Plane sextics via dessins d'enfants

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We develop a geometric approach to the study of plane sextics with a triple singular point. As an application, we give an explicit geometric description of all irreducible maximal sextics with a type $\mathbf{E}_7$ singular point and compute their fundamental groups. All groups found are finite; one of them is nonabelian.

14H45; 14H30, 14H50

1 Introduction

1.1 Motivation

The subject of this paper is singular complex plane projective algebraic curves of degree six (sextics), considered up to equisingular deformation. Throughout the paper we assume that all curves involved have at worst simple singularities. Formally, the classification of plane sextics can be reduced to a purely arithmetical problem (see Degtyarev [6]), which can be solved in many interesting cases (see, eg, I Shimada's list [26] of maximal sextics, the classification of classical Zariski pairs by A Özgüner [23] or the list of special sextics by A Degtyarev [5]); the general impression is that one can answer any reasonable particular question, although the complete classification would require an enormous amount of work. Furthermore, this arithmetical approach, based on the theory of $K3$ surfaces, does solve a number of problems concerning the geometry of plane sextics (see, eg, the solution to M Oka's conjecture [13] in Degtyarev [5] and Ishida and Tokunaga [19], the classification of Z–splitting curves in Shimada [25] or the classification of stable symmetries of irreducible sextics in Degtyarev [7]).

However, more subtle questions, such as the computation of the fundamental group of the complement of a sextic, still remain unanswered, as they require a much more thorough understanding of the topology of the curve. A great deal of effort has been made lately (see Degtyarev [9; 4; 8], Degtyarev and Oka [12] and Eyral and Oka [16; 14; 15]; more references can be found in [15]) in order to compute the fundamental groups of relatively few curves. Each time, the main achievement is discovering a way to visualize a particular curve and its braid monodromy; once this is done, computing the group is a technicality.

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Apart from a few curves given by explicit equations, most approaches to the visualization of plane sextics found in the literature rely, in one way or another, to an elliptic pencil in the covering $K3$ surface. One such approach was suggested in Degtyarev [7]: one uses a stable symmetry and represents the sextic as a double covering of an appropriate trigonal curve. In the present paper, we suggest another approach, which is also based on the study of trigonal curves; in the long run, we anticipate being able to use this correspondence to handle all sextics with a triple singular point. Here, we deal with the type $\mathbf{E}_7$ singular points and prepare the background for types $\mathbf{E}_6$ and $\mathbf{E}_8$, which are to be the subject of a forthcoming paper.

It is worth mentioning that, by now, the approach suggested in [7] is almost exhausted, at least if one tries to confine oneself to irreducible maximal trigonal curves: the only case that has not been considered yet is that of sextics with two type $\mathbf{E}_8$ singular points. In my next paper, it will be shown that any such sextic has abelian fundamental group. Jumping a few steps ahead, I can announce that *the only irreducible maximal sextic with a type* $\mathbf{E}_8$ *singular point and nonabelian fundamental group has the set of singularities* $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$*; its group is a semidirect product of its abelianization* $\mathbb{Z}_6$ *and its commutant* $\mathrm{SL}(2, \mathbb{F}_5)$.

1.2 Principal results

Recall that a plane sextic B is called *maximal* (sometimes, *maximizing*), if the total Milnor number $\mu(B)$ of the singular points of B takes the maximal possible value, which is 19 (see Persson [24], where the term was introduced). Maximal sextics are projectively rigid; they are always defined over algebraic number fields.

1.2.1 Theorem *Up to projective transformation (equivalently, up to equisingular deformation), there are* 19 *maximal irreducible plane sextics* $B \subset \mathbb{P}^2$ *with simple singularities only and with at least one type* $\mathbf{E}_7$ *singular point; they realize* 11 *sets of singularities (see Table 3).*

This theorem is proved in Section 6.1, where all sextics are constructed explicitly using trigonal curves. Alternatively, the statement of the theorem follows from combining the results of J-G Yang [27] (the existence) and I Shimada [26] (the enumeration of the sets of singularities realized by more than one deformation family). In this respect, it is worth emphasizing that our proof is purely geometric; although not writing down explicit equations, we provide a means to completely recover the topology of the pair $(\mathbb{P}^2, B)$ and even the topology of the projection $(\mathbb{P}^2, B) \to \mathbb{P}^1$ from the type $\mathbf{E}_7$ singular point. (For the description of the topology of a trigonal curve in terms of its skeleton, see Degtyarev, Itenberg and Kharlamov [11] or Degtyarev [10].) As an application of this geometric construction, we compute the fundamental groups $\pi_1(\mathbb{P}^2 \smallsetminus B)$ of all curves involved and study their perturbations.

1.2.2 Theorem *With one exception, the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ of a plane sextic $B \subset \mathbb{P}^2$ as in Theorem 1.2.1 is abelian. The exception is the (only) sextic with the set of singularities $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$; its group is given by*

$$G = \big\langle \alpha_1, \alpha_2, \alpha_3 \;\big|\; [\alpha_2, \alpha_3] = [\alpha_i, \rho^3] = [\alpha_i, \alpha_2^2\alpha_3] = 1,\ i = 1, 2, 3,$$
$$\rho^2\alpha_1 = \alpha_2,\ (\alpha_1\alpha_2)^2\alpha_1 = \alpha_2(\alpha_1\alpha_2)^2,\ (\alpha_1\alpha_3)^2\alpha_1 = \alpha_3(\alpha_1\alpha_3)^2\big\rangle,$$

where $\rho = \alpha_1\alpha_2\alpha_3$. One can represent this group G as a semidirect product of its abelianization $\mathbb{Z}_6$ and its commutant $[G, G] \cong \mathrm{SL}(2, \mathbb{F}_{19})$, which is the only perfect group of order 6840.

1.2.3 Theorem *For any proper perturbation B' of any plane sextic B as in Theorem 1.2.1, the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B')$ is abelian.*

Theorems 1.2.2 and 1.2.3 are proved in Section 6.2 and Section 7.2, respectively.

Although we do not treat systematically reducible sextics (the principal reason being the fact that `GAP` [18] does not work well with infinite groups), the following by-product of our calculation seems worth mentioning (see Section 6.4 for the proof).

1.2.4 Proposition *Let B be a plane sextic splitting into two irreducible cubics and having one of the following five sets of singularities Σ:*

$$2\mathbf{E}_7 \oplus \mathbf{A}_5, \qquad \mathbf{E}_7 \oplus \mathbf{D}_{12}, \qquad \mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_7,$$
$$\mathbf{E}_7 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1, \qquad \mathbf{E}_7 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1.$$

Let $G = \pi_1(\mathbb{P}^2 \smallsetminus B)$. Then, for $\Sigma = 2\mathbf{E}_7 \oplus \mathbf{A}_5$ or $\mathbf{E}_7 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$, the commutant $[G, G]$ is a central subgroup of order 3; in the other three cases, G is abelian.

1.2.5 Corollary *For any irreducible sextic B' obtained by a perturbation from a sextic B as in Proposition 1.2.4, the group $\pi_1(\mathbb{P}^2 \smallsetminus B') \cong \mathbb{Z}_6$ is abelian.*

Theorems 1.2.2 and 1.2.3 and Corollary 1.2.5 provide further evidence to substantiate my conjecture that the fundamental group of an irreducible sextic that is not of torus type (ie, not given by a polynomial of the form $p^3 + q^2$) is finite.

Altogether, Theorem 1.2.3 gives rise to about 250 new sets of singularities that are realized by sextics with abelian fundamental groups, and Corollary 1.2.5 adds about 70 more. (Recall that, according to Degtyarev [9], any induced subgraph of the combined

Dynkin graph of a sextic B with simple singularities can be realized by a perturbation of B; in other words, the singular points of B can be perturbed independently.) Of special interest are the eleven (mentioning only the new ones) sets of singularities listed in Table 1. The corresponding curves are included into the so called *classical Zariski pairs*, ie, pairs of plane sextics that share the same set of singularities but differ by the Alexander polynomial. Together with the previously known results (see Degtyarev [8] and Eyral and Oka [13]), this makes 46 out of the 51 (see Özgüner [23]) classical Zariski pairs. In each of these 46 pairs, the groups of the two curves are $\mathbb{Z}_2 * \mathbb{Z}_3$ and $\mathbb{Z}_6$. (For the curves with nonabelian groups, see Degtyarev [8] and references there. Note that, formally, all classical Zariski pairs are known, one of them being in fact a triple [23], but not all curves have been constructed explicitly, hence not all fundamental groups have been computed yet.)

$3\mathbf{E}_6$	$\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$	$\mathbf{A}_{11} \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$
$2\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$	$\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$
$2\mathbf{E}_6 \oplus \mathbf{A}_5$	$\mathbf{E}_6 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_1$	$3\mathbf{A}_5 \oplus \mathbf{A}_1$
	$\mathbf{E}_6 \oplus 2\mathbf{A}_5$	$3\mathbf{A}_5$

Table 1: “New” classical Zariski pairs

Another interesting example is the set of singularities $3\mathbf{A}_6$, which is obtained by a perturbation of $\mathbf{E}_7 \oplus 2\mathbf{A}_6$. The corresponding curve B_{ns} has a so called *special* counterpart, ie, a sextic B_{sp} with the set of singularities $3\mathbf{A}_6$ and $\pi_1(\mathbb{P}^2 \smallsetminus B_{\text{sp}}) \cong \mathbb{Z}_3 \times \mathbb{D}_{14}$, where $\mathbb{D}_{14}$ is the dihedral group of order 14. (The latter group was computed in Degtyarev and Oka [12].) Another, very explicit, construction of a nonspecial sextic B_{ns} with the set of singularities $3\mathbf{A}_6$ was recently discovered in Eyral and Oka [15], where the group of B_{ns} was also shown to be abelian. The pair $(B_{\text{sp}}, B_{\text{ns}})$ constitutes a so called *Alexander equivalent Zariski pair*: the Alexander polynomials of both curves are trivial. We refer to [15] for the further discussion of special sextics and the current state of the subject.

1.3 Tools and further results

Our principal tool is to blow up the type $\mathbf{E}_7$ point of the sextic and, after a series of elementary transformations, to consider the result as a trigonal curve in a ruled rational surface. We show that maximal sextics correspond to maximal trigonal curves, and the latter can be effectively studied using Grothendieck's *dessins d'enfants* (skeletons in the terminology of the paper). In fact, it turns out that a great deal of relevant statements is already scattered across Degtyarev [10; 7], and we merely bring these results together and draw conclusions.

The following intermediate statement seems to be of independent interest: it gives an estimate on the total Milnor number μ of a nonisotrivial trigonal curve and characterizes maximal curves as those maximizing μ. We refer to Section 2 for the terminology and notation, and to Section 2.6 for the proof.

1.3.1 Theorem *Let $\bar{B}$ be a trigonal curve in the Hirzebruch surface Σ_k, and assume that $\bar{B}$ is not isotrivial and that all singularities of $\bar{B}$ are simple. Then the total Milnor number $\mu(\bar{B})$ of the singular points of $\bar{B}$ is subject to the inequality*

$$\mu(\bar{B}) \leqslant 5k - 2 - \#\{\text{unstable fibers of } \bar{B}\},$$

which turns into an equality if and only if $\bar{B}$ is maximal.

It is Theorem 1.3.1 that explains the relation between maximal sextics and maximal trigonal curves: both maximize the total Milnor number.

1.3.2 Remark The estimate given by Theorem 1.3.1 does *not* always hold for isotrivial trigonal curves, where $\mu(\bar{B})$ can be at least as large as $\approx 48k/5$. Besides, each nonsimple singular point increases the upper bound by 1.

Certainly, the crucial property of the sextics in question is the fact that they have a singular point of multiplicity $d-3$, where d is the degree of the curve, and our approach is an immediate generalization of a similar (although much simpler) study of curves with a singular point of multiplicity $d-2$ (see, eg, Degtyarev [2; 3]). The approach should work equally well for all sextics with a triple singular point, and we lay the foundation for a further development by completing the necessary preliminary calculations for the singular points of type $\mathbf{E}_8$ and $\mathbf{E}_6$. The precise statements concerning these two types will appear in a subsequent paper; the case of a $\mathbf{D}$–type point will be considered later.

1.4 Contents of the paper

In Section 2, we recall a few basic facts concerning the classification of maximal trigonal curves in Hirzebruch surfaces. The new result proved here is Theorem 1.3.1.

Section 3 introduces the principal tool used in the paper: the *trigonal model* of a plane sextic with a distinguished $\mathbf{E}$–type singular point. Keeping in mind further development of the subject, in addition to type $\mathbf{E}_7$ appearing in the principal results (Theorems 1.2.1–1.2.3), in the auxiliary Sections 3–5 we consider as well sextic with a singular point of type $\mathbf{E}_6$ or $\mathbf{E}_8$.

In Section 4, we outline the strategy used to compute the fundamental groups, cite a few statements from Degtyarev [10] concerning the braid monodromy of trigonal curves, and compute two "universal" relations present in the group of each curve: the so called relation at infinity and monodromy at infinity. As a further extension of this analysis of local canonical forms, in Section 5 we compute the homomorphism induced by the inclusion of a Milnor ball about an $\mathbf{E}$–type singular point.

Theorems 1.2.1 and 1.2.2 are proved in Section 6: we enumerate the trigonal models of sextics with a type $\mathbf{E}_7$ singular point by listing their skeletons (mainly, the problem is reduced to a previously known classification found in Degtyarev [7]); then, we use the skeletons obtained to compute the fundamental groups.

The concluding Section 7 deals with Theorem 1.2.3: first, we (re-)compute the fundamental groups of the perturbations of a type $\mathbf{E}_7$ singular point (using the same techniques involving trigonal curves and skeletons), and then we apply these results, the inclusion homomorphism of Section 5, and the presentations obtained in Section 6 to prove the theorem.

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2 Trigonal curves

In this section, we cite a few results concerning the classification and properties of maximal trigonal curves in Hirzebruch surfaces. The principal reference is [10].

