Hausdorff dimension and the Weil–Petersson extension to quasifuchsian space

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We consider a natural nonnegative two-form G on quasifuchsian space that extends the Weil–Petersson metric on Teichmüller space. We describe completely the positive definite locus of G, showing that it is a positive definite metric off the fuchsian diagonal of quasifuchsian space and is only zero on the "pure-bending" tangent vectors to the fuchsian diagonal. We show that G is equal to the pullback of the pressure metric from dynamics. We use the properties of G to prove that at any critical point of the Hausdorff dimension function on quasifuchsian space the Hessian of the Hausdorff dimension function must be positive definite on at least a half-dimensional subspace of the tangent space. In particular this implies that Hausdorff dimension has no local maxima on quasifuchsian space.

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1 Statement of results

Let S be a closed hyperbolic surface and $T(S)$ be the associated *Teichmüller space*. Then the *Weil–Petersson metric* w is a Riemannian metric on $T(S)$. For simplicity, we normalize the Weil–Petersson metric to define the *normalized Weil–Petersson metric*

$$g = \left(\frac{2}{3\pi|\chi(S)|}\right) w \,.$$

If $\mathrm{QF}(S)$ is the quasifuchsian space of S, then by Bers simultaneous uniformization, $\mathrm{QF}(S) \simeq T(S) \times T(\overline{S})$ where $\overline{S}$ has opposite orientation to S. This gives the natural diagonal embedding $\Delta\colon T(S) \to T(S) \times T(\overline{S}) \simeq \mathrm{QF}(S)$ given by $\Delta(X) = (X, \overline{X})$. We let $F(S) = \Delta(T(S))$ the diagonal in $\mathrm{QF}(S)$. Then $F(S)$ corresponds to the subspace of fuchsian elements of $\mathrm{QF}(S)$ and is called the *fuchsian subspace*. It is a smooth submanifold of $\mathrm{QF}(S)$ and we have the natural identification $T(S) \simeq F(S)$ via Δ.

Quasifuchsian space $\mathrm{QF}(S)$ has a complex structure arising out of identifying the isometry group $\mathrm{Isom}_+(\mathbb{H}^3)$ with $\mathrm{PSL}(2,\mathbb{C})$ (see Marden [9]). This complex structure is given by a bundle map $J\colon T(\mathrm{QF}(S)) \to T(\mathrm{QF}(S))$ with J a lift of the identity map

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on $\mathrm{QF}(S)$ and $J^2 = -I$ where I the identity map on $T(\mathrm{QF}(S))$. If $v \in T(\mathrm{QF}(S))$ we will write $J.v = J(v)$ for simplicity.

We define $h\colon \mathrm{QF}(S) \to \mathbb{R}$ by setting $h(X)$ equal to the Hausdorff dimension of the limit set of X. Then by Ruelle [14], h is real-analytic. Also, associated with each $X \in \mathrm{QF}(S)$ is the real-analytic length function $L_{\mu_X}\colon \mathrm{QF}(S) \to \mathbb{R}$ of the *unit Patterson–Sullivan geodesic current* μ_X of X.

In [6], the author and Taylor showed that the function (hL_{μ_X}) on $\mathrm{QF}(S)$ is minimum at X. Using this we defined a natural nonnegative two-form G on $\mathrm{QF}(S)$ given by taking the Hessian of (hL_{μ_X}) at X. Thus

$$G_X = (hL_{\mu_X})''(X).$$

We showed that G extends the normalized Weil–Petersson metric g on $F(S)$. Specifically:

Theorem 1.1 (Bridgeman–Taylor [6]) *There exists a continuous nonnegative two-form G on $\mathrm{QF}(S)$ such that for all $X \in F(S) \subseteq \mathrm{QF}(S)$*

$$\langle v, w\rangle_G = \langle v, w\rangle_g \text{ for all } v, w \in T_X(F(S)) \subset T_X(\mathrm{QF}(S)).$$

In this paper we answer the question of whether G is positive-definite on $\mathrm{QF}(S)$. The answer is that G is "almost" a metric, in particular it is a metric off the fuchsian locus $F(S)$. The complete description of the positive-definite locus of G is given by the following Theorem.

Main Theorem *Let $v \in T_X(\mathrm{QF}(S))$, $v \neq 0$. Then $\|v\|_G = 0$ if and only if*

(1) $X \in F(S)$,

(2) $v = J.w$ *where* $w \in T_X(F(S))$.

Although we will not discuss this aspect further, the Main Theorem has a simple description in terms of the geometry of the associated deformations. If $X \in F(S) \subseteq \mathrm{QF}(S)$ then the tangent space at X decomposes into $T_X(\mathrm{QF}(S)) = T_X(F(S)) \oplus J(T_X(F(S)))$ (see Bonahon [2]). If $w \in T_X(F(S))$ then w corresponds to deforming X inside the fuchsian subspace $F(S)$. By the Earthquake Theorem, this deformation is given by shearing (or twisting) X along a certain measured lamination β (see Kerckhoff [7]). Thus the vectors in $T_X(F(S))$ are called *pure shearing vectors*. If $v \in J(T_X(F(S)))$ then $v = J.w$ where w is a pure shearing vector with some corresponding measured lamination β. It can be shown that v then corresponds to deforming the structure X by bending along measured lamination β (see Bonahon [2]). Thus the vectors in $J(T_X(F(S)))$ are called *pure bending vectors*.

Therefore in terms of deformations, the Main Theorem states that the degenerate vectors for G are exactly the pure bending vectors at fuchsian representations.

The proof of the Main Theorem is via the conformal equivalence of the two-form G with another two-form W obtained by taking the pullback of the so-called pressure metric of thermodynamics. Then the proof of positive-definiteness reduces to showing that the pullback is only trivial for the above tangent vectors. This relation between G and W was suggested by McMullen in the paper [11].

Using the Main Theorem we study properties of the critical points of h. In particular if h is critical at X then the Hessian of h at X is a well-defined symmetric bilinear two-form. Thus the Hessian has a well-defined signature. Applying the Main Theorem we obtain:

Theorem 1.2 *If $X \in \mathrm{QF}(S)$ is a critical point of $h\colon \mathrm{QF}(S) \to \mathbb{R}$ then the Hessian of h at X is positive definite on a subspace of $T_X(\mathrm{QF}(S))$ of dimension at least $6g-6$. In particular h has no local maxima in $\mathrm{QF}(S)$.*

1.1 Background

In [6], the complex structure on $\mathrm{QF}(S)$ was used to define a metric H on Teichmüller space. If $X \in T_X(F(S))$ the associated two-form at X is given by

$$\langle v, w\rangle_H = h''(X)(J.v, J.w)$$

for $v, w \in T_X(F(S)) \subseteq T_X(\mathrm{QF}(S))$.

From the definition of G and the fact that it is nonnegative, we obtain:

Theorem 1.3 (Bridgeman–Taylor [6]) *If $X \in F(S)$ and $v \in T_X(F(S)) \subseteq T_X(\mathrm{QF}(S))$ then*

$$0 \leq \|J.v\|_G^2 = \|v\|_H^2 - \|v\|_g^2$$

where g is the normalized Weil–Petersson metric. Thus the two-form H is a positive definite metric on $F(S)$ and satisfies

$$\|v\|_H \geq \|v\|_g.$$

In [11], McMullen showed that the Weil–Petersson metric was equivalent to the second derivative of various well-defined Hausdorff dimension functions at the fuchsian locus. In particular McMullen proved the following Theorem.

Theorem 1.4 (McMullen [11]) *The metrics H and g are equal.*

The results of this paper arise out of combining the methods outlined in the paper [6] with those of the paper of McMullen [11] and applying them in the nonfuchsian case. We note that a consequence of the Main Theorem is that $\|J.v\|_G = 0$ when $v \in T_X(F(S))$ which we will show gives a new proof of McMullen's result above (Theorem 1.4).

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2 Kleinian groups and geodesic currents

2.1 Kleinian groups

Let $\text{Isom}_+(\mathbb{H}^n)$ $n \geq 2$ be the space of orientation preserving isometries of $\mathbb{H}^n$. As usual, we give the space of isometries the topology of uniform convergence on compact sets. We define a *Kleinian group* Γ to be a discrete torsion-free subgroup of $\text{Isom}_+(\mathbb{H}^n)$. As such, Γ acts properly discontinuously on $\mathbb{H}^n$, and the quotient manifold $N = \mathbb{H}^n/\Gamma$ is a complete Riemannian manifold of constant curvature -1.

A Kleinian group Γ also acts as a discrete subgroup of conformal automorphisms of the sphere at infinity $\mathbb{S}^{n-1}_\infty$; this action partitions $\mathbb{S}^{n-1}_\infty$ into two disjoint sets. The *regular set* Ω_Γ is the largest open set in $\mathbb{S}^{n-1}_\infty$ on which Γ acts properly discontinuously, and the *limit set* Λ_Γ is its complement. In the case that Λ_Γ contains more than 2 points, it is characterized as being the smallest closed Γ–invariant subset of $\mathbb{S}^{n-1}_\infty$.

Define the *convex hull* $\text{CH}(\Lambda_\Gamma)$ of the limit set Λ_Γ to be the smallest convex subset of $\mathbb{H}^n$ so that all geodesics with both limit points in Λ_Γ are contained in $\text{CH}(\Lambda_\Gamma)$. We can take the quotient of $\text{CH}(\Lambda_\Gamma)$ by Γ (denoted by $C(\Gamma)$); this is the *convex core*. It is the smallest convex submanifold of $N = \mathbb{H}^n/\Gamma$ so that the inclusion map is a homotopy equivalence.

A Kleinian group is *convex cocompact* if its associated convex core is compact and it is *geometrically finite* if the volume of the unit neighborhood of the convex core is finite (see Thurston [18]). This paper deals specifically with convex cocompact

Kleinian groups. For the basics in the theory of Kleinian groups we refer the reader to Maskit [10].

If Γ is a geometrically finite Kleinian group, we define the space $\mathrm{QC}(\Gamma)$ of quasiconformal deformations of Γ as follows; We consider pairs (f_0, Γ_0) such that $f_0\colon \mathbb{S}^{n-1}_\infty \to \mathbb{S}^{n-1}_\infty$ is a quasiconformal homeomorphism, conjugating Γ to Kleinian group Γ_0, ie $\Gamma_0 = f\ \Gamma\ f^{-1}$. The map f_0 is called the *marking*. We define an equivalence relation by saying $(f_1, \Gamma_1) \equiv (f_2, \Gamma_2)$ if there exists a conformal map α conjugating Γ_1 to Γ_2, ie

$$f_2 \circ \gamma \circ f_2^{-1} = (\alpha \circ f_1) \circ \gamma \circ (\alpha \circ f_1)^{-1} \quad \text{for all } \gamma \in \Gamma.$$

Then $\mathrm{QC}(\Gamma)$ is the set of equivalence classes under this equivalence relation. For convenience, we will often suppress the map f_0 in describing a point of $\mathrm{QC}(\Gamma)$ and just refer to it by the group.

2.2 Geodesic currents

We can identify a geodesic with its endpoints on $\mathbb{S}^{n-1}_\infty$ and therefore we identify the space of geodesics on $\mathbb{H}^n$ by $G(\mathbb{H}^n) \cong (\mathbb{S}^{n-1}_\infty \times \mathbb{S}^{n-1}_\infty - \text{ diagonal})/\mathbb{Z}_2$.

