A finitely generated, locally indicable group with no faithful action by C^1 diffeomorphisms of the interval

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According to Thurston's stability theorem, every group of C^1 diffeomorphisms of the closed interval is locally indicable (that is, every finitely generated subgroup factors through $\mathbb{Z}$). We show that, even for finitely generated groups, the converse of this statement is not true. More precisely, we show that the group $\mathbb{F}_2 \ltimes \mathbb{Z}^2$, although locally indicable, does not embed into $\mathrm{Diff}^1_+(]0,1[)$. (Here $\mathbb{F}_2$ is any free subgroup of $\mathrm{SL}(2,\mathbb{Z})$, and its action on $\mathbb{Z}^2$ is the linear one.) Moreover, we show that for every non-solvable subgroup G of $\mathrm{SL}(2,\mathbb{Z})$, the group $G \ltimes \mathbb{Z}^2$ does not embed into $\mathrm{Diff}^1_+(\mathrm{S}^1)$.

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1 Introduction

Without any doubt, one of the most striking results about groups of diffeomorphisms is Thurston's stability theorem [22]. In the 1–dimensional context, this theorem establishes that $\mathrm{Diff}^1_+([0,1[)$ is *locally indicable*, that is, each of its finitely generated subgroups factors through $\mathbb{Z}$. In the language of the theory of orderable groups, this is equivalent to saying that $\mathrm{Diff}^1_+([0,1[)$ is C–orderable (see for example [20]). This is essentially the only known algebraic obstruction for embedding an abstract left-orderable group into $\mathrm{Diff}^1_+([0,1[)$. (Recall that every countable, left-orderable group embeds into $\mathrm{Homeo}_+([0,1])$: see Ghys [12].)

A good discussion on *dynamical* obstructions for C^1 smoothability of continuous actions on the interval appears in D Calegari's nice work [6]. Most of them concern *resilient orbits*. Indeed, as was cleverly noticed by C Bonatti, S Crovisier, and A Wilkinson, for groups of C^1 diffeomorphisms of the interval, there cannot be a central element without interior fixed points in the presence of resilient orbits [18, Proposition 4.2.25]. In the opposite direction, topologically transversal resilient orbits must appear when the topological entropy of the action is positive (see Hurder [13]) or when some sub-pseudogroup acts without invariant probability measure (see Deroin, Kleptsyn and Navas [10]). A new obstruction which does not involve resilient orbits is

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also given by Calegari [6]. Nevertheless, these four conditions do not seem to complete the list of all possible dynamical obstructions. For instance, none of them seems to apply to groups of piecewise affine homeomorphisms, though it is very likely that, 'in general', the corresponding actions are non C^1 smoothable.

Giving a pure *algebraic* equivalent condition for the existence of a group embedding into $\mathrm{Diff}^1_+([0,1[)$ also seems very hard (see Farb and Franks [11] and Navas [19] for two interesting particular cases). In this work, we show that local indicability, although necessary, is not a sufficient condition, even for finitely generated groups. For this, we deal with a concrete example, namely the group $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ (which is easily seen to be locally indicable), where $\mathbb{F}_2$ is any free subgroup of $\mathrm{SL}(2,\mathbb{Z})$ whose action on $\mathbb{Z}^2$ is the linear one.

Theorem A *The (locally indicable) group* $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ *does not embed into* $\mathrm{Diff}^1_+(]0,1[)$.