2.1 Maximal trigonal curves

Recall that the *Hirzebruch surface* Σ_k, $k \geqslant 0$, is a geometrically ruled rational surface with an *exceptional section* E of square $-k$. (Sometimes, the fibers of the ruling are referred to as *vertical lines* in Σ_k.) A *trigonal curve* is a curve $\bar{B} \subset \Sigma_k$ disjoint from E and intersecting each generic fiber at three points.

In this paper, *we consider trigonal curves with simple singularities only.*

A *singular fiber* (sometimes also called a *vertical tangent*) of a trigonal curve $\bar{B} \subset \Sigma_k$ is a fiber of the ruling of Σ_k intersecting $\bar{B}$ geometrically at less than three points. Locally, $\bar{B}$ is the ramification locus of the Weierstraß model of a Jacobian elliptic surface, and to describe the (topological) type of a singular fiber we use (one of) the standard notation for the singular elliptic fibers, referring to the extended Dynkin graph of the corresponding configuration of the exceptional divisors. The types are as follows:

- $\tilde{\mathbf{A}}_0^*$: a simple vertical tangent;
- $\tilde{\mathbf{A}}_0^{**}$: a vertical inflection tangent;
- $\tilde{\mathbf{A}}_1^*$: a node of $\bar{B}$ with one of the branches vertical;
- $\tilde{\mathbf{A}}_2^*$: a cusp of $\bar{B}$ with vertical tangent;
- $\tilde{\mathbf{A}}_p$, $\tilde{\mathbf{D}}_q$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_7$, $\tilde{\mathbf{E}}_8$: a simple singular point of $\bar{B}$ of the same type with minimal possible local intersection index with the fiber.

For the relation to Kodaira's classification of singular elliptic fibers and a few other details, see Table 2; further details and references are found in [10].

Type of F		$j(F)$	Vertex	Valency
$\tilde{\mathbf{A}}_p(\tilde{\mathbf{D}}_{p+5})$, $p \geqslant 1$	I_{p+1} (I^*_{p+1})	∞	$\times$–	$p+1$
$\tilde{\mathbf{A}}_0^*(\tilde{\mathbf{D}}_5)$	I_1 (I^*_1)	∞	$\times$–	1
$\tilde{\mathbf{A}}_0^{**}(\tilde{\mathbf{E}}_6)$	II (IV*)	0	•–	1 mod 3
$\tilde{\mathbf{A}}_1^*(\tilde{\mathbf{E}}_7)$	III (III*)	1	○–	1 mod 2
$\tilde{\mathbf{A}}_2^*(\tilde{\mathbf{E}}_8)$	IV (II*)	0	•–	2 mod 3

Table 2: Types of singular fibers. Fibers of type $\tilde{\mathbf{A}}_0$ (Kodaira's I_0) are not singular; fibers of type $\tilde{\mathbf{D}}_4$ (Kodaira's I_0^*) are not detected by the j–invariant. Fibers of type $\tilde{\mathbf{A}}_0$ or $\tilde{\mathbf{D}}_4$ with complex multiplication of order 2 (respectively, 3) are over the ○–vertices of valency 0 mod 2 (respectively, over the •–vertices of valency 0 mod 3). The types shown parenthetically in the table are obtained from the corresponding $\tilde{\mathbf{A}}$–types by an elementary transformation (see Section 2.2); the pairs are not distinguishable by the j–invariant.

The type $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^*$, and $\tilde{\mathbf{A}}_2^*$ singular fibers of a trigonal curve are called *unstable*, and all other singular fibers are called *stable*. Informally, a fiber is unstable if its type does not need to be preserved under equisingular, but not necessarily fiberwise, deformations of the curve.

A trigonal curve is called *stable* if all its singular fibers are stable.

The *(functional) j–invariant* $j = j_{\bar{B}}\colon \mathbb{P}^1 \to \mathbb{P}^1$ of a trigonal curve $\bar{B} \subset \Sigma_2$ is defined as the analytic continuation of the function sending a point b in the base $\mathbb{P}^1$ of Σ_2 to the j–invariant (divided by 12^3) of the elliptic curve covering the fiber F over b and ramified at $F \cap (\bar{B} + E)$. The curve $\bar{B}$ is called *isotrivial* if $j_{\bar{B}} = \text{const}$. Such curves can easily be enumerated; see, eg, Degtyarev [10].

2.1.1 Definition A nonisotrivial trigonal curve $\bar{B}$ is called *maximal* if it has the following properties:

(1) $\bar{B}$ has no singular fibers of type $\tilde{\mathbf{D}}_4$;

(2) $j = j_{\bar{B}}$ has no critical values other than 0, 1, and ∞;

(3) each point in the pullback $j^{-1}(0)$ has ramification index at most 3;

(4) each point in the pullback $j^{-1}(1)$ has ramification index at most 2.

The maximality of a nonisotrivial trigonal curve $\bar{B} \subset \Sigma_2$ can easily be detected by applying the Riemann–Hurwitz formula to the map $j_{\bar{B}}\colon \mathbb{P}^1 \to \mathbb{P}^1$; it depends only on the (combinatorial) set of singular fibers of $\bar{B}$; see Degtyarev [10] for details. An alternative criterion, based on the total Milnor number, is given by Theorem 1.3.1 proved in this paper. The classification of such curves reduces to a combinatorial problem (see Theorem 2.3.1 below); a partial classification of maximal trigonal curves in Σ_2 is found in [7]. An important property of maximal trigonal curves is their rigidity [10]: any small deformation of such a curve $\bar{B}$ is isomorphic to $\bar{B}$.

2.2 Elementary transformations

An *elementary transformation* (sometimes, *Nagata elementary transformation*) of Σ_p is a birational transformation $\Sigma_p \dashrightarrow \Sigma_{p+1}$ consisting in blowing up a point P in the exceptional section of Σ_p followed by blowing down (the proper transform of) the fiber F through P. (In the sequel, we often omit the reference to "the proper transform of" when it is understood.) The inverse transformation $\Sigma_{p+1} \dashrightarrow \Sigma_p$ is also called an elementary transformation; it consists in blowing up a point P' *not* in the exceptional section of Σ_{p+1} followed by blowing down the fiber F' through P'.

Pick affine charts (x_p, y_p) in Σ_p and (x_{p+1}, y_{p+1}) in Σ_{p+1} so that the exceptional sections are given by $y_p = \infty$ and $y_{p+1} = \infty$, respectively, the fiber F to be blown down is given by $x_p = 0$, and the image of F is the origin $x_{p+1} = y_{p+1} = 0$. Then, under the appropriate choice of (x_{p+1}, y_{p+1}), the elementary transformation is the change of coordinates

$$x_p = x_{p+1}, \qquad y_p = y_{p+1}/x_{p+1}. \tag{2.2.1}$$

Let $\bar{B} \subset \Sigma_k$ be a *generalized trigonal curve*, ie, a curve intersecting each generic fiber of the ruling at three points but possibly *not* disjoint from the exceptional section E. Then, by a sequence of elementary transformations, one can resolve the points of intersection of $\bar{B}$ and E and obtain a true trigonal curve $\bar{B}' \subset \Sigma_{k'}$, $k' \geqslant k$, birationally equivalent to $\bar{B}$. Alternatively, given a trigonal curve $\bar{B} \subset \Sigma_k$ with triple singular points, one can apply a sequence of elementary transformations to obtain a trigonal curve $\bar{B}' \subset \Sigma_{k'}$, $k' \leqslant k$, birationally equivalent to $\bar{B}$ and with $\tilde{\mathbf{A}}$ type singular fibers only.

The j–invariant $j_{\bar{B}}$, being defined as an analytic continuation, does not change under elementary transformations. One can use this observation to define the j–invariant and all related objects (see next section) for generalized trigonal curves, as well as for trigonal curves with triple points, not necessarily simple.

2.3 Dessins and skeletons

The concept of the dessin of a trigonal curve is a modification of Grothendieck's idea of *dessin d'enfant*; it is due to S Orevkov [22], with a further development in [11; 10].

The *dessin* $\Gamma_{\bar{B}}$ of a nonisotrivial trigonal curve $\bar{B} \subset \Sigma_k$ is defined as the planar map $j_{\bar{B}}^{-1}(\mathbb{P}^1_{\mathbb{R}}) \subset S^2 = \mathbb{P}^1$, enhanced with the following decorations: the pullbacks of 0, 1, and ∞ are called, respectively, $\bullet$–, $\circ$–, and $\times$–vertices of $\Gamma_{\bar{B}}$, and the connected components of the pullbacks of $(0,1)$, $(1,+\infty)$, and $(-\infty,0)$ are called, respectively, bold, dotted, and solid edges of $\Gamma_{\bar{B}}$. Clearly, the dessin is invariant under elementary transformations of the curve and, up to elementary transformation and isomorphism, the dessin determines the curve uniquely (see, eg, Degtyarev [10]; it is essential that the dessin is considered in the topological sphere S^2; the analytic structure on S^2 is recovered using the Riemann existence theorem).

The relation between the vertices of the dessin $\Gamma_{\bar{B}}$ and the singular fibers of $\bar{B}$ is shown in Table 2 (see also Convention 2.3.2 concerning the valencies). The $\bullet$–vertices of valency $0 \bmod 3$ and $\circ$–vertices of valency $0 \bmod 2$ are called *nonsingular*; the other $\bullet$– and $\circ$–vertices are called *singular*, as they correspond to singular fibers of the curve.

The *skeleton* $\mathrm{Sk}_{\bar{B}}$ of a trigonal curve $\bar{B}$ is the planar map obtained from the dessin $\Gamma_{\bar{B}}$ by removing all $\times$–vertices and solid and dotted edges and ignoring all bivalent $\circ$–vertices. (Thus, $\mathrm{Sk}_{\bar{B}}$ is Grothendieck's *dessin d'enfant* $j_{\bar{B}}^{-1}([0,1])$, with the bivalent pullbacks of 1 ignored.) Skeletons are especially useful in the study of maximal curves. The skeleton $\mathrm{Sk}_{\bar{B}}$ of any *maximal* curve $\bar{B}$ has the following properties:

(1) $\mathrm{Sk}_{\bar{B}}$ is connected;

(2) each $\bullet$–vertex of $\mathrm{Sk}_{\bar{B}}$ has valency 1, 2, or 3; each $\circ$–vertex has valency 1 and is connected to a $\bullet$–vertex.

Conversely, any planar map $\mathrm{Sk}_{\bar{B}} \subset S^2$ satisfying (1), (2) above extends to a unique, up to orientation preserving diffeomorphism of S^2, dessin of a *maximal* trigonal curve: one inserts a $\circ$–vertex in the middle of each edge connecting two $\bullet$–vertices, places a $\times$–vertex u_R inside each region R of $\mathrm{Sk}_{\bar{B}}$, and connects u_R by disjoint solid (dotted) edges to all $\bullet$– (respectively, $\circ$–) vertices in the boundary ∂R.

According to the following theorem, proved in [10], skeletons classify maximal trigonal curves. For further applications of this concept, see Section 4.2 and Degtyarev [10].

2.3.1 Theorem *The fiberwise deformation classes (or, equivalently, isomorphism classes) of maximal trigonal curves $\bar{B} \subset \Sigma_k$ with t triple points are in a one-to-one correspondence with the orientation preserving diffeomorphism classes of skeletons (ie, planar maps $\mathrm{Sk} \subset S^2$ satisfying conditions (1), (2) above) satisfying the count given by Corollary 2.5.5.* □

2.3.2 Convention It is important to emphasize that the skeleton Sk is merely a convenient way to encode the dessin Γ of a maximal curve; Sk is a subgraph of Γ, with some vertices and edges removed and some vertices ignored. For this reason, in the further exposition we freely switch between skeletons and dessins; if only a skeleton is given, we extend it to the dessin of a maximal curve as explained above. Convenient in general, this practice may cause a confusion concerning the valencies of the vertices. To avoid this confusion, *by the valency of a vertex v we mean one half of the conventional valency of v regarded as a vertex of Γ*. In other words, we only count the edges of one of the two kinds present at v. The number thus defined is the conventional valency of v in Sk (if v *is* a vertex of Sk); it is also equal to the ramification index of $j_{\bar{B}}$ at v.

2.4 Markings

Recall that a *marking* at a trivalent $\bullet$–vertex v of a skeleton Sk (or dessin Γ) is a counterclockwise ordering $\{e_1, e_2, e_3\}$ of the three edges (respectively, bold edges) attached to v. A marking is uniquely defined by assigning index 1 to one of the three edges. Given a marking, the indices of the edges are considered defined modulo 3, so that $e_4 = e_1$, $e_5 = e_2$, etc.

A marking at v defines as well an ordering $\{e'_1, e'_2, e'_3\}$ of the three solid edges attached to v: we let e'_i to be the solid edge opposite to e_i.

A *marking* of the skeleton Sk is a collection of markings at all its trivalent $\bullet$–vertices. Given a marking, one can assign a type $[i, j]$, $i, j \in \mathbb{Z}_3$, to each edge e of Sk connecting two trivalent vertices, according to the indices of the two ends of e. A marking of a skeleton without singular $\bullet$–vertices is called *splitting* if it satisfies the following two conditions:

(1) the types of all edges are $[1,1]$, $[2,3]$ or $[3,2]$;

(2) an edge connecting a $\bullet$–vertex v and a singular $\circ$–vertex has index 1 at v.

The following statement is proved in [10].

2.4.1 Proposition *A maximal trigonal curve $\bar{B} \subset \Sigma_k$ is reducible if and only if its skeleton has no singular $\bullet$–vertices and admits a splitting marking. (Moreover, each splitting marking defines a component of $\bar{B}$ that is a section of Σ_k.)* □

2.4.2 Remark Proposition 2.4.1 is proved by reducing the braid monodromy (see Section 4.2 below) to the symmetric group $\mathbb{S}_3$. A marking at a vertex v gives rise to a natural ordering $\{p_1, p_2, p_3\}$ of the three points of the intersection $F_v \cap \bar{B}$, where F_v is the fiber over v, and to a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ for the fundamental group π_1 of the curve; see Section 4.2 or Degtyarev [10] for more details. For a splitting marking, the point p_1 over each vertex v belongs to a separate component, and in the abelianization $\pi_1/[\pi_1, \pi_1]$ there is no relation of the form $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_3$. The latter observation, combined with the relation at infinity (see Section 4.3 below) gives one an easy way to find the degree of the corresponding component.