If N is a convex cocompact hyperbolic n–manifold, with $N = \mathbb{H}^n/\Gamma$, then each nontrivial free homotopy class of closed curve corresponds to a unique multiple of a primitive closed geodesic in N. If α is a primitive closed geodesic in N, we lift α to get a discrete subset of $G(\mathbb{H}^n)$ which is Γ invariant. In this way we identify every nontrivial homotopy class of closed curves on $\mathbb{H}^n/\Gamma$ with a Γ invariant discrete subset of $G(\mathbb{H}^n)$ and a certain integral multiplicity. We then obtain a Γ invariant measure on $G(\mathbb{H}^n)$ by taking the Dirac measure on this discrete set times the multiplicity. This measure is the *geodesic current* associated with the closed curve. We have the following generalization:

Definition A *geodesic current* for Kleinian group Γ is a positive measure on $G(\mathbb{H}^n)$ that is invariant under the action of Γ and supported on the set of geodesics with endpoints belonging to limit set Λ_Γ.

As geodesic currents are Borel σ–finite measures, we can add two geodesic currents and also multiply a geodesic current by a positive constant. A geodesic current which is a constant multiple of a closed geodesic is called a *discrete geodesic current*.

If Γ is a Kleinian group, we let $\mathcal{C}(\Gamma)$ be the space of geodesic currents defined for Γ. The natural topology on $\mathcal{C}(\Gamma)$, via the Radon–Riesz Representation Theorem, is the weak*-topology on the space of continuous functions with compact support in $G(\mathbb{H}^n)$.

Below is a basic fact we will need concerning the topology on $\mathcal{C}(\Gamma)$. The proof involves first showing that the geodesic flow on the unit tangent bundle has the specification property [3; 15], and then applying Theorem 1 in [15].

Theorem 2.1 *Let Γ be a convex cocompact Kleinian group. Then the set of discrete geodesic currents is dense in $\mathcal{C}(\Gamma)$.*

We note that if $[f_0, \Gamma_0] \in \mathrm{QC}(\Gamma)$ then $f_0\colon \Lambda_\Gamma \to \Lambda_{\Gamma_0}$ is a homeomorphism. Therefore by pushing forward measures, we obtain a continuous homeomorphism, $f_0\colon \mathcal{C}(\Gamma) \to \mathcal{C}(\Gamma_0)$ (see Bridgeman and Taylor [5]). This map is the *marking* on the geodesic currents.

2.3 Patterson–Sullivan geodesic current

Fix $s \in \mathbb{R}^+$. We define the *Poincaré series* of a Kleinian group Γ by

$$g_s(x,y) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}$$

where $x, y \in \mathbb{H}^n$ and d is the hyperbolic metric on $\mathbb{H}^n$. Let

$$\delta(\Gamma) = \inf\{s : g_s < \infty\};$$

then $\delta(\Gamma)$ is called the *exponent of convergence* of the Poincaré series. We refer the reader to Nicholls [12] for further details on the exponent of convergence.

Following the work of Patterson and Sullivan, a measure can be constructed on $\mathbb{S}^{n-1}_\infty$ which is supported on Λ_Γ. For $x, y \in \mathbb{H}^n$ and $s > \delta(\Gamma)$, we define a measure $\sigma_{x,s}$ supported on the orbit of y by

$$\sigma_{x,s} = \frac{1}{g_s(y,y)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)} D(\gamma y)$$

where $D(p)$ is Dirac measure at p. The *Patterson–Sullivan* measure σ_x is constructed by taking a limit of these measures as $s \to \delta(\Gamma)^+$. The measure σ_x can be used to define a measure $\widetilde{m}$ on $(\mathbb{S}^{n-1}_\infty \times \mathbb{S}^{n-1}_\infty - \text{diagonal})$ given by

$$d\widetilde{m} = \frac{d\sigma_x(a)\,d\sigma_x(b)}{|b-a|^{2\delta(\Gamma)}}. \tag{1}$$

We then obtain a geodesic current m by taking the pushforward of $\widetilde{m}$ under the $\mathbb{Z}_2$ cover $\pi\colon (\mathbb{S}^{n-1}_\infty \times \mathbb{S}^{n-1}_\infty - \text{diagonal}) \to G(\mathbb{H}^n)$ given by $\pi(a,b) = g$ where g is the geodesics with endpoints a, b. This measure $m = \pi_*(\widetilde{m})$ is Γ–invariant and supported on $(\Lambda_\Gamma \times \Lambda_\Gamma - \text{ diagonal})/\mathbb{Z}_2$. Therefore it is a geodesic current and is called a

Patterson–Sullivan geodesic current for Γ. By work of Sullivan [17], for Γ being geometrically finite, m is independent of choice of x up to scalar multiple.

2.4 Length functions

If Γ is convex cocompact Kleinian group then associated to each element $\gamma \in \Gamma$ is a natural length function $L_\gamma\colon \mathrm{QC}(\Gamma) \to \mathbb{R}$ given by letting $L_\gamma([f_0, \Gamma_0])$ be the translation length of the element $f_0 \circ \gamma \circ f_0^{-1} \in \Gamma_0$. This function is naturally a smooth function on $\mathrm{QC}(\Gamma)$. Similarly, if $\mu \in \mathcal{C}(\Gamma)$ is a discrete geodesic current then μ is a multiple r of a closed geodesic α. We then choose $\gamma \in \Gamma$ to be a lift of the action α and define L_μ by letting $L_\mu = rL_\gamma$.

This can be generalized for geodesic currents to obtain the following result.

Length Function Theorem (Bridgeman–Taylor [6]) *Let Γ be a convex cocompact Kleinian group acting on $\mathbb{H}^3$. Then there is a continuous function*

$$L\colon \mathcal{C}(\Gamma) \to C^\infty(\mathrm{QC}(\Gamma), \mathbb{R})$$

such that $L(\mu) = L_\mu$ for μ a discrete geodesic current where $C^\infty(\mathrm{QC}(\Gamma), \mathbb{R})$ is the space of smooth real-valued functions on $\mathrm{QC}(\Gamma)$ with the C^∞–topology.

Given $\mu \in \mathcal{C}(\Gamma)$, we define $L_\mu\colon \mathrm{QC}(\Gamma) \to \mathbb{R}$ by $L_\mu(X) = L(\mu, X)$. The function L_μ is the *length function* for μ.

We note that the continuity of L implies that if $\mu_i \to \mu$ then $L_{\mu_i} \to L_\mu$ uniformly on compacts subsets of $\mathrm{QC}(\Gamma)$.

2.5 Quasifuchsian space

Recall that a *fuchsian group* Γ is a finitely generated Kleinian group which acts invariantly on an open geometric disk in $\mathbb{S}^2_\infty$. Identifying $\mathbb{S}^2_\infty$ with the extended complex plane $\widehat{\mathbb{C}}$, we consider Γ as a group of Möbius transformations on $\widehat{\mathbb{C}}$ with limit set a subset of the extended real line $\overline{\mathbb{R}}$ such that Γ preserves each component of $\widehat{\mathbb{C}} - \overline{\mathbb{R}}$. Then the hyperbolic plane $\mathbb{H}^2$ with boundary $\overline{\mathbb{R}}$ is invariant under Γ and $S = \mathbb{H}^2/\Gamma$ is a hyperbolic surface.

Let Γ be convex cocompact and fuchsian, then the quotient manifold $\mathbb{H}^3/\Gamma$ is homeomorphic to $S \times \mathbb{R}$, where S is the closed hyperbolic surface given by $\mathbb{H}^2/\Gamma$. To emphasize that we are dealing with a special case, $\mathrm{QC}(\Gamma)$ is called the *quasifuchsian space* of S and denoted by $\mathrm{QF}(S)$. Also we denote the space of currents $\mathcal{C}(\Gamma)$ by $\mathcal{C}(S)$. Furthermore we will denote the fuchsian elements of $\mathrm{QF}(S)$ by $F(S)$.

By Bers simultaneous uniformization we have $\mathrm{QF}(S) \simeq T(S) \times T(\bar{S})$ where $T(S)$ is the Teichmüller space of S and $F(S)$ corresponds to the diagonal in $T(S) \times T(\bar{S})$ (see Bers [1]). Thus if $\Delta\colon T(S) \to T(S) \times T(\bar{S})$ is the map $\Delta(X) = (X, \bar{X})$, then $F(S) \simeq T(S)$.

In the quasifuchsian case we have the following extension of the real length function L to a complex length function $\mathcal{L}$.

Complex Length Theorem (Bridgeman–Taylor [6]) *For each $\mu \in \mathcal{C}(S)$ there exists a unique holomorphic function $\mathcal{L}_\mu\colon \mathrm{QF}(S) \to \mathbb{C}$ with real part L_μ and imaginary part satisfying $Im(\mathcal{L}_\mu) = 0$ on $F(S)$. Furthermore the function*

$$\mathcal{L}\colon \mathcal{C}(S) \to C^\omega(\mathrm{QF}(S), \mathbb{C})$$

given by $\mathcal{L}(\mu) = \mathcal{L}_\mu$ is continuous with respect to the topology of uniform convergence (on compacta) on the space $C^\omega(\mathrm{QF}(S), \mathbb{C})$ of holomorphic functions on $\mathrm{QF}(S)$.

Convention If $f\colon X \to Y$ is a smooth function then we will let $f'(x)$ denote the derivative map $f'(x)\colon T_x(X) \to T_{f(x)}(Y)$. To simplify, if $v \in T_x(X)$ we will often write $f'(v) = (f'(x))\,(v)$. Similarly if $f'(x) = 0$ then the Hessian of f is denoted by $f''(x)$ and is the well-defined symmetric bilinear two-form given by

$$(f''(x))(v, w) = \frac{\partial^2 f}{\partial v \partial w}.$$

Once again we will often shorten and write $f''(v, w) = (f''(x))(v, w)$.

3 Weil–Petersson extension G

We now describe the symmetric bilinear form G on $\mathrm{QF}(S)$ given in [6].

Let $X = [f_0, \Gamma_0] \in \mathrm{QF}(S)$, then f_0 gives a natural homeomorphism $f_0\colon \mathcal{C}(S) = \mathcal{C}(\Gamma) \to \mathcal{C}(\Gamma_0)$ between geodesic current spaces coming from the marking. We let $m_{\Gamma_0} \in \mathcal{C}(\Gamma_0)$ be a Patterson–Sullivan geodesic current and pullback to define $m_X = f_0^{-1}(m_{\Gamma_0}) \in \mathcal{C}(S)$. We normalize to define the unit length Patterson–Sullivan geodesic current of X by

$$\mu_X = \frac{m_X}{L(X, m_X)}.$$

Then this geodesic current has unit length in X.

In [6], we show that the function $(hL_{\mu_X})\colon \mathrm{QF}(S) \to \mathbb{R}$ given by $(hL_{\mu_X})(Y) = h(Y)L_{\mu_X}(Y)$ is minimum at X. Using this we defined G to be the symmetric bilinear form at X given by

$$G_X = (hL_{\mu_X})''(X).$$

Finally we proved Theorem 1.1, showing that G is a natural extension of the normalized Weil–Petersson metric on $F(S)$.

4 Thermodynamics and pressure metric

We will now describe the pressure metric for a shift of finite type. This will be a cursory introduction to the elements of Thermodynamic Formalism needed to state and prove our results. For a complete description see the book [13] by Parry and Pollicott and the paper [11] of McMullen.

Let A be a $k \times k$ matrix of zeros and ones then we define the associated *(one-sided) shift of finite type* by (Σ, σ) where Σ is the set of sequences

$$\Sigma = \left\{x = (x_n)_{n=0}^{\infty} : x_n \in \{1, \dots, k\}, A(x_n, x_{n+1}) = 1\right\}$$

and $\sigma \colon \Sigma \to \Sigma$ is the standard shift where $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$. We give $\{i, \dots, k\}$ the discrete topology and Σ the associated product topology.

The space $C(\Sigma)$ is the space of continuous real valued functions on Σ. Two functions $f, g \in C(\Sigma)$ are *cohomologous* ($f \sim g$), if there exists a continuous function $h \in C(\Sigma)$ such that $f(x) - g(x) = h(\sigma(x)) - h(x)$. If $f \sim 0$ then f is a *coboundary*.