The interest in considering the group $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ comes from at least two sources. The first concerns the theory of orderable groups. Indeed, although C–orderable, this group admits no ordering with the stronger property of *right-recurrence*. This is cleverly noticed (and proved) in [15], where D Witte-Morris shows that every finitely generated left-orderable amenable group admits a right-recurrent ordering, and hence every left-orderable amenable group is locally indicable. The second source of interest relies on Kazhdan's property (T). Indeed, from [17, Théorème A] it follows that, if the pair (G,H) has the relative property (T) and H is non-trivial and normal in G (as is the case of $(\mathbb{F}_2 \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ when $\mathbb{F}_2$ has finite index in $\mathrm{SL}(2,\mathbb{Z})$), then G does not embed into the group of $C^{1+\alpha}$ diffeomorphisms of the (closed) interval provided that $\alpha > 1/2$. It is perhaps possible to use the L^p extensions of the (relative) property (T) in Bader et al. [1] to conclude, by a similar method, that $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ does not embed into $\mathrm{Diff}^{1+\alpha}_+([0,1])$ for any $\alpha > 0$. However, it does not seem plausible to deal with the C^1 case (even for the closed interval) using this kind of arguments. (Algebraic obstructions for passing from C^1 to $C^{1+\alpha}$ embeddings exist: see for example [19].)

Our proof of Theorem A is strongly influenced by an argument due to J Cantwell and L Conlon (namely the proof of the second half of [7, Theorem 2.1]). It relies on considerations about 'growth' of orbits (perhaps the right invariant to be considered should be the *topological entropy* associated to all possible actions on the interval). With slight modifications, these techniques also apply to the case of the circle. To motivate the theorem below, notice that $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ embeds into $\mathrm{Homeo}_+(\mathrm{S}^1)$ (see Section 2).

Theorem B *For any non-solvable subgroup* G *of* $\mathrm{SL}(2,\mathbb{Z})$, *the group* $G \ltimes \mathbb{Z}^2$ *does not embed into* $\mathrm{Diff}^1_+(\mathrm{S}^1)$.

This result provides a first obstruction for group embeddings into $\mathrm{Diff}^1_+(\mathrm{S}^1)$ for subgroups of $\mathrm{Homeo}_+(\mathrm{S}^1)$ which does not rely on Thurston's stability theorem. This solves a question raised by J Franks in a different manner from those of Calegari [4] and Parwani [21].

Unfortunately, our approach does not seem to be appropriate to deal with many other interesting groups which do act faithfully on the interval, as for example surface groups or general *limit groups* in the spirit of Breuillard et al. [3] (these groups are bi-orderable, which is stronger than being locally indicable), or the braid groups B_3 and B_4 (these groups are locally indicable, see Dehornoy et al. [9, pages 287–289]). Another interesting question is the possibility of extending Theorem A to the group of *germs* of diffeomorphisms, where Thurston's theorem still applies (compare [19, Remark 2.13]). Finally, the investigation of similar phenomena related to the higher dimensional versions of Thurston's theorem also seems promising. Actually, in this context, even the corresponding topological prior versions are widely open (and interesting). For example, it is unknown whether every finitely generated group which is *locally* $\mathrm{GL}_+(2,\mathbb{R})$ (that is, each of its finitely generated subgroups admits a non-trivial homomorphism into $\mathrm{GL}_+(2,\mathbb{R})$) can be realized as a group of germs of homeomorphisms of the plane fixing the origin. A naturally related problem is the realization of finitely generated, locally $\mathrm{SL}(2,\mathbb{R})$ groups as groups of germs of area-preserving homeomorphisms of $\mathbb{R}^2$.

2 Existence of actions by homeomorphisms

As is well-known (see Calegari [5] or Navas [18]) there exist faithful group actions of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ by (orientation preserving) circle homeomorphisms. Indeed, let us consider the canonical action of $\mathrm{SL}(2,\mathbb{R})$ by real-analytic circle diffeomorphisms, and let $p \in \mathrm{S}^1$ be a point whose stabilizer under the action of the subgroup $\mathrm{SL}(2,\mathbb{Z})$ is trivial. Replace each point $f(p)$ of the orbit of p by an interval I_f (where $f \in \mathrm{SL}(2,\mathbb{Z})$) in such a way that the total sum of these intervals is finite. Doing this, we obtain a topological circle S^1_p provided with a faithful $\mathrm{SL}(2,\mathbb{Z})$–action (we use affine transformations for extending the maps in $\mathrm{SL}(2,\mathbb{Z})$ to the intervals I_f).

