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An elementary construction of Anick's fibration

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Cohen, Moore, and Neisendorfer's work on the odd primary homotopy theory of spheres and Moore spaces, as well as the first author's work on the secondary suspension, predicted the existence of a p–local fibration $S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1}$ whose connecting map is degree p^r. In a long and complex monograph, Anick constructed such a fibration for $p \geq 5$ and $r \geq 1$. Using new methods we give a much more conceptual construction which is also valid for $p = 3$ and $r \geq 1$. We go on to establish an H space structure on T_{2n-1} and use this to construct a secondary *EHP* sequence for the Moore space spectrum.

55P45, 55P40, 55P35

1 Introduction

In [6; 5; 15], Cohen, Moore and Neisendorfer proved a landmark result concerning the exponent of the homotopy groups of spheres localized at an odd prime p. When $p \geq 3$ and $r \geq 1$ they constructed a map $\pi_n\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ such that the composition with the double suspension

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p^r–power map. The existence of such a map for $r = 1$ was used to show that p^n annihilates the p–torsion in $\pi_*(S^{2n+1})$.

In [4], Cohen, Moore and Neisendorfer raised the question of whether the map π_n occurs in a fibration sequence[1]

(A) $$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1}.$$

The first construction of such a fibration was accomplished for $p \geq 5$ by Anick [1] and was the subject of a 270 page book. There has been much interest in finding a simpler

[1] We will follow a convention suggested by Mahowald of indexing a family of infinite complexes by a subscript to denote the least dimension in which the reduced homology is nontrivial.

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construction. It is the purpose of this paper to give an elementary construction of the space T_{2n-1} and the fibration (A) which is valid for all odd primes.

The question of the existence of a fibration as in (A) appeared in another context at about the same time. In trying to understand the secondary suspension (see Cohen [3] and Mahowald [13]), the first author [9; 10] was led to conjecture the existence of $(i-1)$–connected spaces T_i which fit into secondary *EHP* sequences

$$T_{2n-1} \xrightarrow{E} \Omega T_{2n} \xrightarrow{H} BW_n$$
$$T_{2n} \xrightarrow{E'} \Omega T_{2n+1} \xrightarrow{H'} BW_{n+1}$$

where BW_n is the classifying space of the fiber of the double suspension constructed by the first author in [8]. These *EHP* fibrations should fit together in such a way that the resulting spectrum $\{T_i\}$ is equivalent to the Moore spectrum $S^0 \cup_{p^r} e^1$. The T_i's would then give a refinement of the secondary suspension into $2p$ stages. The analysis indicated that T_{2n} is homotopy equivalent to $S^{2n+1}\{p^r\}$, the fiber of the map of degree p^r on S^{2n+1}, and that T_{2n-1} would sit in the fibration sequence (A).

Our first objective is to construct a secondary Hopf invariant $H\colon \Omega T_{2n} \longrightarrow BW_n$ for $p \geq 3$. This lets us define T_{2n-1} as the homotopy fiber of H. It follows easily that T_{2n-1} satisfies the fibration in (A) and the secondary *EHP* fibrations. We also show that the space we construct is homotopy equivalent to Anick's when $p \geq 5$.

The *EHP* viewpoint also predicted that the T_i's should have a rich structure. They should be homotopy associative and homotopy commutative H–spaces enjoying a certain universal property. Together, these properties would imply that the mod–p^r homotopy classes of the T_i's could be represented by multiplicative maps. That is, letting $P^i(p^r)$ be the mod–p^r Moore space of dimension i, there should be a one-to-one correspondence

$$[P^{i+1}(p^r), T_j] \leftrightarrow \{\text{homotopy classes of } H\text{–maps from } T_i \text{ to } T_j\}.$$

The properties were easy to establish when i is even [9]. Subsequent to Anick's work, Anick and the first author [2] constructed an H–space structure on T_{2n-1} by showing that, for each n, there is a $(2n-1)$–connected co–H space G_{2n} with the property that T_{2n-1} is a retract of ΩG_{2n} and G_{2n} is a retract of ΣT_{2n-1}. They also proved a semiuniversal property for T_{2n-1}. This work depended heavily on the analysis of Anick in [1]. The other properties were later established by the second author [19].

Our second objective is to take advantage of our construction of the space T_{2n-1} to give a new, simpler construction of the space G_{2n} and prove all the properties in [2] for $p \geq 3$. Collectively, our results are as follows.

Theorem 1.1 *Suppose $p \geq 3$ and $r \geq 1$. Then the following hold:*

(a) *There is an H–fibration sequence*

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1}$$

where the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is the p^r–power map.

(b) *There is a fibration sequence*

$$\Omega G \xrightarrow{h} T_{2n-1} \longrightarrow R \longrightarrow G_{2n}$$

where h has a right homotopy inverse $g\colon T_{2n-1} \longrightarrow \Omega G_{2n}$ so that

$$\Omega G_{2n} \simeq T_{2n-1} \times \Omega R$$

with R a wedge of mod–p^s Moore spaces for $s \geq r$.

(c) *The adjoint of g,*

$$\widetilde{g}\colon \Sigma T_{2n-1} \longrightarrow G_{2n},$$

has a right homotopy inverse $f\colon G_{2n} \longrightarrow \Sigma T_{2n-1}$ and there is a homotopy equivalence

$$\Sigma T_{2n-1} \simeq G_{2n} \vee W$$

where W is a wedge of mod–p^s Moore spaces for $s \geq r$.

(d) *There are secondary EHP fibrations*

$$W_n \xrightarrow{P} T_{2n-1} \xrightarrow{E} \Omega T_{2n} \xrightarrow{H} BW_n$$

$$W_{n+1} \xrightarrow{P'} T_{2n} \xrightarrow{E'} \Omega T_{2n+1} \xrightarrow{H'} BW_{n+1}$$

where $T_{2n} = S^{2n+1}\{p^r\}$, and there is an equivalence of spectra

$$\{T_i\} \simeq S^0 \cup_{p^r} e^1.$$

Our methods are simpler and more direct than those of Anick. He constructed T_{2n-1} as a retract of a loop space ΩD, where D is an infinite dimensional CW–complex whose bottom two cells are the mod–p^r Moore space $P^{2n+1}(p^r)$ and whose other cells come from iteratively attaching certain Moore spaces in a delicately prescribed fashion. A great deal of his effort was directed towards constructing the attaching maps, and this necessitated the introduction of many new techniques. The restriction to primes strictly larger than 3 was due to a heavy reliance on differential graded Lie algebras which require that the primes 2 and 3 be inverted in order for the Lie identities to be satisfied. By contrast, we construct the space T_{2n-1} directly for all $p \geq 3$ without reference to the space D and without reference to differential graded Lie algebras. Section 2 is devoted to Extension Theorem 2.2 which introduces a new technique for doing obstruction theory in principal fibrations. This is the main tool for all our results. In Section 3 we construct the fibration in Theorem 1.1 (a), without the H–space structure. In Section 4 we use the extension theorem again in an elaborate induction to obtain the spaces G_{2n} and the H space structure on T_{2n-1}. Throughout the induction we reproduce some of the delicate properties of G_{2n} and T_{2n-1} which first appeared in [2].

The new methods may be useful in positively resolving a long-standing conjecture that the fiber W_n of the double suspension is a double loop space at odd primes. Including dimension and torsion parameters, the space $T_{2np-1}(p)$ gives a candidate for a double delooping: potentially $W_n \simeq \Omega^2 T_{2np-1}(p)$. Such a homotopy equivalence would have deep implications in homotopy theory, one of which being a much better understanding of the differentials in the *EHP* spectral sequence calculating the homotopy groups of spheres.

This paper is the result of combining separate efforts by the two authors. The second author discovered the extension theorem and obtained part (a) of Theorem 1.1 without the H–space structure, as well as part (d). The first author later found a simpler application of the extension theorem to obtain the map H and Corollary 3.5, Theorem 3.6, Proposition 3.7, Theorem 3.8 and Corollary 3.9 below, as well as a further application of the extension theorem to obtain parts (b) and (c), and the H–space structure.

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2 The extension theorem

In this section we establish an extension theorem for principal fibrations defined over mapping cones. Let

$$E \xrightarrow{\pi} X \cup_\theta CA$$

be a principal fibration classified by a map $\varphi\colon X\cup_\theta CA\longrightarrow Y$. We compare this to the induced fibration over X:

$$\begin{tikzcd}
\Omega Y \arrow[r, equal] \arrow[d] & \Omega Y \arrow[d] \\
E_0 \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
X \arrow[r] & X\cup_\theta CA.
\end{tikzcd}$$

In Extension Theorem 2.2 we will give conditions for when a map $E_0\longrightarrow B$ extends to a map $E\longrightarrow B$. To motivate the underlying idea, observe that there is a pushout

$$\begin{tikzcd}
A \arrow[r, "\theta"] \arrow[d] & X \arrow[d] \\
CA \arrow[r] & X\cup_\theta CA.
\end{tikzcd}$$

Map all four corners of this square into Y by composing with the map $\varphi\colon X\cup_\theta CA\longrightarrow Y$. Appropriately turning maps into fibrations, we obtain fibration sequences $E\longrightarrow X\cup_\theta CA\longrightarrow Y$, $E_0\longrightarrow X\longrightarrow Y$, $CA\times\Omega Y\longrightarrow CA\longrightarrow Y$, and $A\times\Omega Y\longrightarrow A\longrightarrow Y$. By [8] or Mather's Cube Lemma [14], these fibres fit into a homotopy pushout

$$\begin{tikzcd}
A\times\Omega Y \arrow[r] \arrow[d] & E_0 \arrow[d] \\
CA\times\Omega Y \arrow[r] & E.
\end{tikzcd} \tag{1}$$

We wish to produce an extension of $f_0\colon E_0\longrightarrow B$ as a pushout map $f\colon E\longrightarrow B$. To obtain this, we need to produce a map $CA\times\Omega Y\longrightarrow B$ and show that both it and f_0 are homotopic when restricted to $A\times\Omega Y$. This leads to two issues. One is to identify the maps which appear in the homotopy pushout (1). We do this by constructing the homotopy pushout for E from first principles. The other issue is to impose conditions on the spaces and maps in (1) and on the space B which will guarantee the existence of an extension. We will impose three conditions: A is a co–H space, B is a connected H–space whose q–power map is null homotopic, and the restriction of $A\times\Omega Y$ to E_0 is divisible by q. It may be worth noting that the exponent condition on B will be played off of the divisibility condition for the map $A\longrightarrow E_0$.

