Polyhedral Kähler manifolds

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In this article we introduce the notion of *polyhedral Kähler* manifolds, even dimensional polyhedral manifolds with unitary holonomy. We concentrate on the 4–dimensional case, prove that such manifolds are smooth complex surfaces and classify the singularities of the metric. The singularities form a divisor and the residues of the flat connection on the complement of the divisor give us a system of cohomological equations. A parabolic version of the Kobayshi–Hitchin correspondence of T Mochizuki permits us to characterize polyhedral Kähler metrics of nonnegative curvature on $\mathbb{C}P^2$ with singularities at complex line arrangements.

1 Introduction and results

First, we recall the notion of a *polyhedral metric* and a *polyhedral manifold* and give some basic facts about them. Consider a piecewise linear connected manifold M^d with a fixed simplicial decomposition. Let Δ_i^d be the simplices of highest dimension of this decomposition. Choose a flat metric on every Δ_i^d in such a way that every two simplices that have a common face are glued by an isometry. This gives a metric on M^d, which is called *polyhedral*, and M^d is called a *polyhedral manifold*.

For every point x of a polyhedral manifold M^d we canonically associate its *tangent cone*, ie, a cone with polyhedral metric such that a neighborhood of its origin is isometric to a neighborhood of x. At the nonsingular points of M^d the tangent cone is the Euclidean space $\mathbb{R}^d$. A polyhedral metric has no singularities at faces of codimension 1, but may have singularities at faces of codimension 2. The tangent cone of the points in the interior of such faces is isometric to the direct product of a 2–cone and the flat space $\mathbb{R}^{d-2}$. The angle of the 2–cone is called the *conical angle* at the face.

The singular locus of a polyhedral metric is naturally stratified. A point of M^d is called a *metric singularity of codimension at least k* if its tangent cone is not isometric to the direct product of $\mathbb{R}^{d-k+1}$ and a $(k-1)$–dimensional polyhedral cone. The set of all metric singularities of codimension at least k is denoted by M_s^{d-k}.

The complement to the singular locus of the metric is connected and we can consider the holonomy of the metric on it. This gives us a representation $\pi_1(M^d \setminus M_s^{d-2}) \to \mathrm{SO}(d)$ (we will consider only orientable manifolds). For a generic choice of a polyhedral metric this representation has an everywhere dense image in $\mathrm{SO}(d)$.

In this work we study even dimensional polyhedral manifolds M^{2n} whose holonomy group is contained in a subgroup of $\mathrm{SO}(2n)$ conjugate to $U(n)$. On the complement of the singularities of the metric these manifolds have a complex structure J parallel with respect to the flat metric and compatible with it. In addition to the unitarity of the holonomy we impose one condition. For any face F of codimension 2 consider a simplex Δ^{2n} that contains F in its border. The parallel complex structure J defined in the interior of Δ^{2n} naturally extends to the whole Δ^{2n} and we say that F *has a holomorphic direction* if F is a piece of a holomorphic hyperplane with respect to J.

Definition 1.1 A polyhedral manifold M^{2n} is called a *polyhedral Kähler manifold* (or shortly a *PK manifold*) if the holonomy of its metric belongs to a subgroup of $\mathrm{SO}(2n)$ conjugate to $U(n)$ and every codimension 2 face with conical angle $2k\pi$, $k \geq 2$ ($k \in \mathbb{N}$), has a holomorphic direction.

Remark 1.2 Codimension 2 faces with conical angle different form $2k\pi$ ($k \in \mathbb{N}$) automatically have complex direction (cf Section 3) so we don't need to impose this condition on them. If we don't impose the condition on the faces with conical angle $2k\pi$, $k \geq 2$, we obtain Thurston's (X, G)–cone-manifolds modeled on $X = \mathbb{C}^n$ with G the group of unitary isometries of $\mathbb{C}^n$ [19].

Remark 1.3 In this work simplicial decompositions are used only to define the class of PK manifolds and play a secondary role. We will mostly think about PK manifolds as spaces with a specific metric and will not distinguish manifolds that are isometric but have different simplicial decompositions.

A 2–dimensional oriented polyhedral surface is automatically Kähler (since $\mathrm{SO}(2) = U(1)$) and complete classification of such structures is given by Troyanov [20] (we recall this classification in Section 2). In the rest of this work we deal mostly with 4–dimensional polyhedral Kähler manifolds. In Section 2 several elementary examples of such manifolds are given.

A polyhedral metric is called *nonnegatively curved* if the conical angle at every singular face of codimension 2 is smaller than 2π. The original motivation for our study of PK metrics is due to the following remark of Anton Petrunin. *The holonomy of a nonnegatively curved polyhedral $\mathbb{C}P^n$ preserves a symplectic form* (see a related

discussion in Petrunin [17]). This is a partial case of a vanishing theorem proved by Cheeger [2]. This means that a nonnegatively curved $\mathbb{C}P^n$ is PK; we discuss this subject in Section 2.

Known examples An explicit example of a nonnegatively curved polyhedral $\mathbb{C}P^2$ is provided by Kühnel's 9 vertices triangulation [10]. This polyhedral $\mathbb{C}P^2$ can be obtained as a finite isometric quotient of a flat complex 2–torus, and the holonomy of the metric on $\mathbb{C}P^2$ is finite. In general for any n there exist a series of polyhedral metrics on $\mathbb{C}P^n$ that are obtained as quotients of complex tori; see Kaneko, Tokunaga and Yoshida [9].

Couwenberg, Heckman and Looijenga [3] study geometric structures that are more general than PK metrics. They obtain constant holomorphic curvature metrics on $\mathbb{C}P^n$, having as singular locus complex reflection hyperplane arrangements. Their approach is different; in particular from the very beginning they start with a holomorphic manifold. It should be possible to prove that in the case when curvature is zero their metrics are PK.

1.1 Local properties of PK metrics

A PK manifold has a natural complex structure defined outside the singular locus; it is constant in the local flat coordinates. We will prove that for a 4–dimensional PK manifold M^4 this complex structure can be extended to the whole manifold.

Definition 1.4 Let M^4 be a PK manifold. *Holomorphic functions* on M^4 are defined as continuous functions on M^4 that are holomorphic on the complement to the singularities $M^4 \setminus M_s^2$.

A *holomorphic chart* in M^4 is an open subset U with an injective map $\varphi = (f, g)\colon U \to \mathbb{C}^2$ with f and g holomorphic as above and such that $\varphi(M_s^2 \cap U)$ is an analytic subset of $\varphi(U) \subset \mathbb{C}^2$.

The following theorem justifies this definition by proving that *holomorphic functions* and *holomorphic charts* on M^4 define together a genuine complex structure on M^4.

Theorem 1.5 *Every point of a 4–dimensional PK manifold is contained in a holomorphic chart. Holomorphic charts form together a holomorphic atlas on M^4 and induce on it a structure of a smooth complex surface. The singular set M_s^2 is a complex curve for the defined holomorphic structure.*

Remark 1.6 We don't consider any intermediate smooth structure on M^4 in order to define a complex structure on it. At the same time, it is known that PL manifolds of dimension up to 6 have a canonical smooth structure.

The main step in Theorem 1.5 is the construction of holomorphic charts for the singularities of the PK metric. A neighborhood of every singularity embeds isometrically into its tangent cone, and it is possible to introduce on the tangent cone the structure of a single holomorphic chart. We call these cones *polyhedral Kähler cones*, and denote by C_K^4. All PK cones have a natural *holomorphic Euler vector field e* (cf Section 3.1) that acts by dilatations of the metric, and this field is crucial for us.

Theorem 1.7 *Let C_K^4 be a 4–dimensional polyhedral Kähler cone. There exists a homeomorphism $\varphi\colon C_K^4 \to \mathbb{C}^2$ holomorphic outside the singularities of the cone and satisfying the following property: The Euler field e written in coordinates (z, w) of $\mathbb{C}^2$ is given by*

$$\frac{1}{\alpha}z\frac{\partial}{\partial z} + \frac{1}{\beta}w\frac{\partial}{\partial w},$$

where α and β are positive real numbers. The image of the singular locus of C_K^4 under the map φ is given by a union of curves $c_1 z^\alpha = c_2 w^\beta$.

The singularity is called *irrational* if $\alpha/\beta \in \mathbb{R} \setminus \mathbb{Q}$. In this case its tangent cone is isometric to the direct product of two 2–cones $C_1 \times C_2$ with conical angles $2\pi\alpha$ and $2\pi\beta$.

The singularity is called *rational of type (p, q, α)*, $p, q \in \mathbb{N}$ if its Euler field is equal to $e = (p/\alpha)z(\partial/\partial z) + (q/\alpha)w(\partial/\partial w)$ in coordinates (z, w). Here p and q are relatively prime, $p \leq q$ and α is a positive real number. Sometimes, when the choice of α is not important, we may also say that the singularity is of type (p, q). Since e

acts by dilatation of the metric, it preserves the singular locus of the metric. Thus, in the neighborhood of x any irreducible component of the singular locus is a curve given by one of the equations: $cz^q = w^p$, $c \neq 0$; $z = 0$; or $w = 0$. All these curves are flat with respect to the induced PK metric. Each curve $cz^q = w^p$ has a conical point at the origin with the *same* angle $2\pi\alpha$, the line $z = 0$ has conical angle $(2\pi/p)\alpha$, and the line $w = 0$ has conical angle $(2\pi/q)\alpha$.

The next theorem gives a description of the set $S(p, q, \alpha)$ of equivalence classes of singularities of type (p, q, α). There is a slight difference between the cases $(1 = p = q)$, $(1 = p < q)$, $(1 < p < q)$. The triple (p, q, α) does not determine the singularity uniquely, the singularities of a given type form an infinite-dimensional space.

Theorem 1.8 *On a 2–sphere, consider the set of metrics (up to isometry) of curvature 4, with area $\pi\alpha/(pq)$ and having an arbitrary number of conical points. Moreover if $p > 1$ and $q > 1$ we mark two conical points, while if $p = 1$, $q > 1$ we mark one conical point. This set of metrics on the sphere is in natural 1–to–1 correspondence with the set $S(p, q, \alpha)$.*

Consider a (p, q, α) singularity, let $(2\pi\beta_1, \ldots, 2\pi\beta_n)$ be the conical angles of the PK metric at the singular branches $c_k z^q = w^p$, $(c_k \neq 0)$. Let $2\pi\beta_z$ be the angle at $z = 0$ and $2\pi\beta_w$ the angle at $w = 0$.

Theorem 1.9 *The following relation holds:*

$$\alpha = \frac{pq}{2}\left(\sum_k (\beta_k - 1) + \frac{\beta_z - 1}{p} + \frac{\beta_w - 1}{q} \right) + \frac{p + q}{2}.$$

Theorems 1.5–1.9 are proven in Section 3.

1.2 Flat connection and topological relations

By Theorem 1.5 every 4–dimensional PK manifold is a complex surface and the singular locus of the PK metric forms a complex curve on it. Further on will denote the surface by S and the complex curve by Γ.

The PK metric on S defines a flat meromorphic connection on the tangent bundle of S with first order poles at Γ. In Section 4 we study this connection, especially in the neighborhood of singularities of complex codimension 2. We give a list of conditions that imply that a connection on the tangent bundle to a surface is a connection of a PK metric (Theorem 4.13). Using the residues of the connection we write down a system of topological relations on the pair (S, Γ). This is done in Section 5 .

Let us fix some notation. Irreducible components of Γ will be denoted by Γ_j. For every component Γ_j we denote by $2\pi\beta_j$ the conical angle at Γ_j, ie, the angle of a 2–cone orthogonal to any nonsingular point of Γ_j. The singularities of Γ that are not normal crossings are denoted by x_i and their type is denoted by (p_i, q_i, α_i).

Definition 1.10 For any surface S a collection of divisors Γ_j with positive weights β_j is called a weighted arrangements. In the case when there exists a PK metric on S with singularities at Γ_j of angles $2\pi\beta_j$ we call (Γ_j, β_j) the *weighted arrangement of the PK metric* or the PK *arrangement*. Sometimes we will mean by weighted arrangement the whole data $(\Gamma_j, \beta_j; x_i, p_i, q_i, \alpha_i)$.

Define two numbers related to the behavior of Γ_j in the neighborhood of x_i. Denote by $\widetilde{d}_{ij}$ the number of branches (local irreducible components) of Γ_j at x_i. Additionally let d_{ij} be the number of branches, except counting branches $z = 0$ and $w = 0$ with weights $1/p$ and $1/q$. Denote by B_{jk}, $j \neq k$, the number of intersections of curves Γ_j and Γ_k that represent the normal crossing singularity of Γ and define B_{jj} by

$$B_{jj} = -\Gamma_j \cdot \Gamma_j + \sum_i p_i q_i (d_{ij})^2,$$

where $\Gamma_j \cdot \Gamma_j$ is the self-intersection number of Γ_j.

Theorem 1.11 *Any weighted arrangement* $(\Gamma_j, \beta_j; x_i, p_i, q_i, \alpha_i)$ *of a PK metric satisfies the following relations:*

$$(1\text{-}1) \quad \forall j \quad \sum_k B_{jk}(\beta_k - 1) = -2\chi(\Gamma_j) - K_S \cdot \Gamma_j - \sum_i (d_{ij}(p_i + q_i) - 2\widetilde{d}_{ij}),$$

$$(1\text{-}2) \qquad\qquad -c_1(TS) = K_S = \sum_j (\beta_j - 1)[\Gamma_j] \in H_2(S, \mathbb{R}).$$

Moreover, in the case when for every i $p_i = q_i = 1$ *we have the following expression for the second Chern class:*

$$(1\text{-}3) \qquad\qquad c_2(TS) = \sum_i (\alpha_i - 1)^2 + \sum_{j \neq k} \frac{1}{2} B_{jk}(\beta_j - 1)(\beta_k - 1).$$

Here K_S is the canonical class of S and $\chi(\Gamma_j)$ is the Euler characteristic of the normalization of Γ_j.

These relations have the following nature. Relation (1-1) is a consequence of the Gauss–Bonnet formula applied to the curve Γ_j. For every Γ_j the sum of the defects of its conical points is equal to its Euler characteristic. Relations (1-2) and (1-3)

express the Chern classes of TS in terms of the residues of the flat connection on TS corresponding to the PK metric; see Ohtsuki [15].

Construction of weighted arrangements satisfying equations of Theorem 1.11 is a problem of independent interest, leading to questions of the following type:

Problem Classify arrangements of $3n$ lines on $\mathbb{C}P^2$ such that every line intersect other lines exactly at $n+1$ points.

It is easy to see that such line arrangements with weights $\beta_j = (n-1)/n$ satisfy Equations (1-1), (1-2) and (1-3). This problem appeared previously in Hirzebruch [7] and a list of two infinite series and several exceptional arrangements satisfying the condition was given (all these arrangements are complex reflection arrangements). Further questions of similar nature (about *simplicial* and limit PK arrangements with a *cusp*) are formulated in Section 5.

1.3 Reconstruction of nonnegatively curved PK metrics from weighted arrangements

One of the main results of this paper is the following theorem.