Let $I = I_{id}$ be the interval corresponding to the point p, and let $\{\varphi^t : t \in \mathbb{R}\}$ be a non-trivial topological flow on I. Choose any real numbers t_1, t_2 which are linearly independent over the rationals, and let $h_1 = \varphi^{t_1}$ and $h_2 = \varphi^{t_2}$. Extend h_1, h_2 to S^1_p by letting

$$h_1(x) = f^{-1}\big(h_1^a h_2^c(f(x))\big), \qquad h_2(x) = f^{-1}\big(h_1^b h_2^d(f(x))\big),$$

where $x \in I_{f^{-1}}$ and

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{1}$$

For x in the complement of the union of the I_f's, we simply set $h_1(x) = h_2(x) = x$. The reader will easily check that the group generated by $\langle h_1, h_2 \rangle \sim \mathbb{Z}^2$ and the copy of $\mathrm{SL}(2,\mathbb{Z})$ acting on S^1_p is isomorphic to $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$.

If $\mathbb{F}_2$ is a free subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ is locally indicable. Thus it acts faithfully by homeomorphisms of the interval [20]. Although no such action arises as the restriction of the action constructed above, a faithful action may be constructed by following a similar procedure. For this, fix two (orientation preserving) homeomorphisms f_1, f_2 of $[0,1]$ generating a free group admitting a free orbit. There are many ways to obtain these homeomorphisms. We may take for example a left-ordering on $\mathbb{F}_2$, and next consider its dynamical realization (see the comment after [20, Example 2.6]). Another way is to use the fact that the group generated by $x \mapsto x+1$ and $x \mapsto x^3$ is free (see Cohen and Glass [8]). Denoting by $p \in]0,1[$ a point whose stabilizer under the corresponding $\mathbb{F}_2$–action is trivial, and then proceeding as above, we obtain the desired faithful action of $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ on the interval.

Let us point out that, although the actions constructed above are only by homeomorphisms, they are topologically conjugate to actions by Lipschitz homeomorphisms (see Deroin et al. [10, Théorème D]).

3 Preparation arguments: topological rigidity

Consider a faithful action of $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ by homeomorphisms of the interval $[0,1]$. Let I be an open (non-empty) *irreducible component* for the action of $\mathbb{Z}^2$, that is, a minimal open interval which is invariant by $\mathbb{Z}^2$. Since $\mathbb{Z}^2$ is normal in $\mathbb{F}_2 \ltimes \mathbb{Z}^2$, for every $f \in \mathbb{F}_2$ the interval $f(I)$ is also an open irreducible component for the action of $\mathbb{Z}^2$.

According to [18, Section 2.2.5], the group $\mathbb{Z}^2$ preserves a Radon measure μ on I. Associated to this measure, there is a non-trivial *translation number homomorphism* $\tau_\mu \colon \mathbb{Z}^2 \to \mathbb{R}$ defined by $\tau_\mu(g) = \mu([x, g(x)[)$ for any $x \in I$. One has $\tau_\mu(g) > 0$ if and only if $g(x) > x$ for all $x \in I$. Moreover, if μ' is another invariant Radon measure, then τ_μ and $\tau_{\mu'}$ coincide up to multiplication by a positive real number. We identify $h_1 \sim (1,0)$ and $h_2 \sim (0,1)$, and let $r = \tau_\mu((1,0))$ and $s = \tau_\mu((0,1))$.

Claim 1 *If (r,s) is not an eigenvector of f^T, where $f \in \mathbb{F}_2$, then the interval $f(I)$ is disjoint from I.*

Proof Denoting by $f(1,0)$ the image of $(1,0)$ under the linear action of f, we have

$$\tau_\mu(f(1,0)) = \tau_\mu(f h_1 f^{-1}) = \tau_{f^*(\mu)}(1,0).$$

Similarly, $\tau_\mu(f(0,1)) = \tau_{f^*(\mu)}(0,1)$. If f fixes I, then $f^*(\mu)$ is another Radon measure on I invariant by $\mathbb{Z}^2$. By the discussion above, there exists $\lambda > 0$ so that $\tau_{f^*(\mu)} = \lambda \tau_\mu$. This yields

$$\lambda r = \lambda \tau_\mu((1,0)) = \tau_{f^*(\mu)}((1,0)) = \tau_\mu(f(1,0)) = \tau_\mu((a,c)) = ar + cs.$$

