15** shows that a group that is hyperbolic relative to a lacunary hyperbolic subgroup is lacunary hyperbolic itself. This implies that every lacunary hyperbolic group embeds into a 2-generated lacunary hyperbolic group (quasi-isometrically, malnormally and even as a peripheral subgroup). This result cannot be generalized to several subgroups: we show (**Example 3.16**) that even a free product of two lacunary hyperbolic groups can be non-lacunary hyperbolic.

In **Section 3.4**, we provide several general properties of subgroups of lacunary hyperbolic groups already mentioned above in **Section 1.1**.

In **Section 3.5**, we construct elementary amenable lacunary hyperbolic groups and prove **Theorem 1.11**.

Section 4 is devoted to several small cancellation conditions and their applications. We start by introducing a small cancellation condition $C(\varepsilon, \mu, \rho)$ for presentations over any group H (ie presentations of factor groups of H instead of just factor-groups of the free group as in the classical case). We show (**Lemma 4.6**) that if the group H is hyperbolic and the cancellation parameters are appropriately chosen, then the

factor-group satisfies an analog of the Greendlinger lemma and is hyperbolic again with a nice control on the hyperbolicity constant.

This allows us to use induction and introduce direct limits of groups $G_1 \rightarrow G_2 \rightarrow \dots$ where each G_{i+1} is given by a presentation over G_i which satisfies an appropriate $C(\varepsilon, \mu, \rho)$ -condition so that [Lemma 4.6](#) holds. The union of presentations of all G_i gives us a presentation of the limit group G . We say that such a presentation satisfies a *graded small cancellation* condition. We prove ([Corollary 4.15](#)) that many infinitely presented groups with classical small cancellation conditions have graded small cancellation presentations.

[Theorem 4.17](#) gives an important property of graded small cancellation presentations: every asymptotic cone of a group given by a graded small cancellation presentation is a circle-tree or an \mathbb{R} -tree. Moreover, given the parameters of the cone, one can tell which of these options holds and what are the sizes of the circles in the circle-tree.

In [Section 4.4](#), we apply results of the first author [[42](#)] and show that there are non-virtually cyclic groups with graded small cancellation presentations that have all proper subgroups infinite cyclic or all proper subgroups finite ([Theorem 4.26](#), [Remark 4.27](#)).

In [Section 4.5](#), we notice that existence of cut points in all asymptotic cones follows from the nontriviality of the Floyd boundary of a group. The converse statement does not hold as follows from [Theorem 4.26](#).

In [Section 5](#), we first establish some very general results about asymptotic cones of group extensions. In particular ([Theorem 5.2](#)), if

$$1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$$

is an exact sequence then there exists a continuous map from a cone of G to a cone of H (corresponding to the same parameters) with fibers homeomorphic to the cone of N (considered as a subspace of G). In the case of central extensions, the situation is much nicer, and in particular when the cone of H is an \mathbb{R} -tree, the fibration becomes trivial and the cone of G becomes bi-Lipschitz equivalent to the direct product of the \mathbb{R} -tree and the cone of N ([Theorem 5.6](#)). As applications of these general results, we give proofs of [Theorems 1.4](#) and [1.13](#).

[Section 6](#) is devoted to torsion groups and the proofs of [Theorem 1.14](#) and [Corollary 6.4](#) solving the slow divergence problem.

[Section 7](#) contains some open problems.

The [Appendix](#) written by M Kapovich and B Kleiner contains the proof of [Theorem 8.1](#) (that a finitely presented lacunary hyperbolic group is hyperbolic) and other useful results about asymptotic cones of finitely presented groups.

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2 Preliminaries

2.1 Cayley graphs and van Kampen diagrams

Given a word W in an alphabet S , we denote by $|W|$ its length. We also write $W \equiv V$ to express the letter-for-letter equality of words W and V .

Let G be a group generated by a set S . Recall that the *Cayley graph* $\Gamma(G, S)$ of a group G with respect to the set of generators S is an oriented labeled 1–complex with the vertex set $V(\Gamma(G, S)) = G$ and the edge set $E(\Gamma(G, S)) = G \times S^{\pm 1}$. An edge $e = (g, a)$ goes from the vertex g to the vertex ga and has label $\text{Lab}(e) \equiv a$. As usual, we denote the initial and the terminal vertices of the edge e by e_- and e_+ respectively. Given a combinatorial path $p = e_1 \cdots e_k$ in the Cayley graph $\Gamma(G, S)$, where $e_1, \dots, e_k \in E(\Gamma(G, S))$, we denote by $\text{Lab}(p)$ its label. By definition, $\text{Lab}(p) \equiv \text{Lab}(e_1) \cdots \text{Lab}(e_k)$. We also denote by $p_- = (e_1)_-$ and $p_+ = (e_k)_+$ the initial and terminal vertices of p respectively. The length $|p|$ of p is the number of edges in p .

The (*word*) *length* $|g|$ of an element $g \in G$ with respect to the generating set S is defined to be the length of a shortest word in S representing g in G . The formula $\text{dist}(f, g) = |f^{-1}g|$ defines a metric on G . We also denote by dist the natural extension of the metric to $\Gamma(G, S)$.

Recall that a *van Kampen diagram* Δ over a presentation

$$(1) \quad G = \langle S \mid \mathcal{R} \rangle$$

is a finite oriented connected planar 2–complex endowed with a labeling function $\text{Lab}: E(\Delta) \rightarrow S^{\pm 1}$, where $E(\Delta)$ denotes the set of oriented edges of Δ , such that $\text{Lab}(e^{-1}) \equiv (\text{Lab}(e))^{-1}$. Given a cell Π of Δ , we denote by $\partial\Pi$ the boundary of Π ; similarly, $\partial\Delta$ denotes the boundary of Δ . The labels of $\partial\Pi$ and $\partial\Delta$ are defined up to cyclic permutations. An additional requirement is that the label of any cell Π of Δ is

equal to (a cyclic permutation of) a word $R^{\pm 1}$, where $R \in \mathcal{R}$. Labels and lengths of paths are defined as in the case of Cayley graphs.

The van Kampen Lemma states that a word W over an alphabet S represents the identity in the group given by (1) if and only if there exists a connected simply connected planar diagram Δ over (1) such that $\text{Lab}(\partial\Delta) \equiv W$ [33, Chapter 5, Theorem 1.1].

2.2 Tree-graded spaces

Here we collect all the necessary definitions and basic properties of tree-graded spaces from [13] needed in this paper.

Definition 2.1 Let \mathbb{F} be a complete geodesic metric space and let \mathcal{P} be a collection of closed geodesic nonempty subsets (called *pieces*). Suppose that the following two properties are satisfied:

- (T_1) Every two different pieces have at most one common point.
- (T_2) Every nontrivial simple geodesic triangle (a simple loop composed of three geodesics) in \mathbb{F} is contained in one piece.

Then we say that the space \mathbb{F} is *tree-graded with respect to \mathcal{P}* .

For technical reasons it is convenient to allow \mathcal{P} to be empty. Clearly \mathbb{F} is tree-graded with respect to the empty collections of pieces only if \mathbb{F} is a tree.

By [13, Proposition 2.17], property (T_2) in this definition can be replaced by each of the following two properties.

- (T'_2) For every topological arc $c: [0, d] \rightarrow \mathbb{F}$, where $c(0) \neq c(d)$, and any $t \in [0, d]$, let $c[t-a, t+b]$ be a maximal subarc of c containing $c(t)$ and contained in one piece. Then every other topological arc with the same endpoints as c must contain the points $c(t-a)$ and $c(t+b)$.
- (T''_2) Every simple loop in \mathbb{F} is contained in one piece.

In order to avoid problems with pieces that are singletons, we shall always assume that pieces in a tree-graded space cannot contain each other.

Let us define a partial order relation on the set of tree-graded structures of a space. If \mathcal{P} and \mathcal{P}' are collections of subsets of X , and a space X is tree-graded with respect to both \mathcal{P} and \mathcal{P}' , we write $\mathcal{P} < \mathcal{P}'$ if for every set $M \in \mathcal{P}$ there exists $M' \in \mathcal{P}'$ such that $M \subset M'$. The relation $<$ is a partial order because by our convention pieces of \mathcal{P} (resp. \mathcal{P}') cannot contain each other.

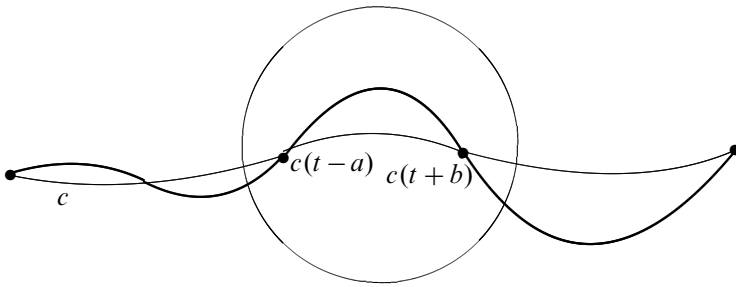


Figure 1: Property (T'_2)

Lemma 2.2 [13, Lemma 2.31] *Let X be a complete geodesic metric space containing at least two points and let \mathcal{C} be a nonempty set of global cut points in X .*

- (a) *There exists the largest in the sense of \prec collection \mathcal{P} of subsets of X such that*
 - *X is tree-graded with respect to \mathcal{P} ;*
 - *any piece in \mathcal{P} is either a singleton or a set with no global cut-point from \mathcal{C} .*

Moreover the intersection of any two distinct pieces from \mathcal{P} is either empty or a point from \mathcal{C} .

- (b) *Let X be a homogeneous space with a cut-point. Then every point in X is a cut-point, so let $\mathcal{C} = X$. Let \mathcal{P} be the set of pieces defined in (a). Then for every $M \in \mathcal{P}$ every $x \in M$ is the projection of a point $y \in X \setminus M$ onto M (ie the closest to y point in M).*

Lemma 2.3 [13, Lemma 2.15] *Let \mathbb{F} be a tree-graded metric space. Let A be a path connected subset of \mathbb{F} without cut points. Then A is contained in a piece. In particular every simple loop in \mathbb{F} is contained in a piece of \mathbb{F} .*

Lemma 2.4 [13, Lemma 2.28] *Let $g = g_1 g_2 \dots g_{2m}$ be a curve in a tree-graded space \mathbb{F} which is a composition of geodesics. Suppose that all geodesics g_{2k} with $k \in \{1, \dots, m-1\}$ are nontrivial and for every $k \in \{1, \dots, m\}$ the geodesic g_{2k} is contained in a piece M_k while for every $k \in \{0, 1, \dots, m-1\}$ the geodesic g_{2k+1} intersects M_k and M_{k+1} only in its respective endpoints. In addition assume that if g_{2k+1} is empty then $M_k \neq M_{k+1}$. Then g is a geodesic.*

- Lemma 2.5** [13, Corollary 2.10] (1) *Every simple path in \mathbb{F} joining two points in a piece is contained in the piece.*
- (2) *Every nonempty intersection between a simple path in \mathbb{F} and a piece is a subpath.*

Lemma 2.6 [13, Corollary 2.11] *Let A be a connected subset (possibly a point) in a tree-graded space \mathbb{F} which intersects a piece M in at most one point.*

- (1) *The subset A projects onto M in a unique point x .*
- (2) *Every path joining a point in A with a point in M contains x .*

2.3 Asymptotic cones

Let us recall the definition of asymptotic cones. A nonprincipal ultrafilter ω is a finitely additive measure defined on all subsets S of \mathbb{N} , such that $\omega(S) \in \{0, 1\}$, $\omega(\mathbb{N}) = 1$, and $\omega(S) = 0$ if S is a finite subset. For a bounded sequence of numbers x_n , $n \in \mathbb{N}$, the limit $\lim^\omega x_n$ with respect to ω is the unique real number a such that $\omega(\{i \in \mathbb{N} : |x_i - a| < \epsilon\}) = 1$ for every $\epsilon > 0$. Similarly, $\lim^\omega x_n = \infty$ if $\omega(\{i \in \mathbb{N} : x_i > M\}) = 1$ for every $M > 0$.

Given two infinite sequences of real numbers (a_n) and (b_n) we write $a_n = o_\omega(b_n)$ if $\lim^\omega a_n/b_n = 0$. Similarly $a_n = \Theta_\omega(b_n)$ (respectively $a_n = O_\omega(b_n)$) means that $0 < \lim^\omega (a_n/b_n) < \infty$ (respectively $\lim^\omega (a_n/b_n) < \infty$).

Let (X_n, dist_n) , $n \in \mathbb{N}$, be a metric space. Fix an arbitrary sequence $e = (e_n)$ of points $e_n \in X_n$. Consider the set \mathcal{F} of sequences $g = (g_n)$, $g_n \in X_n$, such that $\text{dist}_n(g_n, e_n) \leq c$ for some constant $c = c(g)$. Two sequences (f_n) and (g_n) of this set \mathcal{F} are said to be *equivalent* if $\lim^\omega \text{dist}_n(f_n, g_n) = 0$. The equivalence class of (g_n) is denoted by $(g_n)^\omega$. The ω -limit $\lim^\omega (X_n)_e$ is the quotient space of equivalence classes where the distance between $(f_n)^\omega$ and $(g_n)^\omega$ is defined as $\lim^\omega \text{dist}(f_n, g_n)$.

An *asymptotic cone* $\text{Con}^\omega(X, e, d)$ of a metric space (X, dist) where $e = (e_n)$, $e_n \in X$, and $d = (d_n)$ is an unbounded nondecreasing *scaling sequence* of positive real numbers, is the ω -limit of spaces $X_n = (X, \text{dist}/d_n)$. The asymptotic cone is a complete space; it is a geodesic metric space if X is a geodesic metric space [21; 11]. Note that $\text{Con}^\omega(X, e, d)$ does not depend on the choice of e if X is homogeneous (say, if X is a finitely generated group with a word metric), so in that case, we shall omit e in the notation of an asymptotic cone.

If (Y_n) is a sequence of subsets of X endowed with the induced metric, we define $\lim^\omega (Y_n)_e$ to be the subset of $\text{Con}^\omega(X, e, d)$ consisting of $x \in \text{Con}^\omega(X, e, d)$ that can be represented by sequences (x_n) , where $x_n \in Y_n$.

An *asymptotic cone* of a finitely generated group G with a word metric is the asymptotic cone of its Cayley graph (considered as the discrete space of vertices with the word metric). Asymptotic cones corresponding to two different finite generating sets of G (and the same ultrafilters and scaling constants) are bi-Lipschitz equivalent. The

asymptotic cone $\text{Con}^\omega(G, d)$ of a group G is a homogeneous geodesic metric space with transitive group of isometries $G_e^\omega(d)$ consisting of sequences (g_n) , $g_n \in G$ such that $|g_n| \leq Cd_n$ for some constant C depending on the sequence (here $|g_n|$ is the word length of g_n). The action is by multiplication on the left: $(g_n) \circ (h_n)^\omega = (g_n h_n)^\omega$.

Recall that a geodesic p in $\text{Con}^\omega(X; e, d)$ is called a *limit geodesic* if $p = \lim^\omega p_n$, where for every $n \in \mathbb{N}$, p_n is a geodesic in X . The lemma below was proved in [10, Corollary 4.18].

Lemma 2.7 *Assume that in an asymptotic cone $\text{Con}^\omega(X; e, d)$, a collection of closed subsets \mathcal{P} satisfies (T_1) and every nontrivial simple triangle in $\text{Con}^\omega(X; e, d)$ whose sides are limit geodesics is contained in a subset from \mathcal{P} . Then \mathcal{P} satisfies (T_2) , ie, $\text{Con}^\omega(X; e, d)$ is tree-graded with respect to \mathcal{P} .*

2.4 Hyperbolic groups

Recall that a geodesic space X is δ -hyperbolic (or simply *hyperbolic*, for brevity) if for any geodesic triangle Δ in X , each side of Δ is contained in the closed δ -neighborhood of the union of the other two sides. This δ is called the *hyperbolicity constant* of X . A group H is δ -hyperbolic (or simply *hyperbolic*) if it is generated by a finite set S and its Cayley graph $\Gamma(H, S)$ endowed with the combinatorial metric is a hyperbolic metric space.

Recall that a path p is called (λ, c) -quasigeodesic for some $\lambda \in (0, 1]$, $c \geq 0$ if for any subpath q of p , we have

$$\text{dist}(q_-, q_+) \geq \lambda|q| - c.$$

The property of hyperbolic spaces stated below is well-known although it is usually formulated in a slightly different manner (see, for example, Ghys and de la Harpe [18]).

Lemma 2.8 *For any $\lambda \in (0, 1]$, $c > 0$ there exists $\theta(\lambda, c)$ such that any two (λ, c) -quasigeodesic paths p, q in a δ -hyperbolic metric space such that $p_- = q_-$ and $p_+ = q_+$ belong to the closed $\theta(\lambda, c)$ -neighborhoods of each other.*

The next property can easily be derived from the definition of a hyperbolic space by cutting the n -gon into triangles.

Lemma 2.9 *For any $n \geq 3$, any side of a geodesic n -gon in a δ -hyperbolic space belongs to the closed $(n - 2)\delta$ -neighborhood of the other $(n - 1)$ sides.*

3 Lacunary hyperbolic groups: characterization and examples

3.1 A characterization of lacunary hyperbolic groups

We say that a metric space X is *lacunary hyperbolic* if one of the asymptotic cones of X is an \mathbb{R} -tree. In particular, every hyperbolic metric space is lacunary hyperbolic. A group G is *lacunary hyperbolic* if it is finitely generated and the corresponding Cayley graph is lacunary hyperbolic. Clearly this notion is independent of the choice of the finite generating set. We also say that a metric space X is *almost homogeneous* if there is a homogeneous subspace $Y \subseteq X$ such that $\text{dist}_{\text{Hau}}(X, Y) < \infty$, where dist_{Hau} is the Hausdorff distance between X and Y . That is, there exists $\varepsilon > 0$ such that for every $x \in X$, we have $\text{dist}(x, Y) < \varepsilon$. Given a group G , any Cayley graph of G endowed with the combinatorial metric is almost homogeneous. Note also that if X is almost homogeneous, then every asymptotic cone of X is homogeneous.

Lemma 3.1 *Let X be a metric space. Then the following properties are equivalent.*

- (1) *Some asymptotic cone of X is an \mathbb{R} -tree.*
- (2) *Some asymptotic cone of X is lacunary hyperbolic.*

If, in addition, X is almost homogeneous, these properties are equivalent to:

- (3) *Some asymptotic cone of X is locally isometric to an \mathbb{R} -tree.*

Proof It suffices to show that (2) \Rightarrow (1) and (3) \Rightarrow (1). Recall that the set $\mathcal{C}(X)$ of asymptotic cones of X is closed under taking ultralimits [13, Corollary 3.24]. Further let dist denote the standard metric on $\text{Con}^\omega(X; e, d)$. It is straightforward to check that for every $k > 0$, $(\text{Con}^\omega(X; e, d), \frac{1}{k}\text{dist})$ is isometric to the asymptotic cone $\text{Con}(X, e, (kd_n))$. In particular, if $Y \in \mathcal{C}(X)$, then every asymptotic cone of Y belongs to $\mathcal{C}(X)$. This yields (2) \Rightarrow (1).

Further if $\text{Con}^\omega(X; e, d)$ is homogeneous and locally isometric to an \mathbb{R} -tree, there is $c > 0$ such that $\text{Con}^\omega(X; e, d)$ contains no simple nontrivial loops of length at most c . We consider the sequence of cones $C_k = \text{Con}(X, e, (kd_n))$ for $k \rightarrow 0$. Clearly C_k has no simple nontrivial loops of length at most c/k . Then for any nonprincipal ultrafilter ω , $\lim^\omega(C_k)_e$ is an \mathbb{R} -tree. \square

By [Theorem 8.1](#) of Kapovich and Kleiner, if a finitely presented group G is lacunary hyperbolic, then, in fact, it is hyperbolic. Combining this with [Lemma 3.1](#), we obtain the following.

Proposition 3.2 *The following conditions are equivalent for any finitely presented group G .*

- (1) *Some asymptotic cone of G is lacunary hyperbolic.*
- (2) *Some asymptotic cone of G is locally isometric to an \mathbb{R} -tree.*
- (3) *All asymptotic cones of G are \mathbb{R} -trees, ie, G is hyperbolic.*

In contrast, in Sections 4–5 we construct a nonhyperbolic finitely generated group all of whose asymptotic cones are quasi-isometric and locally isometric to an \mathbb{R} -tree.

The next theorem describes the structure of lacunary hyperbolic groups. Given a group homomorphism $\alpha: G \rightarrow H$ and a generating set S of G , we denote by $r_S(\alpha)$ the *injectivity radius of α with respect to S* , ie, the radius of the largest ball B in G such that α is injective on B .

Theorem 3.3 *Let G be a finitely generated group. Then the following conditions are equivalent.*

- (1) *G is lacunary hyperbolic.*
- (2) *There exists a scaling sequence $d = (d_n)$ such that $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree for any nonprincipal ultrafilter ω .*
- (3) *G is the direct limit of a sequence of finitely generated groups and epimorphisms*

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots$$

such that G_i is generated by a finite set S_i , $\alpha_i(S_i) = S_{i+1}$, and each G_i is δ_i -hyperbolic, where $\delta_i = o(r_{S_i}(\alpha_i))$.

Proof We are going to show that (2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (2). The first implication is trivial. Let us prove the second one. Suppose that $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree for some ω and nondecreasing $d = (d_n)$. For every n , we denote by H_n the group given by the presentation $\langle S \mid \mathcal{R}_n \rangle$, where \mathcal{R}_n consists of labels of all cycles in the ball of radius d_n around the identity in $\Gamma(G, S)$. Note that G is a quotient of H_n and the canonical map $H_n \rightarrow G$ is injective on the ball of radius d_n . It follows that the natural epimorphisms $H_n \rightarrow H_m$ are also injective on the balls of radius d_n for arbitrary $m \geq n$.

Observe that by Kapovich and Kleiner’s [Theorem 8.3](#) and [Lemma 8.2](#) from the [Appendix](#), there are constants C_1, C_2, C_3 with the following property. Let H be a group having a finite presentation $\langle S \mid \mathcal{R} \rangle$, $D = \max_{R \in \mathcal{R}} |R|$. Assume that for some δ and $d \geq \max\{C_1 \delta, D\}$, every ball of radius $C_2 d$ in $\Gamma(H, S)$ is δ -hyperbolic. Then $\Gamma(H, S)$ is $C_3 d$ -hyperbolic.

Since $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree, balls of radius d_n in $\Gamma(G, S)$ (and hence in $\Gamma(H_n, S)$) are $o_\omega(d_n)$ -hyperbolic. Recall that any δ -hyperbolic graph endowed with the combinatorial metric becomes 1-connected after gluing 2-cells along all combinatorial loops of length at most 16δ (see, for example, Bridson and Haefliger [7, Chapter III.H, Lemma 2.6]). Hence H_n admits a finite presentation with generating set S and relations of lengths $o_\omega(d_n)$. In particular, for every positive integer k , there exists $n(k)$ such that the sequence $(n(k))$ is strictly increasing and the following conditions hold.

- (H1) $H_{n(k)}$ admits a finite presentation with generating set S and relations of lengths at most $d_{n(k)}/(C_2k)$.
- (H2) Every ball of radius $d_{n(k)}/k$ in the $\Gamma(H_{n(k)}, S)$ is $d_{n(k)}/(C_1C_2k)$ -hyperbolic.

These conditions allow us to apply the above observation to $H = H_{n(k)}$, $d = D = d_{n(k)}/(C_2k)$, and $\delta = d_{n(k)}/(C_1C_2k)$. Thus $H_{n(k)}$ is $C_3d_{n(k)}/(C_2k)$ -hyperbolic. Now setting $G_k = H_{n(k)}$ we obtain a sequence of groups and homomorphisms

$$(2) \quad G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots,$$

where G_k is δ_k -hyperbolic for $\delta_k = C_3d_{n(k)}/(C_2k) = o(d_{n(k)})$ and α_k is injective on the ball of radius $d_{n(k)}$ as desired.

To prove (3) \Rightarrow (2) we fix any sequence $d = (d_n)$ such that

$$(3) \quad \delta_n = o(d_n), \quad d_n = o(r_S(\alpha_n)).$$

Let ω be an arbitrary nonprincipal ultrafilter. According to Lemma 2.7 applied to the collection \mathcal{P} of all one-element subsets of $\text{Con}^\omega(G, d)$, to show that $\text{Con}^\omega(G, d)$ is a tree it suffices to prove that it contains no simple nontrivial limit geodesic triangles.

Suppose that pqs is a nontrivial simple triangle in $\text{Con}^\omega(G, d)$ whose sides are limit geodesics. Clearly $pqs = \lim^\omega H_n$, where $H_n = p_n a_n q_n b_n s_n c_n$ is a geodesic hexagon in $\Gamma(G, S)$ such that $p = \lim^\omega p_n$, $q = \lim^\omega q_n$, $s = \lim^\omega s_n$,

$$(4) \quad |a_n| = o_\omega(d_n), \quad |b_n| = o_\omega(d_n), \quad |c_n| = o_\omega(d_n),$$

and perimeter of H_n satisfies $|H_n| = O_\omega(d_n)$. By (3) we have $|H_n| = o(r_S(\alpha_n))$, hence the label of H_n represents 1 in G_n ω -almost surely. Thus H_n may be considered as a configuration in the Cayley graph of G_n ω -almost surely.

Let p be a nontrivial side of pqs . Lemma 2.9 implies that p_n belongs to the closed $4\delta_n$ -neighborhood of the other sides ω -almost surely. Combining this with (4) and (3), we obtain that the $(O_\omega(\delta_n) + o_\omega(d_n))$ -neighborhood of $q_n \cup r_n$ contains p_n . Since $O_\omega(\delta_n) = o_\omega(d_n)$, p belongs to the union of q and r . \square

Remark 3.4 It is easy to see from the proof of [Theorem 3.3](#) that (3) can be replaced with the following (a priori stronger) condition, which will be useful for some applications.

(4) G is the direct limit of a sequence of finitely generated groups and epimorphisms

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots$$

such that G_i is generated by a finite set S_i , $\alpha_i(S_i) = S_{i+1}$, each G_i is δ_i -hyperbolic, α_i is injective on the ball of radius r_i of the group $G_i = \langle S_i \rangle$, where $\delta_i = o(r_i)$ and

(4a) the sequence of the numbers r_i is nondecreasing;

(4b) the group G_i has a presentation $\langle S_i \mid \mathcal{P}_i \rangle$, where $\max_{P \in \mathcal{P}_i} |P| = o(r_i)$.

Indeed it is easy to see that the proof of the implication (1) \Rightarrow (3) ensures (d1) for $r_i = d_{n(i)}$ and (d2) follows from Condition (H1).

Remark 3.5 The third condition from the theorem implies that every lacunary hyperbolic group (as well as any other limit of hyperbolic groups) embeds into an ultraproduct of hyperbolic groups. Indeed let $\phi: G_1 \rightarrow \prod^\omega G_i$ be the homomorphism defined by the rule $\phi(g) = (\alpha_1(g), \alpha_2 \circ \alpha_1(g), \dots)$. If ω is nonprincipal, it is straightforward to see that $\text{Ker}(\phi) = \bigcup_{i=1}^\infty \text{Ker}(\alpha_i \circ \dots \circ \alpha_1)$ and hence $\phi(G_1) \cong G$. In particular, if a universal sentence holds in the first order group language holds in all hyperbolic groups, then it holds in all lacunary hyperbolic groups.

This observations is similar to Mal'cev's Local Theorems [\[34\]](#). It provides us with a uniform way of proving (nontrivial) universal theorems for lacunary hyperbolic groups. As an example, the reader may verify that the sentence

$$\forall x \forall y (x^{-1}y^2x = y^3 \Rightarrow [y, x^{-1}yx] = 1)$$

is a theorem in the class of all hyperbolic group and hence in the class of all lacunary hyperbolic groups, but it is not a theorem in the class of all groups.

3.2 Lacunary hyperbolic groups and the classical small cancellation condition

We begin with examples of lacunary hyperbolic groups constructed by means of the classical small cancellation theory. Recall the small cancellation condition $C'(\mu)$. Given a set \mathcal{R} of words in a certain alphabet, one says that \mathcal{R} is *symmetrized* if for any $R \in \mathcal{R}$, all cyclic shifts of $R^{\pm 1}$ are contained in \mathcal{R} .

Definition 3.6 Let

$$(5) \quad G = \langle S \mid \mathcal{R} \rangle$$

be a group presentation, where \mathcal{R} is a symmetrized set of reduced words in a finite alphabet S . A common initial subword of any two distinct words in \mathcal{R} is called a *piece*. We say that \mathcal{R} satisfies the $C'(\mu)$ if any piece contained (as a subword) in a word $R \in \mathcal{R}$ has length smaller than $\mu|R|$.

The main property of groups with $C'(\mu)$ -presentations is given by the Greendlinger Lemma below.

Lemma 3.7 [33, Theorem V.4.4.] *Let \mathcal{R} be a symmetrized set of words in a finite alphabet S satisfying a $C'(\mu)$ condition with $\mu \leq 1/6$, $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$. Assume that a reduced van Kampen diagram Δ over \mathcal{P} with cyclically reduced boundary path q has at least one cell. Then q and the boundary path of some cell Π in Δ have a common subpath t with $|t| > (1 - 3\mu)|\partial\Pi|$.*

Given a van Kampen diagram Δ over (5), we denote by $\mathcal{A}(\Delta)$ the sum of the perimeters of all cells in Δ . The next lemma easily follows from the Greendlinger Lemma by induction on the number of cells in the diagram.