Theorem 1.12 *Consider a weighted arrangements of lines (L_j, β_j) in $\mathbb{C}P^2$ satisfying the following conditions:*

$$(1\text{-}4) \qquad \sum_j (\beta_j - 1) = -3, \quad 0 < \beta_j < 1, \quad \sum_j d_{ij}(\beta_j - 1) > -2.$$

Then the following inequality holds:

$$(1\text{-}5) \qquad \sum_i (\alpha_i - 1)^2 - \sum_j \frac{1}{2}(1 - \beta_j)^2 B_{jj} - \frac{3}{2} \le 0.$$

Moreover, if the equality holds then there exists a PK *metric on $\mathbb{C}P^2$ with conical angles $2\pi\beta_j$ at L_j (ie, (L_j, β_j) is a* PK *arrangement.)*

We prove this theorem in Section 7 after recalling (Section 6) the technique of parabolic bundles. To every arrangement that satisfies condition of the theorem we associate a parabolic structure on the pullback of the tangent bundle of $\mathbb{C}P^2$ to the blow up of $\mathbb{C}P^2$ at the multiple points of the arrangement. We prove that constructed parabolic bundle is stable and calculate its parabolic Chern characters. Inequality (1-5) is a consequence of Bogomolov–Gieseker inequality [13] (see also [12]). The existence of a PK metric in the case of equality follows form the parabolic version of Kobayashi–Hitchin correspondence from [13] and additional statements about logarithmic connection the we prove in Section 4.

2 Examples of polyhedral Kähler manifolds

In this section we recall the classification of PK structures on complex curves in Troyanov [20] and give several examples of polyhedral Kähler manifolds of higher dimension.

2.1 Flat metrics on surfaces

Structures on a 2**–dimensional polyhedral cone** A 2–dimensional polyhedral cone is a very simple object but already it supports the majority of geometric structures that are essential for this work. Let us describe these structures. Consider a 2–cone C^2 with conical angle $2\pi\alpha$. Note first that the flat metric on $C^2 \setminus 0$ defines a conformal and hence a *holomorphic structure* on $C^2 \setminus 0$. Moreover $C^2 \setminus 0$ is biholomorphic to $\mathbb{C}^*$, so we can chose a *holomorphic coordinate z* on it (defined up to a multiplicative constant). This coordinate can be used to extend the complex structure from $C^2 \setminus 0$ to C^2. We have a natural action of $\mathbb{R}^*$ on C^2 by homotheties, corresponding vector field can be complexified and we call it *Euler field*. This field is given in the coordinate z by the formula $(z/\alpha)(\partial/\partial z)$, the imaginary part of the field acts by isometries of the cone. The metric induces a *flat meromorphic connection* on the tangent bundle to the cone and it is given by $\nabla = d + (\alpha - 1)(dz/z)$. The multivalued flat coordinate on the cone, ie, a coordinate in which the connection on C^2 is trivial, is given by z^α.

The next theorem classifies polyhedral metrics on surfaces.

Theorem 2.1 (Troyanov [20]) *Consider a complex curve Γ of genus g with pairwise distinct marked points $x_1, \ldots, x_n$. Let $\alpha_1, \ldots, \alpha_n$ be real positive numbers such that $\sum(\alpha_i - 1) = 2g - 2$. Then there is a unique (up to a real multiplication constant) complete flat metric on Γ with conical points of angles $2\pi\alpha_i$ at x_i whose conformal structure on $\Gamma \setminus \{x_1, \ldots, x_n\}$ is the same as of Γ itself.*

For completeness we give here a proof of the theorem.

Lemma 2.2 *For every real $\beta_1, \ldots, \beta_n$ such that $\sum_i \beta_i = 0$, there exists a unique meromorphic 1–form η on Γ with simple poles with residues $\beta_1, \ldots, \beta_n$ at the points $x_1, \ldots, x_n$, having purely imaginary periods (ie, for every closed path $\gamma \in \Gamma$ we have $\int_\gamma \eta \in i\mathbb{R}$).*

Proof By Dirichlet's theorem there exists a unique (up to a constant) real harmonic function f on Γ, satisfying the equation $\Delta f = \sum_i \beta_i \delta_{x_i}$. This function has logarithmic poles at $x_1, \ldots, x_n$. The 1–form η is then given by $\eta(\vec{u}) = df(\vec{u}) + i\,df(J\vec{u})$, where J defines the complex structure on $T\Gamma$. $\square$

Proof of Theorem 2.1 **Existence** Let ξ be a holomorphic differential on Γ with simple zeros $y_1, \ldots, y_{2g-2}$. It defines a flat metric $\xi \otimes \bar{\xi}$ on Γ with conical points of angle 4π at the points y_i. Denote by ∇ the corresponding connection. Consider the 1–form η on Γ with purely imaginary periods that has residue -1 at any point y_i and residue $(\alpha_j - 1)$ at any point x_j.

Let us prove that the connection $\nabla + \eta$ on Γ is unitary. Indeed, the holonomy of $\nabla + \eta$ along a closed path γ is given by the formula

$$\mathrm{hol}_\gamma(\nabla + \eta) = \mathrm{hol}_\gamma(\nabla) \exp\left(-\int_\gamma \eta\right) = \exp\left(-\int_\gamma \eta\right).$$

The first equality follows from the definition of holonomy and the second follows from the fact that the holonomy of ∇ is trivial. The connection $\nabla + \eta$ defines a unique (up to a real multiplication constant) flat metric on Γ. In order to define it one should fix the metric at any point of Γ different from x_j and translate it by means of $\nabla + \eta$ to other points of Γ. The metric constructed this way has singularities exactly at x_j, and the conical angles at x_j are defined by the poles of η.

Uniqueness Suppose that we have two metrics g_1 and g_2 satisfying the conditions of the theorem. Then the 1–form $\nabla_{g_1} - \nabla_{g_2}$ should be holomorphic and it should have purely imaginary periods (since both ∇_{g_1} and ∇_{g_2} are unitary), ie, it is identically zero. Thus g_1 and g_2 coincide. □

2.2 Polyhedral Kähler manifolds of higher dimension

Recall that a polyhedral manifold is called *nonnegatively curved* if the conical angles at all faces of codimension 2 are at most 2π.

Proposition 2.3 *Let M^{2n} be a nonnegatively curved manifold that has a second cohomology class $h \in H^2(M^{2n})$ such that h^n is non zero in $H^{2n}(M^{2n})$. Then the holonomy of M^{2n} is contained in $U(n)$, ie, such a manifold is PK.*

This proposition is a simple corollary of results of J Cheeger [2], which we will now describe. We don't need these results in full generality; instead, we give a version sufficient for our considerations.

Let M^n be a polyhedral manifold, and let M_s^{n-2} be the subset of all its metric singularities. Denote by $H_{L_2}^i(M^n)$ the space of L_2–harmonic forms on $M^n \setminus M_s^{n-2}$ that are closed and coclosed.

Theorem A $\dim(H_{L_2}^i(M^n)) = b^i(M^n)$.

Theorem B *Suppose that the manifold M^n is nonnegatively curved. Then every harmonic form h in $H^i(M^n)$ is parallel, ie, $\nabla h = 0$.*

Remark 2.4 According to [2], Theorem B indicates that nonnegatively curved polyhedral manifolds are analogs of smooth Riemannian manifolds with nonnegative curvature operator (rather then smooth manifolds with nonnegative sectional curvature). In fact, Petrunin [17] proved that analogues curvature inequality in the smooth case is even stronger.

Let us deduce Proposition 2.3 from Theorems A and B. We need a simple fact from linear algebra.

Lemma 2.5 *Consider an Euclidean space V^{2n} with a nondegenerate 2–form w, $w^n \neq 0$. Denote by S_w the subgroup of $\mathrm{SO}(2n)$ that preserves w. Then the group S_w is contained in a subgroup of $\mathrm{SO}(2n)$ conjugate to $U(n)$.*

Proof Take orthonormal coordinates (x_i, y_i) in V^{2n} such that $w = \sum_i a_i dx_i \wedge dy_i$ $(a_i \neq 0)$. It is easy to see that every element of $\mathrm{SO}(2n)$ that preserves w preserves the form $w' = \sum_i dx_i \wedge dy_i$. The stabilizer of w' in $\mathrm{SO}(2n)$ is exactly $U(n)$. □

Proof of Proposition 2.3 Let g be a nonnegatively curved polyhedral metric on M^{2n}. By Theorem A there exists a harmonic 2–form w on M^{2n} such that $\int_M w^n \neq 0$. By Theorem B w parallel in the flat metric. It has constant rank outside of the singularities, and since $\int_M w^n \neq 0$, w should be nondegenerate. The holonomy of g preserves w, thus by Lemma 2.5 the holonomy is contained in a subgroup of $\mathrm{SO}(2n)$ conjugate to $U(n)$, ie, g is a polyhedral Kähler metric. □

This proposition indicates that it should be difficult to construct an explicit simplicial decomposition on $\mathbb{C}P^n$ that defines a nonnegatively curved metric. All examples of PK metrics on $\mathbb{C}P^n$ that we know come from algebraic geometry, and produce a metric without a chosen simplicial decomposition.

Examples of nonnegatively curved polyhedral $\mathbb{C}P^n$

Example 1 Choose any nonnegatively curved polyhedral metric on $\mathbb{C}P^1$. Consider the n–th symmetric power $\mathrm{Sym}^n(\mathbb{C}P^1)$ of $\mathbb{C}P^1$ with induced polyhedral metric. We have $\mathrm{Sym}^n(\mathbb{C}P^1) \simeq \mathbb{C}P^n$, and it is clear that the constructed polyhedral metric on $\mathbb{C}P^n$ is nonnegatively curved. This is the first nontrivial example of a higher-dimensional PK manifold that I learned and it was proposed to me by M Gromov.

For $n = 2$ we obtain a PK metric on $\mathbb{C}P^2$ with singularities at a conic and several lines tangent to it. The conical angle at the conic is equal to π and the sum of defects of the conical angles at the lines is equal to -4π. The conic is the image of the diagonal of $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Example 2 Let T^2 be a *real* 2–torus with a flat metric. Consider the $(n+1)$–th power of T^2, $T^{2n+2} = (T^2)^{n+1}$. Let T^{2n} be a subtorus of T^{2n+2} given by the equation $\sum_i x_i = 0, x_1, \ldots, x_{n+1} \in T^2$. Let S_{n+1} be the permutation group acting on T^{2n+2}.

Lemma 2.6 *The quotient T^{2n}/S_{n+1} is a $\mathbb{C}P^n$ with a nonnegatively curved polyhedral metric.*

Proof Let E be the unique elliptic curve with the same conformal structure as T^2. Let L_n be a complex line bundle over E with first Chern class $n+1$. Then T^{2n}/S_{n+1} can be identified with the space of zero divisors of sections of L_n. □

For $n = 2$ we obtain a PK metric on $\mathbb{C}P^2$ singular along an elliptic curve of degree 6 with 9 cusps. This curve is projectively dual to a smooth cubic.

Remark 2.7 It follows from Theorem B that an orientable 4–dimensional manifold admitting a nonnegatively curved polyhedral metric with *irreducible* holonomy is homeomorphic to S^4 or $\mathbb{C}P^2$ (recall that irreducible means no invariant subspaces). Nonnegative polyhedral metrics on $S^2 \times S^2$ were studied by Orshanskiy [16] using the theory of Alexandrov spaces. The results of that paper can be obtained in a different way using complex geometry.

PK metrics via branched covering One can construct polyhedral Kähler metrics via branched coverings. Let $f\colon S_2 \to S_1$ be a branched covering of a smooth complex surface S_1 by a smooth complex surface S_2. Suppose that S_1 has a polyhedral Kähler metric and f is ramified over a set of flat curves on S_1. Then the pullback of the metric on S_2 is a polyhedral Kähler metric.

Consider the map $f\colon \mathbb{C}P^2 \to \mathbb{C}P^2$, $f(x:y:z) = (x^n : y^n : z^n)$. This map is ramified at the lines $x = 0, y = 0, z = 0$. The following two examples use this map to produce new PK metrics.

Example 3 (7 lines) Consider a PK metric on $\mathbb{C}P^2$ with the singular locus given by the lines $x = 0, y = 0, z = 0$ and a conic tangent to these lines (cf Example 1). The conical angle at the conic is π and the conical angles at the lines are equal to $2\pi\alpha$, $2\pi\beta$, $2\pi\gamma$, $\alpha + \beta + \gamma = 1$. Consider the branched covering $f(x:y:z) = (x^2 : y^2 : z^2)$. Then the singular locus of the pullback metric is composed of 7 lines, 4 of which have conical angle π and three of which have angles $4\pi\alpha$, $4\pi\beta$, $4\pi\gamma$.

Example 4 (A metric on a symmetric $K3$ surface) Consider a PK metric on $\mathbb{C}P^2$ with the singular locus given by the lines $x = 0$, $y = 0$, $z = 0$, $x + y = z$ (all of them having conical angle π) and a conic tangent to these lines (Example 1). Consider the pullback metric on $\mathbb{C}P^2$ under the map $f(x : y : z) = (x^6 : y^6 : z^6)$. The preimage of the line $x + y = z$ is given by the equation $x^6 + y^6 = z^6$. The double cover of $\mathbb{C}P^2$ ramified over the curve $x^6 + y^6 = z^6$ is a $K3$ surface. This construction gives a polyhedral Kähler metric on it.

Example 5 (PK metrics on algebraic Kummer surfaces) Recall that a Kummer $K3$ surface is obtained from a *complex* 2–torus T^2 by the quotient with respect to the involution $I: x \to -x$ and successive blow up of 16 fixed points. If we first blow up the points on T^2 fixed by I, we get a surface that is a double cover of the Kummer surface. So in order to get a PK metric on a Kummer surface it will be sufficient to construct any I–invariant PK metric on T^2 blown up at 16 invariant points.

Let Γ be a genus 2 curve, σ its hyperelliptic involution, and $\mathrm{Jac}_2(\Gamma)$ the Jacobian of degree 2 line bundles on Γ. Let g be a flat metric with conical points on Γ, invariant under σ (we suppose that the conformal structure of g is that of Γ). The metric g induces a PK metric $\widetilde{g}$ on the symmetric square $\mathrm{Sym}^2(\Gamma)$ of Γ.

Recall that $\mathrm{Sym}^2(\Gamma)$ is naturally isomorphic to the blow up of $\mathrm{Jac}_2(\Gamma)$ at the point corresponding to the canonical class of Γ. Moreover the involution σ on Γ induces the involution I on the blown up of $\mathrm{Jac}_2(\Gamma)$. Consider the degree 16 cover of $\mathrm{Sym}^2(\Gamma)$ corresponding to the subgroup $(2\mathbb{Z})^4 \subset H_1(\mathrm{Sym}^2(\Gamma))$. One can check that the involution I lifts to this cover and it fixes 16 exceptional curves. Moreover, I fixes the lift of $\widetilde{g}$. This finishes the construction.

Nonelementary examples All the examples of PK manifolds listed above are obtained by relatively elementary constructions. One of the main goals of this paper is to construct PK metrics on $\mathbb{C}P^2$ with singularities at lines arrangements, and some examples of such arrangements are given in Section 5.3. To construct the corresponding metric on $\mathbb{C}P^2$ we need to use deep results of T Mochizuki.

3 Singularities of 4–dimensional PK manifolds

Starting from this section we deal only with 4–dimensional PK manifolds. In the next two subsections we will prove Theorems 1.5 and 1.7. Before doing this let us explain why existence of holomorphic charts (Definition 1.4) on a polyhedral Kähler manifold M^4 implies immediately that M^4 is a complex surface.