2.5 The vertex count

Given a nonisotrivial trigonal curve $\bar{B}$, denote by $\#_*$ the total number of $*$–vertices (where $*$– stands for either $\bullet$–, or $\circ$–, or $\times$–) in the dessin of $\bar{B}$, let $\#_*(i)$, $i \in \mathbb{N}$, be the number of $*$–vertices of valency i, and let $\#_*(i \bmod N)$, $i \in \mathbb{Z}_N$, be the number of $*$–vertices of valency $i \bmod N$.

Assume that $\bar{B} \subset \Sigma_k$ has double singular points only. Then one has (see Degtyarev [10])

$$\deg j_{\bar{B}} = \sum_{i>0} i\#_\bullet(i) = \sum_{i>0} i\#_\circ(i) = \sum_{i>0} i\#_\times(i), \tag{2.5.1}$$

$$6k = \deg j_{\bar{B}} + 2\#_\bullet(1 \bmod 3) + 3\#_\circ(1 \bmod 2) + 4\#_\bullet(2 \bmod 3), \tag{2.5.2}$$

$$\#_\bullet + \#_\circ + \#_\times \geqslant \deg j_{\bar{B}} + 2, \tag{2.5.3}$$

the latter inequality turning into an equality if and only if $j_{\bar{B}}$ has no critical values other than 0, 1, and ∞, ie, if $\bar{B}$ satisfies condition (2) of Definition 2.1.1.

2.5.4 Remark In [11], the number $3k$ is called the *degree* $\deg \Gamma_{\bar{B}}$ of the dessin. (The reason is the fact that generic dessins of degree 3 correspond to plane cubics, regarded as trigonal curves in Σ_1.) We define the degree of a skeleton as the degree of its extension to the dessin of a maximal curve. In general, for a curve $\bar{B} \subset \Sigma_k$ with t

(simple) triple points, one has $\deg\Gamma_{\bar B}=3(k-t)$, cf Section 2.2. In this notation, one can replace $6k$ with $2\deg\Gamma_{\bar B}$ in (2.5.2) and lift the assumption that $\bar B$ should have double singular points only.

The next statement is an immediate consequence of (2.5.1)–(2.5.3).

2.5.5 Corollary *Let $\bar B\subset\Sigma_k$ be a maximal trigonal curve. Then the numbers of vertices in its skeleton are subject to the identity*

$$\#_\bullet+\#_\circ(1)+\#_\bullet(2)=2(k-t),$$

where t is the number of triple singular points of $\bar B$.

Proof By a sequence of elementary transformations one can convert $\bar B$ to another maximal trigonal curve $\bar B'\subset\Sigma_{k-t}$ (see Section 2.2), which has double singular points only and has the same dessin as B. Then, it suffices to substitute to (2.5.2) the first expression for $\deg j_{\bar B}$ from (2.5.1), collect (partially for $i=2$) the terms with $\#_\bullet(i)$, $i=1,2,3$, and divide by 3. $\square$

2.6 Proof of Theorem 1.3.1

First, assume that $\bar B$ has double singular points only. Then

$$\mu(\bar B)=\sum_{i>0}(i-1)\#_\times(i)+\#_\circ(1\bmod 2)+2\#_\bullet(2\bmod 3)$$

(see Table 2). Substituting to (2.5.3) the third expression for $\deg j_{\bar B}$ from (2.5.1), one obtains the estimate

$$\mu(\bar B)\leqslant\#_\bullet+\#_\circ+\#_\circ(1\bmod 2)+2\#_\bullet(2\bmod 3)-2,\tag{2.6.1}$$

which is sharp if and only if $\bar B$ satisfies condition (2) of Definition 2.1.1. Substituting to (2.5.2) the first two expressions for $\deg j_{\bar B}$ and replacing the valencies of $\bullet$– and $\circ$–vertices with their residues modulo 3 (in the range $\{1,2,3\}$) and 2 (in the range $\{1,2\}$), respectively, one obtains the inequalities

$$2k\geqslant\#_\bullet+\#_\circ(1\bmod 2)+\#_\bullet(2\bmod 3),\tag{2.6.2}$$

$$3k\geqslant\#_\circ+\#_\bullet(1\bmod 3)+\#_\circ(1\bmod 2)+2\#_\bullet(2\bmod 3),\tag{2.6.3}$$

which turn into equalities if and only if the dessin of $\bar B$ has no $\bullet$–vertices of valency greater than 3 (for (2.6.2)) or no $\circ$–vertices of valency greater than 2 (for (2.6.3)), ie, if $\bar B$ satisfies conditions (3) or (4) of Definition 2.1.1, respectively. Combining this with (2.6.1) and taking into account that $\#_\bullet(1\bmod 3)+\#_\circ(1\bmod 2)+\#_\bullet(2\bmod 3)$

is the number of unstable fibers of $\bar{B}$, one obtains the desired inequality. The equality holds if and only if $\bar{B}$ is maximal, as the remaining condition (1) of Definition 2.1.1 holds automatically.

If $\bar{B}$ has triple points, one can remove them one by one and use induction. Let $\bar{P}$ be a triple point of $\bar{B}$ and let $\bar{B}' \subset \Sigma_{k-1}$ be the curve obtained from $\bar{B}$ by the inverse elementary transformation centered at $\bar{P}$. If $\bar{P}$ is of type $\mathbf{D}_4$ (and hence $\bar{B}$ is not maximal), then $\mu(\bar{B}) = \mu(\bar{B}')+4$ and the inequality for $\bar{B}'$ turns into a strict inequality for $\bar{B}$. In all other cases, $\bar{B}$ and $\bar{B}'$ are or are not maximal simultaneously. If $\bar{P}$ is of type $\mathbf{D}_p$, $p \geqslant 5$, then $\mu(\bar{B}) = \mu(\bar{B}') + 5$ and the unstable fibers of $\bar{B}'$ are in a one-to-one correspondence with those of $\bar{B}$. If $\bar{P}$ is of type $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$, then $\mu(\bar{B}) = \mu(\bar{B}') + 6$ and, on the other hand, $\bar{B}'$ has one extra unstable fiber compared to $\bar{B}$ (of type $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^{*}$, or $\tilde{\mathbf{A}}_2^{*}$, respectively). In both cases, the defect in the inequality for $\bar{B}'$ is the same as the defect in the resulting inequality for $\bar{B}$, and the statement follows. □

3 Plane sextics

The bulk of this section deals with the trigonal models of plane sextics, which are the principal tool in both the classification and the computation of the fundamental group. To facilitate the classification, we also prove Proposition 3.1.1, restricting the sets of singularities of irreducible maximal sextics.

3.1 Irreducible maximal sextics

Recall that a plane sextic B is called *maximal* if its total Milnor number $\mu(B)$ takes the maximal possible value 19.

3.1.1 Proposition *An irreducible maximal plane sextic cannot have a singular point of type* $\mathbf{D}_{2k}$, $k \geqslant 2$ *or more than one singular point from the following list:* $\mathbf{A}_{2k+1}$, $k \geqslant 0$, $\mathbf{D}_{2k+1}$, $k \geqslant 2$, *or* $\mathbf{E}_7$.

Proof Formally, one can derive the statement from Yang's list [27] of the sets of singularities realized by irreducible maximal sextics. For a more conceptual proof, consider the double covering of the plane ramified at the sextic and denote by X its minimal resolution of singularities. It is a $K3$ surface. Let $L = H_2(X)$, let $\Sigma \subset L$ be the sublattice spanned by the classes of the exceptional divisors contracted by the projection $X \to \mathbb{P}^2$, and let $S = \Sigma \oplus \langle h \rangle$, where h is the class realized by the pullback of a generic line. One has $h^2 = 2$ and Σ is the direct sum of (negative definite) irreducible root systems of the same type ($\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$) as the singular points of the sextic. Let $\widetilde{S} \supset S$ be the primitive hull of S in L. As shown in [6], the quotient $\widetilde{S}/S$ is free of 2–torsion. Hence, the 2–torsion of the discriminant groups $\operatorname{discr} S = \operatorname{discr} \Sigma \oplus \mathbb{Z}_2$ and $\operatorname{discr} \widetilde{S}$ coincide. On the other hand, $\operatorname{discr} \widetilde{S} \cong \operatorname{discr} S^\perp$ and, since $\operatorname{rk} S^\perp = 2$, the 2–torsion of $\operatorname{discr} \Sigma$ must be a cyclic group. □

3.2 Trigonal models: the statements

In Propositions 3.2.1, 3.2.2 and 3.2.4, we introduce certain trigonal curves birationally equivalent to plane sextics; these curves will be called the *trigonal models* of the corresponding sextics. Proofs are given in Sections 3.3–3.5 below.

3.2.1 Proposition *There is a natural bijection ϕ, invariant under equisingular deformations, between the following two sets:*

(1) *plane sextics B with a distinguished type $\mathbf{E}_8$ singular point P, and*

(2) *trigonal curves $\bar{B} \subset \Sigma_3$ with a distinguished type $\tilde{\mathbf{A}}_1^*$ singular fiber F.*

A sextic B is irreducible if and only if so is $\bar{B} = \phi(B)$ and, with one exception, B is maximal if and only if $\bar{B}$ is maximal and has no unstable fibers other than F. (The exception is the reducible sextic B with the set of singularities $\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{D}_4$; in this case, $\phi(B)$ is isotrivial.) Furthermore, for each pair B, $\bar{B} = \phi(B)$, there is a diffeomorphism

$$\mathbb{P}^2 \smallsetminus (B \cup L) \cong \Sigma_3 \smallsetminus (\bar{B} \cup E \cup F),$$

where L is the line tangent to B at P and E is the exceptional section.

3.2.2 Proposition *There is a natural bijection ϕ, invariant under equisingular deformations, between Zariski open (in each equisingular stratum) subsets of the following two sets:*

(1) *plane sextics B with a distinguished type $\mathbf{E}_7$ singular point P and without linear components through P, and*

(2) *trigonal curves $\bar{B} \subset \Sigma_3$ with a distinguished type $\tilde{\mathbf{A}}_1$ singular fiber F and a distinguished branch at the corresponding type $\mathbf{A}_1$ singular point of $\bar{B}$.*

A sextic B is irreducible if and only if so is $\bar{B} = \phi(B)$, and B is maximal if and only if $\bar{B}$ is maximal and stable. Furthermore, for each pair B, $\bar{B} = \phi(B)$, there is a diffeomorphism

$$\mathbb{P}^2 \smallsetminus (B \cup L) \cong \Sigma_3 \smallsetminus (\bar{B} \cup E \cup F),$$

where L is the line tangent to B at P and E is the exceptional section.

3.2.3 Remark Thus, one should expect that, in many cases, a maximal stable pair $(\bar{B}, F)$ as in Proposition 3.2.2(2) would correspond to *two* deformation classes of sextics. This is indeed the case; see the sets of singularities marked with a * in Table 3 in Section 6.1. Arithmetically, this phenomenon is probably due to the fact that the discriminant group $\operatorname{discr} S$ has two essentially different copies of $\langle\frac{1}{2}\rangle$, namely, those coming from $\operatorname{discr}\mathbf{E}_7$ and from $\operatorname{discr}\langle h\rangle$. For details, see Degtyarev [6].

3.2.4 Proposition *There is a natural bijection ϕ, invariant under equisingular deformations, between Zariski open (in each equisingular stratum) subsets of the following two sets:*

(1) *plane sextics B with a distinguished type $\mathbf{E}_6$ singular point P, and*

(2) *trigonal curves $\bar{B} \subset \Sigma_4$ with a distinguished type $\tilde{\mathbf{A}}_5$ singular fiber F.*

A sextic B is irreducible if and only if so is $\bar{B} = \phi(B)$, and B is maximal if and only if $\bar{B}$ is maximal and stable. Furthermore, for each pair B, $\bar{B} = \phi(B)$, there is a diffeomorphism

$$\mathbb{P}^2 \smallsetminus (B \cup L) \cong \Sigma_4 \smallsetminus (\bar{B} \cup E \cup F),$$

where L is the line tangent to B at P and E is the exceptional section.

3.2.5 Remark There are statements similar to Propositions 3.2.1, 3.2.2, and 3.2.4 for sextics with a distinguished type $\mathbf{D}$ singular point. In this case, one would need to keep track of two (three in the case $\mathbf{D}_4$) singular fibers of $\bar{B}$.

3.2.6 Remark Informally, the relation between maximal sextics and maximal trigonal curves follows from the fact that both objects are rigid, ie, curves are isomorphic to their small equisingular deformations. Formal proofs are given below.

3.3 Proof of Proposition 3.2.1

The bijection ϕ and the diffeomorphism in the statement are given by a birational transformation $\mathbb{P}^2 \dashrightarrow \Sigma_3$, so that $\bar{B}$ is the proper transform of B: one blows up the distinguished type $\mathbf{E}_8$ point P to get a generalized trigonal curve $B' \subset \Sigma_1$ with a cusp tangent to the exceptional section of Σ_1 (the exceptional divisor of the blow-up), and then one applies two elementary transformations to make the curve disjoint from the exceptional section; see Section 2.2.

Pick affine coordinates (u, v) in $\mathbb{P}^2$ centered at the distinguished singular point P and with the v–axis along the line L in the statement. By the Bézout theorem, B

intersects L at one more point $v=a\ne 0$. Hence, up to higher order terms, B is given by a polynomial of the form

$$(u^3-v^5)(v-a). \tag{3.3.1}$$

In appropriate affine coordinates $(x,y)=(x_3,y_3)$ in Σ_3, cf Section 2.2, the transformation is given by the coordinate change

$$u=x^3/y,\qquad v=x^2/y, \tag{3.3.2}$$

(in particular, it restricts to a biholomorphism $\mathbb{P}^2\smallsetminus L\to\Sigma_3\smallsetminus(E\cup F)$), and the proper transform $\bar B$ of B is given by

$$(y^2-x)(x^2-ay). \tag{3.3.3}$$

One can see that $F=\{x=0\}$ is a type $\tilde{\mathbf{A}}_1^*$ singular fiber of $\bar B$.