To set things up, observe as in (1) that the map $A\longrightarrow X$ lifts to E_0. There may be many inequivalent choices of such a lift. By the homotopy lifting property, we can extend this lift to a map of pairs $\phi\colon (CA, A) \longrightarrow (E, E_0)$ such that the composite

$$(CA, A) \xrightarrow{\phi} (E, E_0) \xrightarrow{\pi} (X \cup_\theta CA, X)$$

induces a cohomology isomorphism. Again, note that there may be many inequivalent choices of ϕ with this property. In what follows, all spaces will have the homotopy type of p–local CW–complexes with $p > 2$.

Extension Theorem 2.2 *Let A be a co–H space and B be a connected H–space whose q–power map is null homotopic. Let $\phi\colon (CA, A) \to (E, E_0)$ be a map such that the composition*

$$(CA, A) \xrightarrow{\phi} (E, E_0) \xrightarrow{\pi} (X \cup_\theta CA, X)$$

induces a cohomology isomorphism. Suppose that the restriction $\phi|_A\colon A \longrightarrow E_0$ is divisible by q in the co–H structure on A. Then the restriction

$$[E, B] \longrightarrow [E_0, B]$$

is onto. Consequently, any map $E_0 \longrightarrow B$ extends to a map $E \longrightarrow B$.

Proof Let $s\colon (CA, A) \longrightarrow (X \cup_\theta CA, X)$ be the standard map defining the mapping cone. Observe that $\pi\phi \neq s$, in general. We first modify ϕ so that this does happen. We do this by constructing a pullback diagram

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d] & E' \arrow[r, "\Psi"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
X \arrow[r] & X \cup_{\theta'} CA \arrow[r, "\psi"] & X \cup_\theta CA
\end{tikzcd}$$

and a map $\phi'\colon (CA, A) \longrightarrow (E', E_0)$ such that

(a) $\phi'|_A = \phi|_A$;

(b) $\pi'\phi' = s'\colon (CA, A) \longrightarrow (X \cup_{\theta'} CA, X)$, the standard map;

(c) $\Psi\colon E' \longrightarrow E$ is a homotopy equivalence.

Let $\theta'\colon A \longrightarrow X$ be the restriction of $\pi\phi$ to A. Define $\psi\colon X \cup_{\theta'} CA \longrightarrow X \cup_\theta CA$ extending the identity on X with the map $\pi\phi\colon CA \longrightarrow X \cup_\theta CA$. Define E' as the pullback of π and ψ. By the pullback property of E', we can define ϕ' such that $\Psi\phi' = \phi$ and $\pi'\phi' = s'$. Then $\psi s' = \gamma\pi'\phi' = \pi\Psi\phi' = \pi\phi$. Since $\pi\phi$ induces an isomorphism in cohomology and s' is an excision, $\psi\colon (X \cup_{\theta'} CA, X) \longrightarrow (X \cup_\theta CA, X)$ induces a cohomology isomorphism as well. Hence ψ is a homotopy equivalence and it follows that Ψ is a homotopy equivalence.

We are therefore reduced to considering the case when $\pi\phi = s$. Since both fibrations are principal fibrations, there is an action of ΩY on the total space

$$\Omega Y \times E \longrightarrow E$$

which restricts to an action on E_0

$$\Omega Y \times E_0 \longrightarrow E_0.$$

Consider the composition

$$\gamma\colon (\Omega Y \times CA, \Omega Y \times A) \xrightarrow{1\times\phi} (\Omega Y \times E, \Omega Y \times E_0) \longrightarrow (E, E_0).$$

Identifying $\Omega Y \times A \subset \Omega Y \times CA$ with its image under γ in E_0, define $\bar{E} = E_0 \cup \Omega Y \times CA$ by the pushout

$$\begin{tikzcd}
\Omega Y \times A \arrow[r] \arrow[d] & E_0 \arrow[d] \\
\Omega Y \times CA \arrow[r] & \bar{E}.
\end{tikzcd}$$

From the map γ we obtain a pushout map

$$\Gamma\colon \bar{E} = E_0 \cup \Omega Y \times CA \longrightarrow E.$$

Observe that there is a homotopy commutative square

$$\begin{tikzcd}
\bar{E} = E_0 \cup \Omega Y \times CA \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d] \\
X \cup_\theta CA \arrow[r, equal] & X \cup_\theta CA.
\end{tikzcd}$$

By [8], the left hand vertical map is a quasi-fibration with quasi-fiber ΩY. Since quasi-fibrations induce long exact sequences in homotopy, the 5–lemma implies that Γ is a homotopy equivalence.

Let $f_0\colon E_0 \longrightarrow B$. We will construct an extension $\bar{f}\colon \bar{E} \longrightarrow B$ and hence an extension $f\colon E \longrightarrow B$ via the homotopy equivalence Γ. Since $\bar{E}$ is a homotopy pushout, it suffices to construct a map

$$g\colon \Omega Y \times CA \longrightarrow B$$

such that the composite

$$\Omega Y \times A \xrightarrow{\gamma} E_0 \xrightarrow{f_0} B$$

is homotopic to the composite

$$\Omega Y \times A \longrightarrow \Omega Y \times CA \xrightarrow{g} B.$$

Define g as the composition

$$\Omega Y \times CA \xrightarrow{\pi_1} \Omega Y \xrightarrow{\iota} E_0 \xrightarrow{f_0} B$$

where π_1 is the projection and ι is the map in the given principal fibration $\Omega Y \longrightarrow E_0 \longrightarrow X$. Then we are reduced to showing that the compositions

$$\alpha\colon \Omega Y \times A \xrightarrow{\gamma} E_0 \xrightarrow{f_0} B$$
$$\beta\colon \Omega Y \times A \xrightarrow{\pi_1} \Omega Y \xrightarrow{\iota} E_0 \xrightarrow{f_0} B$$

are homotopic. In Lemma 2.4, we will show in general that if $\alpha, \beta\colon X \times A \longrightarrow B$ are two maps with $\alpha|_{X\times *} \sim \beta|_{X\times *}$, then $\alpha \circ (1 \times q) \sim \beta \circ (1 \times q)$. Assuming this for the moment, apply the lemma in our case. By definition, $\gamma\colon \Omega Y \times A \longrightarrow E_0$ is the composition

$$\Omega Y \times A \xrightarrow{1\times\phi} \Omega Y \times E_0 \longrightarrow E_0,$$

and by hypothesis $\phi|_A\colon A \longrightarrow E_0$ is divisible by q. Let $\phi|_A = q \cdot \phi'$. Consequently, $\gamma = \gamma' \circ (1 \times q)$ (where γ' is constructed by replacing ϕ with ϕ') and $\alpha = \alpha' \circ (1 \times q)$ where$\alpha' = f_0 \circ \gamma'$. Also, the projection in the definition of β implies that $\beta = \beta \circ (1 \times q)$. Thus by Lemma 2.4, $\alpha' \circ (1 \times q)$ is homotopic to β. That is, α is homotopic to β, as required. □

It remain to prove Lemma 2.4. This will rely on Lemma 2.3, which summarizes some well known properties of connected H spaces.

Lemma 2.3 *Let B be a connected H–space and suppose u, v are maps $Z \longrightarrow B$. Then there is a difference map $\delta(u, v)\colon Z \longrightarrow B$ such that*

(a) $\delta(u, v) \sim *$ *if and only if* $u \sim v$;

(b) *if* $h\colon W \longrightarrow Z$, *then* $\delta(u \circ h, v \circ h) \sim \delta(u, v) \circ h$;

(c) *if* $C \subset Z$ *and* $u|_C = v|_C$, *then* $\delta(u, v)|_C \sim *$.

Proof Write $x, y\colon B \times B \longrightarrow B$ for the projections onto the first and second coordinates. Then there is a homotopy equivalence $e\colon B \times B \longrightarrow B \times B$ given by

$e = (x, \mu(x, y))$. Let $\delta\colon B \times B \longrightarrow B$ be the second coordinate of a homotopy inverse for e. Then

$$\delta(x, \mu(x, y)) \sim y \tag{2.3.1}$$

$$\mu(x, \delta(x, y)) \sim y. \tag{2.3.2}$$

Define $\delta(u, v)$ as the composition

$$Z \xrightarrow{u \times v} B \times B \xrightarrow{\delta} B.$$

If $\delta(u, v) \sim *$, then $u \sim \mu(u, *) \sim \mu(u, \delta(u, v)) \sim v$ by (2.3.2). On the other hand, if $u \sim v$, then $\delta(u, v) \sim \delta(u, u) \sim \delta(u, \mu(u, *)) \sim *$ by (2.3.1). This proves part (a). Part (b) follows by naturality. For part (c), apply (b) with $W = C$ and h the inclusion. It follow that $\delta(u, v)|_C \sim *$. $\square$

Lemma 2.4 *Let B be a connected H–space whose q–power map is null homotopic. Suppose $\alpha, \beta\colon X \times A \longrightarrow B$ are two maps with $\alpha|_{X \times *} \sim \beta|_{X \times *}$. Then $\alpha \circ (1 \times q) \sim \beta \circ (1 \times q)$.*