Similarly, $\lambda s = br + ds$. This shows that (r,s) is an eigenvector of f^T with eigenvalue λ. □

Claim 2 *There exists a free subgroup $\langle f_0, f_1 \rangle \subset \mathbb{F}_2$ such that, for every element $f \in \langle f_0, f_1 \rangle$:*

(i) (r,s) *is not an eigenvector of* f^T,

(ii) (r,s) *is not orthogonal to an eigenvector of* f^{-1},

(iii) *neither* $(1,0)$ *nor* $(0,1)$ *are eigenvectors of* f.

Proof Since (the projection in $\mathrm{PSL}(2,\mathbb{R})$ of) $\mathbb{F}_2$ is non-elementary, a well-known property of Möbius groups yields the existence of infinitely many two-by-two disjoint pairs of points u_n, v_n in $\mathbb{PR}^1$ that are fixed by some hyperbolic element $g_n \in \mathbb{F}_2$ (see for instance Beardon [2, Theorem 5.1.3]). For n big enough, these pairs avoid $[1:0]$, $[0:1]$, and the points corresponding to the directions that are orthogonal to that of (r,s). A ping-pong argument then shows that, for m,n,k sufficiently large, the subgroup $F = \langle g_m^k, g_n^k \rangle$ of $\mathbb{F}_2$ is free and all of its non-trivial elements f satisfy (ii) and (iii). In order to ensure (i), one proceeds similarly but starting with F instead of $\mathbb{F}_2$, and considering the (projective) transpose action.

Let us consider the generator f_0 given by Claim 2. By Claim 1, $f_0(I)$ is disjoint from I. Thus, changing f_0 by its inverse if necessary, we may suppose that $f_0(I)$ is to the left of I. Moreover, changing f_0 by f_0^k for $k>0$ sufficiently large, we may suppose that the expanding eigenvalue λ of f_0^{-1} is greater than 2. For certain vectors (α,β) and (α^*,β^*) in the expanding direction of f_0^{-1} we have

$$\lim_{n\to\infty} \left[f_0^{-n}(1,0) - \lambda^n(\alpha,\beta) \right] = 0, \qquad \lim_{n\to\infty} \left[f_0^{-n}(0,1) - \lambda^n(\alpha^*,\beta^*) \right] = 0.$$

In what follows we will only deal with $f_0^{-n}(1,0)$, but the same arguments work with $f_0^{-n}(0,1)$ instead. Notice that, since (r,s) is not orthogonal to any eigenvector of f_0^{-1}, the value of $\alpha r + \beta s$ is nonzero. □

Claim 3 *The value of* $\left|\tau_\mu(f_0^{-n}h_1 f_0^n) - \lambda^n(\alpha r + \beta s)\right|$ *converges to zero as* n *tends to infinite.*

Proof Write $(1,0) = (\alpha,\beta) + (\gamma,\delta)$, where (γ,δ) is in the expanding direction of f_0. Letting $\overline{\lambda} > 1$ be the expanding eigenvalue of f_0, for each $n > 0$ we have

$$f_0^{-n}(1,0) = \lambda^n(\alpha,\beta) + \overline{\lambda}^{-n}(\gamma,\delta) = \big([\lambda^n\alpha],[\lambda^n\beta]\big) + \big(\{\lambda^n\alpha\} + \overline{\lambda}^{-n}\gamma, \{\lambda^n\beta\} + \overline{\lambda}^{-n}\delta\big),$$

where $[\cdot]$ (resp. $\{\cdot\}$) denotes the integer (resp. fractional) part. Notice that both numbers $\gamma_n = \{\lambda^n\alpha\} + \overline{\lambda}^{-n}\gamma$ and $\delta_n = \{\lambda^n\beta\} + \overline{\lambda}^{-n}\delta$ are integers. Hence