The construction is obviously invertible: given a trigonal curve $\bar B\subset\Sigma_3$ with a type $\tilde{\mathbf{A}}_1^*$ singular fiber F, one can apply two elementary transformations centered at (the transform of) the branch of $\bar B$ not tangent to F and then blow down the exceptional section of the resulting Hirzebruch surface Σ_1; the result is a sextic with a type $\mathbf{E}_8$ singular point.

Since the curves $\bar B$ and B are proper transforms of each other, it is immediate that $\bar B$ is irreducible if and only if so is B. The assertion on maximal curves follows from Theorem 1.3.1. Indeed, the total Milnor numbers of B and $\bar B$ are related via $\mu(\bar B)=\mu(B)-7$: the singular points of B are in a one-to-one correspondence with those of $\bar B$, which are of the same type, except that the type $\mathbf{E}_8$ point P corresponds to the type $\mathbf{A}_1$ singular point of $\bar B$ in the fiber F. Hence, B is maximal if and only if $\mu(\bar B)=12$. If $\bar B$ is not isotrivial (the isotrivial case is treated in the next paragraph), then, taking into account the fact that $\bar B$ does have an unstable fiber F, Theorem 1.3.1 implies that the latter equality holds if and only if $\bar B$ is maximal and has no other unstable fibers.

If $\bar B$ is isotrivial, then $j_{\bar B}\equiv 1$ (as $j_F=1$; see Table 2) and, in appropriate affine coordinates (x,y), the curve is given by the Weierstraß equation of the form $y^3-yxp(x)=0$, $\deg p=5$ and $p(0)\ne 0$. (We assume that $x=0$ is the distinguished type $\tilde{\mathbf{A}}_1^*$ fiber F.) Such a curve has singular points of types $\mathbf{A}_1$, $\mathbf{D}_4$, and $\mathbf{E}_7$ (corresponding, respectively, to simple, double, and triple roots of the equation $xp(x)=0$), and the only such set of singularities with the total Milnor number 12 is $\mathbf{E}_7\oplus\mathbf{D}_4\oplus\mathbf{A}_1$, the type $\mathbf{A}_1$ point being located in F. (This set of singularities cannot be realized by a maximal nonisotrivial curve; see Definition 2.1.1(1).) All other statements are straightforward. □

3.4 Proof of Proposition 3.2.2

As in the previous proof, the bijection ϕ and the diffeomorphism are given by a birational transformation $\mathbb{P}^2 \dashrightarrow \Sigma_3$: one blows up the distinguished type $\mathbf{E}_7$ point P to get a generalized trigonal curve $B' \subset \Sigma_1$ that has a node with one of the branches tangent to the exceptional section of Σ_1; then, two elementary transformations centered at this branch produce a trigonal curve $\bar{B} \subset \Sigma_3$; see Section 2.2. In appropriate affine coordinates (u, v) in $\mathbb{P}^2$ and (x, y) in Σ_3, such that L is the v–axis and F is the y–axis, the transformation is given by (3.3.2). Up to higher order terms, the defining polynomial of a typical (see below) sextic B as in the statement has the form

$$(u^2 - v^3)(u - bv^2)(v - a), \tag{3.4.1}$$

$a, b = \text{const}$ (the smooth branch of B at P is tangent to L and B intersects L at one more point $v = a$), and the transform $\bar{B}$ of B is given by the polynomial

$$(y - 1)(y - bx)(x^2 - ay). \tag{3.4.2}$$

One can see that $F = \{x = 0\}$ is a type $\tilde{\mathbf{A}}_1$ singular fiber of $\bar{B}$ and $\bar{P} = (0, 0)$ is a type $\mathbf{A}_1$ singular point. The branch $x^2 = ay$ of $\bar{B}$ at $\bar{P}$ is the transform of the "separate" branch $v = a$ of B; thus, it is distinguished.

The inverse construction consists in applying two elementary transformations centered at (the transform of) the distinguished branch of $\bar{B}$ at $\bar{P}$, followed by blowing down the exceptional section of the resulting Hirzebruch surface Σ_1.

The assertion on the correspondence between irreducible and maximal curves is proved similar to Section 3.3. This time, one has $\mu(\bar{B}) = \mu(B) - 6$ (the type $\mathbf{E}_7$ singular point P is replaced with the type $\mathbf{A}_1$ singular point $\bar{P}$); hence, B is maximal if and only if $\mu(\bar{B}) = 13$, and Theorem 1.3.1 implies that the latter identity holds if and only if $\bar{B}$ is maximal and stable. Note that $\bar{B}$ cannot be isotrivial, as it has a singular fiber F of type $\tilde{\mathbf{A}}_1$ and $j_{\bar{B}}(F) = \infty$; see Table 2.

It remains to show that ϕ is defined on a Zariski open subset of each equisingular stratum. The only extra degeneration that a sextic B within a given stratum may have is that the smooth branch of B at P may become inflection tangent to L. Then, the singular fiber F of $\bar{B}$ is of type $\tilde{\mathbf{A}}_2$ rather than $\tilde{\mathbf{A}}_1$. However, from the theory of trigonal curves it follows that such a fiber can be perturbed to a fiber of type $\tilde{\mathbf{A}}_1$ and a close fiber of type $\tilde{\mathbf{A}}_0^*$; this perturbation can obviously be followed by a one-parameter family of inverse birational transformations $\Sigma_3 \dashrightarrow \mathbb{P}^2$ and hence by an equisingular deformation of sextics. □

3.4.3 Remark At the end of the proof, we essentially show, using deformations of trigonal curves, that a line inflection tangent to the smooth branch of a type $\mathbf{E}_7$ singular point of a plane sextic cannot be stable under equisingular deformations of the sextic. Alternatively, one can argue that, if such a line were stable, it would be a Z–*splitting curve* in the sense of Shimada [25], and refer to the classification of Z–splitting curves found in [25]. A similar observation applies as well to the end of the proof in Section 3.5.

3.5 Proof of Proposition 3.2.4

The bijection ϕ and the diffeomorphism of the complements are given by a birational transformation $\mathbb{P}^2 \dashrightarrow \Sigma_4$: one blows up the distinguished point P to obtain a generalized trigonal curve $B' \subset \Sigma_1$ with a branch inflection tangent to the exceptional section of Σ_1, and then applies three elementary transformations centered at (the transforms of) this branch to make the curve disjoint from the exceptional section; see Section 2.2.

In appropriate affine coordinates (u, v) in $\mathbb{P}^2$ and (x, y) in Σ_3, such that L is the v–axis and F is the y–axis, the transformation is given by

$$u = x^4/y, \qquad v = x^3/y. \tag{3.5.1}$$

A typical (see below) sextic B as in the statement intersects L at two other points $v = a_1$, $v = a_2$, $a_1 \neq a_2$, $a_1, a_2 \neq 0$. Hence, up to higher order terms, its defining polynomial has the form

$$(u^3 - v^4)(v - a_1)(v - a_2), \tag{3.5.2}$$

and its transform $\bar{B} \subset \Sigma_4$ is given by the polynomial

$$(y - 1)(x^3 - a_1 y)(x^3 - a_2 y). \tag{3.5.3}$$

One can see that $F = \{x = 0\}$ is a type $\tilde{\mathbf{A}}_5$ singular fiber of $\bar{B}$. The inverse correspondence is given by three elementary transformations centered at (the transforms of) the type $\mathbf{A}_5$ singular point, followed by blowing down the exceptional section of the resulting Hirzebruch surface Σ_1.

The correspondence between irreducible and maximal curves is established as above: one has $\mu(\bar{B}) = \mu(B) - 1$ (the type $\mathbf{E}_6$ singular point P is replaced with a type $\mathbf{A}_5$ singular point of $\bar{B}$); hence, B is maximal if and only if $\mu(\bar{B}) = 18$, and Theorem 1.3.1 implies that the latter identity holds if and only if $\bar{B}$ is maximal and stable. Note that $\bar{B}$ cannot be isotrivial, as $j_{\bar{B}}(F) = \infty$; see Table 2.

The only extra degeneration that a sextic B may have within a given equisingular stratum is that it may become tangent to L or one of its type $\mathbf{A}_p$, $p \geqslant 1$, singular points may slide into L. Then, the fiber F of the transform $\bar{B}$ is of type $\tilde{\mathbf{A}}_6$ or $\tilde{\mathbf{A}}_{6+p}$, respectively. Such a fiber can be perturbed to a fiber of type $\tilde{\mathbf{A}}_5$ and a close fiber of type $\tilde{\mathbf{A}}_0^*$ or $\tilde{\mathbf{A}}_p$, respectively, and this perturbation is followed by an equisingular deformation of sextics. Thus, the bijection ϕ is well defined on a Zariski open subset of each stratum. (An alternative proof of this fact is explained in Remark 3.4.3.) □

4 The fundamental group

We outline the strategy used to compute the fundamental groups, explain how the braid monodromy can be found, and compute a few "universal" relations, present in the group of any curve in question.

4.1 The strategy

In this section, we consider a plane sextic B with a distinguished type $\mathbf{E}$ singular point P and use Propositions 3.2.1, 3.2.2, and 3.2.4 to transform it to a trigonal curve $\bar{B} \subset \Sigma_k$, $k = 3$ or 4, with a distinguished singular fiber F. (We assume B generic in its equisingular deformation class.) In each case, $\bar{B}$ has a unique singular point in F; it will be denoted by $\bar{P}$. The above cited propositions give a diffeomorphism

$$\mathbb{P}^2 \smallsetminus (B \cup L) \cong \Sigma_k \smallsetminus (\bar{B} \cup E \cup F),$$

where L is the line tangent to B at P. Hence, there is an isomorphism

$$\pi_1(\mathbb{P}^2 \smallsetminus (B \cup L)) \cong \pi_1(\Sigma_k \smallsetminus (\bar{B} \cup E \cup F))$$

of the fundamental groups. According to E R van Kampen [20] (see also Fujita [17]), the passage from $\pi_1(\mathbb{P}^2 \smallsetminus (B \cup L))$ to $\pi_1(\mathbb{P}^2 \smallsetminus B)$ results in adding an extra relation, which can be represented in the form $[\partial\Gamma] = 1$, where $\Gamma \subset \mathbb{P}^2$ is a small holomorphic disk transversal to L and disjoint from B, and $[\partial\Gamma] \subset \pi_1(\mathbb{P}^2 \smallsetminus (B \cup L))$ is the class of the boundary of Γ (more precisely, its conjugacy class). Denoting by $\bar{\Gamma}$ the image of Γ in Σ_k, one has

$$\pi_1(\mathbb{P}^2 \smallsetminus B) \cong \pi_1(\Sigma_k \smallsetminus (\bar{B} \cup E \cup F))/[\partial\bar{\Gamma}]. \tag{4.1.1}$$

The relation $[\partial\bar{\Gamma}] = 1$ is called the *relation at infinity*; the bulk of this section deals with computing this relation.

The group $\pi_1(\Sigma_k \smallsetminus (\bar{B} \cup E \cup F))$ is computed using the classical Zariski–van Kampen method [20]. Pick some coordinates (x', y') in the affine chart $\Sigma_k \smallsetminus (E \cup F)$. For the further exposition, it is convenient to take

(4.1.2) $$x' = 1/x, \qquad y' = y/x^k,$$

where (x, y) are the coordinates about F introduced in Section 3.3–Section 3.5. Let $F_1, \dots, F_r$ be all singular fibers of $\bar{B}$ other than F, and let F_0 be a nonsingular fiber. Pick a closed topological disk Δ in the x–axis containing all F_j, $j = 0, \dots, r$, in its interior and let $\Delta^\circ = \Delta \smallsetminus \{F_1, \dots, F_r\}$. (We identify fibers with their projections to the base of the ruling.) Pick a topological section $s\colon \Delta \to \Sigma_k$ proper in the sense of [4]. (For all practical purposes, it suffices to consider a constant section $y = c$, where c is a constant, $|c| \gg 0$. In [4], one can find a more formal exposition; in particular, it is shown there that the result does not depend on the choice of a proper section. A similar approach is found in [1].) Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis for the free group $\pi_F := \pi_1(F_0 \smallsetminus (\bar{B} \cup E), s(F_0))$, and let $\gamma_1, \dots, \gamma_r$ be a basis for the group $\pi_1(\Delta^\circ, F_0)$. Dragging the nonsingular fiber along a loop γ_j, $j = 1, \dots, r$ and keeping the base point in s, one obtains an automorphism $m_j \in \operatorname{Aut} \pi_F$, which is called the *braid monodromy* along γ_j. (Since the reference section is proper, this automorphism is indeed a braid.) In this notation, the Zariski–van Kampen theorem states that

(4.1.3) $$\pi_1(\Sigma_k \smallsetminus (\bar{B} \cup E \cup F)) = \bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| m_j = \mathrm{id},\ j = 1, \dots, r \bigr\rangle,$$

where each *braid relation* $m_j = \mathrm{id}$, $j = 1, \dots, r$, should be understood as the triple of relations $m_j(\alpha_i) = \alpha_i$, $i = 1, 2, 3$.

4.1.4 Remark Since each m_j is a braid and thus preserves $\alpha_1\alpha_2\alpha_3$, it would suffice to keep the relations $m_j(\alpha_1) = \alpha_1$ and $m_j(\alpha_2) = \alpha_2$ only. Note however that, in a more advanced setting, the braid monodromy does not necessarily take values in the braid group, and all three relations should be kept. Besides, following the principle "the more relations the better", often it is more convenient to restate the braid relations in the form $m_j(\alpha) = \alpha$ for each $\alpha \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

We will also consider the *monodromy at infinity* m_∞, ie, the braid monodromy along the loop $\partial\Delta$ (assuming that the base point F_0 is chosen in $\partial\Delta$).