We can metrize the topology on Σ by choosing any $K > 1$ and then defining $d(x, y) = K^{-N}$ where $N = N(x, y) = \min\{n \mid x_n \neq y_n\}$.

Then given $\theta \in (0, 1)$ we say $f \in F_\theta(\Sigma)$ if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \le C\theta^{N(x,y)}.$$

The set F_θ is the set of Hölder continuous functions with the same Hölder constant, with respect to the metric d.

$F_\theta(\Sigma)$ is given the norm $\|\cdot\|_\theta$ by

$$\|f\|_\theta = \|f(x)\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\theta^{N(x,y)}}.$$

Given a map f we can take the iterated sum $S_n f$ defined by

$$(S_n f)(x) = \sum_{k=0}^{n-1} f(\sigma^k(x)).$$

If $f \sim g$ with $f(x) - g(x) = h(\sigma(x)) - h(x)$ then

$$S_n f(x) - S_n g(x) = h(\sigma^n(x)) - h(x)$$

so shifts of cohomologous functions are cohomologous. Also if $f \in F_\theta(\Sigma)$, the *Ruelle operator* $L_f \colon F_\theta(\Sigma) \to F_\theta(\Sigma)$ is defined by

$$(L_f g)(x) = \sum_{\sigma(y)=x} e^{f(y)} g(y).$$

We note that under iteration of the Ruelle operator we have

$$(L_f^n g)(x) = \sum_{\sigma^n(y)=x} e^{S_n f(y)} g(y).$$

The shift (Σ,σ) is *aperiodic* if there exists an $n > 0$ such that A^n is all positive entries. We have the following generalization of the Perron-Frobenius Theorem for matrices.

Theorem 4.1 (Ruelle–Perron–Frobenius [13]) *Let $f \in F_\theta(\Sigma)$ and let (Σ,σ) be an aperiodic shift of finite type. Then:*

(1) *There is a simple maximal positive eigenvalue β for L_f with corresponding strictly positive eigenvector h.*

(2) *The remainder of the spectrum of L_f is contained in a disk of radius strictly smaller than β.*

(3) *There is a unique probability measure μ such that $L_f^* \mu = \beta \,.\, \mu$.*

(4) *Let h be a maximal eigenvector normalized so that $\mu(h) = 1$. Then*

$$\frac{L_f^n(g)}{\beta^n} \longrightarrow h \int g \, d\mu \qquad \textit{uniformly for all } g \in C(\Sigma).$$

The *pressure* $P(f)$ is defined by $P(f) = \log \beta$. If $f \in F_\theta(\Sigma)$ satisfies $P(f) = 0$ and h is a maximal normalized eigenvector of L_f then the measure $m = h \,.\, \mu$ is an ergodic σ–invariant probability measure and is called the *equilibrium measure* of f.

In [13] the properties of the function $P \colon F_\theta(\Sigma) \to F_\theta(\Sigma)$ are described in detail. In particular it is convex and real-analytic and depends only on cohomology class.

Also if $P(f) = 0$, with equilibrium measure m and $g \in F_\theta(\Sigma)$ then

$$P'(f)(g) = \frac{d}{dt} P(f + tg)\Big|_{t=0} = \int g \, dm.$$

Also if $P'(f)(g) = 0$ then the *variance* $\mathrm{Var}(g,m)$ is defined by

$$P''(f)(g) = \frac{d^2}{dt^2} P(f + tg)\Big|_{t=0} = \mathrm{Var}(g,m).$$

We define $T(\Sigma)$ to be the set of pressure zero, Hölder continuous functions up to coboundary, that is

$$T(\Sigma) = \{f : f \in F_\theta(\Sigma) \text{ for some } \theta,\, P(f) = 0\}/\sim.$$

If $[f] \in T(\Sigma)$ and f has equilibrium measure m, then by the formula for the derivative of pressure P, the tangent space of $T(\Sigma)$ at $[f]$ can be identified with

$$T_{[f]}T(\Sigma) = \left\{g : \int g\,dm = 0\right\}\Big/\sim.$$

The pressure metric $\|\cdot\|_P$ on $T(\Sigma)$ is then defined by

(2) $$\|[g]\|_P = \frac{\mathrm{Var}(g,m)}{-\int f\,dm}.$$

By Theorem 4.2 of [13], $\mathrm{Var}(g,m) = 0$ implies that $g \sim 0$. Thus $\|[g]\|_P = 0$ implies $[g] = 0$ and therefore $\|\cdot\|_P$ is positive definite metric on $T(\Sigma)$.

5 Thermodynamics on QF(S)

Let Γ be a quasifuchsian group with limit set $\Lambda_\Gamma \subset \widehat{\mathbb{C}}$. A conformal *Markov* map for Γ is a piecewise conformal map $f\colon \Lambda_\Gamma \to \Lambda_\Gamma$ such that Λ_Γ has a partition into segments $J_1, \dots, J_m$ so that

(1) $f|_{J_k} = \gamma_k|_{J_k}$ for some $\gamma_k \in \Gamma$,

(2) for each k, $f(J_k)$ is the union of various J_l's.

A Markov map is *expanding* if there is an $n > 0$ such that the n-th iterate $f^n = f \circ f \circ \dots \circ f$ has derivative whose norm in the spherical metric satisfies

$$|(f^n)'(x)| > C > 1$$

and for any $U \subset L_\Gamma$ open, there exists an $m > 0$ such that $f^m(U) = \Lambda_\Gamma$.

If Γ has an expanding Markov map f then we can define a matrix A by $A(i,j) = 1$ if $J_j \subset f(J_i)$ and zero otherwise. Then we have an aperiodic shift (Σ, σ) and we define $\pi\colon \Sigma \to L_\Gamma$ by $\pi(x) = z$ where $f^i(z) \in J_{x_i}$. The map f obviously satisfies $f(\pi(x)) = \pi(\sigma(x))$. The map π is surjective but as the segments J_i may have boundary points in common, the map π is two to one on a countable set of points P. If Q is the finite set of endpoints of the J_i's then P is precisely

$$P = \bigcup_{n=0}^{\infty} f^{-n}(Q).$$

The points of P are called *bad* points and if $z \notin P$ it is called a *good* point. We note that if z is a good point, then there is a unique $x \in \Sigma$ such that $\pi(x) = z$ and for any $n > 0$, there is unique $\gamma_n \in \Gamma$ such that $f^n = \gamma_n$ on an open interval about z.

5.1 Expanding Markov map for quasifuchsian groups

In the following we describe Bowen's results from [4] on expanding Markov maps for quasifuchsian groups.

Bowen first considered the cocompact fuchsian group Γ_r obtained by identifying sides of a regular hyperbolic $4n$–gon in the standard way given by the side labelling

$$x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_n y_n x_n^{-1} y_n^{-1}.$$

He then described an expanding Markov map $f_{\Gamma_r}\colon \mathbb{S}^1 \to \mathbb{S}^1$ for Γ_r which we will describe in detail below.

Then if $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal map conjugating the action of Γ_r to the action of Γ, then this gives the map $f_\Gamma\colon \Lambda_\Gamma \to \Lambda_\Gamma$ by $f_\Gamma = g \circ f_{\Gamma_r} \circ g^{-1}$ and $\pi_\Gamma\colon \Sigma \to \Lambda_\Gamma$ by $\pi_\Gamma = g \circ \pi_{\Gamma_r}$. Then f_Γ is an expanding Markov map for Γ with the same shift space (Σ, σ).

The function $\phi_\Gamma\colon \Sigma \to \mathbb{R}$ defined by $\phi_\Gamma(x) = -\log|f'_\Gamma(\pi_\Gamma(x))|$ is Hölder continuous. By the chain rule for differentiation we have

$$(S_n\phi_\Gamma)(x) = -\log|(f_\Gamma^n)'(\pi_\Gamma(x))|. \tag{3}$$

Then if h_Γ is the Hausdorff dimension of the limit set Λ_Γ, Bowen showed that h_Γ is characterized by the equation

$$P(h_\Gamma \phi_\Gamma) = 0. \tag{4}$$

We now describe the map f_{Γ_r} in more detail. The group Γ_r has fundamental domain D, the regular hyperbolic $4n$–gon. We label the sides of D by $s_i, i = 1, \ldots, 4n$. Each s_i belongs to a unique geodesic g_i with endpoints p_i, q_i on $\mathbb{S}^1$. We let I_i be the interval on $\mathbb{S}^1$ with endpoints p_i, q_i which is smallest in length. We further define $\gamma_i \in \Gamma_r$ to be the element which identifies s_i with another side s_j of D for some j.

For each $\gamma \in \Gamma_r$ we let $D_\gamma = \gamma(D)$ and say D_γ *abuts* D if $D_\gamma \cap D \neq \varnothing$. For each D_γ we define the intervals $I_{\gamma,i} = \gamma(I_i)$. We let $R \subseteq \mathbb{S}^1$ be the union of the endpoints of $I_{\gamma,i}$ for D_γ abutting D. Then R defines a decomposition of $\mathbb{S}^1$ into intervals J_k. We note that each $J_k \subseteq I_j$ for some j, not necessarily unique. We then define

$$f_{\Gamma_r}|_{J_k} = \gamma_j \text{ where } J_k \subseteq I_j.$$

It follows that for any choice of j such that $J_k \subseteq I_j$, the map f_{Γ_r} is a Markov map for Γ_r. Bowen makes a specific choice to define $f_{\Gamma_r}\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that f_{Γ_r} is an expanding Markov map and orbit equivalent to the action of Γ_r on $\mathbb{S}^1$ (see Bowen [4]).

We say a geodesic g *abuts* D if it intersects one of the images of D in the tessellation that abuts D. We now describe an elementary property of the map f_{Γ_r} that follows easily from its definition.

Lemma 5.1 *Let $f_{\Gamma_r}\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the expanding map for the Γ_r described above. Let g be a geodesic with endpoints a, b that abuts D. If $f_{\Gamma_r} = \gamma \in \Gamma_r$ at a, then geodesic $\gamma(g)$ abuts D.*

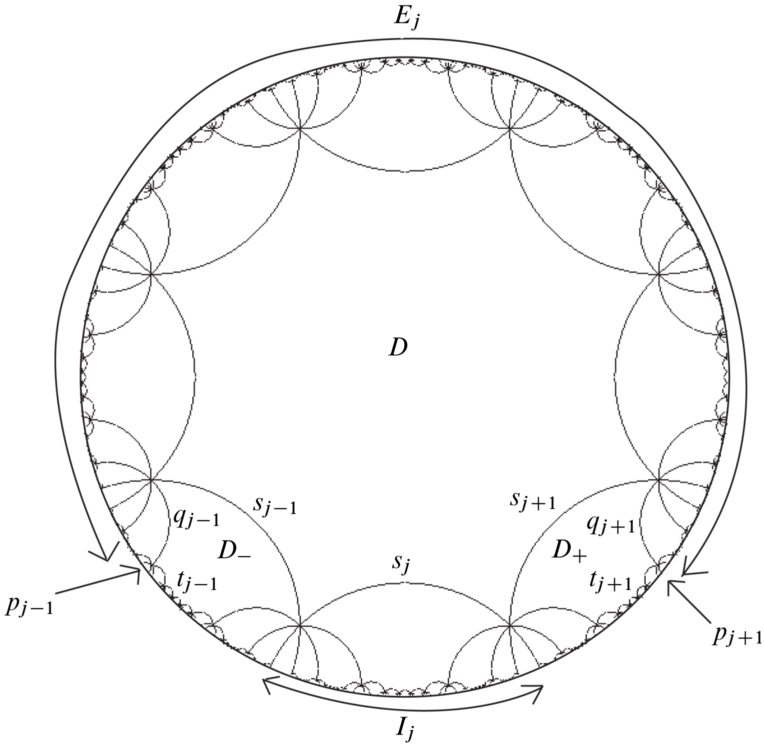

Figure 1: Tesselation by regular $4n$–gons

Proof We place D with the origin in the center. We label the edges e_i of D clockwise for $i = 1, \ldots, 4n$. The edges e_i define geodesics g_i which given overlapping intervals I_i. We further define the half-plane given by I_i, H_i (see Figure 1).