Indeed, suppose that M^4 can be covered by holomorphic charts $(U_\alpha, \varphi_\alpha)$. To prove that $(U_\alpha, \varphi_\alpha)$ is a holomorphic atlas on M^4 we need to show that for every α and β the gluing map

$$\varphi_\alpha \varphi_\beta^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is holomorphic. By Definition 1.4 the map $\varphi_\alpha \varphi_\beta^{-1}$ is continuous on $\varphi_\beta(U_\alpha \cap U_\beta)$ and holomorphic on the complement to an analytic subset. So by standard results it is holomorphic on the whole domain $\varphi_\beta(U_\alpha \cap U_\beta)$. □

The same argument gives us the following lemma.

Lemma 3.1 *For $i = 0, 1$ suppose that every point of $M^4 \setminus M_s^i$ has a holomorphic chart. Then the space $M^4 \setminus M_s^i$ has a well-defined holomorphic structure.*

The proof of Theorem 1.5 will be done in 3 steps. First we show that every point in $M_s^2 \setminus M_s^1$ is contained in a holomorphic chart. Then we prove that singularities of pure codimension 3 don't exit, ie, $M_s^1 = M_s^0$. And finally for singularities of codimension 4 the existence of a holomorphic chart is claimed by Theorem 1.7.

3.1 Complex structure in codimension 4 and the Euler field

Lemma 3.2 *Every point $x \in M_s^2 \setminus M_s^1$ is contained in a holomorphic chart. In particular the space $M^4 \setminus M_s^1$ has a well-defined holomorphic structure.*

Proof It is sufficient to prove this lemma for tangent cones of points in $M_s^2 \setminus M_s^1$, ie for PK manifolds that are direct products of a 2–cone C^2 and the Euclidean plane $\mathbb{R}^2$.

By Definition 1.1 the complex structure on $\mathbb{R}^2 \times (C^2 \setminus 0)$ is constant with respect to the flat connection of the metric and invariant with respect to the holonomy around $(\mathbb{R}^2, 0)$. If the conical angle $2\pi\beta$ of C^2 is not divisible by 2π then this holonomy is nontrivial, it rotates the tangent planes of the horizontal fibers $(*, C^2 \setminus 0)$ by the angle $2\pi\{\beta\}$. So these fibers are holomorphic with respect to the complex structure. The fibers $(\mathbb{R}^2, *)$ are orthogonal to $(*, C^2 \setminus 0)$ and so they are holomorphic, since J preserves the metric. Thus the complex structure on $M^4 = \mathbb{R}^2 \times (C^2 \setminus 0)$ is given by the product of the natural complex structures on $\mathbb{R}^2$ and $C^2 \setminus 0$. Finally we note that $C^2 \setminus 0$ is biholomorphic to $\mathbb{C}^*$, so there is a coordinate z on C^2 holomorphic on $C^2 \setminus 0$ and continuous on C^2. The coordinate z together with a holomorphic coordinate w on $\mathbb{R}^2$ define the structure of a chart on $\mathbb{R}^2 \times C^2$. The existence of the complex structure on $M^4 \setminus M_s^1$ follows now from Lemma 3.1. The case $2\pi\beta = 2\pi k$, $k \geq 2$ is similar, the holonomy is trivial this time but by Definition 1.1 the vertical fiber $\mathbb{R}^2 \times 0$ has a holomorphic direction. □

Remark 3.3 We proved that every 2–face of a 4 dimensional polyhedral Kähler manifold that belongs to the singular locus has a *holomorphic direction*. We need to impose the condition on the faces with conical angle $2k\pi$, $k \geq 2$, in order to be able to extend the complex structure on these faces. Indeed, for a degree k ramified cover of $\mathbb{C}^2$ with a branching of order k over a totally real two-dimensional plane, the complex structure on the cover can not be extended on the branching locus.

Definition 3.4 Let M^4 be a PK manifold and let U be the universal cover of $M^4 \setminus M_s^2$. The *enveloping map* E of M^4 is defined as a locally isometric map $E \colon U \to \mathbb{C}^2$. Equivalently this map can be seen as a multivalued map from M^4 to $\mathbb{C}^2$ that is locally isometric outside of M_s^2 and has infinite ramification at M_s^2. The image of M_s^2 under the map is called *branching set* $B(E)$ of E, it is composed of linear holomorphic faces. Note that $B(E)$ is usually everywhere dense in $\mathbb{C}^2$ but in the case when $B(E)$ is closed the restriction map $E \colon E^{-1}(\mathbb{C}^2 \setminus B(E)) \to \mathbb{C}^2 \setminus B(E)$ is a covering map.

Proposition 3.5 *Any PK cone C_K^4 that is a product of $\mathbb{R}$ with a 3–cone is isometric to the product of $\mathbb{C}$ with a 2–cone. So 4–dimensional PK manifolds can not have singularities of pure codimension 3.*

Proof Suppose that C_K^4 is isometric to $\mathbb{R} \times P^3$. Denote by v the constant vector field on C_K^4 tangent to the vertical lines $(\mathbb{R}, *)$. This field is acting on $\mathbb{R} \times (P^3 \setminus 0)$ preserving the complex structure defined by Lemma 3.2. Consider the field $J(v)$ obtained from v by the complex rotation and let $v_{\mathbb{C}} = v + iJ(v)$ be the complexification of v. The field $v_{\mathbb{C}}$ is constant in the flat holomorphic coordinates on the complement to the singularities. Moreover, since the singularities of C_K^4 are tangent to v and they are holomorphic on $\mathbb{R} \times (P^3 \setminus 0)$, $v_{\mathbb{C}}$ is also tangent to the singularities.

Consider now the enveloping map E of C_K^4 and let us show that its branching locus is contained in a complex line through $E(0)$ (note that the image of the center 0 of C_K^4 is well defined). Indeed, the singularities of C_K^4 are of the form $\mathbb{R} \times r_i$, where r_i is a singular ray of P^3. The image of $\mathbb{R} \times r_i$ under E in $\mathbb{C}^2$ is a complex half-line containing $E(0)$ at its boundary. At the same time it is clear that the field $v_{\mathbb{C}}$ descends to a constant filed $E(v_{\mathbb{C}})$ on $\mathbb{C}^2$, and so all half-lines of $B(E)$ are contained in the line L through $E(0)$ tangent to $E(v_{\mathbb{C}})$.

Since $B(E) \subset L$, the map $E \colon E^{-1}(\mathbb{C}^2 \setminus L) \to \mathbb{C}^2 \setminus L$ is a covering map. But the set $E^{-1}(\mathbb{C}^2 \setminus L)$ is also a cover of the complement in C_K^4 to all half-planes $\mathbb{R} \times r$ tangent to $v_{\mathbb{C}}$ (including all singular half-planes). We deduce that the last complement is isometric to a product of a punctured 2–cone with $\mathbb{C}$ and the proposition follows. $\square$

Let us sketch an alternative proof of this proposition where instead of studying the enveloping map we work directly with the cone P^3. Consider the restriction of the field $J(v)$ on $0 \times P^3 = P^3$. The field $J(v)$ is well defined on $P^3 \setminus 0$, it is preserved by the holonomy of the metric and it is tangent to the singular rays of P^3. We will show that P^3 has at most 2 singular rays.

Let S^2 be the unit sphere centered at the origin of P^3. Consider the following function on S^2:

$$f \colon S^2 \to [-1, 1], \quad f(x) = \cos(\angle(e_r(x), I(v(x)))).$$

We claim that the critical values of this function must be equal to 1 or -1. Indeed, if x is a nonsingular point and $e_r(x)$ is not tangent to $J(v(x))$, then f has nonzero differential at x. If x is singular (ie, $x \in r_i$), then $J(v(x))$ is tangent to r_i, thus $f(x) = \pm 1$. It follows from Morse theory that f must have exactly two critical points. Thus the number of conical points on S^2 is at most 2. A further analysis shows that S^2 is either the unit sphere or a sphere with 2 conical points admitting an isometric S^1 action preserving the points. This proves the proposition. □

Example 3.6 Let S^2 be a unit sphere and p and q be two points on it. Consider a ramified degree n cover of S^2 by a sphere $\widetilde{S}^2$ with ramifications of order n at p and q. Then the pullback metric to $\widetilde{S}^2$ has 2 conical points of angles $2\pi n$. Let P^3 be a cone over $\widetilde{S}^2$ and consider the polyhedral cone $\mathbb{R} \times P^3$. The holonomy of the metric on $\mathbb{R} \times P^3$ is trivial for all choices of p and q, but the cone admits a PK structure only if p and q are opposite points on S^2. Otherwise, the line $\mathbb{R} \times 0$ forms the singular locus of codimension 3.

Definition 3.7 Let C_K^4 be a polyhedral Kähler cone. The group $\mathbb{R}^*$ is acting on C_K^4 by dilatations (since C_K^4 is a cone). It is clear that this action preserves the holomorphic structure on $C_K^4 \setminus 0$ defined by Proposition 3.5. So the vector field e_r generating this action can be complexified and the obtained holomorphic vector field is called *Euler vector field* and denoted by e. The imaginary part of the field e is called the spherical component $e_s = J(e_r)$. It is important that e_s is acting on C_K^4 by *isometries*.

Example 3.8 Consider the cone C_K^4 that is the direct product of two 2–cones with conical angles $2\pi\alpha$ and $2\pi\beta$. Choose holomorphic coordinates z and w on each 2–cone as in the beginning of Section 2.1. Then the Euler field is given by $e = (1/\alpha)z(\partial/\partial z) + (1/\beta)w(\partial/\partial w)$. Let us decompose e as above in the radial and spherical components $e = e_r + ie_s$. If $(\alpha/\beta) \in \mathbb{Q}$, then all orbits of the field e_s are closed, and the field e_s generates an action of S^1 on C_K^4. For (α/β) irrational, the closure of a generic orbit of e_s is a 2–torus.

3.2   Linear coordinates on PK cones

In this subsection we prove Theorem 1.7. In particular, we introduce a holomorphic chart in a neighborhood of every singularity of codimension 4. So, this also finishes the proof of Theorem 1.5.

Let C_K^4 be a 4–dimensional PK cone. We will consider the isometric action generated by e_s on C_K^4 and will distinguish two cases.

   (Irrational)   There exists at least one nonclosed orbit.

   (Rational)   All orbits of the action are closed.

Proposition 3.9 *If at least one of the orbit of the e_s action on C_K^4 is nonclosed then C_K^4 is isometric to the product of two 2–cones.*

Consider the group of isometries of C_K^4 preserving its origin. This is a compact Lie group, and the field e_s generates its subgroup isomorphic to $\mathbb{R}^1$ (because at least one orbit of the action on S^3 is nonclosed). The closure of this subgroup in the group of isometries is a compact connected Abelian group, ie, a torus of dimension at least two. Thus we have a faithful action of T^2 on C_K^4 by isometries.

Let us show that the branching locus of the enveloping map E of C_K^4 is contained in the union of two orthogonal lines in $\mathbb{C}^2$. Indeed, T^2 is acting on C_K^4 and this action induces an action of $\mathbb{R}^2$ on $\mathbb{C}^2$ equivariant with respect to E and fixing the point $E(0)$ in $\mathbb{C}^2$. This action factors through the standard action of T^2 on $\mathbb{C}^2$ and it leaves invariant two orthogonal lines L_1 and L_2 through $E(0)$. The branching locus of E is a union of lines trough $E(0)$ invariant under T^2 action. Thus the map

$$E\colon E^{-1}(\mathbb{C}^2 \setminus L_1 \cup L_2) \to \mathbb{C}^2 \setminus L_1 \cup L_2$$

is a covering map. It follows that C_K^4 is isometric to a product of two 2–cones.   □

Theorem 1.7 holds for PK cones isometric to the direct product of two 2–cones (see the example above). So the first case of the theorem is proved. To treat the second case we will study the action of e_s on the unit sphere S^3 of C_K^4 (S^3 it the set of points lying at distance 1 from the origin). This action is isometric and we suppose this time that all orbits are closed.

Lemma 3.10 *Suppose that all orbits of the action of e_s on S^3 are closed. Then there exists $\alpha > 0$ such that all orbits except at most two have period $2\pi\alpha$, and the exceptional orbits have periods $2\pi\alpha/p$, $2\pi\alpha/q$, where p and q are coprime numbers. Moreover the action is conjugate to the action $(z, w) \to (e^{i\theta p}z, e^{i\theta q}w)$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ on the unit sphere in $\mathbb{C}^2$, $|z|^2 + |w|^2 = 1$.*

Proof This lemma is standard and follows essentially form the fact that e_s is acting on S^3 by isometries, we will just indicate the proof. It is sufficient to show that the action induces on S^3 the structure of a Seifert fibration, in particular all orbits apart from a finite number have period $2\pi\alpha$ and the periods of all exceptional orbits divide $2\pi\alpha$. Then the lemma follows from the classification of Seifert fibrations on S^3.

Let o be an orbit of e_s on S^3, denote by $2\pi\alpha_1$ its length. Consider the flow on S^3 generated by e_s in time $2\pi\alpha_1$. It is identical on o and induces a self-map on an invariant slice transversal to o. This self-map of the slice is an isometry and it has finite period n (otherwise there exist orbits of e_s that are not closed). It follows that the flow generated by e_s in time $2n\pi\alpha_1$ induces the identity map on S^3 thus the period of every orbit divides $2n\pi\alpha_1$. Since every orbit has a neighborhood where all other orbits have period $2n\pi\alpha_1$, the number of exceptional orbits is finite. $\square$

Now, we are ready to give the proof of Theorem 1.7 in the second case. Note first that since the field e_s defines an S^1 action on C_K^4, the field $e = e_r + ie_s$ defines a holomorphic $\mathbb{C}^*$ action on $C_K^4 \setminus 0$.

Suppose first that all orbits of the action of S^1 on S^3 have the same length, ie, the pair (p, q) from Lemma 3.10 is $(1, 1)$. Then the quotient space $(C_K^4 \setminus 0)/\mathbb{C}^*$ is a complex curve homeomorphic to S^2, hence it is $\mathbb{C}P^1$. Thus $C_K^4 \setminus 0$ is isomorphic to a holomorphic $\mathbb{C}^*$ fibration over $\mathbb{C}P^1$. This fibration can be completed in a unique way to a line bundle by adding the zero section. The completed line bundle has first Chern class -1, (indeed, the associate S^1 bundle is homeomorphic to the Hopf fibration of S^3). We conclude that $C_K^4 \setminus 0$ can be identified with $\mathbb{C}^2 \setminus 0$ and the Euler field has the form $(1/\alpha)z(\partial/\partial z) + (1/\alpha)w(\partial/\partial w)$.