4.1.5 Proposition *Let $\{\gamma_1, \dots, \gamma_r\}$ be a basis for the free group $\pi_1(\Delta^\circ, F_0)$ such that $\gamma_1 \cdots \gamma_r = [\partial\Delta]$. Then the group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ has a presentation of the form*

$$\bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| m_j = \mathrm{id},\ j = 1, \dots, r,\ m_\infty = \mathrm{id},\ [\partial\bar{\Gamma}] = 1 \bigr\rangle.$$

Furthermore, in the presence of the last two relations, (any) one of the first r braid relations $m_j = \mathrm{id}$ can be omitted.

Proof The presentation is given by (4.1.1) and (4.1.3); the relation $m_\infty = \mathrm{id}$ holds since $m_\infty = m_1 \cdots m_r$. For the same reason, any monodromy m_{j_0} can be expressed in terms of m_∞ and the other monodromies m_j, $j \ne j_0$; hence, the corresponding relation can be omitted. □

4.1.6 Remark Note that, unlike, eg, [4], where the case of a nonsingular fiber at infinity is considered, here the relation $m_\infty = \mathrm{id}$ does *not* automatically follow from the relation at infinity. Both relations are computed in Sections 4.4–4.6 below.

4.1.7 Remark Usually, it is convenient to take for $\{\gamma_1, \dots, \gamma_r\}$ a *geometric* basis for the group of the punctured plane Δ°: each basis element is represented by the loop composed of the counterclockwise boundary of a small disk surrounding a puncture and a simple arc connecting this disk to the reference point; all disks and arcs are assumed pairwise disjoint except at the common reference point.

4.2 The braid monodromy

The braid monodromy of a trigonal curve $\bar{B}$ can be computed using its dessin (skeleton in the case of a maximal curve); below, we cite a few relevant results of [10].

Recall that the braid group $\mathbb{B}_3$ can be defined as $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$; it acts on the free group $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ via

$$\sigma_1\colon (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1\alpha_2\alpha_1^{-1}, \alpha_1, \alpha_3), \quad \sigma_2\colon (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1, \alpha_2\alpha_3\alpha_2^{-1}, \alpha_2).$$

Introduce also the element $\sigma_3 = \sigma_1^{-1}\sigma_2\sigma_1$ and consider the indices of σ_1, σ_2, σ_3 as residues modulo 3, so that $\sigma_4 = \sigma_1$ etc.

The center of $\mathbb{B}_3$ is generated by $(\sigma_1\sigma_2)^3$. We denote by $\bar{\beta}$ the image of a braid $\beta \in \mathbb{B}_3$ in the *reduced braid group* $\mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong \mathbb{Z}_2 * \mathbb{Z}_3$. A braid β is uniquely recovered from $\bar{\beta}$ and its *degree* $\deg\beta \in \mathbb{B}_3/[\mathbb{B}_3, \mathbb{B}_3] = \mathbb{Z}$. Recall that $\deg\sigma_i = 1$.

4.2.1 Remark To be consistent with [10], we use the left action of $\mathbb{B}_3$ on the free group $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$. It appears, however, that the right action is more suitable for the braid monodromy, as it makes the map $\pi_1(\Delta^\circ) \to \mathbb{B}_3$ a homomorphism rather than an antihomomorphism. One can check that, with one exception, all expressions involving braids are symmetric modulo the central element $(\sigma_1\sigma_2)^3 = (\sigma_2\sigma_1)^3$. Hence, the only change needed to pass to the right action is the definition of σ_3: it should be defined via $\sigma_1\sigma_2\sigma_1^{-1}$.

As in Section 4.1, we fix a disk $\Delta \subset \mathbb{P}^1$ and a proper section s over Δ. All vertices, paths, etc. below are assumed to belong to Δ.

For a trivalent $\bullet$–vertex v of the skeleton Sk of $\bar{B}$, let F_v be the fiber over v and let $\pi_v = \pi_1(F_v \smallsetminus (\bar{B} \cup E), s(v))$. A marking at v (see Section 2.4) gives rise to a natural ordering $\{p_1, p_2, p_3\}$ of the three points of the intersection $F_v \cap \bar{B}$ and, hence, to a *canonical basis* $\{\alpha_1, \alpha_2, \alpha_3\}$ for π_v (see Figure 1), which is well defined up to simultaneous conjugation of the generators by a power of $\alpha_1\alpha_2\alpha_3$, ie, up to the action of the central element $(\sigma_1\sigma_2)^3 \in \mathbb{B}_3$. (In the figure, α_i is a small loop about p_i, $i = 1, 2, 3$.)

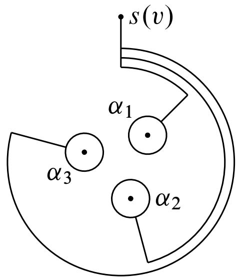

Figure 1: The canonical basis

We extend the notion of canonical basis to the star of v *in the dessin*, ie, to the bold and solid edges incident to v, extending to but not including the $\circ$– and $\times$–vertices. Over these edges, the three points $\{p_1, p_2, p_3\}$ still form a proper triangle (see Degtyarev [10]); hence, they are still ordered by the marking at v and one can construct the loops by combining radial segments and arcs of a large circle. (Alternatively, one can define this basis as the one obtained by translating a canonical basis over v along the corresponding edge of the dessin.)

Given two marked trivalent $\bullet$–vertices u and v of Sk, one can identify π_u and π_v by identifying the canonical bases defined by the marking. This identification is well defined up to the action of the center of $\mathbb{B}_3$ (as so are the canonical bases). If u and v are connected by a path γ, one can drag the nonsingular fiber along γ and define the braid monodromy $m_\gamma\colon \pi_u \to \pi_v$. Combining this with the identification above, one can define the element $\bar{m}_\gamma \in \mathbb{B}_3/(\sigma_1\sigma_2)^3$. In particular, this construction applies if u and v are connected by an edge e of Sk; depending on the type of the edge, $\bar{m}_e$ is given by the following expressions:

(4.2.2) $$\bar{m}_{[i,i+1]} = \bar{\sigma}_i, \quad \bar{m}_{[i+1,i]} = \bar{\sigma}_i^{-1} \quad \text{and} \quad \bar{m}_{[i,i]} = \bar{\sigma}_i\bar{\sigma}_{i-1}\bar{\sigma}_i.$$

Using these relations, one can compute the reduced monodromy $\bar{m}_\gamma$ for any path γ composed of edges of Sk connecting trivalent $\bullet$–vertices. If γ is a loop, the true monodromy m_γ is recovered from $\bar{m}_\gamma$ and the degree $\deg m_\gamma$, which equals the total multiplicity of the singular fibers of $\bar{B}$ encompassed by γ. (The multiplicity of a singular fiber F can be defined as the number of the simplest type $\tilde{\mathbf{A}}_0^*$ fibers into which F can split.)

Now, let v be a marked trivalent $\bullet$–vertex of the dessin of $\bar{B}$, and let u be the $\times$–vertex connected to v by the solid edge e_i'. Assume that the valency of u is d, so that the singular fiber F_u over u is of type $\tilde{\mathbf{A}}_{d-1}$ ($\tilde{\mathbf{A}}_0^*$ if $d=1$) or $\tilde{\mathbf{D}}_{d+4}$; see Table 2. Let γ be the loop composed of a small counterclockwise circle around u connected to v along e_i'. Then, in any canonical basis for π_v defined by the marking at v, the monodromy m_γ along γ is given by

$$
\begin{aligned}
m_\gamma &= \sigma_{i+1}^d, && \text{if } F_u \text{ is a type } \tilde{\mathbf{A}} \text{ fiber, or}\\
m_\gamma &= \sigma_{i+1}^d(\sigma_1\sigma_2)^3, && \text{if } F_u \text{ is a type } \tilde{\mathbf{D}} \text{ fiber.}
\end{aligned}
\tag{4.2.3}
$$

4.2.4 Remark It is obvious geometrically (and can easily be shown formally) that the reduced monodromy given by (4.2.2) along the boundary of a d–gonal region of the skeleton, when lifted to a braid of appropriate degree, coincides with the monodromy about a d–valent $\times$–vertex given by (4.2.3).

4.3 The relation at infinity and the monodromy at infinity

We keep using the notation of Section 4.1. In order to compute the relation at infinity $[\partial\bar{\Gamma}]=1$, assume that Δ is the closure of the projection of the disk $\bar{\Gamma}$ and that the reference fiber F_0 is chosen in $\partial\Delta$.

Dragging a nonsingular fiber along $\partial\bar{\Gamma}$ and keeping two points in s and $\partial\bar{\Gamma}$, one can define the monodromy m along $\partial\bar{\Gamma}$ as an automorphism of the relative homotopy set $\pi_1(F_0 \smallsetminus (\bar{B}\cup E), (F_0\cap\partial\bar{\Gamma})\cup s(F_0), s(F_0))$. Pick a path p connecting $s(F_0)$ to $F_0\cap\partial\bar{\Gamma}$ in $F_0\smallsetminus(\bar{B}\cup E)$. Then one has

$$p\cdot[\partial\bar{\Gamma}]\cdot m(p)^{-1}=[s(\partial\Delta)]^{-1}.$$

This relation holds in any reasonable fundamental group, eg, in the group of the complement of $(\bar{B}\cup E)$ in the pullback of $\partial\Delta$. Indeed, when dragged with the fiber, the arc p spans a square S shown in Figure 2, disjoint from $\bar{B}$ and E, and the product of the four paths forming the boundary ∂S (with appropriate orientations) is a null homotopic loop. Note that the counterclockwise directions of $\partial\bar{\Gamma}$ and $\partial\Delta$ (induced from the complex orientations of the respective disks) are opposite to each other.

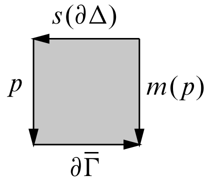

Figure 2: The square spanned by p

Since $[s(\partial\Delta)] = 1$ in $\pi_1(\Sigma_k \smallsetminus (\bar{B} \cup E \cup F))$ (the loop is contractible along $s(\Delta)$), the relation at infinity $[\partial\bar{\Gamma}] = 1$ takes the form

$$p \cdot m(p)^{-1} = 1.$$

The image $m(p)$ of a suitable arc p can easily be found using the local forms given by (3.3.3), (3.4.2), and (3.5.3). One can take for Γ a small disk in the line $v = d$, $d = \text{const}$, so that $\bar{\Gamma}$ is the disk

(4.3.1) $$\{|x| \leqslant \epsilon,\ y = x^{k-1}/d\},$$

and compute the monodromy along the loop $x = \epsilon \exp(2\pi i t)$, $t \in [0, 1]$. (Note that, in the coordinates (x, y), the "constant" section s: $x' \mapsto c = \text{const}$ is given by $x \mapsto cx^k$ (see (4.1.2)); hence, the base point makes k full turns about the origin.) We omit the details, merely stating the result in Sections Section 4.4–Section 4.6 below.

One can use the same local models to compute the monodromy at infinity m_∞. In other words, m_∞ is the local braid monodromy about F (in the clockwise direction) composed with $(\sigma_1\sigma_2)^{3k}$ (due to the fact that the base point makes k full turns about the origin). Below, we compute and simplify the group of relations $[\partial\bar{\Gamma}] = 1$, $m_\infty = \text{id}$, which are present in the fundamental group of any curve in question; see Proposition 4.1.5.

The results of the computation are stated in Sections 4.4–4.6. We take for the reference fiber F_0 the fiber F_v over a trivalent $\bullet$–vertex v of the dessin of $\bar{B}$ connected by an edge to the vertex u corresponding to the distinguished fiber F, and take for $\{\alpha_1, \alpha_2, \alpha_3\}$ a canonical basis in $\pi_F = \pi_v$ defined by a marking at v. The particular choice of the marking in each case is described below.

4.4 The case of type $\mathbf{E}_8$

Assume that P is of type $\mathbf{E}_8$, and hence F is of type $\tilde{\mathbf{A}}_1^*$; see Proposition 3.2.1. Let $\mathfrak{b}$ be the branch of $\bar{B}$ at $\bar{P}$ that is not vertical. Then $\bar{\Gamma}$ is an ordinary tangent to $\mathfrak{b}$

(see (3.3.3) and (4.3.1)), and the relation at infinity is

(4.4.1) $$\rho^3 = \alpha_1\alpha_2^2,$$

assuming that α_2 is represented by a loop about $\mathfrak{b}$ (so that the edge $[v,u]$ is e_2 at v). The monodromy at infinity is $m_\infty = (\sigma_1\sigma_2)^9(\sigma_1\sigma_2\sigma_1)^{-1}$, and the corresponding braid relations are

$$\alpha_1 = \rho^{-3}(\alpha_1\alpha_2)\alpha_3(\alpha_1\alpha_2)^{-1}\rho^3, \quad [\alpha_2, \rho^{-3}\alpha_1] = 1, \quad \alpha_3 = \rho^{-3}\alpha_1\rho^3.$$

In view of (4.4.1), the second relation becomes a tautology and the other two turn into

(4.4.2) $$\alpha_3 = \alpha_2\alpha_1\alpha_2^{-1} \quad \text{and} \quad [\alpha_1, \alpha_2^3] = 1.$$

In particular, α_2^3 is a central element.

4.5 The case of type $\mathbf{E}_7$

Assume that P is of type $\mathbf{E}_7$, and hence F is of type $\tilde{\mathbf{A}}_1$; see Proposition 3.2.1. Then $\bar{\Gamma}$ is tangent to the distinguished branch $\mathfrak{b}$ of $\bar{B}$ at $\bar{P}$. Unless stated otherwise, we will choose the basis α_1, α_2, α_3 so that

($*$) α_2 and α_3 are represented by loops about the two branches of $\bar{B}$ at $\bar{P}$ and α_2 corresponds to the distinguished branch $\mathfrak{b}$. In particular, $[v,u]$ is the edge e_1' at v.