Let g be a geodesic with endpoints a, b which abuts D and $f_{\Gamma_r} = \gamma$ at a. Then $\gamma = \gamma_j$ for some j and $a \in I_j$. We let P be the convex polygon obtained by taking the union of all domains that abut D. Then by assumption g intersects P.

If s_j is the side of D corresponding to the side identification γ_j, we let P_1 be the collection of domains which intersect s_j (ie contain s_j as a side or contain an endpoint of s_j as a vertex). As $\gamma_j(s_j) = s_k$ some other side of D, if g intersects P_1 then $\gamma_j(g)$ abuts D. We will now show that if g intersects P then it intersects P_1 thereby proving the result.

Let P_2 be the set of domains $P \cap H_j$. As $H_j \cap D = s_j$, then $P_2 \cap P_1$. As we are assuming g does not intersect P_1, then g does not intersect P_2. If both endpoints of g are in I_j then by convexity $g \subseteq H_j$ and therefore $\varnothing \neq g \cap P = g \cap (P \cap H_j) = g \cap P_2 \subseteq g \cap P_1$. Thus if g has both endpoints in I_j then g intersects P_1.

We let D_- be the domain that shares the side s_{j-1} with D and D_+ be the domain that shares the side s_{j+1} with D. Then as s_{j-1}, s_{j+1} share a vertex with s_j then $D_-, D_+ \subseteq P_1$.

In domain D_- we label the opposite side to s_{j-1} by t_{j-1}. We let h_{j-1} be the geodesic associated to t_{j-1} and the interval $T_{j-1} \subseteq I_{j-1}$. The interval T_{j-1} does not intersect with any other interval I_k for $k \neq j-1$. Also D_- shares a unique side with a domain of $P - P_1$ (abutting D but not side s_j). We label the geodesic to the side q_{j-1} and its unique endpoint $p_{j-1} \in I_{j-1}$. We define similar quantities for D_+. We let E_j be the interval in $\mathbb{S}^1$ with endpoints p_{j-1}, p_{j+1} and not containing I_j. Then E_j is disjoint from T_{j-1} and T_{j+1}. If g intersects a domain of P but not of P_1 then g must have an endpoint in E_j. Therefore g separates the geodesics h_{j-1}, h_{j+1}. Therefore g must intersect either D_-, D_+ or D. Thus g intersects P_1. □

The above Lemma says that the abutting geodesics are an invariant set under the Möbius map defined by their endpoints.

We now prove an important property of the expanding Markov map f_{Γ_r} that we will need later.

If G is a group, then we say g is commensurable to h if there exists $k \in G$ such that $g^n = kh^mk^{-1}$ for some $n, m \neq 0$. The set of commensurability classes of G is denoted $[G]$. We note that for Γ a cocompact Kleinian group, then $[\Gamma]$ is equivalent to the set of primitive geodesics in $N = \mathbb{H}^n/\Gamma$.

Lemma 5.2 *Let $f_{\Gamma_r}\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the expanding Markov map described above. Then there is a finite set $\mathcal{S} \subseteq [\Gamma_r]$ of commensurability classes such that if $[\gamma] \notin \mathcal{S}$, then:*

(1) *If $\gamma' \in [\gamma]$ then the endpoints of the axis of γ' are good points of f_{Γ_r}.*
(2) *There exists a $\gamma' \in [\gamma]$ whose axis abuts D and has fixed points a, b such that the expanding fixed point a of γ' is a periodic point of f_{Γ_r}.*

Proof We note that as Γ_r does not contain parabolics, then if two elements γ_1, γ_2 have a common fixed point, then they share the same axis and are commensurate. Let D be the fundamental domain for Γ_r given by the regular $4n$–gon in the Poincare disk model with center at 0. We extend the $4n$ sides of D to complete geodesics g_i, $i = 1, \ldots, 4n$, and let P_0 be the union of the endpoints of the g_i's. We then define $P = \Gamma_r P_0$, the orbit of P_0 under the group. The set P is precisely the set of bad points for f_{Γ_r}. Also as D is the regular $4n$–gon, each of the geodesics g_i is the axis for an element of Γ and we let γ_i be the corresponding primitive element. Then we let $\mathcal{S} = \{[\gamma_i]\}_{i=1}^{4n}$. If an element $\gamma \in \Gamma_r$ has a bad endpoint $z \in P$, then $z = \gamma_1.z_0$ for $z_0 \in P_0$ then $\gamma_1^{-1}\gamma\gamma_1$ has fixed point z_0. Therefore $\gamma_1^{-1}\gamma\gamma_1$ shares an endpoint with some γ_i. Therefore γ and γ_i are commensurate. Thus we conclude that if $[\gamma] \notin \mathcal{S}$, then γ has good endpoints.

We now let $[\gamma] \notin \mathcal{S}$ and choose γ such that its axis intersects D. As $[\gamma] \notin \mathcal{S}$, then its axis g has endpoints a_0, b_0. We define geodesic g_n to have endpoints $a_n = f_{\Gamma_r}^n(a)$, and let $b_n \in \mathbb{S}^1$ be the unique point such that the pair $(a_n, b_n) \in \mathbb{S}^1 \times \mathbb{S}^1$ are endpoints of the axis of a conjugate of γ. Then by Lemma 5.1, g_n also abuts D. By compactness of the union of domains abutting D, we have there is an $\epsilon > 0$ such that $|a_n - b_n| > \epsilon$ for all n. But if the sequence $\{(a_n, b_n)\}$ in $\mathbb{S}^1 \times \mathbb{S}^1$ has an infinite number of values, it must have a convergent subsequence. As the orbit of the axis g under Γ_r is discrete in the space of geodesics $G(\mathbb{H}^2)$, any convergent subsequence must converge to a point on the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$ contradicting $|a_n - b_n| > \epsilon$. Therefore the sequence takes a finite set of values and there exists a $k > 0$ and an n such that $(a_n, b_n) = (a_{n+k}, b_{n+k})$ and therefore $f_{\Gamma_r}^k(a_n) = a_n$ and a_n is a periodic point for f_{Γ_r}. We let γ' be the conjugate of γ with endpoints (a_n, b_n), giving the result. □

5.2 Pullback of pressure metric

We first define the *Hausdorff dimension function* h: $\mathrm{QF}(S) \to \mathbb{R}$ by $h([f_0, \Gamma_0]) = h_{\Gamma_0}$, the Hausdorff dimension of the limit set Λ_{Γ_0}. This is well-defined, and by Ruelle [14], h is real-analytic.

We let Γ be a fuchsian group such that $S = \mathbb{H}^2/\Gamma$ with expanding Markov map f_Γ as described in the Section 5.1. We let (Σ, σ) be the associated shift and $\pi_\Gamma; \Sigma \to \mathbb{S}^1$ as before. Then for each $X \in \mathrm{QF}(S)$ we let $X = [g_0, \Gamma_0]$ where g_0 conjugates Γ to Γ_0. Then we define $\phi_X = \phi_{\Gamma_0}$: $\Sigma \to \mathbb{R}$ and $\Phi_X = h_{\Gamma_0}\phi_{\Gamma_0}$.

Note that if $[g_0, \Gamma_0] = [g_1, \Gamma_1]$, then $\phi_{\Gamma_0}, \phi_{\Gamma_1}$ are cohomologous (see McMullen [11]). Also, by Equation (4), $P(\Phi_X) = 0$. Thus we obtain a well-defined map F: $\mathrm{QF}(S) \to T(\Sigma)$ by $F(X) = [\Phi_X]$. We then define W to be the pullback of the pressure metric on $T(\Sigma)$. As the pressure metric is positive definite it follows that W is at least nonnegative.

To obtain a formula for $\|\cdot\|_W$, given $v \in T_X(\mathrm{QF}(S))$, we choose a smooth curve $\alpha\colon (-\epsilon,\epsilon) \to \mathrm{QF}(S)$ with $\alpha(0) = X$ and $\alpha'(0) = v$. Then let $\alpha(t) = X_t = [g_t, \Gamma_t]$ where g_t is a smooth 1–parameter family of quasiconformal maps conjugating Γ to Γ_t.

We let $\Phi_t = \Phi_{\alpha(t)}$ and $\phi_t = \phi_{\alpha(t)}$ and define $\dot{\Phi}_0$ by

$$\dot{\Phi}_0(x) = \frac{d}{dt}\Big|_{t=0}(\Phi_t(x))$$

Then by definition of the pressure metric in Equation (2), $\|v\|_W$ is given by

$$\|v\|_W^2 = \frac{\mathrm{Var}(\dot{\Phi}_0, m)}{-\int \Phi_0\, dm}.$$

We obtain an alternative definition of $\|\cdot\|_W$ by noting that $P(\Phi_t) = 0$ and taking derivatives with respect to t. Taking first derivatives we obtain

$$P'(\Phi_t)(\dot{\Phi}_t) = 0.$$

Then taking derivative again we have

$$P''(\Phi_t)(\dot{\Phi}_t) + P'(\Phi_t)(\ddot{\Phi}_t) = 0.$$

Evaluating at $t = 0$ we have

$$\mathrm{Var}(\dot{\Phi}_0, m) + \int \ddot{\Phi}_0\, dm = 0.$$

Therefore we have

$$\|v\|_W^2 = \frac{\mathrm{Var}(\dot{\Phi}_0, m)}{-\int \Phi_0\, dm} = \frac{\int \ddot{\Phi}_0\, dm}{\int \Phi_0\, dm}. \tag{5}$$

6 Conformal equivalence of G and W

The proof that G and W are conformally equivalent follows by generalizing the argument in [11] of McMullen for the fuchsian subspace $F(S)$ to all of quasifuchsian space $\mathrm{QF}(S)$.

Theorem 6.1 *The pseudometrics G and W are conformally equivalent with*

$$\|v\|_G = \sqrt{h(X)}\,.\,\|v\|_W \text{ for } v \in T_X(\mathrm{QF}(S)).$$

Proof From Equation (5), we have

$$\|v\|_W^2 = \frac{\int \ddot{\Phi}_0 \, dm}{\int \Phi_0 \, dm} = \frac{\frac{d^2}{dt^2}\big(h(X_t)\int \phi_t \, dm\big)\big|_{t=0}}{h(X)\int \phi_0 \, dm} = \frac{1}{h(X)}(hF)''(v)$$

where
$$F(t) = \frac{\int \phi_t \, dm}{\int \phi_0 \, dm}.$$

Therefore the result follows from showing that $F(t) = L_{\mu_X}(X(t))$ where $\mu_X \in \mathcal{C}(S)$ is the unit Patterson–Sullivan geodesic current for X.

By the density of discrete geodesic currents (Theorem 2.1), there exists a sequence of discrete geodesic currents μ_n such that $\mu_n \to \mu_X$. As μ_X is unit length in X we can normalize so that μ_n are unit length in X. Therefore $\mu_n = \alpha_n / l_n$ where α_n is a geodesic current coming from Dirac measure on the lifts of a primitive geodesic (also labeled α_n) and l_n is the length of α_n in X.

We choose as our basepoint for QF(S) the fuchsian group $\Gamma = \Gamma_r$ described in Section 5.1. By Lemma 5.2 for each α_n we can choose a lift $\gamma_n \in \Gamma$ such that the axis g_n of γ_n abuts D and has fixed points a_n, b_n with expanding fixed point a_n being a periodic point for f_Γ with period p_n.