$$f_0^{-n}h_1 f_0^n = h_1^{\alpha_n} h_2^{\beta_n} h_1^{\gamma_n} h_2^{\delta_n},$$

where $\alpha_n = [\lambda^n\alpha]$ and $\beta_n = [\lambda^n\beta]$. Therefore, $\tau_\mu(f_0^{-n}h_1 f_0^n) = r(\alpha_n + \gamma_n) + s(\beta_n + \delta_n)$, which yields

$$\left|\tau_\mu(f_0^{-n}h_1 f_0^n) - \lambda^n(\alpha r + \beta s)]\right| \leq \overline{\lambda}^{-n}\big(|\gamma r| + |\delta s|\big),$$

thus showing the claim. □

Assume throughout that $t = \alpha r + \beta s$ is positive (the case where it is negative is similar). Replacing h_1 and h_2, respectively, by h_1^k and h_2^k for $k > 0$ very large, we can ensure that $t > 0$ is sufficiently large so that we have:

- $\lambda t > 1$,
- there exists an open interval $J \subset I$ with $0 < \mu(J) < t$,
- for all $i \in \mathbb{N}$ one has

$$i \leq t\left[\lambda^i - \frac{\lambda^i - 1}{\lambda - 1}\right]. \tag{2}$$

Moreover, replacing f_0 by f_0^k for $k > 0$ large enough, we may suppose that, for *all* $n \in \mathbb{N}$,

$$\left|\tau_\mu(f_0^{-n}h_1 f_0^n) - \lambda^n t\right| \leq 1. \tag{3}$$

Let a (resp. b) be the fixed point of f_0 to the left (resp. to the right) of I. Since f_0 normalizes $\mathbb{Z}^2$, these points are also fixed by $\mathbb{Z}^2$. In Section 4.1, we will show that the dynamics of the subgroup H of $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ generated by f_0 and h_1 is not C^1–smoothable on $[0,1[$ by showing that, actually, it is not C^1–smoothable on $[a,b[$. The case of the open interval $]0,1[$ needs a supplementary argument and will be treated in Section 4.2.

4 Cantwell–Conlon's argument: smooth rigidity

4.1 The case of the half-closed interval

In the statement of Cantwell–Conlon's theorem, there is an additional hypothesis of tangency to the identity at the endpoints. Nevertheless, such a hypothesis is not necessary, as the argument below shows.

Claim 4 *If the action of $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ is by C^1 diffeomorphisms of $[0,1[$, then the restriction of H to $[a,b[$ is topologically conjugate to a group of C^1 diffeomorphisms which are tangent to the identity at a.*

Proof This follows as a direct application of the Müller–Tsuboi's conjugacy trick: it suffices to conjugate by a C^∞ diffeomorphism of $[a,b[$ whose germ at a is that of $x \mapsto e^{-1/x^2}$ at the origin (see Müller [16] and Tsuboi [23] for the details). □

In what follows, we will consider the dynamics of f_0 and h_1 after the preceding conjugacy, so they are tangent to the identity at a.

Remark Since h_1 has a sequence of fixed points converging to a, its derivative at this point must equal 1 even for the original action; nevertheless, this was not necessarily the case for the original diffeomorphism f_0.

Claim 5 *For each $k > 0$, the intervals of the form*

$$(f_0^{-k} h_1 f_0^k)^{\varepsilon_k} \cdots (f_0^{-2} h_1 f_0^2)^{\varepsilon_2} (f_0^{-1} h_1 f_0)^{\varepsilon_1}(J),$$

where $\varepsilon_i \in \{0,1\}$, are two-by-two disjoint.