Proof Write $\delta = \delta(\alpha, \beta)\colon X \times A \longrightarrow B$ for the difference element defined in Lemma 2.3. Since $\delta|_{X \times *}$ is null homotopic, we obtain a homotopy commutative diagram

$$\begin{tikzcd}
X \times A \arrow[r, "1 \times q"] \arrow[d] & X \times A \arrow[r, "\delta"] \arrow[d] & B \\
X \ltimes A \arrow[r, "1 \ltimes q"] & X \ltimes A \arrow[ur, "\delta'"'] &
\end{tikzcd}$$

for some map δ'. Since $X \ltimes (A_1 \vee A_2)$ is homeomorphic to $(X \ltimes A_1) \vee (X \ltimes A_2)$, the co–$H$ structure on A induces a co–H structure on $X \ltimes A$ and $1 \ltimes q$ is the degree q self-map on $X \ltimes A$. The co–H space structure on $X \ltimes A$ induces the same group structure on $[X \ltimes A, B]$ as the H–space structure on B. Thus the composite

$$X \ltimes A \xrightarrow{1 \ltimes q} X \ltimes A \xrightarrow{\delta'} B$$

is homotopic to the composite

$$X \ltimes A \xrightarrow{\delta'} B \xrightarrow{q} B.$$

By assumption, the q–power map on B is null homotopic. Thus $q \circ \delta'$ is null homotopic and so $\delta' \circ (1 \ltimes q)$ is null homotopic. Therefore $\delta \circ (1 \times q) = \delta(\alpha, \beta) \circ (1 \times q)$ is null homotopic. On the other hand, by Lemma 2.3 (b), $\delta(\alpha, \beta) \circ (1 \times q) \sim \delta(\alpha \circ (1 \times q), \beta \circ (1 \times q))$, implying that the right hand side is null homotopic. Hence Lemma 2.3 (a) implies that $\alpha \circ (1 \times a) \sim \beta \circ (1 \times q)$. $\square$

3 The construction of the space T_{2n-1}

The purpose of this section is to construct the spaces T_{2n-1} and produce several fibration sequences. We begin our discussion with the Moore space

$$P^k(p^r) = S^{k-1} \cup_{p^r} e^k$$

which we will abbreviate as P^k. Let us fix some notation by defining a diagram of fibration sequences induced by the lower right hand corner

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \arrow[r, "\partial"] \arrow[d] & E \arrow[r, "\pi"] \arrow[d, "\sigma"] & F \arrow[r] \arrow[d, "\rho"] & \Omega S^{2n+1} \arrow[d] \\
* \arrow[r] \arrow[d] & P^{2n+1} \arrow[r, equal] \arrow[d] & P^{2n+1} \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega S^{2n+1} \arrow[r] & S^{2n+1}\{p^r\} \arrow[r] & S^{2n+1} \arrow[r, "p^r"] & S^{2n+1}.
\end{tikzcd} \tag{3.1}$$

The spaces E and F were first introduced in [5; 6]. It is easy to see that

$$H^i(F) = \begin{cases} \mathbb{Z} & i = 2kn, \\ 0 & i \neq 2kn \end{cases}$$

from the cohomology Serre spectral sequence for the fibration

$$\Omega S^{2n+1} \xrightarrow{\partial} F \xrightarrow{\rho} P^{2n+1}$$

which is induced from the path space fibration over S^{2n+1}.

In [5], Cohen, Moore and Neisendorfer introduced certain iterated relative Samelson products $x_i\colon P^{2ni-1} \longrightarrow \Omega F$. We will work with their adjoints $\hat{x}_i\colon P^{2ni} \longrightarrow F$ which can be thought of as iterated relative Whitehead products. The following lemma was certainly known by them.

Lemma 3.2 *For $i \geq 1$, there are maps $\hat{x}_i\colon P^{2ni} \longrightarrow F$ such that*

(a) *if Z is an H space and $f\colon P^{2n+1} \to Z$, then $f\rho\hat{x}_i \sim *$ for $i > 1$;*

(b) *$\hat{x}_i\colon P^{2ni} \longrightarrow F$ induces an epimorphism in integral cohomology.*

Proof Let $\hat{x}_1\colon P^{2n} \longrightarrow F$ be a lift of the standard map $P^{2n} \longrightarrow S^{2n} \longrightarrow P^{2n+1}$. Since $\partial\colon \Omega S^{2n+1} \longrightarrow F$ has degree p^n in cohomology, $\hat{x}_1$ is unique up to homotopy and (b) clearly holds. Having constructed $\widehat{x}_i$, consider the map of fibrations

$$\begin{tikzcd}
P^{2ni} \rtimes \Omega P^{2n+1} \arrow[r, "W"] \arrow[d] & F \arrow[d, "\rho"] \\
P^{2ni} \vee P^{2n+1} \arrow[r, "\rho\widehat{x}_i \vee 1"] \arrow[d, "\pi_2"] & P^{2n+1} \arrow[d] \\
P^{2n+1} \arrow[r] & S^{2n+1}
\end{tikzcd}$$

where the fibration on the left is the universal fibration for relative Whitehead products and the induced map of fibres defines W. Let $\widehat{x}_{i+1}$ be the composition

$$P^{2n(i+1)} \longrightarrow P^{2ni} \rtimes P^{2n} \longrightarrow P^{2ni} \rtimes \Omega P^{2n+1} \xrightarrow{W} F$$

where the first map has a left homotopy inverse and projects trivially to P^{2ni} and the second map is the natural inclusion. To prove (a), suppose that $f\colon P^{2n+1} \to Z$ where Z is an H space. Then we can construct a homotopy commutative diagram:

$$\begin{tikzcd}
P^{2ni} \vee P^{2n+1} \arrow[r, "p\widehat{x}_i \vee 1"] \arrow[d] & P^{2n+1} \arrow[d, "f"] \\
P^{2ni} \times P^{2n+1} \arrow[r] & Z.
\end{tikzcd}$$

Now since $f\widehat{x}_{i+1}$ factors through $P^{2ni} \times P^{2n+1}$, it is null homotopic by construction. To prove (b) apply induction and compare the Serre spectral sequences for the two fibrations. $\square$

Since $S^{2n+1}\{p^r\}$ is an H space, the classes $\rho\widehat{x}_i$ lift to classes $y_i\colon P^{2ni} \longrightarrow E$ for $i > 1$. Thus $\sigma y_i \sim \rho\widehat{x}_i$ and $\rho\pi = \sigma$ imply that $\rho(\pi y_i - \widehat{x}_i)$ is null homotopic. Consequently $\pi y_i - \widehat{x}_i = \partial u_i$ for some maps $u_i\colon P^{2ni} \longrightarrow \Omega S^{2n+1}$. Since $\partial^*\colon H^{2ni}(F) \longrightarrow H^{2ni}(\Omega S^{2n+1})$ has degree p^r, $(\pi y_i - \widehat{x}_i)^* = 0$ and we obtain the following.

Lemma 3.3 *For $i > 1$, the composite $H^{2ni}(F) \xrightarrow{\pi^*} H^{2ni}(E) \xrightarrow{y_i^*} H^{2ni}(P^{2ni})$ is an epimorphism.* $\square$

Define $F_{(i)}$ as the $2ni$ skeleton of F. Define $E_{(i)}$ by the homotopy pullback

$$\begin{tikzcd}
E_{(i)} \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
F_{(i)} \arrow[r] & F.
\end{tikzcd}$$

Note that by (3.1) the fibre of π is $\Omega^2 S^{2n+1}$, and so the same is true of the induced map $E_{(i)} \longrightarrow F_{(i)}$. Including $F_{(i-1)}$ as the $(2n(i-1))$–skeleton of $F_{(i)}$, we obtain a homotopy pullback diagram

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \arrow[r, equal] \arrow[d] & \Omega^2 S^{2n+1} \arrow[d] \\
E_{(i-1)} \arrow[r] \arrow[d] & E_{(i)} \arrow[d] \\
F_{(i-1)} \arrow[r] & F_{(i)}.
\end{tikzcd}$$

Let $\mathcal{X}_i\colon (D^{2ni}, S^{2ni-1}) \longrightarrow (P^{2ni}, S^{2ni-1})$ be the characteristic map of the $2ni$ cell. Then $\mathcal{X}_i|_{S^{2ni-1}}$ has degree p^r. The composition

$$\theta\colon (D^{2ni}, S^{2ni-1}) \xrightarrow{\mathcal{X}_i} (P^{2ni}, S^{2ni-1}) \xrightarrow{y_i} (E_{(i)}, E_{(i-1)}) \xrightarrow{\pi} (F_{(i)}, F_{(i-1)})$$

induces a cohomology isomorphism by Lemma 3.3. Thus there is an equivalence

$$F_{(i)} = F_{(i-1)} \cup_\theta e^{2ni}.$$

The restriction of θ to S^{2ni-1} is divisible by p^r so the conditions of Extension Theorem 2.2 are satisfied with $q = p^r$. Therefore we have proved the following.

Theorem 3.4 *If B is a connected H space whose p^r–power map is null homotopic, then for $i > 1$ any map $E_{(i-1)} \longrightarrow B$ extends to a map $E_{(i)} \longrightarrow B$.* □

In [8], a classifying space BW_n of the fiber of the double suspension was constructed, along with a fibration sequence

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n.$$

Corollary 3.5 *There is a map $\nu^E\colon E \longrightarrow BW_n$ such that the composition*

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} E \xrightarrow{\nu^E} BW_n$$

is homotopic to ν.

Proof Since $F_{(1)} = S^{2n}$, we have the fibration

$$\Omega^2 S^{2n+1} \longrightarrow E_{(1)} \longrightarrow S^{2n} \longrightarrow \Omega S^{2n+1}.$$

This fibration was analyzed in [8] and it was shown that $E_{(1)} \simeq S^{4n-1} \times BW_n$ in such a way that the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} S^{4n-1} \times BW_n \xrightarrow{\pi_2} BW_n$$

is homotopic to ν. It was also shown that for $p \geq 5$ BW_n is a homotopy associative H space. The H space structure on BW_n was shown to be homotopy associative for $p = 3$ and that the p–power map on BW_n is null homotopic in [20]. Thus for $i > 1$ we can apply Theorem 3.4 to construct maps $\nu_i \colon E_{(i)} \longrightarrow BW_n$ by induction such that $\nu_i \partial_i \sim \nu$. Since $E = \bigcup E_{(i)}$, we define $\nu^E \colon E \longrightarrow BW_n$ by $\nu^E \mid E_i = \nu_i$. □

Theorem 3.6 *There is a diagram of fibrations*

$$\begin{tikzcd}
S^{2n-1} \arrow[r] \arrow[d, "i"] & \Omega^2 S^{2n+1} \arrow[r, "\nu"] \arrow[d, "\partial"] & BW_n \arrow[d, equal] \\
R_0 \arrow[r] \arrow[d] & E \arrow[r, "\nu^E"] \arrow[d] & BW_n \\
F \arrow[r, equal] & F &
\end{tikzcd}$$

with i null homotopic and so $\Omega F \simeq S^{2n-1} \times \Omega R_0$.