Lemma 3.8 *Suppose that a group presentation (5) satisfies the $C'(\mu)$ -condition for some $\mu \leq 1/6$. Then for any reduced diagram Δ over (5), we have*

- (a) $|\partial\Delta| > (1 - 3\mu)|\partial\Pi| > |\partial\Pi|/2$ for any cell Π in Δ ;
- (b) $|\partial\Delta| > (1 - 6\mu)\mathcal{A}(\Delta)$.

The lemma below was actually proved in [2] although it was not stated explicitly there. The explicit statement is due to Ollivier [38].

Lemma 3.9 *Suppose that there exists $C > 0$ such that for every minimal van Kampen diagram Δ over (5), we have $|\partial\Delta| \geq C\mathcal{A}(\Delta)$. Then provided G is finitely presented, it is δ -hyperbolic, where $\delta \leq 12 \max_{R \in \mathcal{R}} |R|/C^2$.*

Corollary 3.10 *If a finite group presentation (5) satisfies the $C'(\mu)$ -condition for some $\mu < 1/6$. Then G is δ -hyperbolic, where $\delta \leq 12 \max_{R \in \mathcal{R}} |R|/(1 - 6\mu)^2$.*

Definition 3.11 We say that a subset $L \subset \mathbb{N}$ is *sparse*, if for any $\lambda > 0$, there exists a segment $I = [a, b] \subset [1, +\infty)$ such that $I \cap L = \emptyset$ and $a/b < \lambda$.

Given a (not necessary finite) presentation (5), we denote by $L(\mathcal{R})$ the set $\{|R| \mid R \in \mathcal{R}\}$. The following result provides us with a rich source of examples of lacunary hyperbolic groups.

Proposition 3.12 *Let (5) be a group presentation with finite alphabet S , satisfying the $C'(\mu)$ small cancellation condition for some $\mu < 1/6$. Then the group G is lacunary hyperbolic if and only if the set $L(\mathcal{R})$ is sparse.*

Proof Suppose that $L(\mathcal{R})$ is sparse. Then for every $n \in \mathbb{N}$, there exists a segment $I_n = [a_n, b_n] \subset \mathbb{R}$ such that

$$(6) \quad I_n \cap L(\mathcal{R}) = \emptyset,$$

$b_n/a_n > n$, and $a_{n+1} > b_n$. We set $\mathcal{R}_n = \{R \in \mathcal{R} \mid |R| \leq a_n\}$ and $G_n = \langle S \mid \mathcal{R}_n \rangle$. Then G is the limit of the sequence of the groups G_n and the obvious homomorphisms $\alpha_n: G_n \rightarrow G_{n+1}$. By Corollary 3.10, G_n is δ_n -hyperbolic, where $\delta_n = O(a_n) = o(b_n)$. On the other hand, by Lemma 3.8 (a) and (6) we have $r_S(\alpha_n) \geq b_n/2$. Hence G is lacunary hyperbolic by Theorem 3.3.

Now assume that G is lacunary hyperbolic. Let $G = \lim G_i$ as in Remark 3.4. Then, in the notation of Remark 3.4, we have $r_i \geq \lambda(2M_i + 1)$ for given $\lambda > 0$ and all sufficiently large i , where $M_i = \max_{P \in \mathcal{P}_i} |P|$. To prove that $L(\mathcal{R})$ is sparse, it suffices to show that there is no $R \in \mathcal{R}$ with $r_i \geq |R| \geq 2M_i + 1$. By Condition (d1) in Remark 3.4, the natural homomorphism $G_i \rightarrow G$ is also injective on balls of radius r_i . Hence $R = 1$ in G_i . However, by Lemma 3.8 (a), all words from \mathcal{P}_i represent 1 in the group $G'_i = \langle S \mid \mathcal{T}_i \rangle$, where $\mathcal{T}_i = \{R \in \mathcal{R} \mid |R| \leq 2M_i\}$. Hence $R = 1$ in G'_i . As $R \notin \mathcal{T}_i$, this contradicts Lemma 3.7 and the $C'(\mu)$ -condition. \square

3.3 Relative and lacunar hyperbolicity

Recall a definition of relatively hyperbolic groups. There are at least six equivalent definitions (the first one is due to Gromov [21]). We use the definition whose equivalence to the other definitions is proved in [13].

Definition 3.13 [13, Theorem 8.5] *Let G be a finitely generated group, H_1, \dots, H_n subgroups of G . Then G is called (strongly) hyperbolic relative to peripheral subgroups H_1, \dots, H_n if every asymptotic cone $\text{Con}^\omega(G, d)$ of G is tree-graded with respect to the collection of nonempty ultralimits of sequences of left cosets $\lim^\omega(g_j H_i)$, $g_j \in G$, where different sequences of cosets $(g_j H_i), (g'_j H_i)$ define the same piece if they coincide ω -almost surely.*

Remark 3.14 Recall that in a finitely generated relatively hyperbolic group G , every peripheral subgroup is finitely generated and quasi-isometrically embedded in G [46]. This implies that each ultralimit $\lim^\omega (g_j H_i)$ is either empty or bi-Lipschitz equivalent to the asymptotic cone $\text{Con}^\omega(H_i, d)$ with respect to a finite generating set of H_i (see Druţu and Sapir [13] for details).

The first two claims of the following proposition provides us with a way of constructing new lacunary hyperbolic groups from given ones. The third claim is related to the following problem. It is well-known that if a group G is hyperbolic relative to a finitely presented subgroup H , then G is finitely presented itself. However it is still unknown (see [46, Problem 5.1]) whether finite presentability of G implies finite presentability of H (although H is finitely generated whenever G is [46]). An ultimate negation of this implication would be the following statement:

Any finitely generated recursively presented group H embeds into a finitely presented relatively hyperbolic group G as a peripheral subgroup.

Proposition 3.12 and Proposition 3.15 (c) imply that this statement does not hold.

Proposition 3.15 (a) *If a finitely generated group G is hyperbolic relative to a lacunary hyperbolic subgroup H , then G is itself lacunary hyperbolic.*

- (b) *Every lacunary hyperbolic group H embeds into a 2-generated lacunary hyperbolic group G . Moreover one can assume that G is hyperbolic relative to H .*
- (c) *If G is hyperbolic relative to a lacunary hyperbolic subgroup H and H is not finitely presented, then G is not finitely presented.*

Proof (a) Suppose that G is hyperbolic relative to H and $\text{Con}^\omega(H, d)$ is an \mathbb{R} -tree. By Definition 3.13 and Remark 3.14, $\text{Con}^\omega(G, d)$ is tree-graded relative to a collection of \mathbb{R} -trees. In particular, $\text{Con}^\omega(G, d)$ has no nontrivial simple loops, ie, it is an \mathbb{R} -tree.

(b) By [3, Theorem 1.1] applied to the free group of rank 2, there exists a 2-generated group G such that H embeds in G and G is hyperbolic relative to H . It remains to use part (a).

(c) The group G is lacunary hyperbolic by (a). If G was finitely presented, it would be hyperbolic by Kapovich and Kleiner's Theorem 8.1 from the Appendix. Since peripheral subgroups are quasi-isometrically embedded into relatively hyperbolic groups [46] and quasi-isometrically embedded subgroups of hyperbolic groups are hyperbolic [2], H is hyperbolic. Hence H is finitely presented that contradicts our assumption. \square

Observe that the first assertion of [Proposition 3.15](#) cannot be generalized to the case of several peripheral subgroups. Moreover, we have the following.

Example 3.16 The free product $H_1 * H_2$ of lacunary hyperbolic groups is not necessarily lacunary hyperbolic. Indeed it is not hard to construct a set of words $\mathcal{R} = \{R_i, i \in \mathbb{N}\}$ in a finite alphabet S such that $|R_i| = i$ and \mathcal{R} satisfies the $C'(1/7)$ condition. It is also easy to find two subsets $N_1, N_2 \subset \mathbb{N}$ such that both N_1, N_2 are sparse and $N_1 \cup N_2 = \mathbb{N}$. Set

$$H_1 = \langle S \mid R_i = 1, i \in N_1 \rangle, \quad H_2 = \langle S \mid R_i = 1, i \in N_2 \rangle.$$

Then by [Proposition 3.12](#) H_1, H_2 are lacunary hyperbolic while $H_1 * H_2$ is not.

3.4 Subgroups of lacunary hyperbolic groups

The next theorem shows that subgroups of lacunary hyperbolic groups share common properties with subgroups of hyperbolic groups.

Definition 3.17 Let H be a subgroup of a finitely generated group $G = \langle S \rangle$. Then the *growth function* of H (relative to S) is the function

$$f_{H,G}(n) = \#(\text{Ball}_G(n) \cap H)$$

where $\text{Ball}_G(n)$ is the ball of radius n around 1 in the group G (in the word metric related to S). We say that H has *exponential growth in G* if its growth function $f_{H,G}(n)$ is bounded from below by an exponent d^n for some $d > 1$.

Theorem 3.18 Let G be a lacunary hyperbolic group.

- (a) Every finitely presented subgroup of G is a subgroup of a hyperbolic group.
- (b) Every undistorted (ie quasi-isometrically embedded) subgroup of G is lacunary hyperbolic.
- (c) Let H be a (not necessarily finitely generated) bounded torsion subgroup of G . Then the growth of H in G is not exponential.

Proof (a) Indeed, every finitely presented subgroup of a lacunary hyperbolic group G that is a direct limit of hyperbolic groups G_i as in [Theorem 3.3](#) is isomorphic to a subgroup of one of the G_i 's.

Remark 3.19 It is worth noting that for the same reason, every finitely presented subgroup of the free Burnside group B of any sufficiently large odd exponent is cyclic because B is a direct limit of hyperbolic groups [\[25\]](#), periodic subgroups of hyperbolic groups are finite [\[20\]](#), and finite subgroups of B are cyclic [\[1\]](#).

(b) If H is a finitely generated undistorted subgroup of a lacunary hyperbolic group G then every asymptotic cone $\text{Con}^\omega(H, (d_n))$ of H is bi-Lipschitz homeomorphic to a subspace of the corresponding asymptotic cone $\text{Con}^\omega(G, (d_n))$. Since a connected subspace of an \mathbb{R} -tree is an \mathbb{R} -tree, every undistorted subgroup of a lacunary hyperbolic group is lacunary hyperbolic itself.

(c) Let $G = \langle S \rangle$ be a direct limit of hyperbolic groups $G_i = \langle S_i \rangle$ and homomorphisms α_i as in [Remark 3.4](#), $S = S^{-1}$, $S_i = S_i^{-1}$. Note that the volume of the ball $\text{Ball}_{G_i}(r)$ of radius r in G_i is at most a^r for $a = \#S + 1$.

Suppose $H \leq G$ has exponential growth in G and bounded torsion: $h^n = 1$ for all $h \in H$. Denote by H_i the preimage of H in G_i . Since H has exponential growth in G , there is $d > 1$ such that the number of elements from H_i of length at most t in the generators S_i is at least d^t for every integer t . We denote this subset by $B_i(t)$.

Suppose for some i and t_0 , all the elements of $B_i(2t_0)$ have finite orders. Then all elements of both $B_i(t_0)$ and $B_i(t_0)B_i(t_0)$ have finite orders. It is proved in [[26](#), Lemma 17] that under these assumptions the subset $B_i(t_0)$ must be conjugate in the hyperbolic group G_i to a subset of $\text{Ball}_{G_i}(205\delta_i)$. Therefore $d^{t_0} \leq \text{card } B_i(t_0) \leq a^{205\delta_i}$, and so $t_0 \leq C\delta_i$, for $C = 205 \log_d a$. Hence there is a constant $D \geq 2C$ such that in every intersection $H_i \cap \text{Ball}_{G_i}(D\delta_i)$, there exists an element g_i of infinite order.

Since H has exponent $\leq n$, the epimorphism $G_i \rightarrow G$ is not injective on the ball of radius $|g_i^n| \leq nD\delta_i$ for every i . Hence in the notation of [Remark 3.4](#), we have $r_i < nD\delta_i$. Therefore $\delta_i/r_i > 1/(nD)$ for all i and (c) is proved by contradiction. \square

Remark 3.20 Note that both the growth condition and the bounded torsion condition are essential in [Theorem 3.18\(c\)](#). Indeed, [Theorem 5.11](#) gives examples of lacunary hyperbolic groups with infinite (central) subgroups of any given finite exponent ≥ 2 ; [Theorem 3.25](#) below provides an example of lacunary hyperbolic group with infinite locally finite (torsion) normal subgroup, and lacunary hyperbolic groups from [Theorem 4.26](#) (1) and [Theorem 6.3](#) are torsion themselves.

Corollary 3.21 (a) *A lacunary hyperbolic group G cannot contain copies of \mathbb{Z}^2 or Baumslag–Solitar groups.*

(b) *The free Burnside group $B(m, n)$ with sufficiently large exponent n and $m \geq 2$ cannot be a subgroup of a lacunary hyperbolic group.*

(c) *The lamplighter group $(\mathbb{Z}/2\mathbb{Z})\text{wr } \mathbb{Z}$ cannot be a subgroup of a lacunary hyperbolic group.*

Proof (a) It follows from [Theorem 3.18](#) (a). Indeed \mathbb{Z}^2 and Baumslag–Solitar groups are finitely presented and are not subgroups of hyperbolic groups

(b) Indeed, $B(m, n)$ has exponential growth for any $m \geq 2, n \gg 1$ [[25](#)]. Hence it has exponential growth in any group G containing $B(m, n)$ and we can use [Theorem 3.18](#) (c).

(c) Consider any short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

where $G = \langle S \rangle, Q = \langle SN \rangle$. Let $f_{N,G}$ be the growth function of N in G, f_G and f_Q be the growth functions of G and Q relative to the generating sets S and SN respectively. Then the following inequality obviously holds:

$$f_G(n) \leq f_{N,G}(2n) f_Q(n)$$

for every $n \geq 1$. Applying this inequality to the short exact sequence

$$1 \rightarrow N \rightarrow (\mathbb{Z}/2\mathbb{Z}) \text{wr } \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 1$$

where $N = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is the base of the wreath product, we deduce that N has exponential growth in the lamplighter group, and, consequently, in any finitely generated group containing the lamplighter group. Since N has exponent 2, it cannot be a subgroup of a lacunary hyperbolic group by [Theorem 3.18](#) (c). □

Remark 3.22 It is clear that the same argument shows that a lacunary hyperbolic group has no finitely generated subgroup H of exponential growth which is an extension of a bounded torsion subgroup by a nilpotent group.

3.5 Lacunary hyperbolic amenable groups

Recall that the class of elementary amenable groups is defined to be the smallest class containing all finite and Abelian groups and closed under taking directed unions, extensions, quotients, and subgroups. In this section, we will construct an elementary amenable group some of whose asymptotic cones are \mathbb{R} -trees.

Pick a prime number p and a nondecreasing sequence \mathbf{c} of positive integers $c_1 \leq c_2 \leq \dots$. Consider the group $A = A(p, \mathbf{c})$ generated by $a_i, i \in \mathbb{Z}$ subject to the following relations:

$$a_i^p = 1, \quad i \in \mathbb{Z},$$

$$[\dots [a_{i_0}, a_{i_1}], \dots, a_{i_{c_n}}] = 1$$

for every n and all commutators with $\max_{j,k} |i_j - i_k| \leq n$. The group $A = A(p, \mathbf{c})$ is locally nilpotent since arbitrary $a_j, a_{j+1}, \dots, a_{j+n}$ generate a nilpotent subgroup of nilpotency class at most c_n . Since the locally nilpotent group A is generated by elements of order p , it is a p -group [23]. Notice that for $l \geq 0$, there is a retraction π_l of A onto the finite subgroup $A(l)$ generated by a_0, \dots, a_l ($\pi_l(a_j) = 1$ if $j \neq 0, \dots, l$).

The group A admits the automorphism $a_i \rightarrow a_{i+1}$ ($i \in \mathbb{Z}$). Denote by $G = G(p, \mathbf{c})$ the extension of A by this automorphism, ie, G is generated by the normal subgroup A and an element t of infinite order such that $ta_it^{-1} = a_{i+1}$ for every integer i .

Lemma 3.23 *The group $G = G(p, \mathbf{c})$ satisfies the following properties:*

- (a) G is 2-generated.
- (b) G is (locally nilpotent p -group)-by-(infinite cyclic), and so it is elementary amenable.
- (c) G admits an epimorphism onto the wreath product $(\mathbb{Z}/p\mathbb{Z}) \text{ wr } \mathbb{Z}$ and so G is not virtually cyclic.
- (d) G is residually (finite p -group).

Proof (a) It is clear that G is generated by a_0 and t .

(b) and (c) follow from the construction.

(d) Fix $m = p^s$ for some $s > 0$ and consider a ‘‘circular version’’ B of the group A , namely the (finite) group B generated by m elements $b_{[i]}$, where $[i]$ is a residue class modulo m , with defining relations $b_{[i]}^p = 1$ and $[\dots [b_{[i_0]}, b_{[i_1]}], \dots, b_{[i_{c_n}]}] = 1$ for every n and all commutators with $\max_{j,k} |[i_j - i_k]| \leq n$, where $|[i]|$ is the smallest nonnegative integer s such that either $i - s$ or $i + s$ is 0 modulo m . The group B is a finite p -group.

Notice that if $0 \leq l \leq m/2$, then B has a retraction on the subgroup $B(l)$ generated by $b_{[0]}, \dots, b_{[l]}$ which are subject to all the defining relations of B involving these generators. Since $l \leq m/2$, we have $|[i_j - i_k]| = |i_j - i_k|$ for $i_j, i_k \in \{0, \dots, l\}$, and so the group $B(l)$ is naturally isomorphic with $A(l)$.

We have an epimorphism $\alpha_m: A \rightarrow B = B_m$ such that $\alpha_m(a_i) = b_{[i]}$ for all a_i 's, and, by the previous observation, this homomorphism maps the subgroup $A(l) \leq A$ isomorphically onto the subgroup $B(l) \leq B$ provided $l \leq m/2$.

Let H_m be the extension of $B = B_m$ by the automorphism of order $m: t_m^{-1} b_{[i]} t_m = b_{[i+1]}, t_m^m = 1$. We have $|H_m| = m|B_m|$, so H_m is a p -group. Then the epimorphism α_m extends to the epimorphism β_m of G onto H_m such that $\beta_m(t) = t_m$. The

intersection of kernels of arbitrary infinite family of homomorphisms β_m contains no nontrivial elements from $A(l)$, $l = 0, 1, 2, \dots$, and so it is trivial. Hence G is a residually (finite p -group). □

Lemma 3.24 *The groups $G(p, \mathbf{c})$ are limits of hyperbolic (in fact virtually free) groups satisfying all assumptions of Theorem 3.3, provided the sequence \mathbf{c} grows fast enough.*

Proof We chose a sequence $\mathbf{c} = (c_1, c_2, \dots)$ by induction.

Let $C_0 = *_{i=-\infty}^{\infty} \langle a_i \mid a_i^p = 1 \rangle$, $c_1 = 1$. Suppose that the integers $c_1 \leq \dots \leq c_n$ are already chosen. For every $l \leq n$, we denote by U_l the normal subgroup of C_0 generated (as a normal subgroup) by all commutators of the form

$$[\dots [a_{i_0}, a_{i_1}], \dots, a_{i_{c_l}}],$$

where $\max_{j,k} |i_j - i_k| \leq l$. Then we set $V_n = \prod_{l=1}^n U_l$ and $C_n = C_0/V_n$.

The map $a_i \rightarrow a_{i+1}$, $i = 0, 1, \dots$, extends to an automorphism of C_n . Let

$$G_n = \langle C_n, t \mid a_i^t = a_{i+1}, i \in \mathbb{Z} \rangle$$

be the corresponding extension by the automorphism. Clearly G_n is generated by $\{a_0, t\}$, and the set of defining relations of G_{n-1} is a subset of the set of defining relations of G_n (and the set of defining relations of G). Thus the identity map on this set induces epimorphisms

$$G_1 \rightarrow G_2 \rightarrow \dots \rightarrow G_{n-1} \rightarrow G_n \rightarrow G.$$

Observe that G_n splits as an HNN-extension of its subgroup generated by a_0, \dots, a_n . This subgroup is nilpotent of class at most c_n and generated by elements of order p , hence it is a finite p -group [23]. This implies that G_n is virtually free and so the Cayley graph of G_n corresponding to the generators t, a_0 is δ_n -hyperbolic for some δ_n .

Now we are going to explain that choosing $c_{n+1} \gg c_n$ we can always ensure the condition $\delta_n = o(t_n)$ from Theorem 3.3. By the results of Higman [24], C_n is residually (finite p -group), and so it is residually nilpotent.

Now observe that for every $c = c_{n+1}$, the image of $U_{n+1} = U_{n+1}(c)$ in C_n belongs to the $(c+1)$ -st term $\gamma_{c+1}(C_n)$ of the lower central series of C_n and since C_n is residually nilpotent we can make the natural homomorphism $C_n \rightarrow C_n/\gamma_{c+1}(C_n)$ (and, hence, the homomorphism $C_n \rightarrow C_{n+1}$) injective on any given finite subset by

choosing big enough c . Hence the homomorphism $G_n \rightarrow G_{n+1}$ can be made injective on the ball of radius $t_n = \exp(\delta_n)$, for example.

Finally we note that the set of relations of the direct limit group G coincides with the set of relations of the group $G(p, \mathbf{c})$ which is not virtually cyclic by [Lemma 3.23](#) (c). This completes the proof. \square

Combining [Theorem 3.3](#) and [Lemma 3.24](#), we obtain the following.

Theorem 3.25 *There exists a finitely generated elementary amenable group G and a scaling sequence $d = (d_n)$ such that G is not virtually cyclic and for any ultrafilter ω , the asymptotic cone $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree.*

Note that $G(p, \mathbf{c})$ from [Lemma 3.23](#) is clearly not finitely presented (because it is not virtually free and is a direct limit of virtually free groups). Hence it has a non-simply connected asymptotic cone [[11](#)]. Thus we obtain the following corollary.

Corollary 3.26 *There is a finitely generated elementary amenable group having at least two nonhomeomorphic asymptotic cones.*

4 Graded small cancellation and circle-tree asymptotic cones

Recall the definition of circle-trees.

Definition 4.1 We say that a metric space X is a *circle-tree*, if X is tree graded with respect to a collection of circles (with the standard length metric) whose radii are uniformly bounded from below and from above by positive constants. In particular, every circle-tree is locally isometric to an \mathbb{R} -tree.

Note that by [Lemma 3.1](#), any group having a circle-tree asymptotic cone is lacunary hyperbolic. In this Section, we shall show that the class of groups all of whose asymptotic cones are trees or circle-trees is very large and contains all groups given by presentations satisfying certain small cancellation conditions. As a consequence, this class contains groups all of whose proper subgroups are cyclic or finite.

4.1 The Greendlinger Lemma for small cancellation presentations over hyperbolic groups

Let H be a group generated by a finite set S . We will consider quotient groups of H as groups given by presentations over H (ie presentations including all relations of H plus some extra relations). Our goal is to generalize Definition 3.6 for such presentations.

We start with a definition of a piece. In what follows we write $U \equiv V$ for two words U and V if U and V are equal letter-by-letter.

Definition 4.2 Let H be a group generated by a set S . Let \mathcal{R} be a symmetrized set of reduced words in $S^{\pm 1}$. For $\varepsilon > 0$, a subword U of a word $R \in \mathcal{R}$ is called a ε -piece if there exists a word $R' \in \mathcal{R}$ such that

- (1) $R \equiv UV, R' \equiv U'V'$, for some V, U', V' ;
- (2) $U' = YUZ$ in H for some words Y, Z such that $\max\{|Y|, |Z|\} \leq \varepsilon$;
- (3) $YRY^{-1} \neq R'$ in the group H .

Note that if U is an ε -piece, then U' is an ε -piece as well.

Recall that a word W in the alphabet $S^{\pm 1}$ is called (λ, c) -quasigeodesic (respectively geodesic) in H if any path in $\Gamma(H, S)$ labeled by W is (λ, c) -quasigeodesic (respectively geodesic).

Definition 4.3 Let $\varepsilon \geq 0, \mu \in (0, 1)$, and $\rho > 0$. We say that a symmetrized set \mathcal{R} of words over the alphabet $S^{\pm 1}$ satisfies the condition $C(\varepsilon, \mu, \rho)$ for the group H , if

- (C_1) all words from \mathcal{R} are geodesic in H ;
- (C_2) $|R| \geq \rho$ for any $R \in \mathcal{R}$;
- (C_3) the length of any ε -piece contained in any word $R \in \mathcal{R}$ is smaller than $\mu|R|$.

Suppose now that H is a group defined by

$$(7) \quad H = \langle S \mid \mathcal{O} \rangle,$$

where \mathcal{O} is the set of all relators (not only defining) of H . Given a symmetrized set of words \mathcal{R} , we consider the quotient group

$$(8) \quad H_1 = \langle H \mid \mathcal{R} \rangle = \langle S \mid \mathcal{O} \cup \mathcal{R} \rangle.$$

A cell in a van Kampen diagram over (8) is called an \mathcal{R} -cell (respectively, an \mathcal{O} -cell) if its boundary label is a word from \mathcal{R} (respectively, \mathcal{O}). We always consider van

Kampen diagrams over (8) up to some natural elementary transformations. For example we do not distinguish diagrams if one can be obtained from the other by joining two distinct \mathcal{O} -cells having a common edge or by the inverse transformation, etc. (see Ol'shanskii [42, Section 5] for details).

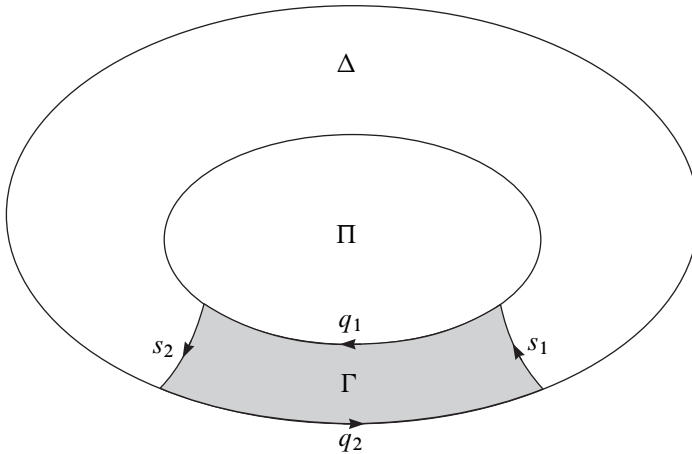


Figure 2: A contiguity subdiagram

Let Δ be a van Kampen diagram over (8), q a subpath of its boundary $\partial\Delta$, Π , Π' some \mathcal{R} -cells of Δ . Suppose that there is a simple closed path $p = s_1q_1s_2q_2$ in Δ , where q_1 (respectively q_2) is a subpath of the boundary $\partial\Pi$ (respectively q or $\partial\Pi'$) and $\max\{|s_1|, |s_2|\} \leq \varepsilon$ for some constant ε . By Γ we denote the subdiagram of Δ bounded by p . If Γ contains no \mathcal{R} -cells, we say that Γ is an ε -contiguity subdiagram of Π to the part q of $\partial\Delta$ or Π' respectively (Figure 2). The subpaths q_1 and q_2 are called *contiguity arcs* of Γ and the ratio $|q_1|/|\partial\Pi|$ is called the *contiguity degree* of Π to $\partial\Delta$ (or Π') and is denoted by $(\Pi, \Gamma, \partial\Delta)$ (or (Π, Γ, Π')).

The following easy observation will often be useful.

Lemma 4.4 *Suppose that the group H is hyperbolic. Let \mathcal{R} be a set of geodesic in H words, Δ a diagram over (8), and q a subpath of $\partial\Delta$ whose label is geodesic in H_1 . Then for any $\varepsilon \geq 0$, no \mathcal{R} -cell Π in Δ have an ε -contiguity subdiagram Γ to q such that $(\Pi, \Gamma, q) > 1/2 + 2\varepsilon/|\partial\Pi|$.*

Proof Let Γ be an ε -contiguity subdiagram of an \mathcal{R} -cell Π to q , $\partial\Gamma = s_1q_1s_2q_2$, where q_2 is a subpath of q and $\partial\Pi = q_1r$. Let also γ denote the contiguity degree (Π, Γ, q) . Then we have $|q_2| \geq |q_1| - |s_1| - |s_2| = \gamma|\partial\Pi| - 2\varepsilon$ since $\text{Lab}(q_1)$ is geodesic in H . On the other hand, $|q_2| \leq |r| + |s_1| + |s_2| \leq (1 - \gamma)|\partial\Pi| + 2\varepsilon$ as $\text{Lab}(q_2)$ is geodesic in H . These two inequalities yield $\gamma \leq 1/2 + 2\varepsilon/|\partial\Pi|$. \square

Given a van Kampen diagram Δ over (8), we call a combinatorial map from the 1-skeleton $\text{Sk}^{(1)}(\Delta)$ to the Cayley graph $\Gamma(H_1, S)$ *natural* if it preserves labels and orientation of edges. The following easy observation will be useful.