Proof Let

$$W = (f_0^{-k} h_1 f_0^k)^{\varepsilon_k} \cdots (f_0^{-2} h_1 f_0^2)(f_0^{-1} h_1 f_0)^{\varepsilon_1},$$
$$W' = (f_0^{-k} h_1 f_0^k)^{\varepsilon'_k} \cdots (f_0^{-2} h_1 f_0^2)^{\varepsilon'_2}(f_0^{-1} h_1 f_0)^{\varepsilon'_1}$$

be such that $W \neq W'$, where all $\varepsilon_i, \varepsilon'_i$ belong to $\{0,1\}$. Let i be the largest index for which $\varepsilon_i \neq \varepsilon'_i$, say $\varepsilon_i = 1$ and $\varepsilon'_i = 0$, and let

$$W_* = (f_0^{-i} h_1 f_0^i)(f_0^{-(i-1)} h_1 f_0^{i-1})^{\varepsilon_{i-1}} \cdots (f_0^{-1} h_1 f_0)^{\varepsilon_1},$$
$$W'_* = (f_0^{-(i-1)} h_1 f_0^{i-1})^{\varepsilon'_{i-1}} \cdots (f_0^{-1} h_1 f_0)^{\varepsilon'_1}.$$

Notice that each of the maps $(f_0^{-j} h_1 f_0^j)^{\varepsilon_j}$ either fixes all the points in I (when $\varepsilon_j = 0$) or moves all of them to the right (when $\varepsilon_j = 1$). In particular, W_* moves the left endpoint u of $J=]u,v[$ to a point u_* which coincides with or is to the right of $f_0^{-i}h_1f_0^i(u)$. By (3), this implies that

$$\mu([u,u_*[) \geq \mu([u,f_0^{-i}h_1f_0^i(u)[) = \tau_\mu(f_0^{-i}h_1f_0^i) \geq \lambda^i t - 1. \tag{4}$$

On the other hand, W'_* moves v to a point v'_* which coincides with or is to the left of

$$(f_0^{-(i-1)}h_1f_0^{i-1})\cdots(f_0^{-1}h_1f_0)(v).$$

Since $\mu(J)<t$ and

$$\begin{aligned}\mu\big([v,(f_0^{-(i-1)}h_1f_0^{i-1})\cdots(f_0^{-1}h_1f_0)(v)[\big) &= \tau_\mu\big((f_0^{-(i-1)}h_1f_0^{i-1})\cdots(f_0^{-1}h_1f_0)\big)\\ &= \sum_{j=1}^{i-1}\tau_\mu(f_0^{-j}h_1f_0^j)\\ &\leq \sum_{j=1}^{i-1}\Big(\lambda^j t+1\Big)\\ &= \left[\frac{\lambda^i-1}{\lambda-1}-1\right]t+(i-1),\end{aligned}$$

inequalities (2) and (4) show that v'_* is to the left of u_*. This implies that $W_*(J)$ and $W'_*(J)$ do not intersect, and hence $W(J)\cap W'(J)=\varnothing$. $\square$

To conclude the proof of the fact that the action of $\mathbb{F}_2\ltimes\mathbb{Z}^2$ is not by C^1 diffeomorphisms of $[0,1[$, fix $N\in\mathbb{N}$ so that, for all $x\in[a,b[$ to the left of $f_0^N(I)$,

$$f_0'(x)\geq\sqrt[3]{3/4},\qquad h_1'(x)\geq\sqrt[3]{3/4}.$$

Fix also a positive lower bound $A<1$ for the derivative of f_0 and h_1 to the left of I. By opening brackets in the next expression, one easily checks that the length of each interval of the form

$$(f_0^{-k}h_1f_0^k)^{\varepsilon_k}\cdots(f_0^{-2}h_1f_0^2)^{\varepsilon_2}(f_0^{-1}h_1f_0)^{\varepsilon_1}(J)$$

is at least

$$A^{3N}\big(\sqrt[3]{3/4}\big)^{3(k-N)}|J|.$$

Since there are 2^k of these intervals this yields, for some constant $C>0$,

$$|[a,b]|\geq C\big(\tfrac{3}{2}\big)^k|J|.$$

However, this is clearly impossible for a large k, thus completing the proof.