As $f_\Gamma^{p_n}(a_n) = a_n$, then $f_\Gamma^{p_n} = \gamma \in \Gamma$ in an open neighborhood of a_n where γ fixes a_n. Then γ and γ_n both fix a_n and therefore are commensurate with axes being equal. As γ_n is primitive, it follows that $\gamma = \gamma_n^{k_n}$ for some nonzero integer k_n. Letting $X(t) = [g_t, \Gamma_t]$, then $g_t(a_n)$ is a fixed point of $f_{\Gamma_t}^{p_n}$. Also if we let $\gamma_{n,t} = g_t \circ \gamma_n \circ g_t^{-1}$ then $\gamma_{n,t}$ has fixed point $g_t(a_n)$ and $f_{\Gamma_t}^{p_n} = (\gamma_{n,t})^{k_n}$ in an open neighborhood of $g_t(a_n)$.

Then for $i = 0, p_n - 1$ we let $g_n(i)$ be the element of the orbit of g_n with endpoints $a_n(i)$ and $b_n(i)$, where $a_n(i) = f_\Gamma^i(a_n)$. Since geodesic g_n abuts D, then by Lemma 5.1, $g_n(i)$ must also abut D. Therefore by compactness of the finite union of domains abutting D, there is an $\epsilon > 0$ such that $|a_n(i) - b_n(i)| > \epsilon$ for all n, i.

We let m_n be the probability measure on $\mathbb{S}^1$ obtained by taking $1/p_n$ dirac measure on the $a_n(i), i = 0, \dots, p_{n-1}$.

We have $\phi_t \colon \Sigma \to \mathbb{R}$ is given by $\phi_t(x) = -\log|f'_{\Gamma_t}(\pi_t(x))|$ where $f_{\Gamma_t} = g_t \circ f_\Gamma \circ g_t^{-1}$ and $\pi_t = g_t \circ \pi_\Gamma$. Therefore $\phi_t(x) = \overline{\phi}_t(\pi_\Gamma(x))$ where $\overline{\phi} \colon \mathbb{S}^1 \to \mathbb{R}$ is the map $\overline{\phi}_t(z) = -\log|f'_{\Gamma_t}(g_t(z))|$. Then

$$m_n(\overline{\phi}_t) = \int_{\mathbb{S}^1} \overline{\phi}_t(z)\, dm_n = -\frac{1}{p_n}\log|(f_{\Gamma_t}^{p_n})'(g_t(a_n))|.$$

As $g_t(a_n)$ is a fixed point of $f_{\Gamma_t}^{p_n}$ with $f_{\Gamma_t}^{p_n} = \gamma_t^{k_n}$ in an open neighborhood of $g_t(a_n)$ we have

$$m_n(\overline{\phi}_t) = \int_{\mathbb{S}^1} \overline{\phi}_t \, dm_n = -\frac{1}{p_n} \log |((\gamma_{n,t})^{k_n})'(g_t(a_n))| = -\frac{k_n}{p_n} L_{\gamma_n}(X_t).$$

In particular we have

$$m_n(\overline{\phi}_0) = -\frac{k_n}{p_n} L_{\gamma_n}(X) = -\frac{k_n l_n}{p_n}.$$

Therefore
$$\frac{m_n(\overline{\phi}_t)}{m_n(\overline{\phi}_0)} = \frac{L_{\gamma_n}(X_t)}{L_{\gamma_n}(X)} = L_{\mu_n}(X_t).$$

We now show that l_n/p_n is bounded. As the map $\overline{\phi}_0$ is bounded on $\mathbb{S}^1$, there exists a C such that $|\overline{\phi}_0| \leq C$. As μ_n is a probability measure and k_n is a nonzero integer,

$$\frac{l_n}{p_n} \leq \left|\frac{k_n l_n}{p_n}\right| \leq \left|\int \overline{\phi}_0 \, dm_n\right| \leq \int |\overline{\phi}_0| \, dm_n \leq C. \tag{6}$$

Let ν_n be the probability measure on $G(\mathbb{H}^3)$ obtained by taking $1/p_n$ times Dirac measure on set of geodesics $g_n(i)$ given by the endpoint pair $(a_n(i), b_n(i))$. As $|a_n(i) - b_n(i)| > \epsilon$ for all n, i, the measures ν_n does not accumulate on the diagonal, therefore the sequence ν_n has convergent subsequences in the weak* topology on $G(\mathbb{H}^3)$. Let ν be a limit with $\nu = \lim_{i\to\infty} \nu_{n_i}$.

We will show that ν is absolutely continuous with respect to μ_X. Let A be a set with $\mu_X(A) = 0$, then as $\mu_n \to \mu_X$

$$\lim_{n\to\infty} \mu_n(A) = \mu_X(A) = 0.$$

We compare μ_n and ν_n. Both are discrete measures and the support of ν_n is contained in the support of μ_n and with measures ν_n, μ_n having point masses $1/p_n, 1/l_n$ respectively. Therefore by Equation (6)

$$\nu_n(A) \leq \frac{l_n}{p_n} \mu_n(A) \leq C\mu_n(A).$$

Thus we have

$$\nu(A) = \lim_{i\to\infty} \nu_{n_i}(A) \leq \lim_{i\to\infty} C\mu_{n_i}(A) \leq C\mu_X(A) = 0.$$

Thus $\mu_X(A) = 0$ implies $\nu(A) = 0$. Thus ν is absolutely continuous with respect to μ_X. We take m_{n_i} to be the probability measures corresponding to the convergent sequence ν_{n_i}. By reducing to subsequence we can assume that μ_{n_i} converge to a probability measure m_f on $\mathbb{S}^1$. Then we have that m_f satisfies

$$m_f = \lim_{i\to\infty} m_{n_i}$$

and
$$\frac{m_f(\overline{\phi}_t)}{m_f(\overline{\phi}_0)} = \lim_{i\to\infty} \frac{m_{n_i}(\overline{\phi}_t)}{m_{n_i}(\overline{\phi}_0)} = \lim_{i\to\infty} L_{\mu_{n_i}}(X_t) = L_{\mu_X}(X_t).$$

Let $g_0\colon \mathbb{S}^2 \to \mathbb{S}^2$ be the quasiconformal homeomorphism conjugating Γ to Γ_0. Then by the definition of the Patterson–Sullivan geodesic current μ_X (see Equation (1)) we have

$$(d\mu_X)(a,b) = \pi_*\left(\frac{dm_X dm_X}{|g_0(a)-g_0(b)|^{2h(X)}}\right).$$

where m_X is the Patterson–Sullivan measure for Γ_0 and π is the $\mathbb{Z}_2$ cover $\pi\colon (\mathbb{S}^2 \times \mathbb{S}^2 - \text{diagonal}) \to G(\mathbb{H}^3)$. Therefore μ_X is absolutely continuous with respect to the measure $\pi_*(m_X \times m_X)$ on $G(\mathbb{H}^3)$.

Now we will show that m_f is absolutely continuous with respect to m_X. Let $m_X(A) = 0$. If $S \subset \mathbb{S}^2 \times \mathbb{S}^2$, we let $[S] = \pi(S - \text{diagonal})$. Then $[S]$ is precisely the set of (unoriented) geodesics in S. Then by definition of m_n we have that

$$m_n(A) = \nu_n([A \times \mathbb{S}^2]).$$

As $m_X(A) = 0$ then $(m_X \times m_X)(A \times \mathbb{S}^2) = (m_X \times m_X)(\mathbb{S}^2 \times A) = 0$. Therefore on $G(\mathbb{H}^3)$ we obtain

$$\begin{aligned}
\pi_*(m_X \times m_X)([A \times \mathbb{S}^2]) &= m_X \times m_X(\pi^{-1}([A \times \mathbb{S}^2])) \\
&= m_X \times m_X\Big(\big((A \times \mathbb{S}^2) \cup (\mathbb{S}^2 \times A)\big) - \text{diagonal}\Big) \\
&\le (m_X \times m_X)(A \times \mathbb{S}^2) + (m_X \times m_X)(\mathbb{S}^2 \times A) = 0.
\end{aligned}$$

Therefore $\pi_*(m_X \times m_X)([A \times \mathbb{S}^2]) = 0$ and as μ_X is absolutely continuous with respect to $\pi_*(m_X \times m_X)$, then $\mu_X([A \times \mathbb{S}^2]) = 0$. Then as ν is absolutely continuous with respect to μ_X we have $\nu([A \times \mathbb{S}^2]) = 0$. As $m_n(A) = \nu_n([A \times \mathbb{S}^2])$ then

$$m_f(A) = \lim_{i\to\infty} m_{n_i}(A) = \lim_{i\to\infty} \nu_{n_i}([A \times \mathbb{S}^2]) = \nu([A \times \mathbb{S}^2]) = 0.$$

Therefore the limit m_f must be absolutely continuous with respect to m_X. By Sullivan [16], the Patterson–Sullivan measure m_X is equal to the Hausdorff measure of dimension $h(X)$ on the limit set. Also by Bowen the pushforward $\overline{m} = (\pi_{\Gamma_0})_*(m)$ of the equilibrium measure m to $\mathbb{S}^1$ is equivalent to the Hausdorff measure of dimension $h(X)$ on the limit set and therefore equivalent to m_X [4, Lemma 10]. Therefore m_f is then absolutely continuous with respect to the measure $\overline{m}$. Also as the m_n are invariant under f_{Γ_0}, then the limit m_f is invariant under f_{Γ_0}. As $\bar{m}$ is ergodic, then m_f is also ergodic. But by the Ruelle-Perron-Frobenius Theorem (see Theorem 4.1), there is a unique f_{Γ_0} invariant ergodic probability measure. Thus $m_f = \bar{m}$. As $\phi_t = \overline{\phi}_t \circ \pi_\Gamma$, then $m(\phi_t) = \bar{m}(\overline{\phi}_t)$ and

$$F(t) = \frac{m(\phi_t)}{m(\phi_0)} = \frac{\bar{m}(\overline{\phi}_t)}{\bar{m}(\overline{\phi}_0)} = \frac{m_f(\overline{\phi}_t)}{m_f(\overline{\phi}_0)} = L_{\mu_X}(X_t). \qquad \square$$

7 Positive-definite locus for G

Before we prove the Main Theorem we characterize the zero vectors of G in terms of derivatives of length functions.

Theorem 7.1 *Let $v \in T_X(\mathrm{QF}(S))$ then $\|v\|_G = 0$ if and only if for every $\gamma \in \Gamma$, the associated length function $L_\gamma\colon \mathrm{QF}(S) \to \mathbb{R}$ satisfies*

$$(hL_\gamma)'(v) = 0.$$

Proof We choose our basegroup Γ to be the fuchsian group described in Lemma 5.2, ie if S is a genus g surface, then Γ is generated by the standard identification of the sides of the regular $4g$–gon.

We first prove that $\|v\|_G = 0$ implies that $(hL_\gamma)'(v) = 0$ for all $\gamma \in \Gamma$. As it is automatically true for $v = 0$, we assume that $v \neq 0$ and choose a smooth curve $\alpha\colon (-\epsilon, \epsilon) \to \mathrm{QF}(S)$ with $\alpha(0) = X$ and $\alpha'(0) = v$ and $\alpha(t) = X_t = [g_t, \Gamma_t]$ as before. Therefore g_t conjugates the action of Γ to the action of Γ_t.

We will show that $(hL_\mu)'(v) = 0$ for all discrete geodesic currents except for the finite set $\mathcal{S}$ considered in Lemma 5.2. Since these are projectively dense in $\mathcal{C}(S)$, and $(hL_\mu)'(v)$ is a continuous function that is homogeneous under scaling of currents, it will then follow that $(hL_\mu)'(v) = 0$ for all geodesic currents.