Lemma 4.5 *Suppose that H is hyperbolic. Let \mathcal{R} be a symmetrized set of words in $S^{\pm 1}$ satisfying the condition $C(\varepsilon, \mu, \rho)$ for some $\varepsilon \geq 0$, $\mu \in (0, 1)$, $\rho > 0$. Suppose that Π, Π' are two \mathcal{R} -cells in a diagram Δ over (8) and Γ is an ε -contiguity subdiagram of Π' to Π such that $\gamma = (\Pi', \Gamma, \Pi) \geq \mu$. Then*

$$(9) \quad ||\partial\Pi| - |\partial\Pi'| || \leq 2\varepsilon$$

and for any natural map $\phi: \text{Sk}^{(1)}(\Delta) \rightarrow \Gamma(H_1, S)$, the Hausdorff distance between $\phi(\partial\Pi)$ and $\phi(\partial\Pi')$ does not exceed $2\varepsilon + 2\delta$.

Proof Let $\partial\Pi = uv$, $(\partial\Pi')^{-1} = u'v'$, and $\partial\Gamma = yuz(u')^{-1}$. Since $\text{Lab}(yuz) = \text{Lab}(u')$ in H , we have

$$(10) \quad \text{Lab}(y)\text{Lab}(uv)\text{Lab}(y)^{-1} = \text{Lab}(u'v')$$

in H by the $C(\varepsilon, \mu, \rho)$ -condition. As labels of $\partial\Pi$ and $\partial\Pi'$ are geodesic in H , (10) implies (9).

Further let $Q = abcd$ be a quadrangle in $\Gamma(H, S)$ such that a, b, c , and d are labeled by $\text{Lab}(y)$, $\text{Lab}(uv)$, $\text{Lab}(y)^{-1}$, and $(\text{Lab}(u'v'))^{-1}$, respectively. By the first assertion of Lemma 2.9, b and d belong to the closed $(\varepsilon + 2\delta)$ -neighborhoods of each other. To finish the proof it remains to note that the map $\Gamma(H, S) \rightarrow \Gamma(H_1, S)$ induced by the homomorphism $H \rightarrow H_1$ does not increase the distance. \square

We call a (disc) van Kampen diagram over (8) *minimal* if it has minimal number of \mathcal{R} -cells among all disc diagrams with the same boundary label. The first part of the following result is an analog of the Greendlinger Lemma 3.7 for presentations over hyperbolic groups.

Lemma 4.6 *Suppose that H is a δ -hyperbolic group having presentation $\langle S \mid \mathcal{O} \rangle$ as in (7), $\varepsilon \geq 2\delta$, $0 < \mu \leq 0.01$, and ρ is large enough (it suffices to choose $\rho > 10^6\varepsilon/\mu$). Let H_1 be given by a presentation*

$$H_1 = \langle H \mid \mathcal{R} \rangle = \langle S \mid \mathcal{O} \cup \mathcal{R} \rangle$$

as in (8) where \mathcal{R} is a finite symmetrized set of words in $S^{\pm 1}$ satisfying the $C(\varepsilon, \mu, \rho)$ -condition. Then the following statements hold.

- (1) Let Δ be a minimal disc diagram over (8). Suppose that $\partial\Delta = q^1 \cdots q^t$, where the labels of q^1, \dots, q^t are geodesic in H and $t \leq 12$. Then, provided Δ has an \mathcal{R} -cell, there exists an \mathcal{R} -cell Π in Δ and disjoint ε -contiguity subdiagrams $\Gamma_1, \dots, \Gamma_t$ (some of them may be absent) of Π to q^1, \dots, q^t respectively such that

$$(\Pi, \Gamma_1, q^1) + \cdots + (\Pi, \Gamma_t, q^t) > 1 - 23\mu.$$

- (2) H_1 is a δ_1 -hyperbolic group with $\delta_1 \leq 4r$ where $r = \max\{|R| \mid R \in \mathcal{R}\}$.

Proof The first part of the lemma is essentially a special case of [42, Lemma 6.6] where the parameters λ and c of the quasigeodesicity of the defining words are equal to 1 and 0, respectively because the words in \mathcal{R} are geodesic in H . The minor corrections in the argument of [42] leading to this special case are the following.

We replace $t \leq 4$ by $t \leq 12$. Note that the proof from [42] works even in the case $t \leq k$, for any fixed k , but then we should replace 23 by $C = C(k)$ in the formulation of the lemma.

As in the proof from [42], we need to consider geodesic quadrangles $s_1q_1s_2q_2$ with “short” sides s_1, s_2 and “long” sides q_1, q_2 . Then if a point $o \in q_1$ is far from the ends of q_1 , say, $\min(\text{dist}(o, q_-), \text{dist}(o, q_+)) \geq \max(|s_1|, |s_2|) + 2\delta$ then in [42], it is proved that the distance from o to q_2 is bounded from above by $c_1 = 13\delta$. In our case, by Lemma 2.9, we can take $c_1 = 2\delta$.

Finally, in [42, Lemma 6.2], we can replace the upper estimate $n\sqrt{\rho}$ by $n\mu\rho$ (this is possible because ρ is large enough).

The proof of the second statement of the lemma is divided into two steps.

Step 1 First we consider a minimal diagram Γ over H_1 whose contour contains a subpath pq where segments p and q are geodesic in Γ . Assume that there is an \mathcal{R} -cell Π with two contiguity subdiagrams Γ_p and Γ_q to p and q , respectively, and with contiguity arcs $v_p \subseteq p$ and $v_q \subseteq q$, such that $(v_p)_- = p_-$ and $(v_q)_+ = q_+$. We claim that the Hausdorff distance between the images \bar{p} and \bar{q} of p and q in the Cayley graph of H_1 does not exceed $2r$. We may assume that the boundary path of Γ is the product of pq , an arc of Π , and two side arcs of Γ_p and Γ_q (see Figure 3).

Let $v_p x u y$ be the boundary path of Γ_p where u is the contiguity arc of Γ_p lying on $\partial\Pi$. Since v_p is geodesic we have $|v_p| \leq |x| + |y| + |u| < 2\varepsilon + r$. Therefore, using Π and the contiguity subdiagrams, we can connect every point of v_p (of v_q) to q (to p) with a path of length $\leq |v_p|/2 + 2\varepsilon + r < 2r$ since, according to the first statement of the lemma, ρ is chosen much greater than ε .

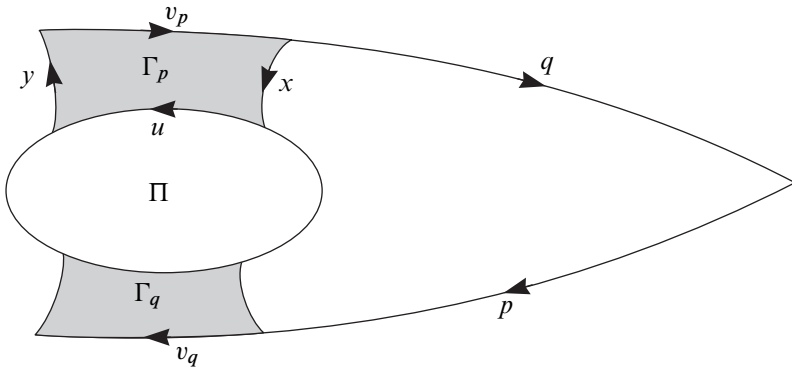


Figure 3: The diagram Γ

Then we consider a maximal set of \mathcal{R} -cells Π, Π', Π'', \dots in Γ having disjoint contiguity subdiagrams $\Gamma_p, \Gamma_q, \Gamma'_p, \Gamma'_q, \dots$ to both p and q . After suitable enumeration, they provide us with decompositions $p = v_p w_p v'_p w'_p \dots$ and $q = \dots w'_q v'_q w_q v_q$ where v_p, v'_p, v''_p, \dots and v_q, v'_q, v''_q, \dots are the contiguity arcs of the above contiguity subdiagrams. As in the previous paragraph, we have that the distance between every point of v'_p, v''_p, \dots (of v'_q, v''_q, \dots) and q (and p) is less than $2r$. Thus it suffices to obtain the same estimate for the distance between a point of one of w_p, w'_p, \dots and q . (More precisely, it suffices to do this with the images $\bar{w}_p, \dots, \bar{q}$ of these paths in the Cayley graph of H_1 .)

For example, w'_p is a section of a loop $w'_p abcw'_q def$ where b and e are arcs of Π' and Π'' , respectively, and a, c, d, f are geodesics of length at most ε . Denote by Ξ the subdiagram bounded by this octagon. If Ξ is a diagram over H , then by Lemma 2.9, every point of \bar{w}'_p is at distance at most 6δ from the union of the remaining 7 sides. Since $\max(|b|, |e|) \leq r$, we have that the distance between a point of w'_p and w'_q is at most $6\delta + 2\varepsilon + r < 2r$ since the parameter ρ is chosen so that $\rho > 6\delta + 2\varepsilon$.

Thus to complete Step 1, it suffices to show that Ξ contains no \mathcal{R} -cells. Arguing by contradiction, we have an \mathcal{R} -cell π and its contiguity subdiagrams $\Gamma_1, \dots, \Gamma_8$ (some of them may be absent) to w'_p, a, \dots, f , respectively, with

$$(\pi, \Gamma_1, w'_p) + \dots + (\pi, \Gamma_8, f) > 1 - 23\mu$$

by the first assertion of the lemma. Here the contiguity degree to b and e are less than μ by the $C(\varepsilon, \mu, \rho)$ -condition. Since the lengths of a, c, d, f are less than $\varepsilon \ll \rho$, the contiguity degree of π to each of these four boundary sections of Ξ is less than $\mu/2$. (The accurate proof of the latter inequality is given in [42, Lemma 6.5(a)].) Hence $(\pi, \Gamma_1, w'_p) + (\pi, \Gamma_5, w'_q) > 1 - 27\mu$.

If, for example, Γ_5 is absent, then w'_p is homotopic in Γ to the path $x'zy'$, where x' and y' are side arcs of contiguity subdiagram and z an arc on the boundary $\partial\pi$ with length $< 27\mu|\partial\pi|$. Since w'_p is geodesic, we have $|w'_p| < 2\varepsilon + 27\mu|\partial\pi|$. On the other hand, w'_p is homotopic to $x'u'y'$ in the diagram Γ_p over H where u' is the arc of π of length at least $(1 - 27\mu)|\partial\pi|$. Therefore $|w'_p| + 2\varepsilon > (1 - 27\mu)|\partial\pi|$. Since $|\partial\pi| \geq \rho$ we obtain $(1 - 54\mu)\rho < 4\varepsilon$ that contradicts $\rho > 10^6\mu^{-1}\varepsilon$.

Thus both Γ_1 and Γ_5 are present, and so the cell π can be added to the set Π, Π', \dots contrary the maximality. This contradiction completes Step 1.

Step 2 Assume that $\delta_1 > 4r$. Then, by Rips' definitions of hyperbolicity (see Gromov [20, 6.6]) there exists a geodesic triangle xyz in the Cayley graph of H_1 such that

$$(\star) \left\{ \begin{array}{l} \text{for arbitrary three points } o_1, o_2, o_3 \text{ chosen on the sides } x, y, \text{ and } z, \\ \text{respectively, we have} \\ \max(\text{dist}(o_1, o_2), \text{dist}(o_2, o_3), \text{dist}(o_3, o_1)) \geq \delta_1 > 4r. \end{array} \right.$$

Let Δ be a minimal diagram over H_1 corresponding to the triangle. We preserve the notation xyz for the boundary of Δ . A subdiagram Γ is said to be an xy -corner of Δ if for some subpaths p of x and q of y such that $p_+ = q_-$ (ie, pq is a subpath of xy), it contains a cell Π with two contiguity subdiagrams Γ_p and Γ_q satisfying the conditions of Step 1, and it is bounded by p, q, Π, Γ_p and Γ_q as in the first paragraph of Step 1. The xy -corner is called maximal, if the sum $|p| + |q|$ is maximal. By definition, it consists of the single vertex $x_+ = y_-$ if there exists no Π as above. Similarly we define yz - and zx -corners.

In this notation, we suppose that the xy -corner $\Gamma = \Gamma_{xy}$ is maximal, x contains the subsegment p , and let p' be a similar segment of a maximal zx -corner Γ_{zx} where p' also lies on x . If p and p' together cover x , then every point of x belongs to the $2r$ -neighborhood of the union of two other sides of the triangle xyz . This implies that there is a point o_1 on x and two points o_2 and o_3 on y and z , respectively, such that $\max(\text{dist}(o_1, o_2), \text{dist}(o_1, o_3)) \leq 2r$, and this contradicts Condition (\star) . Hence p and p' must be disjoint. It follows that all three maximal corners Γ_{xy}, Γ_{yz} and Γ_{zx} can be chosen pairwise disjoint(see Figure 4).

Now consider the diagram Δ' obtained from Δ by cutting off these three corners. The contour of Δ' is of the form $x'a'y'bz'c$ where $x', y',$ and z' belong in $x, y,$ and z , respectively. In turn, $a = a_1a_2a_3$ where a_2 is an arc of the cell Π from the definition of corner, and a_1, a_3 are side arcs of Γ_p and Γ_q . Similarly, $b = b_1b_2b_3$ and $c = c_1c_2c_3$. Thus we have a decomposition of $\partial\Delta'$ in 12 sections. Note that $\sum_{i=1}^3 (|a_i| + |b_i| + |c_i|) \leq 6\varepsilon + 3r$.

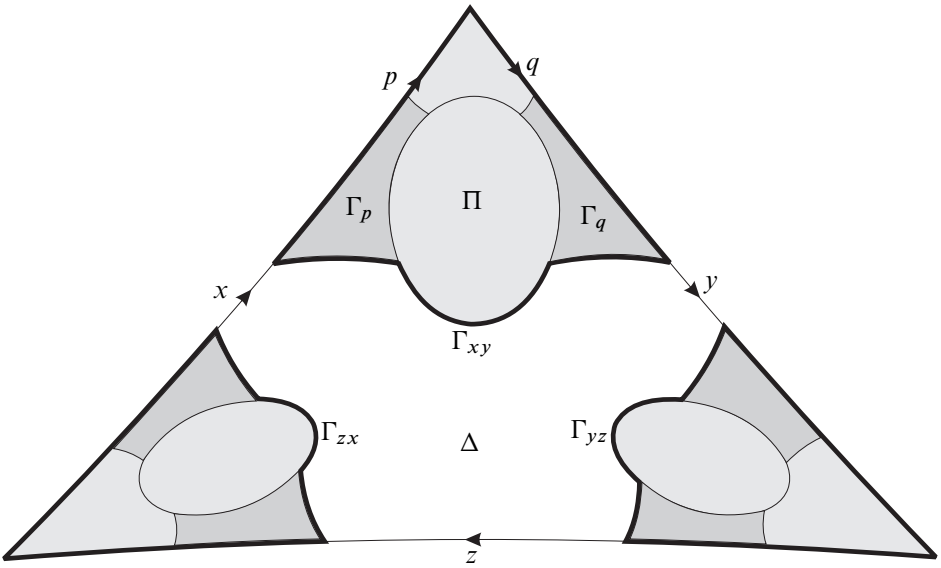


Figure 4: The corners Γ_{xy}, Γ_{yz} and Γ_{zx} in Δ

If Δ' is a diagram over H , then by Lemma 2.9, x' belongs to the 10δ -neighborhood of the union of the remaining 11 sides of the 12-gon, and so x' belongs to the $(10\delta + 6\varepsilon + 3r)$ -neighborhood of $y \cup z$. Since $10\delta + 6\varepsilon < \rho \leq r$, every point of x' belongs to the $4r$ -neighborhood of $y \cup z$. Recall that every point of $x \setminus x'$ is in $p \cup p'$ and belongs in the $2r$ -neighborhood of $y \cup z$. Similarly, every point of y or z belongs in $4r$ -neighborhood of the union of two other sides, which contradicts to the choice of the triangle xyz .

Then we may assume that Δ' has an \mathcal{R} -cell, and so, by the first assertion of the lemma, it has a cell π with contiguity subdiagrams $\Gamma_{x'}, \Gamma_{a_1}, \dots$ to the 12 sections, such that the sum of their contiguity degrees is greater than $1 - 23\mu$. As at the Step 1, the sum of contiguity degrees for all a_i, b_i, c_i ($i = 1, 2, 3$) is less than 6μ . So the sum of the contiguity degrees to x', y' , and z' is greater than $1 - 29\mu$. It is impossible that two of these three subdiagrams be absent because the degree of contiguity of π to a geodesic section cannot be as high as $1 - 29\mu$ by the argument of Step 1 with coefficient 27 replaced by 29.

Therefore we may assume that there are contiguity subdiagrams of π to both x' and y' . But this contradicts the maximality of the corner Γ_{xy} , and the lemma is completely proved. □

4.2 Comparing various small cancellation conditions

The purpose of this section is to compare the small cancellation condition introduced in Section 4.1 with the classical small cancellation condition.

Lemma 4.7 *Let R_1, R_2, \dots be an infinite set of distinct cyclically reduced words in a finite alphabet S , \mathcal{R}_i the set of all cyclic shifts of $R_i^{\pm 1}$, and $\mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{R}_i$. Assume that*

(SC₁) *the set \mathcal{R} satisfies $C'(\lambda)$ for some $\lambda \leq 1/10$;*

(SC₂) *the set \mathcal{R}_n ($n = 1, 2, \dots$) satisfies $C'(\lambda_n)$ where $\lambda_n \rightarrow 0$.*

Let also $H = \langle S \mid R_{i_1}, \dots, R_{i_j} \rangle$ for some $i_1 < \dots < i_j$. Then, for every $\mu > 0$, $\varepsilon \geq 0$, and $\rho > 0$, there is $n > i_j$ such that the set \mathcal{R}_n satisfies the $C(\varepsilon, \mu, \rho)$ -condition over H .

Proof We first prove that any subword V of a cyclic shift of $R_n^{\pm 1}$ is geodesic in the group H . Let U be a geodesic word equal to V in H . Notice that V has no subword W which is also a subword of one of the words R from $\mathcal{T} = \mathcal{R}_{i_1} \cup \dots \cup \mathcal{R}_{i_j}$ with $|V| \geq \lambda|R|$ by Condition (SC₁). Then by Lemma 3.7, either $U \equiv V$, and we are done, or there is a subword W of U which is also a prefix of a word R from \mathcal{T} with $|W| > (1 - 3\lambda - 2\lambda)|R| \geq |R|/2$. Therefore the subword W is equal in H to a shorter word T^{-1} , where $R \equiv WT = 1$ in H . But then U is not geodesic, a contradiction.

Now, it remains to show that a word from \mathcal{R}_n has no ε -pieces of length at least $\mu|R_n|$ if n is large enough. We may further assume by (SC₂) that $\lambda_n < \mu/2$ and $|R_n| > \mu^{-1} \max(|R_{i_j}|, 4\varepsilon)$. (Observe that (SC₂) implies $|R_n| \rightarrow \infty$ and $n \rightarrow \infty$.)

Assume that $R, R' \in \mathcal{R}_n$, $R \equiv UV$, $R' \equiv U'V'$, $|U| \geq \mu|R_n|$, and $U' = YUZ$ in H for some geodesic in H words Y, Z with $\max(|Y|, |Z|) \leq \varepsilon$. We must prove that $YRY^{-1} = R'$ in H .

Consider a diagram Δ over H with boundary path $p_1q_1p_2q_2$, where $\text{Lab}(p_1) \equiv Y$, $\text{Lab}(q_1) \equiv U$, $\text{Lab}(q_2) \equiv Z$, and $\text{Lab}(p_2) \equiv (U')^{-1}$.

If Δ has no cells then there is a common subpath t of q_1 and q_2^{-1} with $|t| \geq \lambda_n|R_n|$ because

$$|p_1| + |p_2| \leq 2\varepsilon < \mu|R_n|/2 < (\mu - \lambda_n)|R_n|,$$

and every edge of q_1 belong to the path $p_2q_2p_1$. From the small cancellation condition (SC₁), we see that $T \equiv \text{Lab}(t)$ is a prefix of a unique word R_0 from \mathcal{R}_n . Since $U \equiv X_1TX_2$ and $U' \equiv X_3TX_4$ for some words X_1, X_2, X_3, X_4 , we obtain that the conjugates $X_1^{-1}RX_1$ and $X_3^{-1}R'X_3$ are both freely equal to R_0 , ie

$R = X_1 X_3^{-1} R' (X_1 X_3^{-1})^{-1}$. Since $YX_1 X_3^{-1}$ is a label of a closed path in Δ , we have $YRY^{-1} = (YX_1 X_3^{-1}) R' (YX_1 X_3^{-1})^{-1} = R'$ in H , as required.

Now assume by contradiction that Δ has a cell. No cell Π has a boundary arc with length $> |\partial\Pi|/2$ contained in p_1 or in p_2 because these segments are geodesic. Since q_1 is a geodesic path by (1), no cell Π can have vertices in both p_1 and p_2 because otherwise

$$\mu|R_n| \leq |q_1| \leq |p_1| + |\partial\Pi|/2 + |p_2| \leq 2\varepsilon + |R_{i_j}|/2 \leq \max(|R_{i_j}|, 4\varepsilon)$$

contrary to the choice of n . Then it follows from Lemma 3.7, that there is a cell Π in Δ , whose boundary has a common subpath t with either q_1 or q_2 , where $|t| > \frac{1}{2}(1 - 3\lambda - 1/2)|\partial\Pi| \geq \lambda|\partial\Pi|$. But this contradicts Condition (SC_1) because $n \notin \{i_1, \dots, i_j\}$. □

Remark 4.8 Infinite sets of words in the alphabet $\{a, b\}$ satisfying (SC_1) and (SC_2) were constructed in various places (see, for example, Erschler and Osin [15]). Moreover, one can find such sets satisfying (SC_1) for arbitrary small $\lambda > 0$.

Our next goal is to relate $C(\varepsilon, \mu, \rho)$ to the small cancellation condition $C(\varepsilon, \mu, \lambda, c, \rho)$ introduced in [42]. Essentially, this condition is obtained from $C(\varepsilon, \mu, \rho)$ by replacing the word “geodesic” by “quasigeodesic”; the additional parameters are the quasigeodesic parameters. We shall show that any $C(\varepsilon, \mu, \lambda, c, \rho)$ -presentation can be transformed into a $C(\varepsilon', \mu', \rho')$ -presentation of the same group for suitable $\varepsilon', \mu', \rho'$.

Definition 4.9 Let H be a hyperbolic group generated by a finite set S . A symmetrized set \mathcal{R} of words in $S^{\pm 1}$ satisfies the condition $C(\varepsilon, \mu, \lambda, c, \rho)$ for some constants $\varepsilon > 0, \mu \in (0, 1), \lambda \in (0, 1], c \geq 0, \rho > 0$, if all words in \mathcal{R} are (λ, c) -quasigeodesic and conditions $(C_2), (C_3)$ from Definition 4.3 hold. Thus $C(\varepsilon, \mu, \rho)$ is equivalent to $C(\varepsilon, 1, 0, \mu, \rho)$.

In the two lemmas below the following notation is used. Let \mathcal{R}_0 be a set of words in $S^{\pm 1}$, \mathcal{R}'_0 the set obtained from \mathcal{R}_0 by replacing each $R \in \mathcal{R}_0$ with a shortest word R' such that R and R' are conjugate in H . Denote by \mathcal{R} (respectively \mathcal{R}') the set of all cyclic shifts of words from $\mathcal{R}_0^{\pm 1}$ (respectively $(\mathcal{R}'_0)^{\pm 1}$). Note that all words in \mathcal{R}' are geodesic in H .

Given $\lambda > 0, c \geq 0$, we define

$$\kappa = 8\delta + 4\theta,$$

where δ is the hyperbolicity constant of the Cayley graph $\Gamma(H, S)$, and $\theta = \theta(\lambda, c)$ is the constant from Lemma 2.8.

Lemma 4.10 Suppose that all words in \mathcal{R} are (λ, c) -quasigeodesic in H for some λ, c and have lengths at least $(\kappa + c)/\lambda$. Then for every $W \in \mathcal{R}'$, there is $R_W \in \mathcal{R}$ such that W and R_W are conjugate by an element of length at most $\kappa/2$ in H .

Proof Any word $W \in \mathcal{R}'$ is conjugate to some word $R \in \mathcal{R}^{\pm 1}$ in H . Let $T_{W,R}$ denote a shortest word conjugating W to R in H . Let also U and S be cyclic shifts of W and R respectively such that

$$(11) \quad |T(U, S)| \leq |T(U', S')|$$

for any cyclic shifts U' and S' of W and R .

There is a 4-gon $asbu^{-1}$ in $\Gamma(H, S)$ such that $\text{Lab}(u) \equiv U$, $\text{Lab}(s) \equiv S$, $\text{Lab}(a) \equiv \text{Lab}(b^{-1}) \equiv T(U, S)$. Clearly s is (λ, c) -quasigeodesic, and a, u, b are geodesic. Moreover

$$(12) \quad \text{dist}(u, s) \geq |a|$$

by (11). There are two cases to consider.

Case 1 If $|a| \leq 2\delta + \theta$, then by Lemma 2.8 and Lemma 2.9, u and s belong to the closed $(4\delta + 2\theta)$ -neighborhoods of each other. Therefore, W is conjugate to a cyclic shift of R by a word of length at most $4\delta + 2\theta$.

Case 2 Now assume that $|a| > 2\delta + \theta$. In particular, $\text{dist}(u, s) > 2\delta + \theta$ by (12). Consider the middle point m of s . Lemma 2.8 and Lemma 2.9 imply that $\text{dist}(m, a \cup b) \leq 2\delta + \theta$. For definiteness, assume that $\text{dist}(m, a) \leq 2\delta + \theta$. Let z be the point on a such that $\text{dist}(m, a) = \text{dist}(m, z)$. Then

$$\text{dist}(z, a_-) \geq \text{dist}(m, a_-) - 2\delta - \theta \geq \text{dist}(s, u) - 2\delta - \theta \geq |a| - 2\delta - \theta$$

by (12). Therefore,

$$\text{dist}(z, s_-) \leq |a| - \text{dist}(z, a_-) \leq 2\delta + \theta.$$

Consequently,

$$\text{dist}(m, s_-) \leq \text{dist}(m, z) + \text{dist}(z, s_-) \leq 2(2\delta + \theta).$$

This means that a cyclic shift of R represents an element of length at most $4(2\delta + \theta)$ in H . Hence $|R| \leq \lambda^{-1}(8\delta + 4\theta + c)$ that contradicts our assumption. Hence this case is impossible. \square

Lemma 4.11 Suppose that \mathcal{R} satisfies $C(\varepsilon, \mu, \lambda, c, \rho)$ for some $\varepsilon > 0$, $\mu \in (0, 1)$, $\lambda \in (0, 1]$, $c > 0$, and

$$(13) \quad \rho > 2(c + 3\kappa)/\lambda.$$

Then \mathcal{R}' satisfies $C(\varepsilon', \mu', \rho')$ for

$$(14) \quad \varepsilon' = \varepsilon - 2\kappa, \quad \mu' = 2\mu/\lambda, \quad \rho' = \lambda\rho - c - \kappa.$$

Proof By (13) and Lemma 4.10 for any word $W \in \mathcal{R}'$, there is a word $R_W \in \mathcal{R}$ that is conjugate to W by a word of length at most $\kappa/2$ in H . In particular, this yields the last inequality in (14).

Suppose now that for some words W_1, W_2 of \mathcal{R}' , we have $W_1 \equiv U_1 V_1$, $W_2 \equiv U_2 V_2$, and $U_1 = Y U_2 Z$ in H , where $|Y|, |Z| \leq \varepsilon'$ and $|U_1| \geq \mu'|W_1|$. Let A_1, A_2 be words of lengths at most $\kappa/2$ such that

$$(15) \quad R_{W_i} = A_i W_i A_i^{-1}, \quad i = 1, 2$$

in H . Using Lemma 2.8 and Lemma 2.9, we can find initial subwords C_i of R_{W_i} , $i = 1, 2$, such that

$$(16) \quad C_i = A_i U_i B_i, \quad i = 1, 2$$

in H , where

$$(17) \quad |B_i| \leq \theta + 2\delta + \kappa/2 < \kappa.$$

Thus $C_1 = A_1 Y A_2^{-1} C_2 B_2^{-1} Z B_1$ in H . Note that

$$\max\{|B_2^{-1} Z B_1|, |A_1 Y A_2^{-1}|\} \leq \varepsilon' + 2\kappa = \varepsilon.$$

Now using subsequently (16), (17), (15), and (13) we obtain

$$\begin{aligned} |C_1| &\geq |U_1| - |A_1| - |B_1| \geq \mu'|W_1| - 2\kappa \geq \mu'(\lambda|R_{W_1}| - c - \kappa) - 2\kappa \\ &\geq 2\mu|R_{W_1}| - 2\mu(c + \kappa)/\lambda - 2\kappa \geq \mu|R_{W_1}|. \end{aligned}$$

Hence
$$R_{W_1} = (A_1 Y A_2^{-1}) R_{W_2} (A_1 Y A_2^{-1})^{-1}$$

in H by the $C(\varepsilon, \lambda, c, \mu, \rho)$ -condition. Combining this with (15), we obtain $W_1 = Y W_2 Y^{-1}$. □

4.3 Groups with circle-tree asymptotic cones

In this section we define the graded small cancellation condition and prove that all asymptotic cones of any group given by a graded small cancellation presentation are circle-trees.