(Occasionally, we will also consider the case when the generator corresponding to $\mathfrak{b}$ is α_3.) Then, the relation at infinity is

$$\rho^3 = \alpha_2\alpha_3\alpha_2 \quad \text{or} \quad \rho^3 = \alpha_2\alpha_3\alpha_2\alpha_3\alpha_2^{-1},$$

assuming that α_2 or, respectively, α_3 corresponds to $\mathfrak{b}$. The monodromy at infinity is $m_\infty = (\sigma_1\sigma_2)^9\sigma_2^{-2}$, the corresponding braid relations being

$$[\alpha_1, \rho^3] = 1 \quad \text{and} \quad [\alpha_i, \rho^{-3}(\alpha_2\alpha_3)] = 1,\ i = 2, 3.$$

Combining the last pair of relations with the relation at infinity, one concludes that (assuming that α_2 corresponds to $\mathfrak{b}$)

(4.5.1) $$[\alpha_2, \alpha_3] = 1 \quad \text{and} \quad [\alpha_i, \rho^3] = [\alpha_i, \alpha_2^2\alpha_3] = 1,\ i = 1, 2, 3.$$

Then the relation at infinity takes the form

(4.5.2) $$\rho^2\alpha_1 = \alpha_2.$$

If the generator corresponding to $\mathfrak{b}$ is α_3, instead of (4.5.1) and (4.5.2) one has

(4.5.3) $$[\alpha_2, \alpha_3] = [\alpha_i, \rho^3] = [\alpha_i, \alpha_2\alpha_3^2] = 1,\ i = 1, 2, 3, \quad \text{and} \quad \rho^2\alpha_1 = \alpha_3.$$

4.6 The case of type $\mathbf{E}_6$

Assume that P is of type $\mathbf{E}_6$, and hence $\bar{P}$ is of type $\mathbf{A}_5$; see Proposition 3.2.1. Then $\bar{\Gamma}$ is inflection tangent to each of the two branches of $\bar{B}$ at $\bar{P}$, and the relation at infinity is

$$\rho^4 = (\alpha_2\alpha_3)^3, \tag{4.6.1}$$

assuming that α_2 and α_3 are represented by loops about the two branches at $\bar{P}$ (so that $[v,u]$ is the edge e_1' at v). The monodromy at infinity is $m_\infty = (\sigma_1\sigma_2)^{12}\sigma_2^{-6}$, and the corresponding braid relations are

$$[\alpha_1, \rho^4] = 1 \quad \text{and} \quad [\alpha_i, \rho^4(\alpha_2\alpha_3)^{-3}] = 1,\ i = 2, 3.$$

These relations follow from (4.6.1).

5 The inclusion homomorphism

Here, we compute the homomorphism of the fundamental groups induced by the inclusion to $\mathbb{P}^2$ of a Milnor ball M about a type $\mathbf{E}$ singular point P of a sextic B. These results are used in Section 7.

5.1 The setup

In order to compute the inclusion homomorphisms, we represent the sextic B by the polynomial given by (3.3.1), (3.4.1), or (3.5.2), assuming all parameters involved real and positive, and generate the group $\pi_1(M \smallsetminus B)$ by the classes of appropriately chosen loops β_1, β_2, β_3 in the complement $\{v = \epsilon\} \smallsetminus B$, where $\epsilon \ll 1$ is a positive real constant. (In what follows, we identify the loops and their classes.) Each loop β_i, $i = 1, 2, 3$, is composed of a small circle C_i about a point of intersection $\{v = \epsilon\} \cap B$ and a path p_i connecting a point $r_i \in C_i$ to the base point, which is a large real number.

The image of the line $\{v = \epsilon\}$ in Σ_k is the parabola $\{x^{k-1} = \epsilon y\}$. It intersects the "constant" section $\{y' = c\} = \{y = cx^k\}$ used to define the braid monodromy at the origin and at the point $r_0 := (x_0, y_0) = (1/\epsilon c, 1/\epsilon^k c^{k-1})$. We assume that c is also a real constant, $0 \ll c \ll 1/\epsilon$, so that $y_0 \gg 0$, and take r_0 for the common base point in both the line $\{v = \epsilon\}$ and the reference fiber $F_0' = \{x = x_0\}$. (This fiber may differ from the reference fiber considered in Section 4; the necessary adjustments are explained below.)

Now, consider the fiber F_i', $i = 1, 2, 3$, passing through r_i. (We keep the same notation C_i, r_i, and p_i for the images of the corresponding elements in Σ_k.) The point r_i is close to a branch of $\bar{B}$; let $\beta_i' \in \pi_1(F_i' \smallsetminus (\bar{B} \cup E), r_i)$ be the element represented by a small circle through r_i encompassing this branch. Dragging F_i' along p_i while keeping the base point in p_i, one defines the braid monodromy

$$m_i' \colon \pi_1(F_i' \smallsetminus (\bar{B} \cup E), r_i) \to \pi_1(F_0' \smallsetminus (\bar{B} \cup E), r_0).$$

(One should make sure that p_i does not pass through the origin in the line $\{v = \epsilon\}$.) It is immediate that $m_i'(\beta_i')$ represents the image of the generator β_i under the inclusion homomorphism (cf Figure 3, where the curve $\bar{B}$, the line $\{v = \epsilon\}$, and the section s are drawn in bold, dashed, and dotted lines, respectively; the two grey lassoes, one in $\{v = \epsilon\}$ and one in the fiber $F_i' = F_0'$, represent the same element of the fundamental group).

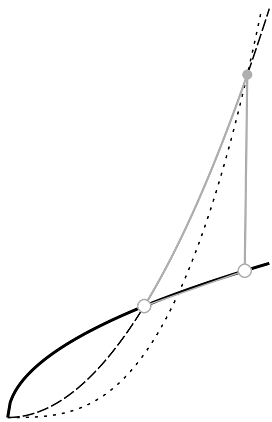

Figure 3: Computing the inclusion homomorphism

5.2 The case of type $\mathbf{E_7}$

The original curve B is given by (3.4.1). All three points of intersection of B and the line $\{v = \epsilon\}$ are real, and we take for $\{\beta_1, \beta_2, \beta_3\}$ a "linear" basis, numbering the intersection points consecutively by the decreasing of the u–coordinates and taking for p_i segments of the real line, circumventing the interfering intersection points and the origin in the counterclockwise direction.

All three points of intersection of $\bar{B}$ and the reference fiber F_0 are also real, and we choose a similar "linear" basis $\{\alpha_1', \alpha_2', \alpha_3'\}$ for the group $\pi_1(F_0' \smallsetminus (\bar{B} \cup E), r_0)$. Then one has

$$\beta_1 \mapsto \alpha_1', \quad \beta_2 \mapsto \alpha_2', \quad \beta_3 \mapsto (\alpha_2'\alpha_3')^{-1}\alpha_1'(\alpha_2'\alpha_3').$$

In order to pass to the reference fiber F_0 considered in Section 4.5 and a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ satisfying $(*)$, one can drag F'_0 along the arc $x = x_0 \exp(\pi i t/2)$, $t \in [0, 1]$. Then $\alpha'_1 = \alpha_1$, $\alpha'_2 = \alpha_2\alpha_3\alpha_2^{-1} = \alpha_3$ (we use the commutativity relation in (4.5.1)), and $\alpha'_3 = \alpha_2$. Finally, the inclusion homomorphism is given by

$$\beta_1 \mapsto \alpha_1, \quad \beta_2 \mapsto \alpha_3, \quad \beta_3 \mapsto (\alpha_2\alpha_3)^{-1}\alpha_1(\alpha_2\alpha_3). \tag{5.2.1}$$

5.3 The case of type $\mathbf{E}_8$

The curve B is given by (3.3.1), and the points of intersection of B and the line $\{v = \epsilon\}$ are the three roots $u = \sqrt[3]{\epsilon^5}$. Let $\{\beta_1, \beta_2, \beta_3\}$ be a basis similar to the one shown in Figure 1, with the paths p_i composed of radial segments and the arcs $u = \mathrm{const}\cdot\exp(\pm 2\pi i t/3)$, $t \in [0, 1]$. We assume that the generator corresponding to the real branch of B is β_2.

All three points of intersection of $\bar{B}$ and the reference fiber F_0 are real, and we choose a "linear" basis $\{\alpha_1, \alpha_2, \alpha_3\}$ as in Section 5.2 for the group $\pi_1(F'_0 \smallsetminus (\bar{B} \cup E), r_0)$. Dragging F_0 along the arc $x = x_0 \exp(\pi i t)$, $t \in [0, 1]$, to the fiber $\{x = -\epsilon\}$, one can see that α_1, α_2, α_3 are indeed equal to the basis elements considered in Section 4.4.

In these bases, the inclusion homomorphism is given by

$$\beta_1 \mapsto (\alpha_1\alpha_2)\alpha_3(\alpha_1\alpha_2)^{-1}, \quad \beta_2 \mapsto \alpha_1, \quad \beta_3 \mapsto \alpha_3. \tag{5.3.1}$$

5.4 The case of type $\mathbf{E}_6$

The curve B is given by (3.5.2), the intersection points of B and $\{v = \epsilon\}$ are the three roots $u = \sqrt[3]{\epsilon^5}$, and we take for $\{\beta_1, \beta_2, \beta_3\}$ the same basis as in Section 5.3. The basis $\{\alpha_1, \alpha_2, \alpha_3\}$ in the reference fiber is chosen "linear" as above; these elements do satisfy the conditions imposed in Section 4.6. In these bases, the inclusion homomorphism is given by

$$\beta_1 \mapsto (\alpha_1\alpha_2\alpha_3)\alpha_1(\alpha_1\alpha_2\alpha_3)^{-1}, \quad \beta_2 \mapsto \alpha_1, \quad \beta_3 \mapsto (\alpha_2\alpha_3)^{-1}\alpha_1(\alpha_2\alpha_3). \tag{5.4.1}$$

6 The computation

In this section, Theorems 1.2.1 and 1.2.2 are proved. Throughout the section, we fix the following notation: B stands for an irreducible maximal plane sextic with a distinguished type $\mathbf{E}_7$ singular point P, and $L \subset \mathbb{P}^2$ is the line tangent to B at P. We denote by $\bar{B}$ and F, respectively, the trigonal curve corresponding to B and its distinguished fiber (see Proposition 3.2.2); Sk stands for the skeleton of $\bar{B}$.

6.1 Proof of Theorem 1.2.1

According to Proposition 3.1.1, all triple points of B other than P are of type $\mathbf{E}_6$ or $\mathbf{E}_8$. Let $t = 0$, 1, or 2 be their number. Then Sk has the following properties:

(1) $\deg \mathrm{Sk} = 9 - 3t$ and Sk has exactly t singular vertices, none of which is ∘–;

(2) if $t = 0$, then Sk does not admit a splitting marking (Proposition 2.4.1).

Conversely, in view of Theorem 2.3.1, any skeleton Sk satisfying (1) and (2) above (for some integer $t \geqslant 0$) represents an irreducible maximal trigonal curve $\bar{B}$ as in Proposition 3.2.2; hence, it represents two irreducible maximal sextics with a distinguished type $\mathbf{E}_7$ singular point.

The distinguished fiber F is located at the center of a bigonal region R of Sk. In the drawings below, we show the boundary of R in grey.

Assume that all •–vertices in the boundary of R are nonsingular. Then R looks as shown in Figure 4. This fragment of the skeleton (if present) is called the *insertion*. The two branches of $\bar{B}$ at the node located in F are in a natural correspondence with the two edges of ∂R; hence, selecting one of the branches (see Proposition 3.2.2) can be interpreted geometrically as selecting one of the two arcs in the boundary of the insertion.

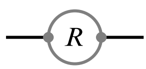

Figure 4: The insertion

Removing the insertion and patching it with an edge, one obtains another valid skeleton Sk' of degree $6 - 3t \leqslant 6$, cf Figure 5. (Note that one has $t \leqslant 1$ in this case.) The •–vertices of Sk' are in a one-to-one correspondence with the same valency vertices of Sk other than the two vertices in ∂R. Conversely, given a skeleton Sk' of degree $6-3t \leqslant 6$ with t singular •–vertices, $t = 0, 1$, one can place the insertion at the middle of any edge of Sk' to obtain a new skeleton Sk satisfying (1) above.

6.1.1 Lemma *If $t = 0$, the skeleton* Sk *admits a splitting marking if and only if so does* Sk'.

Proof It is immediate that any splitting marking of Sk restricts to a splitting marking of Sk' and, vice versa, any splitting marking of Sk' extends (uniquely) to a splitting marking of Sk. □

6.1.2 Remark This trick, replacing a given skeleton Sk by another skeleton Sk′ obtained from Sk by removing a certain fragment, appears on numerous occasions in the classification of plane sextics and, more generally, in the study of extremal elliptic surfaces. It would be interesting to understand if the passage from Sk to Sk′ corresponds to a simple geometric construction defined in terms of trigonal curves or covering elliptic surfaces. At present, I do not know any geometric interpretation.

Thus, the classification of skeletons Sk satisfying conditions (1) and (2) above and containing an insertion can be done in two steps:

- The classification of skeletons of degree 6 without singular vertices and not admitting a splitting marking, and the classification of skeletons of degree 3 with exactly one singular vertex, which is •–. This is done in [7], and the complete list is presented in Figure 5(a)–(e).
- Placing an insertion with one of the two arcs selected to one of the edges of each skeleton Sk′ discovered at step one.

The second step is clearly equivalent to choosing a pair $(e, \mathfrak{o})$, where e is an edge of Sk′ and $\mathfrak{o}$ is a coorientation of e. Such pairs are to be considered up to orientation preserving automorphisms of $\mathrm{Sk}' \subset S^2$. All essentially distinct edges e (with the coorientation $\mathfrak{o}$ ignored) are shown in Figure 5(a)–(e). Taking into account the coorientation, one obtains the list given by Table 3, lines 1–9.