Proof The space R_0 is defined as the fiber of ν^E. Since the fibration

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} E \longrightarrow F$$

is induced by a map to ΩS^{2n+1} which induces an isomorphism in $H_{2n}(\ )$, the map $\Omega F \to S^{2n-1}$ induces an isomorphism in $H_{2n-1}(\ )$ and hence has a right homotopy inverse. □

It is worth noting at this point that the space ΩF_0 is split in [6]; consequently there is a homotopy decomposition

$$\Omega R_0 \simeq \prod_{i \geq 1} S^{2np^i-1}\{p^{r+1}\} \times \Omega P(n,r)$$

where $P(n,r)$ is a complicated wedge of mod–p^r Moore spaces. The fact that the product on the right is a loop space and is mapped to ΩF by a loop map is not obvious from their analysis. The cohomology structure of R_0 is rather simple.

Proposition 3.7 *We have*

$$H^m(R_0) = \begin{cases} \mathbb{Z}/p^r i\mathbb{Z} & \text{if } m = 2ni \text{ and } i > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, there is a choice of generators $e_i \in H^{2mi}(R_0)$ such that $e_i e_j = p^r \binom{i+j}{i} e_{i+j}$ when $i, j > 0$.

Proof Apply the Serre spectral sequence to the fibration $S^{2n-1} \longrightarrow R_0 \longrightarrow F$ in Theorem 3.6. □

We now construct the space T in Theorem 1.1 and prove the existence of the fibrations in parts (a) and (d), leaving the H–structure to the next section. By Diagram (3.1) there is a fibration sequence

$$\Omega S^{2n+1}\{p^r\} \xrightarrow{\tau} E \xrightarrow{\sigma} P^{2n+1} \longrightarrow S^{2n+1}\{p^r\}.$$

Define H by the composition

$$H\colon \Omega S^{2n+1}\{p^r\} \xrightarrow{\tau} E \xrightarrow{\nu^E} BW_n.$$

Note that H can be regarded as a secondary Hopf invariant. Define T as the homotopy fiber of H. Then Theorem 3.6 implies the following.

Theorem 3.8 *There is a diagram of fibrations*

$$\begin{tikzcd} T_{2n-1} \arrow[r] \arrow[d] & \Omega S^{2n+1}\{p^r\} \arrow[r, "H"] \arrow[d, "\tau"] & BW_n \arrow[d, equal] \\ R_0 \arrow[r] \arrow[d] & E \arrow[r, "\nu^E"] \arrow[d, "\sigma"] & BW_n \\ P^{2n+1} \arrow[r, equal] & P^{2n+1}. & \end{tikzcd}$$

□

The connecting maps for the vertical fibrations in Theorem 3.8 immediately give the following.

Corollary 3.9 *There is a homotopy commutative diagram*

$$\begin{tikzcd} \Omega P^{2n+1} \arrow[r, equal] \arrow[d] & \Omega P^{2n+1} \arrow[d] \\ T_{2n-1} \arrow[r] & \Omega S^{2n+1}\{p^r\} \end{tikzcd}$$

where the right map is the loop of the inclusion of the bottom Moore space. □

Continuing the diagram in (3.1), we have

$$\begin{tikzcd} \Omega^2 S^{2n+1} \arrow[r, "\xi"] \arrow[d, equal] & \Omega S^{2n+1}\{p^r\} \arrow[d, "\tau"] \\ \Omega^2 S^{2n+1} \arrow[r, "\partial"] & E. \end{tikzcd}$$

Applying Theorems 3.6 and 3.8 gives

$$H\xi \sim \nu^E \tau\xi \sim \nu^E \partial \sim \nu,$$

and we conclude:

Theorem 3.10 *There is a diagram of fibrations*

$$\begin{tikzcd} \Omega^2 S^{2n+1} \arrow[r, equal] \arrow[d, "\pi_n"] & \Omega^2 S^{2n+1} \arrow[d, "p^r"] & \\ S^{2n-1} \arrow[r, "E^2"] \arrow[d] & \Omega^2 S^{2n+1} \arrow[r, "\nu"] \arrow[d, "\xi"] & BW_n \arrow[d, equal] \\ T_{2n-1} \arrow[r] \arrow[d] & \Omega S^{2n+1}\{p^r\} \arrow[r, "H"] \arrow[d] & BW_n \\ \Omega S^{2n+1} \arrow[r, equal] & \Omega S^{2n+1}. & \end{tikzcd}$$

□

In particular, the top square in Theorem 3.10 is Cohen, Moore, and Neisendorfer's factorization of the p^r–power map on $\Omega^2 S^{2n+1}$. Since π_n has degree p^r, we have the following corollary.

Corollary 3.11 *There is a homotopy commutative diagram*

$$\begin{tikzcd} S^{2n-1} \arrow[r, "p^r"] \arrow[d] & S^{2n-1} \arrow[d] \\ \Omega^2 S^{2n+1} \arrow[r, "p^r"] \arrow[ur, "\pi_n"] & \Omega^2 S^{2n+1} \end{tikzcd}$$

for each $r \geq 1$. □

4 The construction of G_{2n} and the H–space structure on T_{2n-1}

In this section we will fix n and abbreviate T_{2n-1} and G_{2n} as T and G. We will need to filter T by skeleta and write T^m for the m skeleton.

Our purpose is to construct an H–space structure on T. In fact we do more than that. We construct a corresponding co–H space G in the sense of [11]; ie, we construct a $(2n-2)$–connected space G and maps

$$f\colon G \longrightarrow \Sigma T$$
$$g\colon T \longrightarrow \Omega G$$
$$h\colon \Omega G \longrightarrow T$$

such that the compositions

$$G \xrightarrow{f} \Sigma T \xrightarrow{\widetilde{g}} G$$
$$T \xrightarrow{g} \Omega G \xrightarrow{h} T$$

are homotopic to the identity, where $\widetilde{g}$ is the adjoint of g. We go on to derive several interesting results from this structure.

We will write T^m for the m–skeleton of T. We will also reintroduce the torsion parameter for Moore spaces as we will need to consider mod–p^s Moore spaces $P^m(p^s)$ for $s \neq r$. The space G will be filtered by the $2np^k+1$ skeleton which we will abbreviate as G_k. These will be constructed inductively starting with $G_{-1} = *$. We will construct a map

$$\alpha_k\colon P^{2np^k}(p^{r+k}) \longrightarrow G_{k-1}$$

and define G_k as the mapping cone of α_k.

One of the features of [11] is that the spaces G and T come with a fibration

$$T \longrightarrow R \longrightarrow G.$$

Thus we seek to construct G inductively over the subspaces G_k together with the induced fibrations

$$T \longrightarrow R_k \longrightarrow G_k.$$

The fibration in Theorem 3.8 provides the case $k=0$. These fibrations will be induced from corresponding fibrations

$$\Omega S^{2n+1}\{p^r\} \longrightarrow E_k \longrightarrow G_k$$

as in Theorem 3.8. The entire induction involves obtaining key properties of the skeleta of ΣT as well as G and involves a cyclical induction through 14 steps.

Proposition 4.1 *As an algebra, $H^*(T;\mathbb{Z}/p\mathbb{Z})$ is generated by classes u of dimension $2n-1$ and v_i of dimension $2np^i$ for each $i \geq 0$ subject to the relations $v_i^p = 0$ and $u^2 = 0$. For each i define*

$$u_i = uv_0^{p-1}v_1^{p-1}\cdots v_{i-1}^{p-1}.$$

Then $\beta^{(r+i)}u_i = v_i$. As a vector space $\widetilde{H}^*(T;\mathbb{Z}/p\mathbb{Z})$ is generated by classes $v(m)$ of dimension $2mn$ and $u(m)$ of dimension $2mn-1$ for each $m \geq 1$ where

$$v(m) = v_s^{e_s}\cdots v_t^{e_t} = \beta^{(r+s)}u(s)$$
$$u(m) = u_s v_s^{e_s} v_{s+1}^{e_{s+1}}\cdots v_t^{e_t}$$

and $m = \sum_{i=s}^{t} e_i p^i$, $0 \leq e_i < p$, $e_s \neq 0$.

Proof We apply the Serre spectral sequence for the cohomology of the fibration

$$S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}.$$

Using $\mathbb{Z}/p\mathbb{Z}$ coefficients we see that

$$H^*(T;\mathbb{Z}/p\mathbb{Z}) \cong H^*(S^{2n-1};\mathbb{Z}/p\mathbb{Z}) \otimes H^*(\Omega S^{2n+1};\mathbb{Z}/p\mathbb{Z})$$

as algebras. Using integer coefficients we see that $v(m)$ is the reduction of a class of order p^{r+s} so $v(m) = \beta^{(r+s)}u(s) \neq 0$. We define $v_s = \beta^{(r+s)}u_s$. □

Note that dually the homology of T has a very simple description. There is a Hopf algebra isomorphism

$$H_*(T) \cong \Lambda(\overline{u}) \otimes \mathbb{Z}/p\mathbb{Z}[\overline{v}]$$

where $\overline{u}$ and $\overline{v}$ are dual to u and v respectively, and the dual Bocksteins are determined by $\beta^{(r+i)}\overline{v}^{p^i} = \overline{u}\overline{v}^{p^i-1}$ for $i \geq 0$.

Anick [1] introduced the notation $\mathcal{W}_a^b$ for the collection of all spaces that are the homotopy type of simply connected locally finite wedges of mod–p^s Moore spaces for $a \leq s \leq b$. Note that any simply connected Moore space is a suspension, so any space in $\mathcal{W}_a^b$ is a suspension. Recall that the smash of two Moore spaces is homotopy equivalent to a wedge of Moore spaces: if $s \leq t$ then there is a homotopy equivalence

$$P^m(p^s) \wedge P^n(p^t) \simeq P^{m+n}(p^s) \wedge P^{m+n-1}(p^s).$$

In particular, $\mathcal{W}_a^b$ is closed under smash products. Recall also that any retract of a wedge of Moore spaces is homotopy equivalent to a wedge of Moore spaces, so $\mathcal{W}_a^b$ is closed under retracts.