Consider now the case $2 \leq p < q$. Let us reduce it to the case $(p, q) = (1, 1)$. According to Lemma 3.10 there are two orbits of the action of S^1 on S^3 of lengths $2\pi(\alpha/p)$ and $2\pi(\alpha/q)$. Consider the corresponding orbits O_p and O_q of the action of $\mathbb{C}^*$ on C_K^4. It follows from Lemma 3.10 that the triple (C_K^4, O_p, O_q) is homeomorphic to a triple $(\mathbb{C}^2, \mathbb{C}^1, \mathbb{C}^1)$ composed of a complex plane and two transversal lines. Thus there exists a unique ramified covering of the cone C_K^4 by another polyhedral Kähler cone $\widetilde{C_K^4}$ of degree pq that has ramifications of orders p over O_p and q over O_q. It is easy to see that constructed cone $\widetilde{C_K^4}$ has type $(1, 1)$ and that there are holomorphic coordinates (x, y) on $\widetilde{C_K^4}$ such that the Euler field equals $(1/\alpha)x(\partial/\partial x) + (1/\alpha)y(\partial/\partial y)$. The holomorphic coordinates on C_K^4 will then be $z = x^p$, $w = y^q$ and the Euler field is $e = (p/\alpha)z(\partial/\partial z) + (q/\alpha)w(\partial/\partial w)$. $\square$

Definition 3.11 The coordinates z and w constructed above are called *linear coordinates* of a PK cone. A PK cone is called *rational of type* (p, q, α) $(p, q \in \mathbb{N})$ if its Euler

field is equal to $e = (p/\alpha)z(\partial/\partial z) + (q/\alpha)w(\partial/\partial w)$ in the linear coordinates. The number α is called the *conical angle* of the cone. A cone is called *irrational of type* (α_1, α_2), $(\alpha_1/\alpha_2) \in \mathbb{R}/\mathbb{Q}$, if its Euler field is equal to $e = (1/\alpha_1)z(\partial/\partial z) + (1/\alpha_2)w(\partial/\partial w)$.

Remark 3.12   For a rational polyhedral Kähler cone of type (p, q, α), all the orbits of the Euler field action are given by the equations $(z^p/w^q) = $ const. These curves are flat with respect to the PK metric and all of them (except the curves $z = 0$, $w = 0$) have the same conical angle at 0 equal to $2\pi\alpha$.

The proof of the following corollary is contained the second part of the proof of Theorem 1.7.

Corollary 3.13   *For a polyhedral Kähler cone C_1 of type (p, q, α) there exists a unique cone C_2 of type $(1, 1, \alpha)$ with a holomorphic map $f\colon C_2 \to C_1$ of degree pq that is a local isometry outside the branching locus.*

Finally, we describe all PK cones whose singular locus is a union of two lines in linear coordinates.

Lemma 3.14   *Let C_K^4 be a PK cone with linear coordinates (z, w) and such that the singular locus is the union of the lines $z = 0$ and $w = 0$. Then either C_K^4 is isometric to the product of two 2–cones, or the metric on C_K^4 is the pullback of a constant metric on $\mathbb{C}^2$ under the map $\mathbb{C}^2 \to \mathbb{C}^2$, $(z, w) \to (z^n, w^m)$.*

Proof   We can suppose that C_K^4 is a rational cone, irrational case is treated by Proposition 3.9. Using Corollary 3.13 we can assume that the cone is of type $(1, 1)$, ie, conical angles at the lines $z = 0$ and $w = 0$ are both equal $2\pi\alpha$. Now, consider two cases.

(1) α is not integer. Fix a nonsingular point x in C_K^4 and consider the holonomy of the metric based at x. This holonomy is generated by two commuting operators H_z and H_w corresponding to two pathes around lines $z = 0$ and $w = 0$ (both operators are nontrivial, since α is non integer). Then on $C_K^4 \setminus \{zw = 0\}$ we have two holomorphic rank 1 subbundle of TC_K^4 invariant under parallel translation and orthogonal at every point.

Consider the enveloping map $E\colon C_K^4 \to \mathbb{C}^2$. Invariant subbundles are mapped by E to constant orthogonal subbundles of $T\mathbb{C}^2$. It is clear that the ramification locus is composed of two lines through $E(0)$, tangent to one of the constant fields. These lines are orthogonal and so C_K^4 is a direct product of two 2–cones.

(2) α is integer. Then it is clear that the enveloping map from C_K^4 is in fact not multivalued but is a finite degree ramified covering of $\mathbb{C}^2$ with ramifications of degree α at the lines $z = 0$, $w = 0$. Moreover, the images of both lines are lines in $\mathbb{C}^2$ containing $E(0)$, so we are in the second case described by the lemma. $\square$

3.3 $4_\mathbb{R}$ PK cones and $2_\mathbb{R}$ spheres with conical points (proofs of Theorems 1.8 and 1.9)

We start the proof of Theorem 1.8 and associate to every PK cone of type $(1, 1, \alpha)$ a metric on a sphere S^2, of curvature 4, having conical singularities.

Denote by S^3 the unit sphere around the origin of the cone and by S^2 the quotient of S^3 by the action of e_s. Locally, outside the singularities, the action of e_s on S^3 is isometric to the action of the Euler field on the standard (nonsingular) unit sphere. Therefore, locally, outside the singularities, the quotient metric on S^2 is isometric to the quotient of the standard (nonsingular) sphere by e_s. The last quotient obviously has curvature 4. The singularities of the cone correspond to the conical points on S^2. $\square$

Lemma 3.15 *Let Ω be a contractible domain on the standard sphere S^2 of curvature 4 (without conical points). Then, for any positive l, there is a unique metric g of curvature 1 on $\Omega \times S^1$ with the following properties: All the fibers of the product are geodesics of length l; there is an action of S^1 on $\Omega \times S^1$ by isometries; the quotient metric on Ω coincides with the original one.*

Proof Let $\phi \colon S^3 \to S^2$ be the standard Hopf fibration. The universal cover of $\phi^{-1}(\Omega)$ is diffeomorphic to $\Omega \times \mathbb{R}$, and $\mathbb{R}$ acts on it by parallel translations. The quotient of $\Omega \times \mathbb{R}$ by the subgroup $l\mathbb{Z}$ of $\mathbb{R}$ induces on $\Omega \times S^1$ the metric we are looking for. $\square$

There is a natural connection ∇ on the fibration $\Omega \times \mathbb{R} \to \Omega$. Its horizontal distribution is given by the planes orthogonal to the fibers. The following lemma is standard and we omit the proof.

Lemma 3.16 *The holonomy of the connection ∇ along a closed curve $\gamma \subset \Omega$ is equal to the parallel translation by $2\,\mathrm{area}(\gamma)$, where $\mathrm{area}(\gamma)$ is the algebraic area bounded by γ.*

Now, let S^2 be a sphere with a metric of curvature 4 with conical points. We will associate to it a PK cone of type $(1, 1)$.

First, we reconstruct the sphere S^3 of curvature 1 (with singularities) that fibers over S^2. Cut S^2 by geodesic segments with vertices at all the conical points, in order to obtain a contractible polygon P. This polygon can be immersed into the standard sphere of curvature 4 by the enveloping map. Consider the fibration over P from Lemma 3.15 with length $l = 2\,\mathrm{area}(P)$. The holonomy of the fibration along the border of P is trivial (by Lemma 3.16 the circle S^1 makes one full rotation). This means that the original gluing of P, which gives S^2 with conical points, can be lifted to a

gluing of $P \times S^1$. To construct such a gluing, we choose a horizontal section s of $P \times S^1$ over the boundary of P and identify $(x, s(x))$ with $(y, s(y))$ whenever x and y are identified by the gluing of P. Since the border circle turns once, we obtain the sphere S^3.

Now, consider the space $\mathbb{R}_+ \times S^3$ with the metric $(dr)^2 + r(ds)^2$, where $r \in \mathbb{R}_+$ and $(ds)^2$ is the metric constructed on S^3. This space is a PK cone of type $(1, 1)$. □

The general case The 2–sphere with conical and marked points associated to the PK cone is given by the metric quotient of the unit sphere in the PK cone by the action of the field e_s. The marked points correspond to the multiple orbits.

Let S^2 be a sphere of curvature 4 with conical points two of which, x and y, are marked. Let us construct for every $1 < p < q$ the corresponding (p, q)–cone. Take first the $(1, 1)$–cone C associated to the sphere S^2 constructed above. Denote by l_x and l_y the preimages of $x, y \subset C$ under the projection to S^2. Consider the ramified covering of degree pq over C with the branchings of orders p over l_x and of q over l_y. This is the (p, q)–cone we are looking for. □

Proof of Theorem 1.9 Consider the case of PK cones of type $(1, 1, \alpha)$. Let S^2 be the quotient sphere associated to the cone. It has n conical points of angles $2\pi\beta_1, \ldots, 2\pi\beta_n$ and its area is given by the Gauss–Bonnet formula:

$$\text{area}(S^2) = \frac{1}{4}\left(4\pi + \sum_i 2\pi(\beta - 1)\right).$$

From Lemma 3.16 it follows $2\pi\alpha = 2\text{area}(S^2)$. This proves the theorem for $(1, 1, \alpha)$ singularities. Singularities of other types are treated in a same way using Corollary 3.13. □

3.4 PK metrics on singular piecewise linear spaces

Definition 1.1 can be naturally extended to the following class of PL manifolds with singularities. We call a 4–dimensional topological space with a simplicial decomposition a *PL–manifold up to codimension* 2 if every 3–simplex is a border of exactly two 4–simplices. A compatible choice of flat metric on the 4–simplices of such a space defines a polyhedral metric on it. Obtained metric has singularities only in codimension 2 and we can repeat Definition 1.1 saying that this metric is a PK metric if its holonomy is contained in $U(2)$ and all singular 2–faces of conical angles $2\pi k$ ($k \geq 2$) have holomorphic directions. A space with such a structure is called a *singular* PK *manifold*.

Most of the theorems of this section about (nonsingular) $4_{\mathbb{R}}$ dimensional PK manifolds can be restated for singular $4_{\mathbb{R}}$ dimensional PK manifolds. In fact, these singular manifolds are complex surfaces with isolated singularities. We will formulate the result but will skip the proof.

Theorem 3.17 *For a singular PK manifold of dimension $4_{\mathbb{R}}$ its complex structure defined outside singularities can be extended to the whole singular manifold. Obtained complex space is a complex surface with isolated singularities. In the neighborhood of every isolated singularity there is a natural holomorphic field $e = e_r + i e_s$ such that the (real) field e_r acts by dilatation of the metric and the (real) field e_s generates an action of S^1 by isometries.*

4 Polyhedral Kähler metrics via logarithmic connections

For every PK surface S the PK metric induces on $T(S \setminus \text{sing})$ a holomorphic, flat, unitary, torsion free connection. This connection extends to a meromorphic connection on TS with first order poles at the singular locus. In this section we will write explicit formulas of PK connections in *linear coordinates* z, w on 2–dimensional PK cones. We also give a condition for a unitary connection on the tangent bundle of a complex surface that implies that the corresponding metric on S is PK.

4.1 Definitions and first results

Let M be a complex manifold and D be a normal crossing divisor. A meromorphic 1–form ω on M is called *logarithmic* with respect to D if it is holomorphic on $M \setminus D$, and in a neighborhood of any point of D it can be represented as

$$\omega = \sum_{i=1}^{k} f_i \frac{dz_i}{z_i} + \sum_{i=k+1}^{n} f_i dz_i,$$

where f_i are holomorphic functions, z_i are local coordinates, and D is given locally by the equation

$$D = \bigcup_{i=1}^{k} \{z_i = 0\}.$$

The sheaf of logarithmic 1–forms is denoted by $\Omega^1(\log D)$.

Let E be a holomorphic vector bundle over M. A meromorphic connection ∇ on E is called *logarithmic* (with respect to D) if it can be written in local coordinates as

$$\nabla = d + A,$$

where A is a $\Omega^1(\log D)$–valued section of $\mathrm{End}(E)$.

For any irreducible component D_i of D we denote by $\mathrm{Res}_{D_i}(\nabla)$ the *residue* of ∇ with respect to D_i, it is a holomorphic section of $\mathrm{End}(E)|_{D_i}$.

The following proposition is standard, the proof can be found in Section 4 of the article of Malgrange in [1].

Proposition 4.1 *Let ∇ be a flat logarithmic connection on (M, E^k) with poles at a normal crossing divisor D. Suppose that all eigenvalues of $\mathrm{Res}_D(\nabla)$ are contained in $]-1, 0]$.*

For a point $x \in D$ chose local coordinates z_i such that D_i is given by $\bigcup_i\{z_i = 0\}$. Then there exists a neighborhood U of x and holomorphic sections $s_1, \ldots, s_k$ of E^k giving a trivialisation of E^k over U and such that in this trivialisation ∇ is given by

$$\nabla = d + \sum_i B_i \frac{dz_i}{z_i},$$

where B_i are constant matrix-valued functions.

Definition 4.2 For a complex manifold M and a meromorphic connection ∇ on TM its *torsion* is a meromorphic section of $\Omega^2(M) \otimes TM$. It is given by the following formula:

$$T(u, v) = \nabla_u(v) - \nabla_v(u) - [u, v],$$

$u, v \in TM$. A connection with zero torsion is called *torsion-free*.

Now, we will restrict our attention to connections on the tangent bundles of surfaces. Let S be a surface with a weighted arrangement of curves (Γ_j, β_j) and let $x_1, \ldots, x_k$ be the points of the arrangement of multiplicity at least 3.

Definition 4.3 We say that a meromorphic connection ∇ on TS is *partially adapted* to (Γ_j, β_j) if ∇ is logarithmic on $S\setminus\{x_1, \ldots, x_k\}$, $\mathrm{Res}_{\Gamma_j}\nabla$ has eigenvalues $(\beta_j - 1, 0)$ at Γ_j, and $T\Gamma_j \cong \ker(\mathrm{Res}_{\Gamma_j}\nabla)$.

Lemma 4.4 *On every PK surface S the PK connection is partially adapted to the weighted arrangement of S.*

Proof The statement of the lemma clearly holds at smooth points of the singular locus of S, because they can be embedded isometrically in the product of $\mathbb{C}$ with a 2–cone. At the same time the connection on the 2–cone is logarithmic and has residue $\beta - 1$ where $2\pi\beta$ is the cone angle.

From the description of double points of the singular locus (Lemma 3.14) it follows that the connection is also logarithmic at these points. Indeed, the connection on the direct product of two 2–cones is logarithmic, and for the second type of cones that are branched covers of $\mathbb{C}^2$ one can change the metric without changing the connection to make these cones also direct products. □

Lemma 4.5 *Let S be a complex surface with a weighted arrangement (Γ_j, β_j), $\beta_j \neq 1$. Suppose that ∇ is partially adapted, then its torsion is holomorphic.*

Proof By Hartog's theorem it is sufficient to show that the torsion of ∇ is holomorphic outside of the multiple points of the arrangement Γ_j. So it is sufficient to consider the case of a connection on $\mathbb{C}^2$ with a pole at the line $z = 0$. Chose locally the second coordinate w in such a way that the residue of ∇ is given by the formula

$$\mathrm{Res}_{z=0}\nabla = \begin{pmatrix} \beta - 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case the connection ∇ can be written as

(4-1) $$\nabla = d + \frac{dz}{z}\mathrm{Res}_{z=0}\nabla + A,$$

where A is holomorphic. This proves the lemma. □

Corollary 4.6 *Suppose that $\Gamma(\Omega_S^1) = 0$, ie, there is no nontrivial holomorphic 1–forms on S. Then the torsion of ∇ is identically zero.*

Proof The torsion of a meromorphic connection on the tangent bundle to a complex manifold X is a section of $\Omega_X^2 \otimes TX$. In the case when X is a two-dimensional complex surface S the bundle $\Omega_S^2 \otimes TS$ is two-dimensional and isomorphic to the bundle of holomorphic 1–forms Ω_S^1. Thus on S any holomorphic section of $\Omega_S^2 \otimes TS$ is identically zero. □

4.2 Formulas for connections on $(1, 1)$ cones

In the following proposition we describe the connection of a PK metric in a neighborhood of a singular point of type $(1, 1)$.