We note that if M is a loxodromic möbius transformation then the translation distance of M is given by $\log|M'(z)|$ where z is the expanding fixed point of M.

Thus if $\gamma \in \Gamma$, with expanding fixed point z then we let $\gamma_t = g_t \circ \gamma \circ g_t^{-1} \in \Gamma_t$. Then γ_t has expanding fixed point $z_t = g_t(z)$ and

$$L_\gamma(X_t) = \log|\gamma_t'(z_t)|. \tag{7}$$

As $\|v\|_G = 0$, then as G is conformally equivalent to W, $\|v\|_W = 0$. Therefore by Equation (5), $\mathrm{Var}(\dot{\Phi}_0, m) = 0$. But by nondegeneracy of the variance, this gives

$\dot{\Phi}_0 \sim 0$ and is a coboundary. Therefore there is a continuous function $u\colon \Sigma \to \mathbb{R}$ such that $\dot{\Phi}_0(x) = u(\sigma(x)) - u(x)$. Iterating we have

$$(S_n\dot{\Phi}_0)(x) = u(\sigma^n(x)) - u(x).$$

In particular if $\sigma^n(x) = x$ then $(S_n\dot{\Phi}_0)(x) = 0$.

We consider the set of commensurability classes in Γ given by $\mathcal{D} = \{[\gamma] \mid, [\gamma] \notin \mathcal{S}\}$, where $\mathcal{S}$ is the finite set defined in Lemma 5.2. Then the associated set of projective geodesic currents is dense in the space $\mathcal{PC}(S)$ of projective geodesic currents. We now show $(hL_\gamma)'(v) = 0$ for all $[\gamma] \in \mathcal{D}$.

We let $[\gamma] \in \mathcal{D}$. Then by Lemma 5.2, there is an element $\gamma' \in [\gamma]$ whose expanding fixed point z is good and a periodic point of f_Γ. Therefore there is an n such that $f_\Gamma^n(z) = z$. As z is a good point, we let $x \in \Sigma$ be the unique point such that $\pi_\Gamma(x) = z$. As $\pi_\Gamma \circ \sigma = f_\Gamma \circ \pi_\Gamma$, we have

$$\pi_\Gamma(\sigma^n(x)) = f_\Gamma^n(\pi_\Gamma(x)) = f_\Gamma^n(z) = z = \pi_\Gamma(x).$$

As z is a good point, x is the unique preimage of z under π_Γ. Therefore $\sigma^n(x) = x$.

As f_Γ is an expanding Markov map and z is a good point, there exists a $\gamma_z \in \Gamma$ with expanding fixed point z such that $f_\Gamma^n = \gamma_z$ in an open neighborhood of z. Thus elements γ' and γ_z have common fixed point z. As Γ is cocompact, this implies that γ' and γ_z are commensurate. Therefore by transitivity of commensurability, γ and γ_z are commensurate.

We let $\gamma_{z,t} = g_t \circ \gamma_z \circ g_t^{-1} \in \Gamma_t$. By definition $\pi_t(x) = g_t(\pi_\Gamma(x)) = g_t(z)$ and is therefore the expanding fixed point of $\gamma_{z,t}$. Also as $f_{\Gamma_t} = g_t \circ f_\Gamma \circ g_t^{-1}$ we have $f_{\Gamma_t}^n = g_t \circ f_\Gamma^n \circ g_t^{-1}$ and therefore $f_{\Gamma_t}^n = \gamma_{z,t}$ at $g_t(z)$. Thus by the iteration relation in Equation (3) we have

$$(S_n\phi_t)(x) = -\log|(f_{\Gamma_t}^n)'(\pi_t(x))| = -\log|\gamma_{z,t}'(z_t)| = -L_{\gamma_z}(X_t).$$

Also as $\Phi_t(x) = h(X_t)\phi_t(x)$, $(S_n\Phi_t)(x) = h(X_t)(S_n\phi_t)(x)$. Therefore

$$(S_n\dot{\Phi}_0)(x) = \frac{d}{dt}\left((S_n\Phi_t)(x)\right)\Big|_{t=0} = \frac{d}{dt}\left(-h(X_t)L_{\gamma_z}(X_t)\right)\Big|_{t=0} = -(hL_{\gamma_z})'(v).$$

As $(S_n\dot{\Phi}_0)(x) = 0$, we have that $(hL_{\gamma_z})'(v) = 0$. Therefore $(hL_\gamma)'(v) = 0$ for all $[\gamma] \in \mathcal{D}$.

We now prove that if v satisfies $(hL_\gamma)'(v) = 0$ for all $\gamma \in \Gamma$ then $\|v\|_G = 0$. As it is true for $v = 0$, we assume that $v \neq 0$ and as before, choose a smooth curve $\alpha\colon (-\epsilon, \epsilon) \to \mathrm{QF}(S)$ with $\alpha(0) = X$ and $\alpha'(0) = v$ and $\alpha(t) = X_t = [g_t, \Gamma_t]$.

A Theorem of Livsic that states $f \sim g$ if and only if $(S_n f)(x) = (S_n g)(x)$ whenever $\sigma^n(x) = x$ [8]. Therefore we let $\sigma^n(x) = x$. Then for $z = \pi_\Gamma(x)$ we have $f_\Gamma^n(z) = z$ and $f_\Gamma^n = \gamma_z$ at z for some $\gamma_z \in \Gamma$. As above we have $(S_n\phi_t)(x) = -L_{\gamma_z}(X_t)$ and

$$(S_n\dot{\Phi}_0)(x) = -(hL_{\gamma_z})'(v).$$

By the assumption $(hL_{\gamma_z})'(v) = 0$ and therefore $(S_n\dot{\Phi}_0)(x) = 0$. Therefore by the result of Livsic, $\dot{\Phi}_0 \sim 0$ and therefore $\mathrm{Var}(\dot{\Phi}_0, m) = 0$. It follows that $\|v\|_W = 0$. As G is conformally equivalent to W, then $\|v\|_G = 0$. □

Corollary 7.2 *If $\|v\|_W = 0$ then there is a $k \in \mathbb{R}$ such that*

$$L'_\mu(v) = kL_\mu(X) \text{ for all } \mu \in C(S).$$

Proof This follows immediately with $k = -h'(v)/h(X)$ by applying the product rule to $(hL_\mu)'(v) = 0$. □

We now show that pure bending vectors to the fuchsian locus have zero length.

Lemma 7.3 *If $v = J.w$ where $w \in T_X(F(S))$ then $\|v\|_G = 0$.*

Proof Let $v = J.w$ where $w \in T_X(F(S))$. By Theorem 7.1, to show $\|v\|_G = 0$ we only need to prove that $(hL_\mu)'(v) = 0$ for all $\mu \in \mathcal{C}(S)$.

As the complex length functions $\mathcal{L}_g$ are holomorphic on $\mathrm{QF}(S)$,

$$\mathcal{L}'_g(v) = \mathcal{L}'_g(J.w) = i\mathcal{L}'_g(w).$$

As $w \in T_X(F(S))$, $\mathcal{L}'_g(w)$ is real and equal $\mathcal{L}'_g(w) = L'_g(w)$. Therefore $\mathcal{L}'_g(v) = iL'_g(w)$ is purely imaginary giving

$$L'_g(v) = \Re(iL'_g(w)) = 0.$$

Thus $L'_g(v) = 0$ for all $g \in \Gamma$. As h is minimum on the fuchsian locus $F(S)$ then $h'(v) = 0$ and

$$(hL_g)'(v) = h'(v)L_g(X) + h(X)L'_g(v) = 0.$$

Therefore by Theorem 7.1, $\|v\|_W = 0$. As G is conformally equivalent to W we therefore have $\|v\|_G = 0$. □

To complete the proof of the Main Theorem, we need to prove the converse of the above Lemma, that pure bending vectors to the fuchsian locus are the only zero length vectors. We do this by showing that the condition that there exists a $k \in \mathbb{R}$ such that $(hL_\mu)'(v) = kL_\mu(X)$ for all $\mu \in \mathcal{C}(S)$ is sufficient to show that X is fuchsian and

v is a pure bending vector. This will require a technical analysis of two-generator subgroups of Γ.

We let S be a closed hyperbolic surface with $S = \mathbb{H}^2/\Gamma$ as before and let $g \in \Gamma$. Given any $X = [f_0, \Gamma_0] \in \mathrm{QF}(S)$, then g can be identified to a unique element $g(\Gamma_0) = f_0 \circ g \circ f_0^{-1} \in \Gamma_0 \subseteq \mathrm{PSL}(2,\mathbb{C})$. We can conjugate such that $g(\Gamma_0)$ is of the form

$$\pm \begin{pmatrix} \lambda_g(X) & 0 \\ 0 & \lambda_g^{-1}(X) \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}), \text{ where } |\lambda_g(X)| > 1.$$

We note that λ_g is well-defined up to sign and $\lambda_g^2(X)$ is therefore well-defined.

Therefore the element $g(\Gamma_0)$ is conjugate to the fractional linear map $f(z) = c.z$, where $c = \lambda_g^2(X)$. Therefore we have that the length function $L_g \colon \mathrm{QF}(S) \to \mathbb{R}$ is given by $L_g(X) = 2\log|\lambda_g(X)|$. Also the holomorphic length function $\mathcal{L}_g \colon \mathrm{QF}(S) \to \mathbb{C}$ satisfies $L_g = \Re(\mathcal{L}_g)$ and $\lambda_g^2 = e^{\mathcal{L}_g}$.

Let $X \colon (-\epsilon,\epsilon) \to \mathrm{QF}(S)$ be a smooth curve such that $X'(0) = v$. We let $X(t) = [f_t, \Gamma_t]$. Let Γ_t be a smooth parameterization. Therefore for $g \in \Gamma_0$, the map $\gamma_g \colon (-\epsilon,\epsilon) \to \mathrm{PSL}(2,\mathbb{C})$ defined by $\gamma_g(t) = g(\Gamma_t)$ is a smooth function. Also as $g(\Gamma_t) \in \mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\pm I$, we can lift γ_g to a smooth map $\widetilde{\gamma}_g \colon (-\epsilon,\epsilon) \to \mathrm{SL}(2,\mathbb{C})$.

We then can define $\lambda_g \colon (-\epsilon,\epsilon) \to \mathbb{C}$ by letting $\lambda_g(t)$ equal the largest eigenvalue of $\widetilde{\gamma}_g(t)$. Furthermore we define the trace functions

$$t_g(t) = \mathrm{tr}(\widetilde{\gamma}_g(t)) = \lambda_g(t) + \lambda_g^{-1}(t).$$

Lemma 7.4 *Let $v \in T_X(\mathrm{QF}(S))$, $v \neq 0$. If there exists a $k \in \mathbb{R}$ such that*

$$L'_g(v) = kL_g(X) \text{ for all } g \in \Gamma$$

then λ_g^2 and t_g^2 are both real and

$$\Re\left(\frac{\lambda'_g}{\lambda_g}\right) = 0$$

for all $g \in \Gamma$.

Proof As trace functions are holomorphic coordinate function for $\mathrm{QF}(S)$ (see Marden [9]), as $v \neq 0$, there exists $\alpha \in \Gamma$ such that $t'_\alpha(0) \neq 0$. As

$$t'_g = \lambda'_g - \frac{\lambda'_g}{\lambda_g^2} = \lambda'_g\left(\frac{\lambda_g^2 - 1}{\lambda_g^2}\right)$$

then $\lambda'_\alpha(0) \neq 0$.

As Γ is nonelementary, we can choose a $\beta\in\Gamma$ such that α,β do not have the same axis. We note that α,β have the same axes if and only if there exist $n,m\in\mathbb{Z}$, both nonzero, such that $\alpha^n=\beta^m$.