We close this Section by noticing that similar arguments to those above apply to actions by C^1 diffeomorphisms of the interval $]0,1]$ instead of $[0,1[$.

4.2 The case of the open interval

To prove Theorem A in the general case of the *open* interval, we would like to apply the arguments of the preceding Section. For this, we need to ensure that either a or b actually belongs to $]0,1[$. Indeed, if not, we are not allowed to use the procedure of Claim 4.

Thus, we need to find a hyperbolic element $f \in \mathbb{F}_2$ such that:

(i) (r,s) is not an eigenvector of f^T.

(ii) (r,s) is not orthogonal to any eigenvector of f^{-1},

(iii) neither $(1,0)$ nor $(0,1)$ are eigenvectors of f,

(iv) f has fixed points inside $]0,1[$.

For this, notice that Claim 2 provides us with a free subgroup F on two generators whose non-trivial elements are hyperbolic and satisfy properties (i), (ii), and (iii) above. Now F must contain non-trivial elements having fixed points in $]0,1[$; if not, the action of F on $]0,1[$ would be free, which is in contradiction with Hölder's theorem (see Ghys [12] or Navas [18]). Therefore, any element $f \in F$ having fixed points in $]0,1[$ satisfies (i), (ii), (iii), and (iv), and this concludes the proof of Theorem A.

4.3 The case of the circle

Let G be a non-solvable subgroup of $\mathrm{SL}(2,\mathbb{Z})$. To show Theorem B, we would again like to apply similar arguments to those of Section 4.1. However, there are certain technical issues that need a careful treatment.

First of all, notice that, *a priori*, an irreducible component I for the action of $\mathbb{Z}^2$ is not necessarily an interval: it could coincide with the whole circle. We claim, however, that this cannot happen. Indeed, let (r',s') be the point in $\mathbb{T}^2$ whose coordinates are the rotation numbers of $(1,0)$ and $(0,1)$, respectively. Recall that the rotation number function is invariant under conjugacy. Moreover, its restriction to $\mathbb{Z}^2$ is a group homomorphism into $\mathbb{T}^2$ (see for example Ghys [12, Section 6.6] or Navas [18, Section 2.2.2]). Since G normalizes $\mathbb{Z}^2$, for all $f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ we have (modulo $\mathbb{Z}$)

$$r' = \rho(1,0) = \rho(f(1,0)) = \rho(a,c) = ar' + cs'$$

and

$$s' = br' + ds'.$$

This means that (r',s') is a fixed point for the action of f^T on $\mathbb{T}^2$. But since G is non-solvable, this cannot hold for every $f \in G$, thus showing that I does not coincide with S^1.

Now let μ be a $\mathbb{Z}^2$–invariant Radon measure on I. Let $\tau_\mu\colon \mathbb{Z}^2 \to \mathbb{R}$ be the corresponding translation number homomorphism, and let $r = \tau_\mu((1,0))$ and $s = \tau_\mu((0,1))$. Analogously to the case of Section 4.2, we need to find a hyperbolic element $f \in G$ so that the following conditions are fulfilled:

(i) (r,s) is not an eigenvector of f^T,

(ii) (r,s) is not orthogonal to any eigenvector of f^{-1},

(iii) neither $(1,0)$ nor $(0,1)$ are eigenvectors of f,

(iv) f has fixed points on the circle.

To obtain the desired element, we need to consider two cases separately.

If G does not preserve any probability measure on S^1, then Margulis' alternative [14] and its proof provide us with a free subgroup F (in two generators) of G all of whose elements have fixed points. Claim 2 applied to F then yields the desired element.

If G preserves a probability measure on S^1, then the rotation number function $\rho\colon G \to \mathbb{T}^1$ is a group homomorphism (see Ghys [12] or Navas [18]). Therefore, the rotation number of all the elements in $[G,G]$ is zero, and hence these elements must have fixed points. Since G is non-solvable, $[G,G]$ contains free subgroups, which still allows applying Claim 2.

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