In the next two sections we show that many classical small cancellation groups as well as some “monsters” obtained by methods from [42] admit graded small cancellation presentations.

Definition 4.12 Let α, K be positive numbers. We say that the presentation

$$(18) \quad \langle S \mid \mathcal{R} \rangle = \left\langle S \mid \bigcup_{i=0}^{\infty} \mathcal{R}_i \right\rangle$$

of a group G is a $\mathcal{Q}(\alpha, K)$ -presentation if the following conditions hold for some sequences $\varepsilon = (\varepsilon_n)$, $\mu = (\mu_n)$, and $\rho = (\rho_n)$ of positive real numbers ($n = 1, 2, \dots$).

- (Q₀) The group $G_0 = \langle S \mid \mathcal{R}_0 \rangle$ is δ_0 -hyperbolic for some δ_0 .
- (Q₁) For every $n \geq 1$, \mathcal{R}_n satisfies $C(\varepsilon_n, \mu_n, \rho_n)$ over $G_{n-1} = \langle S \mid \bigcup_{i=0}^{n-1} \mathcal{R}_i \rangle$.
- (Q₂) $\mu_n = o(1)$, $\mu_n \leq \alpha$, and $\mu_n \rho_n > K \varepsilon_n$ for any $n \geq 1$.
- (Q₃) $\varepsilon_{n+1} > 8 \max\{|R|, R \in \mathcal{R}_n\} = O(\rho_n)$.

The following lemma shows that if α is small enough and K is big enough, $\mathcal{Q}(\alpha, K)$ -presentations have properties resembling the properties of ordinary small cancellation presentations.

Lemma 4.13 Let (18) be a $\mathcal{Q}(.01, 10^6)$ -presentation. Then the following conditions hold.

- (a) For every $n \geq 1$, Lemma 4.6 applies to $H = G_{n-1}$ and $H_1 = G_n = \langle H \mid \mathcal{R}_n \rangle$. In particular, G_n is δ_n -hyperbolic, where $\delta_n \leq 4 \max_{R \in \mathcal{R}_n} |R|$.
- (b) $\varepsilon_n = o(\rho_n)$.
- (c) $\rho_n = o(\rho_{n+1})$; in particular, $\rho_n \rightarrow \infty$ as $n \rightarrow \infty$ and $\delta_n = o(\rho_{n+1})$.
- (d) $\rho_n = o(r_S(G_n \rightarrow G_{n+1}))$, where r_S is the injectivity radius.

Proof The first assertion easily follows from (Q₀)–(Q₂) by induction. Assertions (b) and (c) follow immediately from (Q₂) and (Q₃). Finally given $g \in \text{Ker}(G_n \rightarrow G_{n+1})$, $g \neq 1$, we consider a geodesic (in G_n) word W representing g and a minimal van Kampen diagram over G_{n+1} with boundary label W . Applying Lemma 4.6 for $r = 1$ and taking into account that words in \mathcal{R}_{n+1} are geodesic in G_n , we obtain $|W| > (1 - 23\mu_{n+1})|\partial\Pi| - 2\varepsilon_n$, where Π is an \mathcal{R}_{n+1} -cell provided by Lemma 4.6. Hence $|g| \geq (1 - o(1))\rho_{n+1}$. Combining this with (c) we obtain (d). \square

Definition 4.14 From now on the condition $\mathcal{Q} = \mathcal{Q}(.01, 10^6)$ will be called the *graded small cancellation condition*.

The following statement is an immediate corollary of [Lemma 4.7](#). It shows, in particular, that the class of groups admitting graded small cancellation presentations is large.

Corollary 4.15 *Let R_1, R_2, \dots be an infinite set of distinct cyclically reduced words in a finite alphabet S , \mathcal{R}_i the set of all cyclic shifts of $R_i^{\pm 1}$, and $\mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{R}_i$. Assume that conditions (SC_1) for $\lambda = 1/100$ and (SC_2) of [Lemma 4.7](#) are satisfied. Then there is an infinite sequence $i_1 < i_2 < \dots$ such that $\langle S \mid \bigcup_{j=1}^{\infty} \mathcal{R}_{i_j} \rangle$ satisfies the graded small cancellation condition.*

Definition 4.16 Given an ultrafilter ω and a scaling sequence $d = (d_n)$, we say that a sequence of real numbers $f = (f_n)$ is (ω, d) -visible if there exists a subsequence (f_{n_i}) of f such that $f_{n_i} = \Theta_{\omega}(d_i)$.

Theorem 4.17 *For any group G having a graded small cancellation presentation, any ultrafilter ω , and any sequence of scaling constants $d = (d_n)$, the asymptotic cone $\text{Con}^{\omega}(G, d)$ is a circle-tree. $\text{Con}^{\omega}(G, d)$ is an \mathbb{R} -tree if and only if the sequence (ρ_n) from [Definition 4.12](#) is not (ω, d) -visible.*

The proof of the theorem is divided into a sequence of lemmas. Throughout the rest of the section we fix arbitrary scaling sequence $d = (d_n)$ and ultrafilter ω .

Let us fix a group G having a graded small cancellation presentation [\(18\)](#). In what follows, we denote by dist_S (respectively dist_{Hau}) the distance (respectively the Hausdorff distance) in the Cayley graph $\Gamma(G, S)$.

Lemma 4.18 *Any subword of any word $R_n \in \mathcal{R}_n$ of length at most $|R_n|/2$ is $(1 - o_{\omega}(1), 0)$ -quasigeodesic.*

Proof Suppose $U_n = V_n$ in G , where U_n is a subword of $R_n \in \mathcal{R}_n$, $|U_n| \leq |R_n|/2$, and V_n is a geodesic word in G . By [Lemma 4.13](#), $U_n = V_n$ in G_n ω -almost surely. Let Δ_n be a minimal diagram over G_n with boundary $p_n q_n$, where $\text{Lab}(p_n) = U_n$, $\text{Lab}(q_n) = V_n^{-1}$. If Δ_n has no \mathcal{R}_n -cells, then $|U_n| = |V_n|$ since R_n is geodesic in G_{n-1} . Thus we may assume that Δ_n has at least one \mathcal{R}_n -cell. Let Π_n be the \mathcal{R}_n -cell, Γ_n^1, Γ_n^2 the contiguity diagrams to p, q , respectively, provided by [Lemma 4.6](#).

Observe that $\lim^{\omega}(\Pi_n, \Gamma_n^2, q_n) \leq 1/2$. (Otherwise $(\Pi_n, \Gamma_n^2, q_n) > 1/2 + 2\varepsilon_n/|\Pi_n|$ ω -almost surely as $2\varepsilon_n/|\Pi_n| = o(1)$ and V_n could not be geodesic by [Lemma 4.4](#).)

Thus $\lim^\omega(\Pi_n, \Gamma_n^1, p_n) \geq 1/2 > \mu_n$. Hence $|\partial\Pi_n| - |R_n| \leq 2\varepsilon_n$ by Lemma 4.5. Now there are two cases to consider.

Case 1 If $\lim^\omega(\Pi_n, \Gamma_n^2, q_n) = 1/2$, then

$$\begin{aligned} |V_n| &\geq (1/2 - o_\omega(1))|\partial\Pi_n| - 2\varepsilon_n \\ &\geq (1/2 - o_\omega(1))(|R_n| - 2\varepsilon_n) - 2\varepsilon_n \geq (1 - o_\omega(1))|U_n| \end{aligned}$$

and we are done.

Case 2 If $\lim^\omega(\Pi_n, \Gamma_n^2, q_n) < \theta < 1/2$, then $\lim^\omega(\Pi_n, \Gamma_n^1, p_n) > 1 - \theta$ and we get a contradiction as

$$|U_n| > (1 - \theta)|\partial\Pi_n| - 2\varepsilon_n \geq (1 - \theta)(|R_n| - 2\varepsilon_n) - 2\varepsilon_n > \frac{1}{2}|R_n|$$

ω -almost surely. □

Definition 4.19 Let \mathcal{A} denote the set of all loops in the Cayley graph $\Gamma(G, S)$ labeled by words from the set of relators \mathcal{R} (see (18)). We say that a sequence (p_n) of elements of \mathcal{A} is *asymptotically visible* (relative to the scaling sequence d and ultrafilter ω) if

$$(19) \quad |p_n| = \Theta_\omega(d_n) \quad \text{and} \quad \text{dist}_S(1, p_n) = O_\omega(d_n).$$

By $\mathcal{C} = \mathcal{C}(d, \omega)$ we denote the collection of all distinct limits $\lim^\omega p_n$, where (p_n) ranges in the set of all asymptotically visible sequences of elements of the set \mathcal{A} .

Lemma 4.20 Every piece from $C = \lim^\omega p_n \in \mathcal{C}$ is isometric to a circle of length $\lim^\omega(|p_n|/d_n)$.

Proof By Lemma 4.18, p_n equipped with the metric induced from $\Gamma(G, S)$ is $(1 - o_\omega(1), 0)$ -quasi-isometric to a circle of length $|p_n|$ and our lemma follows. Checking details is straightforward and we leave this to the reader. □

The next observation is quite trivial and follows immediately from Property (\mathbf{Q}_3) of the graded small cancellation condition (Definition 4.12) and Lemma 4.13 (c).

Lemma 4.21 Suppose that the sequence (ρ_n) is (ω, d) -visible. Let (\mathcal{R}_{i_n}) be a subsequence of (\mathcal{R}_n) , such that $\rho_{i_n} = \Theta_\omega(d_n)$. Then for any asymptotically visible sequence (p_n) where $\text{Lab}(p_n) \in \mathcal{R}$, we have $\text{Lab}(p_n) \in \mathcal{R}_{i_n}$ ω -almost surely.

Lemma 4.22 Suppose that for some asymptotically visible sequences (p_n) and (q_n) , the intersection $\lim^\omega p_n \cap \lim^\omega q_n$ contains at least two distinct points. Then $\lim^\omega p_n = \lim^\omega q_n$.

Proof The assumptions of the lemma imply that for every $n \in \mathbb{N}$, there exist quadrangles $Q_n = p'_n s_n q'_n t_n$ in $\Gamma(G, S)$ such that p'_n and q'_n are subpaths of p_n and q_n , respectively, and s_n, t_n are geodesics such that

$$(20) \quad |p_n| = \Theta_\omega(d_n), \quad |q_n| = \Theta_\omega(d_n),$$

$$(21) \quad |s_n| = o_\omega(d_n), \quad |t_n| = o_\omega(d_n).$$

Without loss of generality we may also assume that

$$(22) \quad |p'_n| \leq \frac{1}{2}|p_n|, \quad |q'_n| \leq \frac{1}{2}|q_n|.$$

Let (\mathcal{R}_{i_n}) be the subsequence of (\mathcal{R}_n) provided by Lemma 4.21. Thus $\text{Lab}(p_n), \text{Lab}(q_n) \in \mathcal{R}_{i_n}$ ω -almost surely. Note that (20) and (21) imply

$$|Q_n| = \Theta_\omega(d_n) = \Theta_\omega(\rho_{i_n}).$$

Therefore, by Lemma 4.13 (d) there is a van Kampen diagram Ξ_n with the boundary label $\text{Lab}(Q_n)$ having no \mathcal{R}_j -cells for $j > i_n$ ω -almost surely. For simplicity we keep the notation p'_n, s_n, q'_n, t_n for the corresponding parts of $\partial\Xi_n$. Let also Δ_n be the diagram obtained from Ξ_n by attaching two \mathcal{R}_{i_n} -cells Σ_n^1 and Σ_n^2 along p'_n and q'_n respectively so that the natural map $\text{Sk}^{(1)}(\Delta_n) \rightarrow \Gamma(G, S)$ sending parts of $\partial\Xi_n$ to the corresponding sides of Q_n , maps $\partial\Sigma_n^1$ to p_n and $\partial\Sigma_n^2$ to q_n . There are two cases to consider.

Case 1 Ξ_n has no \mathcal{R}_{i_n} -cells ω -almost surely and hence $\text{Lab}(Q_n) = 1$ in G_{i_n-1} ω -almost surely. We recall that $\Gamma(G_{i_n-1}, S)$ is δ_{i_n-1} -hyperbolic. Let p''_n be the subpath of p'_n such that $\text{dist}((p''_n)_\pm, (p'_n)_\pm) = \max\{|s_n|, |t_n|\} + 2\delta_{i_n-1}$. (Note that $|p''_n| > 2(\max\{|s_n|, |t_n|\} + 2\delta_{i_n-1})$ ω -almost surely.) By Lemma 2.9, $\text{dist}_{\text{Hau}}(p''_n, q'_n) \leq 2\delta_{i_n-1}$. Thus we may assume that Δ_n contains a $2\delta_{i_n-1}$ -contiguity subdiagram Γ_n of Σ_n^1 to Σ_n^2 such that

$$(\Sigma_n^1, \Gamma_n, \Sigma_n^2) = \frac{|p'_n| - 2(\max\{|s_n|, |t_n|\} + 2\delta_{i_n-1})}{|\partial\Sigma_n^1|} = \frac{\Theta_\omega(d_n) - o_\omega(d_n)}{\Theta_\omega(d_n)} > \mu_{i_n}$$

ω -almost surely (see Figure 5). Now Lemma 4.5 and Lemma 4.13 imply that

$$\text{dist}_{\text{Hau}}(p_n, q_n) \leq \varepsilon_n + 2\delta_{i_n-1} = o_\omega(d_n).$$

Therefore $\lim^\omega p_n = \lim^\omega q_n$.

Case 2 Suppose now that Δ_n has at least one \mathcal{R}_{i_n} -cell ω -almost surely. Then by Lemma 4.6, we may assume that there is an \mathcal{R}_{i_n} -cell Π_n in Ξ_n and disjoint ε_{i_n} -contiguity subdiagrams $\Gamma_n^1, \dots, \Gamma_n^4$ of Π_n to p'_n, s_n, q'_n, t_n , respectively, such that

$$(\Pi_n, \Gamma_n^1, p'_n) + (\Pi_n, \Gamma_n^2, s_n) + (\Pi_n, \Gamma_n^3, q'_n) + (\Pi_n, \Gamma_n^4, t_n) > 1 - 23\mu_{i_n}$$

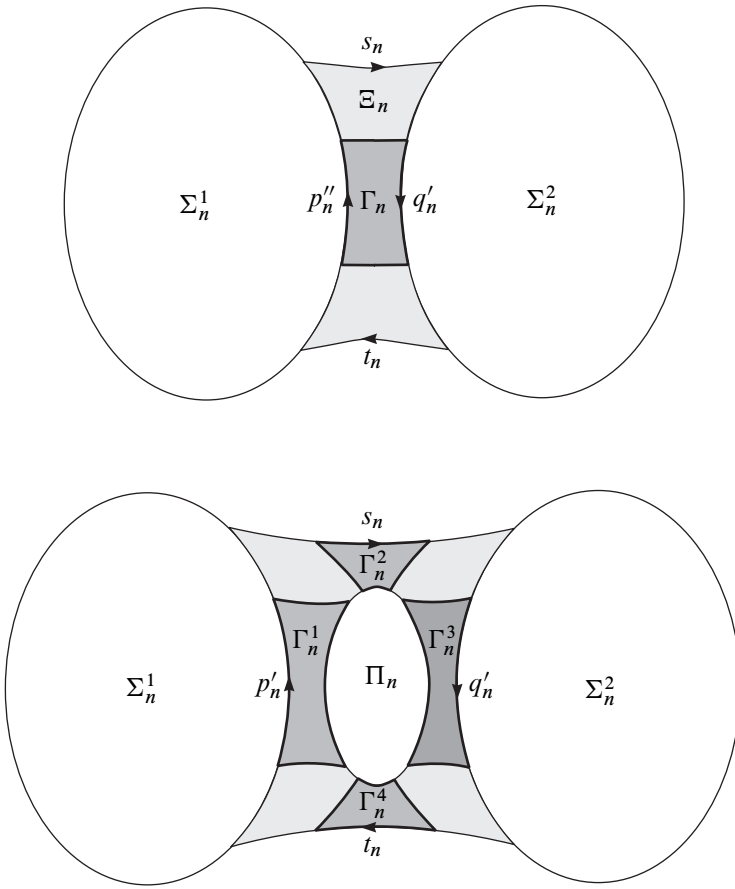


Figure 5: Two cases in the proof of Lemma 4.22

ω -almost surely (Figure 5). Using (21) and Lemma 4.13 one can easily show that

$$(\Pi_n, \Gamma_n^2, s_n) + (\Pi_n, \Gamma_n^4, t_n) = o_\omega(1).$$

Hence

$$(\Pi_n, \Gamma_n^1, p'_n) + (\Pi_n, \Gamma_n^3, q'_n) = 1 - o_\omega(1).$$

Note that Γ_n^3 may also be considered as a contiguity subdiagram of Π_n to Σ_n^2 in Δ_n .

Suppose first that we have $(\Pi_n, \Gamma_n^1, p'_n) = (\Pi_n, \Gamma_n^1, \Sigma_n^1) \leq \mu_{i_n}$ ω -almost surely. Then $(\Pi_n, \Gamma_n^3, \Sigma_n^2) = 1 - o_\omega(1)$. Applying Lemma 4.5 we obtain

$$|\partial\Pi_n| - |\partial\Sigma_n^2| < 2\varepsilon_n.$$

Hence the length of the contiguity arc v_n of Γ_n^3 to Σ_n^2 satisfies

$$|v_n| \geq (1 - o_\omega(1))|\partial\Pi_n| - 2\varepsilon_n = (1 - o_\omega(1))|\partial\Sigma_n^2|.$$

However this contradicts (22).

Therefore $(\Pi_n, \Gamma_n^1, p'_n) > \mu_{i_n}$ and similarly $(\Pi_n, \Gamma_n^3, q'_n) > \mu_{i_n}$. Let w_n be the image of $\partial\Pi_n$ under the natural map $\text{Sk}^{(1)}(\Delta_n) \rightarrow \Gamma(G, S)$ sending $\partial\Sigma_n^1$ and $\partial\Sigma_n^2$ to p_n and q_n respectively. Then using Lemma 4.5 and Lemma 4.13 we obtain

$$\text{dist}_{\text{Hau}}(p_n, q_n) \leq \text{dist}_{\text{Hau}}(p_n, w_n) + \text{dist}_{\text{Hau}}(q_n, w_n) = o_\omega(d_n).$$

Thus $\lim^\omega(p_n) = \lim^\omega(q_n)$ again. □

Lemma 4.23 *Every simple triangle in $\text{Con}^\omega(G, d)$ whose sides are limit geodesics is contained in a subset from \mathcal{C} .*

Proof Suppose that pqs is a simple triangle in $\text{Con}^\omega(G, d)$ whose sides are limit geodesics. As in the proof of Theorem 3.3, $pqs = \lim^\omega H_n$, where H_n is a geodesic hexagon in $\Gamma(G, S)$. Let Δ_n be a van Kampen diagram over (18) with boundary label $\text{Lab}(H_n)$.

First of all we assume that (ρ_n) is (ω, d) -visible and denote by (\mathcal{R}_{i_n}) be the subsequence of (\mathcal{R}_n) provided by Lemma 4.21. Note that $|H_n| = \Theta_\omega(d_{i_n})$. Arguing as in the proof of the previous lemma, one can show that Δ_n has no \mathcal{R}_j -cells for $j > i_n$ ω -almost surely. To simplify our notation we identify the 1-skeleton of Δ_n with its natural image in $\Gamma(G, S)$.

If Δ_n has no \mathcal{R}_{i_n} -cells ω -almost surely, we obtain a contradiction as in the third paragraph of the proof of Theorem 3.3. Thus we may assume that Δ_n has at least one \mathcal{R}_{i_n} -cell ω -almost surely and Δ_n is minimal over G_{i_n} . By Lemma 4.6 there is an \mathcal{R}_{i_n} -cell Π_n in Δ_n and ε_{i_n} -contiguity subdiagrams $\Gamma_n^1, \dots, \Gamma_n^6$ of Π_n to the sides of H_n such that $\partial\Pi_n$ belongs to the closed $(23\mu_{i_n}|\partial\Pi_n|)$ -neighborhood of the union $\bigcup_{i=1}^6 \Gamma_n^i$ in Δ_n ω -almost surely. Let w_n denote the natural image of $\partial\Pi_n$ in $\Gamma(G, S)$. By Lemma 2.9 w_n belongs to the closed $(23\mu_{i_n}|\partial\Pi_n| + \varepsilon_{i_n} + 2\delta_{i_n-1})$ -neighborhood of H_n in $\Gamma(G, S)$. Further Lemma 4.13 implies that $23\mu_{i_n}|\partial\Pi_n| + \varepsilon_{i_n} + 2\delta_{i_n-1} = o_\omega(d_n)$. Hence $\lim^\omega w_n \subseteq pqs$. Since $\lim^\omega w_n$ is a circle by Lemma 4.20 and pqs is simple, we have $pqs = \lim^\omega w_n \in \mathcal{C}$.

Finally we assume that (ρ_n) is not (ω, d) -visible. Let j_n be the maximal number such that Δ_n has at least one \mathcal{R}_{j_n} -cell. Let Π_n and $\Gamma_n^1, \dots, \Gamma_n^6$ be the \mathcal{R}_{j_n} -cell and the ε_{j_n} -contiguity subdiagrams of Π_n to sides of H_n provided by Lemma 4.6. Again we can easily show that the total length of the contiguity arcs of $\Gamma_n^1, \dots, \Gamma_n^6$ to the sides

of H_n is $\Theta_\omega(\rho_{j_n})$. Since (ρ_n) is not (ω, d) -visible, $\rho_{j_n} = \Theta_\omega(d_n)$ is impossible. Therefore $\rho_{j_n} = o_\omega(d_n)$. Hence $\delta_{j_n} = o_\omega(d_n)$ by Lemma 4.13. This again leads to a contradiction as in the third paragraph of the proof of Theorem 3.3. Thus $\text{Con}^\omega(G, d)$ has no nontrivial simple triangles whose sides are limit geodesics, ie, it is tree-graded with respect to the empty collection of pieces (ie, it is an \mathbb{R} -tree) by Lemma 2.7. \square

Proof of Theorem 4.17 Let G have a graded small cancellation presentation (18). For any fixed scaling sequence $d = (d_n)$ and any ultrafilter ω , let $\mathcal{C} = \mathcal{C}(d, \omega)$ be the collection of subsets of $\text{Con}^\omega(G, d)$ described in Definition 4.19. If (ρ_n) is (ω, d) -visible, all elements of \mathcal{C} are circles whose radii are uniformly bounded from below and from above by positive constants by Lemma 4.20 and Lemma 4.21. Further by Lemma 4.22, \mathcal{C} satisfies (T_1) (see Definition 2.1). Applying now Lemma 4.23 and Lemma 2.7 we conclude that $\text{Con}^\omega(G, d)$ is tree-graded with respect to \mathcal{C} , ie, is a circle-tree. If (ρ_n) is not (ω, d) -visible, the same arguments show that $\text{Con}^\omega(G, d)$ is tree-graded with respect to \mathcal{C} . But \mathcal{C} is empty in this case, hence $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree. \square

4.4 Groups without free subgroups

Recall that any torsion-free nonelementary hyperbolic group has an infinite quotient group with finite (or cyclic) proper subgroups [42]. We show in this section that some of these groups have graded small cancellation presentations. In what follows, H denotes a hyperbolic group generated by a finite set S .

The following Lemma 4.24 is an analog of Lemma 4.6 for such presentations (and a particular case of [42, Lemma 6.6]), while Lemma 4.25 is a quasigeodesic analog of Lemma 4.4.

Lemma 4.24 *For any hyperbolic group H and any $\lambda > 0$, there is $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ and any $c \geq 0$, there are $\varepsilon \geq 0$ and $\rho > 0$ with the following property:*

Suppose a finite symmetrized presentation $H_1 = \langle H \mid \mathcal{R} \rangle$ satisfies the condition $C(\varepsilon, \mu, \lambda, c, \rho)$, and Δ is a minimal diagram over H_1 whose boundary is a product of (λ, c) -quasigeodesic paths p and q . Then provided Δ has an \mathcal{R} -cell, there exists an \mathcal{R} -cell Π in Δ and disjoint ε -contiguity subdiagrams Γ_1 and Γ_2 (one of them may be absent) of Π to p and q , respectively, such that $(\Pi, \Gamma_1, p) + (\Pi, \Gamma_2, q) > 1 - 23\mu$.

Lemma 4.25 *Let a presentation $H_1 = \langle H \mid \mathcal{R} \rangle$ satisfy a $C(\varepsilon, \mu, \lambda, c, \rho)$ -condition with $\mu < \lambda^2/100$ and $\rho > 2\mu^{-1}(c + 2\varepsilon)$. Let Δ be a minimal diagram over H_1 with a (λ, c) -quasigeodesic subpath q of the boundary, and Γ a contiguity subdiagram of an \mathcal{R} -cell Π to q . Then $\psi = (\Pi, \Gamma, q) < 1 - 24\mu$.*

Proof Let $p_1q_1p_2q_2$ be the boundary of the contiguity subdiagram of Π to q , where $q_1p = \partial\Pi$ and q_2 is a subpath of q . Since q_2 is a (λ, c) -quasigeodesic subpath of q , we have $\lambda|q_2| - c \leq |p_2| + |p| + |p_1| \leq 2\varepsilon + (1 - \psi)|\partial\Pi|$. On the other hand, $|q_2| \geq \lambda|q_1| - c - |p_1| - |p_2|$ since the path q_1 is (λ, c) -quasigeodesic. Hence $|q_2| \geq \lambda\psi|\partial\Pi| - c - 2\varepsilon$. These two estimates for $|q_2|$ give us the inequality

$$\lambda^{-1}(|\partial\Pi|(1 - \psi) + 2\varepsilon + c) \geq \psi\lambda|\partial\Pi| - c - 2\varepsilon,$$

that is, $\psi \leq (\lambda^{-1}|\partial\Pi| + (1 + \lambda^{-1})(c + 2\varepsilon))(|\partial\Pi|(\lambda^{-1} + \lambda))^{-1}$.

Since $|\partial\Pi| \geq \rho$, we obtain from the assumptions of the lemma

$$\psi \leq \frac{\lambda^{-1}\rho + (1 + \lambda^{-1})(c + 2\varepsilon)}{\rho(\lambda^{-1} + \lambda)} < \frac{1 + \mu}{1 + 100\mu} < 1 - 24\mu. \quad \square$$

Below we say that a bi-infinite path p in the Cayley graph of a group generated by a finite set S is V -periodic (or just periodic), if p is labeled by the bi-infinite power of some word V in $S^{\pm 1}$.

Theorem 4.26 (1) *Let G be an arbitrary nonelementary hyperbolic group. Then there exists an infinite torsion quotient group Q_1 of G admitting a graded small cancellation presentation.*

(2) *Let G be an arbitrary torsion free noncyclic hyperbolic group with a finite set of generators S . Then there exists an infinite non-Abelian torsion free quotient group Q_2 of G admitting a graded small cancellation presentation and such that all proper subgroups of Q_2 are cyclic. Moreover, every periodic bi-infinite path in the Cayley graph $\Gamma(Q_2, S)$ is a Morse quasigeodesic.*

Proof (1) Infinite torsion quotient group Q_1 of an arbitrary nonelementary hyperbolic group G was constructed in [42, Corollary 2] as a direct limit of a sequence of hyperbolic groups $G = G(0) \rightarrow G(1) \rightarrow \dots \rightarrow G(i - 1) \rightarrow G(i) \rightarrow \dots$, where each step is a transition from $H = G(i - 1)$ to $H_1 = G(i) = \langle G(i - 1) \mid V^m \rangle$ for a word $V = V_i$ and a sufficiently large $m = m_i$. Lemmas 4.1 and 6.7 of [42] claim that the set of the cyclic shifts of the words $V^{\pm 1}$ satisfy a $C(\varepsilon, \mu, \lambda, c, \rho)$ -condition, where $\lambda = \lambda_i = \lambda(V, i - 1) > 0$, $c = c_i = c(V, i - 1) \geq 0$, the positive $\mu = \mu_i$ can be selected arbitrary small, then $\varepsilon = \varepsilon_i$ can be chosen arbitrary large, and afterwards m , and therefore $\rho = \rho_i$, can be chosen arbitrary large.

It follows from Lemma 4.11 that one can replace the defining word V_i^m by a conjugate in H word R_i having minimal length in its conjugacy class, so that the set of cyclic shifts of R_i satisfy $C(\varepsilon'_i, \mu'_i, \rho'_i)$ -condition with parameters $\varepsilon'_n, \mu'_n, \rho'_n$ ($n = 1, 2, \dots$) satisfying the Definition 4.12. This proves the first statement of the proposition.