Lemma 4.2 *Suppose $W \in \mathcal{W}_a^b$ is simply connected and $f\colon P^k(p^t) \longrightarrow W$ is divisible by p^b.*

(a) *Write $W = W_1 \vee W_2$ with $W_1 \in \mathcal{W}_a^{b-1}$ and $W_2 \in \mathcal{W}_b^b$. Then f factors through W_2 up to homotopy.*

(b) *Suppose in addition that W_2 is $(d-1)$ connected and $k < pd$. Then $f \sim *$.*

Proof Since W is a wedge, there is a homotopy equivalence

$$\Omega W = \Omega W_2 \times \Omega(W_1 \rtimes \Omega W_2)$$

(see, for example, the first author's work [7]). Since $W_1, W_2 \in \mathcal{W}_a^b$, both spaces are suspensions, and we can write $W_1 = \Sigma \overline{W}_1$ and $W_2 = \Sigma \overline{W}_2$. Since W_1 is a suspension, we have $W_1 \rtimes \Omega W_2 \simeq W_1 \vee (W_1 \wedge \Omega W_2)$. For the right wedge summand, the James splitting of $\Sigma\Omega\Sigma X$ as $\bigvee \Sigma X^{(i)}$ gives

$$W_1 \wedge \Omega W_2 \simeq \Sigma \overline{W}_1 \wedge \Omega\Sigma \overline{W}_2 \simeq \overline{W}_1 \wedge \Big(\bigvee \Sigma \overline{W}_2^{(i)}\Big).$$

Combining, we have

$$W_1 \rtimes \Omega W_2 \simeq W_1 \vee \Big(W_1 \wedge \Big(\bigvee W_2^{(i)}\Big)\Big).$$

In particular, since $\mathcal{W}_a^b$ is closed under smash products, we have $W_1 \rtimes \Omega W_2 \in \mathcal{W}_a^b$. Applying the Hilton–Milnor theorem therefore implies that

$$\Omega(W_1 \rtimes \Omega W_2) \simeq \prod_i \Omega P^{n_i}(p^{s_i})$$

with $a \le s \le b-1$.

By [16], the p^{r+1}–power map on $\Omega^2 P^m(p^r)$ is null homotopic for any $r \ge 1$ and $m \ge 3$. Thus $P^m(p^r)$ admits no nontrivial maps which are divisible by p^{r+1}. In our case, this implies that $\prod_i \Omega P^{n_i}(p^{s_i})$ admits no nontrivial maps which are divisible by p^b. Thus the adjoint of f, which is divisible by p^b, is trivial on $\Omega(W_1 \rtimes \Omega W_2)$ and so factors through the inclusion $\Omega W_2 \longrightarrow \Omega W$. Hence, taking the adjoint, f factors through the inclusion $W_2 \longrightarrow W$, proving part (a).

For part (b), since $W_2 \in \mathcal{W}_b^b$ and W_2 is $(d-1)$–connected, the Hilton–Milnor theorem implies that $\Omega W_2 = \prod \Omega P^{n_i}(p^b)$ where $n_i > d$. By [6; 16], $P^{2m+1}(p^r)$ admits no nontrivial maps which are divisible by p^r from a CW–complex of dimension $t < 2mp$, and $P^{2m}(p^b))$ admits no nontrivial maps which are divisible by p^r from a CW–complex of dimension $t < 2(2m-1)p$. In our case, the CW–complex is $P^k(p^t)$, the domain of f, and the target Moore spaces are the $P^{n_i}(p^b)$ in the decomposition of ΩW_2. Since $n_i > d$ for each i, the hypothesis $k < pd$ guarantees that the component of f on $P^{n_i}(p^b)$, being divisible by p^b, is null homotopic. So f is null homotopic. □

Theorem 4.3 *For each $k \geq 0$ there are spaces G_k and $W_k \in \mathcal{W}_r^{r+k-1}$ satisfying the following conditions:*

(a) $\Sigma T^{2np^k-2} \simeq G_{k-1} \vee W_k$.

(b) *There are maps $g_k\colon T^{2np^k-2} \longrightarrow \Omega G_{k-1}$ and $h_{k-1}\colon \Omega G_{k-1} \longrightarrow T$ such that $h_{k-1}g_k$ is homotopic to the inclusion of T^{2np^k-2} into T.*

(c) *There is a homotopy commutative diagram of cofibration sequences which defines the space G_k*

$$\begin{tikzcd}
P^{2np^k}(p^{r+k}) \arrow[r, "m_k"] \arrow[d, equal] & \Sigma T^{2np^k-2} \arrow[r] \arrow[d, "\widetilde{g}_k"] & \Sigma T^{2np^k} \arrow[d, "g'_k"] \\
P^{2np^k}(p^{r+k}) \arrow[r, "\alpha_k"] & G_{k-1} \arrow[r] & G_k
\end{tikzcd}$$

where $\widetilde{g}_k$ is the adjoint of g_k.

(d) *There is a map $e\colon P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \longrightarrow \Sigma T^{2np^k}$ which induces an epimorphism in mod–p homology in dimensions $2np^k$ and $2np^k+1$.*

(e) *The map $m_k\colon P^{2np^k}(p^{r+k}) \longrightarrow \Sigma T^{2np^k-2}$ is divisible by p^{r+k-1}.*

(f) *There is a map $\varphi_k\colon G_k \longrightarrow S^{2n+1}\{p^r\}$ extending φ_{k-1}.*

(g) $\Sigma G_k \in \mathcal{W}_r^{r+k}$.

(h) *There is a homotopy commutative diagram of fibration sequences*

$$\begin{tikzcd}
\Omega G_k \arrow[r, "h_k"] & T \arrow[r] \arrow[d] & R_k \arrow[r] \arrow[d] & G_k \arrow[d, equal] & \\
 & \Omega S^{2n+1}\{p^r\} \arrow[r] \arrow[d, "H"] & E_k \arrow[r] \arrow[d, "\nu_k"] & G_k \arrow[r] & S^{2n+1}\{p^r\} \\
 & BW_n \arrow[r, equal] & BW_n. & &
\end{tikzcd}$$

(i) $\Sigma^2 \Omega G_{k-1} \in \mathcal{W}_r^{r+k-1}$.

(j) *The equivalence in* (a) *extends to an equivalence $\Sigma T^{2np^k} \simeq G_k \vee W_k$.*

(k) $\Sigma^2 T^{2np^k} \in \mathcal{W}_r^{r+k}$.

(l) $G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$.

(m) $\Sigma T^{2np^k} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$.

(n) *There is a map* $\mu_k\colon T^{2np^k}\times T\longrightarrow T$ *which is the inclusion on the first axis and the identity on the second. Furthermore there is a homotopy commutative square*

$$\begin{tikzcd} T^{2np^k}\times T \arrow[r, "\mu_k"] \arrow[d] & T \arrow[d] \\ \Omega S^{2n+1}\times\Omega S^{2n+1} \arrow[r] & \Omega S^{2n+1}. \end{tikzcd}$$

Proof With $G_{-1}=*$ and $G_0=P^{2n+1}$ these statements are all immediate for $k=0$ with $\varphi_0\colon P^{2n+1}\longrightarrow S^{2n+1}\{p^r\}$ the inclusion, $E_0=E$ from Theorem 3.6, $\nu_0=\nu^E$, $\mu_0\colon P^{2n}\times T\longrightarrow T$ obtained from the action of ΩP^{2n+1} on T defined by the fibration in Theorem 3.6. We now supposed that (a)–(n) are all valid with $k-1$ in the place of k and we proceed to prove them for k.

Proof of (a) We will construct a map

$$f_m\colon P^{2mn+1}(p^{r+s})\longrightarrow \Sigma T^{2np^k-2}$$

which induces a monomorphism in mod–p homology for each m satisfying $p^{k-1}<m<p^k$, where $s=\nu_p(m)$. We then assemble these into a map

$$\Sigma T^{2np^{k-1}}\vee\left(\bigvee_{m=p^{k-1}+1}^{p^k-1}P^{2mn+1}(p^{r+s})\right)\longrightarrow \Sigma T^{2np^k-2}$$

which induces an isomorphism in mod–p homology. By applying (j) in the case $k-1$ we are done.

To construct the maps f_m we appeal to (n) in the case $k-1$ and iterate this to produce a diagram with p factors

$$\begin{tikzcd} T^{2np^{k-1}}\times\cdots\times T^{2np^{k-1}} \arrow[r] \arrow[d] & T^{2np^k} \arrow[d] \\ J(S^{2n})_{p^{k-1}}\times\cdots\times J(S^{2n})_{p^{k-1}} \arrow[r] & J(S^{2n})_{p^k} \end{tikzcd}$$

where $J(S^{2n})_j$ is the $2nj$ skeleton of ΩS^{2n+1}. Write m to the base p. Since $p^{k-1}<m<p^k$, m has coefficients of p^i for $i<k$ and the coefficient of p^{k-1} is not zero

$$m=a_s p^s+\cdots+a_{k-1}p^{k-1} \text{ with } a_s>0 \text{ and } a_{k-1}>0.$$

Let $l = a_s p^s + \cdots + a_{k-2} p^{k-2}$ so that $m = l + a_{k-1} p^{k-1}$ and further restrict the above diagram to one with $a_{k-1}+1$ factors

$$\begin{array}{ccc} T^{2nl} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}} & \xrightarrow{\quad \bar{\mu} \quad} & T^{2nm} \\ \downarrow & & \downarrow \\ J(S^{2n})_l \times J(S^{2n})_{p^{k-1}} \times \cdots \times J(S^{2n})_{p^{k-1}} & \longrightarrow & J(S^{2n})_m. \end{array}$$

Applying the maps in this diagram to a generator of $H^{2mn}(J(S^{2n})_m; \mathbb{Z}/p\mathbb{Z})$ we get

$$(\bar{\mu})^* \left(v(m)\right) = v(l) \otimes v_{k-1} \otimes \cdots \otimes v_{k-1}.$$