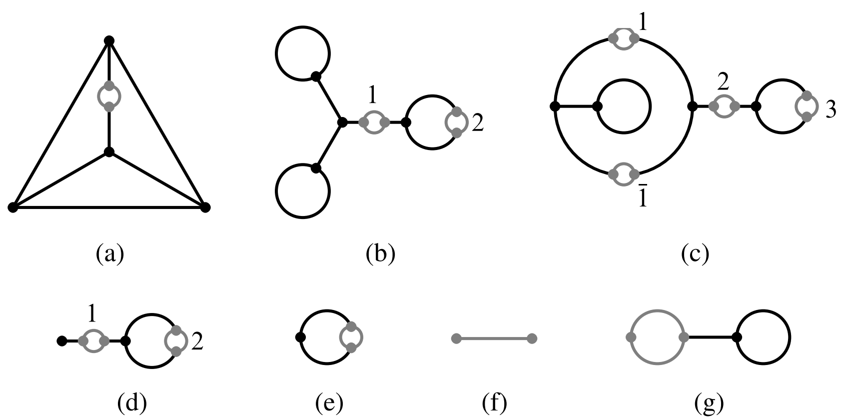

Figure 5: The skeletons Sk′ and Sk

The few remaining cases, when the boundary of R contains singular •–vertices, can easily be treated manually using the vertex count given by Corollary 2.5.5. The two skeletons obtained are shown in Figure 5(f), (g), and the corresponding sets of singularities are listed in Table 3, lines 10, 11.

#	Set of singularities	Figure	Count	π_1	$S^\perp$
1	$\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	5(a)	$(1,0)$	6.8	$(15,0,15)$
2	* $\mathbf{E}_7 \oplus \mathbf{A}_{12}$	5(b)–1	$(0,1)$	6.3	$(7,2,2)$
3	* $\mathbf{E}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	5(b)–2	$(2,0)$	6.5	$(11,0,3)$
4	$\mathbf{E}_7 \oplus 2\mathbf{A}_6$	5(c)–1, $\bar{1}$	$(0,1)$	6.7	$(7,0,7)$
5	* $\mathbf{E}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$	5(c)–2	$(0,1)$	6.3	$(23,2,2)$
6	* $\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	5(c)–3	$(2,0)$	6.5	$(35,0,3)$
7	* $\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_6$	5(d)–1	$(0,1)$	6.3	$(11,2,2)$
8	* $\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	5(d)–2	$(2,0)$	6.5	$(15,0,3)$
9	$\mathbf{E}_7 \oplus \mathbf{E}_8 \oplus 2\mathbf{A}_2$	5(e)	$(1,0)$	6.5	$(3,0,3)$
10	$\mathbf{E}_7 \oplus 2\mathbf{E}_6$	5(f)	$(1,0)$	6.6	$(3,0,3)$
11	* $\mathbf{E}_7 \oplus \mathbf{E}_8 \oplus \mathbf{A}_4$	5(g)	$(0,1)$	6.3	$(3,2,2)$

Table 3: Maximal sets of singularities with a type $\mathbf{E}_7$ point represented by irreducible sextics

The results of the computation are collected in Table 3, where we list the set of singularities, the skeleton Sk of $\bar{B}$, the number of deformation classes (see below), and a reference to the computation of the fundamental group. A set of singularities is marked with a * if it is realized by two equisingular deformation classes which have the same skeleton but differ by the selected branch of the insertion. (In the terminology of Proposition 3.2.2, the two families differ by the distinguished branch of $\bar{B}$ at $\bar{P}$.) For completeness, we also list the lattice $S^\perp$ corresponding to the homological type of the sextic (see Degtyarev [6] for the definitions): the notation (a,b,c) stands for the quadratic form generated by two elements u, v with $u^2 = 2a$, $u \cdot v = b$, and $v^2 = 2c$. The lattice is obtained by comparing two independent classifications, those of curves and of abstract homological types, and taking into account the number of classes obtained (see also Shimada [26]).

The number of deformation classes is listed in the form (n_r, n_c), where n_r is the number of real curves and n_c is the number of pairs of complex conjugate curves. (Thus, the total number of classes is $n_r + 2n_c$.) Real are the curves whose skeletons admit an orientation reversing automorphism of order 2 (with the marked arc taken into account); otherwise, two symmetric skeletons represent a pair of complex conjugate curves. □

6.2 Proof of Theorem 1.2.2

We compute the fundamental groups using the strategy outlined in Section 4.1 and the presentation given by Proposition 4.1.5. As in Section 4.3, we choose for the reference fiber F_0 the fiber F_v over a •–vertex v connected by an edge to the ×–vertex corresponding to F, and consider a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ for the group

$\pi_F = \pi_v$ defined by an appropriate marking. In most cases, we assume that the basis satisfies $(*)$. Then, for all groups, the relations $m_\infty = \mathrm{id}$ and $[\partial\bar\Gamma] = 1$ are given by (4.5.1) and (4.5.2), and the remaining braid relations $m_j = \mathrm{id}$ are computed using the techniques outlined in Section 4.2; in most cases, just a few extra relations suffice to show that the group is abelian. A detailed case by case analysis is given in Sections 6.3–6.8 below.

A great deal of the calculation in Sections 6.3–6.8 was handled using `GAP` [18]. In most cases, we merely input the relations and query the size of the resulting group; having obtained six, we know that the group is $\mathbb{Z}_6$. In the more advanced case in Section 6.8, we quote the `GAP` input/output in Figure 10.

6.3 Sets of singularities numbers 2, 5, 7, and 11

Assume that the distinguished fiber F has a neighborhood shown in Figure 6. (The leftmost •–vertex can be either bi- or trivalent; it is not used in the calculation.)

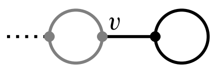

Figure 6: A special fragment of the skeleton

Assume that the distinguished branch is such that the basis in π_v satisfies $(*)$. (The case when the generator corresponding to the distinguished branch is α_3 is treated similarly; alternatively, one can argue that the corresponding fragment is obtained from the one considered by the complex conjugation.) Then, the braid relation about the type $\tilde{\mathbf{A}}_0^*$ singular fiber represented by the rightmost loop of the skeleton is $\alpha_2 = (\alpha_2^{-1}\alpha_1\alpha_2)\alpha_3(\alpha_2^{-1}\alpha_1\alpha_2)^{-1}$. In the presence of (4.5.1), it simplifies to $\alpha_1^{-1}\alpha_2\alpha_1 = \alpha_3$. Let

(6.3.1) $$G = \langle \alpha_1, \alpha_2, \alpha_3 \mid (4.5.1), (4.5.2), \alpha_1^{-1}\alpha_2\alpha_1 = \alpha_3 \rangle.$$

Since α_2, α_3 commute and $\alpha_1^{-1}(\alpha_2^2\alpha_3)\alpha_1 = (\alpha_2^2\alpha_3)$, one has $\alpha_1^{-1}\alpha_3\alpha_1 = \alpha_2^2\alpha_3^{-1}$. Thus, the conjugation by α_1 preserves the abelian subgroup generated by α_2 and α_3, and the map $t\colon w \mapsto \alpha_1^{-1}w\alpha_1$ is given by

$$t\colon \alpha_2 \mapsto \alpha_3 \mapsto \alpha_2^2\alpha_3^{-1} \mapsto \alpha_3^3\alpha_2^{-2} \mapsto \cdots .$$

Using the noncommutativity relations obtained, one can move all three copies of α_1 in (4.5.2) to the left; this gives $\alpha_1^3 = \alpha_2^{-1}\alpha_3^{-2}$. In particular, $t^3 = \mathrm{id}$ and hence $\alpha_2^3 = \alpha_3^3$.

Now, it is easy to see that the commutant $[G,G] \cong \mathbb{Z}_3$ is generated by the central order 3 element $\alpha_2\alpha_3^{-1}$ and the abelianization $\bar{G} := G/[G,G]$ is the group

$$\operatorname{ab}\langle \bar\alpha_1, \bar\alpha_2, \bar\alpha_3 \mid \bar\alpha_2 = \bar\alpha_3,\ 3(\bar\alpha_1 + \bar\alpha_2) = 0\rangle.$$

(Here, the bar stands for the projection of an element to the abelianization.)

For any group π_1 with central commutant (for example, for any quotient group of G above), the map $\bar{x} \wedge \bar{y} \mapsto [x,y] \in [\pi_1, \pi_1]$ is a well defined skew-symmetric bilinear form $\bigwedge^2 \bar\pi_1 \to [\pi_1, \pi_1]$ (where $\bar\pi_1 := \pi_1/[\pi_1,\pi_1]$); clearly, the group is abelian if and only if this form is identically zero. In particular, abelian are the fundamental groups of irreducible sextics with the sets of singularities numbers 2, 5, 7 and 11 in Table 3, as in this case $\bar\pi_1$ is cyclic and $\bigwedge^2 \bar\pi_1 = 0$.

6.4 Proofs of Proposition 1.2.4 and Corollary 1.2.5

The arguments in Section 6.3 apply as well to a *reducible* maximal sextic B, provided that the skeleton of the trigonal model of B has a fragment shown in Figure 6. Such skeletons can be enumerated similar to Section 6.1, placing an insertion to an appropriate edge of one of the skeletons of reducible trigonal curves found in [7]. There are four skeletons with a loop, which result in the five sets of singularities listed in Proposition 1.2.4. (In one case, there is an extra choice of the singular fiber to be converted to a **D**–type point by an elementary transformation.)

Figure 6 implies that the two branches of $\bar{B}$ at $\bar{P}$ are in the same irreducible component; hence, the corresponding sextic B splits into two irreducible cubics (at least one having a cusp). Furthermore, analyzing the other possibilities, one can deduce that each set of singularities listed in Proposition 1.2.4 is realized by a unique, up to complex conjugation, equisingular deformation family of sextics *splitting into two cubics*. (The set of singularities $\mathbf{E}_7 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ is also realized by two other families: one can start from the skeleton marked $\tilde{\mathbf{A}}_7 \oplus \tilde{\mathbf{A}}_1 \oplus 2\tilde{\mathbf{A}}_0^*$ in [7] and place the insertion as shown in Figure 7. In this case, the two branches at $\bar{P}$ are in two distinct components of $\bar{B}$ and, depending on the branch distinguished, the corresponding sextic splits into a quintic and a line or a quartic and a conic.)

As explained in Section 6.3, the fundamental group π_1 of each of the curves obtained is a quotient of the group G given by (6.3.1), and the commutator form $\bigwedge^2 \bar\pi_1 \to [\pi_1, \pi_1]$ is determined by the value $[\bar\alpha_1, \bar\alpha_2]$. Thus, the only question is whether an extra relation in the fundamental group implies $[\bar\alpha_1, \bar\alpha_2] = 0$; if such a relation does exist, π_1 is abelian; otherwise, $\pi_1 = G$. The remaining relations are easily found using the techniques explained in Section 4.2. For example, the monodromy along the boundary of the $2n$–gon adjacent to the insertion (assuming that this region does not contain a $\tilde{\mathbf{D}}$–type fiber) gives the relation $(\alpha_1\alpha_2)^n = (\alpha_2\alpha_1)^n$; in G, it simplifies to $n[\bar\alpha_1, \bar\alpha_2] = 0$; hence, it is a tautology if $n = 0 \bmod 3$, and it implies that the quotient is abelian if $n \neq 0 \bmod 3$. (Note that this region must have an even number of corners, as otherwise the relation would imply $\bar\alpha_1 = \bar\alpha_2$ and the curve would be irreducible.) We omit further details; the final result is stated in Proposition 1.2.4. □

6.5 Sets of singularities numbers 3, 6, 8 and 9

Assume that the distinguished fiber F has a neighborhood shown in Figure 7. (The leftmost •–vertex can be either bi- or trivalent; it is not used in the calculation.) In other words, we assume that one of the regions adjacent to the bigon R containing F is a triangle. Then, from Figure 5 it follows that the other region adjacent to R is a 3–, 5–, 7– or 11–gon.

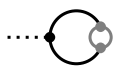

Figure 7: Another special fragment

Over one of the two grey vertices in Figure 7 (depending on the distinguished branch) a canonical basis can be chosen to satisfy $(*)$. In this basis, in addition to (4.5.1) and (4.5.2), the singular fibers inside the two regions adjacent to R give the relations

(6.5.1) $$(\alpha_1\alpha_2)^m\alpha_1 = \alpha_2(\alpha_1\alpha_2)^m, \quad (\alpha_1\alpha_3)^n\alpha_1 = \alpha_3(\alpha_1\alpha_3)^n,$$

where either $m = 1$ and $n = 1, 2, 3, 5$ or $n = 1$ and $m = 1, 2, 3, 5$. (The relations are given by (4.2.3); for the second relation in (6.5.1), we use the commutativity $[\alpha_2, \alpha_3] = 1$.) Using `GAP` [18], one can see that, for each pair (m, n) as above, the group $\langle \alpha_1, \alpha_2, \alpha_3 \mid (4.5.1), (4.5.2), (6.5.1)\rangle$ has order six; hence, it is abelian.

6.6 The set of singularities $\mathbf{E}_7 \oplus 2\mathbf{E}_6$ (number 10)

In this special case, the skeleton of $\bar B$ has no trivalent •–vertices. However, we can choose a basis similar to the one shown in Figure 1 and satisfying $(*)$ in the fiber F_0 over a point in the (open) solid edge connecting the ×–vertex representing F and an appropriate •–vertex. Then, both (4.5.1) and (4.5.2) still hold and, in addition, the monodromy about the type $\tilde{\mathbf{E}}_6$ singular fiber close to F_0 gives the relations

$$\alpha_2\alpha_1\alpha_2\alpha_3 = \alpha_1\alpha_2\alpha_3\alpha_1, \quad \alpha_3\alpha_1\alpha_2\alpha_3 = \alpha_1\alpha_2\alpha_3\alpha_2.$$

Using `GAP` [18], one can conclude that the resulting group has order six; hence, it is abelian.

6.7 The set of singularities $\mathbf{E}_7 \oplus 2\mathbf{A}_6$ (number 4)

After an automorphism and/or complex conjugation, one can assume that the insertion R is as shown in Figure 8 and that a canonical basis over v can be chosen to satisfy $(*)$.