Proposition 4.7 *Consider a PK cone of type $(1, 1, \alpha)$ with linear coordinates z, w. For $i = 1, \ldots, n$ let $l_1 = 0, \ldots, l_n = 0$ be the equations of the singular lines of the cone. Let $2\pi\beta_i$ be the conical angle at $l_i = 0$. Then the following holds:*

(1) *The residue $\mathrm{Res}_{l_i} \nabla$ of ∇ at l_i is given by a constant matrix-valued function A_i and the connection ∇ is given by the following formula:*

$$(4\text{-}2) \qquad \nabla = d + A = d + \sum_{i=1}^{n} A_i \frac{dl_i}{l_i}.$$

(2) *The residues A_i satisfy the following equations:*

$$(4\text{-}3) \quad \text{(a)} \ \sum_{i=1}^{n} A_i = (\alpha - 1)\mathrm{Id}; \quad \text{(b)} \ \mathrm{tr}(A_i) = \beta_i - 1; \quad \text{(c)} \ \{l_i = 0\} = \ker(A_i).$$

Proof of Proposition 4.7(1) We prove first that the residue of ∇ is constant at any l_i. Indeed, the action of $\mathbb{C}^*$ on the cone changes the PK metric by a scalar factor, thus this action preserves the connection ∇. For any $c \in \mathbb{C}^*$ we have

$$A(cz, cw) = A(z, w),$$

ie, the residue $\mathrm{Res}_{l_i}(\nabla)$ is constant on l_i.

Consider now the following connection ∇' on $\mathbb{C}^2$:

$$\nabla' = d + \sum_i \mathrm{Res}_{l_i}(\nabla) \cdot \frac{dl_i}{l_i}.$$

We claim that $\nabla' = \nabla$. Indeed, the matrix-valued 1–form $\nabla' - \nabla$ has no poles at the lines l_i thus it is holomorphic on $\mathbb{C}^2$. Moreover this 1–form is preserved by the $\mathbb{C}^*$ action, ie, it is identically zero. $\square$

Next lemma is essential for the proof of Proposition 4.7(2).

Lemma 4.8 *For the Euler field e of the PK cone we have*

$$\nabla_e e = e.$$

Proof It is sufficient to check this identity for the flat $\mathbb{C}^2$. $\square$

Proof of Proposition 4.7(2) Let us prove (4-3)(a). The Euler field e on the PK cone is given by

$$e = \frac{1}{\alpha}\left(z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}\right).$$

We have

$$\nabla_e e = de(e) + \sum_{i=1}^{n} A_i(e)\frac{dl_i}{l_i}(e) = \frac{1}{\alpha}e + \sum_{i=1}^{n}\frac{1}{\alpha}A_i(e) = e.$$

This means that $\sum_{i=1}^{n} A_i(e) = (\alpha - 1)e$, ie, $A = (\alpha - 1)\mathrm{Id}$.

Statements (4-3)(b)–(c) are proven in Lemma 4.4. □

Remark 4.9 For $i = 1, \ldots, n$ let l_i be lines in $\mathbb{C}^2$ containing the origin and let β_i be complex numbers. Then the space of matrixes A_i satisfying (4-3) has dimension $(n - 3)$. Next formula gives the unique connection for $n = 3$ with poles at the lines $z = 0$, $w = 0$, $z + w = 0$:

$$\nabla = d + \begin{pmatrix} (\beta_1 - 1)\frac{dz}{z} + \frac{\beta_2 + \beta_3 - \beta_1 - 1}{2}\frac{dz + dw}{z + w} & \frac{\beta_2 + \beta_3 - \beta_1 - 1}{2}(\frac{dz + dw}{z + w} - \frac{dw}{w}) \\ \frac{\beta_1 + \beta_3 - \beta_2 - 1}{2}(-\frac{dz}{z} + \frac{dz + dw}{z + w}) & \frac{\beta_1 + \beta_3 - \beta_2 - 1}{2}\frac{dz + dw}{z + w} + (\beta_2 - 1)\frac{dw}{w} \end{pmatrix}.$$

Proposition 4.10 *Any connection ∇ on $\mathbb{C}^2$ given by formula (4-2) with the matrices A_i satisfying (4-3) is flat, torsion-free and thus it defines a singular affine structure on $\mathbb{C}^2$.*

Proof Since $dA = 0$ the curvature of ∇ is given by

$$dA + A \wedge A = A \wedge A.$$

We need to prove that $A \wedge A = 0$. Writing

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix},$$

then the equation $A \wedge A = 0$ is equivalent to the following system:

$$\begin{cases} \sum_i c_i \frac{dl_i}{l_i} \wedge \sum_i b_i \frac{dl_i}{l_i} = 0 \\ \sum_i b_i \frac{dl_i}{l_i} \wedge \sum_i (a_i - d_i)\frac{dl_i}{l_i} = 0 \\ \sum_i c_i \frac{dl_i}{l_i} \wedge \sum_i (a_i - d_i)\frac{dl_i}{l_i} = 0 \end{cases}$$

For the first equation we have

$$\sum_i c_i \frac{dl_i}{l_i} \wedge \sum_i b_i \frac{dl_i}{l_i} = d\log\left(\prod_i l_i^{c_i}\right) \wedge d\log\left(\prod_i l_i^{b_i}\right).$$

Function $f_1 = \log(\prod_i l_i^{c_i})$ and $f_2 = \log(\prod_i l_i^{b_i})$ are homogeneous of degree 0 on $\mathbb{C}^2$ (since $\sum c_i = \sum b_i = 0$ by (4-3)(a)). It follows that $df_1 \wedge df_2 = 0$. The next two equations are completely analogous.

Now we show that A is torsion free. We have

$$T\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial w}\right) = \sum_{i=1}^{n} \frac{dl_i}{l_i}\left(\frac{\partial}{\partial z}\right) A_i\left(\frac{\partial}{\partial w}\right) - \frac{dl_i}{l_i}\left(\frac{\partial}{\partial w}\right) A_i\left(\frac{\partial}{\partial z}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{l_i} A_i\left(\frac{\partial l_i}{\partial z}\frac{\partial}{\partial w} - \frac{\partial l_i}{\partial w}\frac{\partial}{\partial z}\right) = 0.$$

The last equality holds by (4-3)(c). □

Connection on the blow-up

Lemma 4.11 Let ∇ be a connection on $\mathbb{C}^2$ given by formula (4-2) with the matrices A_i satisfying (4-3). Consider the blow-up of $\mathbb{C}^2$ at $(0,0)$, take the pullback of the tangent bundle of $\mathbb{C}^2$ to the blow-up, and consider on it the pullback of ∇. The obtained connection is logarithmic and its residue at the exceptional curve is $(\alpha-1)\mathrm{Id}$.

Proof Let us introduce coordinates u, v on the blow-up $u = (z/w)$, $v = w$. The exceptional line is given by $v = 0$. Let $l_i = s_i z + t_i w = s_i u v + t_i v$. Then the pullback connection is given by

$$\nabla = d + \sum_{i=1}^{n} A_i d(\log l_i) = d + \sum_{i=1}^{n} A_i\big(d(\log v) + d(\log(s_i u + t_i))\big)$$

$$= d + (\alpha-1)\mathrm{Id} \cdot d(\log v) + \sum_{i=1}^{n} A_i \cdot d(\log(s_i u + t_i)).$$

Here we use Equation (4-3)(a). This proves the lemma. □

4.3 Unitary flat logarithmic torsion free connection $\mapsto$ PK metric

In this subsection we consider only arrangements (S, Γ_j) that satisfy the property that Γ_j are smooth and transversal. We give a sufficient criterion for such an arrangement to be the singular locus of a PK metric in terms of an *adapted* connection. Singularities of such an arrangement are normal crossings or singularities of type $(1, 1)$.

Definition 4.12 Let ∇ be a connection partially adapted to (S, Γ_j, β_j). Suppose that Γ_j are smooth and intersect transversally. Consider the blow up $\pi: \widetilde{S} \to S$ at all points x_i of multiplicity at least 3. ∇ is *adapted* to (Γ_j, β_j) if the pullback connection $\pi^*\nabla$ on π^*TS is logarithmic on $\widetilde{S}$ and its residue at the exceptional curve over x_i equals $\sum_j d_{ij}(\beta_j - 1)\mathrm{Id}$.

Theorem 4.13 *Let $(S, \Gamma_j, \beta_j, x_i)$ be a weighted arrangements of curves and ∇ be an adapted flat, unitary, torsion free connection on TS. Suppose that $0 < \beta_j < 1$ and for every x_i, $\sum_j d_{ij}(\beta_j - 1) > -2$. Then the unitary metric on TS corresponding to ∇ is a* PK *metric.*

Proof Note first that since ∇ is unitary, flat, and torsion-free, the metric g corresponding to ∇ is flat on the complement to the curves Γ_j. So to prove that g extends to a PK metric on S it will be sufficient to show the following 3 properties of g.

  (a) For any smooth point of Γ there is a neighborhood U with $U \setminus \Gamma$ isometric to the direct product of a flat punctured 2–cone and a flat disc.

  (b) For any double point of Γ there is a neighborhood U with $U \setminus \Gamma$ isometric to the direct product of two flat punctured 2–cones.

  (c) For any point of Γ of multiplicity greater than 2 a neighborhood of the point is isometric to a $2_{\mathbb{C}}$–dimensional PK cone.

The proofs of (a) and (b) are similar so we will prove only (b) and (c).

Proof of (b) Introduce coordinates z, w in a neighborhood of the double point such that Γ is locally given by $zw = 0$. According to Proposition 4.1 there exist two sections s_1 and s_2 such that the connection ∇ is given by

$$\nabla = d + B_1 \frac{dz}{z} + B_2 \frac{dw}{w},$$

where B_1 and B_2 are constant and commuting. Making an additional linear change in s_1, s_2 we can suppose that $B_1(s_1) = 0$ and $B_2(s_2) = 0$. It is clear then that the subbundles of the tangent bundle generated by s_1 and s_2 are invariant under ∇ and moreover the vector field s_1 is tangent to the line $w = 0$ and s_2 is tangent to the line $z = 0$. Since ∇ is torsion free the integral curves of s_1 and s_2 are flat cones. So we can deduce that locally the neighborhood of $(0, 0)$ is the direct product of two 2–cones. $\square$

Proof of (c) Let 0 be a point of Γ of multiplicity greater than 2. From (a) it follows that the metric g extends continuously to any punctured curve $\Gamma_j \setminus 0$, thus we obtain a polyhedral Kähler metric on $U \setminus 0$. It is necessary to show further that g extends to 0 and the resulting metric on U is a polyhedral Kähler metric. To show this it is sufficient to construct the action of $\mathbb{C}^*$ on $U \setminus 0$ by dilatations, ie, $\mathbb{R}^*$ must act by dilatations of the metric and S^1 must act by isometries. $\square$

Construction of the $\mathbb{C}^*$ action on $U \setminus 0$ Consider the holonomy representation of the group $\pi_1(U \setminus \Gamma, x)$ in the group $U(2, \mathbb{C}) \ltimes \mathbb{C}^2$ of unitary affine transformations of $\mathbb{C}^2$:

$$\mathrm{Hol}_\nabla: \pi_1(U \setminus \Gamma, x) \to U(2, \mathbb{C}) \ltimes \mathbb{C}^2.$$

The linear part of this representation is denoted by hol_∇. Denote by c the generator of the center of $\pi_1(U \setminus \Gamma, x)$ corresponding to an anti-clockwise path around 0 in a complex line through 0. From condition (2) of Definition 4.12 it follows that

$$\mathrm{hol}_\nabla(c) = \exp\left(2\pi i \sum_j (\beta_j - 1) \right) \mathrm{Id}.$$

We see that the affine transformation $\mathrm{Hol}_\nabla(c)$ is a complex rotation of $\mathbb{C}^2$ around some point y on angle $2\pi \sum_j (\beta_j - 1)$.

Since any element h of $\pi_1(U \setminus \Gamma, x)$ commutes with c, the affine transformation $\mathrm{Hol}_\nabla(h)$ fixes the point y. It follows that the representation $\mathrm{Hol}_\nabla(c) \pi_1(U \setminus \Gamma, x)$ commutes with the affine action of $\mathbb{C}^*$ on $\mathbb{C}^2$, given by complex dilatations that fix y. We deduce that the action of $\mathbb{C}^*$ can be pulled back to the action of $\mathbb{C}^*$ on $U \setminus 0$. $\square$

Remark 4.14 The condition $\beta_j < 1$ in this theorem can be replaced by the condition $\beta_j \notin \mathbb{Z}_+$ but we don't prove this here.

5 Topological relations on the singular locus

In this section we prove Theorem 1.11. We use a formula of Ohtsuki [15].

Theorem [15] *Let S be a compact surface and E be a holomorphic vector bundle on it. Let $D = \bigcup_j D_j$ be a normal crossings divisor on S and ∇ a logarithmic connection on E with poles at D. Denote by y_k the double points of D and by D_{k_1} and D_{k_2} the irreducible components of D containing y_k. We also use the notation $R_j = \mathrm{Res}_{D_j}(\nabla)$. The following identities hold:*

$$(5\text{-}1) \qquad c_1(E) = -\sum_j \mathrm{Tr}(R_j) D_j,$$

$$(5\text{-}2) \qquad c_2(E) = \sum_k \big(\mathrm{Det}(R_{k1} + R_{k2}) - \mathrm{Det}(R_{k1}) - \mathrm{Det}(R_{k2})\big)(y_k)$$
$$+ \sum_j \mathrm{Det}(R_j) D_j \cdot D_j.$$

Note that the value of the first summand is defined only at y_k but the function $\mathrm{Det}(R_j)$ is defined on the whole divisor D_j and is constant on it.

5.1 Proof of Theorem 1.11, Equations (1-1) and (1-2)

Lemma 5.1 *For any $j \neq k$ we have*

$$(5\text{-}3) \qquad \Gamma_j \cdot \Gamma_k = B_{jk} + \sum_i (p_i q_i)(d_{ij} d_{ik}).$$

Proof This formula expresses the intersection index of Γ_j and Γ_k as a sum of local multiplicities of their intersections. By definition B_{jk} is the number of transversal intersections of Γ_j and Γ_k. The local multiplicity of the intersection of Γ_j and Γ_k at x_i equals $(p_i q_i)(d_{ij} d_{ik})$. Indeed, the local multiplicity of the intersection of the curves $cz^q = w^p$ and $z^q = w^p$ at 0 equals pq if $1 \neq c \neq 0$; the local multiplicities of intersection of $z^q = w^p$ with lines $z = 0$ and $w = 0$ are equal to p and q correspondingly. Now everything follows from the definition of d_{ij} and d_{ik}. $\qquad\square$

Proof of Theorem 1.11, Equation (1-1)

The Gauss–Bonnet theorem for flat surfaces with conical singularities implies

$$2g(\Gamma_j) - 2 = \sum_{k \neq j} B_{jk}(\beta_k - 1) + \sum_i (d_{ij}\alpha_i - \tilde{d}_{ij}).$$

This formula expresses the Euler characteristics of Γ_j as the sum of the defects of the conical points of Γ_j. The first sum contains the contribution of the normal crossings of Γ_j and the second sum contains the contribution of the singularities x_i.