By conjugation of Γ_t we can put $\alpha(\Gamma_t)$ in the diagonal form with

$$A(t)=\tilde{\gamma}_\alpha(t)=\begin{pmatrix}\lambda_\alpha(t) & 0\\ 0 & \lambda_\alpha^{-1}(t)\end{pmatrix}$$

where $|\lambda_\alpha(t)|>1$, and the corresponding matrix for $\beta(\Gamma_t)$ is

$$B(t)=\tilde{\gamma}_\beta(t)=\begin{pmatrix}a(t) & b(t)\\ c(t) & d(t)\end{pmatrix}$$

where $a(t)d(t)-b(t)c(t)=1$.

We consider the two-generator subgroup $G_t=\langle A(t),B(t)\rangle\subseteq \mathrm{SL}(2,\mathbb{C})$ acting on the upper half space model of $\mathbb{H}^3$. Then $A(t)$ fixes $0,\infty$ and preserves the z–axis. Since α and β have different axes and Γ contains no parabolics, it follows that $a(t),d(t)$ are both nonzero.

If $\|v\|_W=0$, then by Corollary 7.2, $L'_g(v)=kL_g(X)$ for all $g\in\Gamma$. As $L_g=\log|\lambda_g|$, we obtain the equation

$$(\log|\lambda_g|)'(v)=k\log|\lambda_g(X)|. \tag{8}$$

As we are only interested in derivatives at 0 for $X(t)$, we will adopt the notation $f'=f'(0)$.

Therefore for $g=\alpha$ we have $(\log|\lambda_\alpha|)'=k\log|\lambda_\alpha|$ or equivalently

$$(\log|\lambda_\alpha|)'=\frac{|\lambda_\alpha|'}{|\lambda_\alpha|}=k\log|\lambda_\alpha|. \tag{9}$$

Now we consider the element $C_n=A^nB$ with matrix

$$C_n=\begin{pmatrix}\lambda_\alpha^n a & \lambda_\alpha^n b\\ \lambda_\alpha^{-n}c & \lambda_\alpha^{-n}d\end{pmatrix}.$$

Let μ_n,μ_n^{-1} be the eigenvalues of C_n, with $|\mu_n|>1$. Solving for μ_n we have for large n that

$$\begin{aligned}\mu_n&=\frac{\lambda_\alpha^n a+\lambda_\alpha^{-n}d+\sqrt{(\lambda_\alpha^n a+\lambda_\alpha^{-n}d)^2-4}}{2}\\ &=\lambda_\alpha^n a\left(1+\lambda_\alpha^{-2n}\left(\frac{ad-1}{a^2}\right)+O(\lambda_\alpha^{-4n})\right)\end{aligned} \tag{10}$$

Let g_n be the element of Γ corresponding to C_n; then we have

$$\begin{aligned}\log|\lambda_{g_n}| &= n\log|\lambda_\alpha| + \log|a| + \log\left|1+\lambda_\alpha^{-2n}\left(\frac{ad-1}{a^2}\right)+O(\lambda_\alpha^{-4n})\right|.\\ &= n\log|\lambda_\alpha| + \log|a| + \Re\left(\lambda_\alpha^{-2n}\left(\frac{ad-1}{a^2}\right)\right)+O(|\lambda_\alpha|^{-4n}).\end{aligned}$$

Then differentiating and using that $(\log|\lambda_{g_n}|)' = k\log|\lambda_{g_n}|$ we obtain

$$\begin{aligned}0 = n\left(\frac{|\lambda_\alpha|'}{|\lambda_\alpha|} - k\log|\lambda_\alpha|\right) + \left(\frac{|a|'}{|a|} - k\log|a|\right) + \Re\left(-2n\lambda_\alpha^{-2n-1}\lambda_\alpha'\left(\frac{ad-1}{a^2}\right)\right)\\ + \Re\left(\lambda_\alpha^{-2n}\left(\left(\frac{ad-1}{a^2}\right)' - k\left(\frac{ad-1}{a^2}\right)\right)\right) + O(|\lambda_\alpha|^{-4n}).\end{aligned} \tag{11}$$

The first term in the equation above is zero by Equation (9), and the last two go to zero as $n\to\infty$, so taking the limit gives

$$\frac{|a|'}{|a|} = k\log|a|. \tag{12}$$

Therefore Equation (11) becomes a relation between the real parts of the last two terms, which holds for each n. Multiplying by $|\lambda_\alpha|^{2n}/n$ and taking the limit as $n\to\infty$ gives

$$\lim_{n\to\infty}\Re\left(\left(\frac{\lambda_\alpha}{|\lambda_\alpha|}\right)^{-2n}\left(\frac{\lambda_\alpha'}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = 0 \tag{13}$$

We let
$$u = \left(\frac{\lambda_\alpha}{|\lambda_\alpha|}\right)^2.$$

As we can always choose a sequence n_i such that $\lim_{i\to\infty} u^{-n_i} = 1$, we have that

$$\lim_{i\to\infty}\Re\left(\left(\frac{\lambda_\alpha}{|\lambda_\alpha|}\right)^{-2n_i}\left(\frac{\lambda_\alpha'}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = \Re\left(\left(\frac{\lambda_\alpha'}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = 0.$$

Therefore we obtain the equation

$$\Re\left(\left(\frac{\lambda_\alpha'}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = 0. \tag{14}$$

Next we will show that λ_α^2 is real; suppose on the contrary that $u = e^{\pi i\theta}$ for $\theta\in(0,2)$, $\theta\neq 1$. We will derive a contradiction from this assumption by considering two cases.

Case 1 θ is irrational.

If θ is irrational, then we can choose a sequence m_i such that $\lim_{i\to\infty} u^{-m_i} = i$. Then

$$\lim_{i\to\infty} \Re\left(\left(\frac{\lambda_\alpha}{|\lambda_\alpha|}\right)^{-2m_i}\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = \Im\left(\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = 0.$$

Thus both the real and imaginary parts are zero giving

$$\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right) = 0.$$

As $\lambda'_\alpha \neq 0$ we have $ad = 1$. Therefore as $ad - bc = 1$, we have $bc = 0$ and either $b = 0$ or $c = 0$. If $b = 0$, then α, β have common fixed point 0 and if $c = 0$, then α, β have common fixed point ∞. However β was chosen so that it does not share any fixed point with α, so this is a contradiction.

Case 2 θ is positive rational but not an integer.

We let $\theta = p/q$, where $q > 1$ and p, q have no common divisors. Then $u^q = 1$ and $u^{nq+1} = u$. Then let $n_i = i.q - 1$. Then $u^{-n_i} = u$. Thus

$$\lim_{i\to\infty} \Re\left(\left(\frac{\lambda_\alpha}{|\lambda_\alpha|}\right)^{-2n_i}\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = \Re\left(u\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = 0.$$

Let $u = x + iy$ where $y \neq 0$. Then

$$\Re\left(u\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = x\Re\left(\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) - y\Im\left(\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = 0.$$

Therefore by Equation (14), we have

$$y\Im\left(\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = 0.$$

As $y \neq 0$ we obtain the conclusion that both real and imaginary parts are zero giving

$$\left(\frac{\lambda'_\alpha}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right) = 0.$$

This leads to the same contradiction as Case 1.

Thus we conclude that λ_α^2 is real and λ_α is either purely imaginary or purely real. As $t_\alpha = \lambda_\alpha + \lambda_\alpha^{-1}$, then t_α is similarly either purely imaginary or purely real and t_α^2 is real. Therefore we have shown that if $t'_g \neq 0$ then t_g^2 is real.

Also as $t_n = \lambda_\alpha^n . a + \lambda^n . d$ then

$$t_n' = n\lambda_\alpha^{n-1}\lambda_\alpha' a + \lambda_\alpha^n a' - n\lambda_\alpha^{-n-1}\lambda_\alpha' a + \lambda_\alpha^{-n} a'.$$

Thus
$$\lim_{n\to\infty}\left(\frac{t_n'}{n\lambda_\alpha^n}\right) = \frac{\lambda_\alpha'}{\lambda_\alpha} a$$

and therefore for large n, $t_n' \neq 0$. Choose n_0 such that $t_n' \neq 0$ for $n > n_0$.

We let $n > n_0$. By the above, t_n^2 is real and

$$t_n^2 = (\lambda_\alpha^n a + \lambda_\alpha^{-n} d)^2 = \lambda_\alpha^{2n} a^2 + 2ad + \lambda_\alpha^{-2n} d^2.$$

As t_n^2 is real and λ_α^2 is real, we have

$$\Im(t_n^2) = 0 = \lambda_\alpha^{2n}\Im(a^2) + 2\Im(ad) + \lambda_\alpha^{-2n}\Im(d^2).$$

Taking limits we have

$$\lim_{n\to\infty}\frac{\Im(t_n^2)}{\lambda_\alpha^{2n}} = \Im(a^2) = 0.$$

Therefore
$$\lim_{n\to\infty}\Im(t_n^2) = 2\Im(ad) = 0$$

and finally
$$\lim_{n\to\infty}(\lambda_\alpha^{2n}\Im(t_n^2)) = \Im(d^2) = 0.$$

Thus a^2, d^2, ad are all real. Applying this to Equation (14) we have

$$\Re\left(\left(\frac{\lambda_\alpha'}{\lambda_\alpha}\right)\left(\frac{ad-1}{a^2}\right)\right) = \left(\frac{ad-1}{a^2}\right)\Re\left(\frac{\lambda_\alpha'}{\lambda_\alpha}\right) = 0.$$

Therefore we have

$$\Re\left(\frac{\lambda_\alpha'}{\lambda_\alpha}\right) = 0.$$

As the only assumption on α was that t_α' and therefore λ_α' was nonzero, we have

$$\Re\left(\frac{\lambda_g'}{\lambda_g}\right) = 0 \text{ for all } g \in \Gamma.$$

Also as $t_\beta^2 = (a+d)^2 = a^2 + 2ad + d^2$, then we have that t_β^2 is real. As β was arbitrarily chosen, we therefore have that t_g^2 is real for all $g \in \Gamma$. As t_g^2 is real, then λ_g^2 is also real. □

Lemma 7.5 *If $v \in T_X(\mathrm{QF}(S))$ and there exists a $k \in \mathbb{R}$ such that $L_g'(v) = k . L_g(X)$ then $k = 0$.*

Proof If $v = 0$ then $L'_g(v) = 0$ and obviously $k = 0$. Therefore we assume $v \neq 0$. Let $g \in \Gamma$, and $\lambda_g = |\lambda_g| e^{i\theta}$. Then

$$\frac{\lambda'_g}{\lambda_g} = \frac{|\lambda_g|' e^{i\theta} + |\lambda_g| e^{i\theta} i\theta'}{|\lambda_g| e^{i\theta}} = \frac{|\lambda_g|'}{|\lambda_g|} + i\theta'.$$

Then by Equation (9) and Lemma 7.4,

$$0 = \Re\left(\frac{\lambda'_g}{\lambda_g}\right) = \frac{|\lambda_g|'}{|\lambda_g|} = k \log |\lambda_g|.$$

Thus we have $k.\log|\lambda_g| = 0$. As $|\lambda_g| > 1$, $\log|\lambda_g| \neq 0$ and therefore $k = 0$. □

We now are ready to prove the Main Theorem.

Proof of Main Theorem As Lemma 7.3 proves one direction of the Main Theorem, we only need to prove if $v \in T_X(\mathrm{QF}(S)), v \neq 0$, and $\|v\|_G = 0$ then $X \in F(S)$ and $v = J.w$ for some $w \in T_X(F(S))$.