Now $v(m) = \beta^{(r+s)} u(m)$ and $v(l) = \beta^{(r+s)} u(l)$, so

$$v(l) \otimes v_{k-1} \otimes \cdots \otimes v_{k-1} = \beta^{(r+s)} \left(u(l) \otimes v_{k-1} \otimes \cdots \otimes v_{k-1}\right).$$

Applying (a) and (m) in case $k-1$ we see that

$$\Sigma\big(T^{2nl} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}}\big) \simeq V \vee W$$

where V is a wedge of p spaces, all but one being G_{k-1} with the other being the subspace G_l of G_{k-1}, and $W \in \mathcal{W}_r^{r+k-1}$. In particular, the dimension of V is less than $2np^{k-1}+2$. Since $p^{k-1} < m$, $\mu^*(u(m))$ projects trivially to $H^*(v)$. But since $W \in \mathcal{W}_r^{r+k-1}$, for any class $\xi \in H^i(W; \mathbb{Z}/p\mathbb{Z})$ with $\beta^{(j)}\xi \neq 0$, there is a map

$$f_\xi \colon P^{i+1}(p^j) \longrightarrow W$$

with f_ξ^* an epimorphism. Thus for each m satisfying $p^{k-1} < m < p^k$ we may choose such a map corresponding to $\xi = u(l) \otimes v_{k-1} \otimes \cdots \otimes v_{k-1}$. The composition

$$P^{2mn+1}(p^{r+s}) \xrightarrow{f_\xi} \Sigma\big(T^{2nl} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}}\big) \xrightarrow{\Sigma\bar{\mu}} \Sigma T^{2mn}$$

therefore gives the desired map f_m.

Proof of (b) From part (a) we obtain a map $T^{2np^k-2} \longrightarrow \Omega G_{k-1}$ which induces an isomorphism in $\pi_{2n-1}(\,)$. The composition

$$T^{2np^k-2} \longrightarrow \Omega G_{k-1} \xrightarrow{h_{k-1}} T$$

factors through the inclusion of T^{2np^k-2} and provides a self map of T^{2np^k-2} which induces an isomorphism on $\pi_{2n-1}(\,)$. Calculations with cup products and Bocksteins show that this map is a homotopy equivalence, so composing with the inverse provides a possibly different map

$$g_k \colon T^{2np^k-2} \longrightarrow \Omega G_{k-1}$$

such that $h_{k-1} g_k$ is homotopic to the inclusion.

Proof of (c) Using the map g_k from (b) we construct a commutative diagram where the bottom row is the fibration sequence from (h) in case $k-1$ and the middle row is a cofibration sequence

$$\begin{tikzcd}
 & & T^{2np^k}/T^{2np^k-2} \arrow[r, equal] \arrow[d] & P^{2np^k}(p^{r+k}) \arrow[d, "m_k"] \\
T^{2np^k-2} \arrow[r] \arrow[d, "g_k"] & T \arrow[r] \arrow[d, equal] & T/T^{2np^k-2} \arrow[r] \arrow[d] & \Sigma T^{2np^k-2} \arrow[d, "\widetilde{g}_k"] \\
\Omega G_{k-1} \arrow[r, "h_{k-1}"] & T \arrow[r] & R_{k-1} \arrow[r] & G_{k-1}.
\end{tikzcd}$$

Here, m_k is defined as the composite $T^{2np^k}/T^{2np^k-2} \to T/T^{2np^k-2} \to \Sigma T^{2np^k-2}$. Define

$$\alpha_k\colon P^{2np^k}(p^{r+k}) \longrightarrow G_{k-1}$$

as the composite $\widetilde{g}_k \circ m_k$. Define g'_k by the diagram of cofibration sequences

$$\begin{tikzcd}
P^{2np^k}(p^{r+k}) \arrow[r, "m_k"] \arrow[d, equal] & \Sigma T^{2np^k-2} \arrow[r] \arrow[d, "\widetilde{g}_k"] & \Sigma T^{2np^k} \arrow[d, "g'_k"] \\
P^{2np^k}(p^{r+k}) \arrow[r, "\alpha_k"] & G_{k-1} \arrow[r] & G_k.
\end{tikzcd}$$

Proof of (d) As in part (a), we consider the diagram

$$\begin{tikzcd}
T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}} \arrow[r, "\widetilde{\mu}"] \arrow[d] & T^{2np^k} \arrow[d] \\
J(S^{2n})_{p^{k-1}} \times \cdots \times J(S^{2n})_{p^{k-1}} \arrow[r] & J(S^{2n})_{p^k}
\end{tikzcd}$$

with p factors on the left. This is defined by iterated application of part (n) in case $k-1$. Clearly

$$\widetilde{\mu}^*(v_k) = v_{k-1} \otimes \cdots \otimes v_{k-1} = \beta^{(r+k-1)}(uv_1^{p-1} \cdots v_{k-2}^{p-1} \otimes v_{k-1} \otimes \cdots \otimes v_{k-1}).$$

As before there is a map

$$q\colon P^{2np^k+1}(p^{r+k-1}) \longrightarrow \Sigma\big(T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}}\big)$$

such that $(\Sigma(\widetilde{\mu})q)_*$ is an epimorphism in $\mathbb{Z}/p\mathbb{Z}$ homology in dimension $2np^k+1$, obtained by applying (k) and (m) in case $k-1$. Similarly, $v_k = \beta^{(r+k)}u_k$ and

$$(\widetilde{\mu})^* u_k = \sum_{p \text{ terms}} v_{k-1} \otimes \cdots \otimes v_{k-1} \otimes u_{k-1} \otimes v_{k-1} \cdots \otimes v_{k-1}.$$

In particular, the map

$$T^{2np^{k-1}-1} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}} \xrightarrow{\widetilde{\mu}'} T^{2np^k}$$

has the property that

$$\begin{aligned}\left(\widetilde{\mu}'\right)^* (u_k) &= u_{k-1} \otimes v_{k-1} \otimes \cdots \otimes v_{k-1} \\ &= \beta^{(r+k-1)} (u_{k-1} \otimes u_{k-1} \otimes v_k \otimes \cdots \otimes v_k).\end{aligned}$$

It follows, as before, that there is a map

$$r\colon P^{2np^k}(p^{r+k-1}) \longrightarrow \Sigma\big(T^{2np^{k-1}-1} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}}\big)$$

such that $(\Sigma(\widetilde{\mu}')r)_*$ is an epimorphism in $\mathbb{Z}/p\mathbb{Z}$ homology in dimension $2np^k$. We construct e as the wedge sum

$$e = (\Sigma\widetilde{\mu}')r \vee (\Sigma\widetilde{\mu})q\colon P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \longrightarrow \Sigma T^{2np^k}.$$

Proof of (e) We will construct a map $n_k\colon P^{2np^k}(p^{r+k}) \to \Sigma T^{2np^k-2}$ such that $m_k \sim p^{r+k-1} n_k$ out of a commutative diagram obtained from mapping a cofibration sequence into a fibration sequence. Given a Hurewicz fibration

$$F \xrightarrow{\iota} E \xrightarrow{\pi} B$$

and a commutative square

$$\begin{tikzcd} Y \cup_g CX \arrow[r] \arrow[d, "\theta"] & SX \arrow[d, "\theta'"] \\ E \arrow[r, "\pi"] & B, \end{tikzcd}$$

one can construct a commutative ladder

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\theta''"] & Y \arrow[r] \arrow[d, "\theta|_Y"] & Y \cup_g CX \arrow[r, "\mu"] \arrow[d, "\theta"] & SX \arrow[d, "\theta'"] \\ \Omega B \arrow[r] & F \arrow[r] & E \arrow[r, "\pi"] & B \end{tikzcd}$$

where θ'' is adjoint to θ'. (See, for example, Neisendorfer [17, 3.3.1]). We will apply this to the cofibration induced by the map

$$P^{2np^k+1}(p^{r+k}) \xrightarrow{p^{r+k-1}} P^{2np^k+1}(p^{r+k})$$

and the fibration induced by the map

$$\Sigma T^{2np^k} \xrightarrow{\pi} P^{2np^k+1}(p^{r+k})$$

pinching the $2np^k-1$ skeleton to a point. In this case the space $Y\cup_g CX$ is homotopy equivalent to $P^{2np^k}(p^{r+k-1})\vee P^{2np^k+1}(p^{r+k-1})$ and the map μ_* induces an epimorphism in mod p homology. By part (d), $(\pi e)_*$ induces a mod p homology epimorphism, so for some equivalence ξ, $\pi e\xi\sim\mu$. Taking $\theta=e\xi$ and $\theta'=1$, we obtain the diagram

$$\begin{tikzcd}
P^{2np^k}(p^{r+k}) \arrow[r, "p^{r+k-1}"] \arrow[d] & P^{2np^k}(p^{r+k}) \arrow[r] \arrow[d] & \begin{array}{c} P^{2np^k}(p^{r+k-1}) \\ \vee P^{2np^k+1}(p^{r+k-1}) \end{array} \arrow[r] \arrow[d, "\theta"] & P^{2np^k+1}(p^{r+k}) \arrow[d, equal] \\
\Omega P^{2np^k+1}(p^{r+k}) \arrow[r] & F \arrow[r] & \Sigma T^{2np^k} \arrow[r] & P^{2np^k+1}(p^{r+k}).
\end{tikzcd}$$

The $2np^k+2n-1$ skeleton of F is ΣT^{2np^k-2}. Restricting the lefthand square to the $2np^k+1$ skeleton yields a homotopy commutative square:

$$\begin{tikzcd}
P^{2np^k}(p^{r+k}) \arrow[r, "p^{r+k-1}"] \arrow[d, equal] & P^{2np^k}(p^{r+k}) \arrow[d, "n_k"] \\
P^{2np^k}(p^{r+k}) \arrow[r, "m_k"] & \Sigma T^{2np^k-2}
\end{tikzcd}$$

Proof of (f) To show that there is an extension of φ_{k-1} to φ_k,

$$\begin{tikzcd}
P^{2np^k}(p^{r+k}) \arrow[r, "\alpha_k"] & G_{k-1} \arrow[r] \arrow[d, "\varphi_{k-1}"] & G_k = G_{k-1}\cup_{\alpha_k} CP^{2np^k}(p^{r+k}) \arrow[dl, "\varphi_k"'] \\
& S^{2n+1}\{p^r\} &
\end{tikzcd}$$

it suffices to show that α_k is divisible by p^r. This holds by (e) since $r+k-1\geq r$.