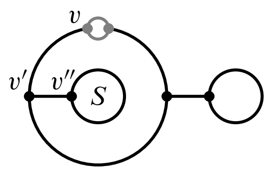

Figure 8: The set of singularities $\mathbf{E}_7 \oplus 2\mathbf{A}_6$

Then, in addition to (4.5.1) and (4.5.2), one has relations (6.5.1) with $m = n = 3$ (given by the two heptagons adjacent to the insertion) and a relation $\alpha_1' = \alpha_2'\alpha_3'\alpha_2'^{-1}$ given by the monodromy along the loop $[\partial S]$, where S is the region shown in Figure 8 and $\alpha_1', \alpha_2', \alpha_3'$ is an appropriate canonical basis in the fiber over v''. Connecting v'' to v via v' and using (4.2.2), one obtains

$$\alpha_1' = (\alpha_1\alpha_2)^{-1}\alpha_1(\alpha_1\alpha_2), \quad \alpha_2' = (\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2), \quad \alpha_3' = \alpha_3,$$

and, in view of the fact that α_2 and α_3 commute, the relation about $[\partial S]$ simplifies to

$$\alpha_1 = (\alpha_2\alpha_1)\alpha_3(\alpha_2\alpha_1)^{-1}.$$

Using `GAP` [18], one finds that the group obtained has order six; hence, it is abelian.

6.8 The set of singularities $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$ (number 1)

After an automorphism and/or complex conjugation, one can assume that the skeleton of $\bar{B}$ is as shown in Figure 9 and that a canonical basis over v can be chosen to satisfy $(*)$.

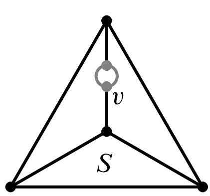

Figure 9: The set of singularities $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$

Then, in addition to (4.5.1) and (4.5.2), one has relations (6.5.1) with $m = n = 2$ (given by the two pentagons adjacent to the insertion) and the relation

(6.8.1) $$(\alpha_1\alpha_3\alpha_1^{-1})\alpha_2(\alpha_1\alpha_3\alpha_1^{-1}) = \alpha_2(\alpha_1\alpha_3\alpha_1^{-1})\alpha_2$$

given by the monodromy along the boundary of the triangle S shown in Figure 9. The monodromy about the remaining singular fiber contained in the remaining triangular region of the skeleton can be ignored due to Proposition 4.1.5. Thus, the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ is given by

(6.8.2) $$\big\langle \alpha_1, \alpha_2, \alpha_3 \;\big|\; \text{(4.5.1), (4.5.2), (6.8.1), (6.5.1) with } m = n = 2 \big\rangle.$$

Denote this group by G. Using `GAP` [18] (see Figure 10), one can see that:

(1) one has $\operatorname{ord} G = 41040 = 2^4 \cdot 3^3 \cdot 5 \cdot 19$, and the commutant $[G, G]$ is a perfect group of order $6840 = 2^3 \cdot 3^2 \cdot 5 \cdot 19$;

(2) the only perfect group of order $6840 = \operatorname{ord}[G, G]$ is $\mathrm{SL}(2, \mathbb{F}_{19})$;

(3) relation (6.8.1) in (6.8.2) follows from the others (as dropping this relation does not change the order of the group);

(4) the order of each generator α_i, $i = 1, 2, 3$, in G equals $6 \cdot 19$;

(5) the group G is generated by α_1 and α_2 only, as well as by α_1 and α_3 only.

Due to (3), one can drop relation (6.8.1); then (6.8.2) turns into the presentation given in the statement of Theorem 1.2.2. The abelianization of G is the cyclic group $\mathbb{Z}_6$ generated by the image $\bar{\alpha}_1$ of α_1. Hence, due to (4), the map $\bar{\alpha}_1 \mapsto \alpha_1^{19}$ splits the exact sequence

$$\{1\} \longrightarrow [G, G] \longrightarrow G \longrightarrow \mathbb{Z}_6 \longrightarrow \{1\},$$

representing G as a semidirect product of its abelianization and its commutant.

This completes the proof of Theorem 1.2.2. □

6.8.3 Proposition *Let B be an irreducible plane sextic with the set of singularities $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$, and let M be a Milnor ball about the type $\mathbf{E}_7$ singular point of B. Then the inclusion homomorphism $\pi_1(M \smallsetminus B) \to \pi_1(\mathbb{P}^2 \smallsetminus B)$ is onto.*

Proof The statement follows from (5) above and from (5.2.1), which implies that α_1 and α_3 are in the image of the inclusion homomorphism. □

7 Perturbations

We compute the fundamental groups of all perturbations of a type $\mathbf{E}_7$ singular point and apply these results to prove Theorem 1.2.3.

```
gap> f := FreeGroup(3);;
gap> r1 := f.2*f.3*f.2^-1*f.3^-1;;
gap> r2 := (f.1*f.2*f.3)^3/(f.2*f.3^2);;
gap> r3 := f.1*(f.1*f.2*f.3)^3*f.1^-1*(f.1*f.2*f.3)^-3;;
gap> r4 := (f.1*f.2)^2*f.1/(f.2*(f.1*f.2)^2);;
gap> r5 := (f.1*f.3)^2*f.1/(f.3*(f.1*f.3)^2);;
gap> r6 := f.1*f.3*f.1^-1*f.2*f.1*f.3*f.1^-1/(f.2*f.1*f.3*f.1^-1*f.2);;
gap> g := f / [r1, r2, r3, r4, r5, r6];;
gap> Size(g);
41040
gap> List(DerivedSeriesOfGroup(g), AbelianInvariants); ## [G,G] is perfect
[ [ 2, 3 ], [  ] ]
gap> List(DerivedSeriesOfGroup(g), Size);              ## order of [G,G]
[ 41040, 6840 ]
gap> NumberPerfectGroups(6840); PerfectGroup(6840);    ## [G,G]=SL(2,19)
1
L2(19) 2^1 = SL(2,19)
gap> Size(f / [r1, r2, r3, r4, r5]);      ## drop r6
41040
gap> Size(Subgroup(g, [g.1]));            ## order of g.1
114
gap> Index(g, Subgroup(g, [g.1, g.2]));   ## g.1, g.2 generate G
1
gap> Index(g, Subgroup(g, [g.1, g.3]));   ## g.1, g.3 generate G
1
```

Figure 10: The GAP output for $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$

7.1 Perturbations of a type $\mathbf{E}_7$ singularity

Let P be a type $\mathbf{E}_7$ singular point of a plane curve B, let M be a Milnor ball about P, and let B_t, $t \in [0,1]$, be a small perturbation of $B = B_0$ which remains transversal to the boundary ∂M. We are interested in the perturbation epimorphism $\pi_1(M \smallsetminus B) \twoheadrightarrow \pi_1(M \smallsetminus B_t)$, $t \neq 0$.

According to E Looijenga [21], the deformation classes (in the obvious sense) of perturbations of a simple singular point P can be enumerated by the induced subgraphs of the Dynkin diagram of P (up to a certain equivalence, which is not important here). Comparing two independent classifications, one can see that any perturbation B_t of a type $\mathbf{E}_7$ singular point can be realized by a family $C_t \subset \mathbb{C}^2$ of affine quartics inflection tangent to the line at infinity (see, eg, Degtyarev [2]), so that $(M, B_t) \cong (\mathbb{C}^2, C_t)$ for each $t \in [0,1]$. The groups $\pi_1(\mathbb{C}^2 \smallsetminus C_t)$ for such quartics are found in [3], and all but five of them are abelian. The sets of singularities (perturbations) with nonabelian fundamental group are those listed in Figure 11 and $\mathbf{A}_2 \oplus 3\mathbf{A}_1$, which is a further perturbation of $\mathbf{D}_5 \oplus \mathbf{A}_1$ not changing the group.

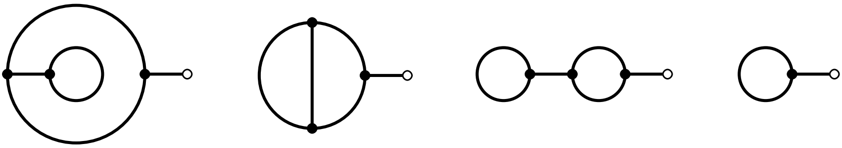

Figure 11: Perturbations of $\mathbf{E}_7$ with nonabelian fundamental group

In order to (re-)compute the four nonabelian groups, we realize each perturbation by a family $\bar{B}_t \subset \Sigma_2$, $t \in [0, 1]$, of trigonal curves with a common type $\tilde{\mathbf{A}}_1^*$ singular fiber F, so that $\bar{B}_0$ is the isotrivial curve given by $y'^3 = x'y'$. (We use the same coordinates (x', y') as in (4.1.2); in fact, $\bar{B}_t$ can be obtained from the family C_t above by a birational transformation, similar to Section 3.3.) Then $M \smallsetminus B_t \cong \Sigma_2 \smallsetminus (\bar{B}_t \cup E \cup F)$, and the group $\pi_1(\Sigma_2 \smallsetminus (\bar{B}_t \cup E \cup F))$ can be found using the techniques outlined in Section 4.1, without adding the relation at infinity.

The skeleton of a typical curve $\bar{B}_t$, $t \neq 0$, which is maximal, is one of those shown in Figure 11. Let u be the $\circ$–vertex representing the fiber at infinity F, and let v be the $\bullet$–vertex connected to u. We take F_v for the reference fiber, and choose a canonical basis $\{\delta_1, \delta_2, \delta_3\}$ for π_v defined by the marking in which $[v, u]$ is the edge e_1 at v. The braid monodromy is computed using Section 4.2. In each case, there are two relations (4.2.3) given by the $\times$–vertices connected to v by the edges e_1' and e_3'. For the third $\times$–vertex, if present, one should connect it to v by a solid edge and a chain of bold edges and use (4.2.2).

The relations obtained are:

- $\mathbf{A}_4 \oplus \mathbf{A}_2$: $\delta_1\delta_2\delta_1 = \delta_2\delta_1\delta_2$, $(\delta_2\delta_3)^2\delta_2 = \delta_3(\delta_2\delta_3)^2$, $\delta_2 = \delta_3\delta_1\delta_3^{-1}$;
- $\mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$: $[\delta_1, \delta_3] = 1$, $(\delta_1\delta_2)^2 = (\delta_2\delta_1)^2$, $\delta_2\delta_3\delta_2 = \delta_3\delta_2\delta_3$;
- $\mathbf{A}_5 \oplus \mathbf{A}_1$: $[\delta_2, \delta_3] = 1$, $(\delta_1\delta_2)^3 = (\delta_2\delta_1)^3$, $\delta_3 = \delta_1\delta_2\delta_1^{-1}$;
- $\mathbf{D}_5 \oplus \mathbf{A}_1$ and $\mathbf{A}_2 \oplus 3\mathbf{A}_1$: $[\delta_1, \delta_2] = [\delta_1, \delta_3] = 1$, $\delta_2\delta_3\delta_2 = \delta_3\delta_2\delta_3$.

7.1.1 Remark The first group is shown in [3] to be isomorphic to $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{F}_5)$. The last group is obviously $\mathbb{Z} \times \mathbb{B}_3$. For the group G corresponding to the set of singularities $\mathbf{A}_5 \oplus \mathbf{A}_1$, one can easily deduce that δ_1^3 is a central element and then obtain a short exact sequence

$$\{1\} \longrightarrow \mathbb{Z}[t]/(t^3 - 1) \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow \{1\},$$

the generator of the quotient $\mathbb{Z}$ acting on the kernel via the multiplication by t. (As a $\mathbb{Z}[t]$–module, the kernel is generated by δ_2.) This result also agrees with [3].

7.2 Proof of Theorem 1.2.3

It suffices to study the perturbations of the only sextic B with nonabelian fundamental group, ie, the one considered in Section 6.8. The set of singularities of B is $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$.

The perturbations of the $\mathbf{A}$–type points can be treated within the framework of the paper, using the trigonal model $\bar{B}$ of B. If one of the two type $\mathbf{A}_4$ points is perturbed, a monovalent $\times$–vertex appears in one of the two regions adjacent to the insertion (see Figure 9); hence, the group gets an additional relation $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_3$ (ie, (6.5.1) with $m = 0$ or $n = 0$) and, in view of the commutativity relation $[\alpha_2, \alpha_3] = 1$, becomes abelian. The two cusps of $\bar{B}$ are interchanged by the complex conjugation, and it suffices to perturb the one represented by the region S in Figure 9. Then, a monovalent $\times$–vertex appears in S and the fundamental group gets an extra relation $\alpha_1\alpha_3\alpha_1^{-1} = \alpha_2$ (instead of (6.8.1)). Thus, the group is a quotient of the group G given by (6.3.1) and hence is abelian; see Section 6.3.

Now, consider a perturbation B' of the type $\mathbf{E}_7$ point P. Let M be a Milnor ball about P. If the group $\pi_1(M \smallsetminus B')$ is abelian, then $\pi_1(\mathbb{P}^2 \smallsetminus B')$ is also abelian due to Proposition 6.8.3. For the four nonabelian groups $\pi_1(M \smallsetminus B')$ (see Section 7.1), we use the description of the inclusion homomorphism found in Section 5.2. The "linear" basis $\{\beta_1, \beta_2, \beta_3\}$ considered in Section 5.2 differs from the canonical basis $\{\delta_1, \delta_2, \delta_3\}$ used in Section 7.1 by "half" the monodromy about a type $\mathbf{E}_7$ singular point: one has

$$\delta_1 = (\beta_1\beta_2\beta_3)\beta_2(\beta_1\beta_2\beta_3)^{-1}, \quad \delta_2 = (\beta_1\beta_2)\beta_3(\beta_1\beta_2)^{-1}, \quad \delta_3 = \beta_1.$$

Hence, in terms of the δ_i, the inclusion homomorphism is given by

$$\delta_1 \mapsto \alpha_1\alpha_2^{-1}\alpha_1\alpha_3\alpha_1^{-1}\alpha_2\alpha_1^{-1}, \quad \delta_2 \mapsto \alpha_1\alpha_2^{-1}\alpha_1\alpha_2\alpha_1^{-1}, \quad \delta_3 \mapsto \alpha_1.$$

(We used the commutativity relations (4.5.1) to simplify the expressions obtained.) Now, it remains to express the extra relations in terms of the α–basis, add them to the presentation given by (6.8.2), and use `GAP` [18]. (In fact, in each of the four cases listed in Section 7.1, the first extra relation alone make the group abelian.) □

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