(2) To construct Q_2 we denote by \mathcal{F} the set of all 2-generated subgroups of G and enumerate all elements p_1, p_2, \dots of the set $\mathcal{P} = S \times \mathcal{F}$. We set $G(0) = G$ and proceed by induction. Suppose that a (torsion-free) hyperbolic group $G(i-1)$ and relators R_1, \dots, R_{i-1} are already constructed. Then we consider the first pair, say $p_k = (s, K) \in \mathcal{P}$, such that the image K' of K in $G(i-1)$ is nonelementary and the image s' of s in $G(i-1)$ does not belong to K' . As in the proof of [42, Corollary 1], we can choose a word R_i of the form $R_i \equiv X_0 U^m X_1 U^m \dots X_l U^m$, where X_0 represents an element of sK and U, X_1, \dots, X_l represent elements of K , such that the set of all cyclic shifts of $R_i^{\pm 1}$ satisfies a $C(\varepsilon, \mu, \lambda, c, \rho)$ -condition. Here $\lambda = \lambda_i > 0$, the positive $\mu = \mu_i$ can be selected arbitrary small, and then $c = c_i \geq 0$ arbitrary large, then $\varepsilon = \varepsilon_i$ can be chosen arbitrary large, and afterwards m , and therefore $\rho = \rho_i$, can be chosen arbitrary large. Such a choice of the parameters is guaranteed by Lemmas 4.2 and 6.7 of [42]. Then $G(i) = \langle G(i-1) \mid R_i \rangle$, and the group Q_2 is defined to be the limit of the sequence $G = G(0) \rightarrow G(1) \rightarrow \dots \rightarrow G(i) \rightarrow \dots$. Hence, as in the first part of the proof, one can choose the parameters so that Q_2 has a graded small cancellation presentation. As in [42, Corollary 1], Q_2 is a non-Abelian torsion free group with cyclic proper subgroups. (The only difference is that now we are adding only one relation R_i when passing from $G(i-1)$ to $G(i)$, while in [42], the set \mathcal{F} was enumerated, and, for given $K \in \mathcal{F}$, one imposed finitely many relations to obtain $G(i)$, namely, one relation for every $s \in S$.)

To ensure the Morse property for bi-infinite periodic paths in $\Gamma(Q_2, S)$ we have to make the following additional changes in the scheme from [42]. For every $i = 1, 2, \dots$, when passing from $G(i-1)$ to $G(i)$ we fix a set of words \mathcal{V}_i in $S^{\pm 1}$ of lengths at most i having infinite order in G_{i-1} . Arbitrary power of a word from \mathcal{V}_i is $(\lambda(i), c(i))$ -quasigeodesic in the hyperbolic group $G(i-1)$ for some $\lambda(i) > 0$ and $c(i) \geq 0$, and one can choose the constants λ_i and c_i in the previous paragraph so that $\lambda_i \leq \lambda(i)$ and $c_i \geq c(i)$. Then [42, Lemma 2.5] allows us to choose the words R_i and the parameters in the previous paragraph so that the following is true.

(*) $\left\{ \begin{array}{l} \text{Let } \Delta \text{ be a minimal diagram over } G(i), q \text{ a part of } \partial\Delta \text{ such that } \text{Lab}(q) = \\ V_i^n \text{ for some } 1 \leq i \leq s, n \in \mathbb{N}. \text{ Then } \Delta \text{ contains no } \varepsilon_i\text{-contiguity} \\ \text{subdiagrams of } R_i\text{-cells to } q \text{ with contiguity degree at least } \mu_i. \end{array} \right.$

Assume that V is a word representing a nontrivial element in Q_2 . Then $V \in \mathcal{V}_i$ for some i . Assume that for some $n > 0$, $V^n = U$ in Q_2 , where U is a geodesic word in Q_2 . Let $j = j(n)$ be the smallest positive integer such that $V^n = U$ in $G(j)$. Suppose that $j \geq i$ (hence $V \in \mathcal{V}_j$). Consider a minimal diagram Δ over $G(j)$ with $\partial\Delta = pq$, where $\text{Lab}(p) \equiv V^n$ and $\text{Lab}(q^{-1}) \equiv U$. Let Π, Γ_1, Γ_2 be the \mathcal{R}_j -cell of Δ and the ε_j -contiguity diagrams of Π to p and q , respectively, provided by Lemma 4.24. Then $(\Pi, \Gamma_1, p) < \mu_j$ by (*), and so $(\Pi, \Gamma_2, q) > 1 - 24\mu_j$ that contradicts

Lemma 4.25. Hence $j < i$, ie, $V^n = U$ in the hyperbolic group $G(i)$, where i is independent of n . This implies that any bi-infinite V -periodic path in $\Gamma(Q_2, S)$ is quasigeodesic.

It remains to prove that any bi-infinite V -periodic path in $\Gamma(Q_2, S)$ is Morse. Let us fix arbitrary $L, C > 0$ and consider any (L, C) -quasigeodesic word W such that $V^n = W$ in Q_2 for some n . As above let j be the smallest positive integer such that $V^n = W$ in $G(j)$.

Since in a hyperbolic group every bi-infinite periodic geodesic is Morse [2], we would finish the proof if we show that j can be bounded from above by some constant $J = J(L, C)$ independent of n . Let us choose J so that $\lambda_J < L$, $C < c_J$, and $V \in \mathcal{V}_J$. Again we consider a minimal diagram Δ over Q_2 with $\partial\Delta = pq$, where $\text{Lab}(p) \equiv V^n$, and $\text{Lab}(q^{-1}) \equiv W$.

By contradiction, assume that $\text{rank}(\Delta) = k > J$. Let Π, Γ_1, Γ_2 be the \mathcal{R}_k -cell of Δ and the ε_j -contiguity diagrams of Π to p and q , respectively, provided by Lemma 4.24. Then $(\Pi, \Gamma_2, q) > 1 - 24\mu_j$ as above, contrary to Lemma 4.25. □

Remark 4.27 In a similar way, one can use methods of [42] to construct an infinite group Q admitting a graded small cancellation presentation, and such that all proper subgroups of Q are finite. In that construction, one would have to use [42, Theorem 4]. Note that there exists a slight error in the formulation of that theorem. Let E_0 be the elementary group and C its infinite cyclic normal subgroup from the formulation of [42, Theorem 4]. Since the group H satisfies the quasi-identity

$$x^2y = yx^2 \rightarrow xy = yx,$$

the center Z of E_0 has a finite odd index in E_0 by [42, Proposition 2]. Hence Z contains the Sylow 2-subgroup P_2 of E_0 and an infinite cyclic subgroup C such that the product CP_2 is of odd index in E_0 . To make the formulation of [42, Theorem 4] correct, one needs to add the condition that the cyclic subgroup C in E_0 is chosen with this additional property, namely, the order of $E_0/(CP_2)$ is odd. (This condition was used in the proof of [42, Theorem 4].)

Also the direct limits of hyperbolic groups in [35] can be chosen satisfying Condition (3) of Theorem 3.3. Hence there exist torsion and torsion free examples of divisible (and even verbally complete) lacunary hyperbolic groups.

4.5 Floyd boundary

Finally we note a relation between cut points in asymptotic cones and the Floyd boundary. Recall that the *Floyd boundary* [16] ∂G of a finitely generated group

$G = \langle S \rangle$ is defined as follows. Let dist_F be the metric on $\Gamma = \Gamma(G, S)$ obtained by setting the lengths of each edge e to be equal to $(1 + \text{dist}(e, 1))^{-2}$. It is easy to see that Γ is bounded with respect to dist_F . Let $\bar{\Gamma}$ be the metric completion of (Γ, dist_F) . Then $\partial G = \bar{\Gamma} \setminus \Gamma$.

The Floyd boundary of a group G is a quasi-isometry invariant. If ∂G consists of 0 (respectively 2) points, G is finite (respectively virtually cyclic). If ∂G consists of 0, 1 or 2 points it is said to be trivial. Otherwise it is uncountable (and, moreover, ∂G is a boundary in the sense of Furstenberg). If ∂G is nontrivial, G contains a free non-Abelian subgroup. In particular, G is trivial for any amenable group. The class of groups with nontrivial Floyd boundary includes nonelementary hyperbolic groups, nonelementary geometrically finite Kleinian groups, groups with infinitely many ends, and many other examples. (For more details we refer to Karlsson [29].)

Proposition 4.28 *Let G be a finitely generated group whose Floyd boundary consists of at least 2 points. Then all asymptotic cones of G have cut points.*

Proof Let us fix a scaling sequence $d = (d_n)$ and an ultrafilter ω . Let also $(x_n), (y_n)$ be sequences of elements of G that converge to distinct points $x, y \in \partial G$. For each x_n we fix a geodesic γ_n in $\Gamma(G, S)$ connecting x_n to 1. Since $\Gamma(G, S)$ is locally finite, there is an infinite ray γ such that $\gamma_- = 1$ and the combinatorial length of the common part of γ and γ_n tends to ∞ as $n \rightarrow \infty$. This means, in particular, that the sequence of vertices of γ converge to x as $n \rightarrow \infty$. Thus we may assume that $|x_n| = n$ and $|y_n| = n$. Consider the subsequences $a_n = x_{[d_n]}$ and $b_n = y_{[d_n]}$ and set $a = (a_n)^\omega$, $b = (b_n)^\omega$. Note that $\text{dist}(a, (1)^\omega) = \text{dist}(b, (1)^\omega) = 1$.

Suppose that $\text{Con}^\omega(G, d)$ has no cut points. Then for some $\varepsilon > 0$, there is a path p in $\text{Con}^\omega(G, d) \setminus \text{Ball}((1)^\omega, \varepsilon)$ of some length L connecting a to b . Now applying standard methods it is easy to show that a_n and b_n can be connected by a path p_n in $\Gamma(G, S)$ such that $|p_n| < 2Ld_n$ and p_n avoids the ball of radius $\varepsilon d_n/2$ centered at 1 in $\Gamma(G, S)$ (with respect to the combinatorial metric) ω -almost surely. Hence we have

$$\text{dist}_F(a_n, b_n) \geq \frac{2Ld_n}{(\varepsilon d_n/2)^2} = o(1).$$

Thus (a_n) and (b_n) converge to the same points of the Floyd boundary and we get a contradiction. \square

We note that the converse to [Proposition 4.28](#) does not hold. Indeed all asymptotic cones of groups constructed in this section have cut points by [Theorem 4.17](#). On the other hand their Floyd boundary consists of a single point since they contain no non-Abelian free subgroups and are not virtually cyclic.

Corollary 4.29 *There exists a lacunary hyperbolic group G such that the Floyd boundary ∂G consists of a single point and all asymptotic cones of G are circle-trees.*

5 Central extensions of lacunary hyperbolic groups

5.1 Asymptotic cones of group extensions

In this section, we obtain some results about asymptotic cones of group extensions. These results are used in the next two sections.

Lemma 5.1 (Asymptotic cones of isometry groups) *Suppose that a finitely generated group G acts isometrically on a metric space X . Fix an arbitrary point $x \in X$. Then the map $\alpha: G \rightarrow X$ defined by $\alpha(g) = gx$ for any $g \in G$ induces a continuous map $\hat{\alpha}: \text{Con}^\omega(G, d) \rightarrow \text{Con}^\omega(X, d)$.*

Proof Note that the map α is C -Lipschitz for

$$C = \max\{\text{dist}(x, sx) \mid s \in S^{\pm 1}\},$$

where S is a finite generating set of G . Indeed if $g = s_1 \cdots s_n$ for some $g \in G$ and $s_1, \dots, s_n \in S^{\pm 1}$, then

$$\text{dist}(x, gx) \leq \sum_{i=1}^n \text{dist}(g_{i-1}x, g_{i-1}s_i x) \leq \sum_{i=1}^n \text{dist}(x, s_i x) \leq Cn.$$

where $g_0 = 1$ and $g_i = s_1 \cdots s_i$ for $1 \leq i \leq n$. Thus for any scaling sequence $d = (d_n)$ and any nonprincipal ultrafilter ω , the map $(g_n)^\omega \mapsto (g_n x)^\omega$ from $\text{Con}^\omega(G, d)$ to $\text{Con}^\omega(X, d)$ is continuous. \square

Given a group G generated by a finite set S and a normal subgroup N of G , we endow the group G and the quotient group G/N by the word metric with respect to the set S and its image in G/N , respectively. We also assume that N is endowed with the metric induced from G . Thus for any d and ω , $\text{Con}^\omega(N, d)$ may be considered as a subset of $\text{Con}^\omega(G, d)$. Set $x = 1 \in G/N$. Then Lemma 5.1 applied to the natural action of G on the quotient group G/N by left multiplications gives us the map $\hat{\alpha}: \text{Con}^\omega(G, d) \rightarrow \text{Con}^\omega(G/N, d)$. Note that α is the natural homomorphism $G \rightarrow G/N$ in this case. Given $b \in \text{Con}^\omega(G/N, d)$, we call the subset $\mathcal{F}_b = \hat{\alpha}^{-1}(b) \subseteq \text{Con}^\omega(G, d)$ fiber.

Recall that the group

$$G_e^\omega(d) = \left\{ (g_n) \in \prod_{n \in \mathbb{N}} G \mid |g_n| = O_\omega(d_n) \right\}$$

acts transitively by isometries on $\text{Con}^\omega(G, d)$ by bigg multiplication. Let $N_e^\omega(d)$ be the subgroup of $G_e^\omega(d)$ defined as follows:

$$N_e^\omega(d) = \left\{ (g_n) \in \prod^\omega N \mid |g_n| = O_\omega(d_n) \right\}.$$

Theorem 5.2 (Asymptotic cones of quotient groups) *Let $G = \langle S \rangle$ be a finitely generated group G , N a normal subgroup of G endowed with the metric induced from G . Then the following conditions hold.*

- The map $\hat{\alpha}$ is surjective.
- For any $b \in \text{Con}^\omega(G/N, d)$, we have $\mathcal{F}_b = \gamma \text{Con}^\omega(N, d)$ for some element $\gamma \in G_e^\omega(d)$.
- The action of $G_e^\omega(d)$ permutes fibers, that is, for any $b \in \text{Con}^\omega(G/N, d)$ and $\gamma \in G_e^\omega(d)$, $\gamma \mathcal{F}_b$ is a fibre.
- The action of $N_e^\omega(d)$ stabilizes each fiber (as a set) and acts on each fiber transitively.
- If $\text{Con}^\omega(N, d)$ is discrete, the map $\hat{\alpha}: \text{Con}^\omega(G, d) \rightarrow \text{Con}^\omega(G/N, d)$ is locally isometric. If $\text{Con}^\omega(N, d)$ consists of a single point, then $\text{Con}^\omega(G, d)$ and $\text{Con}^\omega(G/N, d)$ are isometric.

Proof Let $\sigma: G/N \rightarrow G$ be a section that assigns to each element $x \in G/N$ a shortest preimage of x in G . If $h = (h_n N)^\omega$ is a point in $\text{Con}^\omega(G/N, d)$, then $g = (\sigma(h_n N)^\omega)$ belongs to $\text{Con}^\omega(G, d)$ and $\hat{\alpha}(g) = h$. Thus $\hat{\alpha}$ is surjective.

Further for any $b \in \text{Con}^\omega(G/N, d)$, we have

$$(23) \quad \mathcal{F}_b = \{(g_n)^\omega \in \text{Con}^\omega(G, d) \mid (\alpha(g_n))^\omega = b\}.$$

Let us fix any element $(f_n)^\omega \in \mathcal{F}_b$ and set $\gamma = (f_n)$. Clearly $\gamma \text{Con}^\omega(N, d) \subseteq \mathcal{F}_b$. If $g = (g_n)^\omega \in \mathcal{F}_b$, then $(\alpha(f_n))^\omega = (\alpha(g_n))^\omega$, ie, $|\alpha(g_n^{-1})\alpha(f_n)| = o_\omega(d_n)$. Let s_n be a shortest preimage of $\alpha(g_n^{-1}f_n)$ in G . Then $|s_n| = o_\omega(d_n)$ and $u_n = f_n^{-1}g_n s_n \in N$. Note that $|u_n| \leq |f_n| + |g_n| + |s_n| = O_\omega(d_n)$. Hence $u = (u_n)^\omega \in \text{Con}^\omega(N, d)$. Clearly $\gamma u = (g_n s_n)^\omega = (g_n)^\omega$. Thus $\mathcal{F}_b \subseteq \gamma \text{Con}^\omega(N, d)$ and the second assertion is proved.

Similarly it is easy to show that each subset of the form $\gamma \text{Con}^\omega(N, d)$, where $\gamma \in G_e^\omega(d)$, is a fiber. This and the second assertion imply the third one.

Given two elements $(g_n)^\omega$ and $(h_n)^\omega$ of \mathcal{F}_b , the element $(h_n g_n^{-1})^\omega$ belongs to $N_e^\omega(d)$ and takes $(g_n)^\omega$ to $(h_n)^\omega$. This proves the fourth assertion.

Finally, assume that a ball of radius $\varepsilon \in (0, \infty]$ in $\text{Con}^\omega(N, d)$ consists of a single point. Let $h = (h_n)^\omega$ and $g = (g_n)^\omega$ be two elements of $\text{Con}^\omega(G, d)$ such that $\text{dist}(g, h) < \varepsilon/2$. Note that

$$|\sigma(g_n^{-1}h_nN)| \leq |g_n^{-1}h_n| < \varepsilon d_n/2$$

ω -almost surely. Then for $u_n = g_n^{-1}h_n(\sigma(g_n^{-1}h_nN))^{-1}$, we have $|u_n| < \varepsilon d_n$ ω -almost surely. Since $u_n \in N$, $(u_n)^\omega$ belongs to the ball of radius ε in $\text{Con}^\omega(N, d)$ around $(1)^\omega$. Hence $(u_n)^\omega = (1)^\omega$, ie, $|u_n| = o_\omega(d_n)$. Finally we obtain

$$\begin{aligned} \text{dist}(\hat{\alpha}(g), \hat{\alpha}(h)) &= \lim^\omega d(h_nN, g_nN)/d_n = \lim^\omega d(g_n^{-1}h_nN, N)/d_n \\ &= \lim^\omega |\sigma(g_n^{-1}h_nN)|/d_n \leq \lim^\omega (|g_n^{-1}h_n| + |u_n|)/d_n \\ &= \lim^\omega |g_n^{-1}h_n|/d_n = \text{dist}(g, h). \end{aligned}$$

Thus the restriction of $\hat{\alpha}$ to any ball of radius $\varepsilon/4$ is an isometry. □

Recall that a subspace Y of a metric space X is said to be *convex* if any geodesic path p in X such that $p_\pm \in Y$ belongs to Y . In particular, if X is geodesic, then any convex subspace of X is geodesic.

Proposition 5.3 *Let $d = (d_n)$ a scaling sequence, ω a nonprincipal ultrafilter. Suppose that T is a convex \mathbb{R} -tree in $\text{Con}^\omega(G, d)$. Then for any point $y \in \text{Con}^\omega(G, d)$ such that $\hat{\alpha}(y) \in T$, there is an isometric section $\sigma: T \rightarrow \text{Con}^\omega(G, d)$ of $\hat{\alpha}$ such that $y \in \sigma(T)$.*

To prove the proposition we need an auxiliary result.

Lemma 5.4 *For any point $x \in \text{Con}^\omega(G, d)$ and any fiber \mathcal{F}_b , there is a point $f \in \mathcal{F}_b$ such that $\text{dist}(x, \mathcal{F}_b) = \text{dist}(x, f) = \text{dist}(\hat{\alpha}(x), b)$.*

Proof By Theorem 5.2 (c), it suffices to prove the statement of the lemma for $x = (1)^\omega$. Let $\mathcal{F} = \gamma \text{Con}^\omega(N, d)$, where $\gamma = (g_n) \in G_\varepsilon^\omega(d)$. We fix any section $s: G/N \rightarrow G$ that assigns to every element of G/N a shortest preimage. Take $f = (s(g_nN))^\omega$. By the choice of s , we have $|s(g_nN)| = |\alpha(s(g_nN))|$. Hence

$$\text{dist}(1, f) = \text{dist}(1, (\alpha \circ s(g_nN))^\omega) = \text{dist}(1, \hat{\alpha}(f)) = \text{dist}(1, b).$$

Note that $\gamma^{-1}f = (g_n^{-1}s(g_nN))^\omega \in \text{Con}^\omega(N, d)$ and thus $f \in \mathcal{F}$. Finally for any point $f' \in \mathcal{F}$ we have $f' = (g_nu_n)^\omega$ for some $u_n \in N$. Hence

$$\text{dist}(1, f') = \lim^\omega \frac{|g_nu_n|}{d_n} \geq \lim^\omega \frac{|s(g_nN)|}{d_n} = \text{dist}(1, f).$$

Thus $\text{dist}(1, \mathcal{F}) = \text{dist}(1, f) = \text{dist}(1, b)$. □

Proof of Proposition 5.3 We first consider a segment $I = [a, b] \subset T$ and any preimage x of a . Let f be the point of \mathcal{F}_b provided by Lemma 5.4. Recall that $\text{Con}^\omega(G, d)$ is a geodesic metric space. Let $J_I = [x, f]$ be a geodesic segment in $\text{Con}^\omega(G, d)$. Note that $\hat{\alpha}$ does not increase the distance. If $u, v \in J_I$ and $\text{dist}(\alpha(u), \alpha(v)) < \text{dist}(u, v)$, then

$$\begin{aligned} \text{dist}(a, b) &\leq \text{dist}(a, \alpha(u)) + \text{dist}(\alpha(u), \alpha(v)) + \text{dist}(\alpha(v), b) \\ &< \text{dist}(x, u) + \text{dist}(u, v) + \text{dist}(v, f) = |J_I| \end{aligned}$$

that contradicts the choice of f . Thus $\hat{\alpha}$ isometrically maps J_I to its image in $\text{Con}^\omega(G/N, d)$. Since $|J_I| = \text{dist}(x, f) = \text{dist}(a, b) = |I|$ by the choice of f , $\hat{\alpha}(J_I)$ is a geodesic segment in $\text{Con}^\omega(G/N, d)$. As T is convex, we have $\hat{\alpha}(J_I) \in T$. Hence $\hat{\alpha}(J_I) = I$. Thus for any preimage x of a , there is an isometric section $\sigma_I: I \rightarrow \text{Con}^\omega(G, d)$ such that $\sigma_I(a) = x$.

Let us fix a vertex o of T . Suppose that we have already found an isometric section σ_{T_0} for a subtree $T_0 \subset T$ containing o such that $y \in T_0$. Let $b \in T \setminus T_0$. Then there is a unique point $a \in T_0$ such that $[b, o] \cap T_0 = [a, o]$. Let $x = \sigma_{T_0}(a)$ and let $\sigma_I: I \rightarrow \text{Con}^\omega(G, d)$ be an isometric section for $I = [a, b]$ such that $\sigma_I(a) = x$. We then define an isometric section $\sigma_{T_1}: T_1 \rightarrow \text{Con}^\omega(G, d)$, where $T_1 = T_0 \cup [a, b]$, by the rule

$$\sigma_{T_1}(t) = \begin{cases} \sigma_{T_0}(t), & \text{if } t \in T_0, \\ \sigma_I(t), & \text{if } t \in [a, b]. \end{cases}$$

Now we can complete the proof by transfinite induction. □

Corollary 5.5 *If $\text{Con}^\omega(G/N, d)$ is an \mathbb{R} -tree for some scaling sequence $d = (d_n)$ and nonprincipal ultrafilter ω , then there is an isometric section $\sigma: \text{Con}^\omega(G/N, d) \rightarrow \text{Con}^\omega(G, d)$ of the map $\hat{\alpha}$.*

5.2 Central extensions of lacunary hyperbolic groups

We keep the notation from the previous section here. Given a product $X \times Y$ of metric spaces X and Y , by the *product metric* we mean the metric on $X \times Y$ defined by the rule

$$\text{dist}_{X \times Y}((x_1, y_1), (x_2, y_2)) = \text{dist}_X(x_1, x_2) + \text{dist}_Y(y_1, y_2).$$

Recall that every 2-dimensional cohomology class on a hyperbolic group can be represented by a bounded cocycle by Neumann and Reeves [37] (see also Mineyev [36], where it is proved for all dimensions ≥ 2). This implies that for any finitely generated group G and any finitely generated central subgroup $N \leq G$ such that G/N is hyperbolic, G is quasi-isometric to $N \times G/N$ [37]. (A particular case was also proved

in Gersten [17].) Therefore, for any d and ω , $\text{Con}^\omega(G, d)$ is bi-Lipschitz equivalent to $\text{Con}^\omega(N, d) \times \text{Con}^\omega(G/N, d)$ endowed with the product metric. In this section we generalize this result to the class of lacunary hyperbolic groups as follows.

Theorem 5.6 *Let N be a central subgroup of a finitely generated group G endowed with the induced metric. Suppose that for some nonprincipal ultrafilter ω and some scaling sequence $d = (d_n)$, $\text{Con}^\omega(G/N, d)$ is an \mathbb{R} -tree. Then $\text{Con}^\omega(G, d)$ is bi-Lipschitz equivalent to $\text{Con}^\omega(N, d) \times \text{Con}^\omega(G/N, d)$ endowed with the product metric.*

Proof Let $\sigma: \text{Con}^\omega(G/N, d) \rightarrow \text{Con}^\omega(G, d)$ be the isometric section provided by Corollary 5.5. We define a map $\varkappa: \text{Con}^\omega(N, d) \times \text{Con}^\omega(G/N, d) \rightarrow \text{Con}^\omega(G, d)$ as follows. Suppose that $x \in \text{Con}^\omega(G/N, d)$ and $g = (g_n)^\omega \in \text{Con}^\omega(N, d)$, where $(g_n) \in N_e^\omega(d)$. Then $\varkappa(g, x) = (g_n)\sigma(x)$.

First observe that \varkappa is well-defined. Indeed if $(g_n)^\omega = (h_n)^\omega \in \text{Con}^\omega(N, d)$, then $g_n = h_n u_n$, where $u_n \in N$ and $|u_n| = o_\omega(d_n)$. Since N is central, for any $y = (y_n)^\omega \in \text{Con}^\omega(G, d)$, we have

$$(g_n)(y_n)^\omega = (h_n u_n y_n)^\omega = (h_n y_n u_n)^\omega = (h_n y_n)^\omega = (h_n)(y_n)^\omega.$$

Further observe that for any $y = (y_n)^\omega \in \text{Con}^\omega(G, d)$ and $g_1 = (g_{1n})^\omega, g_2 = (g_{2n})^\omega \in \text{Con}^\omega(N, d)$ we have

$$\text{dist}(g_{1n}, g_{2n}) = \text{dist}(y_n g_{1n}, y_n g_{2n}) = \text{dist}(g_{1n} y_n, g_{2n} y_n).$$

Therefore,

$$(24) \quad \text{dist}(g_1, g_2) = \text{dist}((g_{1n})y, (g_{2n})y).$$

Suppose now that $(g_1, x_1), (g_2, x_2) \in \text{Con}^\omega(N, d) \times \text{Con}^\omega(G/N, d)$, where $g_1 = (g_{1n})^\omega, g_2 = (g_{2n})^\omega \in \text{Con}^\omega(N, d)$, and $k_1 = \varkappa(g_1, x_1), k_2 = \varkappa(g_2, x_2)$. Applying (24) we obtain

$$(25) \quad \begin{aligned} \text{dist}(k_1, k_2) &= \text{dist}((g_{1n})\sigma(x_1), (g_{2n})\sigma(x_2)) \\ &\leq \text{dist}((g_{1n})\sigma(x_1), (g_{2n})\sigma(x_1)) + \text{dist}((g_{2n})\sigma(x_1), (g_{2n})\sigma(x_2)) \\ &= \text{dist}(g_1, g_2) + \text{dist}(\sigma(x_1), \sigma(x_2)) \\ &= \text{dist}(g_1, g_2) + \text{dist}(x_1, x_2). \end{aligned}$$

Note that $\hat{\alpha}(\varkappa(g, x)) = \hat{\alpha}(\sigma(x)) = x$ since the action of $G_e^\omega(d)$ preserves fibers. Hence

$$(26) \quad \text{dist}(x_1, x_2) = \text{dist}(\hat{\alpha}(k_1), \hat{\alpha}(k_2)) \leq \text{dist}(k_1, k_2).$$

Now reversing the inequality and replacing pluses with minuses in (25), we obtain

$$(27) \quad \text{dist}(g_1, g_2) \leq \text{dist}(k_1, k_2) + \text{dist}(x_1, x_2) \leq 2\text{dist}(k_1, k_2).$$

Finally combining (26) and (27) we obtain

$$\text{dist}((g_1, x_1), (g_2, x_2)) = \text{dist}(g_1, g_2) + \text{dist}(x_1, x_2) \leq 3\text{dist}(k_1, k_2).$$

This inequality together with (25) shows that \varkappa is a 3-bi-Lipschitz map. To complete the proof it remains to note that \varkappa is surjective by Theorem 5.2 (d). \square

Our next goal is to prove Theorem 5.8, which will be used in the next section. We start with the following general lemma.

Lemma 5.7 *Let G be a non-virtually cyclic finitely generated group, $\mathcal{C} = \text{Con}^\omega(G, d)$ its asymptotic cone. Let $X \subseteq \mathcal{C}$ be a finite subset. Then for every $x \in \mathcal{C} \setminus X$ there exists a path p in $\mathcal{C} \setminus X$ with $p_- = x$, containing points arbitrary far away from x .*

Proof Since G is finitely generated, there exists an infinite geodesic ray in the Cayley graph of G . Its ultralimit is an infinite geodesic ray r in \mathcal{C} . Since \mathcal{C} is homogeneous, we can assume that $r_- = x$. We can also assume that $x = e = (1)^\omega$.