Proof of (g) By part (g) for $k-1$, there is a homotopy equivalence $\Sigma G_{k-1}\simeq\bigvee_{i=0}^{k-1}P^{2np^i+2}(p^{r+i})$. Also, by definition, $\Sigma G_k=\Sigma G_{k-1}\cup_{\Sigma\alpha_k} CP^{2np^k}(p^{r+k})$.

By part (e) $\alpha_k = \widetilde{\alpha} \circ (p^{r+k-1}\iota)$ so $\Sigma\alpha_k = \Sigma\widetilde{\alpha} \circ (p^{r+k-1}\iota) \sim (p^{r+k-1}\iota) \circ \Sigma\widetilde{\alpha}$. However

$$p^{r+k-1}\iota\colon \bigvee_{i=0}^{k-1} P^{2np^i+2}(p^{r+i}) \to \bigvee_{i=0}^{k-1} P^{2np^i+2}(p^{r+i})$$

is null homotopic since the order of the identity map on a mod–p^r Moore space is p^r.

Proof of (h) In case $k = 0$, this is Theorem 3.8. As we have constructed $\varphi_k\colon G_k \longrightarrow S^{2n+1}\{p^r\}$ in part (f), we have a pullback diagram of principal fibrations

$$\begin{tikzcd}
\Omega S^{2n+1}\{p^r\} \arrow[r, equal] \arrow[d] & \Omega S^{2n+1}\{p^r\} \arrow[d] \\
E_{k-1} \arrow[r] \arrow[d] & E_k \arrow[d] \\
G_{k-1} \arrow[r] \arrow[d, "\varphi_{k-1}"] & G_k \arrow[d, "\varphi_k"] \\
S^{2n+1}\{p^r\} \arrow[r, equal] & S^{2n+1}\{p^r\}.
\end{tikzcd}$$

Let $(CP^{2np^k}(p^{r+k}), P^{2np^k}(p^{r+k})) \xrightarrow{\theta} (G_k, G_{k-1})$ be the relative homeomorphism extending α_k and defining G_k. Since $E_k \longrightarrow G_k$ has the homotopy lifting property, we can find a map $\phi\colon CP^{2np^k}(p^{r+k}) \longrightarrow E_k$ covering θ. As E_{k-1} is a pullback, we get a map of pairs

$$\phi\colon (CP^{2np^k}(p^{r+k}), P^{2np^k}(p^{r+k})) \longrightarrow (E_k, E_{k-1})$$

covering θ. The composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\phi} E_{k-1} \longrightarrow G_{k-1}$$

is α_k which is divisible by p^{r+k-1}. Since $k \geqslant 1$ we may apply Extension Theorem 2.2 to obtain an extension ν_k of ν_{k-1}.

Proof of (i) By part (j) in case $k-1$, $\Sigma^2\Omega G_{k-1}$ is a retract of $\Sigma^2\Omega\Sigma T^{2np^{k-1}}$. The latter space splits since the loop space can be approximated by the James construction [12], giving

$$\Sigma^2\Omega\Sigma T^{2np^{k-1}} \simeq \Sigma^2\left(\bigvee_{i\geq 1}(T^{2np^{k-1}})^{(i)}\right)$$

which is in $\mathcal{W}_r^{r+k-1}$ by (k) and (l) in case $k-1$. Since $\mathcal{W}_r^{r+k-1}$ is closed under retracts we are done.

Proof of (j) By part (a), we have $\Sigma T^{2np^k-2} \simeq G_{k-1} \vee W_k$ and by (e), we have

$$\Sigma T^{2np^k} \simeq \left(\Sigma T^{2np^k-2}\right) \cup_{m_k} CP^{2np^k}(p^{r+k})$$

with m_k divisible by p^{r+k-1}. It suffices to show that the map

$$m_k\colon P^{2np^k}(p^{r+k}) \longrightarrow \Sigma T^{2np^k-2} \simeq G_{k-1} \vee W_k$$

factors though G_{k-1}. To this end, observe that there is a homotopy decomposition

$$\Omega\left(G_{k-1} \vee W_k\right) \simeq \Omega G_{k-1} \times \Omega\left(W_k \rtimes \Omega G_{k-1}\right).$$

We will show that any map $P^{2np^k}(p^{r+k}) \longrightarrow W_k \rtimes \Omega G_{k-1}$ which is divisible by p^{r+k-1} is null homotopic. Since W_k is $(4n-1)$–connected, the Moore spaces in W_k are double suspensions, so $W_k \rtimes \Omega G_{k-1} \in \mathcal{W}_r^{r+k-1}$ by (i). In fact, $W_k \rtimes \Omega G_{k-1} \simeq W_1 \vee W_2$ with $W_1 \in \mathcal{W}_r^{r+k-2}$ and W_2 a retract of

$$\bigvee_{r=2}^{p-1} P^{2np^{k-1}+1}(p^{r+k-1}) \rtimes \Omega G_{k-1}$$

which is $4np^{k-1}-1$ connected. The result follows from Lemma 4.2.

Proof of (k) This follows immediately from (g) and (j).

Proof of (l) This follows from 3 steps based on an analysis which first appeared in [18].

Step 1 $G_k \wedge T^{2np^{k-1}} \in \mathcal{W}_r^{r+k-1}$.

Consider the cofibration sequence

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^{k-1}} \xrightarrow{\alpha_k \wedge 1} G_{k-1} \wedge T^{2np^{k-1}} \to G_k \wedge T^{2np^{k-1}}.$$

We have $P^{2np^k}(p^{r+k}) \wedge T^{2np^{k-1}} \in \mathcal{W}_r^{r+k-1}$ and $\alpha_k \wedge 1$ is divisible by p^{r+k-1}. Consequently, $\alpha_k \wedge 1 \sim *$ and so there is a homotopy decomposition

$$G_k \wedge T^{2np^{k-1}} \simeq (G_{k-1} \wedge T^{2np^{k-1}}) \vee (P^{2np^k+1}(p^{r+k}) \wedge T^{2np^{k-1}})$$

which is in $\mathcal{W}_r^{r+k-1}$ by (k) in case $k-1$.

Step 2 $G_{k-1} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}$.

By (j) in case $k-1$, $G_{k-1} \wedge T^{2np^k}$ is a retract of $\Sigma T^{2np^{k-1}} \wedge T^{2np^k}$. But

$$\Sigma T^{2np^{k-1}} \wedge T^{2np^k} \simeq T^{2np^{k-1}} \wedge \Sigma T^{2np^k} \simeq T^{2np^{k-1}} \wedge (G_k \vee W_k)$$

by (j). By Step 1 and (k) in case $k-1$, the latter space is in $\mathcal{W}_r^{r+k-1}$. Since $\mathcal{W}_r^{r+k-1}$ is closed under retracts, we therefore have $G_{k-1} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}$.

Step 3 $G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$.

Consider here the cofibration sequence

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha_k \wedge 1} G_{k-1} \wedge T^{2np^k} \to G_k \wedge T^{2np^k}.$$

The first space is in $\mathcal{W}_r^{r+k}$ by (k) and the second is in $\mathcal{W}_r^{r+k-1}$ by Step 2. In fact, $G_{k-1} \wedge T^{2np^k} \simeq (P^{2np^{k-1}+1}(p^{r+k-1}) \wedge T^{2np^k}) \vee W'$ with $W' \in \mathcal{W}_r^{r+k-2}$. Here, the projection onto the first factor is $\rho_{k-1} \wedge 1$, where ρ_{k-1} is obtained by collapsing G_{k-2} to a point. Applying Lemma 4.2(b), we see that if $\alpha_k \wedge 1$ is nontrivial, so is the composition

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha_k \wedge 1} G_{k-1} \wedge T^{2np^k} \xrightarrow{\rho_{k-1} \wedge 1} P^{2np^{k-1}}(p^{r+k-1}) \wedge T^{2np^k}.$$

We will show that this composition is null homotopic. Let $\delta = \rho_{k-1}\alpha_k$, which is divisible by p^{r+k-1} because α_k is. According to [15], the p^{r+k-1}–power map on $S^{2np^{k-1}+1}\{p^{r+k-1}\}$ is null homotopic. Therefore the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\delta} P^{2np^{k-1}+1}(p^{r+k-1}) \longrightarrow S^{2np^{k-1}+1}\{p^{r+k-1}\}$$

is null homotopic. It follows that the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\delta} P^{2np^{k-1}+1}(p^{r+k-1}) \xrightarrow{\rho} S^{2np^{k-1}+1}$$

is null homotopic. Since the map

$$P^{2np^{k-1}+1}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\rho \wedge 1} S^{2np^{k-1}+1} \wedge T^{2np^k}$$

has a left homotopy inverse, the map

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\delta \wedge 1} P^{2np^{k-1}+1}(p^{r+k}) \wedge T^{2np^k}$$

is null homotopic. Since $\alpha_k \wedge 1$ is the composition

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha \wedge 1} G_{k-1} \wedge T^{2np^k} \xrightarrow{\delta \wedge 1} P^{2np^{k-1}+1}(p^{r+k}) \wedge T^{2np^k},$$

it is null homotopic as well. Consequently, there is a homotopy decomposition

$$G_k \wedge T^{2np^k} \simeq (G_{k-1} \wedge T^{2np^k}) \vee (P^{2np^k+1}(p^{r+k}) \wedge T^{2np^k}).$$

Both terms on the right are in $\mathcal{W}_r^{r+k}$ by (k) and Step 2.

Proof of (m) By (j), $\Sigma T^{2np^k} \wedge T^{2np^k} \simeq (G_k \vee W_k) \wedge T^{2np^k}$. By (l), $G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$, and as W_k is a wedge of Moore spaces which are at least $(4n-1)$–connected, it is a double suspension, so $W_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$ by (k). So $\Sigma T^{2np^k} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$.