Let $v \in T_X(\mathrm{QF}(S)), v \neq 0$, and $\|v\|_G = 0$. As W is a multiple of G, $\|v\|_W = 0$. Then by Corollary 7.2, there is a $k \in \mathbb{R}$ such that $L'_g(v) = kL_g(X)$ for all $g \in \Gamma$. Then by Lemma 7.5, $k = 0$ giving $L'_g(v) = 0$ for all $g \in \Gamma$.

We pick α, β as in Lemma 7.4. For the group $G_0 = \langle A(0), B(0)\rangle$ we have t_α^2, λ_α^2, a^2, ad, d^2 are all real. Therefore the fractional linear map given by A is $f_A(z) = \lambda_\alpha^2 . z \in \mathrm{PSL}(2,\mathbb{R})$.

As $ad - bc = 1$, we therefore have that $bc = ad - 1$ is real. Therefore $b = re^{i\theta}$ and $c = se^{-i\theta}$ where r, s are real.

If a, d are both real, we conjugate G_0 by rotation R about the axis of A by angle θ. Then as R, A commute, $RAR^{-1} = A$ and

$$RBR^{-1} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} = \begin{pmatrix} a & r \\ s & d \end{pmatrix}.$$

Therefore the fractional linear map given by RBR^{-1} is in $\mathrm{PSL}(2,\mathbb{R})$.

If a, d are both imaginary, we conjugate by a rotation R about the axis of A by angle $\pi + \theta$. Then

$$RBR^{-1} = \begin{pmatrix} a & ir \\ is & d \end{pmatrix}.$$

Thus as each entry is imaginary, the fractional linear map is in $\mathrm{PSL}(2,\mathbb{R})$.

Therefore we have conjugated G_0 to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Thus G_0 has limit set contained in a Euclidean line L_0 through the origin and G_0 preserves a hyperbolic plane H_0 containing the axis of A. We conclude that if $\alpha \in \Gamma_0$ has $\lambda'_\alpha \neq 0$ then for any $\beta \in \Gamma_0$, the axes of α and β are contained in the same geometric circle.

If $X \notin F(S)$ then there is an element $\gamma \in \Gamma$ such that the associated fractional linear map $C \in \Gamma_0$ does not preserve H_0. Then we have as before that the group $G_1 = \langle A, C\rangle$ can be conjugated to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Therefore Γ_1 preserves a line L_1, and hyperbolic plane H_1 containing the axis of A. As by assumption C does not preserve H_0 then $H_0 \neq H_1$ and therefore $L_1 \neq L_0$. Thus $L_1 \cap L_0 = \{0,\infty\}$. By conjugation, we assume that L_0 is the real axis.

We note that if g, h are loxodromic hyperbolic elements, then the axis of ghg^{-1} is the image of the axis of h under g.

Thus we conjugate α by $\beta\alpha^n$ to get

$$\alpha_n = (\beta\alpha^n)^{-1}\alpha(\beta\alpha^n) = \alpha^{-n}(\beta^{-1}\alpha\beta)\alpha^n.$$

Then as α_n is a conjugate of α we have $\lambda'_{\alpha_n} = \lambda'_\alpha \neq 0$. Let $A_n = (BA^n)^{-1})A(BA^n) = A^{-n}(B^{-1}AB)A^n \in \Gamma_0$. Then the endpoints of the axes of A_n and C must be contained in a geometric circle. Also the axis of A_n is the image of the axis of BAB^{-1} under A^{-n}. Therefore we let a, b be the endpoints of the axis of $B^{-1}AB$. As A, B are noncommensurate, their axes do not have common endpoints. Therefore $a, b \in \mathbb{R}$ and are not equal 0 or ∞. Then the endpoint of the axis of A_n are a_n, b_n where $a_n = A^{-n}(a) = \lambda_\alpha^{-2n}a, b_n = A^{-n}(b) = \lambda_\alpha^{-2n}b$.

Let $z, w \in L_1$ be the endpoints of the axis of C. As L_1 is not the real axis, then $z = re^{i\theta}, w = se^{i\theta}$ where $r, s \in \mathbb{R}$, and $e^{i\theta}$ is not real. As the axes of C and A_n are on the same geometric circle, the cross ratio $(a_n, z; b_n, w)$ is real for all n:

$$(a_n, z; b_n, w) = \frac{(a_n - b_n)(z-w)}{(a_n - w)(z - b_n)} = \frac{\lambda_\alpha^{-2n}(a-b)(r-s)e^{i\theta}}{(\lambda_\alpha^{-2n}a - se^{i\theta})(re^{i\theta} - \lambda_\alpha^{-2n}b)}$$

Therefore as $\Im(a_n, z; b_n, w) = 0$ and λ_α^2 is real then

$$\begin{aligned} 0 = \lim_{n\to\infty} \lambda_\alpha^{2n}.\Im(a_n, z; b_n, w) &= \lim_{n\to\infty} \Im\left(\frac{(a-b)(r-s)e^{i\theta}}{(\lambda_\alpha^{-2n}a - se^{i\theta})(re^{i\theta} - \lambda_\alpha^{-2n}b)}\right) \\ &= \Im\left(\frac{(a-b)(r-s)e^{i\theta}}{-sre^{2i\theta}}\right) = \frac{(a-b)(r-s)}{-rs}\Im(e^{-i\theta}). \end{aligned}$$

Thus $\Im(e^{-i\theta}) = 0$, and therefore $e^{i\theta}$ is real. But by assumption $e^{i\theta}$ is not real, which gives us our contradiction. Thus $X \in F(S)$.

Finally as $X \in F(S)$, we have the decomposition [6]

$$T_X(\mathrm{QF}(S)) = T_X(F(S)) \oplus J(T_X(F(S))).$$

If $v \in T_X(F(S))$ then $\mathcal{L}'_\alpha(v) = L'_\alpha(v)$ and is real. Therefore if $v \in T_X(\mathrm{QF}(S))$, then $v = v_1 + J.v_2$ where $v_i \in T_X(F(S))$. Therefore

$$\begin{aligned} L'_g(v) = \Re(\mathcal{L}'_g(v)) &= \Re(\mathcal{L}'_g(v_1) + \mathcal{L}'_g(J.v_2)) \\ &= \Re(\mathcal{L}'_g(v_1) + i\mathcal{L}'_g(v_2)) = \Re(L'_g(v_1) + iL'_g(v_2)) = L'_g(v_1). \end{aligned}$$

Therefore if $L'_g(v) = 0$ for all $g \in \Gamma$, then $L'_g(v) = L'_g(v_1) = 0$ for all $g \in \Gamma$. But this implies that $v_1 = 0$ and therefore $v = J.v_2$. Thus $X \in F(S)$ and $v = J.w$ for some $w \in T_X(F(S))$. □

8 Critical points of Hausdorff dimension

We will now use the description of the positive definite locus of G to obtain information about the critical points of $h\colon \mathrm{QF}(S) \to \mathbb{R}$.

If $f\colon X \to \mathbb{R}$ is a smooth map, then $x \in X$ is a critical point of the differential $f'(x)\colon T_x(X) \to \mathbb{R}$ is the trivial linear function.

If x is a critical point of f then the Hessian of f is a well-defined two-form which we label $f''(x)$. Then we can decompose $T_x(X) = V_+ \oplus V_0 \oplus V_-$ where the subspaces are mutually orthogonal and $f''(x)$ is positive-definite (respectively, zero, negative-definite) on V_+ (resp. V_0, V_-). The positive definite (respectively zero, negative-definite) dimension of $f''(x)$ is the dimension of V_+ (resp. V_0, V_-).

As $h \geq 1$ and $h = 1$ on the fuchsian subspace $F(S)$ it follows that each h is minimum (and therefore critical) at each point of $F(S)$. Thus for $X \in F(S)$, $h''(X)$ has negative definite dimension zero and trivial dimension at least $\dim(F(S)) = 6g - 6$. In [6], we show that $h''(X)$ has positive definite dimension $6g-6$. We generalize this to all critical points of h to prove Theorem 1.2.

Theorem 1.2 *If $X \in \mathrm{QF}(S)$ is a critical point of $h\colon \mathrm{QF}(S) \to \mathbb{R}$ then X has positive definite dimension at least $6g-6$. In particular h has no local maxima.*

Proof As the Theorem is true for $X \in F(S)$ [6], we assume that $X \notin F(S)$. By [5], if μ_X is the unit Patterson–Sullivan geodesic current for $X \in \mathrm{QF}(S)$ then the real valued function $Y \to h(Y)L_{\mu_X}(Y)$ on $\mathrm{QF}(S)$ is minimum at X. Therefore $(hL_{\mu_X})'(X) = 0$.

If X is a critical point of h then $h'(X) = 0$ and therefore by the product rule

$$h'(X)L_{\mu_X}(X) + h(X)L'_{\mu_X}(X) = h(X)L'_{\mu_X}(X) = 0.$$

As $h(X) \neq 0$ then $L'_{\mu_X}(X) = 0$ and therefore L_{μ_X} has a critical point at X. We note that the holomorphic length function $\mathcal{L}_{\mu_X}$ satisfies

$$\Re(\mathcal{L}_{\mu_X}) = L_{\mu_X}.$$

Therefore as $L'_{\mu_X}(X) = 0$, then for all $v \in T_X(\mathrm{QF}(S))$,

$$\Re(\mathcal{L}'_{\mu_X}(v)) = L'_{\mu_X}(v) = 0.$$

Therefore applying this to $J.v$ we have

$$0 = \Re(\mathcal{L}'_{\mu_X}(J.v)) = \Re(i\mathcal{L}'_{\mu_X}(v)) = -\Im(\mathcal{L}'_{\mu_X}(v)).$$

Thus $\mathcal{L}'_{\mu_X}(v)$ has real and imaginary part zero and therefore $\mathcal{L}'_{\mu_X}(v) = 0$ for all $v \in T_X(\mathrm{QF}(S))$. Thus $\mathcal{L}'_{\mu_X}(X) = 0$ and we have a well-defined complex bilinear 2–form $\mathcal{L}''_{\mu_X}(X)$.

As the two-form G_X is given by $G_X = (hL_{\mu_X})''(X)$ and both h, L_{μ_X} are critical at X we have

$$G_X = h''(X)L_\mu(X) + 2h'(X)L'_{\mu_X}(X) + h(X)L''_\mu(X) = h''(X) + h(X)L''_{\mu_X}(X).$$

We write $T_X(\mathrm{QF}(S)) = V_+ \oplus V_0 \oplus V_-$ where L''_{μ_X} is positive-definite (respectively, zero, negative-definite) on V_+ (resp. V_0, V_-). Since L''_{μ_X} is the real part of a complex bilinear form, the complex structure J is an isomorphism from V_+ to V_-, and each of these has dimension at most $(6g-6)$. Therefore $\dim(V_- \oplus V_0) \geq 6g-6$.

As $X \notin F(S)$ then G_X is positive definite. As $G_X = h''(X) + h(X)L''_{\mu_X}(X)$, then $h''(X)$ must be positive-definite on $V_0 \oplus V_-$. Therefore h has positive definite dimension at least $6g-6$ at X. □

We now give a proof of McMullen's result (Theorem 1.4) that the Weil–Petersson and Hausdorff norms are equal on $J(TF(S))$.

Proof If $w = J.v$ for $v \in T_X(F(S))$ then by the Main Theorem we have $\|w\|_G = 0$. As $h(X) = 1$, and by holomorphicity, $L''_{\mu_X}(J.v, J.v) = -L''_{\mu_X}(v)$ we have

$$0 = \|w\|_G^2 = h''(J.v, J.v) + h(X)\,.\,L''_{\mu_X}(J.v, J.v) = \|v\|_H^2 - L''_{\mu_X}(v, v).$$

Thus
$$\|v\|_H^2 = L''_{\mu_X}(v, v).$$

In [19], Wolpert showed that $L''_{\mu_X}(v, v) = \|v\|_g$ and the Theorem follows. □

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