Let $X = \{x_1, \dots, x_m\}$, $x_i = (x_i(n))^\omega$. Let $2l = \min\{\text{dist}(e, x_i), i = 1, \dots, m\}$, $l_i = \text{dist}(e, x_i)$. For every $g = (g_n) \in G_e^\omega(d)$ with $|g_n| \leq ld_n$ consider the ray $g^{-1}r$. The union of $g^{-1}r$ and any geodesic $[e, g^{-1}e]$ is a path $p(g)$ starting at e and containing points arbitrary far from e .

If one of these paths does not contain any x_i , we are done. Assume that every $p(g)$ contains x_i from X . For every $i = 1, \dots, m$ let M_i be the set of all $g = (g_n)^\omega$ with $|g_n| \leq ld_n$ such that $p(g)$ contains x_i . Then the union of M_i contains the limit $\lim^\omega \text{Ball}_G(e, ld_n)$.

For every $g = (g_n)^\omega \in M_i$ $g^{-1}r$ must contain x_i because $[e, g^{-1}e]$ is too short to contain x_i . Hence $gx_i \in r$. Since $\text{dist}(e, ge) \leq l$, $\text{dist}(gx_i, ge) = \text{dist}(x_i, e) = l_i$, we conclude that gx_i belongs to the subgeodesic $r[l_i - l, l_i + l]$ of the ray r . Pick a number $N \geq 2$ and divide the interval $r[l_i - l, l_i + l]$ into $N + 1$ equal subintervals. Let a_1, \dots, a_N be the division points, $a_j = (a_j(n))^\omega$. Then for every $g = (g_n)^\omega \in M_i$ the point gx_i is within distance $2l/(N + 1)$ from one of a_j . Hence

$$\text{dist}_G(g_n x_i(n), a_j(n)) \leq \frac{2ld_n}{N + 1} + o_\omega(d_n) \leq \frac{2ld_n}{N}$$

ω -almost surely. Therefore $g_n x_i(n)$ is in the ball B_j of radius $2ld_n/N$ around $a_j(n)$ in G ω -almost surely. Hence, for any fixed $i \in \{1, \dots, m\}$, the number of elements

g_n such that $g_n x_i(n) \in B_j$ does not exceed N times the number of elements of the ball $\text{Ball}_G(2ld_n/N)$.

Let $f(n)$ be the growth function of the group G . Since the union of M_i contains the limit $\lim^\omega \text{Ball}_G(ld_n)$, we proved in the previous paragraph that for every $N \geq 2$,

$$f(ld_n) \leq mNf\left(\frac{ld_n}{N}\right)$$

ω -almost surely.

By Gromov [19, page 68] (see also van den Dries and Wilkie [9]), then the asymptotic cone $\text{Con}(G, (ld_n))$ is locally compact. As shown by Point [48], that asymptotic cone has Minkovski dimension 1 (for the definition, see [48]). Then by [48] G is virtually nilpotent and $\text{Con}^\omega(G, (ld_n))$ is homeomorphic to \mathbb{R} . But then [13, Proposition 6.1] implies that G is virtually cyclic, which contradicts the assumption of the lemma. \square

For every metric space X , we define a *connectedness degree* $c(X)$ as the minimal number of points of X whose removal disconnects X . If X can not be disconnected by removing finitely many points, we set $c(X) = \infty$. In particular, X has cut points if and only if $c(X) = 1$. By a *cut set* of X we mean any subset of X whose removal disconnects X .

Theorem 5.8 *Let N be a central subgroup of a finitely generated group G . Suppose that $\text{Con}^\omega(N, d)$ consists of $m < \infty$ points for some nonprincipal ultrafilter ω and some scaling sequence $d = (d_n)$. Then*

$$c(\text{Con}^\omega(G, d)) = mc(\text{Con}^\omega(G/N, d)).$$

Moreover, a finite subset $C \subset \text{Con}^\omega(G, d)$ disconnects $\text{Con}^\omega(G, d)$ if and only if C contains a full preimage of a cut set of $\text{Con}^\omega(G/N, d)$ under the map $\hat{\alpha}: \text{Con}^\omega(G, d) \rightarrow \text{Con}^\omega(G/N, d)$.

Proof First note that G is not virtually cyclic. Indeed otherwise either N is finite or N contains an infinite cyclic group. In the first case $\text{Con}^\omega(G, d)$ is \mathbb{R} , $m = 1$, and the proposition is obvious. In the second case $\text{Con}^\omega(N, d)$ contains infinitely many points that contradicts $m < \infty$.

Take a finite set C in $\text{Con}^\omega(G, d)$. Let \hat{C} be the full preimage of $\hat{\alpha}(C)$ under $\hat{\alpha}$. Then \hat{C} is finite.

We shall need the following statement.

Lemma 5.9 *Let $a, a' \in \text{Con}^\omega(G, d) \setminus \hat{C}$ and $\hat{\alpha}(a) = \hat{\alpha}(a')$. Then a and a' are in the same connected component of $\text{Con}^\omega(G, d) \setminus \hat{C}$.*

Proof Let $\gamma \in N_e^\omega(d)$ be such that $\gamma a = a'$ (such γ exists by [Theorem 5.2](#)). By [Lemma 5.7](#), there exists a path r in $\text{Con}^\omega(G, d) \setminus \widehat{C}$ with $r_- = a$ containing points arbitrary far away from a . Consider the path $r' = \gamma r$. Then $r'_- = a'$. Note that \widehat{C} is closed under the action of $N_e^\omega(d)$. Hence r' does not contain points from \widehat{C} . Consider a point z on r such that $\text{dist}(z, C) > \text{dist}(a, a')$. Let $z' = \gamma z$ (using [Theorem 5.2](#) again). Since $\text{Con}^\omega(G, d)$ is homogeneous, there exists $\beta \in G_e^\omega(d)$ such that $z = \beta a$. Since N is central in G , we have

$$\text{dist}(z, z') = \text{dist}(\beta a, \gamma \beta a) = \text{dist}(\beta a, \beta \gamma a) = \text{dist}(a, a').$$

Therefore any geodesic path from z to z' avoids \widehat{C} .

Now consider the path w that goes first from a to z along r , then from z to z' along any geodesic $[z, z']$, then goes back to a' along r' . That path avoids points from \widehat{C} . \square

Let us continue the proof of [Theorem 5.8](#).

Suppose the finite set C does not contain a full preimage of a cut set of $\text{Con}^\omega(G/N, d)$ under $\widehat{\alpha}$. We need to show that C is not a cut set of $\text{Con}^\omega(G, d)$ that is any two points $u, v \in \text{Con}^\omega(G, d) \setminus C$ can be connected by a path avoiding C . Take any geodesic p connecting $u, v \in \text{Con}^\omega(G, d) \setminus C$. If this geodesic does not contain points in C , we are done. Suppose that p contains a point from C . Then it is enough to show how to replace subpaths of p connecting points close enough to points in C . Thus without loss of generality we can assume that u and v are from $\text{Ball}(c, \zeta) \setminus \widehat{C}$ for some $c \in C$ and some small enough ζ .

Pick any ζ such that $\text{Ball}(c, \zeta) \cap \widehat{C} = \{c\}$. Pick two points $u, v \in \text{Ball}(c, \zeta) \setminus \{c\}$. Then $u, v \notin \widehat{C}$.

Suppose that $\widehat{\alpha}(C)$ is not a cut set of $\text{Con}^\omega(G/N, d)$. Then $\widehat{\alpha}(u)$ can be connected with $\widehat{\alpha}(v)$ by a path q in $\text{Con}^\omega(G/N, d) \setminus \widehat{\alpha}(C)$. Since $\widehat{\alpha}$ is a covering map by [Theorem 5.2](#) (e), we can lift the path q to a path \widehat{p} in $\text{Con}^\omega(G, d)$ avoiding \widehat{C} and such that $\widehat{p}_- = u$. Note that $\widehat{\alpha}(\widehat{p}_+) = \widehat{\alpha}(v)$. Hence by [Lemma 5.9](#), we can connect \widehat{p}_+ with v by a path p_1 in $\text{Con}^\omega(G, d) \setminus \widehat{C}$. The composition of \widehat{p} and p_1 connects u and v and avoids C as required.

Now suppose that $\widehat{\alpha}(C)$ is a cut set in $\text{Con}^\omega(G/N, d)$. Let C' be the union of all fibers of $\widehat{\alpha}$ contained in C . We can assume that $\widehat{\alpha}(C')$ is not a cut set. Therefore there exists a path from $\widehat{\alpha}(u)$ to $\widehat{\alpha}(v)$ in $\text{Con}^\omega(G/N, d) \setminus \widehat{\alpha}(C')$. That path must contain points $\widehat{\alpha}(c_1)$ for some $c_1 \in C$ with $c_1 \notin C'$. Lifting this path to $\text{Con}^\omega(G, d)$ and using [Lemma 5.9](#), we can obtain a path connecting u and v in $\text{Con}^\omega(G, d) \setminus C'$ but containing points in $C \setminus C'$. If we could replace parts of this path connecting points close the points in $C \setminus C'$ by paths avoiding these points we would show that C is not

a cut set. Thus without loss of generality we can assume that u, v are very close to a point in $C \setminus C'$, ie we can assume that $c \in C \setminus C'$.

Then there exists a point $c' \notin C$ with $\hat{\alpha}(c') = \hat{\alpha}(c)$. Let $\gamma \in N_e^\omega(d)$ be such that $\gamma c = c'$. We can assume that ζ is small enough so that $\gamma \text{Ball}(c, \zeta)$ does not intersect C . By Lemma 5.9, there exist paths r, r' in $\text{Con}^\omega(G, d) \setminus \hat{C}$ with $r_- = u, r_+ = \gamma u, r'_- = v, r'_+ = \gamma v$. Then one can travel from u to v by first going to γu along r then by a path from γu to γv inside $\gamma \text{Ball}(c, \zeta)$, then back to v along r' . That path avoids C . Hence C is not a cut set in $\text{Con}^\omega(G, d)$.

We have proved that every cut set of $\text{Con}^\omega(G/N, d)$ contains a full preimage of a cut set of $\text{Con}^\omega(G/N, d)$. This implies

$$c(\text{Con}^\omega(G, d)) \geq mc(\text{Con}^\omega(G/N, d)).$$

In order to prove the opposite inequality, note that if a finite set $\{b_1, \dots, b_c\}$ disconnects $\text{Con}^\omega(G/N, d)$ then the union of fibers $\mathcal{F}_{b_1}, \dots, \mathcal{F}_{b_c}$ disconnect $\text{Con}^\omega(N, d)$ because $\hat{\alpha}$ is continuous.

5.3 Applications

Let us consider a family of central extensions of a lacunary hyperbolic group constructed as follows. By Corollary 4.15 and Remark 4.8 there is a presentation

$$H = \left\langle a, b \mid \bigcup_{i=1}^{\infty} \mathcal{R}_i \right\rangle$$

that simultaneously satisfies the graded small cancellation condition and the classical small cancellation condition $C'(1/24)$, where for every i , \mathcal{R}_i consists of cyclic shifts of a single word R_i and its inverses. Throughout this section we fix any such presentation and denote by r_n the length of the word R_n . Given a sequence of integers $k = (k_n)$, where $k_n \geq 2$, we consider the central extension of H defined as follows.

$$(28) \quad G(k) = \langle a, b \mid [R_n, a] = 1, [R_n, b] = 1, R_n^{k_n} = 1, n = 1, 2, \dots \rangle$$

We begin with auxiliary results.

Lemma 5.10 *Let U be a subword of a word $R_n^{k_n}$ of length at most $k_n r_n / 2$. Then the length of the element represented by the word U in G is at least $|U|/8$.*

Proof Suppose that $U = V$ in G where the word V is geodesic in G . Let Δ be a diagram over G corresponding to this equality. Let $\partial\Delta = pq^{-1}$ be the decomposition of the boundary of Δ , where $U \equiv \text{Lab}(p), V \equiv \text{Lab}(q)$. We can turn Δ into a

diagram Δ' over H with the same boundary label by the following procedure. Every cell corresponding to the relation $R_i^{k_i}$ is replaced by a union of k_i cells each labeled by R_i connected by a point. Every cell labeled by $[R_i, a]$ or $[R_i, b]$ is replaced by the union of two cells labeled by R_i connected by an edge labeled by a or b . In both cases the union has the same boundary label as the original cell (see Figure 6).

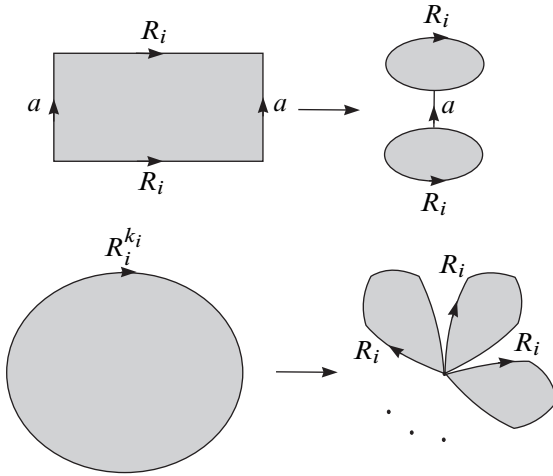


Figure 6: Transforming cells in Δ

To any cell Π labeled by $R_n^{\pm 1}$ in Δ' , we assign the number -1 if R_n reads along $\partial\Pi$ in the clockwise direction and 1 otherwise. By the *algebraic number* of R_n -cells in Δ' we mean the sum of the assigned numbers over all cells labeled by $R_n^{\pm 1}$. Notice that the algebraic number of R_n -cells in Δ' is divisible by k_n . If we reduce Δ' by canceling pairs of cells having a common edge and being mirror images of each other, we obtain a diagram Δ'' with the same property. We keep the notation pq^{-1} for the boundary of Δ'' . We have to show that

$$(29) \quad |q| \geq |p|/8.$$

Assume that the boundary of some cell Π in Δ'' has a common subpath of length at least $|\partial\Pi|/24$ with p . Then $\partial\Pi$ is labeled by R_n by the $C'(1/24)$ -condition, and since p is labeled by a power of R_n we may assume without loss of generality that $\partial\Pi \subset p$. After cutting such a cell off we obtain a new diagram with boundary $p'q^{-1}$, where $\text{Lab}(q')$ is again a power of R_n . Continuing this process we obtain a subdiagram Σ of Δ'' with boundary tq^{-1} , where t is labeled by a power of R_n , such that no R_n -cell Π of Σ has a common subpath of lengths at least $|\partial\Pi|/24$ with t . Now there are two cases to consider.

Case 1 Suppose that $|t| \geq |p|/4$. Observe that any cell Π in Σ satisfies the following condition:

- (+) $\left\{ \begin{array}{l} \text{For any two vertices } x, y \in t \cap \partial\Pi, \text{ there is a common subpaths of } \partial\Pi \\ \text{and } t \text{ connecting } x \text{ and } y. \end{array} \right.$

Indeed otherwise there is a subdiagram Ξ of Σ such that Ξ contains at least one cell and $\partial\Xi = s_1s_2$, where s_1 is a subpath of t and s_2 is a subpath of $\partial\Pi$. Note that a common subpath of the boundary of any cell Ω in Ξ and s_1 (respectively s_2) has length less than $|\partial\Omega|/24$ by the construction of Σ (respectively since Δ'' is reduced). This contradicts [Lemma 3.7](#).

In particular, (+) and the $C'(1/24)$ -condition imply that for any cell Π of Σ , the intersection $\partial\Pi \cap t$ is a path of length less than $|\partial\Pi|/24$. Note that if the number of common edges of t and q is at least $|p|/8$, the inequality (29) is obvious. Hence we may assume that more than $|t| - |p|/8 \geq |p|/8$ edges of t belong to cells of Σ . Therefore the sum of perimeters of all cells in Σ is greater than $24|p|/8 = 3|p|$. Applying [Lemma 3.8](#) to Δ'' we obtain $|p| + |q| \geq 9|p|/4$, which yields (29).

Case 2 Suppose that $|t| \leq |p|/4$. This means that we have to cut at least $k = 3|p|/(4r_n)$ R_n -cells to get Σ from Δ'' . Note that $k \leq |p|/r_n \leq k_n/2$ and all these cells have the same orientation by the $C'(1/24)$ -condition. Since the algebraic number of R_n -cells in Δ'' should be divisible by k_n , the total number of R_n -cells in Δ'' is at least $2k$. Applying [Lemma 3.8](#) again, we obtain $|p| + |q| \geq 6kr_n/4 \geq 9|p|/8$. Hence $|q| \geq |p|/8$. □

Theorem 5.11 *For every $m \geq 2$ there exists a finitely generated group G such that for any ultrafilter ω and any scaling sequence $d = (d_n)$, exactly one of the following possibilities occurs and both of them can be realized for suitable ω and d .*

- (a) $\text{Con}^\omega(G, d)$ is an m -fold cover of a circle-tree and $c(\text{Con}^\omega(G, d)) = m$. Moreover, a finite subset $C \subset \text{Con}^\omega(G, d)$ disconnects $\text{Con}^\omega(G, d)$ if and only if C contains a fiber of the map $\hat{\alpha}: \text{Con}^\omega(G, d) \rightarrow \text{Con}^\omega(G/N, d)$.
- (b) $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree.

In particular, in both cases $\text{Con}^\omega(G, d)$ is locally isometric to an \mathbb{R} -tree.

Proof Let $G = G(k)$ be the group corresponding to the sequence $k_n = m \geq 2$ for all n . The central subgroup $N = \langle R_1, R_2, \dots \rangle$ inherits a metric from G . Let

$g = (g_n)^\omega \in \text{Con}^\omega(N, d)$. If $g_n \neq 1$, then $g_n = R_1^{\varepsilon_1} R_2^{\varepsilon_2} \dots R_{j(n)}^{\varepsilon_{j(n)}}$, where $0 \leq \varepsilon_i \leq m-1$ for all i and $\varepsilon_{j(n)} \neq 0$. Note that

$$\text{dist}(g_n, R_{j(n)}^{\varepsilon_{j(n)}}) \leq \sum_{i=1}^{j(n)-1} |R_i^{\varepsilon_i}| \leq (m-1) \sum_{i=1}^{j(n)-1} |R_i| = o_\omega(R_{j(n)})$$

by (Q_3) , Lemma 4.13 (c), and Lemma 5.10. Hence

$$(30) \quad (g_n)^\omega = (R_{i_n}^s)^\omega$$

for some $s \in \{0, 1, \dots, m-1\}$. Now there are two cases to consider.

Case 1 Suppose that (r_n) is not (ω, d) -visible. Then (30) implies $(g_n)^\omega = (1)^\omega$, ie, $\text{Con}^\omega(N, d)$ is a point. Applying the last assertion of Theorem 5.2 and Theorem 4.17, we obtain (b).

Case 2 Assume now that (r_n) is (ω, d) -visible (see Definition 4.16). Let (R_{i_n}) be the sequence such that $r_{i_n} = \Theta_\omega(d_n)$. Note any sequence (i'_n) satisfying $r_{i'_n} = \Theta_\omega(d_n)$ is ω -equal to (i_n) (that is, $i_n = i'_n$ ω -almost surely) by (Q_4) and Lemma 4.13 (c). Therefore $\text{Con}^\omega(N, d)$ contains at most m points.

In contrast, the points $(R_{i_n}^s)^\omega$ are different for different values of $s \in \{0, 1, \dots, m-1\}$. Indeed if $(R_{i_n}^s)^\omega = (R_{i_n}^t)^\omega$ for some $s \neq t$, $s, t \in \{0, 1, \dots, m-1\}$, then $|R_{i_n}^{s-t}| = o_\omega(d_n)$. Passing from $s-t$ to $m-(s-t)$ if necessary, we obtain $|R_{i_n}^l| = o_\omega(d_n) = o_\omega(r_{i_n})$ for some $0 < l \leq m/2$. However this contradicts Lemma 5.10. Hence $\text{Con}^\omega(N, d)$ consists of exactly m points. Applying Theorem 5.2 again, we obtain that the map $\text{Con}^\omega(G, d) \rightarrow \text{Con}^\omega(G/N, d)$ induced by the natural homomorphism is locally isometric and each fiber consists of m points, ie, $\text{Con}^\omega(G, d)$ is an m -fold cover of $\text{Con}^\omega(G/N, d)$. Note that by Theorem 4.17 $\text{Con}^\omega(G/N, d)$ is a circle-tree. Hence it contains cut points by Lemma 2.3. Therefore a finite subset $C \subset \text{Con}^\omega(G, d)$ disconnects $\text{Con}^\omega(G, d)$ if and only if C contains a fiber of the map $\hat{\alpha}: \text{Con}^\omega(G, d) \rightarrow \text{Con}^\omega(G/N, d)$ by Theorem 5.8. In particular, $c(\text{Con}^\omega(G, d)) = m$.

Finally we note that (Q_4) and Lemma 4.13 (c) guarantees that the second case occurs. The first case occurs when taking $d_n = r_n$ and any ω . □

Below we denote by \mathbb{S}^1 the unit circle with the lengths metric.

Theorem 5.12 *There exists a finitely generated group G and a scaling sequence $d = (d_n)$ such that for any ultrafilter ω , $\text{Con}^\omega(G, d)$ is bi-Lipschitz equivalent to the product of an \mathbb{R} -tree and \mathbb{S}^1 . In particular, $\pi_1(\text{Con}^\omega(G, d)) = \mathbb{Z}$.*

Proof Let $G(k)$ be the group corresponding to a sequence $k = (k_n)$ such that

$$(31) \quad k_n \rightarrow \infty \quad \text{and} \quad k_n r_n = o(r_{n+1}).$$

The existence of such a sequence is guaranteed by the equality $r_n = o(r_{n+1})$, which follows from (Q_4) and Lemma 4.13 (c). Set $d_n = k_n r_n$, $d = (d_n)$. As in the proof of the previous theorem let $g = (g_n)^\omega \in \text{Con}^\omega(N, d)$. If $g_n \neq 1$, then $g_n = R_1^{\varepsilon_1} R_2^{\varepsilon_2} \cdots R_{j(n)}^{\varepsilon_{j(n)}}$, where $0 \leq \varepsilon_i \leq k_i - 1$ for all i and $\varepsilon_{j(n)} \neq 0$. Note that

$$\text{dist}(g, R_{j(n)}^{\varepsilon_{j(n)}}) \leq \sum_{i=1}^{j(n)-1} |R_i^{\varepsilon_i}| \leq \sum_{i=1}^{j(n)-1} k_i |R_i| = o_\omega(R_{j(n)})$$

by (31) and Lemma 5.10. Therefore $(g_n)^\omega = (R_{j(n)}^{s(n)})^\omega$ where

$$s(n) \in \{0, 1, \dots, k_n - 1\}.$$

Note that if $(g_n)^\omega \neq (1)^\omega$, then $r_{j_n} = \Theta_\omega(|g_n|) = \Theta_\omega(d_n)$ and hence $j_n = n$ ω -almost surely. Thus $(g_n)^\omega \in \lim^\omega p_n$, where p_n is the cycle in $\Gamma(G, \{a, b\})$ that begins and ends at 1 and has label $R_n^{k_n}$. Since $|R_n| = r_n = o(d_n)$, $\lim^\omega(p_n)$ coincides with the set of points of type $(R_n^{s(n)})^\omega \in \text{Con}^\omega(N, d)$ and we obtain $\text{Con}^\omega(N, d) = \lim^\omega p_n$.

Observe that by Lemma 5.10 p_n is $(1/4, 1)$ -quasi-isometric to a circle of lengths $|p_n| = k_n r_n = d_n$. Hence $\lim^\omega p_n$ is bi-Lipschitz equivalent to the unit circle. Note that by (31) the sequence r_n is not (ω, d) -visible. Hence $\text{Con}^\omega(G/N, d)$ is an \mathbb{R} -tree by Theorem 4.17. Applying now Theorem 5.6, we obtain that $\text{Con}^\omega(G, d)$ is bi-Lipschitz equivalent to the product of an \mathbb{R} -tree and \mathbb{S}^1 . □

Remark 5.13 Arguing as in the proof of Theorem 5.11 it is not hard to classify all asymptotic cones of the group $G(k)$ from the proof of Theorem 5.12 as follows. For any scaling sequence d_n and any ultrafilter ω , exactly one of the conditions (a)–(c) below holds and all possibilities can be realized.

- (a) (r_n) is (ω, d) -visible, $\text{Con}^\omega(G, d)$ is an infinite degree cover of a circle-tree.
- (b) $(k_n r_n)$ is (ω, d) -visible and $\text{Con}^\omega(G, d)$ is bi-Lipschitz equivalent to the product of an \mathbb{R} -tree and a unit circle.
- (c) There exists an (ω, d) -visible sequence (c_n) such that $c_n = o(k_n r_n)$ and $r_n = o(c_n)$. In this case $\text{Con}^\omega(G, d)$ is bi-Lipschitz equivalent to the product of an \mathbb{R} -tree and \mathbb{R} .
- (d) There is no (ω, d) -visible sequence (c_n) such that $r_n \leq c_n \leq k_n r_n$. In this case $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree.

In particular, even the finiteness of the connectedness number $c(\text{Con}^\omega(G, d))$ for a given group G is not invariant under changing d and ω .

6 Lacunar hyperbolicity and divergence

6.1 Divergence of nonconstricted groups

The following general statement allows one to estimate the divergence function of a group with no cut points in some of its asymptotic cones. Recall that given a path p in a metric space, we denote by p_- and p_+ the beginning and the ending points of p respectively. The length of p is denoted by $|p|$.

Theorem 6.1 *Let G be a finitely generated group. Suppose that for some sequence of scaling constants d_n and every ultrafilter ω , the asymptotic cone $\text{Con}^\omega(G, (d_n))$ does not have cut points. Let $f(n) \geq n$ be a nondecreasing function such that $d_n \leq f(d_{n-1})$ for all sufficiently large n . Then the divergence function $\text{Div}(n)$ of G does not exceed $Cf(n)$ for some constant C (and all n).*

Proof Since the asymptotic cone does not change if we change a finite subsequence of (d_n) , we can assume without loss of generality that $d_1 = \frac{1}{4}$. Taking a constant multiple of f if necessary, we can assume that $d_n \leq f(d_{n-1})$ for all $n \geq 2$.

Since \mathcal{C} does not have cut points for any choice of ω , by [12, Theorem 2.1], we can conclude that there exists a constant C_1 such that $\text{Div}(n)$ is bounded by C_1n for every n in any interval $[d_k/18, 18d_k]$ for every $k \geq 2$.

Let $\delta = \frac{1}{4}$, $\lambda = 2$. For every $n \geq 1$ choose elements a_n, b_n, c_n , in G with $\text{dist}(a_n, b_n) \leq n$ and such that $\text{Div}_\lambda(a_n, b_n, c_n; \delta)$ is maximal possible, ie

$$\text{Div}(n) = \text{Div}_\lambda(a_n, b_n, c_n; \delta).$$

Suppose, by contradiction, that $\text{Div}(n)$ is not smaller than $Cf(n)$ for some constant C and all n . Then for every $m \geq 1$ there exists $n = n(m)$ such that $\text{Div}(n) > mf(n)$. Pick $m > 12 + 18C_1$. Let $a = a_n, b = b_n, c = c_n$ where $n = n(m)$. Let $r = \text{dist}(c, \{a, b\})$. Let B be the ball of radius δr around c . Without loss of generality assume that $\text{dist}(c, a) = r$.

Note that any geodesic h connecting a and b passes through B since $\text{Div}_\gamma(a, b, c; \delta) > n$. Hence $r \leq 2n$. (Indeed, if $r > 2n$ every point in h is at distance at least $r - |h| \geq r - n > \frac{r}{2} > \delta r$ from c , and cannot belong to B .) Let c' be a point in $h \cap B$, so $c' \in h$ is at distance at most δr from c . Then $\text{dist}(a, c') \leq r(1 + \delta) \leq 2r$.

Let b' be either b or the point between c' and b at distance $2r$ from c' . Let h' be the part of h between c' and b . Then h' does not intersect B (any point in h' is at distance $\geq (1 - \delta)r > 2\delta r$ from B).

Since $\delta < \frac{1}{3}$, $\text{dist}(a, B), \text{dist}(b, B) > 2\delta r$ which exceeds the diameter of B .

Since the Cayley graph Γ of G is infinite, homogeneous and locally finite, for every vertex x in Γ there exists a bi-infinite geodesic $q(x)$ passing through x .

Consider the geodesic $q(a)$. The point a cuts $q(a)$ into two geodesic rays $l(a)$ and $l(a)'$. Since $\text{dist}(a, B)$ is greater than the diameter of B , one of these rays does not pass through the ball B . Let it be $l(a)$. Similarly, let $l(b')$ be a geodesic ray starting at b' and not passing through B .

Choose the smallest $k \geq 2$ such that $r/2 \leq d_k$. Note that then $d_{k-1} \leq r/2$ (even if $k = 2$ since $d_1 = \frac{1}{4}$ and $r \geq 1$). Let x be the point in $l(a)$ at distance $5d_k$ from a , and let y be the point in $l(b')$ at distance $5d_k$ from b' .