Proof of (n) Since the composite $R_k \longrightarrow E_k \longrightarrow G_k \longrightarrow S^{2n+1}\{p^r\} \longrightarrow S^{2n+1}$ is null homotopic by (h), there is a commutative diagram of principal fibrations

$$\begin{tikzcd}
\Omega G_k \arrow[r] \arrow[d, "h_k"] & \Omega S^{2n+1} \arrow[d, equal] \\
T \arrow[r] \arrow[d] & \Omega S^{2n+1} \arrow[d] \\
R_k \arrow[r] \arrow[d] & PS^{2n+1} \arrow[d] \\
G_k \arrow[r] & S^{2n+1}
\end{tikzcd}$$

where PS^{2n+1} is the path space on S^{2n+1}. Consequently the actions are compatible:

$$\begin{tikzcd}
\Omega G_k \times T \arrow[r] \arrow[d] & \Omega S^{2n+1} \times \Omega S^{2n+1} \arrow[d] \\
T \arrow[r] & \Omega S^{2n+1}.
\end{tikzcd}$$

Using (j) we construct a map $g_k\colon T^{2np^k} \longrightarrow \Omega G_k$ such that the composition

$$T^{2np^k} \xrightarrow{g_k} \Omega G_k \xrightarrow{h_k} T$$

is homotopic to the inclusion as in (b). This gives a homotopy commutative diagram

$$\begin{tikzcd}
T^{2np^k} \times T \arrow[r, "g_k \times 1"] \arrow[d, "\mu_k"] & \Omega G_k \times T \arrow[d, "a"] \\
T \arrow[r, equal] & T.
\end{tikzcd}$$

Combining the preceding two diagrams gives the result and completes the induction. □

We now consider the limiting case. Write $G = \bigcup G_k$, $R = \bigcup R_k$ and $E_\infty = \bigcup E_k$.

Theorem 4.4 *There is a diagram of fibration sequences*

$$\begin{tikzcd}
\Omega G \arrow[r, "h"] & T \arrow[r, "i"] \arrow[d, "E"] & R \arrow[r] \arrow[d] & G \arrow[d, equal] & \\
& \Omega S^{2n+1}\{p^r\} \arrow[r] \arrow[d, "H"] & E_\infty \arrow[r] \arrow[d] & G \arrow[r, "\varphi"] & S^{2n+1}\{p^r\} \\
& BW_n \arrow[r, equal] & BW_n & &
\end{tikzcd}$$

and there are maps $\tilde{g}\colon T \longrightarrow \Omega G$ *and* $f\colon G \longrightarrow \Sigma T$ *such that the composites*

$$G \xrightarrow{f} \Sigma T \xrightarrow{g} G$$
$$T \xrightarrow{\tilde{g}} \Omega G \xrightarrow{h} T$$

are homotopic to the identity maps.

Proof The diagram is the direct limit of the diagrams in Theorem 4.3 (h) with

$$h = \varinjlim h_k, \quad g = \varinjlim g_k \quad \text{and} \quad f = \varinjlim f_k,$$

where $f_k\colon G_k \longrightarrow \Sigma T^{2np^k}$ is a right inverse for g_k given by Theorem 4.3 (j). □

Theorem 4.5 *The following space belong to* $\mathcal{W}_r^\infty$*:* $\Sigma^2\Omega G$, ΣG, $G \wedge T$, $\Sigma T \wedge T$, *and* W *where* $\Sigma T \simeq G \vee W$.

Proof This follows immediately from the results in Theorem 4.3 by taking limits. □

The retraction of T off ΩG in Theorem 4.4 induces an H–structure on T by the composite

$$m\colon T \times T \xrightarrow{\tilde{g}\times\tilde{g}} \Omega G \times \Omega G \longrightarrow \Omega G \xrightarrow{h} T.$$

The following proposition establishes the H–fibration property in Theorem 1.1 (a) as a consequence of a slightly stronger result.

Proposition 4.6 *The map* $T \xrightarrow{E} \Omega S^{2n+1}\{p^r\}$ *is an* H *map with respect to the* H–*space structure* m *on* T. *Consequently, there is an* H–*fibration sequence* $S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$.

Proof Filling in the fibration diagram in Theorem 4.4 on the right, we obtain a homotopy commutative square

$$\begin{tikzcd} \Omega G \arrow[r, "h"] \arrow[d, equal] & T \arrow[d, "E"] \\ \Omega G \arrow[r, "\Omega\varphi"] & \Omega S^{2n+1}\{p^r\}. \end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd} T\times T \arrow[r, "\tilde{g}\times\tilde{g}"] \arrow[dr, "E\times E"'] & \Omega G\times\Omega G \arrow[rr] \arrow[d, "\Omega\varphi\times\Omega\varphi"] & & \Omega G \arrow[r, "h"] \arrow[d, "\Omega\varphi"] & T \arrow[dl, "E"'] \\ & \Omega S^{2n+1}\{p^r\}\times\Omega S^{2n+1}\{p^r\} \arrow[rr] & & \Omega S^{2n+1}\{p^r\}. & \end{tikzcd}$$

The middle square commutes as $\Omega\varphi$ is an H–map and we have just seen that the right triangle commutes. The left triangle commutes since $\varphi\sim Eh$, so $\varphi\tilde{g}\sim E$. As the top row is the definition of the multiplication m on T, the commutativity of the diagram implies that E is an H–map.

Consequently, the composition $T\xrightarrow{E}\Omega S^{2n+1}\{p^r\}\longrightarrow\Omega S^{2n+1}$ is an H–map as it is a composite of H–maps, and so the homotopy fibration $S^{2n-1}\longrightarrow T\longrightarrow\Omega S^{2n+1}$ is of H–spaces and H–maps. □

The next proposition and the following corollary give structural properties of the spaces T, G, and R.

Proposition 4.7 *The spaces T and G are atomic.*

Proof It is easy to see that T is atomic using the product structure and the Bockstein relations. The case of G is more difficult. We first show that α_k is essential for all k. Suppose not, so that for some k we have a homotopy equivalence

$$G_k\simeq G_{k-1}\vee P^{2np^k+1}(p^{r+k}).$$

Using the retraction $G_k\to G_{k-1}$ we construct a composition

$$T^{2np^{k+1}-2}\xrightarrow{g_{k+1}}\Omega G_k\to\Omega G_{k-1}\xrightarrow{h_{k-1}}T$$

which induces an isomorphism in H^{2n}. Using the cup product and Bockstein structure, we infer that this composition is a cohomology isomorphism in dimensions less than $2np^{k+1}-1$. Since $H^{2np^k}(T)\simeq Z/p^{r+k}$, there must be an element of $H^{2np^k}(\Omega G_{k-1})$ of order p^{r+k}. This contradicts Theorem 4.3 (i).

Now suppose $\gamma\colon G \to G$ is a cellular map inducing and isomorphism in H_{2n}. We will show, by induction on k, that $\gamma_p\colon H_{2np^k}(G) \to H_{2np^k}(G)$ is an isomorphism. Assuming that γ_* is an isomorphism in degrees less that $2np^k$, consider the diagram

$$\begin{tikzcd} P^{2np^k}(p^{r+k}) \arrow[r, "\alpha_k"] \arrow[d, "d"] & G_{k-1} \arrow[r] \arrow[d, "\gamma"] & G_k \arrow[d, "\gamma"] \\ P^{2np^k}(p^{r+k}) \arrow[r, "\alpha_k"] & G_{k-1} \arrow[r] & G_k \end{tikzcd}$$

where d is an integer multiple of the identity map. Since $d\alpha_k = \alpha_k$ and α_k is nontrivial, d is a p local unit and γ_* is an isomorphism in dimension $2np^k$ by the 5–lemma. □

Corollary 4.8 $R \in \mathcal{W}_r^\infty$.

Proof Since G and T are atomic, Theorem 4.4 implies that (G, T) is a corresponding pair in the sense of [11]. According to [11, Theorem 3.2], R is a retract of $\Sigma T \wedge T$. However $\Sigma T \wedge T \in \mathcal{W}_r^\infty$ by Theorem 4.3(m). □

The next proposition implies that the space T constructed in this paper is homotopy equivalent to the space Anick constructed in [1] when $p \geq 5$ (the primes for which Anick's construction holds).

Proposition 4.9 *Suppose X is an H space and there is a fibration sequence*

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{i} X$$

such that the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p^r power map. Then $X \simeq T$.

Proof Consider the diagram of fibrations

$$\begin{tikzcd} \Omega W_n \arrow[r] \arrow[d] & \Omega X \arrow[r] \arrow[d] & \Omega^2 S^{2n+1}\{p^r\} \arrow[d] \\ P W_n \arrow[r] \arrow[d] & \Omega^2 S^{2n+1} \arrow[r, equal] \arrow[d, "\varphi"] & \Omega^2 S^{2n+1} \arrow[d, "p^r"] \\ W_n \arrow[r] & S^{2n-1} \arrow[r, "E^2"] \arrow[d] & \Omega^2 S^{2n+1} \\ & X. & \end{tikzcd}$$

Since $p\cdot\pi_*(W_n)=0$ and $p^r\cdot\pi_*\left(S^{2n+1}\{p^r\}\right)=0$ we conclude that $p^{r+1}\cdot\pi_*(X)=0$. Since $\pi_{2np-1}(W_n)=0$, we also see that $p^r\cdot\pi_{2np-1}(X)=0$. According to [2, Corollary 4.2] this is sufficient to construct a map

$$\varphi\colon G\longrightarrow \Sigma X$$

which induces an isomorphism in π_{2n}. The construction given in [2] depends only on the co–H space structure on G and the fact that α_k is divisible by p^{r+k-1}, so the proof works in this context as well. From this we construct the composition

$$T\xrightarrow{g}\Omega G\xrightarrow{\Omega\varphi}\Omega\Sigma X\longrightarrow X.$$

It is an easy calculation with the Serre spectral sequence that $H^*(X;\mathbb{Z}/p\mathbb{Z})\cong H^*(T;\mathbb{Z}/p\mathbb{Z})$, so this map is a homotopy equivalence. □

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