Now, using Theorem 1.9, we obtain the following expression for the right term of the previous equation:

$$\sum_{k \neq j} B_{jk}(\beta_k - 1) + \sum_i \left(d_{ij}\frac{p_i q_i}{2} \sum_k d_{ik}(\beta_k - 1) + d_{ij}\frac{p_i + q_i}{2} - \tilde{d}_{ij} \right).$$

Using Lemma 5.1, this equals

$$\sum_{k \neq j} B_{jk}\frac{\beta_k - 1}{2} + \sum_k \Gamma_j \cdot \Gamma_k \frac{\beta_k - 1}{2} - \Gamma_j \cdot \Gamma_j \frac{\beta_j - 1}{2} +$$

$$+ \sum_i \frac{p_i q_i}{2}(d_{ij})^2(\beta_j - 1) + \sum_i \left(d_{ij}\frac{p_i + q_i}{2} - \tilde{d}_{ij} \right).$$

Using relation (1-2), this becomes

$$\sum_k B_{jk}\frac{\beta_k - 1}{2} + \frac{K_S \cdot \Gamma_j}{2} + \sum_i \left(d_{ij}\frac{p_i + q_i}{2} - \tilde{d}_{ij} \right).$$

This proves Theorem 1.11, Equation (1-1). $\qquad\square$

Proof of Theorem 1.11, Equation (1-2) Let S be a PK surface and (Γ_j, β_j, x_i) be the corresponding weighted arrangement. By Lemma 4.4 the corresponding connection ∇ on TS is logarithmic on $S \setminus \bigcup_i x_i$. Furthermore ∇ induces a flat connection ∇_K on K_S logarithmic on $S \setminus \bigcup_i x_i$, and with residue $1 - \beta_j$ at Γ_j. Consider a resolution of singularities of Γ on S, $p: S' \to S$ such that $p^*(\Gamma)$ is a curve with normal crossing singularities. It is not hard to show that the pullback connection $p^* \nabla_K$ on the pullback line bundle $p^* K_S$ is logarithmic on the whole S'. To prove the statement of the theorem it is sufficient to show that

$$p^*[K_s] - \sum_j (\beta_j - 1) p^*[\Gamma_j] = 0 \in H_2(S').$$

Now notice that by (5-1) this homology class is expressed as a combination of homology classes of exceptional divisors (since the residue of $p^* \nabla_K$ at $p^* \Gamma_j$ is $(1 - \beta_j)$). But since at the same time this class is a pullback from S it must vanish. □

5.2 Proof of Theorem 1.11, Equation (1-3)

Consider a PK surface S with the weighted arrangement $(\Gamma_j, \beta_j; x_i, \alpha_i)$ such that all multiple points of the arrangement are either normal crossings or singularities of type $(1, 1)$. Let S_b be the blow-up of S at the points x_i and let $\pi: S_b \to S$ be the blow-down. Denote by P_i the exceptional curve over x_i and denote by $\widetilde{\Gamma}_j$ the proper transform of Γ_j. Consider the pullback $\widetilde{\nabla}$ of the PK connection ∇ to $\pi^*(TS)$. By Lemma 4.11 $\widetilde{\nabla}$ is logarithmic with poles at the divisor $\bigcup_i P_i \bigcup_j \widetilde{\Gamma}_j$ (further we call this divisor by D), and the residue of $\widetilde{\nabla}$ at P_i is equal to $(\alpha_i - 1)\mathrm{Id}$.

Proof of Theorem 1.11, Equation (1-3) According to (5-2) the number $c_2(\pi^* TS)$ can be expressed as the sum of the contributions of the irreducible components of D and the sum over their pairwise intersections. The first sum is the following:

$$\sum_i \mathrm{Det}(\mathrm{Res}_{P_i} \widetilde{\nabla}) P_i \cdot P_i + \sum_j \mathrm{Det}(\mathrm{Res}_{\widetilde{\Gamma}_j} \widetilde{\nabla}) \widetilde{\Gamma}_j \cdot \widetilde{\Gamma}_j = -\sum_i (\alpha_i - 1)^2.$$

Here we use $\mathrm{Det}(\mathrm{Res}_{\widetilde{\Gamma}_j} \widetilde{\nabla}) = 0$.

The sum of the contributions of the double points on P_i is the following:

$$(\alpha_i - 1) \sum_j d_{ij} (\beta_j - 1) = 2(\alpha_i - 1)^2.$$

Any intersection of $\widetilde{\Gamma}_j$ with $\widetilde{\Gamma}_k$ contributes $(\beta_j - 1)(\beta_k - 1)$ thus the sum over all intersections of curves $\widetilde{\Gamma}_j$ is given by

$$\sum_{j>k} B_{jk}(\beta_j - 1)(\beta_k - 1).$$

Finally, taking the sum of all contribution we obtain

$$c_2(TS) = c_2(\pi^*(TS)) = \sum_i (\alpha_i - 1)^2 + \sum_{j>k} B_{jk}(\beta_j - 1)(\beta_k - 1). \qquad \square$$

5.3 Line arrangements in $\mathbb{C}P^2$

In this subsection we consider weighted arrangements of lines on $\mathbb{C}P^2$ that satisfy Equations (1-1)–(1-3). Let $(L_1, \beta_1; \ldots; L_n, \beta_n)$ be such an arrangement. Any singularity of this arrangement is either a normal crossing or a $(1, 1)$–type singularity. So Equations (1-1) and (1-2) simplify and take the form:

$$\sum_k B_{jk}(\beta_k - 1) = -1, \quad \sum_k (\beta_k - 1) = -3.$$

Here by definition the number B_{ij} $(i \neq j)$ is equal to 1 if the point of the intersection of L_i and L_j is a double point (ie, other lines of the arrangement don't contain it) and $B_{ij} = 0$ otherwise. The number $(B_{jj} + 1)$ is equal to the number of points of multiplicity at least 3 on the line L_j.

Symmetric case Consider the most symmetric case when all angles β_j are equal. Then we have

$$\beta_j - 1 = -\frac{3}{n}, \quad \sum_k B_{jk} = \frac{n}{3}.$$

The number $\sum_k B_{jk} + 1$ is equal to the number of all intersections of L_j with other lines. Thus we obtain the following condition:

The arrangement contains $3m$ lines and any line intersects the other lines exactly at $m + 1$ points ($m = n/3$).

One can show that such arrangements satisfy as well Equation (1-3). These arrangements were considered first by Hirzebruch in [7] and we recall several examples (all arrangements apart from the first one are PK).

(1) (3 lines) A generic configuration of 3 lines on $\mathbb{C}P^2$.

(2) (6 lines) The configuration of 6 lines $x - y = 0$, $x \pm z = 0$, $y \pm z = 0$ $z = 0$.

(3) (3($m + 1$) lines, $m > 1$) Consider the ramified covering of $\mathbb{C}P^2$ by itself given by $(x : y : z) \to (x^m : y^m : z^m)$. The preimage of the configuration of 6 lines is an arrangement of $3(m + 1)$ lines and any line has $m + 2$ intersections with the other lines.

(4) (Hesse arrangement) Consider a nonsingular cubic in $\mathbb{C}P^2$. It has 9 points of inflections. There exist 12 lines in $\mathbb{C}P^2$ that intersect the cubic exactly at the points of inflections.

Criterion   There exists one geometric condition that often permits to decide quickly that a given line arrangement is not PK.

Proposition 5.2   *Every PK arrangement (L_j, β_j) in $\mathbb{C}P^2$ satisfy the following criterium. For every L_j there exists a point in $\mathbb{C}P^2$ that belongs to all lines L_k such that $L_k \cap L_j$ is a double point of the arrangement.*

Proof   Consider the subbundle of $T\mathbb{C}P^2|_{L_j}$ of directions orthogonal to L_j with respect to PK metric (these directions are eigenvectors of the residue map $\mathrm{Res}_{L_j}\nabla$). This subbundle is holomorphic outside of the multiple points of the arrangement and it extends holomorphically on the whole line L_j. Indeed, at double points the PK metric is a direct product of two 1–cones, and at every point of multiplicity more than 2 the eigenvectors of $\mathrm{Res}_{L_j}\nabla$ are constant in the local linear coordinates (Proposition 4.7). The defined subbundle of $T\mathbb{C}P^2|_{L_j}$ is transversal to L_j and so it has degree 1. It follows that there exists a point y_j in $\mathbb{C}P^2$ such that this subbundle of $T\mathbb{C}P^2|_{L_j}$ is given by directions tangent to the lines through y_j. This proves the proposition.   □

5.4 Limit PK arrangements with a cusp

Definition 5.3   An arrangement of lines on $\mathbb{R}P^2$ is called *simplicial* if it subdivides $\mathbb{R}P^2$ in triangles.

Simplicial arrangements often occur as solutions to some extreme (combinatorial) problems (see Grünbaum [6]) and so it is not very surprising that some of simplicial arrangements are PK (PK arrangements give an extremum for the Bogomolov–Gieseker Inequality (1-5) in Theorem 1.12). For the moment 3 infinite families of simplicial plane arrangements and 91 sporadic examples are know; 90 sporadic examples are listed by Grünbaum in [5] and one additional in [6]. It will be interesting to find out what sporadic arrangements are PK, the criterium from Proposition 5.2 rules out some of them.

For three infinite series of arrangements from [5] there exists a system of weights such that all *equalities* of Theorem 1.11 and Theorem 1.12 are satisfied. The first series is a union of a pencil of n lines and a line that does not belong to the pencil. For this series the weight of the line should be 0 and the weights of the lines from the pencil can be arbitrary (but we impose of course $\sum_j (\beta_j - 1) = -3$). Second series is called R_{2k} and consists of the lines formed by extending the sides of a regular k–gon together with an additional k lines formed be the axes of symmetry of the k–gon. We associate to the axes weight $(k-1)/k$ and to the sides weight $(2k-1)/(2k)$ (the choice of wights is unique for $k > 3$). The third series R_{4k+1} is the union of R_{4k} with the line at infinity of weight 1. We see that for the point of the highest multiplicity of these arrangements the inequality $\sum_j d_{ij}(\beta_j - 1) > -2$ does not hold strictly but instead of this the equality holds. So these arrangements are not PK. We conjecture instead that there is a different geometric structure related to these arrangements.

Definition 5.4 A weighted arrangement (L_j, β_j) in $\mathbb{C}P^2$ is called a *limit PK arrangement with a cusp* if it satisfies all conditions of Theorem 1.12 apart from one inequality. We impose that there is a multiple point x of the arrangement called the *cusp* such that $\sum_{j|x \in L_j}(\beta_j - 1) = -2$.

Arrangements R_{2k} satisfy this definition, R_{4k+1} and a pencil of lines plus a line formally don't satisfy (because the first arrangement has a line of weight 1 and the second has a line of weight 0). But the following should hold for all 3 series.

Conjecture 5.5 For every limit PK arrangement with a cusp there exists a flat torsion-free connection on $\mathbb{C}P^2$ with holonomy in the upper triangular subgroup of $SL(2, \mathbb{C})$ and with the poles of residues $(0, \beta_j - 1)$ at the lines L_j. This connection should preserve the sub–line bundle of $T\mathbb{C}P^2$ tangent to the pencil of lines through the cusp of the arrangement.

For a pencil of lines plus a line the connection on $\mathbb{C}P^2$ should be given by a formula from Proposition 4.10. Here $\mathbb{C}P^2$ is the completion of $\mathbb{C}^2$, the line at infinity belongs to the arrangement and has weight $\beta = 0$.

If Conjecture 5.5 holds it should be possible to deduce that limit PK arrangements with a cusp satisfy the following restrictive properties.

Conjectural properties For every multiple point y of the arrangement the line $[x, y]$ belongs to the arrangement (here x is the cusp). If $\mathrm{mult}(y) \geq 3$ then sum of the defects of the lines L_j that contain y but don't contain x equals the defect of the line $[x, y]$.

It will be very interesting to classify all weighted arrangements that satisfy the two conjectural properties (they hold for R_{2k} and R_{4k+1}). This will help to classify *nonrigid* PK arrangements for which the admissible collection of weights β_j have moduli. Of course the weights β_j belong to a certain open polyhedron and the weights corresponding to limit PK arrangements can appear on the boundary of the polyhedron.

6   Parabolic bundles and Kobayashi–Hitchin correspondence

The goal of this section is to recall the notion of parabolic bundles and to formulate in a handy way several results from Mochizuki [13] that we use in the proof of Theorem 1.12. In particular we formulate the parabolic version of Kobayashi–Hitchin correspondence from [13]. A systematic and thorough treatment of parabolic bundles can be found in Iyer and Simpson [8]. Parabolic Chern character is also defined in [8] (but we will not use it here). We adopt partially the notation of these articles.

We will discuss only parabolic bundles on complex surfaces. A good reference for usual two-dimensional bundles on surfaces is Friedman [4].

Definition 6.1   Let X be a complex surface and D be a simple normal crossing divisor with the irreducible decomposition $D = \bigcup_{i \in S} D_i$. A *parabolic bundle E_** on X is given by a bundle E with a collection of increasing filtrations by subsheaves F_a^i, indexed by $i \in S$, $a \in]0, 1]$ and satisfying the following properties:

  (1)   Every subsheaf F_a^i is locally free.

  (2) $E(-D_i) \subset F_a^i(S)$ for any $a \in]0, 1]$.

  (3)   The sets $\{a \mid F_a^i(E)/F_{<a}^i(E) \neq 0\}$ are finite for any i in S.

Remark 6.2   A parabolic bundle E_* on a complex surface X with a simple normal crossing divisor D induces natural filtrations on the restrictions $E|_{D_i}$ by their vector subbundles. These filtrations are indexed by $a \in]0, 1]$ and defined by the formula:

$$F_a^i/E(-D) \subset E|_{D_i}.$$

Parabolic structure can be reconstructed from these filtrations (see Iyer and Simpson [8]).

Definition 6.3   The *parabolic first Chern class* of a parabolic bundle E_* is given by the following formula:

(6-1) $$\mathrm{par\text{-}ch}_1(E_*) = \mathrm{ch}_1(E) - \sum_i \sum_{a_i} a_i \cdot \mathrm{rank}_{D_i}(F_{a_i}^i/F_{<a_i}^i) \cdot [D_i].$$

Let L be an ample line bundle on S. Then the *parabolic degree* of E_* with respect to L is given by

$$(6\text{-}2) \qquad \operatorname{pardeg}_L(E_*) = \int_S \operatorname{par-ch}_1(E_*) \cdot c_1(L).$$

The parabolic second Chern characteristic number Let (S, D) be a complex surface with a normal crossing divisor $D = \bigcup_{i=1}^n D_i$. Let E_* be a parabolic vector bundle. We will recall now the formula for the parabolic second character of E_* given by Mochizuki in [13]. His formula works in much larger generality but we need only the case of surfaces.

The parabolic second Chern character of E_* is given as a sum of $c_2(E)$, the contributions of the divisors D_i, and the points of their intersections $D_i \cap D_j$.