Case 1 Suppose that $\text{dist}(x, y) < d_k/2$. Then consider a geodesic p connecting x and y . Any point in p is at distance at least $5d_k - d_k/2 > 2r$ from a . Hence any point in p is at distance at least $2r - r > \delta r$ from c . Thus p does not intersect B , so we found a path $[a, x] \cup p \cup [y, b'] \cup h'$ of length at most $11d_k + n$ connecting a and b . Then

$$11d_k + n \geq \text{Div}_\lambda(a, b, c; \delta) = \text{Div}(n) \geq mf(n).$$

But $11d_k + n \leq 11f(d_{k-1}) + n \leq 11f(r/2) + n \leq 12f(n)$ since f is a nondecreasing function and $f(n) \geq n$. Thus $12f(n) \geq mf(n)$, a contradiction since $m > 12$.

Case 2 Suppose that $\text{dist}(x, y) \geq d_k/2$. Since $\text{dist}(x, y) \leq 10d_k + \text{dist}(a, b') \leq 10d_k + 4r \leq 18d_k$, the distance $\text{dist}(x, y)$ is in the interval $[d_k/18, 18d_k]$, and so there exists a path p of length at most $C_1 \text{dist}(x, y)$ avoiding the ball of radius $\delta \text{dist}(c, \{x, y\}) > \delta(5d_k - r) \geq \delta(5d_k - 2d_k) > \delta r$ around c . Then the path $[a, x] \cup p \cup [y, b'] \cup h'$ connects a and b , avoids B , and has length at most

$$10d_k + C_1(18d_k) + n \leq (10 + 18C_1)f(d_{k-1}) + f(n) \leq (11 + 18C_1)f(n),$$

a contradiction since $m > 11 + 18C_1$. □

6.2 Torsion groups with slow nonlinear divergence

Let $F_2 = \langle a, b \rangle$ be the free group of rank 2. We fix an arbitrary odd prime p and a large odd power n_0 of p , say, $n_0 > 10^{80}$. Let $G(0) = F_2$, ie the set of relators \mathcal{R}_0 of rank 0 is empty. The set of *periods* of rank 0 is empty by definition. Below we define the sets \mathcal{R}_i of defining relations of groups $G(i)$ and an increasing sequence $d = (d_r)$ by induction. This sequence depends on a nondecreasing function ϕ such that $\phi(0) = 0$, $\phi(1) = 1$, $\phi(r) \geq 2$ for every $r = 2, 3, \dots$, and $\lim_{r \rightarrow \infty} \phi(r) = \infty$.

Two arbitrary segments from the set $\{(d_r/\phi(r), \phi(r)d_r] \mid r = 1, 2, \dots\}$ will have empty intersection, and $d_0 = 1$. After d_{r-1} is defined we introduce d_r and then define all groups $G(i)$ for $\phi(r-1)d_{r-1} < i \leq \phi(r)d_r$.

Assume $r > 0$ and d_0, \dots, d_{r-1} are already defined along with hyperbolic groups

$$G(0), G(1), \dots, G(i_{r-1}),$$

where $i_{r-1} = [\phi(r-1)d_{r-1}]$. Let $\delta_{i_{r-1}}$ be the hyperbolicity constant of $G_{i_{r-1}}$. We chose a minimal integer d_r such that

$$(32) \quad d_r \geq \max\{\phi(r)^2 d_{r-1}, \phi(r)^2 \delta_{i_{r-1}}, 2\}.$$

For example, $d_1 = 2$. We also define $i_r = [\phi(r)d_r]$.

Then we argue by induction on i ($i_{r-1} < i \leq i_r$). A word A is called *simple in rank* $i - 1$ if it is not conjugate in rank $i - 1$ (that is in $G(i - 1)$) to B^m , where $|B| < |A|$ or B is a period of rank $j \leq i - 1$.

Let \mathcal{X}_i be a maximal set of simple in rank $i - 1$ words of length i such that for two different $A, B \in \mathcal{X}_i$, we have that A is not conjugate of $B^{\pm 1}$ in rank $i - 1$. All the words from \mathcal{X}_i are called *periods* of rank i .

For every period A of rank i ($i_{r-1} < i \leq i_r$), we introduce a large odd exponent n_A , where n_A is a minimal power of p such that

$$\begin{aligned} n_A &\geq \max(n_0, d_r/i) && \text{for } i_{r-1} < i < d_r/\phi(r), \\ n_A &= n_0 && \text{for } d_r/\phi(r) \leq i \leq i_r. \end{aligned}$$

The set \mathcal{R}_i is, by definition, equal to $\mathcal{R}_{i-1} \cup \{A^{n_A} \mid A \in \mathcal{X}_i\}$, and $G(i) = \langle a, b \mid \mathcal{R}_i \rangle$. We will show in [Lemma 6.6](#) that the group $G(i)$ is hyperbolic. Finally,

$$G = G(n_0, \phi) = \langle a, b \mid \bigcup_{i=0}^{\infty} \mathcal{R}_i \rangle.$$

Since $d_0 = 1$ and $d_1 = 2$ we can chose $\mathcal{R}_1 = \{a^{n_0}, b^{n_0}\}$, and $G(1)$ is the free product of two cyclic groups of order n_0 . Since $i_1 \geq d_1 = 2$, one can set $\mathcal{R}_2 = \{a, b, ab, ab^{-1}\}$. Thus a and b are periods of rank 1, and ab and ab^{-1} are periods of rank 2. Hence, for every word w of length at most 2, we have $w^{n_0} = 1$ in G .

The proof of the following Proposition is based on [\[41\]](#) and is contained in the next section.

Proposition 6.2 *The 2-generated group G satisfies the following properties:*

- (a) *The natural homomorphism $G(i_{r-1}) \rightarrow G(i_r)$ is injective on the ball of radius $Kd_r/\phi(r)$, for a nonzero constant K .*
- (b) *(Bounded torsion up to a small deformation) There is a constant $c > 1$ such that for every large enough integer r and every word W with $cd_r/\phi(r) < |W|_G < d_r\phi(r)/c$, there exists a word U of length ≤ 1 such that $(WU)^{n_0} = 1$ in G .*
- (c) *(Relations are locally quasigeodesic) For every large enough r there exists a relation $u_r^{n_0} = 1$ in $\bigcup_{i=0}^{\infty} \mathcal{R}_i$ with $|u_r| = d_r$, such that no nonempty subword w of $u_r^{n_0}$ of length $< n_0|u_r|/2$ can be equal to a word of length at most $|w|/2$ in G .*
- (d) *(Finite cyclic centralisers) G is an infinite p -group in which the centralizer of every nontrivial element is cyclic.*
- (e) *The hyperbolic constant δ_{i_r} of the group $G(i_r)$ is $O(\phi(r)d_r)$.*
- (f) *The order of arbitrary word X in the generators of G is $O(\phi(|X|)^3)$.*

Theorem 6.3 *Let G and $d = (d_n)$ be the group and the scaling sequence constructed above. Then G is lacunary hyperbolic, but for any nonprincipal ultrafilter ω , the asymptotic cone $\mathfrak{C} = \text{Con}^\omega(G, d)$ does not have cut points.*

First we note that by (32) we have $\delta_{r-1} = o(d_r/\phi(r))$. Together with the first assertion of Proposition 6.2 and Theorem 3.3 this implies the first assertion of the Theorem.

Let us now prove that \mathfrak{C} does not have cut points. Indeed suppose this is not so. Then \mathfrak{C} is a tree-graded space with respect to the collection \mathcal{C}_1 of maximal subsets without cut points (see Lemma 2.2). Let $u_r, r = 1, 2, \dots$ be the words given by Proposition 6.2 (c). Then the limit of the loops p_r in the Cayley graph of \mathcal{G} corresponding to the relations $u_r^{n_0} = 1$ is a nontrivial loop in \mathfrak{C} (of length n_0). Indeed, if $\lim^\omega p_r$ is not a simple loop then ω -a.s. there are two points x_r and y_r on p_r that are distance $\Theta(d_r)$ apart along the loop p_r but $o(d_r)$ -close in the Cayley graph of G . But this would contradict Proposition 6.2 (c). Therefore \mathfrak{C} is not an \mathbb{R} -tree.

Hence some of the pieces in \mathfrak{C} contain infinitely many points.

Since \mathfrak{C} is homogeneous, one of these pieces, M , contains $O = (1)^\omega$. Let A be another point in M .

As in the proof of [13, Lemma 6.10] consider two cases.

Case 1 Suppose that there are two pieces from \mathcal{C}_1 that intersect. Then every point is in two distinct pieces. Then we can construct a geodesic $g: [0, s] \rightarrow \mathfrak{C}_1$ such that

$s = \sum_{i=1}^{\infty} s_i$ with $0 < s_i < 1/i^2$ and $g[\sum_{i=0}^r s_i, \sum_{i=0}^{r+1} s_i] \subset M_r$ for some pieces M_r , where $M_r \neq M_{r+1}$ for all $r \in \mathbb{N} \cup \{0\}$. Here $s_0 = 0$. Such a geodesic exists by Lemma 2.4. We call such a geodesic *fractal at the arrival point*. That geodesic with reverse orientation will be called *fractal at the departure point*. If g is fractal at the departure point, g' is fractal at the arrival point, $g_+ = g'_-$, we can construct (using Lemma 2.4) a geodesic p which is a composition of an initial piece of g and the terminal piece of g' . The geodesic p is then fractal at the departure and arrival points or *bifractal*. By homogeneity, every point in \mathcal{C} is the endpoint of a bifractal geodesic.

Let $[A, B]$ be a bifractal geodesic. Lemma 2.5, part (2), implies that $[A, B]$ can intersect a piece M containing A in A or in a nontrivial subgeodesic $[A, B']$. Since $[A, B]$ is fractal at the departure point the latter case cannot occur. It follows that the intersection of $[A, B]$ and M is $\{A\}$. There exists an isometry $\gamma = (x_r) \in G_e^\omega(d)$ such that $\gamma(O) = B$. Since $[A, B]$ is fractal at the arrival point also, it follows that $[A, B] \cap \gamma M = \{B\}$. For every $Z \in \gamma M$ we have that $[A, B] \cup [B, Z]$ is a geodesic, by Lemma 2.4. In particular A is the projection of γM onto M . A symmetric argument gives that $B = \gamma(O)$ is the projection of M onto γM .

Note that in the argument of the previous paragraph we only used the fact that $\gamma O = B$. Let us change γ a little bit preserving the property. Then the conclusions of the previous paragraph will still be true.

Let w_r be the shortest word representing x_r^{-1} in G . Note that $\frac{1}{C}d_r \leq |w_r| \leq Cd_r$ for ω -almost every r where C is a constant. By Proposition 6.2 (b), ω -a.s. there exist words u_r with $|u_r| \leq 1$ such that $(w_r u_r)^{n_0} = 1$ is in \mathcal{R} . Let $\beta = (y_r x_r) \in G_e^\omega(d)$ where $y_r \in G$ is represented by u_r^{-1} . Then again $\beta(O) = B$. Notice that $\beta^{n_0} = 1$.

For every $k \geq 2$ consider the piecewise geodesic path from $\beta^k O \in \beta^k M$ to A

$$(33) \quad g = [\beta^k O, \beta^{k-1} A] \cup [\beta^{k-1} A, \beta^{k-1} O] \cup [\beta^{k-1} O, \beta^{k-2} A] \cup \dots \cup [\beta O, A]$$

where every odd numbered segment is a bifractal geodesic and every even segment is a nontrivial geodesic inside a piece. By Lemma 2.4, g is a geodesic which is not inside M . By the strong convexity of pieces in a tree-graded space (Lemma 2.5), we conclude that $\beta^k O \notin M$, so $\beta^k M \neq M$. This contradicts the equality $\beta^{n_0} = 1$.

Case 2 Now suppose that all pieces in \mathcal{C}_1 are disjoint. Note that we could repeat the argument from Step 1 if we found an isometry δ from $G_e^\omega(d)$ such that $B = \delta(O) \notin M$, the projection of M onto $\delta(M)$ is B and the projection of $\delta(M)$ onto M is A . Indeed, by slightly changing δ , we can find an isometry β with the same property and, in addition, $\beta^{n_0} = 1$. On the other hand, for every k , consider the piecewise geodesic curve (33). Every even numbered geodesic segment g_i in it is nontrivial and inside

a piece M_i , and every odd numbered geodesic segment connects a point U_i in a piece M_i and a point U_{i+1} in a piece M_{i+1} such that U_i is the projection of M_{i+1} onto M_i and U_{i+1} is a projection of M_i onto M_{i+1} . Hence by Lemma 2.6, the odd numbered geodesic segments g_i intersect M_i (resp. M_{i+1}) in exactly one point. By Lemma 2.4 the curve (33) is a geodesic, and so $\beta^k(O) \neq O$ for any k , a contradiction.

Thus our goal is to find such δ .

Lemma 2.2, part (b), implies that A is the projection of a point $B \in \mathfrak{C} \setminus M$. Let $\gamma = (x_r)$ be an isometry from G_ϵ^ω such that $\gamma(O) = B = [x_r^{-1}]$. If $[A, B]$ intersects $\gamma(M)$ in B then we have found the desired $\delta = \gamma$ by Lemma 2.6 (since the isometries of \mathfrak{C} permute the pieces of \mathcal{C}_1 , $\gamma(M) \in \mathcal{C}_1$).

Assume $[A, B] \cap \gamma(M) = [B', B]$, $B' \neq B$. Since all the pieces are disjoint, $B' \neq A$. We have $B' = \gamma(A')$ for some $A' \in M$. Since the space \mathfrak{C} is homogeneous, and all pieces of \mathcal{C}_1 are disjoint, the stabilizer of $\gamma(M)$ in $G_\epsilon^\omega(d)$ acts transitively on $\gamma(M)$. Then there exists γ' in it such that $\gamma'(B) = B'$. We have that $\gamma'\gamma(M) = \gamma(M)$ projects onto M in A and M projects onto $\gamma'\gamma(M)$ in $B' = \gamma'\gamma(O)$, so we can take $\delta = \gamma'\gamma$.

Proposition 6.2 and Theorem 6.3 imply that the divergence and the orders of elements of $G(n_0, \phi)$ can grow arbitrarily slow.

Corollary 6.4 *For any positive function f with $f(r)/r$ nondecreasing satisfying $\lim_{r \rightarrow \infty} f(r)/r = \infty$, and for any nondecreasing function $g(r)$ $\lim_{r \rightarrow \infty} g(r) = \infty$, there is a function ϕ such that*

- (a) *for some λ , the divergence function $\text{Div}(r, \lambda)$ of the group $G = G(n_0, \phi)$ is $O(f(r))$, is not linear, but does not exceed a linear function on an infinite subset of \mathbb{N} ;*
- (b) *the order of any element $x \in G(n_0, \phi)$ is $O(g(|x|))$.*

Proof Consider the functions f and g from the formulation of the corollary. Then we can choose a nondecreasing function ϕ such that $\lim_{r \rightarrow \infty} \phi(r) = \infty$, $\phi(r) \geq 2$ for any $r \geq 2$, and $\phi(r)^3 < \min(cf(r-1)/(r-1), g(r))$ for every $r \geq 2$ and a constant c . Then, by (32) and Condition (e) of Proposition 6.2, we have $d_r \leq c'\phi(r)^3 d_{r-1}$ for some constant c' and every $r > 0$. The right-hand side is less than $c'cf(r-1)d_{r-1}/(r-1) \leq Cf(d_{r-1})$ for $C \geq c'c$ since the function $f(r)/r$ is nondecreasing and $d_{r-1} \geq r-1$ by (32), if $r \geq 2$. Besides, $d_1 < Cf(d_0)$ if C is large enough. Now the statement on the divergency follows from Theorem 6.1. Condition (f) obviously implies the second statement of the corollary. □

6.3 The proof of Proposition 6.2

Here we present the proof of Proposition 6.2. We can apply lemmas from [41] to the construction of group G from that proposition because it obviously satisfies the R -condition from [41, Section 25] since we do not use relations of the “second type” here. In particular, every reduced diagram of rank i arising below is a B -map by [41, Lemma 26.5]. (See the definition in [41, Subsection 20.4].) The contiguity diagrams we use now are more particular than those in the previous sections. (See their definition in Subsection 20.1 of [41]).

Notation 6.5 As in [41, Chapter 7] we fix certain positive numbers $\eta \ll \zeta \ll \epsilon \ll \gamma \ll \beta \ll \alpha$ between 0 and 1 where \ll means “much smaller”. Here “much” means enough to satisfy all the inequalities in Chapters 7 and 8 of [41]. We also have $n_0^{-1} \ll \eta$.

Denote by $P(i)$ the maximum of $n_A|A|$ for the periods A of rank at most i , that is the maximum length of relations of rank at most i .

Lemma 6.6 *The group $G(i)$ is δ_i -hyperbolic for arbitrary $i \leq i_r = [\phi(r)d_r]$, where $\delta_i = n_0 P(i)$.*

Proof Step 1 First we want to prove that a geodesic subpath p of a boundary $\partial\Delta$, where Δ is a reduced diagram of rank i , is a smooth section of rank k (see the formulations of the smoothness conditions $S1$ – $S5$ in Subsection 20.4 of [41]) if we define $\text{rank}(p) = k = [(1 - 2\beta)^{-1} P(i) + 1]$.

Condition $S1$ holds since p is geodesic.

Assume that Γ is a contiguity diagram of a cell Π to a geodesic subpath p in a reduced diagram Δ over $G(i)$ with $(\Pi, \Gamma, p) \geq \epsilon$. Let $\partial(\Pi, \Gamma, p) = p_1 q_1 p_2 q_2$. Then by [41, Lemma 21.2], $|q_1| > (1 - 2\beta)|q_2|$. Indeed, the proof of [41, Lemma 21.2] does not change if one replaces “smooth p ” by “geodesic p ”.

Let A^{n_A} be the label of $\partial\Pi$. Since $|q_1| \leq P(i)$, we have $|q_2| < (1 - 2\beta)^{-1} P(i)$. This implies condition $S2$ from [41, Section 20.4] for p since then $|q_2| < (1 + \gamma)k$. Since our relations are of the first type (ie, of the form A_A^n , and so the boundary of any cell is just one “long section” [41, Section 25]), $S2$ implies $S3$, and $S4$ is obvious because $1 < \alpha^{-1}$. Condition $S5$ automatically holds since $k > i$, and therefore there are no cells of rank k in Δ . We conclude that p is a smooth section of rank k in $\partial\Delta$.

Step 2 Now let xyz be a triangle in the Cayley graph of $G(i)$ with geodesic paths x , y , and z . To prove that x belongs to $n_0 P(i)$ -neighborhood of $y \cup z$, we introduce an inscribed geodesic hexagon $\Psi = t_1 x' t_2 y' t_3 z'$, where (1) x' , y' , and z' are subpaths of

$x, y,$ and $z,$ respectively, (2) $\max(|t_1|, |t_2|, |t_3|) \leq \eta^{-1}k,$ and the sum $|x'| + |y'| + |z'|$ is minimal for hexagons satisfying (1) and (2). (It follows that if $|x'| > 0,$ then the distance between any point of x' and $y' \cup z'$ is at least $\eta^{-1}k.$ Similar properties hold for y' and $z'.$)

Then we have decompositions $x = x_1x'x_2,$ $y = y_1y'y_2,$ and $z_1z'z_2.$ If $|x_1| < 2\gamma^{-1}\eta^{-1}k,$ then every point o of x_1 is at the distance at most $2\gamma^{-1}\eta^{-1}k$ from $z_2.$ If $|x_1| \geq 2\gamma^{-1}\eta^{-1}k \geq 2\gamma^{-1}|t_1|,$ then we can apply [41, Lemma 22.4] to a reduced diagram with boundary $x_1t_1^{-1}z_2$ where, according to Step 1, sections x_1 and z_2 are smooth of rank $k.$ By [41, Lemma 22.4], again, every point o of x_1 can be connected to a point of z_2 by a path of length $< 2\gamma^{-1}\eta^{-1}k < 3\gamma^{-1}\eta^{-1}P(i) < \frac{1}{2}n_0P(i).$

Similarly, the distance from every point of x_2 to y_1 is less than $\frac{1}{2}n_0P(i).$ Thus to complete the proof, it suffices to show that $|x'| < n_0P(i).$ Proving by contradiction, we suppose $|x'| \geq n_0P(i) > \frac{1}{2}n_0k$ and consider a reduced diagram Δ of rank i with boundary $z't_1x't_2y't_3.$ First let us check that Δ is a C -map in the meaning of [41, Section 23.1] with 3 long sections of the first type having rank $k,$ namely, $s_0 = z',$ $s_1 = x', s_2 = y',$ with short sections $t_1, t_2, t_3,$ and with paths p_1, p_2, q of zero length (in the notation of [41]).

Since $|s_1| = |x'| \geq \frac{1}{2}n_0k,$ the diagram Δ satisfies condition $C1$ and $C2$ with $j = k$ because $l = 3 - 1 = 2$ for 3 sections $s_1, s_2, s_3.$ Conditions $C3$ and $C4$ hold since $|q| = |p_1| = |p_2| = 0$ and s_1, s_2, s_3 are geodesic in $\Delta.$ Condition $C5$ holds since $\max\{|t_1|, |t_2|, |t_3|\} < \eta^{-1}k$ (and $\eta^{-1} = d$ in the book [41]). It follows from the choice of Ψ that there are no contiguity subdiagrams between either x' and y' or x' and $z',$ or y' and $z',$ since, by [41, Lemma 21.1(1)], the side arcs of such a subdiagram would be of length $< \eta^{-1}k.$ Therefore Δ satisfy condition $C6.$ The condition $C7$ holds for the same reason as $S4$ at Step 1. Thus Δ is a C -map.

Since Δ is a C -map, there must be a contiguity subdiagram between a pair of sides from $\{x', y', z'\}$ by [41, Lemma 23.15]. But this is impossible as was shown in the previous paragraph, which gives a contradiction. □

Lemma 6.7 *The natural homomorphism $G(i_{r-1}) \rightarrow G(i_r)$ is injective on the ball of radius $0.4d_r/\phi(r).$*

Proof Let w be a word equal to 1 in $G(i_r)$ but not in $G(i_{r-1}).$ Then there is a reduced diagram Δ such that its boundary label is w and it contains a cell Π of a rank $j > i_{r-1}.$ It follows from the construction of defining words of rank $j > i_{r-1}$ that the perimeter $|\partial\Pi|$ of Π is at least $d_r/\phi(r).$ Therefore, by Lemma 23.16 of [41],

$$|w| = |\partial\Delta| > (1 - \alpha)|\partial\Pi| \geq (1 - \alpha)d_r/\phi(r) > 0.8d_r/\phi(r).$$

The inequality $|w| > 0.8d_r/\phi(r)$ gives the injectivity radius at least $0.4d_r/\phi(r).$ □

The following lemma seems to be known. The short proof of it has been communicated to the authors by Ian Agol.

Lemma 6.8 *Let o_1 and o_2 be two distinct points on the boundary of a double punctured disk D , and x the boundary cycle starting (and ending) at o_1 . Let y be a simple path connecting o_1 and o_2 in D and separating the punctures, and z a simple loop starting at o_1 and going around exactly one of the punctures. Then every simple path connecting o_1 and o_2 in D is homotopic to $x^s p_1 x^{-s} p_2$ where each of p_1, p_2 is a product of at most 2 factors from $\{x, y, z^{\pm 1}\}$.*

Proof Let \mathcal{D} be the diffeomorphism group of D . (The elements of \mathcal{D} leave the boundary ∂D invariant and may permute the punctures.) Denote by \mathcal{D}_0 the subgroup of \mathcal{D} that fixes every point on the boundary of the disc ∂D . It is well known that, modulo diffeomorphisms isotopic to the identity element, \mathcal{D}_0 is the cyclic braid group $B_2 = \langle \sigma \rangle$.

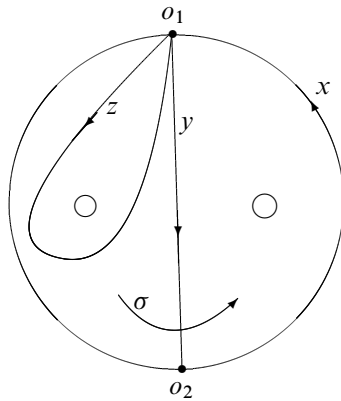


Figure 7: The paths x, y, z and the diffeomorphism σ

Up to the action of \mathcal{D} , there are only two arcs with boundary on the disk: one inessential isotopic into the boundary, and one essential separating the two punctures. It is not hard to see that any essential arc connecting o_1 and o_2 is the image of y under some diffeomorphism from \mathcal{D}_0 .

One computes that: $\sigma^{2k}(z) = x^{-k} z x^k, \sigma^{2k}(x) = x$ (which is the well-known action of B_2 on the fundamental group of the punctured disk). There are two boundary parallel arcs, zy and xzy , both of which are fixed by σ . We may also compute that $\sigma(y) = zxzy$.

Now, we compute that

$$\sigma^{2k}(y) = \sigma^{2k}(z^{-1}zy) = \sigma^{2k}(z^{-1})zy = x^{-k}z^{-1}x^kzy.$$

We may also compute

$$\sigma^{2k+1}(y) = \sigma^{2k}(zxzy) = \sigma^{2k}(z)xzy = x^{-k}zx^kxzy = x^{-k-1}xzx^{k+1}zy.$$

In each case, we see that the arcs are in the normal form $x^k p_1 x^{-k} p_2$, where each of p_1, p_2 is a product of at most 2 factors from $\{x, y, z^{\pm 1}\}$. □

Lemma 6.9 *Let W be a word with $|W|_G > 12n_0$. Then for at most one word U of length ≤ 1 , the word WU is conjugate in G to a word of length $\leq |W|_G/15$.*

Proof By contradiction, assume that there are two distinct words U_1 and U_2 of length at most 1 such that WU_1 and WU_2 are conjugate in G to some V_1 and V_2 , respectively, with $|V_1|, |V_2| \leq |W|_G/15$. Then we have two annular diagrams Δ_k ($k = 1, 2$) over G with contours $w_k u_k$ and v_k such that $\text{Lab}(w_k) \equiv W, \text{Lab}(u_k) \equiv U_k,$ and $\text{Lab}(v_k) \equiv V_k$.

Since $\text{Lab}(w_1) \equiv \text{Lab}(w_2)$ we can identify the paths w_1 and w_2 and obtain a diagram Γ on a disk with two holes. The holes are bounded by paths v_1 and v_2^{-1} , and the third boundary component of Γ is $x = u_1 u_2^{-1}$. There are two vertices o_1 and o_2 on the boundary of Γ connected by three paths, namely, by u_1^{-1} , by u_2 , and by $w = w_1 = w_2$. Let Δ be a reduced diagram with the same boundaries obtained from Γ .

Note that the words WU_k are not nontrivial in G since $|W|_G > |U_k|_G$. So are the words V_1 and V_2 . The word $X \equiv U_1 U_2^{-1}$ is also nontrivial since every nontrivial in F_2 word of length at most 2 is also nontrivial in G by [41, Lemma 23.16]. Therefore [41, Lemma 22.2] is applicable to Δ , and there is a simple path t in Δ connecting some vertices O_1 on x and O_2 on v_1 such that $|t| < (1/2 + 4\gamma)(|v_1| + |v_2| + |x|)$. Hence o_1 is connected with O_2 by a simple path s of length $|s| < (1/2 + 4\gamma)(2|W|_G/15 + 2) + 1$ because $|V_k| \leq |W|_G/15$ and $|U_k| \leq 1$. Therefore there is a simple loop z starting at o_1 and surrounding the hole bounded by v_1 such that

$$|z| \leq 2|s| + |v_1| < (1 + 8\gamma)(2|W|_G/15 + 2) + 2 + |W|_G/15 < 0.205|W|_G$$

since $|W|_G > 12n_0$ and γ is very small.

On the one hand, vertices o_1 and o_2 are connected by the simple path w labeled by W in Γ . The reduction process (the cancellations of cells) preserves these properties of w in the following sense (see [41, Section 13]): There is a simple path w' in Δ connecting o_1 and o_2 such that its label W' is equal to W in G .

On the other hand, $y = u_1^{-1}$ also connects o_1 and o_2 in Δ , and $|y| \leq 1$. The path w' is homotopic for some e , to $p_1 x^e p_2 x^{-e}$ where $|p_1|, |p_2| < 2 \times 0.205 |W|_G = 0.41 |W|_G$, by Lemma 6.8.

Hence $|W|_G = |W'|_G \leq |p_1| + |p_2| + 2|x^e|_G < 0.82 |W|_G + 2|X^e|_G$. But $|X^e|_G \leq n_0$ since we have $X^{n_0} = 1$ in G for every word X of length at most 2. Therefore we obtain $0.18 |W|_G \leq 2n_0$ against the assumption of the lemma. The lemma is proved. \square

The following lemma gives Condition (b) of Proposition 6.2:

Lemma 6.10 *There is a constant $c > 1$ such that for every integer $r \geq 1$ and every word W with $\max(12n_0, cd_r/\phi(r)) < |W|_G < d_r\phi(r)/c$, there exists a word U of length ≤ 1 such that $(WU)^{n_0} = 1$ in G .*

Proof We set $c = 150\zeta^{-1}$ where ζ is the small positive number from Chapter 7 of the book [41]. Then we may assume that W is a geodesic word in G , that is $|W| = |W|_G$.