To define the contributions of the points in $D_i \cap D_j$, for every $a_i, a_j \in]0, 1]$, consider the skyscraper sheaf $\operatorname{Gr}_{(a_i,a_j)}^F$:

$$F_{a_i}^i \cap F_{a_j}^j / ((F_{a_i}^i \cap F_{<a_j}^j) + (F_{<a_i}^i \cap F_{a_j}^j)).$$

This sheaf is supported at the points in $D_i \cap D_j$, and it is nontrivial only for finite set of (a_i, a_j). Consider the sum

$$v(i, j) = \sum_{p \in D_i \cap D_j; a_i, a_j} a_i \cdot a_j \cdot \operatorname{rank}_p(\operatorname{Gr}_{(a_i,a_j)}^F), \quad (v(i, j) = v(j, i)).$$

Definition 6.4 The second parabolic Chern character of the parabolic bundle E_* is given by the formula

$$\operatorname{par-ch}_2(E_*) = \operatorname{ch}_2(E) - \sum_{i; a_i} a_i \cdot c_1(F_{a_i}^i / F_{<a_i}^i)$$

$$+ \sum_{i; a_i} \frac{1}{2} a_i^2 \operatorname{rank}_{D_i}(F_{a_i}^i / F_{<a_i}^i) \cdot [D_i \cdot D_i] + \frac{1}{2} \sum_{i \neq j} v(i, j).$$

Stable bundles and Bogomolov–Gieseker inequality Here again we consider a surface S with a parabolic bundle E_*. For any subsheaf V of E the filtration on E induces a structure of a parabolic sheaf on V. Recall that a subsheaf V of E is called *saturated* is the quotient E/V is torsion-free.

Definition 6.5 Let L be an ample bundle on S. The bundle E_* is called μ_L–*stable* (or *slope stable*) if for every saturated subsheaf V of E,

$$\operatorname{pardeg}_L V_* < \operatorname{pardeg}_L E_*.$$

Remark 6.6 When E is a rank 2 bundle in order to check its stability it is sufficient to consider only *saturated locally free rank one subsheaves* of E. Following [4] we call such subsheaves of E *sub–line bundles*.

The following inequality (called *Bogomolov–Gieseker* inequality) is proven in [13] and was also proven in different terms in [12].

Theorem 6.7 Let E_* be a μ_L–stable parabolic bundle on the surface S. Then following inequality holds:

$$(6\text{-}3) \qquad \text{par-ch}_2(E_*) - \frac{1}{2}\text{par-ch}_1^2(E_*) \le 0.$$

Definition 6.8 Let (X, D, E_*) be a complex surface with a simple normal crossing divisor D with a parabolic bundle E_*, and let ∇ be a unitary flat logarithmic connection on E with poles at D. We say that ∇ is *compatible* with E_* if the following conditions hold.

(1) For every i and $a \in\,]0, 1]$ the subbundle $F_a^i/E(-D_i)$ of $E|_{D_i}$ is preserved by the residue map $\text{Res}_{D_i}(\nabla)$.

(2) The eigenvalue of $\text{Res}_{D_i}(\nabla)$ on the bundle $F_a^i/F_{<a}^i$ equals $-a$ (recall that this bundle is nontrivial only for finite number of values of a).

(3) The connection induced by ∇ on $F_a^i(X \setminus D_i)$ extends to a logarithmic connection on $F_a^i(X)$.

Finally we can formulate the version of parabolic Kobayashi–Hitchin correspondence that we use later.

Theorem 6.9 Let (X, D, E_*) be a complex projective surface with a simple normal crossing divisor D and a parabolic bundle E_*. Suppose that E_* is μ_L–stable (with L ample), has zero parabolic degree, and has zero second parabolic Chern number. Then there exists a flat unitary logarithmic connection on E compatible with E_*.

This statement can be deduced from Mochizuki [13; 14] (see also Li [12]). We explain this very briefly skipping all details. In [13] Mochizuki works with parabolic Higgs bundles and proves Kobayashi–Hitchin correspondence for stable Higgs bundles with vanishing first and second Chern characters. The case that we are interested in is the particular case when the Higgs field is zero. So by [13] there exists a unitary flat metric on $E(X \setminus D)$ adapted to the parabolic structure. It follows further from [14] that the flat unitary connection corresponding to the flat metric constructed on $E(X \setminus D)$ extends to a logarithmic connection on E and moreover this connection is compatible with the parabolic structure E_*.

7 Theorem of existence

In this section we prove Theorem 1.12.

7.1 A description of the proof

Let (L_j, β_j) be a weighted arrangement of lines in $\mathbb{C}P^2$. Recall that by x_i we denote the multiple points of the arrangement of multiplicity at least 3; $d_{ij} = 1$ if x_i belongs to L_j and $d_{ij} = 0$ otherwise. Suppose that (L_j, β_j) satisfies the three conditions of Theorem 1.12.

To prove Theorem 1.12 we make the blow up $\pi \colon S \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ at the points x_i and consider on S the pull back E of the tangent bundle $E = \pi^* T\mathbb{C}P^2$. Using the weights β_j we define a parabolic structure on E. We get a parabolic bundle E_* with zero parabolic first Chern class and prove that E_* is stable for a certain polarisation. Inequality (1-5) is just the Bogomolov–Gieseker inequality (Theorem 6.7).

In the case when the second parabolic Chern class of E_* equals 0 using Theorem 6.9 we prove that there exists a logarithmic flat unitary connection on $T\mathbb{C}P^2$ and combining this with results of Section 4 we conclude the proof of Theorem 1.12.

7.2 The parabolic bundle on the blown up $\mathbb{C}P^2$

Let (L_j, β_j) be an arrangement of lines in $\mathbb{C}P^2$, satisfying the inequalities (1-4). In this subsection we construct the parabolic structure on a 2–bundle on the blowup of $\mathbb{C}P^2$ and we calculate its parabolic Chern character. Let us fix some notation.

Notation Denote by S the blow up of $\mathbb{C}P^2$ at the points $x_1, \ldots, x_k$ of multiplicity at least 3 and let $\pi \colon S \to \mathbb{C}P^2$ be the corresponding projection map.

Denote by E the pullback bundle $\pi^* T\mathbb{C}P^2$.

For $j \in \{1, \ldots, n\}$ denote by D_j the proper transform of L_j.

For $j \in \{n+1, \ldots, n+k\}$ denote by D_j the exceptional line P_{j-n}:

$$D_j = P_{j-n}, \quad \pi(P_{j-n}) = x_{j-n}.$$

The parabolic structure on the bundle (S, E) For any $j \in \{1, \ldots, n\}$ the restriction of E to D_j contains a distinguished rank 1 subbundle — the pullback $\pi^* T L_j$ of the tangent bundle of L_j. Denote by E_j the subsheaf of E, generated by the sections that

are contained in π^*TL_j, being restricted to D_j. This subsheaf fits into the following exact sequence:

$$0 \to E_j \to E \to E|_{D_j}/\pi^*TL_j \to 0.$$

Now, for $1 \leq j \leq n$ we put

$$F_a^j = E^j \quad \text{for } 0 < a < 1 - \beta_j, \qquad F_a^j = E \quad \text{for } 1 - \beta_j \leq a \leq 1,$$

and for $0 < i \leq k$

$$F_a^{n+i} = E(-D_{n+i}) \quad \text{for } 0 < a < 1 - \alpha_i, \qquad F_a^{n+i} = E \quad \text{for } 1 - \alpha_i \leq a \leq 1.$$

Here, $1 - \alpha_i = \frac{1}{2}\sum_j d_{ij}(1 - \beta_j)$.

Proposition 7.1 *The first and second parabolic Chern characters of (E_*) are given by the following formulas:*

$$(7\text{-}1) \qquad \text{par-ch}_1(E_*) = \text{ch}_1(E) - \sum_{j=1}^{n}(1-\beta_j)D_j - 2\sum_{i=1}^{k}(1-\alpha_i)D_{n+i},$$

$$(7\text{-}2) \qquad \text{par-ch}_2(E_*) = \frac{3}{2} - \sum_{j=1}^{n}(1-\beta_j) - \sum_{j=1}^{n}\frac{1}{2}(1-\beta_j)^2 B_{jj} + \sum_{i=1}^{k}(1-\alpha_i)^2.$$

Proof The calculation of par-ch$_1$(E$_*$) is a straight-forward application of Definition 6.3. For $j \in \{1,\ldots,n\}$ the quotient sheaf $F_a^j/F_{<a}^j$ has rank 1 for $a = 1 - \beta_j$ and is trivial otherwise. For $j \in \{n+1,\ldots,n+k\}$ the sheaf $F_a^j/F_{<a}^j$ has rank 2 for $a = 1 - \alpha_j$ and is trivial otherwise.

In order to calculate par-ch$_2$(E$_*$) we need the following lemma.

Lemma 7.2 (1) *For $1 \leq j_1 < j_2 \leq n$ we have $v(j_1, j_2) = 0$.*

(2) *For $1 < i_1 < i_2 \leq k$ we have $v(n + i_1, n + i_2) = 0$.*

(3) *For $1 \leq j \leq n$, $1 \leq i \leq k$ we have $v(j, n+i) = d_{ij}(1 - \beta_j)(1 - \alpha_j)$.*

Proof (1) The sheaf $\text{Gr}_{(a_{j_1}, a_{j_2})}^F$ can be nontrivial only when $a_{j_1} = 1 - \beta_{j_1}$ and $a_{j_2} = 1 - \beta_{j_2}$. But in this case by construction the sheaf is equal to $E/(E_{j_1} + E_{j_2})$. At the same time $E_{j_1} + E_{j_2} = E$.

(2) $\text{Gr}_{(a_{n+i_1}, a_{n+i_2})}^F$ is trivial because $D_{n+i_1} \cap D_{n+i_2} = \emptyset$.

(3) The sheaf $\text{Gr}_{(a_j, a_{n+i})}^F$ is nontrivial only when $a_j = 1 - \beta_j$, $a_{n+i} = 1 - \alpha_i$. In this case it is equal to $E/(E(-D_{n+i}) \cup E_j)$. It is supported at the points $D_j \cap D_{n+i}$ and has rank 1 at each point. □

Calculation of par-ch$_2$(E$_*$)   According to Definition 6.4 we have

$$\text{par-ch}_2(E_*) = \frac{3}{2} - \sum_{j=1}^{n}(1-\beta_j) - \sum_{j=1}^{n}\frac{1}{2}(1-\beta_j)^2 B_{jj}$$

$$- \sum_{i=1}^{k}(1-\alpha_i)^2 + \sum_{i=1,j=1}^{k,n} d_{ij}(1-\beta_j)(1-\alpha_i).$$

Here we use the following facts:

(1) For $j \in (1,\ldots,n)$ we have $c_1(E/E_j) = 1$, $[D_j \cdot D_j] = -B_{jj}$.

(2) For $i \in (1,\ldots,k)$ we have $E|D_{n+i}$ is trivial, $[D_{n+i} \cdot D_{n+i}] = -1$.

Finally, by Theorem 1.9 (use $p = q = 1$) the last term is equal to $2\sum_{i=1}^{k}(1-\alpha_i)^2$. This concludes the proof.   □

7.3 Theorem on stability and additional lemmas

In this subsection we prove that the parabolic bundle E_* constructed above is stable with respect to an appropriate polarisation on S. Take $N \in \mathbb{Z}_+$,

$$N > \max\left\{\frac{k}{\min_i \alpha_i}, \frac{2k}{\min_{j,k}(\beta_j + \beta_k)}, \frac{3k}{1 - \max_j \beta_j}\right\},$$

and define the following line bundle:

$$L_N = O_S\left(-\sum_{i=1}^{k} P_i\right) \otimes p^* O(N).$$

Note, that L_N is ample since $N > k$.

Theorem 7.3   *The parabolic bundle* (E_*, S) *is* μ_{L_N}–*stable.*

Let us first give a plan of the proof. The parabolic degree of E_* is zero, so we need to show that the parabolic degree of any saturated sub–line bundle V of E is negative. Every line bundle on S is of the form $O_S(\sum_i d_i P_i) \otimes \pi^* O(d)$. For sub–line bundles of E we have $d \leq 1$ and the constant N is chosen in such a way that the degree of $V \subset E$ can be positive only for $d \geq 0$. Since the parabolic weights are in $[0,1]$, pardeg $V_* \leq \deg V$, so we only need to consider line subbundles of E with $d = 1$, 0. In the case $d = 1$ the pushdown $\pi_* V$ is contained in a sub–line bundle of $T\mathbb{C}P^2$ generated by sections tangent to a pencil of lines. We prove that pardeg(V_*) is negative

comparing the degree of V with the parabolic contribution, coming from the behavior of the pencil of lines with respect to the line arrangement on $\mathbb{C}P^2$.

Let us introduce some notation. For a point x in $\mathbb{C}P^2$ denote by L_x the sub–line bundle of $T\mathbb{C}P^2$ generated by the sections tangent to the pencil of lines containing x. For a section v of $T\mathbb{C}P^2$ with isolated zeros denote by L_v the sub–line bundle of $T\mathbb{C}P^2$ generated by v. The following lemma is standard, we omit the proof.

Lemma 7.4 *Sub–line bundles of $T\mathbb{C}P^2$ have degree at most 1. Every saturated sub–line bundle of $T\mathbb{C}P^2$ of degree 1 equals L_x for some x. Every saturated sub–line bundle of $T\mathbb{C}P^2$ of degree 0 equals L_v for some v.*

Lemma 7.5 *Let $L = O_S(\sum_{i=1}^k d_i P_i) \otimes \pi^*(O(d))$ be a saturated sub–line bundle of E. Then $d \le 1$. Suppose $d = 1$ or $d = 0$.*

  (1) *If $d = 1$ then $\pi_*(V) \subset L_x$ for some $x \in \mathbb{C}P^2$.*

  (2) *If $d = 0$ then $\pi_*(V) \subset L_v$ for some vector filed v with isolated zeros.*

  (3) *$\pi_*(V)$ coincides with L_x or L_v outside of the set $\{x_1, \ldots, x_k\}$.*

Proof Consider $(\pi_* V)^{\vee\vee}$, this is a saturated sub–line bundle of $T\mathbb{C}P^2$. Its degree equals to d, so $d \le 1$. In the case $d = 1$ by the previous lemma $(\pi_* V)^{\vee\vee}$ is L_x for some x, and if $d = 0$ it is L_v for some v. The sheaf $\pi_* V$ is a subsheaf of $(\pi_* V)^{\vee\vee}$ and it differs from it only at points x_i for which $d_i < 0$.   □

Lemma 7.6 *Let $V = O_S(\sum_{i=1}^k d_i P_i) \otimes \pi^*(O(d))$ be a sub–line bundle of E. Then for any i we have $d_i \le 2 - d$. In particular we have an upper bound on degree of V:*

$$(7\text{-}3) \qquad \deg_{L_N}(V) = c_1(V) \cdot c_1(L_N) \le (2 - d)k + Nd.$$

Proof Let us prove that for any i it holds $d_i \le 2 - d$. For any line P in $\mathbb{C}P^2$ we have $T\mathbb{C}P^2_{|P} \simeq O(1) \oplus O(2)$. Take a line P that contains a point x_i and doesn't contain any point x_j for $j \ne i$. Let P' be the proper transform of P. Then again $E_{|P'} \simeq O(1) \oplus O(2)$. Since $\mathrm{Hom}_{O_S}(V, E) \ne 0$, there is a line P through x_i for which $\mathrm{Hom}_{O_{P'}}(V_{|P'}, E_{|P'}) \ne 0$. At the same time, by definition of V

$$V_{|P'} = O(d\pi^* H + d_i P_i)_{|P'} = O(d + d_i).$$

It follows that

$$(d + d_i) \le 2.$$

Now we conclude the proof:

$$\deg_{L_N}\left(O_S\left(\sum_i d_i P_i + d\pi^* H\right)\right) = \sum_i d_i + Nd \le (2 - d)k + Nd. \qquad □$$

Lemma 7.7 *Let (L_j, β_j) be a weighted line arrangement satisfying the conditions* [(1-4)](#) *of* [Theorem 1.12](#).