By Lemma 6.9, there exist four different words U_k ($1 \leq k \leq 4$) of length at most 1 such that WU_k is not conjugate in G to a word of length $\leq |W|/15$. On the other hand, since $|WU| < 2|W|$, it follows from the definition of $G(i)$ that WU_k must be conjugate in rank $i = 2|W|$ to a power $A_k^{m_k}$ where A_k is either period of some rank $j_k \leq i$ or a simple in rank i word. Respectively, we have four reduced annular diagrams Δ_k of rank i with boundary paths $w_k u_k$ and p_k where $\text{Lab}(w_k) \equiv W$, $\text{Lab}(u_k) \equiv U_k$, and $\text{Lab}(p_k) \equiv A_k^{m_k}$.

Further one may assume that if A_k is a period of some rank, then Δ_k has no cells compatible with p_k , since, by [41, Lemma 13.3], one may delete such a cell and replace $A_k^{m_k}$ by $A_k^{m_k \pm n_{A_k}}$. Then we may assume that $A_k \equiv A_l$ if A_k is conjugate to $A_l^{\pm 1}$ in G . Since $|W| > |U_k|$, we also have $m_k \neq 0$.

Case 1 Assume that $m = |m_k| \leq 10\zeta^{-1}$ for some k . Then $|A_k| = |A_k^m|/m \geq 0.1\zeta |W|/15 = |W|/c > d_r/\phi(r)$. On the other hand, by Theorem 22.4 and Lemma 26.5 of [41], $|WU_k| \geq \bar{\beta} m |A_k|$, where $\bar{\beta}$ is the constant from Chapter 7 of [41] which is close to 1. Hence $|A_k| \leq \bar{\beta}^{-1} (|W| + 1) < 2|W| = 2|W|_G < \phi(r)d_r$.

Thus $|A_k| \in (d_r/\phi(r), d_r\phi(r))$ and $|A_k| < 2|W| = i$. It follows that A_k cannot be simple in rank i . (Indeed otherwise it must be simple in all smaller ranks, and so it is conjugate in rank $j - 1 = |A_k| - 1$ to a period (or to its inverse) of rank j by the definition of the set \mathcal{X}_j .) Hence A_k is a period of rank $j_k = |A_k|$, and $n_A = n_0$ because $|A_k| \in (d_r/\phi(r), d_r\phi(r))$. By definition of \mathcal{R}_{j_k} , we have $A_k^{n_0} = 1$ in G . Since WU_k is conjugate to a power of A_k in G , we also have $(WU_k)^{n_0} = 1$, as desired.

Case 2 We may now assume that $|m_k| > 10\zeta^{-1}$ for $k = 1, \dots, 4$. Then for every pair (k, l) ($1 \leq k < l \leq 4$), we identify diagram Δ_k with the mirror copy of Δ_l along the subpaths labeled by W as we did this in the proof of Lemma 6.9. We obtain diagrams Γ_{kl} with 3 boundary components $x_{kl}, v_{kl}^1, v_{kl}^2$ labeled by $X_{kl} \equiv U_k U_l^{-1}, A_k^{m_k},$ and $A_l^{-m_l}$, respectively. Denote by Δ_{kl} the reduced forms of diagrams Γ_{kl} .

Diagrams Δ_{kl} satisfy all conditions of E -maps defined in [41, Subsection 24.2] since $|v_{kl}^1| > 10\zeta^{-1}|A_k|, |v_{kl}^2| > 10\zeta^{-1}|A_l|,$ and $|x_{kl}| \leq 2 < \zeta \min(|v_{kl}^1|, |v_{kl}^2|)$. Lemma 24.6 of [41] says that for some $s, s' \in \{1, 2\}$, the E -map Δ_{kl} contains a contiguity submap Γ^{kl} of v_{kl}^s to $v_{kl}^{s'}$ with contiguity degree $(v_{kl}^s, \Gamma^{kl}, v_{kl}^{s'}) > 0.1,$ and also $(v_{kl}^{s'}, \Gamma^{kl}, v_{kl}^s) > 0.1$ if $s \neq s'$.

If $s = s'$ then using Lemmas 21.1(1) and 25.8 from [41] for Γ^{kl} we obtain a contradiction since $0.1|m_l| > \zeta^{-1}$ for $t = k, l$. Then we may assume that $s = 1$ and $s' = 2$. Lemma 25.10 of [41] implies in turn that $A_k \equiv A_l$ and the cycles v_{kl}^1 and v_{kl}^2 are A_k -compatible, ie, the word $X_{kl} = \text{Lab}(x_{kl})$ is conjugate in G to a power of A_k . These nontrivial in F_2 words X_{kl} are nontrivial in G by [41, Lemma 23.16].

Now we have $A_1 \equiv \dots \equiv A_4 \equiv A,$ and all X_{kl} are conjugate to some powers $A^{m_{kl}}$ in G . By Lemma 26.5 and Theorem 22.4 of [41], $|m_{kl}||A| < \bar{\beta}^{-1}|X_{kl}| \leq 2\bar{\beta}^{-1} < 3$. It follows from this estimate and [41, Lemmas 22.1 and 23.16] that all X_{kl} are conjugate to the powers of the same word A in the free group F_2 . But this is impossible because it is easy to see, that for four different words U_k of length at most 1, there are two words in the set $\{X_{kl} = U_k U_l^{-1}\}$ which are conjugate to $(ab)^{\pm 1}$ and $(ab^{-1})^{\pm 1}$, respectively. Hence Case 2 is also impossible, and the lemma is proved. □

Since every word in \mathcal{X}_i has length i and the set of defining relations of G contains $\{A^{n_0} \mid A \in \mathcal{X}_{d_r}\},$ the following lemma gives Condition (c) of Proposition 6.2.

Lemma 6.11 *The set \mathcal{X}_{d_r} is nonempty for every large enough r . If $A \in \mathcal{X}_{d_r},$ then there exists no nonempty subword w of the word A^{n_0} such that $|w| \leq n_0 d_r / 2$ and w is equal in G to a word of length $\leq |w|/2$.*

Proof The number of positive words of length i in $\{a, b\}$ containing no nonempty subwords of the form B^6 is at least $(3/2)^i$ for $i \geq 1$ [41, Theorem 4.6]. Since the number of all words of length i in $\{a, b\}$ is 2^i there is such 6-aperiodic word A of arbitrary length $i \gg 1$ which is not a proper power. (Indeed, for all large enough $i, (3/2)^i > 2^{\lfloor i/2 \rfloor} + 2^{\lfloor i/3 \rfloor} + 2^{\lfloor i/4 \rfloor} + 2^{\lfloor i/5 \rfloor}.$) Let $i = d_r$. It suffices to prove that A is simple in rank $d_r - 1$.

Arguing by contradiction, we have from the inductive definition that A is conjugate in rank $i - 1$ to a power of a period B or some rank $j < i$ or to a power of a simple

in rank $i - 1$ word B with $|B| < |A|$. In both cases we have a reduced diagram Δ of rank $i - 1$ whose contours p and q are labeled by words A and B^s , respectively. In the second case Δ has no cell compatible with q [41, Section 13.3]. Hence q is a smooth section of the B-map Δ by [41, Lemma 26.5].

If Δ has at least one cell, then there is a cell Π in Δ with contiguity degree to p greater than $1/2 - \alpha - \gamma > \varepsilon$ [41, Lemma 21.7 and Corollary 22.2]). According to [41, Theorem 22.2] this implies that there is a contiguity subdiagram Γ of rank 0 with $(\Pi, \Gamma, p) \geq \varepsilon$. This means that A has a subword B^t , where $t = \lceil \varepsilon n_B / 2 \rceil$. This contradicts the 6-aperiodicity of A because $\varepsilon n_B \geq \varepsilon n_0 > 12$.

Thus Δ has no cells, that is the positive word A is a power of some word of length $|B| < |A|$ in the free group F_2 . This contradicts the choice of A , and the first statement of the lemma is proved.

Then assume that the word w is equal to v in G . If a reduced diagram Δ for this equality has rank $\geq d_r$, then its perimeter $|w| + |v|$ is at least $(1 - \alpha)n_0 d_r > \frac{3}{4}n_0 d_r$ by [41, Lemma 23.16]. Since $|w| \leq n_0 d_r / 2$ we obtain $|v| \geq |w|/2$ as desired.

If $\text{rank}(\Delta) < d_r$, then the section of the boundary $\partial\Delta$ labeled by w is smooth by [41, Lemma 26.5]. It follows from [41, Theorem 22.4] that $|v| > (1 - \beta)|w| > |w|/2$, and the lemma is proved. \square

The group G is infinite by [41, Theorem 26.1]. The order of arbitrary element of G divides some n_A by [41, Theorem 26.2], and so G is a p -group according to our choice of the exponents n_A . The centralizers of nontrivial elements of G are cyclic by [41, Theorem 26.5]. Thus we obtain Condition (d) of Proposition 6.2.

By Lemma 6.6, $\delta_{i_r} \leq n_0 P(i_r)$. If $i_{r-1} < i < d_r / \phi(r)$, then, for a period A of rank (and length) i , we have by definition of n_A , that $n_A |A| \leq p \max\{n_0, d_r\}$. If $d_r / \phi(r) \leq i \leq i_r = \lceil d_r \phi(r) \rceil$, then $n_A |A| = n_0 |A| \leq n_0 d_r \phi(r)$. Hence the obvious induction on r shows that $P(i_r) = O(d_r \phi(r))$. Thus $n_0 P(i_r) = O(\phi(r) d_r)$, and property (e) of Proposition 6.2 is obtained too.

It is shown in [41, Theorem 26.2], that the order of every word X does not exceed the order n_A of a period A of rank (and length) $i \leq |X|$. As above, we have that if $i_{r-1} < i < d_r / \phi(r)$, then $n_A = O(d_r / i_{r-1}) = O(d_r / d_{r-1}) = O(\phi(r)^3)$ by the definition of d_r and Lemma 6.6. If $d_r / \phi(r) \leq i \leq i_r = \lceil d_r \phi(r) \rceil$, then $n_A = n_0 = O(1)$. Since $i \geq i_{r-1} \geq r$, we have by induction on r that $n(X) = n_A = o(\phi(|X|)^3)$, and property (f) is obtained. Since the Conditions (a), (b) and (c) are provided by Lemmas 6.7, 6.10 and 6.11, respectively, the proof of Proposition 6.2 is complete.

7 Open problems

7.1 Algebraic properties of lacunary hyperbolic groups

Since the class of lacunary hyperbolic groups is very large, it would be interesting to establish more common properties of the groups in this class except those established in Section 3.1 and Section 3.4.

Here is a concrete problem.

Problem 7.1 Is it true that the growth of every nonelementary lacunary hyperbolic group is (a) exponential? (b) uniformly exponential?

Inspired by Theorem 3.18 and Corollary 3.21, it is natural to ask what kind of subgroups can lacunary hyperbolic groups have. In particular, we formulate the following:

Problem 7.2 Can a finitely generated non virtually cyclic subgroup of exponential growth of a lacunary hyperbolic group satisfy a nontrivial law?

Remark 7.3 The answer to Problem 7.2 is “no” for lacunary hyperbolic groups for which, using the notation of Remark 3.4, the injectivity radii r_i are “much larger” than the hyperbolicity constants δ_i . More precisely, let G be a direct limit of groups G_i and homomorphisms $\alpha_i: G_i \rightarrow G_{i+1}$ such that $G_i = \langle S_i \rangle$, $\alpha_i(S_i) = S_{i+1}$, $\Gamma(G_i, S_i)$ is δ_i -hyperbolic, and the induced homomorphism $G_i \rightarrow G$ is injective on a ball of radius $r_i = \exp \exp(C\delta_i)$ for a large enough constant C . We claim that then a subgroup of exponential growth in G cannot satisfy a nontrivial law.

Proof Indeed, let $H = \langle x_1, \dots, x_n \rangle$ be a subgroup of G having exponential growth. Let b be the maximal length of an element x_i in generators S of G . Let $H_i = \langle x_1(i), \dots, x_n(i) \rangle$ be a preimage of H in G_i . We can assume that $x_j(i)$ have length at most b in G_i .

By [31, Propositions 3.2 and 5.5], for every $i \geq 1$, there exists a pair of elements u_i, v_i of length at most $\exp \exp(C_0\delta_i)$ (for some uniform constant C_0) in the non virtually cyclic subgroup H_i , generating a free subgroup of H_i . Let a_i, b_i be the images of u_i, v_i in G . Let l_i be the length of the shortest word in $\{a_i, b_i\}^{\pm 1}$ that is equal to 1 in G . Then

$$l_i > \frac{r_i}{\exp \exp(C_0\delta_i)}.$$

Hence if we assume that $C > C_0$ we deduce that $\lim l_i = \infty$. Hence H cannot satisfy any nontrivial law. □

Remark 7.4 It is easy to see that the lacunary hyperbolic groups from examples in Sections 3–5 can be chosen to satisfy the growth condition of Remark 7.3. Thus there are elementary amenable lacunary hyperbolic groups as well as groups with proper subgroups cyclic, torsion groups, groups with nontrivial centers, etc. satisfying this condition.

Problem 7.5 It is easy to construct a lacunary hyperbolic group with undecidable word problem (one can use a small cancellation nonrecursive presentation as in Proposition 3.12). But suppose that the word problem in a lacunary hyperbolic group G is decidable. Does it imply that the conjugacy problem is decidable as well?

Using the known facts about solvability of the conjugacy problem in hyperbolic groups [20; 2] it is easy to deduce that the answer is “yes” if the growth condition of Remark 7.3 holds.

It is also interesting to study linearity of lacunary hyperbolic groups. We do not know the answer to the following basic question.

Problem 7.6 Is every linear lacunary hyperbolic group hyperbolic?

7.2 Asymptotic cones and finitely presented groups

Theorem 4.17 and Theorem 5.11 proved in this paper provide us with a reach source of finitely generated groups all of whose asymptotic cones are locally isometric, but not all of them are isometric. Similar methods can be used to show that the groups from [51; 13, Section 7] also satisfy this property. However all these groups are infinitely presented. Moreover, in all our examples asymptotic cones are locally isometric to an \mathbb{R} -tree, which implies hyperbolicity for finitely presented groups by Proposition 3.2. However the following problem is still open.

Problem 7.7 Does there exist a finitely presented group all of whose asymptotic cones are locally isometric, but not all of them are isometric?

Note that finitely presented groups with different asymptotic cones were constructed in [44] (earlier, in [32], such groups were found under the assumption that the Continuum Hypothesis does not hold).

7.3 Asymptotic cones and amenability

Another interesting problem is to find a characterization of groups all of whose asymptotic cones are locally isometric to an \mathbb{R} -tree in the spirit of [Theorem 3.3](#). In particular, do such groups satisfy a suitable small (graded) cancellation condition? The affirmative answer to this question and the Kesten–Grigorchuk criterion for amenability would give an approach to the following.

Problem 7.8 Suppose that all asymptotic cones of a non–virtually cyclic group G are locally isometric to an \mathbb{R} -tree. Does it follow that G is nonamenable?

Below is another problem about asymptotic cones of amenable groups, which is still open.

Problem 7.9 Is there a finitely generated (resp. finitely presented) amenable non–virtually cyclic group all (resp. some) of whose asymptotic cones have cut-points?

In particular, we do not know whether our groups from [Section 3.5](#) have cut points in all asymptotic cones (for some choice of parameters).

7.4 Divergence and Floyd boundary

Problem 7.10 Is there a finitely presented group with divergence function $\text{Div}(n, \delta)$ strictly between linear and quadratic for some δ ?

Recall that if the Floyd boundary ∂G of a finitely generated group G is nontrivial, G acts on ∂G as a convergence group [\[29\]](#). On the other hand, geometrically finite convergence groups acting on nonempty perfect compact metric spaces are hyperbolic relative to the set of the maximal parabolic subgroups [\[52\]](#).

Problem 7.11 Suppose that a finitely generated group G has a nontrivial Floyd boundary. Is G hyperbolic relative to a collection of proper subgroups?

Note that if G is hyperbolic relative to a collection of proper subgroups, then all asymptotic cones of G are tree-graded with respect to some proper subsets. In particular, all asymptotic cones of G have cut points. Thus [Proposition 4.28](#) may be considered as an evidence towards the positive solution of [Problem 7.11](#).

7.5 Fundamental groups of asymptotic cones

The example of a group G such that $\pi_1(\text{Con}^\omega(G, d)) = \mathbb{Z}$ for some d and ω allows us to realize any finitely generated free Abelian group as the fundamental group of $\text{Con}^\omega(G, d)$ for a suitable G by taking direct products of groups. On the other hand if $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$ is a finitely generated central extension and N is endowed with the metric induced from G , then $\text{Con}^\omega(N, d)$ has the structure of an Abelian topological group. Hence $\pi_1(\text{Con}^\omega(N, d))$ is Abelian. Thus there is no hope to construct asymptotic cones with countable non-Abelian groups by generalizing our methods. This leads to the following.

Problem 7.12 Does there exist a finitely generated group G such that $\pi_1(\text{Con}^\omega(G, d))$ is countable (or, better, finitely generated) and non-Abelian for some (any) d and ω ? Can $\pi_1(\text{Con}^\omega(G, d))$ be finite and nontrivial?

Note that for every countable group C there exists a finitely generated group G and an asymptotic cone $\text{Con}^\omega(G, d)$ such that $\pi_1(\text{Con}^\omega(G, d))$ is isomorphic to the uncountable free power of C [13, Theorem 7.33].

Appendix: Finitely presented groups whose asymptotic cones are \mathbb{R} -trees

MICHAEL KAPOVICH AND BRUCE KLEINER¹

The main result of this appendix is the following:

Theorem 8.1 *Suppose that G is a finitely presented group such that some asymptotic cone of G is an \mathbb{R} -tree. Then G is Gromov-hyperbolic.*

This theorem will be an easy application of (a slightly modified version of) Gromov's local-to-global characterization of hyperbolic spaces.

Before proving [Theorem 8.1](#), we will need several definitions and auxiliary results.

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Metric notions

Given a metric space Z , let $B_R(z)$ denote the closed R -ball centered at z in Z . A geodesic triangle $\Delta \subset Z$ is called R -thin if every side of Δ is contained in the R -neighborhood of the union of two other sides. A geodesic metric space Z is called δ -Rips-hyperbolic if each geodesic triangle in Z is δ -thin. (Rips was the first to introduce this definition.)

Let Z be a metric space (not necessarily geodesic). For a basepoint $p \in Z$ define a number $\delta_p \in [0, \infty]$ as follows. For each $x \in Z$ set $|x|_p := d(x, p)$ and

$$(x, y)_p := \frac{1}{2}(|x|_p + |y|_p - d(x, y)).$$

Then $\delta_p := \inf_{\delta \in [0, \infty]} \{ \delta \mid \forall x, y, z \in Z, (x, y)_p \geq \min((x, z)_p, (y, z)_p) - \delta \}$.

We say that Z is δ -Gromov-hyperbolic, if $\infty > \delta \geq \delta_p$ for some $p \in X$. We note that if Z a geodesic metric space which is δ -Gromov-hyperbolic then Z is 4δ -Rips-hyperbolic and vice-versa (see Gromov [20, 6.3C]).

A metric space Z is said to have bounded geometry if there exists a function $\phi(r)$ such that every r -ball in Z contains at most $\phi(r)$ points. For instance, every finitely generated group G with a word-metric has bounded geometry.

Rips complexes

Given a metric space Z , let $P_d(Z)$ denote the d -Rips complex, ie, the complex whose k -simplices are $k + 1$ -tuples of points in Z , which are within distance $\leq d$ from each other. We equip the Rips complex $P_d(Z)$ with a path metric for which each simplex is path-isometric to a regular Euclidean simplex of side length d .

Given a cell complex X , we let X^i denote the i -skeleton of X .

Lemma 8.2 *Let $G = \langle A \mid \mathcal{R} \rangle$ be a finitely presented group, D the length of the longest relation in \mathcal{R} . Then $P_d(G)$ is simply connected for all $d \geq D$.*

Proof Let Y be the Cayley complex of this presentation, ie the universal cover of the presentation complex of $\langle A \mid \mathcal{R} \rangle$. Then $Y^0 = G$ and Y^1 is the Cayley graph of G (with respect to the generating set A).

First of all, $P_d(G)$ is connected for each $d \geq 1$. We note that $Y^1 = P_1(G)$. Since $\pi_1(Y^1)$ is generated by the boundaries of the 2-cells in Y , it is clear that the map

$$\pi_1(Y^1) \rightarrow \pi_1(P_d(G))$$

is trivial for $d \geq D$. Vanishing of $\pi_1(P_d(G))$ however is slightly less obvious.

Let $d \geq 1$. Consider a loop $\gamma: S^1 \rightarrow P_d^1(G)$. After homotoping γ if necessary, we may assume that it is a simplicial map with respect to some triangulation \mathcal{T} of S^1 . Define a map $\gamma_1: S^1 \rightarrow Y^1$ as follows. For each vertex v of \mathcal{T} , let $\gamma_1(v) \in G = P_d^0(G)$ be equal to $\gamma(v)$. For each edge $e = [v_1 v_2]$ of \mathcal{T} , let $\gamma_1|_e$ be a geodesic in Y^1 between $\gamma_1(v_1)$ and $\gamma_1(v_2)$. There is a natural map

$$Y^1 \xrightarrow{i_1} P_d^1(G)$$

which takes each $v \in G = Y^0$ to the corresponding vertex of $P_d^0(G)$ and maps each edge of Y^1 at constant speed to the corresponding edge of $P_d^1(G)$. Let $\gamma_2 := i_1 \circ \gamma_1$.

If $d \geq D$ then i_1 can be extended to a map

$$Y \xrightarrow{i_2} P_d^2(G).$$

Since Y is simply connected, this implies that γ_2 is null-homotopic in $P_d^2(G)$.

On the other hand, we claim that γ_2 is homotopic to γ in $P_d^2(G)$. To see this, for each edge $e = [v_1 v_2]$ of \mathcal{T} , let $y_0 = \gamma(v)$, $y_1, \dots, y_m = \gamma(w)$ be the vertices of Y^1 on $\gamma_1(e)$ so that $\gamma_2(e)$ is the concatenation of the edges

$$[y_0 y_1], \dots, [y_{m-1} y_m] \subset P_d^2(G).$$

Since $\gamma_1(e)$ is a geodesic between y_0, y_m and $d_{Y^1}(y_0, y_m) \leq d$, we get:

$$d_{Y^1}(y_0, y_i) \leq d, \quad i = 1, \dots, m-1.$$

Hence each triple of vertices y_0, y_i, y_m spans a 2-simplex Δ_i in $P_d^2(G)$. Together these simplices define a homotopy between $\gamma(e)$ and $\gamma_2(e)$ (rel. the endpoints). Thus the loops γ and γ_2 are homotopic. \square

Coarse Cartan–Hadamard theorem

Our main technical result is the following coarse Cartan–Hadamard theorem for Gromov-hyperbolic spaces:

Theorem 8.3 (cf [20], [6, Theorem 8.1.2]) *There are constants C_1, C_2 , and C_3 with the following property. Let Z be a metric space of bounded geometry. Assume that for some δ , and $d \geq C_1 \delta$, every ball of radius $C_2 d$ in Z is δ -Gromov-hyperbolic, and $P_d(Z)$ is 1-connected. Then Z is $C_3 d$ -Gromov-hyperbolic.*

One can give a direct proof of this theorem modeled on the proof of the Cartan–Hadamard theorem. Instead of doing this, we will use 6.8.M and 6.8.N from [20]. In

brief, the idea of the proof is to translate Gromov’s local-to-global result in [20], which is expressed using isoperimetric information, into one using δ -hyperbolicity.

Consider the d -Rips complex $P_d(Z)$ of Z . Given a polygonal loop c

$$c: S^1 \rightarrow P_d^1(Z),$$

let $L(c)$ denote the length of c and let $A(c)$ be the least area of a simplicial disk

$$f: D^2 \rightarrow P_d^2(Z)$$

so that $f|\partial D^2 = c$. If such disk does not exist, we set $A(c) = \infty$. Note that in order to retain the proper scaling behavior, the length and area are computed here using the metric on $P_d^1(Z)$ and $P_d^2(Z)$ rather than the combinatorial length and area.

Taking $A'_0 = 500d^2$ in [20, 6.8.M] we get:

Proposition 8.4 ([20, 6.8.M], adapted version) *Suppose that Z is a metric space of bounded geometry, such that for some $d \geq 0$ every simplicial circle S' in $P_d^1(Z)$ with*

$$500d^2 \leq A(S') \leq 64(500d^2)$$

satisfies

$$(34) \quad L(S') \geq d \sqrt{(4000)(64)(500)}$$

and $P_d(Z)$ is 1-connected. Then $P_d^1(Z)$ is $(400)\sqrt{500d}$ -Rips-hyperbolic [20, 6.8.J] and Z is $(400)\sqrt{500d}$ -Gromov-hyperbolic.

Theorem 6.8.N from [20] states:

Proposition 8.5 [20, Theorem 6.8.N] *If Z is δ -Gromov-hyperbolic and $d \geq 8\delta$, then every simplicial circle $S' \subset P_d^1(Z)$ satisfies $L(S') \geq (d/(4\sqrt{3}))A(S')$.*

Proof of Theorem 8.3 Since the statement of the theorem is scale invariant, after rescaling the metric we may assume that

$$(35) \quad \frac{500d^2}{4\sqrt{3}} \geq \sqrt{(4000)(64)(500)}.$$

Let $C_1 := 32$ and $C_2 := 64 \cdot 500$. Let $S' \subset P_d^1(Z)$ be a simplicial circle with

$$(36) \quad 500d^2 \leq A(S') \leq 64(500d^2)$$

and let $f: D \rightarrow P_d^2(Z)$ be a least area simplicial 2–disk filling S' . By (36), there are at most $(64)(500)$ triangles in the triangulated 2–disk D which are mapped isomorphically by f . Therefore, if we look at $\text{Im}(f) \subset P_d^2(Z)$, and let $W \subset \text{Im}(f)$ be the closure of the union of 2–simplices contained in $\text{Im}(f)$, then connected components W_i of W have diameter $\leq (64)(500)d$. This means that we can decompose D along disjoint arcs as the union of disks D_i , $1 \leq i \leq k + 1$ and regions E_j , so that each $f(E_j)$ is at most 1–dimensional and the diameter of each “minimal 2–disk” $f(D_i)$ is at most $(64)(500)d$. This decomposition corresponds to the van Kampen diagram associated with f .

By assumption, every ball of radius $C_2d = 64 \cdot 500d$ is δ –hyperbolic and

$$d \geq C_1\delta = 32\delta,$$

so by applying Proposition 8.5 to $f(\partial D_1), \dots, f(\partial D_{k+1})$ and adding up the results, we obtain

$$L(S') \geq \frac{d}{4\sqrt{3}}A(S') \geq \frac{d}{4\sqrt{3}}500d^2 \geq d\sqrt{(4000)(64)(500)}$$

where the last inequality comes from (35). By Proposition 8.4 we conclude that Z is C_3d –Gromov-hyperbolic where $C_3 := (400)(\sqrt{500})$. □

Corollary 8.6 *There exist a constant $0 < c < \infty$ such that for each finitely presented group G there exists a constant ρ (depending on the presentation) with the property: Suppose that for some $R \geq \rho$, each ball $B_R(y) \subset Y^1$ is cR –hyperbolic, where Y^1 is the Cayley graph of G . Then G is Gromov-hyperbolic.*

Proof The complex $P_d^2(G)$ is simply connected for each $d \geq D$ (see Lemma 8.2), where D is the length of the longest relator in the presentation of G . Let C_1, C_2 be the constants from Theorem 8.3, where $Z = G$ with the word metric. Choose ρ so that $\rho/C_2 = D$. Let $c := 1/(4C_1C_2)$.

For $R \geq \rho$ set $d := R/C_2$ and $\delta := cR$. Since

$$d = \frac{R}{C_2} \geq \frac{\rho}{C_2} = D,$$

the complex $P_d^2(G)$ is 1–connected; see Lemma 8.2. We now verify the assumptions of Theorem 8.3.

First, by our choice of the constant c ,

$$d = \frac{R}{C_2} \geq 4C_1cR = C_1(4\delta).$$

In fact, the equality holds.

Next, by the assumption of [Corollary 8.6](#), each ball $B_R(y) \subset Y^1$ is $cR = \delta$ -Rips-hyperbolic. Therefore for each $x \in G$, the ball $B_R(x) \subset G$ is 4δ -Gromov-hyperbolic. Since $C_2d = R$, every C_2d -ball in G is 4δ -Gromov-hyperbolic.

[Theorem 8.3](#) now implies that Y^1 is Gromov-hyperbolic. □

Proof of [Theorem 8.1](#)

Let ω be a nonprincipal ultrafilter on \mathbb{N} , let R_j be a sequence of positive real numbers such that $\lim^\omega R_j = \infty$. Let Y be the Cayley complex of a finite presentation of G .

Let $y_j \in G$ be a sequence. By our assumption, the asymptotic cone $\lim^\omega (1/R_j)(Y^1, y_j)$ is a tree for some choice of ω and (R_j) . Thus each geodesic triangle in an R_j -ball $B(y_j, R_j) \subset Y^1$ is δ_j -thin, where $\lim^\omega (\delta_j/R_j) = 0$. Hence the same is true for each ball $B(y, R_j) \subset Y^1$, $y \in G$. For sufficiently large j , $R_j \geq \rho = \rho(Y)$ and $\delta_j/R_j < c$, where ρ, c are the constants from the previous corollary. Hence, by [Corollary 8.6](#), the graph Y^1 is Gromov-hyperbolic and therefore G is too. □

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