(1) *For any point x in $\mathbb{C}P^2$ the following inequality holds:*

$$(7\text{-}4) \qquad \sum_{j \,|\, x \notin L_j} (1 - \beta_j) > 1 + \frac{2k}{N}.$$

(2) *A holomorphic vector field v with isolated zeros can be tangent to at most 3 lines of the arrangement.*

Proof (1) It is clear that the sum attains its minimum for a point that is a multiple point of the arrangement. Since $\sum_j (1 - \beta_j) = 3$, for a double point of the arrangement the sum in [(7-4)](#) is at least $1 + \min_{j,k}(\beta_j + \beta_k)$. For a point x_i of multiplicity more than 2 we have:

$$\sum_{j \,|\, x \notin L_j} (1 - \beta_j) = 3 - \sum_j d_{ij}(1 - \beta_j) = 3 - 2(1 - \alpha_i) \geq 1 + 2\min_i(\alpha_i).$$

(2) This is standard, the field v has 3 zeros and it is tangent only to the lines that join these zeros. $\qquad \square$

7.4 Proof of stability

Proof of [Theorem 7.3](#) By [Lemma 7.5](#) and [Lemma 7.6](#) any sub–line bundle V of E is of the form $O_S(\sum_{i=1}^k d_i P_i) \otimes \pi^*(O(d))$ with $d \leq 1$, and $\deg_{L_N}(V) < 0$ if $d < 0$. Since in our situation the parabolic weights are contained in $]0, 1]$, we have an inequality $\mathrm{pardeg}_{L_N}(V_*) \leq \deg_{L_N}(V)$. So it is necessary only to consider the cases when $d = 1$ and $d = 0$.

To calculate the parabolic first Chern class $\mathrm{par\text{-}ch}_1(V_*)$ we need to find for every $j \in \{1, \ldots, n+k\}$ and $a \in]0, 1]$ the rank of the following quotient sheaf supported on D_j:

$$(V \cap F^j_{a_j})/(V \cap F^j_{<a_j}).$$

Consider the case $d = 1$. Then according to [Lemma 7.5](#) there exists $x \in \mathbb{C}P^2$ such that $\pi_*(V) \subset L_x$.

In the case $j \in \{1, \ldots, n\}$ there are two possibilities.

If $x \in L_j$ then $V \subset F^j_{a_j}$ for all $0 < a_j \leq 1$ so the corresponding quotient sheaf is always trivial. If $x \notin L_j$ then the quotient sheaf is nontrivial for $a_j = 1 - \beta_j$ and has rank one (this follows from [Lemma 7.5](#)).

In the case $j \in \{n+1, \ldots, n+k\}$ the quotient sheaf is nontrivial for $a_j = 1 - \alpha_{j-n}$ and has rank 1. This gives us the formula

$$\text{par-ch}_1(V_*) = \text{ch}_1(V) - \sum_{j \mid x \notin L_j} (1 - \beta_j) D_j - \sum_{i=1}^{n} (1 - \alpha_i) D_{n+i}.$$

We have the following sequence of inequalities

$$\text{pardeg}_{L_N}(V_*) = \deg_{L_N}(V) - c_1(L_N) \cdot \left(\sum_{j \mid x \notin L_j} (1 - \beta_j) D_j + \sum_{i=1}^{n} (1 - \alpha_i) D_{n+i} \right).$$

using Inequality (7-3) and the equality $2(1 - \alpha_i) = D_{n+i} \cdot \sum_j (1 - \beta_j) D_j$. This is less than or equal to

$$N + k - N \sum_{j \mid x \notin L_j} (1 - \beta_j) + \sum_i 2(1 - \alpha_i) - \sum_i (1 - \alpha_i).$$

using $0 < \alpha_i < 1$ and Inequality (7-4). Finally, this is less than

$$N + 2k - N\left(1 + \frac{2k}{N}\right) = 0.$$

The case $d = 0$ is analogous. By Lemma 7.5 there exists a vector-field v with isolated zeros such that $\pi_*(V) \subset L_v$. The arrangement contains more than 3 lines so there is at least one line that is not tangent to v. Making the same calculation as in the case $d = 1$ and using $N > 3k/(1 - \max_j \beta_j)$ we get

$$\text{pardeg}_{L_N}(V_*) \leq 2k - N(1 - \max_j(\beta_j)) + \sum_i (1 - \alpha_i) < 0. \qquad \square$$

Example 7.8 Consider the arrangement of 6 lines that pass through 4 generic points $x_1, \ldots, x_4$. For every $\beta \in {]}0, 1{[}$ we can associate the weight β to the lines $x_1 x_i$ and $1 - \beta$ to the lines $x_i x_j$, $i, j > 1$. We get a stable parabolic bundle on $\mathbb{C}P^2$ blown up at $x_1, \ldots, x_4$. When β tends to 0 the parabolic degree of the sheaf corresponding to L_{x_1} tends to zero and as a result for $\beta = 0$ we don't get a PK metric on $\mathbb{C}P^2$.

7.5 Proof of Theorem 1.12 and an application of the Bogomolov–Gieseker inequality

Proof of Theorem 1.12 Let us sum up what we have done. We started with a weighted arrangement (L_j, β_j) that satisfies the conditions of Theorem 1.12. We introduced a structure of parabolic bundle E_* on the pullback E of the tangent bundle $T\mathbb{C}P^2$ to the blow up of $\mathbb{C}P^2$. We proved that E_* is stable (Theorem 7.3) and has zero first parabolic

Chern class (Proposition 7.1). So inequality (1-5) follows from the calculation of second parabolic Chern number (Proposition 7.1) and Bogomolov–Gieseker inequality (Theorem 6.7). This proves the first part of the theorem.

If the equality is attained in (1-5) then the second parabolic Chern number of E_* vanishes, so we can use parabolic Kobayashi–Hitchin correspondence (Theorem 6.9). Namely, there exists a unitary flat logarithmic connection on E compatible with the parabolic structure of E_*. By Corollary 4.6 the constructed connection is torsion-free. Finally, using Theorem 4.13 we conclude that the corresponding flat connection on $T\mathbb{C}P^2$ defines a PK metric. $\square$

Let us give one corollary of Theorem 1.12. For a multiple point x_i of a line arrangement $L_1, \ldots, L_N$ denote by μ_i the number of lines through x_i.

Corollary 7.9 *Suppose that the multiplicity of every point of the arrangement* $(L_1, \ldots, L_N)$ *is less than* $2N/3$. *Then the following inequality holds:*

$$\text{(7-5)} \qquad \sum_i \mu_i \geq \frac{N^2}{3} + N.$$

In the case of equality N *is divisible by* 3, *every line intersects other lines in* $N/3 + 1$ *points, and there is a* PK *metric on* $\mathbb{C}P^2$ *with conical angles* $2\pi(N-3)/N$ *at the lines* L_j.

Note that for a generic arrangement the total multiplicity is $N(N-1)$ while for the most degenerate arrangement it is N.

Proof We have the following equalities:

$$N^2 = \left(\sum_j L_j \right)^2 = N + \sum_{j \neq k} L_j \cdot L_k = N + \sum_i \mu_i(\mu_i - 1),$$

$$\text{(7-6)} \qquad \sum_i \mu_i^2 = N^2 - N + \sum_i \mu_i.$$

Associate to each L_j weight $\beta_j = (N-3)/N$, then the arrangement satisfies the conditions of Theorem 1.12. Since all weights are equal we can treat double points of the arrangements as points of type $(1,1)$ and we get the following equalities:

$$1 - \alpha_i = \frac{3\mu_i}{2N}, \quad 1 - \beta_j = \frac{3}{N}, \quad \sum_j B_{jj} = \sum_i \mu_i - N.$$

Applying the Bogomolov–Gieseker inequality to the corresponding stable parabolic bundle E_* and using (7-6) we get

$$0 \geq \text{par-ch}_2(E_*) = \frac{\sum_i 9\mu_i^2}{4N^2} - \frac{9\sum_i \mu_i - 9N}{2N^2} - \frac{3}{2}$$

$$= \frac{9\sum_i \mu_i^2 - 18\sum_i \mu_i + 18N - 6N^2}{4N^2} = \frac{3N^2 + 9N - 9\sum_i \mu_i}{4N^2}. \qquad \square$$

Inequality (7-5) was proven previously by Langer [11] using different methods. We finish with the following lemma.

Lemma 7.10 *For line arrangements satisfying condition of* Theorem 1.12 *the system of equations* (1-1) *from* Theorem 1.11 *(ie,* $\sum_k B_{jk}(\beta_k - 1) + 1 = 0$*) can be obtained by differentiating the Bogomolov–Gieseker inequality.*

Proof Let (L_j, β_j) be an arrangement satisfying conditions of Theorem 1.12. For a small deformation β'_j of β_j the deformed parabolic bundle E'_* (defined in the same way as E_* but using the weights β'_j instead β_j) is stable and satisfies the Bogomolov–Gieseker inequality. The inequality is quadratic in β_j and it attains its maxima (zero) for the initial data. So its derivatives in all directions vanish. This produces a system of linear equations on β_j. We will show that this system is equivalent to the system of equations (1-1) from Theorem 1.11. Consider the derivative

$$\frac{\partial}{\partial\beta_j}\text{par-ch}_2(E_*) - \frac{\partial}{\partial\beta_j}\frac{1}{2}\text{par-ch}_1^2(E_*).$$

Let us restrict this expression to the plane $\sum_j (\beta_j - 1) = -3$. Then $\text{par-ch}_1(E_*) = 0$, and so the second term vanishes. $\text{par-ch}_2(E_*)$ is given by Equation (7-2), and using $(1 - \alpha_i) = \frac{1}{2}\sum_k d_{ik}(1 - \beta_k)$ we get

$$0 = \frac{\partial}{\partial\beta_j}(\text{par-ch}_2(E_*)) = 1 + (1 - \beta_j)B_{jj} - \sum_i d_{ij}(1 - \alpha_i).$$

Expanding $(1 - \alpha_i)$, and using the definitions of B_{jj} and d_{ij} we rewrite the last term to obtain

$$1 + (1 - \beta_j)B_{jj} - \frac{1}{2}\left((B_{jj} + 1)(1 - \beta_j) + \sum_{k,i|d_{ij}d_{ik}=1}(1 - \beta_k)\right).$$

Now using $\sum_j (\beta_j - 1) = -3$, this equals

$$\frac{1}{2}\left(2 + (1 - \beta_j)B_{jj} - \left(3 + \sum_{k\neq j}(\beta_k - 1)\right) + \sum_{k,i|d_{ij}d_{ik}=1}(1 - \beta_k)\right).$$

Finally using the definition of B_{jk}, this is

$$\frac{1}{2}\left(-1 + \sum_k (1 - \beta_k) B_{jk}\right). \qquad \square$$

8 Further results, questions and directions

In a subsequent paper we will use Theorem 1.12 to construct several infinite families of aspherical complex surfaces. Some of these families of surfaces admit a metric of type CAT(0). In particular some smooth compact quotients of the complex ball admit a PK metric of type CAT(0), this answers a question of Gromov and Davis.

For every PK surface its PK metric induces a positive $(1,1)$ current on it, so it should not be difficult to prove that every PK surface is a Kähler surface (in principle, one should be able to smoothen a bit the PK metric to get a smooth Kähler metric on the surface). It should be possible to show that nonalgebraic $K3$ surfaces don't admit a PK metric, but we don't know any obstruction for the existence of PK metrics on algebraic surfaces. At the same time the set of examples of PK surfaces that we have is rather limited.

We hope that Theorem 1.5 extends to higher dimensions. Namely, that for a polyhedral Kähler manifold of any dimension the complex structure on the complement of the metric singularities extends to a complex analytic structure on the whole manifold. In particular the metric singularities should not have odd (real) codimension and all singularities of even codimension should have holomorphic directions. Note that in higher dimensions we can obtain complex manifolds with singularities even if we start with a topological polyhedral Kähler manifold (ie, the link of every point is a topological sphere). Indeed, by Brieskorn the link of the hypersurfaces $z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0$, $1 \leq k \leq 28$ in $\mathbb{C}^5$ is S^7 with one of 28 smooth structures. At the same time these hypersurfaces have a PK metric, induced by an obvious degree $24(6k-1)$ ramified cover of the hyperplane $\sum_i z_i = 0$.

The notion of polyhedral Kähler manifolds can be generalised in several directions. A *polyhedral affine structure* on a manifold is a choice of a simplicial decomposition and an affine structure on the complement of codimension 2 faces that restricts to the standard affine structure on the interior of every face of the top dimension. We say that a manifold M^{2n} is *polyhedral complex affine* if the holonomy is contained in $GL(n, \mathbb{C})$, and singular faces of codimension 2 at which the holonomy is trivial have holomorphic directions (cf Definition 1.1). For complex dimension 2 we expect to get a theory similar to the one developed in this article. It should be possible to classify the singularities of complex codimension 2 but the list will be longer.

If the holonomy of a polyhedral affine manifold is contained in the symplectic group $SP(2n)$ we call the manifold *polyhedral nearly symplectic*. It is not hard to see that every symplectic manifold admits a polyhedral nearly symplectic structure. But the converse should be wrong already for 4–manifolds, so we adjust the definition.

Definition 8.1 Let M^4 be a polyhedral nearly symplectic manifold and let $M_1^4(\varepsilon)$ be a neighborhood of the union of all edges. The PL symplectic structure on $M^4 \setminus M_1^4(\varepsilon)$ can be smoothen along the faces of codimension 2 to a genuine symplectic form w. Let c_1 be the first Chern class of an almost complex structure tamed by w on $M_1^4(\varepsilon)$. We call M^4 *polyhedral symplectic* if for every surface S contained in $M_1^4(2\varepsilon) \setminus M_1^4(\varepsilon)$ we have $c_1 \cdot S = 0$.

It is not hard to prove that every symplectic 4–manifold admits a polyhedral symplectic structure. More importantly, we conjecture that every polyhedral symplectic 4–manifold admits a symplectic smoothing.

Another interesting direction to generalise PK manifolds is to consider complex manifolds with Kähler metric of constant holomorphic curvature and conical singularities at holomorphic geodesic divisors. In the case of negative curvature these manifolds will be generalisations of complex hyperbolic orbifolds, ie quotients of the complex ball B^n by a lattice of $SU(n, 1)$. These manifolds presumably are the same as Thurston $(\mathbb{C}H^n, SU(n, 1))$–cone manifolds [19]. In the case of surfaces it is sufficient to ask that singularities of the metric form a complex curve, at points that are not multiple there is a local isometric action of the group $SU(1, 1) \times S^1$, and at the multiple points there is an holomorphic isometric action of $\mathbb{R}^1$. It should be possible to generalise Theorem 1.12 to this setting using parabolic Kobayashi–Hitchin correspondence for parabolic Higgs bundles [13], and using ideas from [18]. It would be interesting to reprove (or even generalise) results of [3] using this approach.

Finally we hope to prove the following conjecture.

Conjecture 8.2 For every arrangement satisfying conditions of Theorem 1.12 its complement is of the type $K(\pi, 1)$.

The converse to this conjecture is wrong because Proposition 5.2 permits us to check that some simplicial arrangements are not PK. At the same time by a theorem of Deligne all (complexified) simplicial arrangement have a complement of the type $K(\pi, 1)$.

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